学位論文

A bound for the least Gaussian prime ω with $\alpha < \arg(\omega) < \beta$

($\alpha < \arg(\omega) < \beta$ を満たすノルム最小のガウス素数の評価)

松井 一

主論文

A bound for the least Gaussian prime ω with $\alpha < \arg(\omega) < \beta$
($\alpha < \arg(\omega) < \beta$ を満たすノルム最小のガウス素数の評価)

to appear in Archiv der Mathematik.

A bound for the least Gaussian prime ω with $\alpha < \arg(\omega) < \beta$

By

HAJIME MATSUI

Abstract

We give an explicit function $B(\theta)$ such that there is a Gaussian prime ω with $\omega\overline{\omega} < B(\beta - \alpha)$ and $\alpha < \arg(\omega) < \beta$.

1. Introduction. In the present paper, we consider the following problem; for given (α, β) with $\alpha < \beta \leq \alpha + \frac{\pi}{2}$, estimate the minimum of norms of Gaussian primes whose arguments are within (α, β). (An element $\omega \in \mathbf{Z}[i]$ is called a Gaussian prime if $(\omega) = \omega\mathbf{Z}[i]$ is a prime ideal of $\mathbf{Z}[i]$.) We can give an answer for the problem under "GRH."

Theorem 1. *Assume the truth of the Generalized Riemann Hypothesis for* $L(s, \psi^r) = \frac{1}{4}\sum_a \frac{\psi^r((a))}{|a|^{2s}}$ *with* $\psi^r((a)) = \exp(4ir\arg(a))$ *and* $r \in \mathbf{Z}$, *where* a *runs over non-zero elements in* $\mathbf{Z}[i]$. *Then for any real numbers* α, β, *with* $\alpha < \beta \leq \alpha + \frac{\pi}{2}$, *there exists a Gaussian prime* ω *with* $\alpha < \arg(\omega) < \beta$ *such that*

$$\omega\overline{\omega} < \frac{A_1}{(\beta-\alpha)^2}\log^4\frac{1}{\beta-\alpha},$$

where A_1 *is a positive absolute constant.*

The proof of Theorem 1 employs classical analytic methods for the Hecke L-functions with Grössencharacters, using a special integral kernel in [2]. Moreover, we make use of certain trigonometric polynomials in [4], [5], which are majorants or minorants of the characteristic function of interval (α, β) on the unit circle.

Next, we consider whether one can say something without GRH.

Theorem 2. *For any real numbers* α, β, *with* $\alpha < \beta \leq \alpha + \frac{\pi}{2}$, *there exists a Gaussian prime* ω *with* $\alpha < \arg(\omega) < \beta$ *such that*

$$\omega\overline{\omega} < \exp\left(\frac{A_2}{\sqrt{\beta-\alpha}}\log^{\frac{3}{2}}\frac{1}{\beta-\alpha}\right),$$

where A_2 *is a positive absolute constant.*

Mathematics Subject Classification(1991): 11R44.

As a matter of fact, we can get similar results for any imaginary quadratic field (Theorem 3, 4, in the text).

2. Hecke L-functions and a special integral kernel. First we summarize Hecke's results which are used in this paper. From now on, we argue about not only the Gaussian field but also imaginary quadratic number fields. Let $\mathbf{Q}\left(\sqrt{-d}\right)$ be an imaginary quadratic field, and $-d$ its discriminant. Let χ be a Grössencharacter of conductor (1) such that

$$\chi((a)) = \left(\frac{a}{|a|}\right)^u \quad \text{for all} \quad a \in \mathbf{Q}\left(\sqrt{-d}\right)^\times$$

with an integer u. If $u=0$, then χ is a character of the ideal class group. We define for a complex number with $\mathrm{Re}(s) > 1$

$$L(s,\chi) = \sum_I \frac{\chi(I)}{NI^s} = \prod_P \left(1-\chi(P)NP^{-s}\right)^{-1},$$

where I runs over all integral ideals, P runs over all prime ideals and NI is the norm of I. We set

$$\Lambda(s,\chi) = \left\{s(1-s)\right\}^{\delta_\chi}\left(\frac{\sqrt{d}}{2\pi}\right)^s \Gamma\left(s+\frac{|u|}{2}\right)L(s,\chi),$$

where $\delta_\chi = 1$ if $\chi \equiv 1$, and 0 otherwise. Hecke showed that $\Lambda(s,\chi)$ is analytically continued to an entire function on the whole s-plane, and satisfies the functional equation

$$\Lambda(s,\chi) = i^{-|u|}W(\chi)\Lambda(1-s,\chi^{-1}),$$

where $W(\chi)$ is the Gaussian sum of χ, and $|W(\chi)| = 1$.

