

Approximation Method by a
Generalization of the Bernstein Polynomial
and its Applications

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NOTATIONS

Throughout this thesis, we adopt the following notations. We define the empty sum and product as zero and unity, respectively.

$\sum_{k=P}^Q$	$= \sum_{k=\max P}^{\min Q}$ if P, Q are finite sets of integers.
\mathbf{N}_0	$= \mathbf{N} \cup \{0\}$; i.e., the set of all nonnegative integers.
\mathbf{P}_n	The set of all polynomials of degree at most $n \in \mathbf{N}_0$ with real coefficients.
$(a)_n$	$= \prod_{k=0}^{n-1} (a+k)$ if $a \in \mathbf{R}, n \in \mathbf{N}_0$; i.e., the Pochhammer symbol.
$a^{(n)}$	$= \prod_{k=0}^{n-1} (a-k)$ if $a \in \mathbf{R}, n \in \mathbf{N}_0$.
$\binom{a}{n}$	$= \frac{a^{(n)}}{n!}$ if $a \in \mathbf{R}, n \in \mathbf{N}_0$; i.e., the generalized binomial coefficient.
$n!!$	$= \prod_{k=0}^{[(n-1)/2]} (n-2k)$ if $n \in \mathbf{Z}, n \geq -1$.
$f^{[n]}$	$= \frac{f^{(n)}}{n!}$ if f is a function and $n \in \mathbf{N}_0$.
$\ \cdot\ $	The uniform functional norm on $C[0, 1]$ or the operator norm subordinate to it.
$\ \cdot\ _2$	The functional L_2 -norm associated with $[0, 1]$ or the operator norm subordinate to it.

Δ_h, ∇_h	The forward and backward difference operators, respectively, of stepsize h ($h \in \mathbf{R}, h > 0$).
$\delta_{i,j}$	The Kronecker delta: $\delta_{i,i} = 1, \delta_{i,j} = 0$ if $i \neq j$.
$s(n, m)$	The Stirling number of the first kind ($n, m \in \mathbf{N}_0, m \leq n$).
$S(n, m)$	The Stirling number of the second kind ($n, m \in \mathbf{N}_0, m \leq n$) with the conventional definition $S(-1, -1) = 1, S(n, -1) = 0$ ($n \geq 0$).
$\mathcal{B}_\nu^n(\cdot)$	The (generalized) Bernoulli polynomial of order n and degree ν .
\mathcal{B}_ν^n	$= \mathcal{B}_\nu^n(0)$; i.e., the (generalized) Bernoulli number of order n and degree ν .
P_n	The Legendre polynomial of degree n .
$C_n^{(\lambda)}$	The Gegenbauer polynomial of degree n .
e_n	The polynomial of degree n defined as $e_{2m}(x) = (x(1-x))^m, e_{2m+1}(x) = (1-2x)e_{2m}(x)$ for every $m \in \mathbf{N}_0$.
B_n	The Bernstein operator (\rightarrow p.3, p.7).
L_n	The Lagrange operator (\rightarrow p.1, p.7).
$B_n^{(K)}$	The left Bernstein quasi-interpolant operator; Sablonnière's operator (\rightarrow p.7).
$P_n^{(s)}$	Stancu's operator (\rightarrow p.30).
${}_a B_n$	The modified Bernstein operator (\rightarrow p.31).

CHAPTER 1

INTRODUCTION

Approximation theory can be considered to have the following two purposes:

- (1) to approximate a known function f by a more treatable function \tilde{f} ;
- (2) to estimate an unknown function f from finite given values f_0, f_1, \dots, f_n related to f .

It is the second part that is directly required by engineering, but it has a close relation to the first. If we find some effective way of approximating known functions, we can readily apply it to estimation of unknown functions.

We consider how to construct an approximating function from given functional values, where we suppose that the sampling points are equidistant. (Our ultimate goal is to handle arbitrary sampling points, but it is too difficult at least theoretically.) By considering a suitable linear transformation for the variable, we can regard without loss of generality that the sampling points are the ones that equally divide the interval $[0, 1]$; namely, ν/n ($n \in \mathbf{N}, \nu = 0, 1, \dots, n$). Hence we devote ourselves to such a case.

Though the case of equidistant sampling points is the most elementary for theoretical treatments and the most necessary for practical applications, it seems that there is no traditional way of approximation that is satisfactory in every sense.

First, we consider the Lagrange interpolation polynomial, which is represented as

$$L_n f(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{nx}{\nu} \binom{n(1-x)}{n-\nu}.$$

This does not converge uniformly to the approximated function f as $n \rightarrow \infty$ even when it is analytic on the interval $[0, 1]$, if f (regarded as a complex function) is not regular in the domain

$$D = \{z \in \mathbf{C} \mid |z^z/(z-1)^{z-1}| < 1\}.$$

(In the condition defining D , we specify the branches as $-\pi < \arg z, \arg(z-1) \leq \pi$.) This fact is well known as "Runge's phenomenon" and the function

$$f(x) = \frac{1}{1 + 25(2x-1)^2}$$

is often discussed (see [9, p.78], [20, pp.275-279], [32, pp.37,38], and [42, p.102]) as a simple example that causes Runge's phenomenon because f has the poles $\zeta = 1/2 \pm i/10$ that belong to the domain D . (In fact, $|\zeta/(\zeta-1)^{\zeta-1}| = 0.671 \dots < 1$.) Furthermore, the norm of the Lagrange operator L_n becomes extremely large as n increases, because the estimation

$$\frac{2^n}{4n(n-1)} \leq \|L_n\| \leq 2^n \quad (n \geq 2)$$

holds. (See [32, p.42], [36, p.99], and [37, p.27].) Therefore the Lagrange polynomial is numerically unstable; that is, it is very sensitive with errors involved in the values of the approximated function. For these reasons, the Lagrange polynomial is thought to be of no practical use.

Probably, the only utility approximation method is interpolation by a spline function, which is a combination of low-degree polynomials connected smoothly at every knot. The spline function converges to the approximated function if it is sufficiently smooth. For example, let $S_n f$ be the natural cubic spline function of f associated with the sampling points ν/n ($\nu = 0, 1, \dots, n$). Then we have

$$\|(S_n f)^{(r)} - f^{(r)}\| \leq M_r n^{-2+r} \|f''\| \quad \text{for all } f \in C^2[0, 1],$$

where $r = 0, 1$ and M_0, M_1 are suitable constants. This is indeed a good property, and in fact, spline functions have been applied practically in many fields of science and engineering. However, spline functions also have disadvantages. Though they look smooth, they are differentiable only finite times at each knot. Therefore, even if the approximated function has good analytic properties, spline functions cannot make the best use of them to reduce errors. Of course, it is impossible to apply a spline function to approximation of derivatives of higher order than that of the differentiability of the spline function itself.

Therefore, we return to the problem of approximation by a single polynomial. Though the Lagrange polynomial is determined as the only polynomial of degree at most n that interpolates a function, there are many other approximating polynomials that are not necessarily interpolating. Among these polynomials, we remark the so-called Bernstein polynomial, which is defined as

$$B_n f(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} x^\nu (1-x)^{n-\nu}.$$

(See, e.g., [26].) This has the good property that it can approximate derivatives of a given function simultaneously if the given function is sufficiently differentiable; namely,

$$\lim_{n \rightarrow \infty} \|(B_n f)^{(r)} - f^{(r)}\| = 0 \quad \text{for all } f \in C^r[0, 1], \quad (1.1)$$

where r is an arbitrary nonnegative integer. Furthermore, it is numerically stable because the norm of the Bernstein operator B_n always satisfies

$$\|B_n\| = 1.$$

However, the Bernstein polynomial has a serious disadvantage for practical applications; its convergence is very slow. In fact, we have

$$\lim_{n \rightarrow \infty} n(B_n f(x) - f(x)) = \frac{1}{2} x(1-x) f''(x) \quad \text{for all } f \in C^2[0, 1]. \quad (1.2)$$

This is known as Voronovskaya's theorem, which indicates that the convergence rate of $B_n f$ is $O(n^{-1})$ and cannot be improved even if f increases its smoothness, except when f is a linear function. This is why the Bernstein polynomial is not used practically.

Though we cannot improve the convergence rate of $B_n f$ itself, there are various attempts to modify the Bernstein polynomial so that its convergence is accelerated. However, most of them require some extra data besides the given values $f(\nu/n)$. (See, e.g., [5].) It is desirable to achieve the same goal without using such extra data.

We try to introduce the new polynomial $\tilde{B}_n f$ as

$$\tilde{B}_n f(x) = B_n f(x) - \frac{1}{2n} x(1-x) (B_n f)''(x),$$

then (1.1) and (1.2) imply

$$\begin{aligned}
 & n|\tilde{B}_n f(x) - f(x)| \\
 &= |n(B_n f(x) - f(x)) - \frac{1}{2}x(1-x)f''(x) - \frac{1}{2}x(1-x)((B_n f)''(x) - f''(x))| \\
 &\leq |n(B_n f(x) - f(x)) - \frac{1}{2}x(1-x)f''(x)| + \frac{1}{2}x(1-x)|((B_n f)''(x) - f''(x))| \\
 &\rightarrow 0 \quad (n \rightarrow \infty).
 \end{aligned}$$

This is an example showing that we can construct an improved Bernstein polynomial using only the values $f(\nu/n)$.

This kind of research has recently been developed by P. Sablonnière [38, 39, 40]. He introduced the *left Bernstein quasi-interpolant* operator and investigated its properties. Inspired by his work, we have considered the problem of improving convergence rate of the Bernstein polynomial from a more general standpoint, and we found that there are many generalizations of the Bernstein polynomial whose convergence rates can be arbitrarily improved if the approximated function is sufficiently differentiable. Furthermore, among such generalized Bernstein polynomials, we found a certain specific polynomial that is defined very simply. We call this polynomial the *modified Bernstein polynomial*. It is, in a sense, an intermediate between the Bernstein and the Lagrange polynomials.

Compared with a spline function, the modified Bernstein polynomial has the advantage that it can approximate derivatives of a given function simultaneously in arbitrarily high order. Furthermore, it is numerically stable in the sense that the norm of the modified Bernstein operator is bounded with respect to n .

This thesis consists of seven chapters. In this chapter, we described the background and the motivation of our research.

In Chapter 2, we will discuss the work of Sablonnière and provide a general framework of our theory.

In Chapter 3, we will introduce the modified Bernstein polynomial and investigate its advantages. We also consider its application to numerical quadrature.

Since we will have obtained enough results for theoretical treatments in Chapters 2 and 3, after these chapters we will consider practical applications of the modified Bernstein polynomials.

In Chapter 4, we will provide an algorithm to evaluate the modified Bernstein polynomial of a given function.

In Chapter 5, we will discuss how we should choose the parameter called *sharpness degree* to minimize errors for general functions.

In Chapter 6, we will present some numerical examples, which will clarify the advantages of our method compared with the interpolation by spline functions.

In Chapter 7, we will give a conclusion and further aspects.

Our work is a breakthrough in approximation theory because we exploded the established thought that any polynomial approximation with equidistant sampling points is of no practical use, and we demonstrated that there certainly exists a very convenient method of approximation both theoretically and practically.

CHAPTER 2

GENERALIZATION OF THE LEFT BERNSTEIN QUASI-INTERPOLANTS

As we noticed in Chapter 1, P. Sablonnière introduced the so-called left Bernstein quasi-interpolant, and proved that the sequence of the approximating polynomials converges pointwise in high-order rate to each sufficiently smooth approximated function. On the other hand, Z.-C. Wu proved that the sequence of the norms of the operators is bounded. In this chapter, we extract the essence why Sablonnière's operator exhibits good convergence and stability properties, and we clarify a sufficient condition for general operators to have similar properties. Moreover, regarding the family of the general operators, we derive detailed results about the derivatives of the approximating polynomials that estimate their uniform convergence degree, using a convenient differentiability condition on approximated functions.

2.1. Introduction

The Bernstein operator B_n of order $n \in \mathbf{N}$ is defined as

$$B_n f(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \quad (f: [0, 1] \rightarrow \mathbf{R}, x \in [0, 1]),$$

while the Lagrange (interpolation) operator L_n of the same sampling points as B_n is represented as

$$L_n f(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{nx}{\nu} \binom{n(1-x)}{n-\nu} \quad (f: [0, 1] \rightarrow \mathbf{R}, x \in [0, 1]).$$

There are many classical results on the Bernstein operator [2, 3, 7, 9, 11, 18, 26, 27, 28, 32, 36]. In this thesis, we particularly notice Sablonnière's work [38, 39, 40]. He defined the *left Bernstein quasi-interpolant* operator $B_n^{(K)}$ ($K \in \mathbf{N}_0$, $K \leq n$) as

$$B_n^{(K)} f = \sum_{k=0}^K \alpha_k^n (B_n f)^{(k)} \quad (f: [0, 1] \rightarrow \mathbf{R}),$$

where α_k^n are polynomials of degree at most k satisfying

$$L_n f = \sum_{k=0}^n \alpha_k^n (B_n f)^{(k)} \quad (f: [0, 1] \rightarrow \mathbf{R});$$

and he proved in [39] that

$$\lim_{n \rightarrow \infty} n^{l+1} (B_n^{(2l)} f(x) - f(x)) = \frac{(-1)^l X^l (4l(l+1)(1-2x)f^{(2l+1)}(x) + 3Xf^{(2l+2)}(x))}{3 \cdot 2^{l+1}(l+1)!},$$

$$\lim_{n \rightarrow \infty} n^{l+1} (B_n^{(2l+1)} f(x) - f(x)) = \frac{(-1)^l X^{l+1} f^{(2l+2)}(x)}{2^{l+1}(l+1)!},$$

where $l \in \mathbf{N}_0$, $f \in C^{2l+3}[0, 1]$, $x \in [0, 1]$, and $X = x(1-x)$. Moreover, Wu [53] proved that the sequence $\{\|B_n^{(K)}\|\}_{n=K}^\infty$ is bounded for each K .

The aim of this chapter is to extract the essence of the above-mentioned facts on $B_n^{(K)}$, to clarify the structure of general operators that have similar properties to those of Sablonnière's operator, and to derive more general and more detailed results than the preceding ones, which imply their theorems as a part of a "corollary."

The content of this chapter has been published as [22], furthermore which is cited in [41].

2.2. Main Results

Our main results are summed up in the following four theorems, whose kernel is Theorem 2.4.

THEOREM 2.1. *Let $n \in \mathbf{N}$ and T be an operator on $\{f \mid f: [0, 1] \rightarrow \mathbf{R}\}$. Then the following two conditions are equivalent:*

- (1) *T is represented as the form $Tf = \sum_{\nu=0}^n f(\nu/n) \tau_\nu$ ($\tau_\nu \in \mathbf{P}_n$, $f: [0, 1] \rightarrow \mathbf{R}$) and $T\mathbf{P}_m \subseteq \mathbf{P}_m$ ($0 \leq m \leq n$);*
- (2) *there exist unique $V_k \in \mathbf{P}_k$ ($0 \leq k \leq n$) such that*

$$Tf = \sum_{k=0}^n V_k (B_n f)^{(k)} \quad (f: [0, 1] \rightarrow \mathbf{R}),$$

where $V_{k,l}$ ($k, l \in \mathbf{N}_0$, $k \leq n$, $l \leq n-k$) are determined by the following recursion formula and V_k can be identified with $V_{k,0}$:

$$V_{-1,l} = 0 \quad (0 \leq l \leq n-1),$$

$$V_{0,l}(x) = T(\cdot - x)^l(x) = \sum_{\nu=0}^n (\nu/n - x)^l \tau_\nu(x) \quad (x \in [0, 1]) \quad (0 \leq l \leq n),$$

$$(n-k)V_{k+1,l} = nV_{k,l+1} - k(e_1 V_{k,l} + e_2 V_{k-1,l}) \quad (0 \leq k \leq n-1, 0 \leq l \leq n-k-1).$$

THEOREM 2.2. *For each $n \in \mathbf{N}$, there exist unique $U_k^n \in \mathbf{P}_k$ ($0 \leq k \leq n$) such that*

$$L_n f = \sum_{k=0}^n U_k^n (B_n f)^{(k)} \quad (f: [0, 1] \rightarrow \mathbf{R}),$$

where U_k^n are determined by the recursion formula

$$U_{-1}^n = 0, \quad U_0^n = 1,$$

$$(n-k)U_{k+1}^n = -k(e_1 U_k^n + e_2 U_{k-1}^n) \quad (0 \leq k \leq n-1).$$

Remark. We use the notation U_k^n throughout this thesis.

THEOREM 2.3. *For each $n \in \mathbf{N}$, $k \in \mathbf{N}_0$ ($k \leq n$), we expand U_k^n as the form*

$$U_k^n = \sum_{l=0}^k u_{k,l}(n) e_l.$$

Then the coefficients are estimated asymptotically as follows for every $k, l \in \mathbf{N}_0$:

$$u_{2k,2l+1}(n) = 0 \quad \text{for all } n \geq 2k \quad \text{if } l \leq k-1,$$

$$u_{2k,2l}(n) = O(n^{l-2k}) \quad (n \rightarrow \infty) \quad \text{if } l \leq k-1,$$

$$\lim_{n \rightarrow \infty} n^k u_{2k,2k}(n) = (-1)^k (2k-1)!!;$$

$$u_{2k+1,2l}(n) = 0 \quad \text{for all } n \geq 2k+1 \quad \text{if } l \leq k,$$

$$u_{2k+1,2l+1}(n) = O(n^{l-2k-1}) \quad (n \rightarrow \infty) \quad \text{if } l \leq k-1,$$

$$\lim_{n \rightarrow \infty} n^{k+1} u_{2k+1,2k+1}(n) = \frac{2}{3} (-1)^{k+1} k(2k+1)!!.$$

Accordingly, they are roughly estimated as

$$u_{k,l}(n) = O(n^{[l/2]-k}) \quad (n \rightarrow \infty) \quad \text{for every } k, l \in \mathbf{N}_0 \quad (l \leq k).$$

In addition,

$$\|U_k^n\| = O(n^{[k/2]-k}) \quad (n \rightarrow \infty) \quad \text{for every } k \in \mathbf{N}_0.$$

Remark. We use the notation $u_{k,l}(n)$ throughout this thesis.

THEOREM 2.4. Let $\{T_n\}_{n=1}^\infty$ be a sequence of operators on $\{f \mid f: [0, 1] \rightarrow \mathbf{R}\}$ such that for each $n \in \mathbf{N}$, T_n is represented as the form $T_n f = \sum_{\nu=0}^n f(\nu/n) \tau_{n,\nu}$ ($\tau_{n,\nu} \in \mathbf{P}_n$, $f: [0, 1] \rightarrow \mathbf{R}$) and $T_n \mathbf{P}_m \subseteq \mathbf{P}_m$ ($0 \leq m \leq n$). According to Theorem 2.1, we expand

$$T_n f = \sum_{k=0}^n V_k^n (B_n f)^{[k]} \quad (f: [0, 1] \rightarrow \mathbf{R})$$

and furthermore

$$V_k^n = \sum_{l=0}^k v_{k,l}(n) e_l.$$

Let $\alpha \in \mathbf{N}_0$ and suppose there exists a $K \in \mathbf{N}_0$ ($K \geq 2\alpha$) such that for every $k, l \in \mathbf{N}_0$ the following conditions are satisfied:

- (a) $V_k^n = 0$ ($K < k \leq n$) for all $n > K$;
- (b) $v_{k,l}(n) = O(n^{[l/2]-k})$ ($n \rightarrow \infty$) if $l \leq k \leq K$;
- (c) $\|V_k^n - U_k^n\| = o(n^{-\alpha})$ ($n \rightarrow \infty$) if $k \leq K$.

Then $\{T_n\}_{n=1}^\infty$ has the following properties:

- (1) for all $p, q, r \in \mathbf{N}_0$, there exists a constant M such that for all $n \in \mathbf{N}$ and for all $f \in C^r[0, 1]$,

$$\|e_{2p}(T_n f)^{(q+r)}\| \leq M n^{q-\min\{p, [q/2]\}} \|f^{(r)}\|;$$

- (2) for all $\beta, \gamma \in \mathbf{N}_0$ ($\beta \leq \alpha$) and for all $f \in C^{2\beta+\gamma}[0, 1]$,

$$\|(T_n f)^{(\gamma)} - f^{(\gamma)}\| = o(n^{-\beta}) \quad (n \rightarrow \infty);$$

- (3) if $\lim_{n \rightarrow \infty} n^{\alpha+1}(V_k^n - U_k^n) = R_k$ in the sense of $\|\cdot\|$ ($0 \leq k \leq 2\alpha + 2$), then for all $\gamma \in \mathbf{N}_0$ and for all $f \in C^{2\alpha+\gamma+2}[0, 1]$,

$$\lim_{n \rightarrow \infty} n^{\alpha+1}((T_n f)^{(\gamma)} - f^{(\gamma)}) = \left(\sum_{k=0}^{2\alpha+2} R_k f^{[k]} \right)^{(\gamma)} \quad \text{in the sense of } \|\cdot\|.$$

Proofs of these theorems will be given in the later sections.

Recall α_k^n in Section 2.1 and note that $\alpha_k^n = U_k^n/k!$ because Theorem 2.2 guarantees the uniqueness of U_k^n . Though they were given in [38, 39, 53] by very complicated recurrence relations, now we can calculate them from the simple three-term recursion formula in Theorem 2.2.

The left Bernstein quasi-interpolant operators $B_n^{(K)}$ were not defined when $n < K$, however, now we can redefine them for all $K \in \mathbf{N}_0$ and for all $n \in \mathbf{N}$ as

$$B_n^{(K)} f = \sum_{k=0}^{\{K, n\}} U_k^n (B_n f)^{[k]} \quad (f: [0, 1] \rightarrow \mathbf{R}).$$

The above theorems imply the following corollary regarding $B_n^{(K)}$.

COROLLARY 2.5. Let $K \in \mathbf{N}_0$ and $\alpha = [K/2]$. Then $\{B_n^{(K)}\}_{n=1}^\infty$ has the following properties:

- (1) for all $p, q, r \in \mathbf{N}_0$, there exists a constant M such that for all $n \in \mathbf{N}$ and for all $f \in C^r[0, 1]$,

$$\|e_{2p}(B_n^{(K)} f)^{(q+r)}\| \leq M n^{q-\min\{p, [q/2]\}} \|f^{(r)}\|;$$

- (2) for all $\beta, \gamma \in \mathbf{N}_0$ ($\beta \leq \alpha$) and for all $f \in C^{2\beta+\gamma}[0, 1]$,

$$\|(B_n^{(K)} f)^{(\gamma)} - f^{(\gamma)}\| = o(n^{-\beta}) \quad (n \rightarrow \infty);$$

- (3) for all $\gamma \in \mathbf{N}_0$ and for all $f \in C^{2\alpha+\gamma+2}[0, 1]$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{\alpha+1}((B_n^{(K)} f)^{(\gamma)} - f^{(\gamma)}) \\ &= \begin{cases} (-1)^\alpha (2\alpha + 1)!! \left(\frac{2}{3} \alpha e_{2\alpha+1} f^{[2\alpha+1]} + e_{2\alpha+2} f^{[2\alpha+2]} \right)^{(\gamma)} & \text{if } K = 2\alpha, \\ (-1)^\alpha (2\alpha + 1)!! (e_{2\alpha+2} f^{[2\alpha+2]})^{(\gamma)} & \text{if } K = 2\alpha + 1, \end{cases} \end{aligned}$$

in the sense of $\|\cdot\|$.

Proof. Let $n \in \mathbf{N}$ and suppose $n > K$. We substitute $B_n^{(K)}$ into T_n of Theorem 2.4 and identify the given K with K in the theorem. Then $K \geq 2\alpha$ and for every $k \in \mathbf{N}_0$,

$$V_k^n = \begin{cases} U_k^n & \text{if } k \leq K, \\ 0 & \text{if } K < k \leq n. \end{cases}$$

Thus the conditions (a) and (c) are trivial. We can also verify (b) using Theorem 2.3. Therefore, Theorem 2.4 implies the properties (1) and (2) in this corollary. The property (3) is also derived by calculating R_k in Theorem 2.4 with the aid of Theorem 2.3. ■

Now we compare this corollary with the preceding results. When $p = q = r = 0$, (1) reduces to

(1') there exists a constant M such that for all $n \in \mathbf{N}$ and for all $f \in C[0, 1]$

$$\|B_n^{(K)} f\| \leq M \|f\|.$$

This is nothing but the result of [53]. Besides, when $\gamma = 0$, we can rewrite (3) as

(3') for all $f \in C^{2\alpha+2}[0, 1]$,

$$\lim_{n \rightarrow \infty} n^{\alpha+1} (B_n^{(K)} f - f) = \begin{cases} \frac{(-1)^\alpha (4\alpha(\alpha+1)e_{2\alpha+1} f^{(2\alpha+1)} + 3e_{2\alpha+2} f^{(2\alpha+2)})}{3 \cdot 2^{\alpha+1} (\alpha+1)!} & \text{if } K = 2\alpha, \\ \frac{(-1)^\alpha e_{2\alpha+2} f^{(2\alpha+2)}}{2^{\alpha+1} (\alpha+1)!} & \text{if } K = 2\alpha+1, \end{cases}$$

in the sense of $\|\cdot\|$.

Here we used the identities $(2\alpha+1)! = (2\alpha+1)!!(2\alpha)!! = (2\alpha+1)!!2^\alpha \alpha!$ and $(2\alpha+2)! = (2\alpha+2)!!(2\alpha+1)!! = 2^{\alpha+1}(\alpha+1)!(2\alpha+1)!!$. As the class $C^{2\alpha+3}[0, 1]$ can be embedded into $C^{2\alpha+2}[0, 1]$, by regarding the sense of convergence as pointwise, (3') reduces further to the result of [39]. As we see from these facts, Corollary 2.5 itself is a much more general and detailed result than the preceding ones, and as to the theorems, therefore, all the more.

2.3. Proofs of Theorems 2.1–2.3

In this section, we prove the first three theorems in the previous section.

Proof of Theorem 2.1. Suppose the condition (2) holds. It is trivial that

$$Tf = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \tau_\nu \quad (\tau_\nu \in \mathbf{P}_n).$$

Let $f \in \mathbf{P}_m$ ($0 \leq m \leq n$). Then, as is well known, $B_n f \in \mathbf{P}_m$. Thus (2) \Rightarrow (1) is immediate.

Suppose the condition (1) holds. We fix $x \in [0, 1]$ for a while and expand with respect to $\xi \in [0, 1]$

$$B_n f(\xi) = \sum_{k=0}^n (\xi - x)^k (B_n f)^{[k]}(x).$$

Since it is well known that B_n is invertible on \mathbf{P}_n (e.g., [38, 39, 53]), we can calculate

$$\begin{aligned} Tf(\xi) &= TL_n f(\xi) = TB_n^{-1} B_n L_n f(\xi) = TB_n^{-1} B_n f(\xi) \\ &= \sum_{k=0}^n TB_n^{-1}(\cdot - x)^k(\xi) (B_n f)^{[k]}(x). \end{aligned}$$

Putting $\xi = x$ gives

$$Tf(x) = \sum_{k=0}^n V_k(x) (B_n f)^{[k]}(x),$$

where $V_k(x) = TB_n^{-1}(\cdot - x)^k(x)$. Thus the existence of V_k satisfying the above formula is guaranteed.

Let $x \in [0, 1]$, $t \in (-1, 1)$ and fix them for a while. We consider the case

$$f(\xi) = (1 + (1-x)t)^{n\xi} (1 - xt)^{n(1-\xi)} \quad (\xi \in [0, 1]).$$

Then

$$\begin{aligned} B_n f(\xi) &= \sum_{\nu=0}^n (1 + (1-x)t)^\nu (1 - xt)^{n-\nu} \binom{n}{\nu} \xi^\nu (1 - \xi)^{n-\nu} \\ &= \sum_{\nu=0}^n \binom{n}{\nu} (\xi + (1-x)\xi t)^\nu (1 - \xi - x(1-\xi)t)^{n-\nu} \\ &= (1 + (\xi - x)t)^n. \end{aligned}$$

For all $k \leq n$,

$$\begin{aligned} (B_n f)^{[k]}(\xi) &= \binom{n}{k} (1 + (\xi - x)t)^{n-k} t^k, \\ (B_n f)^{[k]}(x) &= \binom{n}{k} t^k. \end{aligned}$$

Therefore the relation $Tf = \sum_{k=0}^n V_k(B_n f)^{[k]}$ implies

$$\sum_{\nu=0}^n (1 + (1-x)t)^\nu (1-xt)^{n-\nu} \tau_\nu(x) = \sum_{k=0}^n \binom{n}{k} V_k(x) t^k.$$

This means the V_k are obtained by expanding the left-hand side with respect to t , and consequently the V_k are unique. Generalizing the above formula, we expand for every $l \in \mathbf{N}_0$

$$\sum_{\nu=0}^n \left(\frac{\nu}{n} - x\right)^l (1 + (1-x)t)^\nu (1-xt)^{n-\nu} \tau_\nu(x) = \sum_{k=0}^n \binom{n}{k} V_{k,l}(x) t^k.$$

Here we can identify V_k with $V_{k,0}$. Differentiating by t and multiplying by

$(1 + (1-x)t)(1-xt)$ both sides of the above equation, we get

$$(1 + e_1(x)t - e_2(x)t^2) \sum_{k=1}^n k \binom{n}{k} V_{k,l}(x) t^{k-1} = n \sum_{k=0}^n \binom{n}{k} V_{k,l+1}(x) t^k - n e_2(x) t \sum_{k=0}^n \binom{n}{k} V_{k,l}(x) t^k$$

by virtue of

$$(1 + (1-x)t)(1-xt) \frac{d}{dt} [(1 + (1-x)t)^\nu (1-xt)^{n-\nu}] = ((\nu - nx) - n e_2(x)t) (1 + (1-x)t)^\nu (1-xt)^{n-\nu}.$$

Rearrangement of the above formula with the conventional definition $V_{-1,l}(x) = 0$ gives

$$\sum_{k=0}^{n-1} \binom{n}{k} (n-k) V_{k+1,l}(x) t^k = \sum_{k=0}^n \binom{n}{k} (n V_{k,l+1}(x) - k e_1(x) V_{k,l}(x) - k e_2(x) V_{k-1,l}(x)) t^k.$$

Equating coefficients of t^k on both sides yields

$$(n-k) V_{k+1,l} = n V_{k,l+1} - k(e_1 V_{k,l} + e_2 V_{k-1,l}) \quad (0 \leq k \leq n-1, l \geq 0).$$

Since we need $V_{k,0}$ only, we may restrict the region where l moves, to $0 \leq l \leq n-k-1$. In addition, the initial condition is

$$V_{-1,l} = 0 \quad (0 \leq l \leq n-1)$$

and

$$V_{0,l}(x) = \sum_{\nu=0}^n \left(\frac{\nu}{n} - x\right)^l \tau_\nu(x) = T(\cdot - x)^l(x) \quad (x \in [0, 1]) \quad (0 \leq l \leq n),$$

derived by putting $t = 0$ on both sides of the formula generating $V_{k,l}$.

Finally, we let $\varphi(\xi) = \xi$ ($\xi \in [0, 1]$) and expand

$$V_{0,l}(x) = \sum_{m=0}^l \binom{l}{m} T\varphi^m(x) \cdot (-x)^{l-m}. \quad (2.1)$$

Then $\varphi^m \in \mathbf{P}_m$ and $TP_m \subseteq \mathbf{P}_m$ ($0 \leq m \leq n$) imply $V_{0,l} \in \mathbf{P}_l$ ($0 \leq l \leq n$). Using the recursion formula, we obtain $V_k \in \mathbf{P}_k$ ($0 \leq k \leq n$). Thus (1) \Rightarrow (2) is proved. ■

Proof of Theorem 2.2. Obviously, $T = L_n$ satisfies the condition (1) in Theorem 2.1, therefore it also satisfies (2). We define U_k^n as V_k in the case $T = L_n$. Then this theorem is immediate except the recursion formula.

When $T = L_n$, recalling (2.1), we can expand

$$\begin{aligned} V_{0,l}(x) &= \sum_{m=0}^l \binom{l}{m} L_n \varphi^m(x) \cdot (-x)^{l-m} = \sum_{m=0}^l \binom{l}{m} x^m (-x)^{l-m} \\ &= (x + (-x))^l = \delta_{l,0}. \end{aligned}$$

Thus, from the recursion formula in Theorem 2.1, the identities

$$V_{k,l+1} = 0 \quad (0 \leq k \leq n-1, 0 \leq l \leq n-k-1)$$

hold. Then it suffices to consider the case $l = 0$. ■

Proof of Theorem 2.3. We prove this theorem by induction with the recursion formula in Theorem 2.2. It is valid when $k = 0$ because $U_0^n = 1$ ($n \geq 0$) and $U_1^n = 0$ ($n \geq 1$). Assume this theorem is valid for a fixed $k \in \mathbf{N}_0$. Then for all $n \geq 2(k+1)$,

$$\begin{aligned} (n-2k-1)U_{2(k+1)}^n &= -(2k+1)(e_1 U_{2k+1}^n + e_2 U_{2k}^n) \\ &= -(2k+1) \left(e_1 \sum_{l=0}^k u_{2k+1,2l+1}(n) e_{2l+1} + e_2 \sum_{l=0}^k u_{2k,2l}(n) e_{2l} \right). \end{aligned}$$

Since $e_1 e_{2l+1} = e_{2l} - 4e_{2(l+1)}$ and $e_2 e_{2l} = e_{2(l+1)}$,

$$\begin{aligned} (n-2k-1)U_{2(k+1)}^n &= -(2k+1) \left(\sum_{l=0}^k u_{2k+1,2l+1}(n) (e_{2l} - 4e_{2(l+1)}) + \sum_{l=0}^k u_{2k,2l}(n) e_{2(l+1)} \right) \\ &= -(2k+1) \left(\sum_{l=0}^k u_{2k+1,2l+1}(n) e_{2l} - 4 \sum_{l=1}^{k+1} u_{2k+1,2(l-1)+1}(n) e_{2l} + \sum_{l=1}^{k+1} u_{2k,2(l-1)}(n) e_{2l} \right). \end{aligned}$$

Here we compare the coefficients on both sides. It is obvious that

$$u_{2(k+1),2l+1}(n) = 0 \quad \text{if } l \leq k.$$

$$\begin{aligned}
& (n - 2k - 1)u_{2(k+1),2l}(n) \\
&= -(2k + 1) \begin{cases} u_{2k+1,1}(n) & \text{if } l = 0, \\ (u_{2k+1,2l+1}(n) - 4u_{2k+1,2(l-1)+1}(n) + u_{2k,2(l-1)}(n)) & \text{if } 1 \leq l \leq k, \\ (-4u_{2k+1,2k+1}(n) + u_{2k,2k}(n)) & \text{if } l = k + 1. \end{cases}
\end{aligned}$$

This recursion formula and the assumption of induction imply

$$u_{2(k+1),2l}(n) = O(n^{-1})(O(n^{l-2k-1}) + O(n^{l-2k-2}) + O(n^{l-2k-1})) = O(n^{l-2(k+1)}) \quad \text{if } l \leq k.$$

Furthermore,

$$\begin{aligned}
& \lim_{n \rightarrow \infty} n^{k+1} u_{2(k+1),2(k+1)}(n) \\
&= \lim_{n \rightarrow \infty} n^k (n - 2k - 1) u_{2(k+1),2(k+1)}(n) \\
&= \lim_{n \rightarrow \infty} (-(2k + 1)) (-4n^{-1} n^{k+1} u_{2k+1,2k+1}(n) + n^k u_{2k,2k}(n)) \\
&= -(2k + 1)(0 + (-1)^k (2k - 1)!!) = (-1)^{k+1} (2(k + 1) - 1)!! .
\end{aligned}$$

Thus the estimation of $u_{2(k+1),l}(n)$ ($0 \leq l \leq 2(k+1)$) is demonstrated and that of $u_{2(k+1)+1,l}(n)$ ($0 \leq l \leq 2(k+1) + 1$) is similarly shown by using the assumption of induction and the consequence on $u_{2(k+1),l}(n)$. ■

2.4. Preliminary Lemmas for the Proof of Theorem 2.4

This section is devoted to preparation of lemmas indispensable to prove Theorem 2.4.