Our first aim is to compute the following integral in two ways. We set

$$I_\chi = \frac{1}{2\pi i}\int_{(2)} -\frac{L'}{L}(s,\chi)k(s;x,y)ds$$

with $y > x > 1$, where $\int_{(2)} = \lim_{T\to+\infty}\int_{2-iT}^{2+iT}$ and

$$k(s;x,y) = k(s) = \left(\frac{y^{s-1}-x^{s-1}}{s-1}\right)^2,$$

which is one of the integral kernels used in [2]. The inverse Mellin transform of $k(s)$ is given for $v > 0$ by

$$\widehat{k}(v;x,y) = \widehat{k}(v) = \frac{1}{2\pi i}\int_{(2)} k(s;x,y)v^{-s}ds = \begin{cases} 0 & \text{if } v \le x^2, \\ \frac{1}{v}\log\frac{v}{x^2} & \text{if } x^2 \le v \le xy, \\ \frac{1}{v}\log\frac{y^2}{v} & \text{if } xy \le v \le y^2, \\ 0 & \text{if } y^2 \le v. \end{cases}$$

One method to compute I_χ is based on the logarithmic derivative formula of L-function:

$$-\frac{L'}{L}(s,\chi) = \sum_P \sum_{n=1}^{\infty} \chi(P^n) \log NP \cdot NP^{-ns}.$$

Thus we have

$$I_\chi = \sum_P \sum_{n=1}^{\infty} \chi(P^n) \log NP \cdot \widehat{k}(NP^n; x, y),$$

and this is a finite sum by the above property of $\widehat{k}$. Here the sum over those ideals P^n for which NP^n is not a rational prime is evaluated in [2]; For $y > x \geq 2$,

$$\sum_{NP^n:\,\text{not prime}} \chi(P^n) \log NP \cdot \widehat{k}(NP^n; x, y) \ll \log y \cdot \log \frac{y}{x} \cdot (x \log x)^{-1}.$$

Hence we have

$$I_\chi = \sum_{NP:\,\text{prime}} \chi(P) \log NP \cdot \widehat{k}(NP; x, y) + \mathrm{O}\Big(\log y \cdot \log \frac{y}{x} \cdot (x \log x)^{-1}\Big). \tag{1}$$

Next, using Cauchy's theorem, we can show

$$I_\chi = \delta_\chi k(1; x, y) + \sum_\rho -k(\rho; x, y), \tag{2}$$

where ρ runs over all zeros of $L(s,\chi)$. The case in which χ is a character of the ideal class group is shown in [2], and our case is proved by the similar methods. To estimate the right-hand side of (2), we first compute the sum over trivial zeros, that is, the zeros where $\mathrm{Re}(s) = \beta < 0$. Since $\Lambda(s,\chi)$ is an entire function, $L(s,\chi)$ has simple zeros at the following points:

$$\begin{array}{ll} \rho = -\frac{|u|}{2} - n, \quad n = 0, 1, 2, \ldots & \text{if} \quad u \neq 0, \\ \rho = -1, -2, -3, \ldots & \text{if} \quad u = 0. \end{array}$$

Hence we obtain

$$\left| \sum_{\beta<0} k(\rho; x, y) \right| \leq \sum_{\beta<0} \frac{4x^{2\beta-2}}{(\beta-1)^2} \ll x^{-4}.$$

We next compute the rest, which is the sum over non-trivial zeros, that is, the zeros where $0 \leq \mathrm{Re}(s) \leq 1$. We need the following lemma, which is proved by the similar method as [3].

Lemma 1. *For any $T \in \mathbf{R}$, we have*

$$\sum_{|\gamma-T|\leq 1} 1, \quad \sum_{|\gamma-T|>1} \frac{1}{(\gamma-T)^2} \ll \log(d(|T| + |u| + 1)),$$

where $\rho = \beta + i\gamma$ runs over zeros of $L(s,\chi)$ with $0 \leq \beta \leq 1$, $\gamma \in \mathbf{R}$, subject to each condition.

I. The case where we assume GRH. The hypothesis asserts that any non-trivial zero $\rho = \beta + i\gamma$ of $L(s,\chi)$ satisfies $\beta = \frac{1}{2}$, and thereby

$$\left| \sum_{0 \le \beta \le 1} k(\rho; x, y) \right| \le 4x^{-1} \sum_{\beta = \frac{1}{2}} \frac{1}{|\rho - 1|^2} \ll x^{-1} \sum_{|\gamma| \le 1} 1 + x^{-1} \sum_{|\gamma| > 1} \frac{1}{\gamma^2}.$$

Hence by Lemma 1 we obtain

$$\sum_{0 \le \beta \le 1} k(\rho; x, y) \ll x^{-1} \log(d(|u| + 1)).$$

II. The case where we do not assume GRH.

Lemma 2. *There exists an absolute constant $c > 0$ with the following property. In the region defined by*

$$\mathrm{Re}(s) \ge 1 - \frac{c}{\log(d(|u| + 1))}, \quad |\mathrm{Im}(s)| \le 1,$$

$L(s,\chi)$ has no zeros with the possible exception of a real simple zero when $\chi \equiv 1$.

Proof. If $u \ne 0$, the proof is similar to [2]. Suppose that $u = 0$. The case that $\chi^2 \not\equiv 1$ or $\chi \equiv 1$ is proved in [2]. Thus we can assume $\chi^2 \equiv 1$ and $\chi \not\equiv 1$. In [2], it is shown that $L(s,\chi)$ has no zeros on the above region with the exception of at most one real simple zero. Under the more general condition $\chi \not\equiv 1$, it is a well-known fact that $L(s,\chi)$ is essentially the L-function of a holomorphic cusp form with respect to $\Gamma_0(d)$, and the nonexistence of the zeros outside the region is proved in [1]. This completes the proof.

We denote the zero out of the region of Lemma 3 by β_0 if it exists; then we have

$$\left| \sum_{\substack{\rho \ne \beta_0 \\ 0 \le \beta \le 1}} k(\rho; x, y) \right| \le \sum_{\substack{\rho \ne \beta_0 \\ 0 \le \beta \le 1}} \frac{4x^{2\beta - 2}}{|\rho - 1|^2} \le 4 \sum_{|\gamma| > 1} \frac{1}{\gamma^2} + 4 \sum_{\substack{|\gamma| \le 1 \\ |\rho - 1| > \frac{1}{\sqrt{3}}}} 3 + 4 \sum_{\substack{\rho \ne \beta_0 \\ |\rho - 1| \le \frac{1}{\sqrt{3}}}} \frac{1}{|\rho - 1|^2}.$$

Because of Lemma 1, the first and second terms are $\mathrm{O}\Big(\log(d(|u| + 1))\Big)$. To estimate the third term we need a lemma, which can be shown with the same method as Lemma 2.2 of [2].

Lemma 3. *Let $n_\chi(l, s)$ denote the number of zeros ρ of $L(s,\chi)$ with $|s - \rho| \le l$. Then for $\mathrm{Re}(s) \ge 1$ and $0 < l \le \frac{1}{\sqrt{3}}$*

$$n_\chi(l, s) \ll 1 + l \log(d(|t| + |u| + 1)).$$

If ρ satisfies $|\rho - 1| \le \frac{1}{\sqrt{3}}$, $\rho \ne \beta_0$, it follows from Lemma 2 that $|\rho - 1| \ge 1 - \beta > \frac{c}{\log(d(|u|+1))}$. Let σ denote $\frac{c}{\log(d(|u|+1))}$; then by Lemma 3 we obtain

$$\begin{aligned}\sum_{\substack{\rho\neq\beta_0\\|\rho-1|\le\frac{1}{\sqrt3}}}\frac{1}{|\rho-1|^2} &= \int_\sigma^{\frac{1}{\sqrt3}}\frac{1}{l^2}dn_\chi(l,1)\\ &= \left[\frac{n_\chi(l,1)}{l^2}\right]_\sigma^{\frac{1}{\sqrt3}}+2\int_\sigma^{\frac{1}{\sqrt3}}\frac{n_\chi(l,1)}{l^3}dl\\ &\ll \log^2(d(|u|+1)).\end{aligned}$$