LEMMA 2.6. Let $\{V_k^n\}_{n=k}^\infty$ be a sequence of polynomials of degree at most $k \in \mathbf{N}_0$. For each $n \in \mathbf{N}$, we expand

$$V_k^n = \sum_{l=0}^k v_{k,l}(n) e_l,$$

and furthermore for each $r \in \mathbf{N}_0$ ($r \leq k$),

$$(V_k^n)^{(r)} = \sum_{l=0}^{k-r} v_{k,l}^r(n) e_l.$$

We suppose

$$v_{k,l}(n) = O(n^{[l/2]-k}) \quad (n \rightarrow \infty) \quad (0 \leq l \leq k).$$

Then

$$v_{k,l}^r(n) = O(n^{\min\{[l/2]+r, [k/2]\}-k}) \quad (n \rightarrow \infty) \quad (0 \leq l \leq k - r).$$

Proof. We prove this lemma by induction. It is trivial when $r = 0$. Assume this lemma is valid for a fixed $r \in \mathbf{N}_0$ ($r \leq k - 1$). Then

$$(V_k^n)^{(r+1)} = \sum_{l=0}^{k-r} v_{k,l}^r(n) e_l' = \sum_{l=0}^{[(k-r)/2]} v_{k,2l}^r(n) e_{2l}' + \sum_{l=0}^{[(k-r-1)/2]} v_{k,2l+1}^r(n) e_{2l+1}'.$$

Since

$$e_0' = 0, \quad e_{2l}' = l e_{2l-1} \quad (l \geq 1)$$

and

$$e_1' = -2e_0, \quad e_{2l+1}' = l e_{2(l-1)} - 2(2l+1)e_{2l} \quad (l \geq 1),$$

we have

$$(V_k^n)^{(r+1)} = \sum_{l=1}^{[(k-r)/2]} l v_{k,2l}^r(n) e_{2l-1} - 2v_{k,1}^r(n) e_0 + \sum_{l=1}^{[(k-r-1)/2]} v_{k,2l+1}^r(n) (l e_{2(l-1)} - 2(2l+1)e_{2l}).$$

Therefore,

$$\begin{aligned}
& \sum_{l=0}^{[(k-r-1)/2]} v_{k,2l}^{r+1}(n) e_{2l} + \sum_{l=0}^{[(k-r)/2]-1} v_{k,2l+1}^{r+1}(n) e_{2l+1} \\
&= \sum_{l=0}^{[(k-r-1)/2]-1} (l+1) v_{k,2l+3}^r(n) e_{2l} - 2 \sum_{l=0}^{[(k-r-1)/2]} (2l+1) v_{k,2l+1}^r(n) e_{2l} \\
&+ \sum_{l=0}^{[(k-r)/2]-1} (l+1) v_{k,2l+2}^r(n) e_{2l+1}.
\end{aligned}$$

Equating coefficients of e_{2l} and e_{2l+1} on both sides yields

$$v_{k,2l}^{r+1}(n) = \begin{cases} (l+1) v_{k,2l+3}^r(n) - 2(2l+1) v_{k,2l+1}^r(n) & (0 \leq l \leq [(k-r-1)/2] - 1) \\ -2(2l+1) v_{k,2l+1}^r(n) & (l = [(k-r-1)/2]) \end{cases}$$

and

$$v_{k,2l+1}^{r+1}(n) = (l+1) v_{k,2l+2}^r(n) \quad (0 \leq l \leq [(k-r)/2] - 1).$$

These recursion formulas and the assumption of induction imply

$$v_{k,l}^{r+1}(n) = O(n^{\min\{[l/2]+(r+1), [k/2]\}-k}) \quad (n \rightarrow \infty) \quad (0 \leq l \leq k - (r+1)). \quad \blacksquare$$

LEMMA 2.7. For all $n \in \mathbf{N}$, $k \in \mathbf{N}_0$ ($k \leq n$) and for all $f: [0, 1] \rightarrow \mathbf{R}$,

$$(B_n f)^{(k)} = n^{(k)} \sum_{\nu=0}^{n-k} \Delta_{1/n}^k f\left(\frac{\nu}{n}\right) b_{n-k,\nu},$$

where

$$b_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \quad (x \in [0, 1]).$$

Remark. We use the notation $b_{n,\nu}$ throughout this thesis.

A proof of this lemma appears in [26, p.12].

LEMMA 2.8. For all $p, q, r \in \mathbf{N}_0$, there exists a constant M such that for all $n \in \mathbf{N}$ and for all $f \in C^r[0, 1]$,

$$\|e_p(B_n f)^{(q+r)}\| \leq M n^{q-\min\{[p/2], [q/2]\}} \|f^{(r)}\|.$$

Proof. It was shown in [12, Lemma 3.5] or [13, Theorem 9.4.1] that for all $p \in \mathbf{N}_0$, there exists a constant M such that for all $n \in \mathbf{N}$ and for all $f \in C[0, 1]$

$$\|e_{2p}(B_n f)^{(2p)}\| \leq M n^p \|f\|;$$

that is,

$$\|e_{2p} D^{2p} B_n\| \leq M n^p, \quad (2.2)$$

where D is the differential operator. By considering the Lebesgue constant of the operator $e_p D^q B_n$, we get

$$\|e_p D^q B_n\| = \left\| e_p \sum_{\nu=0}^n |b_{n,\nu}^{(q)}| \right\| \quad \text{for all } p, q \in \mathbf{N}_0 \text{ and } n \in \mathbf{N}. \quad (2.3)$$

We can assume $n > s \in \mathbf{N}_0$ without loss of generality. Applying Lemma 2.7, (2.3), and (2.2), we can estimate

$$\begin{aligned} \|e_{2p} D^{2p+s} B_n\| &= \max_{\|g\|=1} \|e_{2p} ((B_n g)^{(s)})^{(2p)}\| \\ &= \max_{\|g\|=1} \left\| e_{2p} \cdot n^{(s)} \sum_{\nu=0}^{n-s} \Delta_{1/n}^s g\left(\frac{\nu}{n}\right) b_{n-s,\nu}^{(2p)} \right\| \\ &\leq 2^s n^{(s)} \left\| e_{2p} \sum_{\nu=0}^{n-s} |b_{n-s,\nu}^{(2p)}| \right\| \\ &= 2^s n^{(s)} \|e_{2p} D^{2p} B_{n-s}\| \leq M' n^{s+p}. \end{aligned} \quad (2.4)$$

Let $f \in C^r[0, 1]$. We can assume $n > r$ without loss of generality. Applying Lemma 2.7, the mean value theorem, and (2.3), we can calculate

$$(B_n f)^{(q+r)} = ((B_n f)^{(r)})^{(q)} = n^{(r)} \sum_{\nu=0}^{n-r} \Delta_{1/n}^r f\left(\frac{\nu}{n}\right) b_{n-r,\nu}^{(q)},$$

$$\|e_p(B_n f)^{(q+r)}\| \leq \|f^{(r)}\| \left\| e_p \sum_{\nu=0}^{n-r} |b_{n-r,\nu}^{(q)}| \right\| = \|f^{(r)}\| \cdot \|e_p D^q B_{n-r}\|.$$

Replacing p by $\min\{[p/2], [q/2]\}$ and putting $s = q - 2\min\{[p/2], [q/2]\}$ in (2.4) imply

$$\|e_p D^q B_{n-r}\| \leq \|e_{p-2\min\{[p/2], [q/2]\}}\| \cdot \|e_{2\min\{[p/2], [q/2]\}} D^q B_{n-r}\| \leq M n^{q-\min\{[p/2], [q/2]\}},$$

where M is a suitable constant. ■

LEMMA 2.9. Let $r, s \in \mathbf{N}_0$, $f \in C^{r+s}[0, 1]$, and for each $x \in [0, 1]$,

$$g_x(\xi) = \sum_{j=0}^{r+s} f^{[j]}(x) (\xi - x)^j \quad (\xi \in [0, 1]), \quad h_x = f - g_x.$$

Then

$$\max_{x \in [0, 1]} |(B_n h_x)^{(r)}(x)| = o(n^{-s/2}) \quad (n \rightarrow \infty).$$

Proof. Let $n \in \mathbf{N}$. We can assume $n > r$ without loss of generality. Lemma 2.7 and the mean value theorem imply

$$\begin{aligned} (B_n h_x)^{(r)}(x) &= n^{(r)} \sum_{\nu=0}^{n-r} \Delta_{1/n}^r h_x \left(\frac{\nu}{n} \right) b_{n-r,\nu}(x) \\ &= \frac{n^{(r)}}{n^r} \sum_{\nu=0}^{n-r} h_x^{(r)} \left(\frac{\nu + r\theta_\nu}{n} \right) b_{n-r,\nu}(x) \quad (0 < \theta_0, \theta_1, \dots, \theta_{n-r} < 1). \end{aligned}$$

Applying Taylor's theorem to $f^{(r)}$, we obtain

$$\begin{aligned} (h_x)^{(r)}(\xi) &= f^{(r)}(\xi) - g_x^{(r)}(\xi) \\ &= f^{(r)}(\xi) - \sum_{j=0}^s (f^{(r)})^{[j]}(x) (\xi - x)^j \\ &= \frac{f^{(r+s)}(x + \lambda(\xi - x)) - f^{(r+s)}(x)}{s!} (\xi - x)^s \quad \text{for some } \lambda \in (0, 1], \end{aligned}$$

where we noticed that $s = 0$ yields $\lambda = 1$. Since $f^{(r+s)}$ is continuous on $[0, 1]$, it is uniformly continuous on $[0, 1]$. Take an arbitrary $\varepsilon > 0$. We can find a $\delta > 0$ such that for all $x_1, x_2 \in [0, 1]$

$$|x_2 - x_1| < \delta \quad \text{implies} \quad |f^{(r+s)}(x_2) - f^{(r+s)}(x_1)| < \varepsilon.$$

Therefore, when $|(\nu + r\theta_\nu)/n - x| < \delta$,

$$\left| h_x^{(r)} \left(\frac{\nu + r\theta_\nu}{n} \right) \right| \leq \frac{\varepsilon}{s!} \left| \frac{\nu + r\theta_\nu}{n} - x \right|^s.$$

When $|(\nu + r\theta_\nu)/n - x| \geq \delta$,

$$\left| h_x^{(r)} \left(\frac{\nu + r\theta_\nu}{n} \right) \right| \leq \frac{H}{s! \delta} \left| \frac{\nu + r\theta_\nu}{n} - x \right|^{s+1},$$

where $H = 2\|f^{(r+s)}\|$. Hence in either case,

$$\left| h_x^{(r)} \left(\frac{\nu + r\theta_\nu}{n} \right) \right| \leq \frac{\varepsilon}{s!} \left| \frac{\nu + r\theta_\nu}{n} - x \right|^s + \frac{H}{s! \delta} \left| \frac{\nu + r\theta_\nu}{n} - x \right|^{s+1}.$$

Now we can calculate

$$\begin{aligned} |(B_n h_x)^{(r)}(x)| &\leq \sum_{\nu=0}^{n-r} \left| h_x^{(r)} \left(\frac{\nu + r\theta_\nu}{n} \right) \right| b_{n-r,\nu}(x) \\ &\leq \frac{\varepsilon}{s!} \sum_{\nu=0}^{n-r} \left| \frac{\nu + r\theta_\nu}{n} - x \right|^s b_{n-r,\nu}(x) + \frac{H}{s! \delta} \sum_{\nu=0}^{n-r} \left| \frac{\nu + r\theta_\nu}{n} - x \right|^{s+1} b_{n-r,\nu}(x). \end{aligned}$$

Since $0 < \theta_\nu < 1$ and $0 \leq \nu/(n-r) \leq 1$ imply $|\theta_\nu - \nu/(n-r)| < 1$,

$$\left| \frac{\nu + r\theta_\nu}{n} - x \right| = \left| \left(\frac{\nu}{n-r} - x \right) + \frac{r}{n} \left(\theta_\nu - \frac{\nu}{n-r} \right) \right| \leq \left| \frac{\nu}{n-r} - x \right| + \frac{r}{n}.$$

It was shown in [26, pp.13-15] that

$$\max_{x \in [0,1]} \sum_{\nu=0}^n \left| \frac{\nu}{n} - x \right|^s b_{n,\nu}(x) = O(n^{-s/2}) \quad (n \rightarrow \infty).$$

Using this fact, we can estimate

$$\begin{aligned} &\max_{x \in [0,1]} \sum_{\nu=0}^{n-r} \left| \frac{\nu + r\theta_\nu}{n} - x \right|^s b_{n-r,\nu}(x) \\ &\leq \max_{x \in [0,1]} \sum_{\nu=0}^{n-r} \left(\left| \frac{\nu}{n-r} - x \right| + \frac{r}{n} \right)^s b_{n-r,\nu}(x) \\ &= \max_{x \in [0,1]} \sum_{\nu=0}^{n-r} b_{n-r,\nu}(x) \sum_{m=0}^s \binom{s}{m} \left| \frac{\nu}{n-r} - x \right|^m \left(\frac{r}{n} \right)^{s-m} \\ &\leq \sum_{m=0}^s \binom{s}{m} \left(\frac{r}{n} \right)^{s-m} \max_{x \in [0,1]} \sum_{\nu=0}^{n-r} \left| \frac{\nu}{n-r} - x \right|^m b_{n-r,\nu}(x) \\ &= \sum_{m=0}^s O(n^{-s+m}) O(n^{-m/2}) = \sum_{m=0}^s O(n^{-s+m/2}) = O(n^{-s/2}). \end{aligned}$$

Therefore,

$$\begin{aligned} \max_{x \in [0,1]} |(B_n h_x)^{(r)}(x)| &\leq \frac{\varepsilon}{s!} M_1 n^{-s/2} + \frac{H}{s! \delta} M_2 n^{-(s+1)/2} \quad \text{for some } M_1, M_2 > 0, \\ &< M n^{-s/2} \varepsilon \quad \text{for all sufficiently large } n, \end{aligned}$$

where M is a suitable constant. ■

Note that some special cases of Lemmas 2.8 and 2.9 are in Theorems 9.4.1 and 9.7.1 and in Lemma 9.5.2 in [13].

2.5. Proof of Theorem 2.4

Now we are to prove Theorem 2.4. Here the notations Theorem 2.4 (1), (2), and (3) stand for the properties (1), (2), and (3), respectively, in Theorem 2.4.

Proof of Theorem 2.4 (1). We can assume $n > K$ without loss of generality. From the relation $T_n f = \sum_{k=0}^K V_k^n (B_n f)^{[k]}$, we expand

$$\begin{aligned} e_{2p}(T_n f)^{(q+r)} &= e_{2p} \sum_{k=0}^K \sum_{m=0}^{q+r} \binom{q+r}{m} (V_k^n)^{(m)} ((B_n f)^{[k]})^{(q+r-m)} \\ &= \sum_{m=0}^{q+r} \binom{q+r}{m} \sum_{k=m}^K \frac{1}{k!} \left(\sum_{l=0}^{k-m} v_{k,l}^m(n) e_l \right) e_{2p}(B_n f)^{(k+q+r-m)} \\ &= \sum_{m=0}^{q+r} \binom{q+r}{m} \sum_{k=m}^K \frac{1}{k!} \sum_{l=0}^{k-m} v_{k,l}^m(n) e_{2p+l}(B_n f)^{(q+k-m+r)}. \end{aligned}$$

Applying Lemma 2.6, we have

$$v_{k,l}^m(n) = O(n^{[l/2]+m-k}).$$

Replacing p by $2p+l$ and q by $q+k-m$ in Lemma 2.8 implies

$$\|e_{2p+l}(B_n f)^{(q+k-m+r)}\| \leq M n^{q+k-m-\min\{p+[l/2], [(q+k-m)/2]\}} \|f^{(r)}\|.$$

Thus

$$\begin{aligned} \|e_{2p}(T_n f)^{(q+r)}\| &= \|f^{(r)}\| \sum_{m=0}^{q+r} \sum_{k=m}^K \sum_{l=0}^{k-m} O(n^{[l/2]+m-k}) O(n^{q+k-m-\min\{p+[l/2], [(q+k-m)/2]\}}) \\ &= \|f^{(r)}\| \sum_{m=0}^{q+r} \sum_{k=m}^K \sum_{l=0}^{k-m} O(n^{q-\min\{p, [(q+k-m)/2]-[l/2]\}}) \\ &= \|f^{(r)}\| \sum_{m=0}^{q+r} \sum_{k=m}^K O(n^{q-\min\{p, [(q+k-m)/2]-[(k-m)/2]\}}) \\ &= \|f^{(r)}\| \sum_{m=0}^{q+r} \sum_{k=m}^K O(n^{q-\min\{p, [q/2]\}}) \\ &= \|f^{(r)}\| O(n^{q-\min\{p, [q/2]\}}) \quad (n \rightarrow \infty); \end{aligned}$$

that is,

$$\|e_{2p}(T_n f)^{(q+r)}\| \leq M n^{q-\min\{p, [q/2]\}} \|f^{(r)}\|,$$

where M is a suitable constant and we used the inequality $[q/2] + [(k-m)/2] \leq [(q+k-m)/2]$ in the above calculation. ■

Proof of Theorem 2.4 (2). First, we give the proof in the case $f \in C^{K+\gamma}[0, 1]$. We define the functions g_x, h_x dependent of $x \in [0, 1]$ as

$$g_x(\xi) = \sum_{j=0}^{K+\gamma} f^{[j]}(x) (\xi - x)^j \quad (\xi \in [0, 1]), \quad h_x = f - g_x.$$

We can assume $n > K + \gamma$ without loss of generality. Since $\deg g_x \leq K + \gamma$,

$$(L_n g_x)^{(\gamma)}(\xi) = g_x^{(\gamma)}(\xi) = \sum_{j=\gamma}^{K+\gamma} f^{[j]}(x) j^{(\gamma)} (\xi - x)^{j-\gamma}.$$

Thus

$$(L_n g_x)^{(\gamma)}(x) = f^{(\gamma)}(x) \gamma! = f^{(\gamma)}(x).$$

Using this relation, we can estimate

$$\begin{aligned} \|(T_n f)^{(\gamma)} - f^{(\gamma)}\| &= \max_{x \in [0, 1]} |(T_n f)^{(\gamma)}(x) - f^{(\gamma)}(x)| \\ &= \max_{x \in [0, 1]} |(T_n g_x)^{(\gamma)}(x) + (T_n h_x)^{(\gamma)}(x) - f^{(\gamma)}(x)| \\ &\leq \max_{x \in [0, 1]} |(T_n g_x)^{(\gamma)}(x) - f^{(\gamma)}(x)| + \max_{x \in [0, 1]} |(T_n h_x)^{(\gamma)}(x)| \\ &= \max_{x \in [0, 1]} |(T_n g_x)^{(\gamma)}(x) - (L_n g_x)^{(\gamma)}(x)| + \max_{x \in [0, 1]} |(T_n h_x)^{(\gamma)}(x)| \\ &\leq \max_{x \in [0, 1]} \|(T_n g_x)^{(\gamma)} - (L_n g_x)^{(\gamma)}\| + \max_{x \in [0, 1]} |(T_n h_x)^{(\gamma)}(x)|. \end{aligned}$$

Here

$$T_n g_x = \sum_{k=0}^K V_k^n (B_n g_x)^{[k]}$$

implies

$$(T_n g_x)^{(\gamma)} = \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=m}^K \frac{1}{k!} (V_k^n)^{(m)} (B_n g_x)^{(k+\gamma-m)}.$$

Since $\deg g_x \leq K + \gamma$ implies $\deg B_n g_x \leq K + \gamma$,

$$L_n g_x = \sum_{k=0}^{K+\gamma} U_k^n (B_n g_x)^{[k]},$$

and consequently,

$$(L_n g_x)^{(\gamma)} = \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=m}^{K+\gamma} \frac{1}{k!} (U_k^n)^{(m)} (B_n g_x)^{(k+\gamma-m)}.$$

Therefore,

$$\begin{aligned} & \max_{x \in [0,1]} \|(T_n g_x)^{(\gamma)} - (L_n g_x)^{(\gamma)}\| \\ & \leq \sum_{m=0}^{\gamma} \binom{\gamma}{m} \left(\sum_{k=m}^K \frac{1}{k!} \| (V_k^n)^{(m)} - (U_k^n)^{(m)} \| \max_{x \in [0,1]} \| (B_n g_x)^{(k+\gamma-m)} \| \right. \\ & \quad \left. + \sum_{k=K+1}^{K+\gamma} \frac{1}{k!} \| (U_k^n)^{(m)} \| \max_{x \in [0,1]} \| (B_n g_x)^{(k+\gamma-m)} \| \right). \end{aligned}$$

It follows from the condition (c) and Markov's inequality (see, e.g., [2], [4, Chapter 5], [7, pp.91,228], and [11, Chapter 4]) that

$$\| (V_k^n)^{(m)} - (U_k^n)^{(m)} \| = o(n^{-\alpha}).$$

It follows from Theorem 2.3 and Markov's inequality that

$$\| (U_k^n)^{(m)} \| = O(n^{[k/2]-k}).$$

Furthermore, applying Lemma 2.8 with $p = q = 0$ and $r = k + \gamma - m$, we get

$$\| (B_n g_x)^{(k+\gamma-m)} \| \leq M \| g_x^{(k+\gamma-m)} \| \quad \text{for some constant } M.$$

Since

$$g_x^{(k+\gamma-m)}(\xi) = \sum_{j=k+\gamma-m}^{K+\gamma} f^{[j]}(x) j^{(k+\gamma-m)} (\xi - x)^{j-k-\gamma+m},$$

$$\| g_x^{(k+\gamma-m)} \| \leq \sum_{j=k+\gamma-m}^{K+\gamma} j^{(k+\gamma-m)} |f^{[j]}(x)| \leq \sum_{j=k+\gamma-m}^{K+\gamma} j^{(k+\gamma-m)} \| f^{[j]} \|.$$

Thus

$$\max_{x \in [0,1]} \| (B_n g_x)^{(k+\gamma-m)} \| \leq M' \quad \text{for some constant } M'.$$

Consequently,

$$\begin{aligned} \max_{x \in [0,1]} \|(T_n g_x)^{(\gamma)} - (L_n g_x)^{(\gamma)}\| &= o(n^{-\alpha}) + \sum_{k=K+1}^{K+\gamma} O(n^{[k/2]-k}) \\ &= o(n^{-\alpha}) + O(n^{[(K+1)/2]-(K+1)}) = o(n^{-\alpha}), \end{aligned}$$

where we used the assumption $K \geq 2\alpha$. On the other hand,

$$T_n h_x = \sum_{k=0}^K V_k^n (B_n h_x)^{[k]}$$

implies

$$(T_n h_x)^{(\gamma)} = \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=m}^K \frac{1}{k!} (V_k^n)^{(m)} (B_n h_x)^{(k+\gamma-m)}.$$

Therefore,

$$\max_{x \in [0,1]} |(T_n h_x)^{(\gamma)}(x)| \leq \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=m}^K \frac{1}{k!} \| (V_k^n)^{(m)} \| \max_{x \in [0,1]} | (B_n h_x)^{(k+\gamma-m)}(x) |.$$

It follows from the condition (b) — accordingly $\|V_k^n\| = O(n^{[k/2]-k})$ — and Markov's inequality that

$$\| (V_k^n)^{(m)} \| = O(n^{[k/2]-k}).$$

Furthermore, applying Lemma 2.9 with $r = k + \gamma - m$ and $s = K - k + m$, we get

$$\max_{x \in [0,1]} | (B_n h_x)^{(k+\gamma-m)}(x) | = o(n^{-(K-k+m)/2}).$$

Consequently,

$$\begin{aligned} \max_{x \in [0,1]} |(T_n h_x)^{(\gamma)}(x)| &= \sum_{m=0}^{\gamma} \sum_{k=m}^K O(n^{[k/2]-k}) o(n^{-(K-k+m)/2}) \\ &= \sum_{m=0}^{\gamma} o(n^{-(K+m)/2}) = o(n^{-K/2}) = o(n^{-\alpha}). \end{aligned}$$

Hence we obtain

$$\|(T_n f)^{(\gamma)} - f^{(\gamma)}\| = o(n^{-\alpha}) \quad (n \rightarrow \infty) \quad \text{for all } f \in C^{K+\gamma}[0,1]. \quad (2.5)$$

Next, we give the proof in the case $f \in C^{2\beta+\gamma}[0,1]$. It is well known (see [26, pp.25–26]) that for all $r \in \mathbb{N}_0$ and for all $f \in C^r[0,1]$

$$\lim_{n \rightarrow \infty} \| (B_n f)^{(r)} - f^{(r)} \| = 0. \quad (2.6)$$

(We can also prove it by applying (2.5) with $T_n = B_n$, $\alpha = 0$, $K = 0$, $\gamma = r$.) Take an arbitrary $\varepsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that

$$\| (B_N f)^{(r)} - f^{(r)} \| < \varepsilon \quad (r \leq 2\beta + \gamma).$$

Let $\varphi = B_N f$ and $\rho = f - \varphi$. Then

$$\|\rho^{(r)}\| < \varepsilon \quad (r \leq 2\beta + \gamma).$$

We define the new operator ${}_{\beta}T_n$ as

$${}_{\beta}T_n f = \sum_{k=0}^{2\beta} V_k^n (B_n f)^{[k]} \quad (f: [0, 1] \rightarrow \mathbf{R}).$$

Since

$$T_n f - f = T_n \rho - {}_{\beta}T_n \rho + {}_{\beta}T_n \rho - \rho + T_n \varphi - \varphi,$$

we can estimate

$$\|(T_n f)^{(\gamma)} - f^{(\gamma)}\| \leq \|(T_n \rho)^{(\gamma)} - ({}_{\beta}T_n \rho)^{(\gamma)}\| + \|({}_{\beta}T_n \rho)^{(\gamma)} - \rho^{(\gamma)}\| + \|(T_n \varphi)^{(\gamma)} - \varphi^{(\gamma)}\|.$$

Since φ is a polynomial, it is immediate from (2.5) that

$$\|(T_n \varphi)^{(\gamma)} - \varphi^{(\gamma)}\| = o(n^{-\alpha}).$$

Applying (2.5) with replacing T_n by ${}_{\beta}T_n$, α by β , and K by 2β , we have

$$\|({}_{\beta}T_n \rho)^{(\gamma)} - \rho^{(\gamma)}\| = o(n^{-\beta}).$$

Therefore, it suffices to estimate the first term of the right-hand side in the above inequality.

Since

$$\begin{aligned} T_n \rho - {}_{\beta}T_n \rho &= \sum_{k=2\beta+1}^K V_k^n (B_n \rho)^{[k]}, \\ (T_n \rho)^{(\gamma)} - ({}_{\beta}T_n \rho)^{(\gamma)} &= \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=\{m, 2\beta+1\}}^K \frac{1}{k!} (V_k^n)^{(m)} (B_n \rho)^{(k+\gamma-m)} \\ &= \sum_{m=0}^{\gamma} \binom{\gamma}{m} \left(\sum_{k=\{m, 2\beta+1\}}^{2\beta+m} \frac{1}{k!} (V_k^n)^{(m)} (B_n \rho)^{(k+\gamma-m)} \right. \\ &\quad \left. + \sum_{k=2\beta+m+1}^K \frac{1}{k!} (V_k^n)^{(m)} (B_n \rho)^{(k+\gamma-m)} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \|(T_n \rho)^{(\gamma)} - ({}_{\beta}T_n \rho)^{(\gamma)}\| &\leq \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=\{m, 2\beta+1\}}^{2\beta+m} \frac{1}{k!} \|(V_k^n)^{(m)}\| \|(B_n \rho)^{(k+\gamma-m)}\| \\ &\quad + \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=2\beta+m+1}^K \frac{1}{k!} \|(V_k^n)^{(m)}\| \|(B_n \rho)^{(k+\gamma-m)}\|. \end{aligned}$$

Applying Lemma 2.8 with $p = q = 0$ and $r = k + \gamma - m$, we get

$$\begin{aligned} \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=\{m, 2\beta+1\}}^{2\beta+m} \frac{1}{k!} \|(V_k^n)^{(m)}\| \|(B_n \rho)^{(k+\gamma-m)}\| \\ = \sum_{m=0}^{\gamma} \sum_{k=\{m, 2\beta+1\}}^{2\beta+m} O(n^{[k/2]-k}) \|\rho^{(k+\gamma-m)}\| = O(n^{-\beta-1})\varepsilon. \end{aligned}$$

On the other hand, when $k \geq 2\beta + m + 1$,

$$(V_k^n)^{(m)} (B_n \rho)^{(k+\gamma-m)} = \sum_{l=0}^{k-m} v_{k,l}^m(n) e_l (B_n \rho)^{(k+\gamma-m)}.$$

As we mentioned in the proof of Theorem 2.4 (1), we have

$$v_{k,l}^m(n) = O(n^{[l/2]+m-k}).$$

Applying Lemma 2.8 with $p = l$, $q = k - 2\beta - m$, and $r = 2\beta + \gamma$, we get

$$\|e_l (B_n \rho)^{(k+\gamma-m)}\| = O(n^{k-2\beta-m-\min\{[l/2], [(k-m)/2]-\beta\}}) \|\rho^{(2\beta+\gamma)}\|.$$

Therefore,

$$\begin{aligned} \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=2\beta+m+1}^K \frac{1}{k!} \|(V_k^n)^{(m)}\| \|(B_n \rho)^{(k+\gamma-m)}\| \\ = \sum_{m=0}^{\gamma} \sum_{k=2\beta+m+1}^K \sum_{l=0}^{k-m} O(n^{[l/2]+m-k}) O(n^{k-2\beta-m-\min\{[l/2], [(k-m)/2]-\beta\}}) \|\rho^{(2\beta+\gamma)}\| \\ = \sum_{m=0}^{\gamma} \sum_{k=2\beta+m+1}^K \sum_{l=0}^{k-m} O(n^{[l/2]-2\beta-\min\{[l/2], [(k-m)/2]-\beta\}}) \varepsilon \\ = \sum_{m=0}^{\gamma} \sum_{k=2\beta+m+1}^K O(n^{[(k-m)/2]-2\beta-\min\{[(k-m)/2], [(k-m)/2]-\beta\}}) \varepsilon = O(n^{-\beta})\varepsilon. \end{aligned}$$

Thus the proof is completed. ■

Proof of Theorem 2.4 (3). We define the new operator \tilde{T}_n as

$$\tilde{T}_n f = \sum_{k=0}^n \tilde{V}_k^n (B_n f)^{[k]} \quad (f: [0, 1] \rightarrow \mathbf{R}),$$

where

$$\tilde{V}_k^n = \begin{cases} V_k^n - n^{-(\alpha+1)} R_k & \text{if } 0 \leq k \leq 2\alpha + 2, \\ V_k^n & \text{if } 2\alpha + 2 < k \leq n. \end{cases}$$

Let $\tilde{K} = \max\{K, 2\alpha + 2\}$. In Theorem 2.4 (2), we replace T_n by \tilde{T}_n , α by $\alpha + 1$, and K by \tilde{K} . Then we can easily verify that all the preconditions are satisfied. Therefore, we obtain for all $f \in C^{2\alpha+2+\gamma}$,

$$\|(\tilde{T}_n f)^{(\gamma)} - f^{(\gamma)}\| = o(n^{-(\alpha+1)}).$$

Now we can estimate

$$\begin{aligned} & \left\| n^{\alpha+1} ((T_n f)^{(\gamma)} - f^{(\gamma)}) - \left(\sum_{k=0}^{2\alpha+2} R_k f^{[k]} \right)^{(\gamma)} \right\| \\ &= \left\| n^{\alpha+1} ((\tilde{T}_n f)^{(\gamma)} - f^{(\gamma)}) + n^{\alpha+1} ((T_n f)^{(\gamma)} - (\tilde{T}_n f)^{(\gamma)}) - \left(\sum_{k=0}^{2\alpha+2} R_k f^{[k]} \right)^{(\gamma)} \right\| \\ &\leq n^{\alpha+1} \|(\tilde{T}_n f)^{(\gamma)} - f^{(\gamma)}\| + \sum_{k=0}^{2\alpha+2} \|(R_k (B_n f)^{[k]})^{(\gamma)} - (R_k f^{[k]})^{(\gamma)}\| \end{aligned}$$

As we mentioned above, the first term converges to zero when n tends to infinity. It suffices to estimate the second term. This is equal to

$$\begin{aligned} & \sum_{k=0}^{2\alpha+2} \|(R_k ((B_n f)^{[k]} - f^{[k]}))^{(\gamma)}\| \\ &= \sum_{k=0}^{2\alpha+2} \left\| \sum_{m=0}^{\gamma} \binom{\gamma}{m} \frac{R_k^{(\gamma-m)}}{k!} ((B_n f)^{(k+m)} - f^{(k+m)}) \right\| \\ &\leq \sum_{k=0}^{2\alpha+2} \sum_{m=0}^{\gamma} \binom{\gamma}{m} \frac{\|R_k^{(\gamma-m)}\|}{k!} \|(B_n f)^{(k+m)} - f^{(k+m)}\| \rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

where we used (2.6). ■

CHAPTER 3

A NEW CLASS OF MODIFIED BERNSTEIN OPERATORS

The left Bernstein quasi-interpolant operator introduced by Sablonnière is a kind of modified Bernstein operator that has good stability and convergence rate properties. However, we recently found that it is not very convenient for practical applications. Fortunately, we showed in Chapter 2 that there exist many operators having stability and convergence rate properties similar to those of Sablonnière's operator. In this chapter, we introduce another specific class of such operators generated from the operator introduced by D. D. Stancu. We present detailed results about this class, some of which can be applied to numerical quadrature. Finally, we clarify its advantages and assert that it is more natural and more convenient both theoretically and practically, than that of Sablonnière.