Thus we obtain a final estimate without GRH:

$$\sum_{\substack{\rho\neq\beta_0\\0\le\beta\le1}} k(\rho;x,y) \ll \log^2(d(|u|+1)).$$

Summarizing the previous results, we have

$$I_\chi = \delta_\chi k(1)+\mathrm{O}(x^{-4})+\begin{cases}\mathrm{O}\Big(x^{-1}\log(d(|u|+1))\Big) & \text{case } I,\\ \mathrm{O}\Big(\log^2(d(|u|+1))\Big)-\delta_\chi k(\beta_0) & \text{case } II.\end{cases}$$

Since it follows from the Taylor series expansion for $k(s)$ about $s=1$ that $k(1;x,y)=\log^2\frac{y}{x}$, we finally conclude from (1) and the above that for $y>x\ge2$

$$\sum_{NP:\,\text{prime}}\chi(P)\log NP\cdot\widehat{k}(NP;x,y)-\delta_\chi\log^2\frac{y}{x}+\delta_\chi\left(\frac{y^{\beta_0-1}-x^{\beta_0-1}}{\beta_0-1}\right)^2$$

$$\ll \log y\cdot\log\frac{y}{x}\cdot(x\log x)^{-1}+\begin{cases}x^{-1}\log(d(|u|+1)) & \text{case } I,\\ \log^2(d(|u|+1)) & \text{case } II.\end{cases}\qquad(3)$$

3. Extremal trigonometric polynomials. In this section, we make use of specific extremal trigonometric polynomials, namely $S_K^-(x)$ below. (Here we say that $S(t)$ is a trigonometric polynomial of degree K if $S(t)$ is in the form of $\sum_{r=0}^K a_r\sin2\pi rt+b_r\cos2\pi rt$, where a_r, b_r are real numbers.) Notations are those in [4]. Let $\tau(t)$ denote the saw-tooth function

$$\tau(t) = \begin{cases}t-[t]-\frac12 & t\notin\mathbf{Z},\\ 0 & t\in\mathbf{Z},\end{cases}$$

where $[t]$ denotes the largest integer not exceeding a real number t, and

$$\begin{aligned}B_K(t) &= -\frac{1}{K+1}\sum_{r=1}^K\left\{\left(1-\frac{r}{K+1}\right)\cot\left(\frac{\pi r}{K+1}\right)+\frac1\pi\right\}\sin2\pi rt\\ &\quad+\frac{1}{2(K+1)^2}\left\{\frac{\sin\pi(K+1)t}{\sin\pi t}\right\}^2,\end{aligned}$$

where K denotes a positive integer. Vaaler showed in [5] that $B_K(t)\ge\tau(t)$ for all t and that if $T(t)$ is a trigonometric polynomial of degree $\le K$ such that $T(t)\ge\tau(t)$ for all t, then $\int_0^1 T(t)dt\ge\frac{1}{2(K+1)}$ with equality if and only if $T(t)=B_K(t)$.

Let $C_{(a,b)}(t)$ denote the characteristic function of open interval (a,b) with $a<b\le a+1$ in $\mathbf{R}/\mathbf{Z}$. This satisfies

$$\begin{aligned} C_{(a,b)}(t) &= b-a+\tau(t-b)+\tau(a-t) \\ &= b-a-\tau(b-t)-\tau(t-a) \end{aligned}$$

except when t coincides with a or b; in fact the both right-hand sides equal $1/2$ at $t = a, b$. We put

$$\begin{aligned} S_K^+(t) &= b-a+B_K(t-b)+B_K(a-t), \\ S_K^-(t) &= b-a-B_K(b-t)-B_K(t-a). \end{aligned}$$

It is clear that $S_K^{\pm}(t)$ is a trigonometric polynomial of degree at most K, that $S_K^-(t) \leq C_{(a,b)}(t) \leq S_K^+(t)$ for all t, and that $\int_0^1 S_K^{\pm}(t)dt = b - a \pm \frac{1}{K+1}$. Defining the r-th Fourier coefficients of $S_K^{\pm}(t)$ by $\widetilde{S_K^{\pm}}(r) = \int_0^1 S_K^{\pm}(t)e^{-2\pi irt}dt$, one can prove from the above properties that

$$\begin{aligned} \widetilde{S_K^{\pm}}(0) &= b-a \pm \frac{1}{K+1}, \\ \left|\widetilde{S_K^{\pm}}(r)\right| &\leq \frac{1}{K+1} + \min\Big(b-a, \frac{1}{\pi|r|}\Big) \quad \text{with} \quad r \neq 0. \end{aligned} \tag{4}$$

Let λ, μ be real numbers with $\lambda < \mu \leq \lambda + 2\pi$, namely $0 < \mu - \lambda \leq 2\pi$, and let (ω) denote a principal prime ideal; then $C_{(a,b)} \geq S_K^-$ implies

$$\begin{aligned} &\sum_{\lambda < \arg\chi((\omega)) < \mu} \log N(\omega) \cdot \widehat{k}(N(\omega); x, y) \\ &\geq \sum_{N(\omega):\,\text{prime}} S_K^-\left(\frac{\arg\chi((\omega))}{2\pi}\right) \log N(\omega) \cdot \widehat{k}(N(\omega)) \\ &= \sum_{N(\omega):\,\text{prime}} \left\{\sum_{r=-K}^{K} \widetilde{S_K^-}(r) \cdot e^{ir\arg\chi((\omega))}\right\} \log N(\omega) \cdot \widehat{k}(N(\omega)) \\ &= \sum_{r=-K}^{K} \widetilde{S_K^-}(r) \left\{\sum_{N(\omega):\,\text{prime}} \chi^r((\omega)) \log N(\omega) \cdot \widehat{k}(N(\omega))\right\}. \end{aligned}$$

Since $S_K^{\pm}(x)$ is real valued, we have $\widetilde{S_K^-}(-r) = \overline{\widetilde{S_K^-}(r)}$. Thus we have

$$\begin{aligned} &\sum_{\lambda < \arg\chi((\omega)) < \mu} \log N(\omega) \cdot \widehat{k}(N(\omega); x, y) \\ &\geq \widetilde{S_K^-}(0) \sum_{N(\omega):\,\text{prime}} \log N(\omega) \cdot \widehat{k}(N(\omega)) \\ &\quad - 2\sum_{r=1}^{K} \left|\widetilde{S_K^-}(r)\right| \left|\sum_{N(\omega):\,\text{prime}} \chi^r((\omega)) \log N(\omega) \cdot \widehat{k}(N(\omega))\right|. \end{aligned} \tag{5}$$

This inequality with (4) is a variation of the Erdős-Túran inequality ([4], [5]). An upper bound of the left-hand side can be similarly obtained by using S_K^+, but it is no use for our purpose.

4. Proofs of theorems.

Theorem 3. *Let F be an imaginary quadratic field $\mathbf{Q}\left(\sqrt{-d}\right)$ with discriminant $-d$, and let w, h be the number of roots of unity in F and the class number. Assume the truth of the Generalized Riemann Hypothesis for $L(s,\phi_j\psi^r)$ where ϕ_j $(j=1,2,\cdots,h)$ are all characters of the ideal class group of F, $\psi((a)) = \exp(iwh\arg(a))$ for all $a \in F^\times$, and $r \in \mathbf{Z}$. Then for any real numbers α, β, with $\alpha < \beta \le \alpha + \frac{2\pi}{wh}$, there exists a prime element ω in F with $\alpha < \arg(\omega) < \beta$ such that*

$$N(\omega) \; < \; \frac{C_3 \log^2 d}{(\beta-\alpha)^2} \log^4 \frac{1}{h(\beta-\alpha)},$$

where C_3 is a positive absolute constant.