3.1. Introduction

The left Bernstein quasi-interpolant operator $B_n^{(K)}$ ($n \in \mathbf{N}$, $K \in \mathbf{N}_0$) was introduced by Sablonnière in [38, 39, 40] and redefined by us in Chapter 2 as

$$B_n^{(K)} f = \sum_{k=0}^{\{K, n\}} U_k^n (B_n f)^{[k]} \quad (f: [0, 1] \rightarrow \mathbf{R}),$$

where B_n is the Bernstein operator of order n and U_k^n are the unique polynomials satisfying

$$L_n f = \sum_{k=0}^n U_k^n (B_n f)^{[k]} \quad (f: [0, 1] \rightarrow \mathbf{R}), \quad (3.1)$$

where L_n is the Lagrange operator of the same sampling points as B_n . It was shown in [39] that $B_n^{(K)} f - f = O(n^{-(\lfloor K/2 \rfloor + 1)})$ (pointwise) for every f sufficiently smooth, while the boundedness of $\{\|B_n^{(K)}\|\}_n$ was guaranteed in [53]. Furthermore, we generalized and refined these results in Chapter 2.

However, $B_n^{(K)}$ is not very convenient for practical applications. The sequence $\{B_n^{(K)}\}_n$ is indeed stable in the sense that it is uniformly bounded, but the value of $\|B_n^{(K)}\|$ grows

extremely fast as K increases, especially when $n(\geq K)$ is near to K . In fact, when $n = K$, the operator $B_n^{(K)}$ reduces to L_K , whose norm grows exponentially with respect to K , as we mentioned in Chapter 1. We will give details as Table 3 in the final section.

There is another defect regarding $B_n^{(K)}$. Though it was introduced for the purpose of accelerating convergence of Bernstein polynomials, the correspondence between the parameter K and the order of its convergence rate is not one-to-one. In fact, for every $\alpha \in \mathbf{N}_0$, the convergence rates of $\{B_n^{(2\alpha)}f\}_n$ and $\{B_n^{(2\alpha+1)}f\}_n$ are both $O(n^{-(\alpha+1)})$. The operator $B_n^{(K)}$ was constructed by truncating at $k = K$ the expansion (3.1) of L_n , but the above fact suggests that the mode of truncation is not essential for our original purpose.

Fortunately, we showed in Chapter 2 that there exist many operators having stability and convergence rate properties similar to those of Sablonnière's operator. In this chapter, we will introduce another specific class of such operators. The new operator ${}_aB_n$ ($n \in \mathbf{N}$, $\alpha \in \mathbf{N}_0$) is generated from the operator introduced by Stancu [43, 44, 45, 46, 47], and also can be regarded as a truncated operator of L_n , but the mode of truncation is distinct from that of the preceding operator. We will present some detailed results about our new operator and several propositions about Stancu's operator. Finally, we will compare it with Sablonnière's operator, clarifying advantages of our operator.

The content of this chapter has been published as [23], furthermore which is cited in [41].

3.2. Main Results

3.2.1. Definition

Stancu [43, 44] introduced the operator $P_n^{(s)}$ for every $n \in \mathbf{N}$ and for every $s \in \mathbf{R}$ satisfying $\prod_{\mu=0}^{n-1}(1+\mu s) \neq 0$, as

$$P_n^{(s)}f(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} \frac{[\prod_{\mu=0}^{\nu-1}(x+\mu s)][\prod_{\mu=0}^{n-\nu-1}(1-x+\mu s)]}{\prod_{\mu=0}^{n-1}(1+\mu s)} \\ (f: [0, 1] \rightarrow \mathbf{R}, \quad x \in [0, 1]).$$

(The notation $P_n^{(s)}$ is due to [47].) This operator has the two identities

$$P_n^{(0)}f(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} x^\nu (1-x)^{n-\nu} = B_nf(x), \\ P_n^{(-1/n)}f(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} \left(\frac{nx}{n-\nu}\right) \binom{n(1-x)}{n-\nu} = L_nf(x),$$

which mean that the class of Stancu operators contains the Bernstein and the Lagrange ones. Stancu investigated in particular the case $s \geq 0$ as a class of positive linear operators. However, here we treat Stancu's operator from quite a different standpoint. We use it to introduce a new class of operators as follows.

DEFINITION 3.1. We define the modified Bernstein operator ${}_aB_n$ (of order $n \in \mathbf{N}$ and sharpness degree $\alpha \in \mathbf{N}_0 \cup \{\infty\}$) as

$${}_aB_nf(x) = \sum_{j=0}^{\alpha} \frac{(-1)^j}{n^j j!} \left. \frac{\partial^j P_n^{(s)}f(x)}{\partial s^j} \right|_{s=0} \quad (f: [0, 1] \rightarrow \mathbf{R}, \quad x \in [0, 1]);$$

that is, ${}_aB_nf(x)$ (for fixed f, x) is generated by putting $s = -1/n$ in the Maclaurin series truncated at degree α of $P_n^{(s)}f(x)$ regarded as a function of s .

Remark. The function $P_n^{(s)}f(x)$ (with respect to s) is analytic where $(n-1)|s| < 1$ because

$$|1+\mu s| \geq 1-\mu|s| \geq 1-(n-1)|s| > 0 \quad (\mu = 0, 1, \dots, n-1),$$

and $s = -1/n$ belongs to the region $(n-1)|s| < 1$.

Note that the identities ${}_0B_n = P_n^{(0)} = B_n$ and ${}_{\infty}B_n = P_n^{(-1/n)} = L_n$ hold. In this context, ${}_aB_n$ can be regarded as an "intermediate" operator between B_n and L_n . Though Sablonnière's operator $B_n^{(K)}$ is also an intermediate one between them, our new operator is substantially distinct from it.

3.2.2. Representations and properties of ${}_aB_n$

Now we provide two kinds of representations of our operator.

THEOREM 3.1. The modified Bernstein operator ${}_αB_n$ has the representation

$${}_αB_nf = \sum_{k=0}^{2α} (B_nf)^{[k]} \sum_{j=0}^α \frac{\Upsilon_{j,k}}{n^j} = \sum_{j=0}^α \frac{1}{n^j} \sum_{k=0}^{2j} \Upsilon_{j,k} (B_nf)^{[k]} \quad (f: [0, 1] \rightarrow \mathbf{R}),$$

where $\Upsilon_{j,k}$ are the polynomials of degree at most k determined by the recursion formula

$$\Upsilon_{j,-1} = 0 \quad (j \geq 0),$$

$$\Upsilon_{0,0} = 1,$$

$$\Upsilon_{j,0} = 0 \quad (j \geq 1),$$

$$\Upsilon_{0,k} = 0 \quad (1 \leq k \leq n),$$

$$\Upsilon_{j,k+1} = k(\Upsilon_{j-1,k+1} - e_1 \Upsilon_{j-1,k} - e_2 \Upsilon_{j-1,k-1}) \quad (j \geq 1, 0 \leq k \leq n-1).$$

Remark. Since $\Upsilon_{j,k}$ are independent of n and we can take n arbitrarily large, we can understand that $\Upsilon_{j,k}$ are defined for all $j, k \in \mathbf{N}_0$.

THEOREM 3.2. The modified Bernstein operator ${}_αB_n$ has the explicit representation

$${}_αB_nf(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu} \left(\frac{k}{n}\right)^α$$

$$(f: [0, 1] \rightarrow \mathbf{R}, \quad x \in [0, 1]).$$

Note that we can extend the definition of ${}_αB_n$ for all nonnegative real numbers $α$ by using this theorem. This is a surprising fact.

The following theorem, which concerns stability and convergence rate, is the theoretically most important result in this chapter.

THEOREM 3.3. For each $α \in \mathbf{N}_0$, the sequence $\{{}_αB_n\}_{n=1}^\infty$ has the following properties:

- (1) for all $p, q, r \in \mathbf{N}_0$, there exists a constant M such that for all $n \in \mathbf{N}$ and for all $f \in C^r[0, 1]$,

$$\|e_{2p}({}_αB_nf)^{(q+r)}\| \leq Mn^{q-\min\{p, [q/2]\}} \|f^{(r)}\|;$$

- (2) for all $β, γ \in \mathbf{N}_0$ ($β \leq α$) and for all $f \in C^{2β+γ}[0, 1]$,

$$\|({}_αB_nf)^{(γ)} - f^{(γ)}\| = o(n^{-β}) \quad (n \rightarrow \infty);$$

- (3) for all $γ \in \mathbf{N}_0$ and for all $f \in C^{2α+γ+2}[0, 1]$,

$$\lim_{n \rightarrow \infty} n^{α+1} (({}_αB_nf)^{(γ)} - f^{(γ)}) = - \left(\sum_{k=0}^{2α+2} \Upsilon_{α+1,k} f^{[k]} \right)^{(γ)} \quad \text{in the sense of } \|\cdot\|.$$

3.2.3. Application to numerical quadrature

We denote by I the integration operator defined as

$$If = \int_0^1 f(x) dx \quad (f \in C[0, 1]),$$

and we define the integrating operator ${}_αI_n$ as

$${}_αI_n = I {}_αB_n.$$

Theorem 3.3 readily implies the following corollary.

COROLLARY 3.4. For each $α \in \mathbf{N}_0$, the sequence $\{{}_αI_n\}_{n=1}^\infty$ has the following properties:

- (1) there exists a constant M such that for all $n \in \mathbf{N}$ and for all $f \in C[0, 1]$,

$$|{}_αI_nf| \leq M \|f\|;$$

- (2) for all $β \in \mathbf{N}_0$ ($β \leq α$) and for all $f \in C^{2β}[0, 1]$,

$$|{}_αI_nf - If| = o(n^{-β}) \quad (n \rightarrow \infty);$$

- (3) for all $f \in C^{2α+2}[0, 1]$,

$$\lim_{n \rightarrow \infty} n^{α+1} ({}_αI_nf - If) = -I \sum_{k=0}^{2α+2} \Upsilon_{α+1,k} f^{[k]}.$$

Now let us consider how to calculate ${}_αI_nf$. It is a detour to represent ${}_αB_nf$ as the form in Theorem 3.1 and then integrate it on $[0, 1]$. In fact, there is a more direct way to calculate ${}_αI_nf$.

THEOREM 3.5. The integrating operator ${}_αI_n$ has the representation

$${}_αI_nf = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) {}_αw_{n,\nu},$$

where

$$\begin{aligned} {}_0w_{n,\nu} &= \frac{1}{n+1}, \\ {}_\alpha w_{n,\nu} &= \frac{1}{n} \left[1 - \sum_{k=0}^{\alpha-1} (-1)^k \gamma_{k+1} \left((-1)^\nu \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) n^{(k)} \sum_{j=k}^{\alpha-1} \frac{S(j-1, k-1)}{n^j} \right] \\ &\quad (\alpha \geq 1), \end{aligned}$$

where $\gamma_k = \int_0^1 \binom{x}{k} dx$ ($k \in \mathbb{N}_0$) and $S(j, k)$ are defined in "Notations."

Remark. There are various notations for the Stirling numbers of the second kind [1, p.822], but we adopt the symbol $S(\cdot, \cdot)$ because it has often been used recently (e.g., [34, 35] and [51, Chapter 13]).

As we mentioned in "Notations," we denote by $\mathcal{B}_\nu^{(n)}(\cdot)$ and $\mathcal{B}_\nu^{(n)}$ the Bernoulli polynomial and number, respectively, which appear in [29, pp.124-127]. (We use the symbol \mathcal{B} instead of B to distinguish the Bernoulli polynomials and numbers from the Bernstein operators.) Then the identities $\gamma_k = \mathcal{B}_k^{(k)}(1)/k!$ and $S(j, k) = \binom{j}{k} \mathcal{B}_{j-k}^{(-k)}$ hold [29, pp.130, 133]. It is interesting that these two systems of numbers are unified in terms of the Bernoulli polynomials.

Though we can calculate γ_k 's by way of the Bernoulli polynomials, there is a more direct way to calculate them. Let $0 < |t| < 1$ and consider their generating function

$$\sum_{k=0}^{\infty} \gamma_k t^k = \int_0^1 (1+t)^x dx = \frac{t}{\log(1+t)}.$$

(An equivalent identity for $\mathcal{B}_k^{(k)}(1)$ appears in [29, p.135].) Since we can expand

$$\frac{\log(1+t)}{t} = \sum_{l=0}^{\infty} \frac{(-t)^l}{l+1},$$

we can calculate

$$\left(\sum_{l=0}^{\infty} \frac{(-t)^l}{l+1} \right) \left(\sum_{k=0}^{\infty} \gamma_k t^k \right) = \sum_{k=0}^{\infty} t^k \sum_{l=0}^k \frac{(-1)^l \gamma_{k-l}}{l+1} = 1.$$

Equating coefficients of t^k , we obtain the recursion formula

$$\gamma_0 = 1, \quad \gamma_k = \sum_{l=1}^k \frac{(-1)^{l-1} \gamma_{k-l}}{l+1} \quad (k \geq 1).$$

Some γ_k 's are evaluated as follows:

$$\gamma_1 = \frac{1}{2}, \quad \gamma_2 = -\frac{1}{12}, \quad \gamma_3 = \frac{1}{24}, \quad \gamma_4 = -\frac{19}{720}, \quad \gamma_5 = \frac{3}{160}.$$

The above theorem and the identity

$$\begin{aligned} (-1)^k \Delta_{1/n}^k f(0) + \nabla_{1/n}^k f(1) &= (-1)^k \sum_{\nu=0}^k (-1)^{k-\nu} \binom{k}{\nu} f\left(\frac{\nu}{n}\right) + \sum_{\nu=0}^k (-1)^\nu \binom{k}{\nu} f\left(1 - \frac{\nu}{n}\right) \\ &= \sum_{\nu=0}^k (-1)^\nu \binom{k}{\nu} f\left(\frac{\nu}{n}\right) + \sum_{\nu=n-k}^n (-1)^{n-\nu} \binom{k}{n-\nu} f\left(\frac{\nu}{n}\right) \\ &= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \left((-1)^\nu \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \end{aligned}$$

imply the following corollary.

COROLLARY 3.6. When $\alpha \geq 1$, the integrating operator ${}_a I_n$ can be represented as

$${}_a I_n f = \frac{1}{n} \left[\sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) - \sum_{k=0}^{\alpha-1} \gamma_{k+1} \left(\Delta_{1/n}^k f(0) + (-1)^k \nabla_{1/n}^k f(1) \right) n^{(k)} \sum_{j=k}^{\alpha-1} \frac{S(j-1, k-1)}{n^j} \right].$$

This corollary means that ${}_a I_n$ ($\alpha \geq 1$) brings us a kind of trapezoidal rule with end modifications. Particularly when $\alpha = 1$, it coincides with the ordinary trapezoidal rule.

Now let us consider positivity of the linear operator ${}_a I_n$. Proposition 3.8 (in the next section) and Definition 3.1 immediately imply ${}_a B_n f = f$ for all $f \in \mathbf{P}_1$, and in particular, ${}_a I_n 1 = 1$. Therefore, as long as the linear operator ${}_a I_n$ is positive, the identity $\|{}_a I_n\| = 1$ holds, which means ${}_a I_n$ is the best for stability. On the other hand, from the viewpoint of convergence rate, it is desirable to choose α as large as possible. For this reason, we naturally become interested in the problem of determining the maximum of α that preserves positivity of ${}_a I_n$ for each n . We have the following result.

THEOREM 3.7. For each $n \in \mathbf{N}$, we define $\alpha(n) = \max\{\alpha \in \mathbf{N}_0 \cup \{\infty\} \mid {}_\alpha I_n \text{ is positive}\}$.

Then

$$\alpha(n) = \begin{cases} \infty & (1 \leq n \leq 7, n = 9) \\ 13 & (n = 8, 10, 11) \\ 12 & (n = 12) \\ 11 & (13 \leq n \leq 15) \\ 10 & (16 \leq n \leq 23) \\ 9 & (24 \leq n \leq 104) \\ 8 & (n \geq 105). \end{cases}$$

Furthermore, all the ${}_\alpha I_n$ ($\alpha < \alpha(n)$, $\alpha \in \mathbf{N}_0$) are positive.

Remark. The operator ${}_\infty I_n$ means that

$${}_\infty I_n f = \int_0^1 {}_\infty B_n f(x) dx = \int_0^1 L_n f(x) dx;$$

that is to say, ${}_\infty I_n$ is the operator corresponding to the Newton-Cotes rule.

We present Table 1, which is a result of numerical experiments on the Runge function $f(x) = 1/(1 + 25(2x - 1)^2)$. This suggests that our "intermediate" rule between the trapezoidal and the Newton-Cotes ones is very effective for numerical quadrature.

3.3. Propositions Regarding Stancu's Operator

Before proving the main results, here we collect several propositions about Stancu's operator, most of which are new.

TABLE 1. Approximate Values of $|{}_n I_n f - I f|$ When $f(x) = 1/(1 + 25(2x - 1)^2)$

n	α			
	1	8	9	∞
4	5.39×10^{-2}	1.30×10^{-2}	1.89×10^{-2}	3.73×10^{-2}
8	3.77×10^{-3}	4.61×10^{-3}	1.08×10^{-2}	1.25×10^{-1}
16	6.90×10^{-5}	1.83×10^{-5}	4.11×10^{-5}	8.99×10^{-1}
32	2.41×10^{-5}	1.48×10^{-9}	1.22×10^{-9}	1.51×10^2
64	6.02×10^{-6}	3.34×10^{-13}	3.75×10^{-14}	1.51×10^7
128	1.50×10^{-6}	4.48×10^{-16}	2.73×10^{-17}	—
256	3.76×10^{-7}	7.24×10^{-19}	2.21×10^{-20}	—

PROPOSITION 3.8. Stancu's operator $P_n^{(s)}$ has the following representation in terms of the difference operators:

$$P_n^{(s)} f(x) = \sum_{r=0}^n \Delta_{1/n}^r f(0) \binom{n}{r} \prod_{\mu=0}^{r-1} \frac{x + \mu s}{1 + \mu s} \quad (3.2)$$

$$= \sum_{r=0}^n (-1)^r \nabla_{1/n}^r f(1) \binom{n}{r} \prod_{\mu=0}^{r-1} \frac{1 - x + \mu s}{1 + \mu s}. \quad (3.3)$$

Remark. In particular, by considering the cases $s = 0$ and $s = -1/n$, we obtain

$$B_n f(x) = \sum_{r=0}^n \Delta_{1/n}^r f(0) \binom{n}{r} x^r = \sum_{r=0}^n (-1)^r \nabla_{1/n}^r f(1) \binom{n}{r} (1-x)^r,$$

$$L_n f(x) = \sum_{r=0}^n \Delta_{1/n}^r f(0) \binom{nx}{r} = \sum_{r=0}^n (-1)^r \nabla_{1/n}^r f(1) \binom{n(1-x)}{r}.$$

The latter is nothing but Newton's forward (backward) interpolation formula.

Proofs of this proposition appear in [44, 45], but here we give a more direct one.

Proof of Proposition 3.8. Suppose $s \neq 0$ and let $a = -1/s$. Then the right-hand side of (3.2) can be calculated as

$$\sum_{r=0}^n \Delta_{1/n}^r f(0) \binom{n}{r} \frac{(ax)^{(r)}}{a^{(r)}}$$

$$\begin{aligned}
&= \sum_{r=0}^n \sum_{\nu=0}^r (-1)^{r-\nu} \binom{r}{\nu} f\left(\frac{\nu}{n}\right) \binom{n}{r} \frac{(ax)^{(r)}}{a^{(r)}} \\
&= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} \sum_{r=\nu}^n (-1)^{r-\nu} \binom{n-\nu}{r-\nu} \frac{(ax)^{(r)}}{a^{(r)}} \\
&= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} \sum_{r=0}^{n-\nu} (-1)^r \binom{n-\nu}{r} \frac{(ax)^{(r+\nu)}}{a^{(r+\nu)}} \\
&= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} \frac{(ax)^{(\nu)}}{a^{(n)}} \sum_{r=0}^{n-\nu} (-1)^r \binom{n-\nu}{r} (ax-\nu)^{(r)} (a-r-\nu)^{(n-r-\nu)}.
\end{aligned}$$

By using the identities $a^{(m)} = (-1)^m (-a+m-1)^{(m)}$ and $\sum_{r=0}^m \binom{m}{r} a^{(r)} b^{(m-r)} = (a+b)^{(m)}$, the inner sum can be calculated as

$$\begin{aligned}
&\sum_{r=0}^{n-\nu} (-1)^r \binom{n-\nu}{r} (ax-\nu)^{(r)} (-1)^{n-r-\nu} (-a+n-1)^{(n-r-\nu)} \\
&= (-1)^{n-\nu} \sum_{r=0}^{n-\nu} \binom{n-\nu}{r} (ax-\nu)^{(r)} (-a+n-1)^{(n-r-\nu)} \\
&= (-1)^{n-\nu} (-a(1-x) + n - \nu - 1)^{(n-\nu)} = (a(1-x))^{(n-\nu)}.
\end{aligned}$$

Thus the right-hand side of (3.2) equals $P_n^{(s)} f(x)$ if $s \neq 0$. Since both sides of (3.2) are continuous at $s = 0$, (3.2) is valid also when $s = 0$. Letting $\tilde{f}(x) = f(1-x)$ and replacing f by \tilde{f} and x by $1-x$ imply (3.3). ■

PROPOSITION 3.9. *Stancu's operator $P_n^{(s)}$ has the degree-preserving property*

$$P_n^{(s)} \mathbf{P}_m \subseteq \mathbf{P}_m \quad (0 \leq m \leq n).$$

Proof. Let $f \in \mathbf{P}_m$. Then $\Delta_{1/n}^r f(0) = 0$ if $r > m$. Proposition 3.8 implies $P_n^{(s)} f \in \mathbf{P}_m$. ■

PROPOSITION 3.10. *There exist unique $U_k^{(s)} \in \mathbf{P}_k$ ($0 \leq k \leq n$) such that*

$$P_n^{(s)} f = \sum_{k=0}^n U_k^{(s)} (B_n f)^{[k]} \quad (f: [0, 1] \rightarrow \mathbf{R}),$$

where $U_k^{(s)}$ are determined by the recursion formula

$$\begin{aligned}
U_{-1}^{(s)} &= 0, \quad U_0^{(s)} = 1, \\
(1+ks)U_{k+1}^{(s)} &= ks(e_1 U_k^{(s)} + e_2 U_{k-1}^{(s)}) \quad (0 \leq k \leq n-1).
\end{aligned}$$

Remark 1. Put $s = -1/n$ to verify that this proposition is a generalization of Theorem 2.2. (We can identify $U_k^{(-1/n)}$ with U_k^n in (3.1).)

Remark 2. We use the notation $U_k^{(s)}$ throughout this thesis.

Proof. The unique existence of $U_k^{(s)} \in \mathbf{P}_k$ is guaranteed by Theorem 2.2 and Proposition 3.9. It suffices to derive the recursion formula. Since it is obviously valid when $s = 0$, we suppose $s \neq 0$ and let $a = -1/s$.

Let $x \in [0, 1]$, $t \in (-1, 1)$ and fix them for a while. We consider the case

$$f(\xi) = (1 + (1-x)t)^{n\xi} (1 - xt)^{n(1-\xi)} \quad (\xi \in [0, 1]).$$

Then, as we did in the proof of Theorem 2.1, we get

$$(B_n f)^{[k]}(x) = \binom{n}{k} t^k \quad \text{for all } k \leq n.$$

Therefore the relation $P_n^{(s)} f = \sum_{k=0}^n U_k^{(s)} (B_n f)^{[k]}$ implies

$$\sum_{\nu=0}^n (1 + (1-x)t)^\nu (1 - xt)^{n-\nu} \binom{n}{\nu} \frac{(ax)^{(\nu)} (a(1-x))^{(n-\nu)}}{a^{(n)}} = \sum_{k=0}^n \binom{n}{k} U_k^{(s)}(x) t^k. \quad (3.4)$$

The left-hand side can be expanded as

$$\sum_{\nu=0}^n \left[\sum_{l=0}^{\nu} \binom{\nu}{l} (1-x)^l t^l \right] \left[\sum_{m=0}^{n-\nu} \binom{n-\nu}{m} (-x)^m t^m \right] \binom{ax}{\nu} \binom{a(1-x)}{n-\nu} \binom{a}{n}^{-1}.$$

Since the region $0 \leq \nu \leq n$, $0 \leq l \leq \nu$, $0 \leq m \leq n-\nu$ corresponds to the region $0 \leq k \leq n$, $0 \leq l \leq k$, $0 \leq \nu' \leq n-k$ when we let $k = l + m$, $\nu' = \nu - l$, the above formula equals

$$\begin{aligned}
&\sum_{k=0}^n \sum_{l=0}^k \sum_{\nu=0}^{n-k} t^k (1-x)^l (-x)^{k-l} \binom{\nu+l}{l} \binom{ax}{\nu+l} \binom{n-\nu-l}{k-l} \binom{a(1-x)}{n-\nu-l} \binom{a}{n}^{-1} \\
&= \sum_{k=0}^n t^k \binom{a}{n}^{-1} \sum_{l=0}^k \binom{ax}{l} \binom{a(1-x)}{k-l} (1-x)^l (-x)^{k-l} \sum_{\nu=0}^{n-k} \binom{ax-l}{\nu} \binom{a(1-x)-k+l}{n-k-\nu} \\
&= \sum_{k=0}^n t^k \binom{a}{n}^{-1} \sum_{l=0}^k \binom{ax}{l} \binom{a(1-x)}{k-l} (1-x)^l (-x)^{k-l} \binom{a-k}{n-k} \\
&= \sum_{k=0}^n t^k \binom{n}{k} \binom{a}{k}^{-1} \sum_{l=0}^k \binom{ax}{l} \binom{a(1-x)}{k-l} (1-x)^l (-x)^{k-l}.
\end{aligned}$$

Therefore, equating coefficients of t^k on both sides of (3.4) yields

$$\binom{a}{k} U_k^{(s)}(x) = \sum_{l=0}^k \binom{ax}{l} \binom{a(1-x)}{k-l} (1-x)^l (-x)^{k-l} \quad (0 \leq k \leq n).$$

Here we let $\tilde{U}_k(x)$ be the right-hand side for all $k \in \mathbf{N}_0$ and consider its generating function

$$\begin{aligned} \sum_{k=0}^{\infty} \tilde{U}_k(x) t^k &= \sum_{k=0}^{\infty} t^k \sum_{l=0}^k \binom{ax}{l} \binom{a(1-x)}{k-l} (1-x)^l (-x)^{k-l} \\ &= \left[\sum_{l=0}^{\infty} \binom{ax}{l} (1-x)^l t^l \right] \left[\sum_{m=0}^{\infty} \binom{a(1-x)}{m} (-x)^m t^m \right] \\ &= (1 + (1-x)t)^{ax} (1 - xt)^{a(1-x)}. \end{aligned} \quad (3.5)$$

Putting $t = 0$ gives

$$\tilde{U}_0 = U_0^{(s)} = 1. \quad (3.6)$$

Differentiating (3.5) by t and multiplying it by $(1 + (1-x)t)(1 - xt)$, we get

$$(1 + e_1(x)t - e_2(x)t^2) \sum_{k=1}^{\infty} k \tilde{U}_k(x) t^{k-1} = -ae_2(x)t \sum_{k=0}^{\infty} \tilde{U}_k(x) t^k$$

by virtue of

$$(1 + (1-x)t)(1 - xt) \frac{d}{dt} [(1 + (1-x)t)^{ax} (1 - xt)^{a(1-x)}] = -ae_2(x)t(1 + (1-x)t)^{ax} (1 - xt)^{a(1-x)}.$$

Rearrangement of the above formula gives

$$\sum_{k=0}^{\infty} (k+1) \tilde{U}_{k+1}(x) t^k = -e_1(x) \sum_{k=0}^{\infty} k \tilde{U}_k(x) t^k - e_2(x) \sum_{k=1}^{\infty} (a-k+1) \tilde{U}_{k-1}(x) t^k.$$

Equating coefficients of t^k ($0 \leq k \leq n-1$) on both sides yields

$$\tilde{U}_1 = 0, \quad (k+1) \tilde{U}_{k+1} = -ke_1 \tilde{U}_k - (a-k+1)e_2 \tilde{U}_{k-1} \quad (1 \leq k \leq n-1).$$

Recalling that $\tilde{U}_k = \binom{a}{k} U_k^{(s)}$ ($0 \leq k \leq n$) and remarking that the stipulation $\prod_{\mu=0}^{n-1} (1 + \mu s) \neq 0$ implies $\binom{a}{k} \neq 0$, we obtain

$$U_1^{(s)} = 0, \quad (a-k)U_{k+1}^{(s)} = -k(e_1 U_k^{(s)} + e_2 U_{k-1}^{(s)}) \quad (1 \leq k \leq n-1);$$

that is,

$$\begin{aligned} U_{-1}^{(s)} &= 0, \quad U_0^{(s)} = 1, \\ (1 + ks)U_{k+1}^{(s)} &= ks(e_1 U_k^{(s)} + e_2 U_{k-1}^{(s)}) \quad (0 \leq k \leq n-1), \end{aligned}$$

where $U_{-1}^{(s)} = 0$ is a conventional definition and we used (3.6). ■

PROPOSITION 3.11. Stancu's operator $P_n^{(s)}$ can be represented as

$$P_n^{(s)} f(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \left[\binom{nx}{\nu} \binom{n(1-x)}{n-\nu} + (1+ns) \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{1+ks} \binom{n}{k} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu} \right]. \quad (3.7)$$

Remark. This proposition signifies that we can take out the factor $1+ns$ from $P_n^{(s)} - L_n$, as is expected from the identity $P_n^{(-1/n)} = L_n$.

Proof. Assume $s \neq 0$ and let $a = -1/s$. Then

$$P_n^{(s)} f(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{ax}{\nu} \binom{a(1-x)}{n-\nu} \binom{a}{n}^{-1}. \quad (3.8)$$

Since

$$\lim_{a \rightarrow \infty} \binom{ax}{\nu} \binom{a(1-x)}{n-\nu} \binom{a}{n}^{-1} = \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu}$$

and

$$\begin{aligned} \lim_{a \rightarrow k} (a-k) \binom{ax}{\nu} \binom{a(1-x)}{n-\nu} \binom{a}{n}^{-1} &= \frac{n!}{[\prod_{\mu=0}^{k-1} (k-\mu)] [\prod_{\mu=k+1}^{n-1} (k-\mu)]} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu} \\ &= -(-1)^{n-k} (n-k) \binom{n}{k} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu}, \end{aligned}$$

we can decompose into partial fractions with respect to a ,

$$\binom{ax}{\nu} \binom{a(1-x)}{n-\nu} \binom{a}{n}^{-1} = \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu} - \sum_{k=0}^{n-1} \frac{(-1)^{n-k} (n-k)}{a-k} \binom{n}{k} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu}.$$

Putting $a = n$ on both sides yields

$$\binom{nx}{\nu} \binom{n(1-x)}{n-\nu} = \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu} - \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n}{k} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu}. \quad (3.9)$$

Eliminating $\binom{n}{\nu} x^\nu (1-x)^{n-\nu}$ from the above two formulas gives

$$\binom{ax}{\nu} \binom{a(1-x)}{n-\nu} \binom{a}{n}^{-1} = \binom{nx}{\nu} \binom{n(1-x)}{n-\nu} + \sum_{k=0}^{n-1} \frac{(-1)^{n-k} (a-n)}{a-k} \binom{n}{k} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu}.$$

Equation (3.8), this identity, and $a = -1/s$ imply (3.7) when $s \neq 0$. It is valid also when $s = 0$ because both sides of (3.7) are continuous at $s = 0$. ■

Let $\varphi(k) = \binom{kx}{\nu} \binom{k(1-x)}{n-\nu}$. Then $\varphi \in \mathbf{P}_n$ and its leading coefficient is $x^\nu (1-x)^{n-\nu} / (\nu!(n-\nu)!)$. Hence we have

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu} = \Delta_1^n \varphi(0) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}.$$

Note that this identity gives another proof of (3.9) and indicates that Theorem 3.2 is valid when $\alpha = 0$.

PROPOSITION 3.12. *Stancu's operator $P_n^{(s)}$ satisfies the identity*

$$\begin{aligned} \int_0^1 P_n^{(s)} f(x) dx &= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \left[\frac{1-s}{n+1} + \sum_{k=0}^n \gamma_{k+1} \left((-1)^\nu \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \frac{n^{(k)} s^{k+1}}{\prod_{\mu=0}^{k-1} (1+\mu s)} \right]. \end{aligned} \quad (3.10)$$

Remark. When $s = -1/n$, the above identity reduces to

$$\int_0^1 L_n f(x) dx = \frac{1}{n} \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \left[1 - \sum_{k=0}^n (-1)^k \gamma_{k+1} \left((-1)^\nu \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \right]. \quad (3.11)$$

This is nothing but the Newton-Cotes rule, all the weights in which are represented explicitly. (Compare this with the asymptotic expression in [10, p.79] and [14, p.273] due to Uspensky [49, 50].)