P r o o f. From now on, $c_1, c_2, \cdots$ denote absolute positive constants. Since we have $u = whr$ for ψ^r and $w \le 6$ for all imaginary quadratic fields, it follows from (3) that for $y > x \ge 2$

$$\left| \sum_{NP:\,\text{prime}} \phi_j\psi^r(P) \log NP \cdot \widehat{k}(NP;x,y) - \delta_{\phi_j\psi^r} \log^2 \frac{y}{x} \right|$$
$$\le c_1 \Big\{ \log y \cdot \log \frac{y}{x} \cdot (x\log x)^{-1} + x^{-1}\log(d(h|r|+1)) \Big\}.$$

Using this formula and the orthogonality relation of ϕ_j, we have

$$\left| \sum_{N(\omega):\,\text{prime}} \psi^r((\omega)) \log N(\omega) \cdot \widehat{k}(N(\omega);x,y) - h^{-1}\delta_{\psi^r} \log^2 \frac{y}{x} \right|$$
$$\le c_1 \Big\{ \log y \cdot \log \frac{y}{x} \cdot (x\log x)^{-1} + x^{-1}\log(d(h|r|+1)) \Big\}.$$

Since we have $-K \le r \le K$, we obtain the inequality $\left|\widetilde{S_K^-}(r)\right| \le \dfrac{3}{2|r|}$ with $r \ne 0$ by (4). It is trivial that $wh\alpha < \arg\psi((\omega)) < wh\beta$ holds if and only if $\alpha < \arg(\omega) < \beta$. Now applying (5) for $\lambda = wh\alpha$ and $\mu = wh\beta$, we obtain

$$\begin{aligned}
&\sum_{\alpha<\arg(\omega)<\beta} \log N(\omega) \cdot \widehat{k}(N(\omega);x,y) \\
&\quad \ge \left\{ \frac{w}{2\pi}(\beta-\alpha) - \frac{1}{h(K+1)} \right\} \log^2 \frac{y}{x} \\
&\qquad - c_2 \left\{ \log y \cdot \log \frac{y}{x} \cdot (x\log x)^{-1} + x^{-1}\log d \right\} \\
&\qquad - 3\sum_{r=1}^{K} \frac{1}{r} \cdot c_1 \left\{ \log y \cdot \log \frac{y}{x} \cdot (x\log x)^{-1} + x^{-1}\log(d(hr+1)) \right\} \\
&\quad \ge \left\{ \frac{w}{2\pi}(\beta-\alpha) - \frac{1}{h(K+1)} \right\} \log^2 \frac{y}{x} \\
&\qquad - c_3 \left\{ \log y \cdot \log \frac{y}{x} \cdot (x\log x)^{-1} + x^{-1}\log(d(hK+1)) \right\} \log(K+1).
\end{aligned}$$

We set $y = ex$; then the right-hand side becomes

$$\geq \frac{w}{2\pi}(\beta - \alpha) - \frac{1}{h(K+1)} - c_4 x^{-1} \log(d(hK+1)) \cdot \log(K+1).$$

It is well-known that

$$h = \frac{w\sqrt{d}}{2\pi} \sum_{n\geq 1} \frac{(-d/n)}{n} \ll \sqrt{d} \log d,$$

where $(-d/\cdot)$ is the Kronecker symbol, and thereby we have $\log h \ll \log d$. We set $x = x' \log d$ with $x' \geq h$; then the right-hand side becomes

$$\geq \frac{w}{2\pi}(\beta - \alpha) - \frac{1}{h(K+1)} - c_5 x'^{-1} \log^2(K+1).$$

We set $K = [x'/h]$; then we have $\frac{1}{h(K+1)} < 1/x'$. Moreover it follows from the assumption $x' \geq h$ that $\log^2(K+1) \leq \log^2(2x'/h)$. Hence we have

$$\begin{aligned} \sum_{\alpha<\arg(\omega)<\beta} & \log N(\omega) \cdot \widehat{k}(N(\omega); x' \log d, ex' \log d) \\ & \geq \frac{w}{2\pi}(\beta - \alpha) - x'^{-1} - c_5 x'^{-1} \log^2 \frac{2x'}{h} \\ & \geq \frac{w}{2\pi}\left(\beta - \alpha - c_6 x'^{-1} \log^2 \frac{2x'}{h}\right). \end{aligned}$$