Proof. We first prove that for all $a \in \mathbf{R}$ and for all $\nu, \mu \in \mathbf{N}_0$,

$$\int_0^a \binom{x}{\nu} \binom{a-x}{\mu} dx = \binom{a+1}{\nu+\mu+1} - \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+1} \binom{a-k}{\nu+\mu-k} \left((-1)^\nu \binom{k}{\nu} + (-1)^\mu \binom{k}{\mu} \right) \quad (3.12)$$

by using the double generating functions of both sides. Let $|u|, |v| < 1/3$, and $u \neq v$. Then we can calculate

$$\begin{aligned} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} u^\nu v^\mu \int_0^a \binom{x}{\nu} \binom{a-x}{\mu} dx &= \int_0^a (1+u)^x (1+v)^{a-x} dx \\ &= (1+v)^a \int_0^a \left(\frac{1+u}{1+v} \right)^x dx \\ &= \frac{(1+u)^a - (1+v)^a}{\log(1+u) - \log(1+v)}; \end{aligned} \quad (3.13)$$

$$\begin{aligned} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} u^\nu v^\mu \binom{a+1}{\nu+\mu+1} &= \sum_{n=1}^{\infty} \sum_{\nu=0}^{n-1} u^\nu v^{n-\nu-1} \binom{a+1}{n} \\ &= \sum_{n=1}^{\infty} \binom{a+1}{n} \frac{u^n - v^n}{u - v} \\ &= \frac{(1+u)^{a+1} - (1+v)^{a+1}}{u - v}; \end{aligned} \quad (3.14)$$

$$\begin{aligned} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} u^\nu v^\mu \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+1} \binom{a-k}{\nu+\mu-k} (-1)^\nu \binom{k}{\nu} &= \sum_{k=0}^{\infty} \sum_{\nu=0}^k \sum_{\mu=0}^{\infty} u^\nu v^{n+k-\nu} (-1)^k \gamma_{k+1} \binom{a-k}{n} (-1)^\nu \binom{k}{\nu} \\ &= \sum_{k=0}^{\infty} \gamma_{k+1} \left[\sum_{\nu=0}^k \binom{k}{\nu} u^\nu (-v)^{k-\nu} \right] \left[\sum_{n=0}^{\infty} \binom{a-k}{n} v^n \right] \\ &= \sum_{k=0}^{\infty} \gamma_{k+1} (u-v)^k (1+v)^{a-k} \\ &= \sum_{k=1}^{\infty} \gamma_k (u-v)^{k-1} (1+v)^{a-k+1} \\ &= \frac{(1+v)^{a+1}}{u-v} \int_0^1 \left[\sum_{k=1}^{\infty} \binom{x}{k} \left(\frac{u-v}{1+v} \right)^k \right] dx \end{aligned}$$

$$\begin{aligned}
&= \frac{(1+v)^{a+1}}{u-v} \int_0^1 \left[\left(1 + \frac{u-v}{1+v} \right)^x - 1 \right] dx \\
&= \frac{(1+v)^a}{\log(1+u) - \log(1+v)} - \frac{(1+v)^{a+1}}{u-v},
\end{aligned} \tag{3.15}$$

where we noticed $|(u-v)/(1+v)| \leq (|u|+|v|)/(1-|v|) < 1$. Applying this identity, we have also

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} u^{\nu} v^{\mu} \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+1} \binom{a-k}{\nu+\mu-k} (-1)^{\mu} \binom{k}{\mu} \\
&= \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} v^{\nu} u^{\mu} \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+1} \binom{a-k}{\nu+\mu-k} (-1)^{\nu} \binom{k}{\nu} \\
&= \frac{(1+u)^a}{\log(1+v) - \log(1+u)} - \frac{(1+u)^{a+1}}{v-u}.
\end{aligned} \tag{3.16}$$

Since (3.14) - ((3.15) + (3.16)) is equal to (3.13), equating coefficients of $u^{\nu} v^{\mu}$ yields (3.12).

Putting $\mu = n - \nu$, we obtain

$$\int_0^a \binom{x}{\nu} \binom{a-x}{n-\nu} dx = \binom{a+1}{n+1} - \sum_{k=0}^n (-1)^k \gamma_{k+1} \binom{a-k}{n-k} \left((-1)^{\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right).$$

If $a^{(n)} \neq 0$ then we get

$$\begin{aligned}
&\int_0^1 \binom{ax}{\nu} \binom{a(1-x)}{n-\nu} a dx \\
&= \frac{a+1}{n+1} \binom{a}{n} - \sum_{k=0}^n (-1)^k \gamma_{k+1} \left((-1)^{\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \frac{n^{(k)}}{a^{(k)}} \binom{a}{n};
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_0^1 \binom{ax}{\nu} \binom{a(1-x)}{n-\nu} \binom{a}{n}^{-1} dx \\
&= \frac{a+1}{(n+1)a} - \sum_{k=0}^n (-1)^k \gamma_{k+1} \left((-1)^{\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \frac{n^{(k)}}{a \cdot a^{(k)}}.
\end{aligned}$$

Therefore, putting $a = -1/s$ and (3.8) imply (3.10) when $s \neq 0$. It is valid also when $s = 0$ because both sides of it are continuous at $s = 0$. ■

3.4. Proofs of the Main Results

Now we are to prove all the theorems given in Section 3.2.

Proof of Theorem 3.1. When $(n-1)|s| < 1$, we can expand for each $x \in [0, 1]$

$$U_k^{(s)}(x) = \sum_{j=0}^{\infty} \Upsilon_{j,k}(x) (-s)^j \quad (-1 \leq k \leq n). \tag{3.17}$$

The recursion formula in Proposition 3.10 immediately implies

$$\Upsilon_{j,-1} = 0 \quad (j \geq 0), \quad \Upsilon_{0,0} = 1, \quad \Upsilon_{j,0} = 0 \quad (j \geq 1).$$

Furthermore, for every k satisfying $0 \leq k \leq n-1$, it also implies

$$(1+ks) \sum_{j=0}^{\infty} \Upsilon_{j,k+1}(x) (-s)^j = ks \left(e_1(x) \sum_{j=0}^{\infty} \Upsilon_{j,k}(x) (-s)^j + e_2(x) \sum_{j=0}^{\infty} \Upsilon_{j,k-1}(x) (-s)^j \right).$$

Rearrangement of this formula gives

$$\sum_{j=0}^{\infty} \Upsilon_{j,k+1}(x) (-s)^j = k \sum_{j=1}^{\infty} (\Upsilon_{j-1,k+1}(x) - e_1(x) \Upsilon_{j-1,k}(x) - e_2(x) \Upsilon_{j-1,k-1}(x)) (-s)^j.$$

Equating coefficients of $(-s)^j$ on both sides yields

$$\Upsilon_{0,k} = 0 \quad (1 \leq k \leq n),$$

$$\Upsilon_{j,k+1} = k(\Upsilon_{j-1,k+1} - e_1 \Upsilon_{j-1,k} - e_2 \Upsilon_{j-1,k-1}) \quad (j \geq 1, 0 \leq k \leq n-1).$$

Thus the recursion formula in this theorem is proved. Obviously, it gives

$$\Upsilon_{j,k} = 0 \quad (2j < k \leq n). \tag{3.18}$$

Therefore we obtain from Proposition 3.10 and Definition 3.1 that

$${}_a B_n f = \sum_{k=0}^n (B_n f)^{[k]} \sum_{j=0}^{\alpha} \frac{\Upsilon_{j,k}}{n^j} = \sum_{k=0}^{2\alpha} (B_n f)^{[k]} \sum_{j=0}^{\alpha} \frac{\Upsilon_{j,k}}{n^j} = \sum_{j=0}^{\alpha} \frac{1}{n^j} \sum_{k=0}^{2j} \Upsilon_{j,k} (B_n f)^{[k]}.$$

In addition, it is trivial that $\Upsilon_{j,k} \in \mathbf{P}_k$. ■

Note that the definition of the polynomials $\Upsilon_{j,k}$ is give by (3.17) though they are determined by the recursion formula in Theorem 3.1. We use the notation $\Upsilon_{j,k}$ throughout this thesis.

Proof of Theorem 3.2. We can calculate

$$\frac{1+ns}{1+ks} = 1 + \frac{(n-k)s}{1+ks} = 1 + (n-k)s \sum_{j=0}^{\infty} (-ks)^j.$$

If we truncate this series at degree α with respect to s , it becomes

$$1 + (n-k)s \sum_{j=0}^{\alpha-1} (-ks)^j = 1 + \frac{(n-k)s}{1+ks} (1 - (-ks)^\alpha).$$

If we put $s = -1/n$, this formula reduces to $(k/n)^\alpha$. Therefore Proposition 3.11 and Definition 3.1 imply

$$\begin{aligned} {}_\alpha B_n f(x) &= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \left[\binom{n}{\nu} \binom{n(1-x)}{n-\nu} + \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n}{k} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu} \left(\frac{k}{n}\right)^\alpha \right] \\ &= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{kx}{\nu} \binom{k(1-x)}{n-\nu} \left(\frac{k}{n}\right)^\alpha. \quad \blacksquare \end{aligned}$$

Proof of Theorem 3.3. We deduce this theorem from Theorem 2.4. It suffices to verify that $T_n = {}_\alpha B_n$ satisfies all the conditions required in the theorem.

We identify the given α with α in the theorem and let $K = 2\alpha$. Then Theorem 3.1 shows that we can let

$$V_k^n = \sum_{j=0}^{\alpha} \frac{\mathcal{Y}_{j,k}}{n^j} \quad (n \in \mathbf{N}, 0 \leq k \leq n),$$

which clearly satisfies the condition (a) in Theorem 2.4. Since (3.17) implies

$$U_k^n(x) = U_k^{(-1/n)}(x) = \sum_{j=0}^{\infty} \frac{\mathcal{Y}_{j,k}(x)}{n^j},$$

we can regard $V_k^n(x)$ as the asymptotic series truncated at $j = \alpha$ of $U_k^n(x)$ with respect to n . This relationship is parallel for $v_{k,l}(n)$ and $u_{k,l}(n)$. Therefore, Theorem 2.3 implies the conditions (b) and (c). In addition, it is trivial that

$$\lim_{n \rightarrow \infty} n^{\alpha+1} (V_k^n - U_k^n) = -\mathcal{Y}_{\alpha+1,k}$$

in the sense of $\|\cdot\|$ ($0 \leq k \leq 2\alpha + 2$). \blacksquare

Proof of Theorem 3.5. It is presented in [1, p.824] and its proof is given in [21, Section 60] that the Stirling numbers of the second kind have the generating function

$$\frac{t^k}{\prod_{\mu=0}^k (1 - \mu t)} = \sum_{j=k}^{\infty} S(j, k) t^j \quad (k \in \mathbf{N}_0, t \in \mathbf{R}, k|t| < 1).$$

This identity and the conventional definition $S(-1, -1) = 1$, $S(j, -1) = 0$ ($j \geq 0$) imply

$$\frac{s^k}{\prod_{\mu=0}^{k-1} (1 + \mu s)} = (-1)^k \sum_{j=k}^{\infty} S(j-1, k-1) (-s)^j.$$

Hence Proposition 3.12 becomes

$$\begin{aligned} \int_0^1 P_n^{(s)} f(x) dx &= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \left[\frac{1-s}{n+1} - \sum_{k=0}^n (-1)^k \gamma_{k+1} \right. \\ &\quad \left. \times \left((-1)^\nu \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) n^{(k)} \sum_{j=k}^{\infty} S(j-1, k-1) (-s)^{j+1} \right]. \end{aligned}$$

Immediately we obtain the theorem from Definition 3.1. \blacksquare

Now we prepare the following two lemmas for the proof of Theorem 3.7.

LEMMA 3.13. Let $n, k, \nu \in \mathbf{N}_0$. If $0 \leq k \leq n$ and $0 \leq \nu \leq n/2$ then

$$\binom{k}{\nu} \geq \binom{k}{n-\nu}.$$

The proof is clearly given by the inequality

$$\binom{k}{\nu} - \binom{k}{n-\nu} = \frac{k^{(\nu)}}{(n-\nu)!} ((n-\nu)^{(n-2\nu)} - (k-\nu)^{(n-2\nu)}) \geq 0.$$

LEMMA 3.14. For every $n \in \mathbf{N}$ and $\alpha \in \mathbf{N}_0$, if ${}_\alpha I_n$ is not positive then none of the ${}_\beta I_n$ ($\beta > \alpha$, $\beta \in \mathbf{N}_0 \cup \{\infty\}$) is positive.

Proof. Suppose ${}_\alpha I_n$ is not positive. Then it is obvious that $\alpha \geq 1$ because ${}_0 I_n$ is positive. Moreover, there exists some ν such that ${}_\alpha w_{n,\nu} < 0$, and we can assume $\nu \leq n/2$ without loss of generality because ${}_\alpha w_{n,n-\nu} = {}_\alpha w_{n,\nu}$.

Lemma 3.13 shows

$$\binom{k}{\nu} + (-1)^n \binom{k}{n-\nu} \geq 0 \quad (0 \leq k \leq n).$$

Furthermore, for every $k \geq 0$

$$(-1)^k \gamma_{k+1} = \frac{1}{(k+1)!} \int_0^1 x \prod_{\mu=1}^k (\mu - x) dx > 0.$$

Moreover, all the Stirling numbers of the second kind are nonnegative, as is well known.

Therefore, Theorem 3.5 implies that

$${}_{{\alpha}}w_{n,\nu} \geq {}_{\alpha+1}w_{n,\nu} \quad \text{for all } \alpha \geq 1 \quad \text{if } \nu \text{ is even;}$$

$${}_{{\alpha}}w_{n,\nu} \leq {}_{\alpha+1}w_{n,\nu} \quad \text{for all } \alpha \geq 1 \quad \text{if } \nu \text{ is odd.}$$

Thus ${}_1w_{n,\nu} > 0$ and ${}_{{\alpha}}w_{n,\nu} < 0$ imply that ν must be even and consequently

$${}_{{\beta}}w_{n,\nu} \leq \cdots \leq {}_{\alpha+1}w_{n,\nu} \leq {}_{\alpha}w_{n,\nu} < 0.$$

(Note that this inequality is valid even if $\beta = \infty$.) Therefore ${}_{{\beta}}I_n$ is not positive. ■

Proof of Theorem 3.7. When $1 \leq n \leq 7$ or $n = 9$, since all the weights in the Newton-Cotes rule are positive (see [1, p.886], [14, p.268], or calculate them using (3.11)), ${}_{{\infty}}I_n$ is positive. Thus $\alpha(n) = \infty$.

When $n = 8$ or $n \geq 10$, ${}_{{\infty}}I_n$ is not positive (see [33, p.156] and apply a technique similar to what is used in [30, pp.145,146]), and it must be possible to find such an $\alpha \in \mathbf{N}_0$ that ${}_{{\alpha}}I_n$ is positive but ${}_{\alpha+1}I_n$ is not positive. Then Lemma 3.14 guarantees that the found α is nothing but $\alpha(n)$.

Furthermore, for all $n \in \mathbf{N}$, Lemma 3.14 guarantees that all the ${}_{{\alpha}}I_n$ ($\alpha < \alpha(n)$, $\alpha \in \mathbf{N}_0$) are positive.

To complete the proof, we only have to determine $\alpha(n)$ for each n . We have obtained all the necessary data with the computer algebra system *Mathematica*. Here we list some of them; details are available from the author. (We let ${}_{{\alpha}}W_{n,\nu} = n {}_{\alpha}w_{n,\nu}$ in the list. Since ${}_{{\alpha}}W_{n,n-\nu} = {}_{\alpha}W_{n,\nu}$, it suffices to examine ${}_{{\alpha}}W_{n,\nu}$ only in the case $0 \leq \nu \leq [n/2]$.)

$$({}_{13}W_{8,\nu})_{\nu=0}^4 = (206412613269, 1027581059398, 212716917982,$$

$$1258456018206, 87224921170)/687194767360,$$

$${}_{14}W_{8,4} = -1200668453/206158430208.$$

$$({}_{13}W_{10,\nu})_{\nu=0}^5 = (3529060133449, 18442747661870, 2368669823883,$$

$$23813383005996, 117973634934, 23456331479736)/12000000000000,$$

$${}_{14}W_{10,4} = -4318979540557/20000000000000.$$

$$({}_{13}W_{11,\nu})_{\nu=0}^5 = (998147479495, 5327297957586, 478794941918,$$

$$7017006075333, 414657467478, 4594666338516)/3423740047332,$$

$${}_{14}W_{11,4} = -383569519901/12553713506884.$$

Similarly we can obtain the necessary data for every $n \leq 104$.

When $n \geq 105$, it is of course impossible to calculate all the data, which are infinite. However, there are general and skillful expressions of them, which clearly indicate the signs of the weights. Letting $a = 1/n (> 0)$ and $b = 1/105 - 1/n (\geq 0)$, we present them as follows:

$$\begin{aligned} ({}_8W_{n,\nu})_{\nu=0}^7 = & \left(\frac{1070017}{3628800} + \frac{22418548323091}{114354828000000}a + \frac{25359736267}{544546800000}ab \right. \\ & + \frac{5969355437}{93350880000}a^2b + \frac{12887747}{444528000}a^2b^2 + \frac{6151}{529200}a^3b^2 + \frac{1}{210}a^3b^3, \\ & \frac{638662806978248}{422130126796875} + \frac{57436491956239}{42883060500000}b + \frac{1833683798503}{980184240000}b^2 \\ & + \frac{27475395977}{18670176000}b^3 + \frac{3127027}{4939200}b^4 + \frac{289943}{2116800}b^5 + \frac{23}{1680}b^6, \\ & \frac{103613}{403200} + \frac{4145056375238327}{1029193452000000}a + \frac{33409341839701}{3267280800000}ab \\ & + \frac{1395652211053}{93350880000}ab^2 + \frac{5753926289}{444528000}ab^3 + \frac{546353}{88200}ab^4 + \frac{3193}{2520}ab^5, \\ & \frac{976909501271513}{562840169062500} + \frac{30521491285739}{4764784500000}b + \frac{23553221988539}{980184240000}b^2 \\ & + \frac{926729577581}{18670176000}b^3 + \frac{875979869}{14817600}b^4 + \frac{80715931}{2116800}b^5 + \frac{51881}{5040}b^6, \\ & \frac{298951}{725760} + \frac{1351609746105581}{205838690400000}a + \frac{62380941727861}{1960368480000}ab \\ & + \frac{1551304931951}{18670176000}ab^2 + \frac{10858579523}{88905600}ab^3 + \frac{2511179}{26460}ab^4 + \frac{3785}{126}ab^5, \\ & \frac{419377838606459}{337704101437500} + \frac{472640006070649}{128649181500000}b + \frac{22235854586693}{980184240000}b^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{151076675203}{2074464000}b^3 + \frac{5677877201}{44452800}b^4 + \frac{48719261}{423360}b^5 + \frac{206453}{5040}b^6, \\
& \frac{3349879}{3628800} + \frac{88142235701683}{68612896800000}a + \frac{90175372176901}{9801842400000}ab \\
& + \frac{3193508098591}{93350880000}ab^2 + \frac{10144953041}{148176000}ab^3 + \frac{36465791}{529200}ab^4 + \frac{67273}{2520}ab^5, \\
& \frac{113426694758929}{112568033812500} + \frac{21499631502649}{128649181500000}b + \frac{473440190611}{326728080000}b^2 \\
& + \frac{117865768367}{18670176000}b^3 + \frac{640930781}{44452800}b^4 + \frac{2274851}{141120}b^5 + \frac{33953}{5040}b^6 \Big),
\end{aligned}$$

$${}_8W_{n,\nu} = 1 \quad (8 \leq \nu \leq [n/2]),$$

$$\begin{aligned}
{}_9W_{n,4} = & - \left(\frac{7712181462239}{6331951901953125} + \frac{1666745232581279}{120608607656250}b + \frac{379336221741227}{4594613625000}b^2 \right. \\
& + \frac{3232816157663}{11668860000}b^3 + \frac{375083492797}{666792000}b^4 + \frac{21824429909}{31752000}b^5 \\
& \left. + \frac{35095393}{75600}b^6 + \frac{31717}{240}b^7 \right). \quad \blacksquare
\end{aligned}$$

3.5. Comparison between the Two Kinds of Operators

This final section is devoted to comparing our new operator with Sablonnière's.

Equations (3.17) and (3.18) being applied, the Lagrange, the Sablonnière, and our operators can be represented as

$$L_n f(x) = \sum_{k=0}^n (B_n f)^{[k]}(x) \sum_{j=0}^{\infty} \frac{\gamma_{j,k}(x)}{n^j} \quad (3.19)$$

$$= \sum_{j=0}^{\infty} \frac{1}{n^j} \sum_{k=0}^n \gamma_{j,k}(x) (B_n f)^{[k]}(x) = \sum_{j=0}^{\infty} \frac{1}{n^j} \sum_{k=0}^{2j} \gamma_{j,k}(x) (B_n f)^{[k]}(x), \quad (3.20)$$

$$\begin{aligned}
B_n^{(K)} f(x) &= \sum_{k=0}^{\{K,n\}} (B_n f)^{[k]}(x) \sum_{j=0}^{\infty} \frac{\gamma_{j,k}(x)}{n^j}, \\
{}_a B_n f(x) &= \sum_{j=0}^a \frac{1}{n^j} \sum_{k=0}^n \gamma_{j,k}(x) (B_n f)^{[k]}(x) = \sum_{j=0}^a \frac{1}{n^j} \sum_{k=0}^{2j} \gamma_{j,k}(x) (B_n f)^{[k]}(x) \\
&= \sum_{k=0}^{2a} (B_n f)^{[k]}(x) \sum_{j=0}^a \frac{\gamma_{j,k}(x)}{n^j}. \quad (3.21)
\end{aligned}$$

As we can see from the above formulas, $B_n^{(K)}$ is obtained by truncating at $k = K$ the first sum in (3.19). On the other hand, ${}_a B_n$ is obtained by truncating at $j = a$ the first sum in (3.20). Therefore both can be regarded as truncated operators of L_n , but the modes of truncation are distinct. It is interesting that, as (3.21) shows, ${}_a B_n$ is truncated also at $k = 2a$ automatically, not compulsorily.

Now we itemize the advantages of our operator as follows.

- (1) The value of its norm is much smaller.
- (2) The parameter a corresponds exactly to the order of its convergence rate.
- (3) It is simply defined with Stancu's operator and we can readily derive formulas about our operator from ones about Stancu's operator.

Let us compare Table 2 with Table 3, remarking that the convergence rates of $\{{}_a B_n f\}_n$, $\{B_n^{(2a)} f\}_n$, and $\{B_n^{(2a+1)} f\}_n$ are of the same order. (A table similar to Table 3 appears in [38]. We omit the values in the case $n < K$ because they are not defined originally, and according to our new definition, $B_n^{(K)}$ trivially reduces to $B_n^{(n)} (= L_n)$. See also Table 4.5 in [32, p.42].) Here we explain the general procedure of calculating $\|{}_a B_n\|$ and $\|B_n^{(K)}\|$.

Let T be an operator represented as the form

$$Tf = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \tau_{\nu} \quad (\tau_{\nu} \in \mathbf{P}_n - \{0\}, f: [0, 1] \rightarrow \mathbf{R})$$

and Λ be the Lebesgue function of T ; that is,

$$\Lambda(x) = \sum_{\nu=0}^n |\tau_{\nu}(x)| \quad (x \in [0, 1]).$$

By solving algebraic equations numerically with computer, we can determine

$$X_0 = \bigcup_{\nu=0}^n \{x \in (0, 1) \mid \tau_{\nu}(x) = 0\},$$

and furthermore,

$$X = X_0 \cup \{0, 1\} \cup \{x \in (0, 1) - X_0 \mid \Lambda'(x) = 0\}.$$

Then

$$\|T\| = \max_{x \in [0, 1]} \Lambda(x) = \max_{x \in X} \Lambda(x).$$

TABLE 2. Approximate Values of $\|{}_a B_n\|$

n	α							
	1	2	3	4	5	6	7	8
2	1.083	1.161	1.204	1.227	1.238	1.244	1.247	1.249
3	1.089	1.212	1.334	1.427	1.493	1.538	1.569	1.589
4	1.125	1.234	1.405	1.567	1.708	1.824	1.915	1.986
5	1.125	1.250	1.451	1.665	1.879	2.078	2.255	2.408
6	1.134	1.261	1.483	1.735	2.012	2.296	2.574	2.834
7	1.133	1.269	1.506	1.789	2.118	2.482	2.862	3.247
8	1.133	1.275	1.524	1.831	2.204	2.639	3.120	3.637
9	1.136	1.280	1.538	1.865	2.276	2.775	3.351	4.000
10	1.135	1.283	1.549	1.893	2.337	2.891	3.556	4.335
16	1.138	1.296	1.589	1.992	2.563	3.349	4.415	5.849
32	1.140	1.306	1.622	2.081	2.777	3.819	5.379	7.739

TABLE 3. Approximate Values of $\|B_n^{(K)}\|$

n	K							
	2	3	4	5	6	7	8	16
2	1.250	—	—	—	—	—	—	—
3	1.250	1.631	—	—	—	—	—	—
4	1.250	1.529	2.208	—	—	—	—	—
5	1.204	1.482	2.100	3.106	—	—	—	—
6	1.196	1.454	2.038	2.980	4.549	—	—	—
7	1.189	1.436	1.997	2.899	4.389	6.930	—	—
8	1.180	1.424	1.969	2.843	4.279	6.714	10.946	—
9	1.175	1.415	1.948	2.801	4.199	6.557	10.639	—
10	1.172	1.407	1.932	2.769	4.137	6.438	10.408	—
16	1.158	1.385	1.887	2.676	3.951	6.078	9.713	934.534
32	1.150	1.372	1.854	2.607	3.817	5.820	9.218	832.241

Tables 2 and 3 were obtained by applying the above procedure to the cases $T = {}_a B_n$ and $T = B_n^{(K)}$, respectively.

The two tables strongly suggest that

$$\|{}_a B_n\| \leq \|B_n^{(2\alpha)}\| \leq \|B_n^{(2\alpha+1)}\| \quad \text{for all } n \in \mathbf{N} \text{ and } \alpha \in \mathbf{N}_0.$$

This inequality is very difficult to prove and open so far, but we can explain its plausibility qualitatively, not quantitatively. Recall that $B_n^{(K)}$ ($K = 2\alpha, 2\alpha + 1$) is obtained by putting $s = -1/n$ in $\sum_{k=0}^{K,n} U_k^{(s)}(B_n f)^{[k]}$, which has poles at $s = -1/\mu$ ($\mu = 1, 2, \dots, \min\{K, n\} - 1$). (See the recursion formula for $U_k^{(s)}$ in Proposition 3.10.) We consider that these poles cause $B_n^{(K)}$ to be unstable. Therefore we should remove the poles for the sake of stability, while preserving the good convergence property of $\{B_n^{(K)}\}_n$. Fortunately, this is in fact possible by truncating at degree α the Maclaurin series of $P_n^{(s)}f$ (regarded as a function of s). Furthermore, the number of the terms in the truncated series is minimum under the constraint that the condition (c) in Theorem 2.4 is satisfied when we let $s = -1/n$. Consequently, we infer that ${}_a B_n$ is more stable than $B_n^{(2\alpha)}$, which means the advantage (1). At the same time, we also speculate that $B_n^{(2\alpha+1)}$ is more unstable than $B_n^{(2\alpha)}$ (when $2\alpha + 1 \leq n$) because $\sum_{k=0}^{2\alpha+1} U_k^{(s)}(B_n f)^{[k]}$ has the one extra pole $s = -1/(2\alpha)$ compared with $\sum_{k=0}^{2\alpha} U_k^{(s)}(B_n f)^{[k]}$.

The advantage (2) is clear from Theorem 3.3. Furthermore, recall that the advantage (3) played an essential role in the proofs of Theorems 3.2 and 3.5. (On the other hand, Sablonnière's operator has no corresponding procedures of calculation. For example, if we want to integrate $B_n^{(K)}f$ on $[0, 1]$, we cannot help using repeated integration by parts, which is complicated.)

Therefore, we conclude that our new class of modified Bernstein operators is more natural and essential for our purpose, and more convenient both theoretically and practically, than that of Sablonnière.

CHAPTER 4

ALGORITHM FOR THE MODIFIED BERNSTEIN POLYNOMIALS

In Chapter 3, we introduced a new class of modified Bernstein operators and asserted that it is very convenient both theoretically and practically. In this chapter, we present the Legendre expansion of the modified Bernstein polynomial to develop a useful algorithm for its evaluation.

4.1. Introduction

In Chapter 3, we introduced the modified Bernstein operator ${}_aB_n$ and asserted that it is very convenient both theoretically and practically. Now we are in a phase to develop the algorithm. In this chapter, we consider how to evaluate the modified Bernstein polynomial ${}_aB_nf(x)$ for given $f: [0, 1] \rightarrow \mathbf{R}$ and $x \in [0, 1]$.

Though we have already provided two kinds of representations as Theorems 3.1 and 3.2, both are inconvenient for practical calculations. To begin with, we should consider the case $\alpha = 0$; the ordinary Bernstein polynomial. Its original form is inconvenient, of course, and the so-called *de Casteljau algorithm* (see, e.g., [15]) is well known as a simple and numerically stable one. However, from the viewpoint of computational complexity, it is not very convenient because it requires $O(n^2)$ additions and multiplications for each x .

Since ${}_aB_nf$ is a polynomial of degree at most n , we should expand it into a treatable form in advance. The simplest way is to use powers of x or $2x - 1$, but this is known to be numerically unstable. (See [52, p.172] with [17].) On the other hand, expansions in terms of orthogonal polynomials are known to be much more stable. (Compare [16] with [17].) Furthermore, every system of orthogonal polynomials has a three-term recursion formula, and consequently, we can apply the Clenshaw method (see [52, Section 10.2]), which requires only $O(n)$ additions and multiplications for each x .

Among all the Jacobi polynomials, here we select the Legendre polynomials because the weighted integral

$$\begin{aligned} \int_0^1 B_n f(x) x^\alpha (1-x)^\beta dx &= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} \int_0^1 x^{\nu+\alpha} (1-x)^{n-\nu+\beta} dx \\ &= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} B(\nu+\alpha+1, n-\nu+\beta+1) \\ &= \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)} \frac{\Gamma(n-\nu+\beta+1)}{\Gamma(n-\nu+1)} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+\beta+2)} \end{aligned}$$

is simplest when $\alpha = \beta = 0$. (The B and Γ are the beta and gamma functions, respectively.) In fact, we have obtained a harmonized expansion of $B_n f$ in terms of the Legendre polynomials, which will be presented as Corollary 4.2.

Therefore, also for the general ${}_a B_n f$, we decide to adopt the Legendre expansion. The most part of this chapter is devoted to determination of the coefficients in the expansion.

The content of this chapter has been submitted as [24].

4.2. Legendre Expansion of the Modified Bernstein Polynomials

Before presenting the main result of this chapter, we list several definitions.

For every $l \in \mathbf{Z}$ and $m \in \mathbf{N}_0$, we define

$$\begin{aligned} P_m^*(x) &= P_m(2x-1), \\ G_m^{(l)}(x) &= C_m^{(l+1/2)}(2x-1), \\ \hat{P}_{n,m}^*(\nu) &= \int_0^1 P_m^*(x) b_{n,\nu}(x) dx, \\ \hat{G}_{n,m}^{(l)}(\nu) &= \int_0^1 G_m^{(l)}(x) b_{n,\nu}(x) dx, \end{aligned}$$

where P_m and $C_m^{(\lambda)}$ are the m th Legendre and Gegenbauer polynomials, respectively, and $b_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$; as we mentioned in "Notations" and Lemma 2.7. Note that $G_m^{(0)}(x) = P_m^*(x)$ and $\hat{G}_{n,m}^{(0)}(\nu) = \hat{P}_{n,m}^*(\nu)$ holds.

For every $k, l \in \mathbf{N}_0$, we define the numbers $\gamma_k^{(l)}$ by the generating function

$$\left[\frac{t}{\log(1+t)} \right]^l = \sum_{k=0}^{\infty} \gamma_k^{(l)} t^k \quad (0 < |t| < 1). \quad (4.1)$$

Note that $\gamma_k^{(1)}$ is identical with γ_k used in Theorem 3.5. Furthermore, referring to [29, p.135], we have the relationship

$$\gamma_k^{(l)} = \frac{\mathcal{B}_k^{(k-l+1)}(1)}{k!}. \quad (4.2)$$

Now we present the main result.

THEOREM 4.1. *The modified Bernstein operator ${}_a B_n$ has the following representation in terms of the shifted Legendre polynomials:*

$${}_a B_n f(x) = \sum_{m=0}^n (2m+1) P_m^*(x) \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) ({}_a H_{n,m}(\nu) + {}_a K_{n,m}(\nu)),$$

where

$$\begin{aligned} {}_a H_{n,m}(\nu) &= \sum_{l=0}^{\{\alpha, [m/2]\}} \hat{G}_{n,m-2l}^{(l)}(\nu) \sum_{r=0}^{2(\alpha-l)} \binom{m-2l+2}{r} \sum_{j=l+\{(r+1)/2\}}^{\{\alpha, 2l+\max\{r-1, 0\}\}} \frac{a_{j,l,r}}{n^j}, \\ {}_a K_{n,m}(\nu) &= -\frac{1}{n} \sum_{k=0}^{\alpha-1} (-1)^k n^{(k)} \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \\ &\quad \times \sum_{l=0}^{\alpha-1-k} \gamma_{k+l+1}^{(l+1)} \binom{m}{l} (m+l)^{(l)} \sum_{j=k+l}^{\alpha-1} \frac{S(j-l-1, k-1)}{n^j}, \end{aligned}$$

where we determine $a_{j,k,l,r}$ by the following recursion formula and identify the constants $a_{j,l,r}$ with $a_{j,0,l,r}$:

$$\begin{aligned} a_{j,k,-1,r} &= 0 \quad (j \geq 0, -1 \leq k \leq 2j+1, 0 \leq r \leq 2(j+2)), \\ a_{j,k,j+1,r} &= 0 \quad (j \geq 0, -1 \leq k \leq 2j+1, -2 \leq r \leq 0), \\ a_{j,-1,l,r} &= a_{j,2j+1,l,r} = 0 \quad (j \geq 0, 0 \leq l \leq j, -2 \leq r \leq 2(j-l+1)), \\ a_{j,k,l,-2} &= a_{j,k,l,-1} = a_{j,k,l,2(j-l)+1} = a_{j,k,l,2(j-l+1)} = 0 \quad (j \geq 0, 0 \leq k \leq 2j, 0 \leq l \leq j), \\ a_{0,0,0,0} &= 1, \quad a_{1,0,0,0} = a_{1,0,0,1} = 0, \quad a_{1,0,0,2} = 1, \quad a_{1,0,1,0} = 1, \end{aligned}$$

$$\begin{aligned}
a_{j,k,l,r} = & (k-1)(r(r-1)a_{j-1,k-2,l,r-2} + 2r(r+k-3)a_{j-1,k-2,l,r-1} + (r+k-2)^{(2)}a_{j-1,k-2,l,r}) \\
& - (r(r-1)a_{j-1,k-1,l,r-2} + 2r(r+k-2)a_{j-1,k-1,l,r-1} + (r+k-1)^{(2)}a_{j-1,k-1,l,r}) \\
& + 2(2l-1)\left[(k-1)(k-l)(a_{j-1,k-2,l-1,r} + 2a_{j-1,k-2,l-1,r+1} + a_{j-1,k-2,l-1,r+2}) \right. \\
& \left. - (k-l+1)(a_{j-1,k-1,l-1,r} + 2a_{j-1,k-1,l-1,r+1} + a_{j-1,k-1,l-1,r+2})\right] \\
& (j \geq 1, 1 \leq k \leq 2j, 0 \leq l \leq j, 0 \leq r \leq 2(j-l)), \\
a_{j,0,l,r} = & \sum_{k=0}^{2(j-1)} \frac{1}{(k+2)!} \left[r(r-1)a_{j-1,k,l,r-2} + 2r(r+k-1)a_{j-1,k,l,r-1} + (r-2)(r+2k+1)a_{j-1,k,l,r} \right. \\
& \left. + 2(2l-1)(k-l+2)(a_{j-1,k,l-1,r} + 2a_{j-1,k,l-1,r+1} + a_{j-1,k,l-1,r+2}) \right] \\
& (j \geq 2, 0 \leq l \leq j, 0 \leq r \leq 2(j-l)).
\end{aligned}$$

In addition, we can calculate $\hat{G}_{n,m}^{(l)}(\nu)$ with the three-term recursion formula

$$\begin{aligned}
\hat{G}_{n,-1}^{(l)}(\nu) = 0, \quad \hat{G}_{n,0}^{(l)}(\nu) = \frac{1}{n+1}, \\
(m+1)(n+m+2)\hat{G}_{n,m+1}^{(l)}(\nu) = (2m+2l+1)(2\nu-n)\hat{G}_{n,m}^{(l)}(\nu) - (m+2l)(n-m-2l+1)\hat{G}_{n,m-1}^{(l)}(\nu).
\end{aligned}$$

Furthermore, we can calculate the constants $\gamma_k^{(l)}$ with the recursion formula

$$\begin{aligned}
\gamma_{-1}^{(l)} = 0 \quad (l \geq 1), \\
\gamma_0^{(1)} = 1, \\
\gamma_k^{(1)} = \sum_{l=1}^k \frac{(-1)^{l-1} \gamma_{k-l}^{(1)}}{l+1} \quad (k \geq 1), \\
l\gamma_k^{(l+1)} = (l-k)\gamma_k^{(l)} + (l-k+1)\gamma_{k-1}^{(l)} \quad (l \geq 1, k \geq 0).
\end{aligned}$$

Remark 1. The three-term recursion formula for $\hat{G}_{n,m}^{(l)}(\nu)$ indicates that $\hat{G}_{n,m}^{(l)}$ is a polynomial of degree m and it is parallel with the formula (see (4.7.17) in [48] or (22.7.3) in [1, p.782])

$$\begin{aligned}
G_{-1}^{(l)}(x) = 0, \quad G_0^{(l)}(x) = 1, \\
(m+1)G_{m+1}^{(l)}(x) = (2m+2l+1)(2x-1)G_m^{(l)}(x) - (m+2l)G_{m-1}^{(l)}(x). \quad (4.3)
\end{aligned}$$

Remark 2. Put $m = 0$ to verify that this theorem is a generalization of Theorem 3.5. (Investigate the cases $\alpha = 0$ and $\alpha \geq 1$ separately, using the values $a_{0,0,0} = a_{1,0,2} = 1$.)

Remark 3. The term ${}_{\alpha}K_{n,m}(\nu)$ is almost negligible because it vanishes when $\alpha \leq \nu \leq n - \alpha$; the term ${}_{\alpha}H_{n,m}(\nu)$ is leading in the computation.

Remark 4. Since ${}_{\alpha}H_{n,m}(n-\nu) = (-1)^m {}_{\alpha}H_{n,m}(\nu)$ and ${}_{\alpha}K_{n,m}(n-\nu) = (-1)^m {}_{\alpha}K_{n,m}(\nu)$ hold, it suffices to calculate ${}_{\alpha}H_{n,m}(\nu)$, ${}_{\alpha}K_{n,m}(\nu)$ only for $\nu \leq [n/2]$.

Remark 5. Though both ${}_{\alpha}H_{n,m}(\nu)$ and ${}_{\alpha}K_{n,m}(\nu)$ require triple summations, their computational complexities are independent of n ; that is, $O(1)$ with respect to n .

The above theorem seems to be complicated, but note that it is a generalization of the following corollary, which corresponds to the case $\alpha = 0$.

COROLLARY 4.2. The Bernstein operator B_n has the following representation in terms of the shifted Legendre polynomials:

$$B_n f(x) = \sum_{m=0}^n (2m+1)P_m^*(x) \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \hat{P}_{n,m}^*(\nu),$$

where we can calculate $\hat{P}_{n,m}^*(\nu)$ with the three-term recursion formula

$$\begin{aligned}
\hat{P}_{n,-1}^*(\nu) = 0, \quad \hat{P}_{n,0}^*(\nu) = \frac{1}{n+1}, \\
(m+1)(n+m+2)\hat{P}_{n,m+1}^*(\nu) = (2m+1)(2\nu-n)\hat{P}_{n,m}^*(\nu) - m(n-m+1)\hat{P}_{n,m-1}^*(\nu).
\end{aligned}$$

Remark. The above recursion formula is parallel with the formula for $P_m^*(x)$, of course.

Once we obtain the coefficients in the Legendre expansion of ${}_{\alpha}B_n f$ by Theorem 4.1, it is easy to evaluate its derivatives and indefinite integrals. In fact, the relationship (see (4.7.14) in [48])

$$\frac{d}{dx} C_m^{(\lambda)}(x) = 2\lambda C_{m-1}^{(\lambda+1)}(x)$$

implies

$$(G_m^{(l)})^{(r)}(x) = 2^{2r}(l+1/2)_r G_{m-r}^{(l+r)}(x) \quad (4.4)$$

for every $l \in \mathbf{Z}$ and $m, r \in \mathbf{N}_0$ ($r \leq m$). This and (4.3) imply the three-term recursion formula

$$(P_{r-1}^*)^{(r)}(x) = 0, \quad (P_r^*)^{(r)}(x) = 2^r(2r-1)!!,$$

$$(m-r+1)(P_{m+1}^*)^{(r)}(x) = (2m+1)(2x-1)(P_m^*)^{(r)}(x) - (m+r)(P_{m-1}^*)^{(r)}(x) \quad (m \geq r).$$

Furthermore, if we define for every $r \in \mathbf{N}$,

$$(P_m^*)^{(-r)}(x) = \frac{(-1)^r}{2^r(2r-1)!!} G_{m+r}^{(-r)}(x),$$

then we can verify from (4.4) that $(P_m^*)^{(-r)}$ is an r th indefinite integral of P_m^* . Thus we can apply (4.3) similarly to the case of the derivatives. Consequently, in any case, we can use the Clenshaw method.

4.3. Preliminary Results for the Proof of Theorem 4.1

The proof of Theorem 4.1 requires much preparation. We provide many preliminary results for it in this section.

4.3.1. The first stage

LEMMA 4.3. *The shifted Gegenbauer polynomials are represented as*

$$C_m^{(\lambda)}(2x-1) = \frac{1}{m!} \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \frac{(2\lambda)_{m+l}}{(\lambda+1/2)_l} x^l = \frac{1}{m!} \sum_{l=0}^m \binom{m}{l} \frac{(2\lambda)_{m+l}}{(\lambda+1/2)_l} (x-1)^l.$$

In particular, the shifted Legendre polynomials are

$$\begin{aligned} P_m^*(x) &= C_m^{(1/2)}(2x-1) \\ &= \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} x^l = \sum_{l=0}^m \binom{m}{l} \binom{m+l}{l} (x-1)^l. \end{aligned}$$

The proof is immediate from (4.7.4) and (4.7.6) in [48].

LEMMA 4.4. *Let $a \neq 0$ and $|w| < 1/2$. Then the following identity holds for every $m \in \mathbf{N}_0$:*

$$\begin{aligned} &\int_0^1 (1+w)^{ax} P_m^*(x) dx \\ &= - \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \sum_{r=0}^{\infty} w^r \left[\sum_{k=0}^{r+l+1} (-1)^{l+1-k} \gamma_k^{(l+1)} \binom{a+l+1-k}{r+l+1-k} + (-1)^m \gamma_{r+l+1}^{(l+1)} \right]. \end{aligned}$$

Proof. We can assume $w \neq 0$ without loss of generality because both sides of the equation are continuous at $w = 0$. Then we can calculate with repeated integration by parts and Lemma 4.3,

$$\begin{aligned} &\int_0^1 (1+w)^{ax} P_m^*(x) dx \\ &= \left[\sum_{l=0}^m (-1)^l \frac{P_m^{*(l)}(x) (1+w)^{ax}}{(a \log(1+w))^{l+1}} \right]_{x=0}^{x=1} \\ &= \sum_{l=0}^m (-1)^l \binom{m}{l} \frac{(m+l)^{(l)} (1+w)^a}{(a \log(1+w))^{l+1}} - \sum_{l=0}^m (-1)^m \binom{m}{l} \frac{(m+l)^{(l)}}{(a \log(1+w))^{l+1}} \\ &= - \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \left[\frac{(1+w)^a}{\log^{l+1}(1-\frac{w}{1+w})} + \frac{(-1)^m}{\log^{l+1}(1+w)} \right]. \end{aligned}$$

Since $0 < |w| < 1/2$ implies $0 < |-w/(1+w)| \leq |w|/(1-|w|) < 1$, we can expand applying (4.1),

$$\begin{aligned} &\frac{(1+w)^a}{\log^{l+1}(1-\frac{w}{1+w})} + \frac{(-1)^m}{\log^{l+1}(1+w)} \\ &= (1+w)^a \sum_{k=0}^{\infty} \gamma_k^{(l+1)} \left(-\frac{w}{1+w} \right)^{k-l-1} + (-1)^m \sum_{k=0}^{\infty} \gamma_k^{(l+1)} w^{k-l-1} \\ &= \sum_{k=0}^{\infty} \gamma_k^{(l+1)} w^{k-l-1} ((-1)^{l+1-k} (1+w)^{a+l+1-k} + (-1)^m) \\ &= \sum_{k=0}^{\infty} \gamma_k^{(l+1)} w^{k-l-1} \left[(-1)^{l+1-k} \sum_{j=0}^{\infty} \binom{a+l+1-k}{j} w^j + (-1)^m \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} (-1)^{l+1-k} \gamma_k^{(l+1)} \sum_{r=k-l-1}^{\infty} \binom{a+l+1-k}{r+l+1-k} w^r + (-1)^m \sum_{r=-l-1}^{\infty} \gamma_{r+l+1}^{(l+1)} w^r \\
&= \sum_{r=-l-1}^{\infty} w^r \left[\sum_{k=0}^{r+l+1} (-1)^{l+1-k} \gamma_k^{(l+1)} \binom{a+l+1-k}{r+l+1-k} + (-1)^m \gamma_{r+l+1}^{(l+1)} \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^1 (1+w)^{ax} P_m^*(x) dx \\
&= - \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \sum_{r=-l-1}^{\infty} w^r \left[\sum_{k=0}^{r+l+1} (-1)^{l+1-k} \gamma_k^{(l+1)} \binom{a+l+1-k}{r+l+1-k} + (-1)^m \gamma_{r+l+1}^{(l+1)} \right].
\end{aligned}$$

Note that the coefficients of w^r ($-m-1 \leq r \leq -1$) must be zero because the left-hand side is analytic at $w = 0$. Thus we can replace $\sum_{r=-l-1}^{\infty}$ by $\sum_{r=0}^{\infty}$. ■

LEMMA 4.5. Let $a \neq 0$. Then the following identity holds for every $m, \nu, \mu \in \mathbb{N}_0$:

$$\begin{aligned}
&\int_0^1 \binom{ax}{\nu} \binom{a(1-x)}{\mu} P_m^*(x) dx \\
&= \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \left[\sum_{k=0}^l (-1)^{m-k} \gamma_{l-k}^{(l+1)} \binom{\nu+k}{k} \binom{a+k+1}{\nu+\mu+k+1} \right. \\
&\quad \left. - \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+l+1}^{(l+1)} \binom{a-k}{\nu+\mu-k} \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{\mu} \binom{k}{\mu} \right) \right].
\end{aligned}$$

In addition,

$$\begin{aligned}
&\sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \sum_{k=0}^l (-1)^{m-k} \gamma_{l-k}^{(l+1)} \binom{\nu+k}{k} \binom{a+k+1}{\nu+\mu+k+1} \\
&= \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \sum_{k=0}^l (-1)^k \gamma_{l-k}^{(l+1)} \binom{\mu+k}{k} \binom{a+k+1}{\nu+\mu+k+1}. \quad (4.5)
\end{aligned}$$

Proof. Let $a \neq 0$, $|u|, |v| < 1/5$ and $w = (1+u)/(1+v) - 1$. Then $|w| = |u-v|/(1+v) \leq (|u| + |v|)/(1-|v|) < 1/2$. Hence applying Lemma 4.4 gives

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} u^{\nu} v^{\mu} \int_0^1 \binom{ax}{\nu} \binom{a(1-x)}{\mu} P_m^*(x) dx \\
&= \int_0^1 (1+u)^{ax} (1+v)^{a(1-x)} P_m^*(x) dx \\
&= (1+v)^a \int_0^1 (1+w)^{ax} P_m^*(x) dx \\
&= - \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \sum_{r=0}^{\infty} A_{l,r} (1+v)^{a-r} (u-v)^r,
\end{aligned}$$

where

$$A_{l,r} = \sum_{k=0}^{r+l+1} (-1)^{l+1-k} \gamma_k^{(l+1)} \binom{a+l+1-k}{r+l+1-k} + (-1)^m \gamma_{r+l+1}^{(l+1)}.$$

Furthermore,

$$\begin{aligned}
&\sum_{r=0}^{\infty} A_{l,r} (1+v)^{a-r} (u-v)^r \\
&= \sum_{r=0}^{\infty} A_{l,r} \sum_{j=0}^{\infty} \binom{a-r}{j} v^j \sum_{\nu=0}^r \binom{r}{\nu} u^{\nu} (-v)^{r-\nu} \\
&= \sum_{r=0}^{\infty} A_{l,r} \sum_{\nu=0}^r (-1)^{r-\nu} \binom{r}{\nu} u^{\nu} \sum_{j=0}^{\infty} \binom{a-r}{j} v^{j+r-\nu} \\
&= \sum_{r=0}^{\infty} A_{l,r} \sum_{\nu=0}^r (-1)^{r-\nu} \binom{r}{\nu} u^{\nu} \sum_{\mu=0}^{\infty} \binom{a-r}{\nu+\mu-r} v^{\mu} \\
&= \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} u^{\nu} v^{\mu} \sum_{r=\nu}^{\nu+\mu} (-1)^{r-\nu} \binom{r}{\nu} \binom{a-r}{\nu+\mu-r} A_{l,r}.
\end{aligned}$$

Therefore, paying attention to the coefficients of $u^{\nu} v^{\mu}$, we obtain

$$\begin{aligned}
&\int_0^1 \binom{ax}{\nu} \binom{a(1-x)}{\mu} P_m^*(x) dx \\
&= - \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \sum_{r=\nu}^{\nu+\mu} (-1)^{r-\nu} \binom{r}{\nu} \binom{a-r}{\nu+\mu-r} A_{l,r}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
 & \sum_{r=\nu}^{\nu+\mu} (-1)^{r-\nu} \binom{r}{\nu} \binom{a-r}{\nu+\mu-r} A_{l,r} \\
 &= \sum_{r=\nu}^{\nu+\mu} (-1)^{r-\nu} \binom{r}{\nu} \binom{a-r}{\nu+\mu-r} \left[\left(\sum_{k=0}^l + \sum_{k=l+1}^{r+l+1} \right) (-1)^{l+1-k} \gamma_k^{(l+1)} \binom{a+l+1-k}{r+l+1-k} \right. \\
 & \quad \left. + (-1)^m \gamma_{r+l+1}^{(l+1)} \right] \\
 &= (I) + (II) + (III),
 \end{aligned}$$

where

$$\begin{aligned}
 (I) &= \sum_{r=\nu}^{\nu+\mu} (-1)^{r-\nu} \binom{r}{\nu} \binom{a-r}{\nu+\mu-r} \sum_{k=0}^l (-1)^{l+1-k} \gamma_k^{(l+1)} \binom{a+l+1-k}{r+l+1-k} \\
 &= - \sum_{r=0}^{\mu} (-1)^r \binom{r+\nu}{\nu} \binom{a-r-\nu}{\mu-r} \sum_{k=0}^l (-1)^k \gamma_{l-k}^{(l+1)} \binom{a+k+1}{r+\nu+k+1} \\
 & \quad (r \leftarrow r-\nu, k \leftarrow l-k) \\
 &= - \sum_{k=0}^l (-1)^k \gamma_{l-k}^{(l+1)} \binom{a+k+1}{\nu+\mu+k+1} \sum_{r=0}^{\mu} \binom{-\nu-1}{r} \binom{\nu+\mu+k+1}{\mu-r} \\
 & \quad ((r+\nu) = \binom{r+\nu}{r} = (-1)^r \binom{-\nu-1}{r}) \\
 &= - \sum_{k=0}^l (-1)^k \gamma_{l-k}^{(l+1)} \binom{a+k+1}{\nu+\mu+k+1} \binom{(-\nu-1) + (\nu+\mu+k+1)}{\mu} \\
 &= - \sum_{k=0}^l (-1)^k \gamma_{l-k}^{(l+1)} \binom{\mu+k}{k} \binom{a+k+1}{\nu+\mu+k+1}, \\
 (II) &= \sum_{r=\nu}^{\nu+\mu} (-1)^{r-\nu} \binom{r}{\nu} \binom{a-r}{\nu+\mu-r} \sum_{k=l+1}^{r+l+1} (-1)^{l+1-k} \gamma_k^{(l+1)} \binom{a+l+1-k}{r+l+1-k} \\
 &= \sum_{r=0}^{\nu+\mu} (-1)^{r-\nu} \binom{r}{\nu} \binom{a-r}{\nu+\mu-r} \sum_{k=0}^r (-1)^k \gamma_{k+l+1}^{(l+1)} \binom{a-k}{r-k} \\
 &= \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+l+1}^{(l+1)} \sum_{r=k}^{\nu+\mu} (-1)^{r-\nu} \binom{r}{\nu} \binom{a-k}{r-k} \binom{a-r}{\nu+\mu-r}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+l+1}^{(l+1)} \sum_{r=0}^{\nu+\mu-k} (-1)^{r+k-\nu} \binom{r+k}{\nu} \binom{a-k}{r} \binom{a-r-k}{\nu+\mu-r-k} \\
 &= \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+l+1}^{(l+1)} \binom{a-k}{\nu+\mu-k} (-1)^{\mu} \sum_{r=0}^{\nu+\mu-k} (-1)^{\nu+\mu-k-r} \binom{\nu+\mu-k}{r} \binom{r+k}{\nu} \\
 &= \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+l+1}^{(l+1)} \binom{a-k}{\nu+\mu-k} (-1)^{\mu} \Delta_1^{\nu+\mu-k} \binom{\cdot+k}{\nu} (0) \\
 &= \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+l+1}^{(l+1)} \binom{a-k}{\nu+\mu-k} (-1)^{\mu} \binom{k}{\mu}, \\
 (III) &= \sum_{r=\nu}^{\nu+\mu} (-1)^{r-\nu} \binom{r}{\nu} \binom{a-r}{\nu+\mu-r} (-1)^m \gamma_{r+l+1}^{(l+1)} \\
 &= \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+l+1}^{(l+1)} \binom{a-k}{\nu+\mu-k} (-1)^{m+\nu} \binom{k}{\nu}.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 & \int_0^1 \binom{ax}{\nu} \binom{a(1-x)}{\mu} P_m^*(x) dx \\
 &= \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \left[\sum_{k=0}^l (-1)^k \gamma_{l-k}^{(l+1)} \binom{\mu+k}{k} \binom{a+k+1}{\nu+\mu+k+1} \right. \\
 & \quad \left. - \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+l+1}^{(l+1)} \binom{a-k}{\nu+\mu-k} \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{\mu} \binom{k}{\mu} \right) \right].
 \end{aligned}$$

In addition,

$$\begin{aligned}
 & \int_0^1 \binom{ax}{\nu} \binom{a(1-x)}{\mu} P_m^*(x) dx \\
 &= \int_0^1 \binom{a(1-x)}{\nu} \binom{ax}{\mu} P_m^*(1-x) dx \\
 &= (-1)^m \int_0^1 \binom{ax}{\mu} \binom{a(1-x)}{\nu} P_m^*(x) dx \\
 &= \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \left[\sum_{k=0}^l (-1)^{m-k} \gamma_{l-k}^{(l+1)} \binom{\nu+k}{k} \binom{a+k+1}{\nu+\mu+k+1} \right. \\
 & \quad \left. - \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+l+1}^{(l+1)} \binom{a-k}{\nu+\mu-k} \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{\mu} \binom{k}{\mu} \right) \right].
 \end{aligned}$$

Comparing the above two equations, we obtain (4.5). ■

LEMMA 4.6. *The relationship*

$$\gamma_{l-k}^{(l+1)} = \frac{k!}{l!} S(l+1, k+1)$$

holds for every $l, k \in \mathbf{N}_0$ ($k \leq l$).

Proof. The identities $\mathcal{B}_\nu^{(n+\nu+1)}(1) = \frac{n}{n+\nu} \mathcal{B}_\nu^{(n+\nu)}$ and $x^\nu = \sum_{s=0}^\nu \binom{\nu}{s} x^{(s)} \mathcal{B}_{\nu-s}^{(-s)}$; that is, $\binom{\nu}{s} \mathcal{B}_{\nu-s}^{(-s)} = S(\nu, s)$ appear in [29, pp.129, 133]. Applying these identities and (4.2) readily imply this lemma. ■

LEMMA 4.7. *Stancu's operator $P_n^{(s)}$ satisfies the identity*

$$\int_0^1 P_n^{(s)} f(x) P_m^*(x) dx = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) (H_{n,m}^{(s)}(\nu) + K_{n,m}^{(s)}(\nu)),$$

where

$$H_{n,m}^{(s)}(\nu) = \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \sum_{k=0}^l S(l+1, k+1) \frac{(\nu+k)^{(k)} s^{l-k} \prod_{\mu=1}^{k+1} (1-\mu s)}{(n+k+1)^{(k+1)}}, \quad (4.6)$$

$$K_{n,m}^{(s)}(\nu) = \sum_{l=0}^m (-1)^l \binom{m}{l} (m+l)^{(l)} \sum_{k=0}^n \gamma_{k+l+1}^{(l+1)} \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \frac{n^{(k)} s^{k+l+1}}{\prod_{\mu=0}^{k-1} (1+\mu s)}.$$

Remark 1. Put $m = 0$ to verify that this proposition is a generalization of Proposition 3.12.

Remark 2. We use the notations $H_{n,m}^{(s)}$ and $K_{n,m}^{(s)}$ throughout this chapter.

Proof. We can assume $s \neq 0$ without loss of generality because both sides are continuous at $s = 0$. Let $a = -1/s$. Then (3.8) implies

$$\int_0^1 P_n^{(s)} f(x) P_m^*(x) dx = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{a}{n}^{-1} \int_0^1 \binom{ax}{\nu} \binom{a(1-x)}{n-\nu} P_m^*(x) dx.$$

Applying Lemma 4.5 with $\mu = n - \nu$ and Lemma 4.6 imply

$$\begin{aligned} & \binom{a}{n}^{-1} \int_0^1 \binom{ax}{\nu} \binom{a(1-x)}{n-\nu} P_m^*(x) dx \\ &= \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{a^{l+1}} \left[\sum_{k=0}^l (-1)^{m-k} \gamma_{l-k}^{(l+1)} \binom{\nu+k}{k} \binom{a+k+1}{n+k+1} \binom{a}{n}^{-1} \right. \\ & \quad \left. - \sum_{k=0}^n (-1)^k \gamma_{k+l+1}^{(l+1)} \binom{a-k}{n-k} \binom{a}{n}^{-1} \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \right] \\ &= \sum_{l=0}^m \binom{m}{l} \binom{m+l}{l} \sum_{k=0}^l (-1)^{m-k} S(l+1, k+1) \frac{(\nu+k)^{(k)} (a+k+1)^{(k+1)}}{(n+k+1)^{(k+1)} a^{l+1}} \\ & \quad - \sum_{l=0}^m \binom{m}{l} (m+l)^{(l)} \sum_{k=0}^n (-1)^k \gamma_{k+l+1}^{(l+1)} \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \frac{n^{(k)}}{a^{l+1} a^{(k)}}. \end{aligned}$$

Putting $a = -1/s$ complete the proof. ■

As an application of Lemma 4.7, we will give Proposition 4.9 by virtue of the following lemma.

LEMMA 4.8. *For every $l \in \mathbf{N}_0$, the equation*

$$x^l = \sum_{k=0}^l S(l+1, k+1) (x-1)^{(k)}$$

holds as polynomials in x .

Proof. The definition of the Stirling numbers of the second kind gives

$$x^{l+1} = \sum_{k=1}^{l+1} S(l+1, k) x^{(k)} = \sum_{k=0}^l S(l+1, k+1) x^{(k+1)},$$

where we noticed $S(l+1, 0) = 0$. Since this equation holds as polynomials in x , we can divide both sides by x . ■

PROPOSITION 4.9. The Lagrange operator L_n satisfies the identity

$$\begin{aligned} \int_0^1 L_n f(x) P_m^*(x) dx &= \frac{1}{n} \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \left[P_m^*\left(\frac{\nu}{n}\right) \right. \\ &\quad \left. - \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{n^l} \sum_{k=0}^n (-1)^k \gamma_{k+l+1}^{(l+1)} \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \right]. \end{aligned}$$

Remark. Put $m = 0$ to verify that this proposition is a generalization of (3.11).

Proof. Putting $s = -1/n$ in Lemma 4.7 (or $a = n$ in its proof), we get

$$\begin{aligned} \int_0^1 L_n f(x) P_m^*(x) dx &= \frac{1}{n} \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \left[\sum_{l=0}^m \binom{m}{l} \binom{m+l}{l} \frac{1}{n^l} \sum_{k=0}^l (-1)^{m-k} S(l+1, k+1) (\nu+k)^{(k)} \right. \\ &\quad \left. - \sum_{l=0}^m \binom{m}{l} \frac{(m+l)^{(l)}}{n^l} \sum_{k=0}^n (-1)^k \gamma_{k+l+1}^{(l+1)} \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \right]. \end{aligned}$$

Applying Lemmas 4.8 and 4.3 implies

$$\begin{aligned} &\sum_{l=0}^m \binom{m}{l} \binom{m+l}{l} \frac{1}{n^l} \sum_{k=0}^l (-1)^{m-k} S(l+1, k+1) (\nu+k)^{(k)} \\ &= \sum_{l=0}^m \binom{m}{l} \binom{m+l}{l} \frac{(-1)^m}{n^l} \sum_{k=0}^l S(l+1, k+1) (-\nu-1)^{(k)} \\ &= \sum_{l=0}^m \binom{m}{l} \binom{m+l}{l} \frac{(-1)^m (-\nu)^l}{n^l} \\ &= \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \left(\frac{\nu}{n}\right)^l = P_m^*\left(\frac{\nu}{n}\right). \quad \blacksquare \end{aligned}$$

Though we have become able to calculate all the coefficients in the Legendre expansion of $P_n^{(s)} f$ by Lemma 4.7, this is not satisfactory yet. For example, the expression (4.6) involves the Stirling numbers of the second kind and we cannot see the symmetric relationship $H_{n,m}^{(s)}(n-\nu) = (-1)^m H_{n,m}^{(s)}(\nu)$ expected from (4.5).

However, if we put $s = 0$, then $H_{n,m}^{(s)}(\nu)$ reduces to

$$\begin{aligned} H_{n,m}^{(0)}(\nu) &= \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \frac{(\nu+l)^{(l)}}{(n+l+1)^{(l+1)}} \\ &= \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \int_0^1 x^l b_{n,\nu}(x) dx \\ &= \int_0^1 P_m^*(x) b_{n,\nu}(x) dx = \hat{P}_{n,m}^*(\nu) \end{aligned} \quad (4.7)$$

where we used Lemma 4.3 and the identity

$$\begin{aligned} \int_0^1 x^l b_{n,\nu}(x) dx &= \binom{n}{\nu} \int_0^1 x^{\nu+l} (1-x)^{n-\nu} dx = \binom{n}{\nu} B(\nu+l+1, n-\nu+1) \\ &= \binom{n}{\nu} \frac{\Gamma(\nu+l+1) \Gamma(n-\nu+1)}{\Gamma(n+l+2)} = \frac{(\nu+l)^{(l)}}{(n+l+1)^{(l+1)}}. \end{aligned} \quad (4.8)$$

Equation (4.7) is very lucid and in fact we can clearly see $H_{n,m}^{(0)}(n-\nu) = (-1)^m H_{n,m}^{(0)}(\nu)$. Hence, from the next subsection, we devote ourselves to consider how to generalize (4.7) for all s .

4.3.2. The second stage

The aim of this subsection is to prove the following lemma.

LEMMA 4.10. The relationship

$$H_{n,m}^{(s)}(\nu) = \int_0^1 \sum_{j=0}^{m+1} (-s)^j \sum_{k=0}^{2j} (-1)^k (\gamma_{j,k} P_m^*)^{[k]}(x) b_{n,\nu}(x) dx$$

holds for every $n \in \mathbf{N}$, $m, \nu \in \mathbf{N}_0$ ($m, \nu \leq n$).

To prove this, we prepare still more lemmas.

LEMMA 4.11. The polynomial $U_k^{(s)}$ has the explicit representation

$$U_k^{(s)}(x) = \sum_{r=0}^k \binom{k}{r} (-x)^{k-r} \prod_{\mu=0}^{r-1} \frac{x + \mu s}{1 + \mu s}.$$

In particular,

$$U_k^n(x) = U_k^{(-1/n)}(x) = \binom{n}{k}^{-1} \sum_{r=0}^k \binom{n-r}{k-r} \binom{nx}{r} (-x)^{k-r}.$$

Proof. Combining Propositions 3.8 and 3.10, we can calculate

$$\begin{aligned} & \sum_{r=0}^n \Delta_{1/n}^r f(0) \binom{n}{r} \prod_{\mu=0}^{r-1} \frac{x + \mu s}{1 + \mu s} \\ &= \sum_{k=0}^n U_k^{(s)}(x) (B_n f)^{[k]}(x) \\ &= \sum_{k=0}^n U_k^{(s)}(x) \sum_{r=k}^n \Delta_{1/n}^r f(0) \binom{n}{r} \binom{r}{k} x^{r-k} \\ &= \sum_{r=0}^n \Delta_{1/n}^r f(0) \binom{n}{r} \sum_{k=0}^r \binom{r}{k} x^{r-k} U_k^{(s)}(x). \end{aligned}$$

By considering the cases $f(x) = \binom{nx}{r}$, we obtain

$$\prod_{\mu=0}^{r-1} \frac{x + \mu s}{1 + \mu s} = \sum_{k=0}^r \binom{r}{k} x^{r-k} U_k^{(s)}(x).$$

We can easily verify that this linear equation (with respect to $U_k^{(s)}(x)$) is uniquely solvable and that the $U_k^{(s)}(x)$ given in this lemma satisfies this equation. ■

Now we denote by $s(\cdot, \cdot)$ the Stirling numbers of the first kind (recall "Notations" and the remark for Theorem 3.5), which should not be confused with the parameter s .

LEMMA 4.12. The polynomial $\Upsilon_{j,k}$ has the explicit representation

$$\Upsilon_{j,k}(x) = \sum_{r=0}^k \binom{k}{r} (-x)^{k-r} \sum_{p=\{r-j,0\}}^r s(r,p) S(p+j-1, r-1) x^p.$$

Proof. Let $s \neq 0$. In Lemma 4.11, we can expand

$$\frac{1}{\prod_{\mu=0}^{r-1} (1 + \mu s)} = \sum_{j=r}^{\infty} S(j-1, r-1) (-s)^{j-r}$$

and

$$\begin{aligned} \prod_{\mu=0}^{r-1} (x + \mu s) &= (-s)^r \prod_{\mu=0}^{r-1} \left(-\frac{x}{s} - \mu \right) = (-s)^r \left(-\frac{x}{s} \right)^{(r)} \\ &= (-s)^r \sum_{p=0}^r s(r,p) \left(-\frac{x}{s} \right)^p = \sum_{p=0}^r s(r,p) x^p (-s)^{r-p}. \end{aligned}$$

Therefore,

$$\begin{aligned} U_k^{(s)}(x) &= \sum_{r=0}^k \binom{k}{r} (-x)^{k-r} \sum_{p=0}^r s(r,p) x^p (-s)^{r-p} \sum_{j=r}^{\infty} S(j-1, r-1) (-s)^{j-r} \\ &= \sum_{r=0}^k \binom{k}{r} (-x)^{k-r} \sum_{p=0}^r s(r,p) x^p \sum_{j=r-p}^{\infty} S(p+j-1, r-1) (-s)^j \\ &= \sum_{j=0}^{\infty} (-s)^j \sum_{r=0}^k \binom{k}{r} (-x)^{k-r} \sum_{p=\{0, r-j\}}^r s(r,p) S(p+j-1, r-1) x^p \end{aligned}$$

Comparing this with (3.17) completes the proof. ■

PROPOSITION 4.13. For every $j, k, l \in \mathbf{N}_0$ satisfying $k \leq l \leq j+k$, the following identity holds:

$$\begin{aligned} & \sum_{r=l-k}^{2j} (-1)^{r-j} s(r, r+k-l) S(r+j+k-l-1, r-1) \binom{r+k-1}{r} \binom{2j+k}{2j-r} \\ &= \begin{cases} s(k+1, l+1-j) S(l, k) & \text{if } l \geq j, \\ 0 & \text{if } l \leq j-1. \end{cases} \end{aligned}$$

Remark. Let $m, n \in \mathbf{N}_0$ ($m \leq n$) and put $j = n - m$, $k = m$, $l = n$. Then the above identity reduces to

$$\sum_{r=n-m}^{2(n-m)} (-1)^{r-(n-m)} s(r, r+m-n) \binom{r+m-1}{r} \binom{2n-m}{2n-2m-r} = S(n, m);$$

that is,

$$S(n, m) = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} s(n-m+k, k).$$

This identity is presented in [1, p.825] and proved in [6, 19]. We can regard the above proposition as a generalization of this identity.