We set $x' = \frac{C}{\beta-\alpha} \log^2 \frac{1}{h(\beta-\alpha)}$ and C sufficiently large. If we put $\theta = \beta - \alpha$, then we can make the following satisfying for all θ;

$$\theta - \frac{c_6\theta\left(\log 2C + \log \frac{1}{h\theta} + 2\log\log \frac{1}{h\theta}\right)^2}{C \log^2 \frac{1}{h\theta}} > 0.$$

Thus we can conclude that there exists a prime element ω with $\alpha < \arg(\omega) < \beta$ such that $N(\omega) < \frac{e^2 C^2 \log^2 d}{(\beta-\alpha)^2} \log^4 \frac{1}{h(\beta-\alpha)}$. This implies Theorem 3.

Theorem 4. *Let F, w and h be as Theorem 3. Then for any real numbers α, β, with $\alpha < \beta \leq \alpha + \frac{2\pi}{wh}$, there exists a prime element ω in F with $\alpha < \arg(\omega) < \beta$ such that*

$$N(\omega) < \exp\left(\frac{A_4}{\sqrt{\beta-\alpha}} \log^{\frac{3}{2}} \frac{1}{\beta-\alpha}\right),$$

where A_4 is a positive constant depending only on F.

P r o o f. From now on, $a_1, a_2, \cdots$ denote positive constants depending only on F. We put $y = x^2$; then we have $\log y \cdot \log \frac{y}{x} \cdot (x \log x)^{-1} \ll 1$. Thus it follows from (3) that

$$\left| \sum_{N(\omega):\,\text{prime}} \psi^r((\omega)) \log N(\omega) \cdot \widehat{k}(N(\omega)) - h^{-1} \delta_{\psi^r} \log^2 x \right| \leq a_1 \log^2(|r| + e).$$

Similarly as the proof of Theorem 3, we obtain

$$\sum_{\alpha<\arg(\omega)<\beta} \log N(\omega)\cdot \widehat{k}(N(\omega);x,x^2)$$
$$\geq \left\{\frac{w}{2\pi}(\beta-\alpha)-\frac{1}{h(K+1)}\right\}\log^2 x - a_2\log^3(K+1).$$

We set $x=\exp\sqrt{\{(K+1)\log^3(K+1)\}}$; then we obtain $\frac{1}{h(K+1)}\log^2 x \leq \log^3(K+1)$. Hence we have

$$\sum_{\alpha<\arg(\omega)<\beta} \log N(\omega)\cdot \widehat{k}(N(\omega);x,x^2) \geq \frac{w}{2\pi}\log^3(K+1)\Big\{(\beta-\alpha)(K+1)-a_3\Big\}.$$

We set $K=[a_3/(\beta-\alpha)]$; then the right-hand side is positive. Since $K+1\leq a_4/(\beta-\alpha)$, we can conclude that there exists a prime element ω with $\alpha<\arg(\omega)<\beta$ such that $N(\omega)<\exp\left(\frac{a_5}{\sqrt{\beta-\alpha}}\log^{\frac{3}{2}}\frac{1}{(\beta-\alpha)}\right)$. This implies Theorem 4.

References

[1] J. Hoffstein, D. Ramakrishnan, Siegel zeros and cusp forms. Internat. Math. Res. Notices 1995, no. 6, 279-308.