Proof. We can generalize the Bernoulli numbers into $\mathcal{B}_n^{(x)}$, which is a polynomial in x of degree n for each $n \in \mathbf{N}_0$. (See [31, p.146] and [6].) Let

$$\varphi(x) = \binom{x-1}{l-k} \binom{x+j+k-l-1}{j+k-l} \mathcal{B}_{l-k}^{(x)} \mathcal{B}_{j+k-l}^{(1-x)}.$$

Then we have $\varphi \in \mathbf{P}_{2j}$. Hence the Lagrange interpolation formula gives

$$\sum_{r=0}^{2j} \varphi(r) \binom{x}{r} \binom{2j-x}{2j-r} = \varphi(x).$$

Putting $x = -k$ implies

$$\begin{aligned} \sum_{r=0}^{2j} \binom{r-1}{l-k} \binom{r+j+k-l-1}{j+k-l} \mathcal{B}_{l-k}^{(r)} \mathcal{B}_{j+k-l}^{(1-r)} \binom{-k}{r} \binom{2j+k}{2j-r} \\ = \binom{-k-1}{l-k} \binom{j-l-1}{j+k-l} \mathcal{B}_{l-k}^{(-k)} \mathcal{B}_{j+k-l}^{(k+1)}. \end{aligned}$$

Since $\binom{-k}{r} = (-1)^r \binom{r+k-1}{r}$, $\binom{-k-1}{l-k} = (-1)^{l-k} \binom{l}{l-k}$, $\binom{j-l-1}{j+k-l} = (-1)^{j+k-l} \binom{k}{j+k-l}$, we get

$$\begin{aligned} \sum_{r=0}^{2j} (-1)^{r-j} \binom{r-1}{l-k} \mathcal{B}_{l-k}^{(r)} \binom{r+j+k-l-1}{j+k-l} \mathcal{B}_{j+k-l}^{(1-r)} \binom{r+k-1}{r} \binom{2j+k}{2j-r} \\ = \binom{k}{j+k-l} \mathcal{B}_{j+k-l}^{(k+1)} \binom{l}{l-k} \mathcal{B}_{l-k}^{(-k)}. \end{aligned}$$

Referring to [29, p.133] (or "Notations for the Stirling Numbers" in [1, p.822]), we can understand that the following equations hold:

$$\begin{aligned} \binom{r-1}{l-k} \mathcal{B}_{l-k}^{(r)} &= \begin{cases} s(r, r+k-l) & \text{if } r \geq l-k, \\ 0 & \text{if } 0 \leq r \leq l-k-1, \end{cases} \\ \binom{r+j+k-l-1}{j+k-l} \mathcal{B}_{j+k-l}^{(1-r)} &= S(r+j+k-l-1, r-1), \\ \binom{k}{j+k-l} \mathcal{B}_{j+k-l}^{(k+1)} &= \begin{cases} s(k+1, l+1-j) & \text{if } l \geq j, \\ 0 & \text{if } l \leq j-1, \end{cases} \\ \binom{l}{l-k} \mathcal{B}_{l-k}^{(-k)} &= S(l, k). \end{aligned}$$

Using these equations, we complete the proof. ■

Now we are ready to prove Lemma 4.10.

Proof of Lemma 4.10. Combining Lemmas 4.3 and 4.12, we have

$$\begin{aligned} (\Upsilon_{j,k} P_m^*)(x) &= \Upsilon_{j,k}(x) P_m^*(x) \\ &= \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \sum_{r=0}^k \binom{k}{r} (-1)^{k-r} \sum_{p=\{r-j,0\}}^r s(r,p) S(p+j-1, r-1) x^{l+k-r+p}. \end{aligned}$$

Therefore,

$$\begin{aligned} (-1)^k (\Upsilon_{j,k} P_m^*)^{[k]}(x) &= \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \\ &\quad \times \sum_{r=0}^k (-1)^r \sum_{p=\{r-l, r-j, 0\}}^r s(r,p) S(p+j-1, r-1) \binom{k}{r} \binom{k+l-r+p}{k} x^{l-r+p}. \end{aligned}$$

The second sum becomes

$$\begin{aligned} \sum_{r=0}^k (-1)^r \sum_{p=\{r-l, r-j, 0\}}^r s(r,p) S(p+j-1, r-1) \binom{k+l-r+p}{l+p} \binom{l+p}{r} x^{l-r+p} \\ = \sum_{r=0}^k (-1)^r \sum_{q=\{0, l-j, l-r\}}^l s(r, r+q-l) S(r+j+q-l-1, r-1) \binom{k+q}{r+q} \binom{r+q}{r} x^q \\ = \sum_{q=\{0, l-j\}}^l x^q \sum_{r=l-q}^k (-1)^r s(r, r+q-l) S(r+j+q-l-1, r-1) \binom{r+q}{r} \binom{k+q}{r+q}. \end{aligned}$$

Since $\sum_{k=0}^{2j} \sum_{r=l-q}^k = \sum_{r=l-q}^{2j} \sum_{k=r}^{2j}$ and $\sum_{k=r}^{2j} \binom{k+q}{r+q} = \binom{2j+q+1}{r+q+1} = \binom{2j+q+1}{2j-r}$, replacing p by k , we obtain

$$\begin{aligned} \sum_{k=0}^{2j} (-1)^k (\Upsilon_{j,k} P_m^*)^{[k]}(x) &= \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \sum_{k=\{0, l-j\}}^l x^k \\ &\quad \times \sum_{r=l-k}^{2j} (-1)^r s(r, r+k-l) S(r+j+k-l-1, r-1) \binom{r+k}{r} \binom{2j+k+1}{2j-r} \\ &= (-1)^j \sum_{l=\{0, j-1\}}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \sum_{k=\{0, l-j\}}^l s(k+2, l+2-j) S(l+1, k+1) x^k. \end{aligned} \tag{4.9}$$

This and (4.8) imply

$$\begin{aligned}
 & \int_0^1 \sum_{j=0}^{m+1} (-s)^j \sum_{k=0}^{2j} (-1)^k (\Upsilon_{j,k} P_m^*)^{[k]}(x) b_{n,\nu}(x) dx \\
 &= \sum_{j=0}^{m+1} s^j \sum_{l=\{0,j-1\}}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \\
 & \quad \times \sum_{k=\{0,l-j\}}^l s(k+2, l+2-j) S(l+1, k+1) \frac{(\nu+k)^{(k)}}{(n+k+1)^{(k+1)}} \\
 &= \sum_{l=0}^m (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} \sum_{k=0}^l S(l+1, k+1) \frac{(\nu+k)^{(k)}}{(n+k+1)^{(k+1)}} \sum_{j=l-k}^{l+1} s(k+2, l+2-j) s^j.
 \end{aligned} \tag{4.10}$$

On the other hand, in (4.6), we can expand

$$\begin{aligned}
 s^{l-k} \prod_{\mu=1}^{k+1} (1 - \mu s) &= s^{l+2} \prod_{\mu=0}^{k+1} \left(\frac{1}{s} - \mu \right) = s^{l+2} \left(\frac{1}{s} \right)^{(k+2)} \\
 &= s^{l+2} \sum_{j=1}^{k+2} s(k+2, j) s^{-j} = \sum_{j=l-k}^{l+1} s(k+2, l+2-j) s^j,
 \end{aligned}$$

where we noticed $s(k+2, 0) = 0$. Therefore, $H_{n,m}^{(s)}(\nu)$ becomes the right-hand side of (4.10)

4.3.3. The third stage

The aim of this subsection is to deform $H_{n,m}^{(s)}(\nu)$ into a more treatable form.

LEMMA 4.14. For every $j, m \in \mathbb{N}_0$, the deformation

$$\sum_{k=0}^{2j} (\Upsilon_{j,k} P_m^*)^{[k]}(x) (t-1)^k = \sum_{k=0}^{2j} \frac{F_{j,k,m}(x)}{k!} t^k$$

holds, where $F_{j,k,m}(x)$ are determined by the recursion formula

$$F_{j,-1,m}(x) = F_{j,2j+1,m}(x) = 0 \quad (j \geq 0),$$

$$F_{0,0,m}(x) = P_m^*(x),$$

$$F_{j,0,m}(x) = \delta_{j,1} P_m^*(x) + \sum_{k=0}^{2(j-1)} \left[\frac{(2x-1)F'_{j-1,k,m}(x)}{(k+1)!} + \frac{x(x-1)F''_{j-1,k,m}(x)}{(k+2)!} \right] \quad (j \geq 1),$$

$$\begin{aligned}
 F_{j,k,m}(x) &= k(k-1)^2 F_{j-1,k-2,m}(x) - k(k+1) F_{j-1,k-1,m}(x) \\
 & \quad + (2x-1)(k(k-1)F'_{j-1,k-2,m}(x) - (k+1)F'_{j-1,k-1,m}(x)) \\
 & \quad + x(x-1)((k-1)F''_{j-1,k-2,m}(x) - F''_{j-1,k-1,m}(x)) \quad (j \geq 1, 1 \leq k \leq 2j).
 \end{aligned}$$

Remark. The purpose of this lemma is to reduce $\sum_{k=0}^{2j} (-1)^k (\Upsilon_{j,k} P_m^*)^{[k]}(x)$ to $F_{j,0,m}(x)$.

Proof. We may fix m throughout this proof and we denote $F_{j,k,m}$ simply by $F_{j,k}$. Let $u = t-1$ and $\varphi_j(x, t) = \sum_{k=0}^{2j} (\Upsilon_{j,k} P_m^*)^{[k]}(x) u^k$. Then it is obvious from the recursion formula in Theorem 3.1 that

$$\varphi_0(x, t) = \Upsilon_{0,0}(x) P_m^*(x) = P_m^*(x). \tag{4.11}$$

Furthermore, when $j \geq 1$, using the Leibniz formula, we can calculate

$$\begin{aligned}
 \varphi_j(x, t) &= \sum_{k=2}^{2j} (\Upsilon_{j,k} P_m^*)^{[k]}(x) u^k \\
 &= \sum_{k=2}^{2j} ((k-1)(\Upsilon_{j-1,k} - e_1 \Upsilon_{j-1,k-1} - e_2 \Upsilon_{j-1,k-2}) P_m^*)^{[k]}(x) u^k \\
 &= \sum_{k=2}^{2j-2} \frac{k-1}{k!} (\Upsilon_{j-1,k} P_m^*)^{(k)}(x) u^k \\
 & \quad + \sum_{k=2}^{2j-1} \frac{k-1}{k!} [(2x-1)(\Upsilon_{j-1,k-1} P_m^*)^{(k)}(x) + 2k(\Upsilon_{j-1,k-1} P_m^*)^{(k-1)}(x)] \\
 & \quad + \sum_{k=2}^{2j} \frac{k-1}{k!} [x(x-1)(\Upsilon_{j-1,k-2} P_m^*)^{(k)}(x) + k(2x-1)(\Upsilon_{j-1,k-2} P_m^*)^{(k-1)}(x) \\
 & \quad + k(k-1)(\Upsilon_{j-1,k-2} P_m^*)^{(k-2)}(x)].
 \end{aligned}$$

Rearrangement of this formula gives

$$\begin{aligned}\varphi_j(x, t) &= \delta_{j,1} P_m^*(x) + \sum_{k=0}^{2(j-1)} (\Upsilon_{j-1,k} P_m^*)^{[k]}(x) [(k-1)u^k + 2ku^{k+1} + (k+1)u^{k+2}] \\ &\quad + (2x-1) \frac{\partial}{\partial x} \sum_{k=0}^{2(j-1)} (\Upsilon_{j-1,k} P_m^*)^{[k]}(x) \left[\frac{k}{k+1} u^{k+1} + u^{k+2} \right] \\ &\quad + x(x-1) \frac{\partial^2}{\partial x^2} \sum_{k=0}^{2(j-1)} (\Upsilon_{j-1,k} P_m^*)^{[k]}(x) \frac{u^{k+2}}{k+2}.\end{aligned}$$

Here we let

$$\begin{aligned}\Phi_j(x, t) &= \int_1^t \varphi_j(x, \tau) d\tau = \sum_{k=0}^{2j} (\Upsilon_{j,k} P_m^*)^{[k]}(x) \frac{u^{k+1}}{k+1}, \\ \tilde{\Phi}_j(x, t) &= \int_1^t (\tau-1) \varphi_j(x, \tau) d\tau = \sum_{k=0}^{2j} (\Upsilon_{j,k} P_m^*)^{[k]}(x) \frac{u^{k+2}}{k+2}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\varphi_j(x, t) &= \delta_{j,1} P_m^*(x) \\ &\quad + u \frac{\partial}{\partial t} \varphi_{j-1}(x, t) - \varphi_{j-1}(x, t) + 2u^2 \frac{\partial}{\partial t} \varphi_{j-1}(x, t) + u^3 \frac{\partial}{\partial t} \varphi_{j-1}(x, t) + u^2 \varphi_{j-1}(x, t) \\ &\quad + (2x-1) \frac{\partial}{\partial x} [u \varphi_{j-1}(x, t) - \Phi_{j-1}(x, t) + u^2 \varphi_{j-1}(x, t)] + x(x-1) \frac{\partial^2}{\partial x^2} \tilde{\Phi}_{j-1}(x, t) \\ &= \delta_{j,1} P_m^*(x) + (t^3 - t^2) \frac{\partial}{\partial t} \varphi_{j-1}(x, t) + (t^2 - 2t) \varphi_{j-1}(x, t) \\ &\quad + (2x-1) \frac{\partial}{\partial x} [(t^2 - t) \varphi_{j-1}(x, t) - \Phi_{j-1}(x, t)] + x(x-1) \frac{\partial^2}{\partial x^2} \tilde{\Phi}_{j-1}(x, t).\end{aligned}\quad (4.12)$$

Since $\varphi_j(x, t)$ is a polynomial in t of degree $2j$, we can expand it into the form

$$\varphi_j(x, t) = \sum_{k=0}^{2j} \frac{F_{j,k}(x)}{k!} t^k.$$

From this and (4.11), we have immediately

$$F_{0,0}(x) = \varphi_0(x, t) = P_m^*(x).$$

Next, we consider the case $j \geq 1$. Since

$$\begin{aligned}\Phi_j(x, t) &= \sum_{k=0}^{2j} \frac{F_{j,k}(x)}{(k+1)!} t^{k+1} - \sum_{k=0}^{2j} \frac{F_{j,k}(x)}{(k+1)!}, \\ \tilde{\Phi}_j(x, t) &= \sum_{k=0}^{2j} \frac{F_{j,k}(x)}{k!} \left(\frac{t^{k+2}}{k+2} - \frac{t^{k+1}}{k+1} \right) + \sum_{k=0}^{2j} \frac{F_{j,k}(x)}{(k+2)!},\end{aligned}$$

(4.12) implies

$$\begin{aligned}\sum_{k=0}^{2j} \frac{F_{j,k}(x)}{k!} t^k &= \delta_{j,1} P_m^*(x) + \sum_{k=1}^{2(j-1)} \frac{F_{j-1,k}(x)}{(k-1)!} (t^{k+2} - t^{k+1}) + \sum_{k=0}^{2(j-1)} \frac{F_{j-1,k}(x)}{k!} (t^{k+2} - 2t^{k+1}) \\ &\quad + (2x-1) \frac{\partial}{\partial x} \left[\sum_{k=0}^{2(j-1)} \frac{F_{j-1,k}(x)}{k!} (t^{k+2} - t^{k+1}) - \sum_{k=0}^{2(j-1)} \frac{F_{j-1,k}(x)}{(k+1)!} t^{k+1} + \sum_{k=0}^{2(j-1)} \frac{F_{j-1,k}(x)}{(k+1)!} \right] \\ &\quad + x(x-1) \frac{\partial^2}{\partial x^2} \left[\sum_{k=0}^{2(j-1)} \frac{F_{j-1,k}(x)}{k!} \left(\frac{t^{k+2}}{k+2} - \frac{t^{k+1}}{k+1} \right) + \sum_{k=0}^{2(j-1)} \frac{F_{j-1,k}(x)}{(k+2)!} \right].\end{aligned}$$

Rearrangement of this formula with the conventional definition

$$F_{j,-1}(x) = F_{j,2j+1}(x) = 0 \quad (j \geq 0)$$

gives

$$\begin{aligned}\sum_{k=0}^{2j} \frac{F_{j,k}(x)}{k!} t^k &= \delta_{j,1} P_m^*(x) + \sum_{k=0}^{2(j-1)} \left[\frac{(2x-1)F'_{j-1,k}(x)}{(k+1)!} + \frac{x(x-1)F''_{j-1,k}(x)}{(k+2)!} \right] \\ &\quad + \sum_{k=1}^{2j} \frac{t^k}{k!} \left[k(k-1)^2 F_{j-1,k-2}(x) - k(k+1) F_{j-1,k-1}(x) \right. \\ &\quad \left. + (2x-1)(k(k-1)F'_{j-1,k-2}(x) - (k+1)F'_{j-1,k-1}(x)) \right. \\ &\quad \left. + x(x-1)((k-1)F''_{j-1,k-2}(x) - F''_{j-1,k-1}(x)) \right].\end{aligned}$$

Equating coefficients of t^k ($0 \leq k \leq 2j$) on both sides completes the proof. ■

LEMMA 4.15. *The identities*

$$(2x-1)(G_m^{(l)})'(x) = 2(mG_m^{(l)}(x) + (2l+1)G_{m-2}^{(l+1)}(x)),$$

$$x(x-1)(G_m^{(l)})''(x) = m(m-1)G_m^{(l)}(x) - 2(l+1)(2l+1)G_{m-2}^{(l+1)}(x)$$

hold for every $l, m \in \mathbf{N}_0$, where we define $G_{-2}^{(l)}(x) = G_{-1}^{(l)}(x) = 0$.

Proof. Suppose $0 \leq x \leq 1$, $|t| < 1$ and consider the generating function

$$(1 - 2(2x-1)t + t^2)^{-l-1/2} = \sum_{m=0}^{\infty} G_m^{(l)}(x)t^m.$$

We can easily verify

$$\begin{aligned} (2x-1) \frac{\partial}{\partial x} (1 - 2(2x-1)t + t^2)^{-l-1/2} \\ = 2t \frac{\partial}{\partial t} (1 - 2(2x-1)t + t^2)^{-l-1/2} + 2(2l+1)t^2(1 - 2(2x-1)t + t^2)^{-l-3/2}. \end{aligned}$$

Therefore,

$$(2x-1) \sum_{m=0}^{\infty} (G_m^{(l)})'(x)t^m = 2 \sum_{m=0}^{\infty} mG_m^{(l)}(x)t^m + 2(2l+1) \sum_{m=2}^{\infty} G_{m-2}^{(l+1)}(x)t^m.$$

Equating the coefficients of t^m on both sides with the conventional definition $G_{-2}^{(l)}(x) = G_{-1}^{(l)}(x) = 0$ yields the first identity.

The differential equation (see (4.7.5) in [48] or (22.6.5) in [1])

$$(1-x^2)(C_m^{(\lambda)})''(x) - (2\lambda+1)x(C_m^{(\lambda)})'(x) + m(m+2\lambda)C_m^{(\lambda)}(x) = 0$$

implies

$$x(x-1)(G_m^{(l)})''(x) = m(m+2l+1)G_m^{(l)}(x) - (l+1)(2x-1)(G_m^{(l)})'(x).$$

This and the first identity imply the second identity. ■

LEMMA 4.16. *For every $j, k, m \in \mathbf{N}_0$ ($k \leq 2j$), we can decompose*

$$F_{j,k,m}(x) = \sum_{l=0}^{\{j, [m/2]\}} A_{j,k,l}(m) G_{m-2l}^{(l)}(x),$$

where we define $A_{j,k,l}(m)$ by the recursion formula

$$A_{j,-1,l}(m) = A_{j,2j+1,l}(m) = 0 \quad (j \geq 0, -1 \leq l \leq j+1),$$

$$A_{j,k,-1}(m) = A_{j,k,j+1}(m) = 0 \quad (j \geq 0, 0 \leq k \leq 2j),$$

$$A_{0,0,0}(m) = 1,$$

$$\begin{aligned} A_{j,0,l}(m) = \sum_{k=0}^{2(j-1)} \frac{(m-2l)(m+2k-2l+3)A_{j-1,k,l}(m) + 2(2l-1)(k-l+2)A_{j-1,k,l-1}(m)}{(k+2)!} \\ + \delta_{j,1}\delta_{l,0} \quad (j \geq 1, 0 \leq l \leq j), \end{aligned}$$

$$\begin{aligned} A_{j,k,l}(m) = (m+k-2l)[(k-1)(m+k-2l-1)A_{j-1,k-2,l}(m) - (m+k-2l+1)A_{j-1,k-1,l}(m)] \\ + 2(2l-1)[(k-1)(k-l)A_{j-1,k-2,l-1}(m) - (k-l+1)A_{j-1,k-1,l-1}(m)] \\ (j \geq 1, 1 \leq k \leq 2j, 0 \leq l \leq j). \end{aligned}$$

Proof. We may fix m throughout this proof and we denote $F_{j,k,m}$ and $A_{j,k,l}(m)$ simply by $F_{j,k}$ and $A_{j,k,l}$, respectively. We prove this lemma by induction with respect to j . In the following calculations, we apply Lemmas 4.14 and 4.15 frequently.

First, we can verify

$$F_{0,0}(x) = P_m^*(x) = \sum_{l=0}^{\{0, [m/2]\}} A_{0,0,l} G_{m-2l}^{(l)}(x).$$

Take $j \geq 1$ and assume that this lemma is valid for $j-1$. Then we can calculate

$$\begin{aligned} F_{j,0}(x) &= \delta_{j,1}P_m^*(x) + \sum_{k=0}^{2(j-1)} \left[\frac{1}{(k+1)!} \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k,l} (2x-1)(G_{m-2l}^{(l)})'(x) \right. \\ &\quad \left. + \frac{1}{(k+2)!} \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k,l} x(x-1)(G_{m-2l}^{(l)})''(x) \right] \\ &= \delta_{j,1}P_m^*(x) + \sum_{k=0}^{2(j-1)} \left[\frac{1}{(k+1)!} \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k,l} 2((m-2l)G_{m-2l}^{(l)}(x) + (2l+1)G_{m-2l-2}^{(l+1)}(x)) \right. \\ &\quad \left. + \frac{1}{(k+2)!} \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k,l} ((m-2l)(m-2l-1)G_{m-2l}^{(l)}(x) - 2(l+1)(2l+1)G_{m-2l-2}^{(l+1)}(x)) \right]. \end{aligned}$$

Rearrangement of this formula gives

$$\begin{aligned}
 F_{j,0}(x) &= \sum_{l=0}^{\{j, [m/2]\}} G_{m-2l}^{(l)}(x) \sum_{k=0}^{2(j-1)} \frac{(m-2l)(m+2k-2l+3)A_{j-1,k,l}(m) + 2(2l-1)(k-l+2)A_{j-1,k,l-1}(m)}{(k+2)!} \\
 &\quad + \delta_{j,1} G_m^{(0)}(x) \\
 &= \sum_{l=0}^{\{j, [m/2]\}} A_{j,0,l} G_{m-2l}^{(l)}(x).
 \end{aligned}$$

Furthermore, when $1 \leq k \leq 2j$,

$$\begin{aligned}
 F_{j,k}(x) &= k(k-1)^2 \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-2,l} G_{m-2l}^{(l)}(x) - k(k+1) \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-1,l} G_{m-2l}^{(l)}(x) \\
 &\quad + k(k-1) \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-2,l} (2x-1)(G_{m-2l}^{(l)})'(x) - (k+1) \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-1,l} (2x-1)(G_{m-2l}^{(l)})'(x) \\
 &\quad + (k-1) \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-2,l} x(x-1)(G_{m-2l}^{(l)})''(x) - \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-1,l} x(x-1)(G_{m-2l}^{(l)})''(x) \\
 &= k(k-1)^2 \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-2,l} G_{m-2l}^{(l)}(x) - k(k+1) \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-1,l} G_{m-2l}^{(l)}(x) \\
 &\quad + k(k-1) \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-2,l} 2((m-2l)G_{m-2l}^{(l)}(x) + (2l+1)G_{m-2l-2}^{(l+1)}(x)) \\
 &\quad - (k+1) \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-1,l} 2((m-2l)G_{m-2l}^{(l)}(x) + (2l+1)G_{m-2l-2}^{(l+1)}(x)) \\
 &\quad + (k-1) \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-2,l} ((m-2l)(m-2l-1)G_{m-2l}^{(l)}(x) - 2(l+1)(2l+1)G_{m-2l-2}^{(l+1)}(x)) \\
 &\quad - \sum_{l=0}^{\{j-1, [m/2]\}} A_{j-1,k-1,l} ((m-2l)(m-2l-1)G_{m-2l}^{(l)}(x) - 2(l+1)(2l+1)G_{m-2l-2}^{(l+1)}(x)).
 \end{aligned}$$

Rearrangement of this formula gives

$$F_{j,k}(x) = \sum_{l=0}^{\{j, [m/2]\}} A_{j,k,l} G_{m-2l}^{(l)}(x). \quad \blacksquare$$

The recursion formula in the above lemma indicates that $A_{j,k,l}(m)$ is a polynomial in m of degree $2(j-l)$. In practical calculations, we should avoid evaluating $A_{j,k,l}(m)$ with that complicated recursion formula whenever m moves. We should expand the polynomial $A_{j,k,l}(m)$ appropriately in advance and use some nesting procedure to evaluate it for each m . Thus we provide the following lemma.

LEMMA 4.17. For every $j, k, l \in \mathbf{N}_0$ ($k \leq 2j, l \leq j$), we can expand

$$A_{j,k,l}(m) = \sum_{r=0}^{2(j-l)} a_{j,k,l,r} \binom{m-2l+2}{r},$$

where we define $a_{j,k,l,r}$ by the recursion formula in Theorem 4.1.

Proof. We prove this lemma by induction with respect to j . In the following calculations, we apply the recursion formula in Lemma 4.16 frequently.

First, we can verify

$$A_{0,0,0}(m) = 1 = \sum_{r=0}^{2(0-0)} a_{0,0,0,r} \binom{m-2 \cdot 0+2}{r}.$$

Take $j \geq 1$ and assume that this lemma is valid for $j-1$. Then we can calculate

$$\begin{aligned}
 A_{j,0,l}(m) &= \sum_{k=0}^{2(j-1)} \frac{1}{(k+2)!} \left[\sum_{r=0}^{2(j-l-1)} a_{j-1,k,l,r} (m-2l)(m+2k-2l+3) \binom{m-2l+2}{r} \right. \\
 &\quad \left. + 2(2l-1)(k-l+2) \sum_{r=0}^{2(j-l)} a_{j-1,k,l-1,r} \binom{m-2l+4}{r} \right] + \delta_{j,1} \delta_{l,0}.
 \end{aligned}$$

By virtue of the identities

$$\begin{aligned}
 &(m-2l)(m+2k-2l+3) \binom{m-2l+2}{r} \\
 &= (r+1)(r+2) \binom{m-2l+2}{r+2} + 2(r+1)(r+k) \binom{m-2l+2}{r+1} + (r-2)(r+2k+1) \binom{m-2l+2}{r}
 \end{aligned}$$

and

$$\binom{m-2l+4}{r} = \binom{m-2l+2}{r} + 2\binom{m-2l+2}{r-1} + \binom{m-2l+2}{r-2}$$

(we define $\binom{*}{-1} = \binom{*}{-2} = 0$), we can proceed as

$$\begin{aligned} A_{j,0,l}(m) = & \sum_{k=0}^{2(j-l-1)} \frac{1}{(k+2)!} \left\{ \sum_{r=0}^{2(j-l-1)} (r+1)(r+2)a_{j-1,k,l,r} \binom{m-2l+2}{r+2} + \sum_{r=0}^{2(j-l-1)} 2(r+1)(r+k)a_{j-1,k,l,r} \binom{m-2l+2}{r+1} \right. \\ & + \sum_{r=0}^{2(j-l-1)} (r-2)(r+2k+1)a_{j-1,k,l,r} \binom{m-2l+2}{r} + 2(2l-1)(k-l+2) \left[\sum_{r=0}^{2(j-l)} a_{j-1,k,l-1,r} \binom{m-2l+2}{r} \right. \\ & \left. \left. + \sum_{r=1}^{2(j-l)} 2a_{j-1,k,l-1,r} \binom{m-2l+2}{r-1} + \sum_{r=2}^{2(j-l)} a_{j-1,k,l-1,r} \binom{m-2l+2}{r-2} \right] \right\} + \delta_{j,1}\delta_{l,0}. \end{aligned}$$

Rearrangement of this formula gives

$$A_{j,0,l}(m) = \sum_{r=0}^{2(j-l)} a_{j,0,l,r} \binom{m-2l+2}{r}.$$

(Verify this fact separately for $j=1$ and for $j \geq 2$.) Furthermore, when $1 \leq k \leq 2j$,

$$\begin{aligned} A_{j,k,l}(m) = & (k-1) \sum_{r=0}^{2(j-l-1)} a_{j-1,k-2,l,r} (m+k-2l)^{(2)} \binom{m-2l+2}{r} - \sum_{r=0}^{2(j-l-1)} a_{j-1,k-1,l,r} (m+k+1-2l)^{(2)} \binom{m-2l+2}{r} \\ & + 2(2l-1) \left[(k-1)(k-l) \sum_{r=0}^{2(j-l)} a_{j-1,k-2,l-1,r} \binom{m-2l+4}{r} - (k-l+1) \sum_{r=0}^{2(j-l)} a_{j-1,k-1,l-1,r} \binom{m-2l+4}{r} \right]. \end{aligned}$$

Since the identity

$$\begin{aligned} (m+k-2l)^{(2)} \binom{m-2l+2}{r} & = (r+1)(r+2) \binom{m-2l+2}{r+2} + 2(r+1)(r+k-2) \binom{m-2l+2}{r+1} + (r+k-2)^{(2)} \binom{m-2l+2}{r} \end{aligned}$$

holds, similarly to the case $k=0$, we obtain

$$A_{j,k,l}(m) = \sum_{r=0}^{2(j-l)} a_{j,k,l,r} \binom{m-2l+2}{r}. \blacksquare$$

LEMMA 4.18. For every $j, l, r \in \mathbf{N}_0$ ($l \leq j$),

$$a_{j,0,l,r} = 0 \quad \text{if } j \geq 2l+1 \text{ and } r \leq j-2l.$$

Proof. Lemmas 4.14, 4.16, and 4.3 imply

$$\begin{aligned} \sum_{k=0}^{2j} (-1)^k (\Upsilon_{j,k} P_m^*)^{[k]}(x) & = F_{j,0,m}(x) = \sum_{l=0}^{\{j, [m/2]\}} A_{j,0,l}(m) G_{m-2l}^{(l)}(x) \\ & = \sum_{l=0}^{\{j, [m/2]\}} \frac{A_{j,0,l}(m)}{(m-2l)!} \sum_{k=0}^{m-2l} (-1)^{m-k} \binom{m-2l}{p} \frac{(m+k)^{(m-2l+k)}}{(l+k)^{(k)}} x^k \\ & = \sum_{k=0}^m (-1)^{m-k} x^k \sum_{l=0}^{\{j, [(m-k)/2]\}} A_{j,0,l}(m) \frac{l!(m+k)!}{k!(2l)!(l+k)!(m-2l+k)!}. \end{aligned} \quad (4.13)$$

On the other hand, (4.9) implies

$$\begin{aligned} \sum_{k=0}^{2j} (-1)^k (\Upsilon_{j,k} P_m^*)^{[k]}(x) & = (-1)^j \sum_{k=0}^m x^k \sum_{l=\{k, j-1\}}^{\{m, j+k\}} (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} s(k+2, l+2-j) S(l+1, k+1). \end{aligned}$$

Therefore, comparing the coefficients of x^k in the above two equations, we obtain

$$\begin{aligned} \sum_{l=0}^{\{j, [(m-k)/2]\}} A_{j,0,l}(m) \frac{l!(m+k)!}{k!(2l)!(l+k)!(m-2l-k)!} & = (-1)^{m-k+j} \sum_{l=\{k, j-1\}}^{\{m, j+k\}} (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} s(k+2, l+2-j) S(l+1, k+1). \end{aligned}$$

Replacing k by $m-2k$ and supposing $0 \leq k \leq j$ and $2k \leq m$, we have

$$\begin{aligned} \sum_{l=0}^k A_{j,0,l}(m) \frac{l!(2m-2k)!}{(m-2k)!(2l)!(l+m-2k)!(2k-2l)!} & = (-1)^j \sum_{l=\{m-2k, j-1\}}^{\{m, m-2k+j\}} (-1)^{m-l} \binom{m}{l} \binom{m+l}{l} s(m-2k+2, l+2-j) S(l+1, m-2k+1) \\ & = \sum_{l=\{0, j-m+2k-1\}}^{\{2k, j\}} (-1)^{j-l} \binom{m}{2k-l} \binom{2m-2k+l}{m} \check{s}(m-2k+2, j-l) \check{S}(m-2k+l+1, l), \end{aligned} \quad (4.14)$$

where we define $\tilde{s}(m, j) = s(m, m - j)$ and $\tilde{S}(m, j) = S(m, m - j)$ for every $m, j \in \mathbf{N}_0$ ($j \leq m$). The identities

$$\tilde{s}(m, j) = \begin{cases} 1 & (j = 0), \\ \sum_{r=0}^{j-1} C_{j,r} \binom{m}{2j-r} & (j \geq 1), \end{cases}$$

$$\tilde{S}(m, j) = \begin{cases} 1 & (j = 0), \\ \sum_{r=0}^{j-1} \bar{C}_{j,r} \binom{m}{2j-r} & (j \geq 1), \end{cases}$$

appear in [21, pp.149,151,171,172]. The right-hand sides of the above two equations are polynomials in m of degree $2j$. Thus we withdraw the stipulation $j \leq m$ on the definition of \tilde{s} and \tilde{S} . Then the $\tilde{s}(m - 2k + 2, j - l)$ in (4.14) naturally vanishes when $l \leq j - m + 2k - 2$. Multiplying both sides of (4.14) by $(m - 2k)!(2k)!(m - k)!/(k!(2m - 2k)!)$, we get

$$\sum_{l=0}^k A_{j,0,l}(m) \binom{m-k}{k-l} \binom{2k}{2l} \binom{k}{l}^{-1} \\ = \sum_{l=0}^{\{2k,j\}} (-1)^{j-l} \frac{(2k)^{(l)}(m-k)!(2m-2k+l)^{(l)}}{k!(m-2k+l)!(m-2k+l)^{(l)}} \tilde{s}(m-2k+2, j-l) \tilde{S}(m-2k+l+1, l).$$

Since we can write $\tilde{S}(m - 2k + l + 1, l) = (m - 2k + l)^{(l)} \tilde{S}_l(m - 2k + l + 1)$ using some polynomial \tilde{S}_l of degree l , we can proceed as

$$A_{j,0,k}(m) \\ = - \sum_{l=0}^{k-1} A_{j,0,l}(m) \binom{m-k}{k-l} \binom{2k}{2l} \binom{k}{l}^{-1} \\ + \sum_{l=0}^{\{2k,j\}} (-1)^{j-l} \frac{(2k)^{(l)}(m-k)!(2m-2k+l)^{(l)}}{k!(m-2k+l)!} \tilde{S}_l(m-2k+l+1) \tilde{s}(m-2k+2, j-l) \\ = - \sum_{l=0}^{k-1} A_{j,0,l}(m) \binom{m-k}{k-l} \binom{2k}{2l} \binom{k}{l}^{-1} \\ + \sum_{l=0}^k (-1)^{j-l} \frac{(2k)^{(l)}(m-k)^{(k-l)}(2m-2k+l)^{(l)}}{k!} \tilde{S}_l(m-2k+l+1) \tilde{s}(m-2k+2, j-l)$$

$$+ \sum_{l=k+1}^{\{2k,j\}} (-1)^{j-l} \frac{(2k)^{(l)}(2m-2k+l)^{(2k-l)}(2m-4k+2l)^{(2l-2k)}}{k!(m-2k+l)^{(l-k)}} \tilde{S}_l(m-2k+l+1) \tilde{s}(m-2k+2, j-l).$$

Since $(2m - 4k + 2l)^{(2l-2k)}$ is divisible by $(m - 2k + l)^{(l-k)}$, both sides of the above formula are polynomials in m , and here we can remove the stipulation $2k \leq m$. Now suppose $j \geq 2k + 1$. Then from the above recursion formula and the fact that $\tilde{s}(m - 2k + 2, j - l)$ has the factor $(m - 2k + 2)^{(j-l+1)}$ when $j \geq l + 1$, we can see by induction with respect to k that $A_{j,0,k}(m)$ has the factor $(m - 2k + 2)^{(j-2k+1)}$. Replacing k by l and Lemma 4.17 complete the proof. ■

The following proposition itself is a useful formula for evaluation of $P_n^{(s)} f(x)$ as well as the main preparation for the proof of Theorem 4.1.

PROPOSITION 4.19. *Stancu's operator $P_n^{(s)}$ has the following representation in terms of the shifted Legendre polynomials:*

$$P_n^{(s)} f(x) = \sum_{m=0}^n (2m+1) P_m^*(x) \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) (H_{n,m}^{(s)}(\nu) + K_{n,m}^{(s)}(\nu)),$$

where

$$H_{n,m}^{(s)}(\nu) = \sum_{l=0}^{\lfloor m/2 \rfloor} \hat{G}_{n,m-2l}^{(l)}(\nu) \sum_{r=0}^{m-2l+2} \binom{m-2l+2}{r} \sum_{j=l+\lfloor(r+1)/2\rfloor}^{2l+\max\{r-1,0\}} a_{j,l,r} (-s)^j, \\ K_{n,m}^{(s)}(\nu) = \sum_{k=0}^n \left((-1)^{m+\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu} \right) \sum_{l=0}^m (-1)^l \gamma_{k+l+1}^{(l+1)} \binom{m}{l} \frac{(m+l)^{(l)} n^{(k)} s^{k+l+1}}{\prod_{\mu=0}^{k-1} (1 + \mu s)},$$

where we identify $a_{j,l,r}$ with $a_{j,0,l,r}$ defined in Theorem 4.1.

Remark. Put $s = 0$ in the expression of $H_{n,m}^{(s)}(\nu)$ to verify that it is a generalization of (4.7). Furthermore, we can clearly see the relationships $H_{n,m}^{(s)}(n - \nu) = (-1)^m H_{n,m}^{(s)}(\nu)$ and $K_{n,m}^{(s)}(n - \nu) = (-1)^m K_{n,m}^{(s)}(\nu)$ through this proposition.

Proof. Lemma 4.10, (4.13), and Lemma 4.17 imply

$$\begin{aligned} H_{n,m}^{(s)}(\nu) &= \sum_{j=0}^{m+1} (-s)^j \sum_{l=0}^{\{j, [m/2]\}} A_{j,0,l}(m) \int_0^1 G_{m-2l}^{(l)}(x) b_{n,\nu}(x) dx \\ &= \sum_{j=0}^{m+1} (-s)^j \sum_{l=0}^{\{j, [m/2]\}} \widehat{G}_{n,m-2l}^{(l)}(\nu) \sum_{r=0}^{2(j-l)} a_{j,l,r} \binom{m-2l+2}{r} \\ &= \sum_{l=0}^{[m/2]} \widehat{G}_{n,m-2l}^{(l)}(\nu) \sum_{r=0}^{m-2l+2} \binom{m-2l+2}{r} \sum_{j=l+[(r+1)/2]}^{m+1} a_{j,l,r} (-s)^j. \end{aligned}$$

Since Lemma 4.18 indicates $a_{j,l,r} = 0$ for $j \geq 2l + \max\{r, 1\}$, we can replace $\sum_{j=l+[(r+1)/2]}^{m+1}$ by $\sum_{j=2l+\max\{r-1,0\}}^{2l+\max\{r-1,0\}}$. (Note that $2l + \max\{r-1, 0\} \leq m+1$ holds in the above formula.)

Therefore, Lemma 4.7 and the obvious equation

$$P_n^{(s)} f(x) = \sum_{m=0}^n (2m+1) P_m^*(x) \int_0^1 P_n^{(s)} f(x) P_m^*(x) dx$$

complete the proof. ■

4.4. Proof of Theorem 4.1

At last, we have come to the position to prove Theorem 4.1.

Proof of Theorem 4.1. Let $(n-1)|s| < 1$ and $s \neq 0$. As we did in the proof of Theorem 3.5, we can expand

$$\frac{s^{k+l+1}}{\prod_{\mu=0}^{k-1} (1 + \mu s)} = (-1)^k s^{l+1} \sum_{j=k}^{\infty} S(j-1, k-1) (-s)^j = (-1)^{k+l} s \sum_{j=k+l}^{\infty} S(j-l-1, k-1) (-s)^j.$$

Proposition 4.19 and Definition 3.1 imply the main part of this theorem.

Next, we consider the three-term recursion formula for $\widehat{G}_{n,m}^{(l)}(\nu)$. Clearly, we have

$$\widehat{G}_{n,0}^{(l)}(\nu) = \int_0^1 b_{n,\nu}(x) dx = \frac{1}{n+1}.$$

Now we let $|t| < 1$ and

$$\Phi = (1 - 2(2x-1)t + t^2)^{-l-1/2} b_{n,\nu}(x).$$

Then Φ satisfies the partial differential equation

$$\begin{aligned} (t-t^3) \frac{\partial^2 \Phi}{\partial t^2} + (n+2-2(2\nu-n)t + (n-4l-2)t^2) \frac{\partial \Phi}{\partial t} + (2l+1)(n-2\nu+(n-2l)t) \Phi \\ = -2(2l+1)(1-t^2) \frac{\partial}{\partial x} \frac{x(1-x)\Phi}{1-2(2x-1)t+t^2}. \end{aligned}$$

Integrating both sides by x on $[0, 1]$ and letting $\varphi(t) = \int_0^1 \Phi dx$, we get

$$(t-t^3)\varphi''(t) + (n+2-2(2\nu-n)t + (n-4l-2)t^2)\varphi'(t) + (2l+1)(n-2\nu+(n-2l)t)\varphi(t) = 0.$$

Putting

$$\varphi(t) = \int_0^1 \sum_{m=0}^{\infty} G_m^{(l)}(x) t^m b_{n,\nu}(x) dx = \sum_{m=0}^{\infty} \widehat{G}_{n,m}^{(l)}(\nu) t^m$$

in the above equation and equating the coefficients of t^m on both sides (with the convention $\widehat{G}_{n,-1}^{(l)}(\nu)$) yield the recursion formula for $\widehat{G}_{n,m}^{(l)}(\nu)$.

Furthermore, we consider the recursion formula for $\gamma_k^{(l)}$. Since $\gamma_k^{(1)} = \gamma_k$, we have already shown the formula for $\gamma_k^{(1)}$. (Recall the comment after Theorem 3.5.) Now we cite from [29, p.129] the identity

$$\mathcal{B}_\nu^{(n+1)}(x) = \left(1 - \frac{\nu}{n}\right) \mathcal{B}_\nu^{(n)}(x) + \nu \left(\frac{x}{n} - 1\right) \mathcal{B}_{\nu-1}^{(n)}(x) \quad (\nu \geq 1).$$

Putting $x = 1$, $\nu = k$, and $n = k-l$, we obtain

$$(k-l)\mathcal{B}_k^{(k-l+1)}(1) = -l\mathcal{B}_k^{(k-l)}(1) + k(1-k+l)\mathcal{B}_{k-1}^{(k-l)}(1) \quad (k \geq 1);$$

that is,

$$l \frac{\mathcal{B}_k^{(k-l)}(1)}{k!} = (l-k) \frac{\mathcal{B}_k^{(k-l+1)}(1)}{k!} + (l-k+1) \frac{\mathcal{B}_{k-1}^{(k-l)}(1)}{(k-1)!} \quad (k \geq 1).$$

On account of the relationship (4.2), this is identical with the desired recursion formula, which is justified also for $k = 0$ by $\gamma_0^{(l)} = 1$ and the conventional definition $\gamma_{-1}^{(l)} = 0$. ■

4.5. Implementation

Now we express the numerical algorithm for the evaluation of ${}_a B_n f(x)$ based on Theorem 4.1 as follows.

- (1) Give an upper bound of α denoted by α^* ; that is, suppose that $\alpha \leq \alpha^*$.

(2) Make the tables of

$$a_{j,l,r} \quad (0 \leq j \leq \alpha^*, 0 \leq l \leq j, \max\{0, j-2l\} \leq r \leq 2(j-l)),$$

$$\gamma_k^{(l)} \quad (1 \leq l \leq \alpha^*, l \leq k \leq \alpha^*),$$

$$S(j, k) \quad (-1 \leq j \leq \alpha^* - 2, -1 \leq k \leq j),$$

$$\binom{k}{\nu} \quad (0 \leq k \leq \alpha^* - 1, 0 \leq \nu \leq k),$$

where Lemma 4.18 is helpful to determine $a_{j,l,r}$, and the $S(j, k)$ can be calculated with the recursion formula (see [1, p.825] or [21, p.169])

$$S(j, j) = 1 \quad (-1 \leq j \leq \alpha^* - 2),$$

$$S(j, -1) = 0 \quad (0 \leq j \leq \alpha^* - 2),$$

$$S(j, k) = S(j-1, k-1) + kS(j-1, k) \quad (1 \leq j \leq \alpha^* - 2, 0 \leq k \leq j-1).$$

(3) Give values of n and $\alpha(\leq \alpha^*)$.

(4) Calculate

$$A_{l,r} = \sum_{j=l+\lfloor(r+1)/2\rfloor}^{\{\alpha, 2l+\max\{r-1, 0\}\}} \frac{a_{j,l,r}}{n^j} \quad (0 \leq l \leq \min\{\alpha, [n/2]\}, 0 \leq r \leq 2(\alpha-l)),$$

$$C_{k,l} = \gamma_{k+l+1}^{(l+1)} \sum_{j=0}^{\alpha-k-l-1} \frac{S(j+k-1, k-1)}{n^j} \quad (0 \leq k \leq \alpha-1, 0 \leq l \leq \alpha-1-k),$$

$$N_k = \frac{(-1)^k n^{(k)}}{n^k} \quad (0 \leq k \leq \alpha-1).$$

(We use the Horner method for $A_{l,r}$ and $C_{k,l}$, and we calculate N_k with $N_0 = 1$ and

$$N_{k+1} = -(1 - k/n)N_k.)$$

(5) Calculate

$$L_{m,l} = \sum_{r=0}^{2(\alpha-l)} A_{l,r} \binom{m-2l+2}{r} \quad (0 \leq m \leq n, 0 \leq l \leq \min\{\alpha, [m/2]\}),$$

$$M_{m,k} = \sum_{l=0}^{\alpha-1-k} C_{k,l} \binom{m}{l} \frac{(m+l)^{(l)}}{n^l} \quad (0 \leq m \leq n, 0 \leq k \leq \alpha-1).$$

(Since

$$\binom{m-2l+2}{r+1} = \frac{m-2l+2-r}{r+1} \binom{m-2l+2}{r}$$

and

$$\binom{m}{l+1} \frac{(m+l+1)^{(l+1)}}{n^{l+1}} = \frac{(m-l)(m+l+1)}{(l+1)n} \cdot \binom{m}{l} \frac{(m+l)^{(l)}}{n^l}$$

hold, we can use some nesting procedures similar to the Horner method.)

(6) Calculate

$$p_m^{(l)} = \frac{2m+2l+1}{(m+1)(n+m+2)}, \quad q_m^{(l)} = \frac{(m+2l)(n-m-2l+1)}{(m+1)(n+m+2)}$$

$$(0 \leq l \leq \min\{\alpha, [(n-1)/2]\}, 0 \leq m \leq n-2l-1).$$

(7) Execute the following procedures for every ν ($0 \leq \nu \leq [n/2]$).

(i) Calculate $\hat{G}_m^{(l)}$ ($0 \leq l \leq \min\{\alpha, [n/2]\}, 0 \leq m \leq n-2l$) with the three-term recursion formula

$$\hat{G}_{-1}^{(l)} = 0, \quad \hat{G}_0^{(l)} = \frac{1}{n+1},$$

$$\hat{G}_{m+1}^{(l)} = p_m^{(l)}(2\nu-n)\hat{G}_m^{(l)} - q_m^{(l)}\hat{G}_{m-1}^{(l)}.$$

(ii) Calculate for every m ($0 \leq m \leq n$),

$$w_m(\nu) = \sum_{l=0}^{\{\alpha, [m/2]\}} L_{m,l} \hat{G}_{m-2l}^{(l)} - \frac{1}{n} \left(\sum_{k=\nu}^{\alpha-1} (-1)^{m+\nu} \binom{k}{\nu} N_k M_{m,k} + \sum_{k=n-\nu}^{\alpha-1} (-1)^{n-\nu} \binom{k}{n-\nu} N_k M_{m,k} \right).$$

(8) Calculate $p_m = (2m+1)/(m+1)$ and $q_m = m/(m+1)$ for every m ($0 \leq m \leq n-1$).

(9) Give the values $f(\nu/n)$ ($0 \leq \nu \leq n$) of a function $f: [0, 1] \rightarrow \mathbf{R}$.

(10) Calculate for every m ($0 \leq m \leq n$),

$$W_m = (2m+1) \left[\sum_{\nu=0}^{[(n-1)/2]} \left(f\left(\frac{\nu}{n}\right) + (-1)^m f\left(1 - \frac{\nu}{n}\right) \right) w_m(\nu) + \frac{1+(-1)^n}{2} f\left(\frac{1}{2}\right) w_m\left(\frac{n}{2}\right) \right].$$

(If n is odd, neither $f(1/2)$ nor $w_m(n/2)$ is defined, but the factor $1+(-1)^n = 0$ covers up this matter.)

(11) Give a value of $x \in [0, 1]$.

(12) Calculate

$${}_αB_n f(x) = \sum_{m=0}^n W_m P_m^*(x)$$

by the Clenshaw method with the three-term recursion formula

$$P_{-1}^*(x) = 0, \quad P_0^*(x) = 1,$$

$$P_{m+1}^*(x) = p_m(2x-1)P_m^*(x) - q_m P_{m-1}^*(x).$$

Now let us consider the computational complexity — the number of multiplications and divisions — of the above algorithm.

It is unnecessary to count the computational complexity of Step (2) because we only have to make the tables once.

In Step (4), since the computational complexity is independent of n , it is $O(1)$ with respect to n .

In Steps (5) and (6), it is $O(n)$.

In Step (7), for each ν ($0 \leq \nu \leq [n/2]$), the computational complexity is $3(\alpha+1)n + O(1)$ for (i) and $(\alpha+1)n + O(1)$ for (ii). Therefore, the total computational complexity is $(4(\alpha+1)n + O(1))(n/2 + O(1)) = 2(\alpha+1)n^2 + O(n)$.

In Step (8), the computational complexity is $O(n)$.

In Step (10), it is $n^2/2 + O(n)$. Consequently, the computational complexity for the Legendre coefficients is $(2\alpha + 5/2)n^2 + O(n)$, but if we prepare $w_m(\nu)$ beforehand, we can reduced it to $n^2/2 + O(n)$.

Step (12) indicates that the computational complexity to evaluate the modified Bernstein polynomial is $3n + O(1)$ for each x , if we obtain the coefficients in advance.

Finally, we raise a remained problem. To obtain the constants $a_{j,l,r}$, we cannot help calculating the unnecessary numbers $a_{j,k,l,r}$ ($1 \leq k \leq 2j$) from the four-dimensional recursion formula, at present. Is there any three-dimensional recursion formula for $a_{j,l,r}$ that does not involve $a_{j,k,l,r}$?

CHAPTER 5

HOW TO DETERMINE THE SHARPNESS DEGREE

In this chapter, we establish a criterion to select the sharpness degree of the modified Bernstein operator by using the Peano kernel theorem.

5.1. Introduction

In Chapter 3 we presented Theorem 3.3, which indicates that, if the given function $f: [0, 1] \rightarrow \mathbf{R}$ is sufficiently smooth, the convergence of $\{{}_αB_n f\}_n$ to f is accelerated as the sharpness degree α increases. However, in practical calculations, we use a finite n . It is not guaranteed that for a fixed n , the larger the sharpness degree is, the smaller the error is. In fact, Table 2 suggests that the norm $\|{}_αB_n\|$ grows as α increases, and the large norm may amplify the error.

Therefore, we need to establish some criterion to select the optimal α for practical applications. For the purpose, we apply the Peano kernel theorem, which will be discussed in the following section.

5.2. Application of the Peano Kernel Theorem

As we mentioned just before Theorem 3.7, we have ${}_αB_n f = f$ for all $f \in \mathbf{P}_1$. Thus the Peano kernel theorem (see, e.g., [9, Section 3.7], [25, Section 7.6], and [32, Section 22.2]) gives for all $f \in C^2[0, 1]$,

$${}_αB_n f(x) - f(x) = \int_0^1 K(x, t) f''(t) dt, \quad (5.1)$$

where

$$K(x, t) = \frac{1}{2} [{}_αB_n(\cdot - t)_+(x) - (x - t)_+] \quad (5.2)$$

and we define $x_+ = \max\{x, 0\}$. Now we define the operator U as

$$U\varphi(x) = \int_0^1 K(x, t)\varphi(t) dt \quad (\varphi \in C[0, 1], x \in [0, 1]).$$

Then the Cauchy-Schwarz inequality gives

$$\|U\varphi\|_2^2 \leq \left[\int_0^1 \int_0^1 K(x, t)^2 dt dx \right] \|\varphi\|_2^2,$$

where $\|\cdot\|_2$ is defined in "Notations." Since the double integral in the right-hand side is finite, U is bounded with respect to $\|\cdot\|_2$. (Consequently, it is called a *Hilbert-Schmidt operator*. See [11, p.32].) Thus the norm $\|U\|_2$ exists and the inequality

$$\|U\|_2^2 \leq \int_0^1 \int_0^1 K(x, t)^2 dt dx \quad (5.3)$$

holds. In terms of $\|U\|_2$, we can estimate from (5.1),

$$\|\alpha B_n f - f\|_2 = \|U f''\|_2 \leq \|U\|_2 \|f''\|_2.$$

Note that $\|U\|_2$ is independent of f and is the best-possible constant that satisfies the above inequality. However, it is in general very difficult to determine $\|U\|_2$; we have to solve an eigenvalue problem. (See [11, p.32].) Therefore, in practice, we use the right-hand side of (5.3) instead of $\|U\|_2^2$. (This idea is parallel with using the Frobenius norm in numerical linear algebra. See [8, p.54] and [33, p.8].)

Therefore, recalling (5.2), we call $\alpha \in \mathbf{N}_0$ *optimal* if it minimizes

$$\int_0^1 \int_0^1 [\alpha B_n(\cdot - t)_+(x) - (x - t)_+]^2 dt dx. \quad (5.4)$$

To calculate this integral practically, we provide the following theorem.

THEOREM 5.1. Let $n \in \mathbf{N}$ and T be an operator given as the form

$$Tf = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \tau_\nu \quad (\tau_\nu \in \mathbf{P}_n, f: [0, 1] \rightarrow \mathbf{R})$$

and suppose that $Tf = f$ for all $f \in \mathbf{P}_1$. If we let

$$w_m(\nu) = \int_0^1 \tau_\nu(x) P_m^*(x) dx \quad (0 \leq m, \nu \leq n),$$

then we can calculate

$$\begin{aligned} & \int_0^1 \int_0^1 [T(\cdot - t)_+(x) - (x - t)_+]^2 dt dx \\ &= \frac{1}{12} \sum_{m=0}^n (2m+1) \sum_{\nu=0}^n w_m(\nu) \left(\frac{1}{n^3} \sum_{\mu=0}^n w_m(\mu) |\mu - \nu|^3 - \frac{1}{70} G_{m+4}^{(-4)}\left(\frac{\nu}{n}\right) \right). \end{aligned}$$

In addition, if $w_m(n - \nu) = (-1)^m w_m(\nu)$, we have

$$\begin{aligned} & \int_0^1 \int_0^1 [T(\cdot - t)_+(x) - (x - t)_+]^2 dt dx \\ &= \frac{1}{6} \sum_{m=0}^n (2m+1) \left\{ \sum_{\nu=0}^{[(n-1)/2]} w_m(\nu) \left[\frac{1}{n^3} \sum_{\mu=0}^{[n/2]} w_m(\mu) (|\mu - \nu|^3 + (-1)^m |\mu + \nu - n|^3) \right. \right. \\ & \quad \left. \left. - \frac{1}{70} G_{m+4}^{(-4)}\left(\frac{\nu}{n}\right) \right] - \frac{1 + (-1)^n}{280} w_m\left(\frac{n}{2}\right) G_{m+4}^{(-4)}\left(\frac{1}{2}\right) \right\}. \end{aligned}$$

Remark. If n is odd, $w_m(n/2)$ is not defined, but the factor $1 + (-1)^n = 0$ covers up this matter.

Proof. Since $x_+ = (|x| + x)/2$ and $Tf = f$ ($f \in \mathbf{P}_1$) hold, we can calculate

$$\begin{aligned} 2(T(\cdot - t)_+(x) - (x - t)_+) &= T(|\cdot - t| + (\cdot - t))(x) - (|x - t| + (x - t)) \\ &= T|\cdot - t|(x) - |x - t| \\ &= \sum_{\nu=0}^n \left| \frac{\nu}{n} - t \right| \tau_\nu(x) - |x - t|, \end{aligned}$$

$$\begin{aligned} & 4[T(\cdot - t)_+(x) - (x - t)_+]^2 \\ &= \sum_{\nu=0}^n \tau_\nu(x) \sum_{\mu=0}^n \tau_\mu(x) \left| \frac{\nu}{n} - t \right| \left| \frac{\mu}{n} - t \right| - 2 \sum_{\nu=0}^n \tau_\nu(x) \left| \frac{\nu}{n} - t \right| |x - t| + |x - t|^2. \end{aligned}$$

Here for all $a, b \in [0, 1]$, if $a \leq b$,

$$\begin{aligned} \int_0^1 |a - t| |b - t| dt &= \int_0^a (a - t)(b - t) dt + \int_a^b (t - a)(b - t) dt + \int_b^1 (t - a)(t - b) dt \\ &= \frac{1}{3} - \frac{a+b}{2} + ab + \frac{(b-a)^3}{3}. \end{aligned}$$

If $a > b$, commuting a and b , we have

$$\int_0^1 |b-t||a-t| dt = \frac{1}{3} - \frac{b+a}{2} + ba + \frac{(a-b)^3}{3}.$$

Hence in either case,

$$\int_0^1 |a-t||b-t| dt = \frac{1}{3} - \frac{a+b}{2} + ab + \frac{|b-a|^3}{3}.$$

Applying this identity, we can calculate

$$\begin{aligned} & 4 \int_0^1 [T(\cdot-t)_+(x) - (x-t)_+]^2 dt \\ &= \sum_{\nu=0}^n \tau_\nu(x) \sum_{\mu=0}^n \tau_\mu(x) \left(\frac{1}{3} - \frac{\nu}{2n} - \frac{\mu}{2n} + \frac{\nu\mu}{n^2} + \frac{|\mu-\nu|^3}{3n^3} \right) \\ &\quad - 2 \sum_{\nu=0}^n \tau_\nu(x) \left(\frac{1}{3} - \frac{\nu}{2n} - \frac{x}{2} + \frac{\nu x}{n} + \frac{|nx-\nu|^3}{3n^3} \right) + \left(\frac{1}{3} - x + x^2 \right) \\ &= \sum_{\nu=0}^n \tau_\nu(x) \left(\frac{1}{3} - \frac{\nu}{2n} - \frac{x}{2} + \frac{\nu x}{n} + \frac{1}{3n^3} \sum_{\mu=0}^n \tau_\mu(x) |\mu-\nu|^3 \right) \\ &\quad - 2 \left(\frac{1}{3} - x + x^2 + \frac{1}{3n^3} \sum_{\nu=0}^n \tau_\nu(x) |nx-\nu|^3 \right) + \left(\frac{1}{3} - x + x^2 \right) \\ &= \frac{1}{3n^3} \sum_{\nu=0}^n \left(\sum_{\mu=0}^n \tau_\nu(x) \tau_\mu(x) |\mu-\nu|^3 - 2\tau_\nu(x) |nx-\nu|^3 \right). \end{aligned} \quad (5.5)$$

Now let $a \in [0, 1]$ and suppose $0 < |t| < 1$. We calculate

$$\begin{aligned} & \int_0^1 \frac{|x-a|^3}{\sqrt{1-2(2x-1)t+t^2}} dx \\ &= \int_0^a \frac{(a-x)^3}{\sqrt{1-2(2x-1)t+t^2}} dx + \int_a^1 \frac{(x-a)^3}{\sqrt{1-2(2x-1)t+t^2}} dx \\ &= \frac{1}{140t^4} ((1-2(2a-1)t+t^2)^{7/2} - 1 - 7(1-2a)t - 7(3-10a+10a^2)t^2 \\ &\quad - 35(1-4a+6a^2-4a^3)t^3) \\ &= \frac{1}{140t^4} \left(\sum_{m=0}^{\infty} G_m^{(-4)}(a) t^m - G_0^{(-4)}(a) - G_1^{(-4)}(a)t - G_2^{(-4)}(a)t^2 - G_3^{(-4)}(a)t^3 \right) \\ &= \frac{1}{140t^4} \sum_{m=4}^{\infty} G_m^{(-4)}(a) t^m = \frac{1}{140} \sum_{m=0}^{\infty} G_{m+4}^{(-4)}(a) t^m. \end{aligned}$$

Therefore,

$$\int_0^1 |x-a|^3 P_m^*(x) dx = \frac{1}{140} G_{m+4}^{(-4)}(a).$$

Consequently, we can expand

$$|x-a|^3 = \frac{1}{140} \sum_{m=0}^{\infty} (2m+1) G_{m+4}^{(-4)}(a) P_m^*(x).$$

Applying this and the obvious identity

$$\tau_\nu(x) = \sum_{m=0}^n (2m+1) w_m(\nu) P_m^*(x),$$

we can calculate

$$\begin{aligned} \int_0^1 \tau_\nu(x) \tau_\mu(x) dx &= \sum_{m=0}^n (2m+1) w_m(\nu) w_m(\mu), \\ \int_0^1 \tau_\nu(x) |x-a| dx &= \frac{1}{140} \sum_{m=0}^n (2m+1) w_m(\nu) G_{m+4}^{(-4)}(a). \end{aligned}$$

Applying these equations with $a = \nu/n$, we can proceed from (5.5) as

$$\begin{aligned} & \int_0^1 \int_0^1 [T(\cdot-t)_+(x) - (x-t)_+]^2 dt dx \\ &= \frac{1}{12n^3} \sum_{\nu=0}^n \left(\sum_{\mu=0}^n \sum_{m=0}^n (2m+1) w_m(\nu) w_m(\mu) |\mu-\nu|^3 - \frac{n^3}{70} \sum_{m=0}^n (2m+1) w_m(\nu) G_{m+4}^{(-4)}\left(\frac{\nu}{n}\right) \right) \\ &= \frac{1}{12} \sum_{m=0}^n (2m+1) \sum_{\nu=0}^n w_m(\nu) \left(\frac{1}{n^3} \sum_{\mu=0}^n w_m(\mu) |\mu-\nu|^3 - \frac{1}{70} G_{m+4}^{(-4)}\left(\frac{\nu}{n}\right) \right). \end{aligned} \quad (5.6)$$

Furthermore, if $w_m(n-\nu) = (-1)^m w_m(\nu)$, then

$$\begin{aligned} & \sum_{\nu=0}^n w_m(\nu) \sum_{\mu=0}^n w_m(\mu) |\mu-\nu|^3 \\ &= 2 \sum_{\nu=0}^{[(n-1)/2]} w_m(\nu) \sum_{\mu=0}^n w_m(\mu) |\mu-\nu|^3 + \frac{1+(-1)^n}{2} w_m\left(\frac{n}{2}\right) \sum_{\mu=0}^n w_m(\mu) \left| \mu - \frac{n}{2} \right|^3 \\ &= \sum_{\nu=0}^{[(n-1)/2]} w_m(\nu) \left[2 \sum_{\mu=0}^{[(n-1)/2]} w_m(\mu) (|\mu-\nu|^3 + |\mu+\nu-n|^3) + \frac{1+(-1)^n}{2} w_m\left(\frac{n}{2}\right) (1+(-1)^m) \left| \frac{n}{2} - \nu \right|^3 \right] \\ &\quad + \frac{1+(-1)^n}{2} w_m\left(\frac{n}{2}\right) \sum_{\mu=0}^{[(n-1)/2]} (1+(-1)^m) w_m(\mu) \left| \mu - \frac{n}{2} \right|^3 \end{aligned}$$

$$\begin{aligned}
&= 2 \sum_{\nu=0}^{[(n-1)/2]} w_m(\nu) \left[\sum_{\mu=0}^{[(n-1)/2]} w_m(\mu) (|\mu-\nu|^3 + |\mu+\nu-n|^3) + \frac{1+(-1)^n}{2} w_m\left(\frac{n}{2}\right) (1+(-1)^m) \left|\frac{n}{2}-\nu\right|^3 \right] \\
&= 2 \sum_{\nu=0}^{[(n-1)/2]} w_m(\nu) \sum_{\mu=0}^{[n/2]} w_m(\mu) (|\mu-\nu|^3 + |\mu+\nu-n|^3)
\end{aligned}$$

and

$$\sum_{\nu=0}^n w_m(\nu) G_{m+4}^{(-4)}\left(\frac{\nu}{n}\right) = 2 \sum_{\nu=0}^{[(n-1)/2]} w_m(\nu) G_{m+4}^{(-4)}\left(\frac{\nu}{n}\right) + \frac{1+(-1)^n}{2} w_m\left(\frac{n}{2}\right) G_{m+4}^{(-4)}\left(\frac{1}{2}\right).$$

The above two equations and (5.6) imply the second identity in this theorem. ■

Now we apply Theorem 5.1 with $T = {}_{\alpha}B_n$ to determine the quantity (5.4). Theorem 4.1 indicates that we can put $w_m(\nu) = {}_{\alpha}H_{n,m}(\nu) + {}_{\alpha}K_{n,m}(\nu)$, and accordingly, we have $w_m(n-\nu) = (-1)^m w_m(\nu)$. Hence we can use the second identity in Theorem 5.1. In this way, we have carried out numerical experiments for all $n \leq 64$. The result suggests that the optimal α is

$$\left\{ \begin{array}{ll} 3, 4 & (n = 2) \\ 4 & (3 \leq n \leq 5) \\ 5 & (6 \leq n \leq 10) \\ 6 & (11 \leq n \leq 19) \\ 7 & (20 \leq n \leq 34) \\ 8 & (35 \leq n \leq 58) \\ 9 & (59 \leq n \leq 64). \end{array} \right.$$

From this result, the optimal α seems to be $O(\sqrt{n})$, but this problem is open so far.