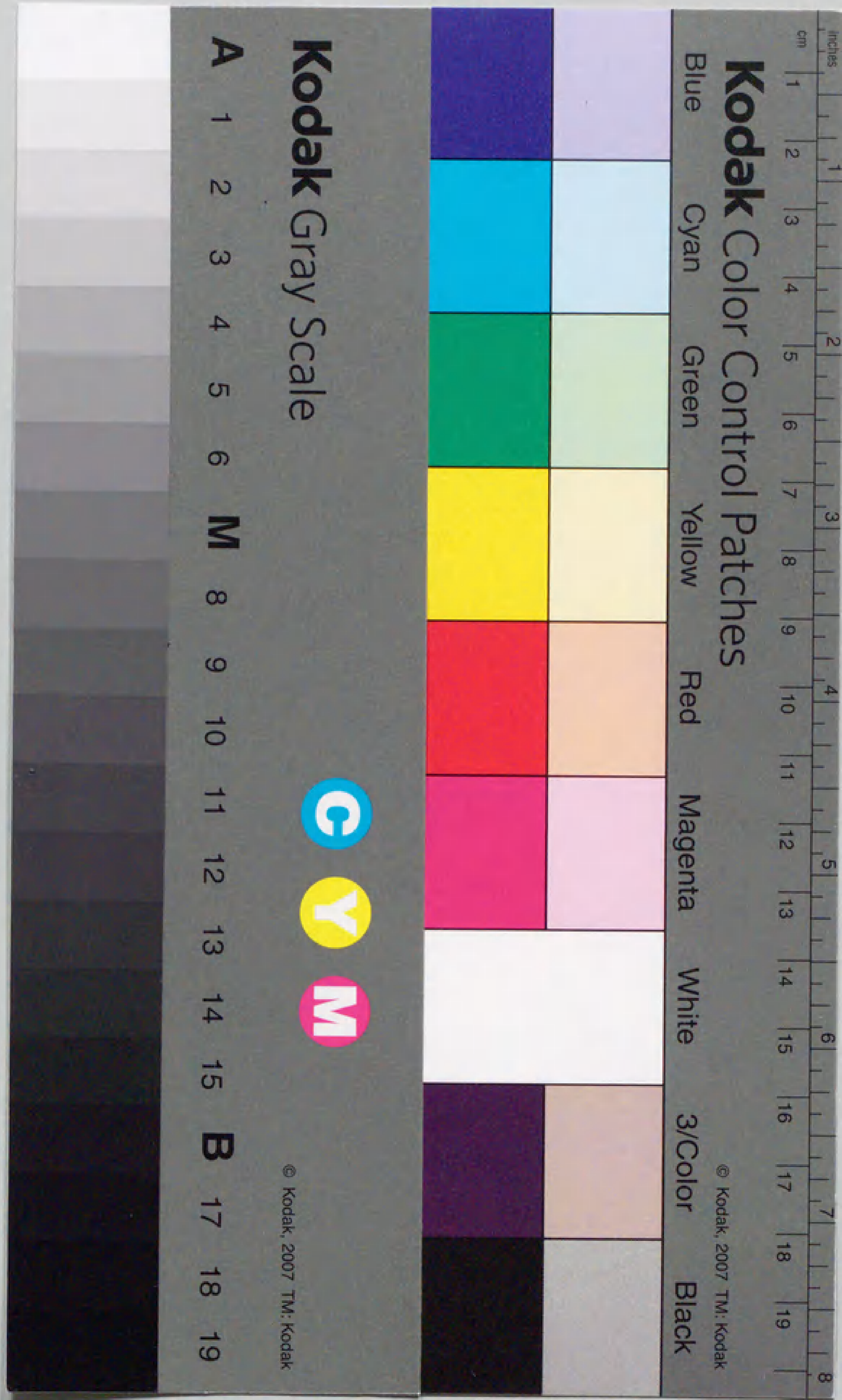
[2] J. C. Lagarias, H. L. Montgomery, A. M. Odlyzko, A Bound for the Least Prime Ideal in the Chebotarev Density Theorem. Inventiones Math. 54 (1979), 271-296.

[3] J. C. Lagarias, A. M. Odlyzko, Effective Versions of the Chebotarev Density Theorem. In: Algebraic Number Fields, L-Functions and Galois Properties (A. Fröhlich, ed.), 409-469, New York, London: Academic Press 1977.

[4] H. Montgomery, Ten lectures on the interface between analytic number theory and harmonic analysis, No.84, CBMS, Amer. Math. Soc. 1994.

[5] J. D. Vaaler, Some extremal functions in Fourier analysis. Bull. Amer. Math. Soc. 12 (1985), 183-216.

Hajime Matsui
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602,
Japan
e-mail adress: d96004z@math.nagoya-u.ac.jp


Kodak Color Control Patches
Blue
Cyan
Green
Yellow
Red
Magenta
White
3/Color
Black
© Kodak, 2007 TM: Kodak
Kodak Gray Scale
A 1 2 3 4 5 6 M 8 9 10 11 12 13 14 15 B 17 18 19
C Y M
© Kodak, 2007 TM: Kodak