CHAPTER 6

NUMERICAL EXAMPLES

Finally, we present several results of numerical experiments through the graphs to demonstrate our theory. We investigated errors of the modified Bernstein polynomials ${}_{\alpha}B_n f$ in the case $n = 32$ and $\alpha = 7$. We selected

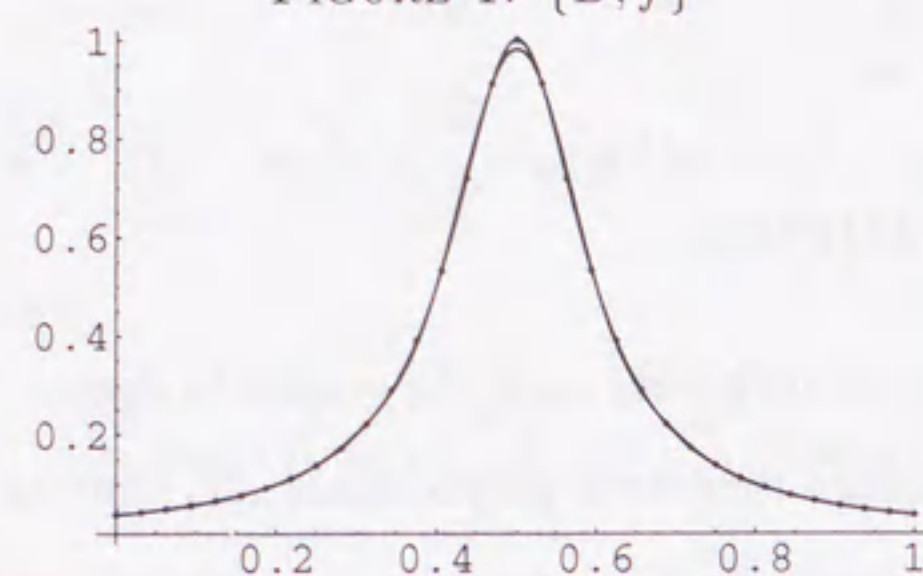
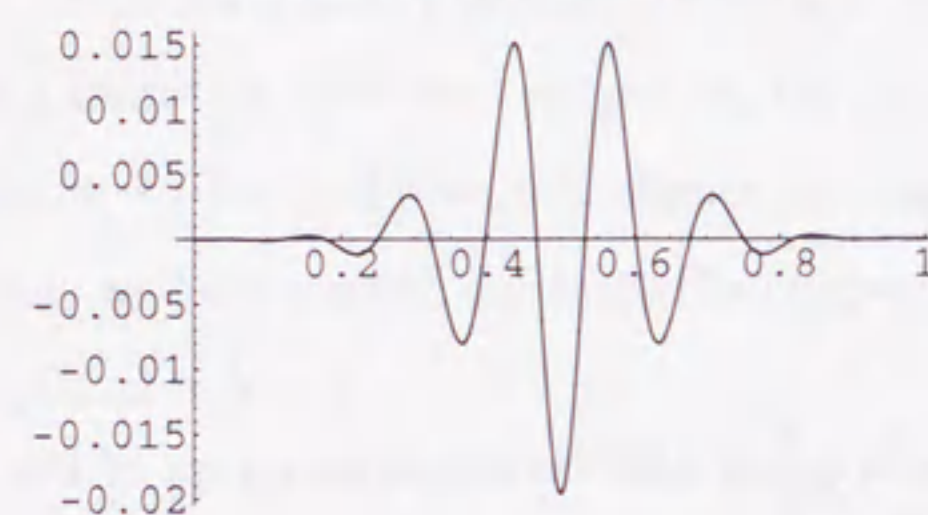
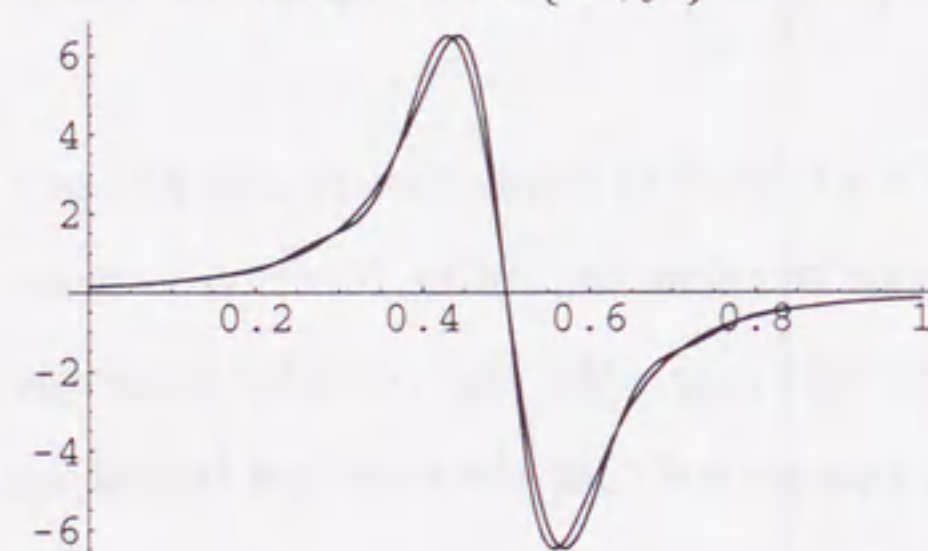
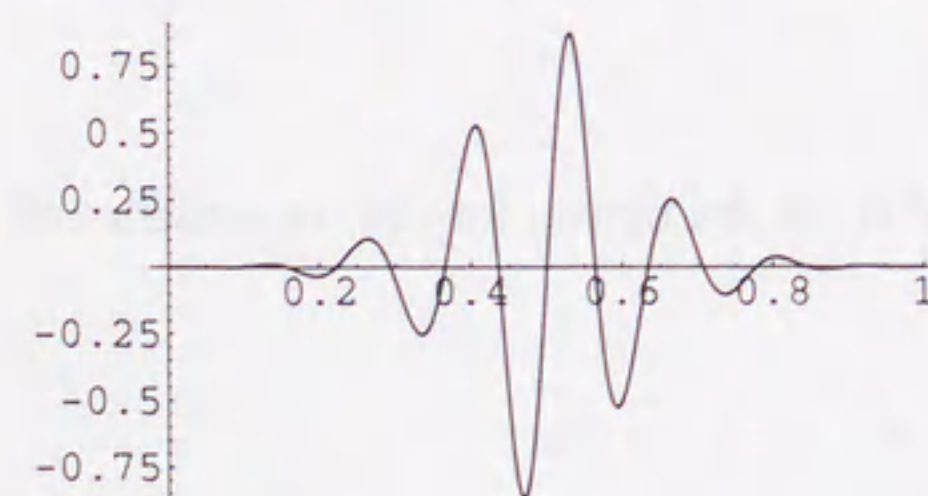
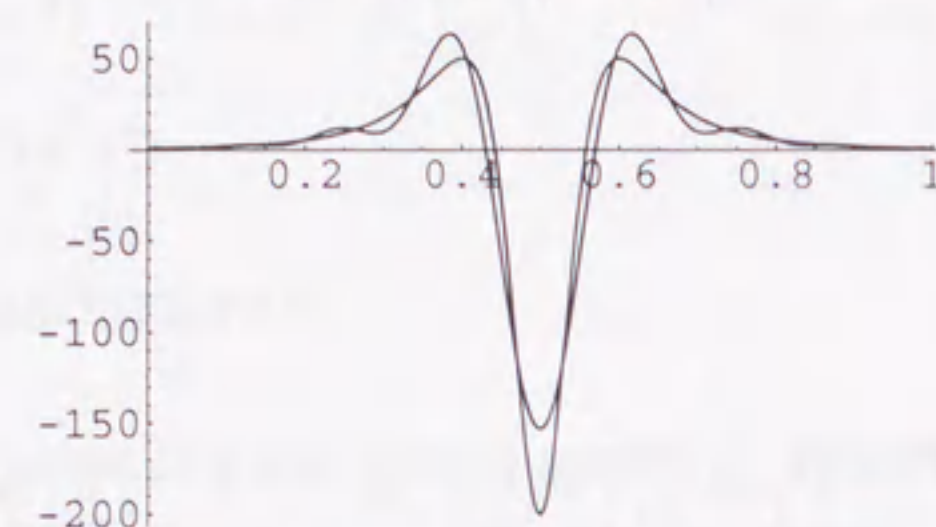
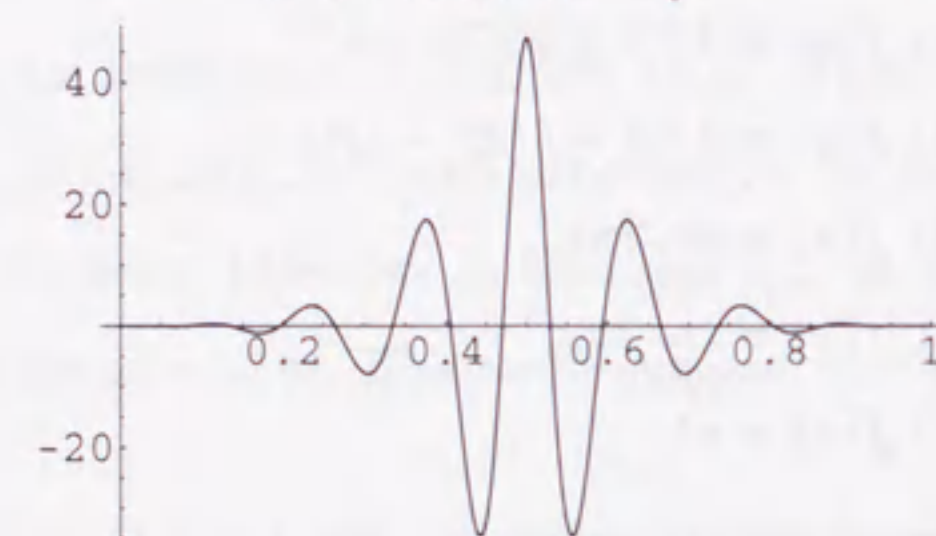
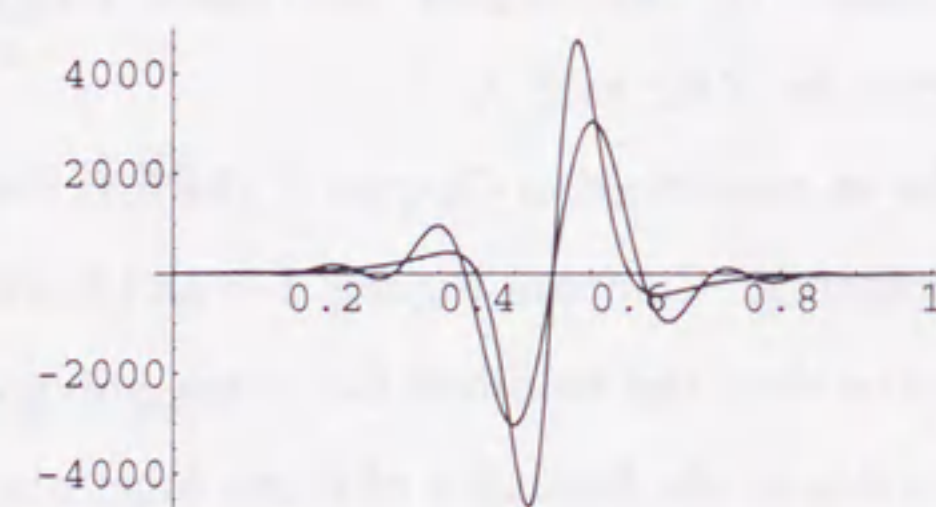
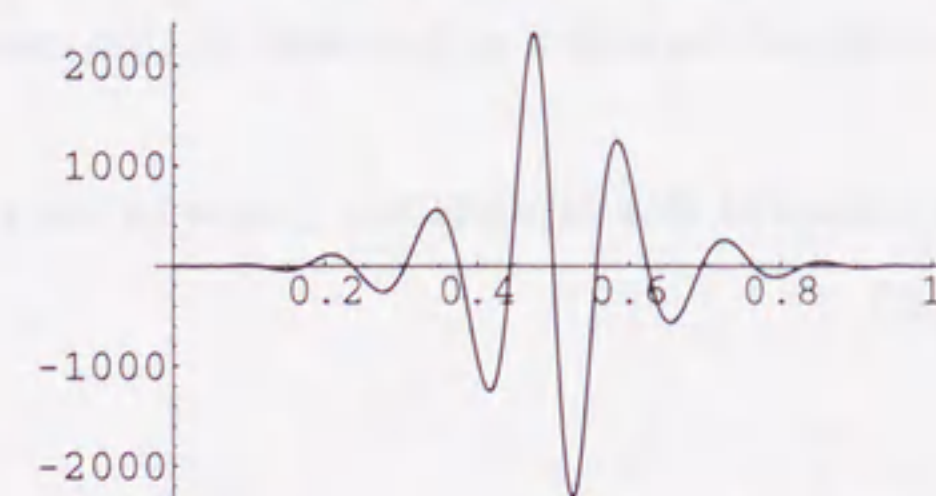
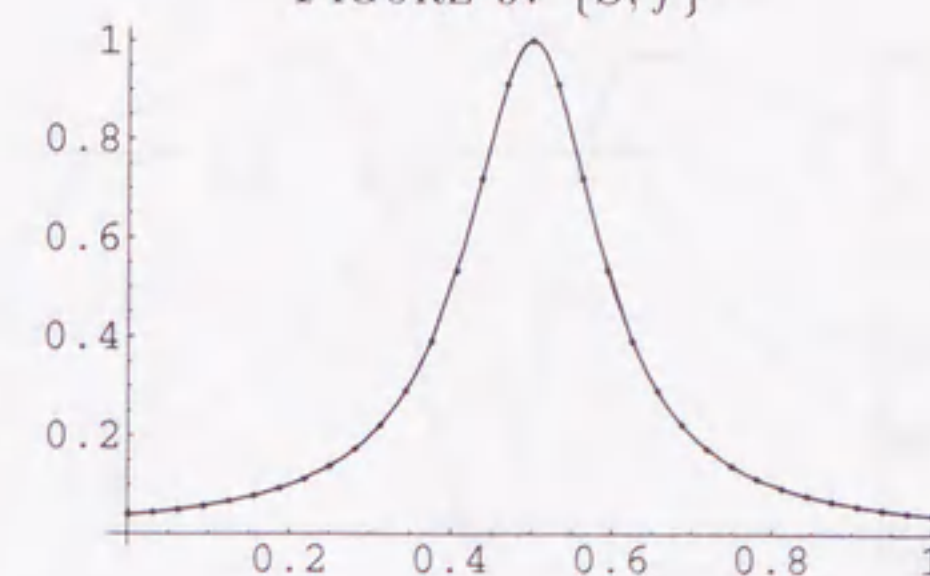
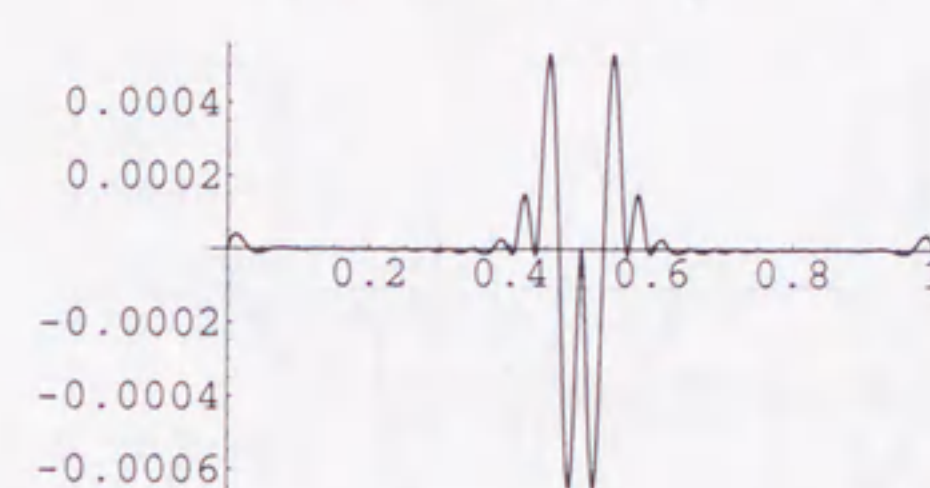
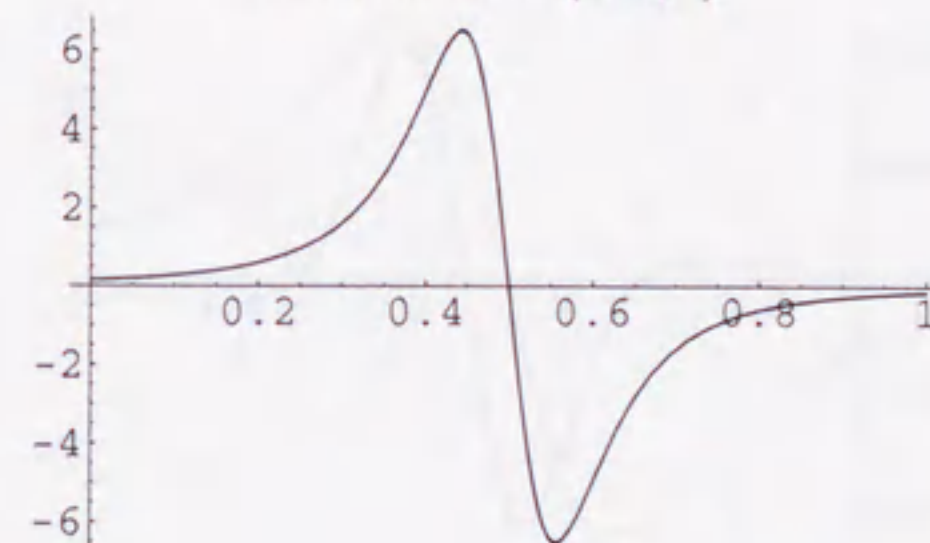
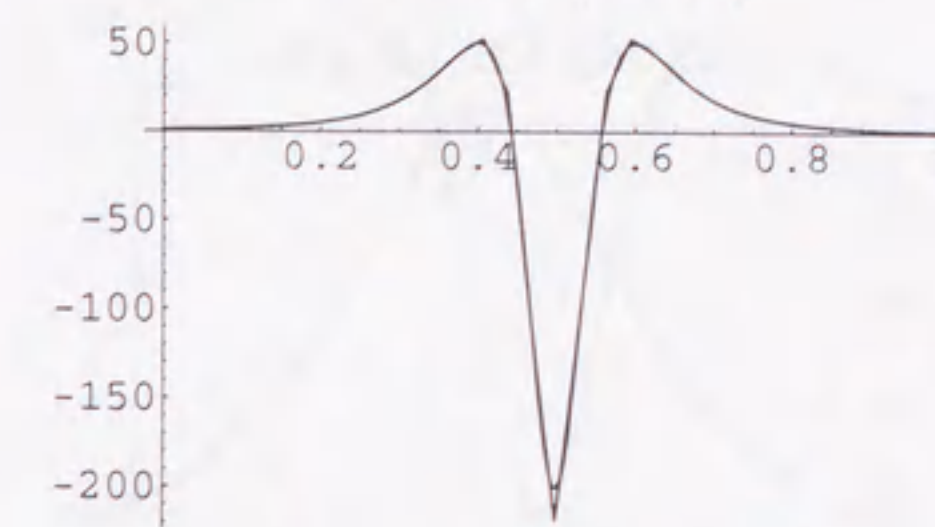
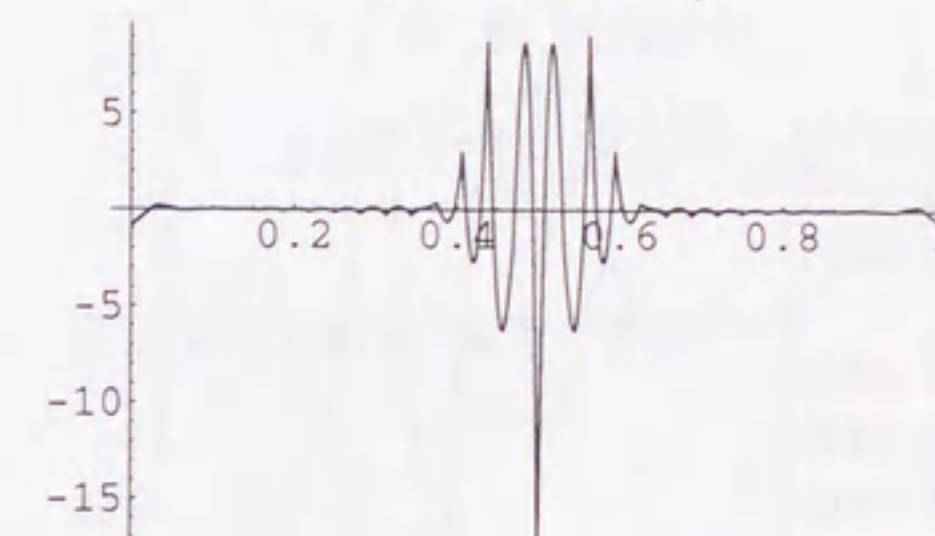
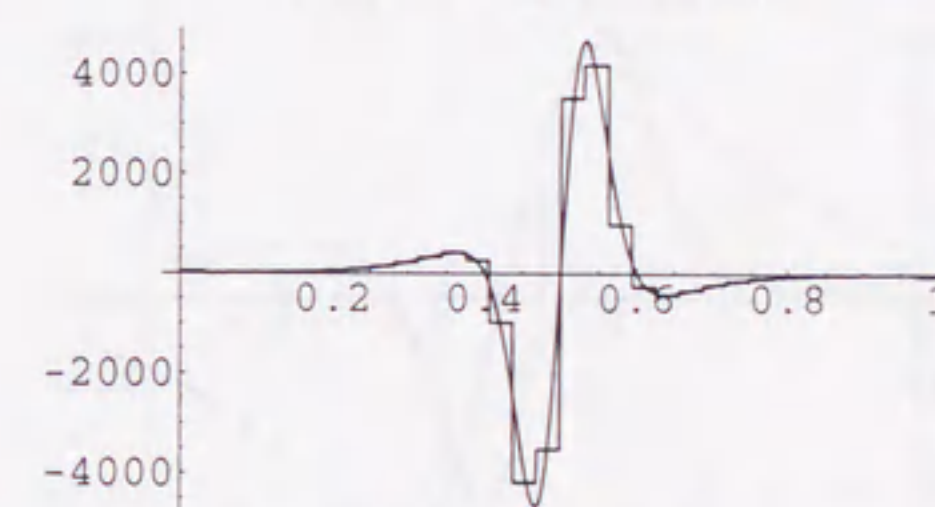
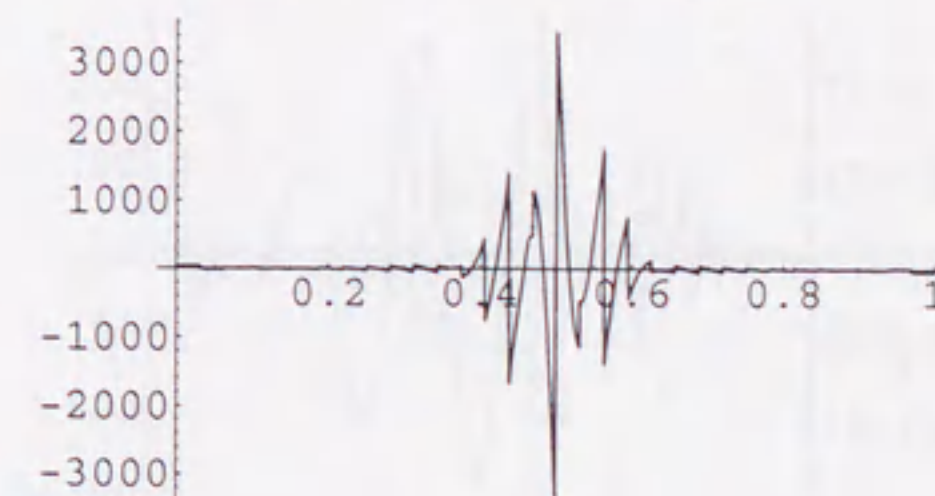
- (1) $f(x) = 1/(1 + 25(2x - 1)^2)$,
- (2) $f(x) = 1/(1 + 5(2x - 1)^2)$,
- (3) $f(x) = \sin 2\pi x$,
- (4) $f(x) = \cos 2\pi x$,
- (5) $f(x) = e^x$

as approximated functions. For the sake of comparison, we also investigated errors of the natural cubic spline functions of the same sampling points and the same approximated functions. In this chapter, we denote ${}_7B_{32}f$ simply by B and the corresponding spline function by S for each f .

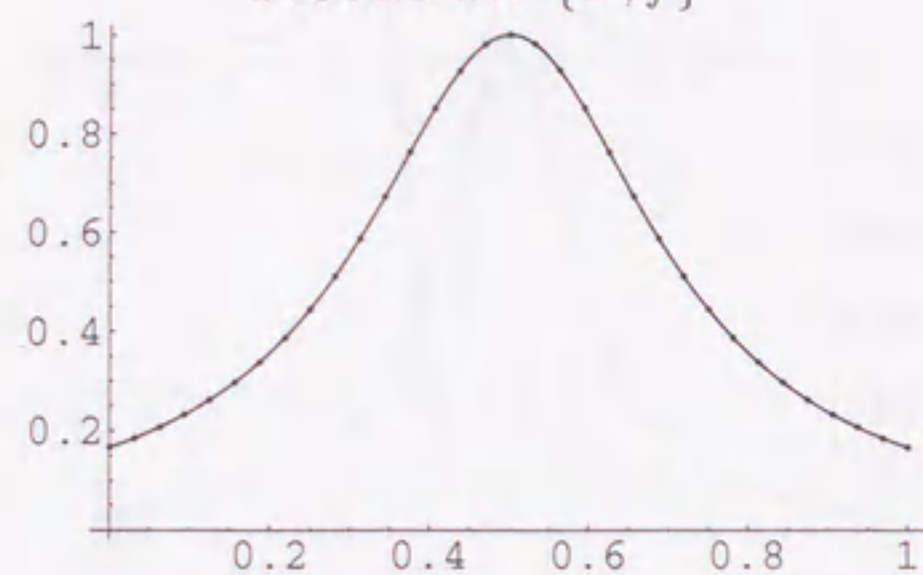
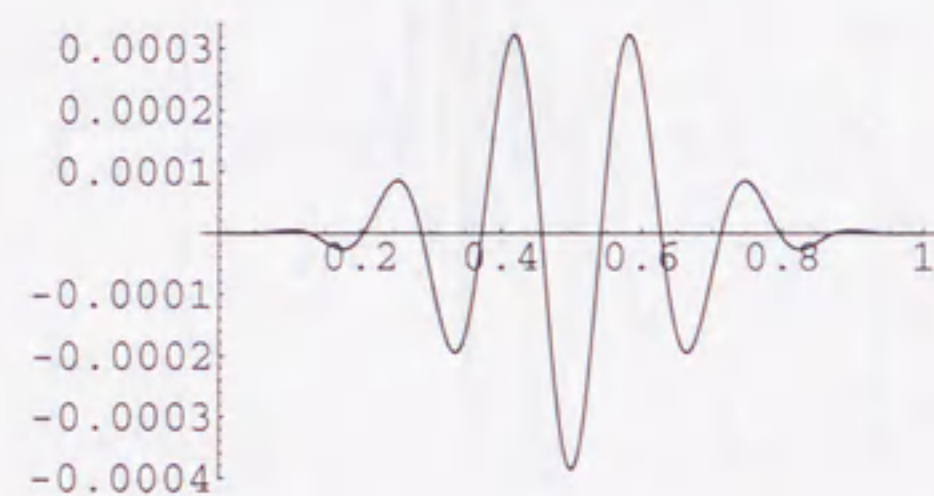
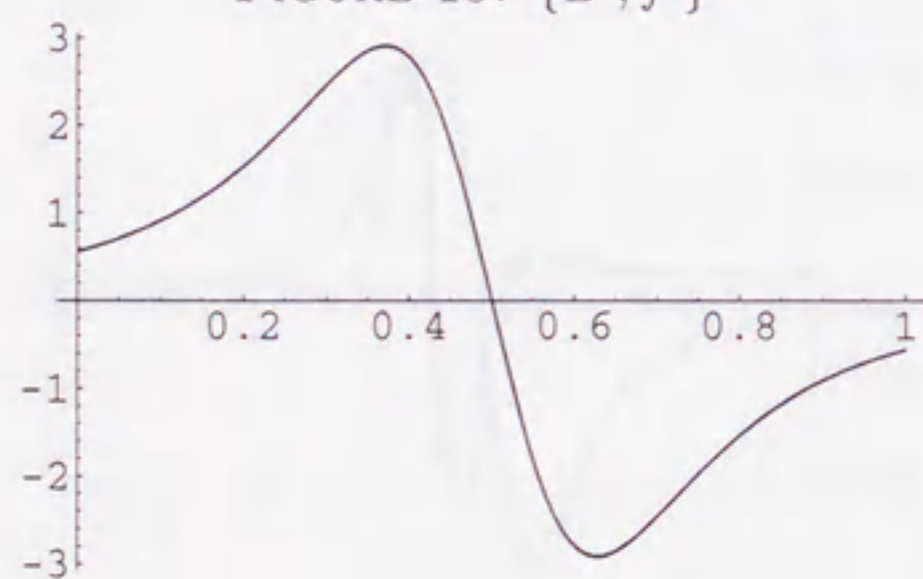
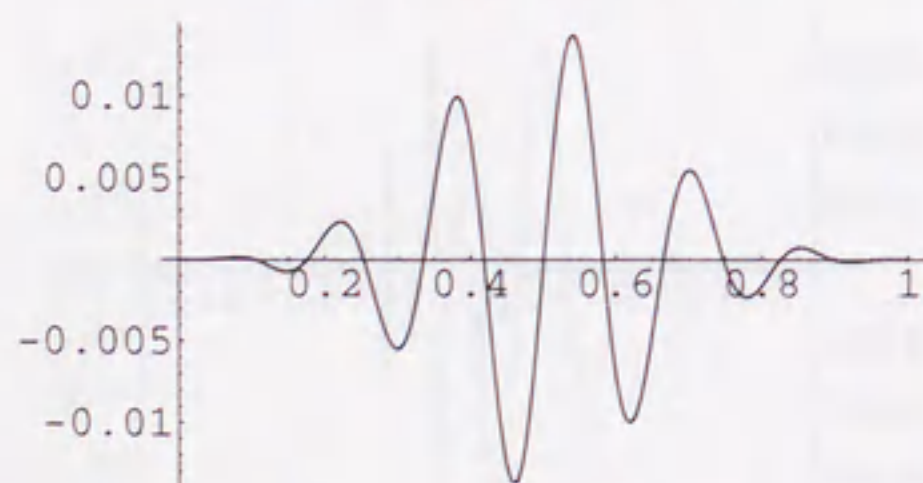
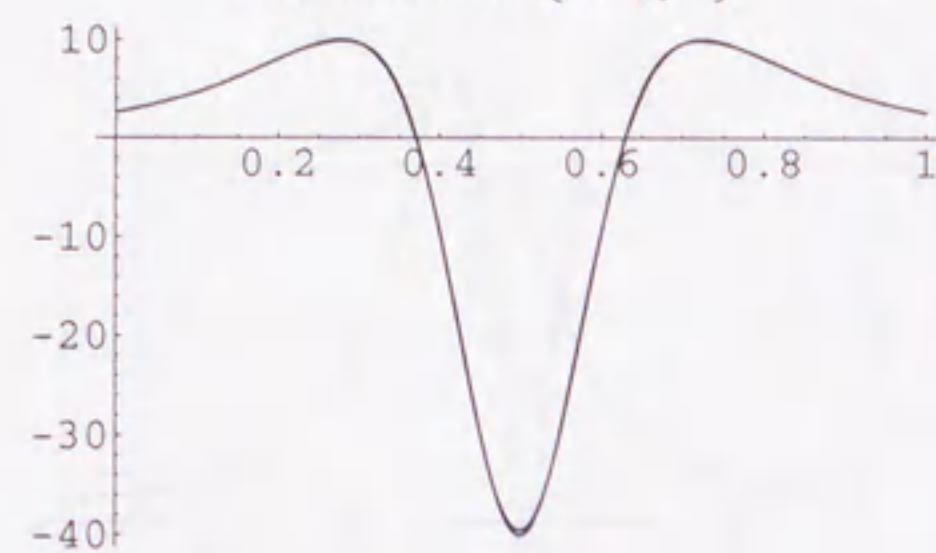
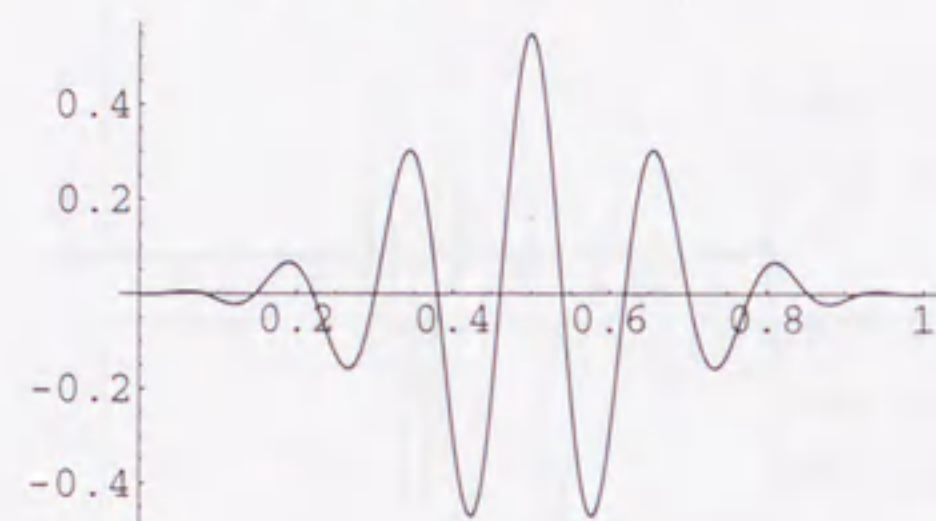
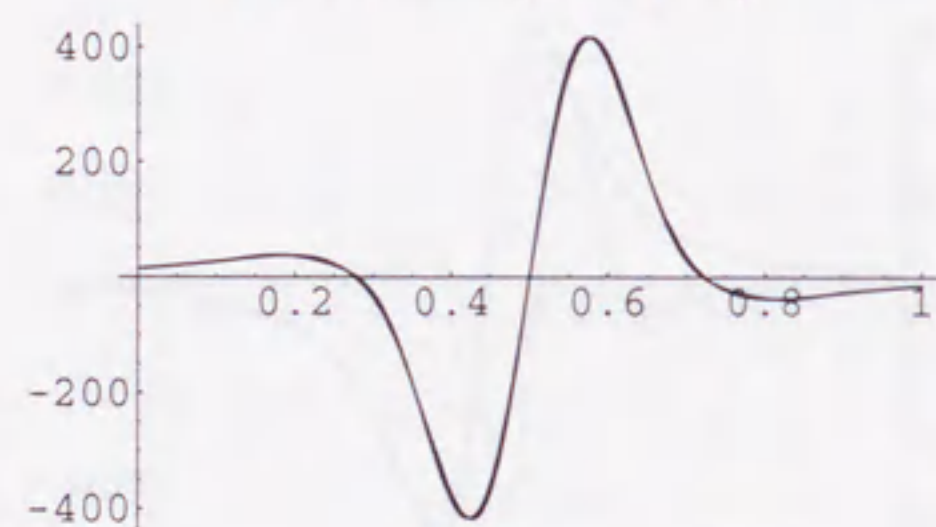
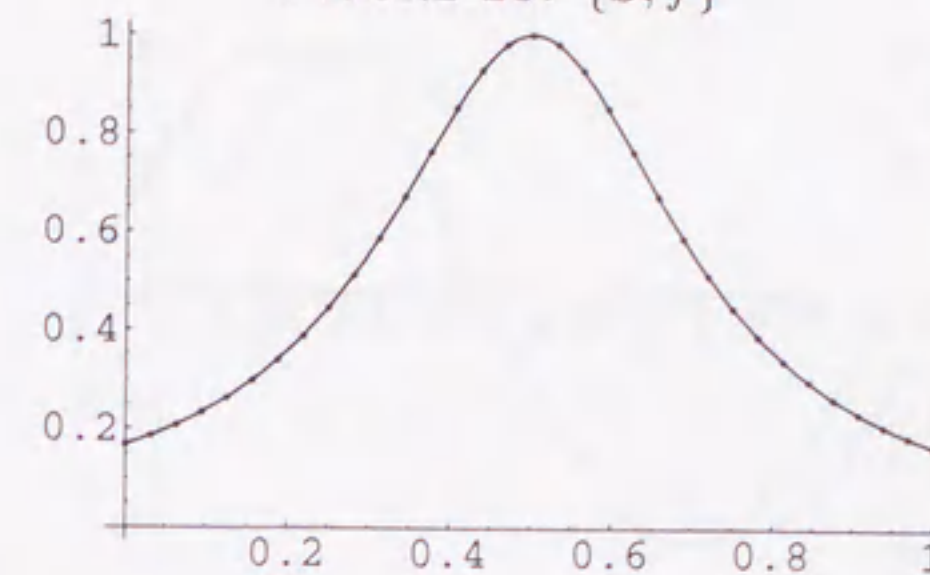
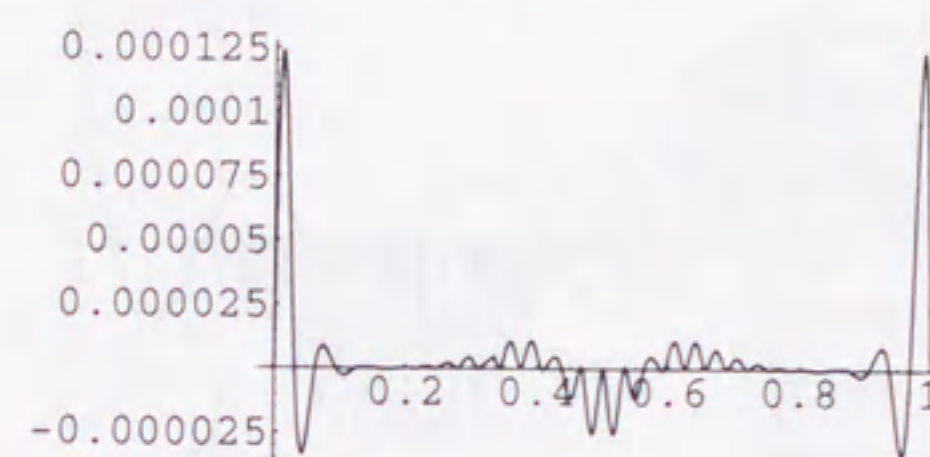
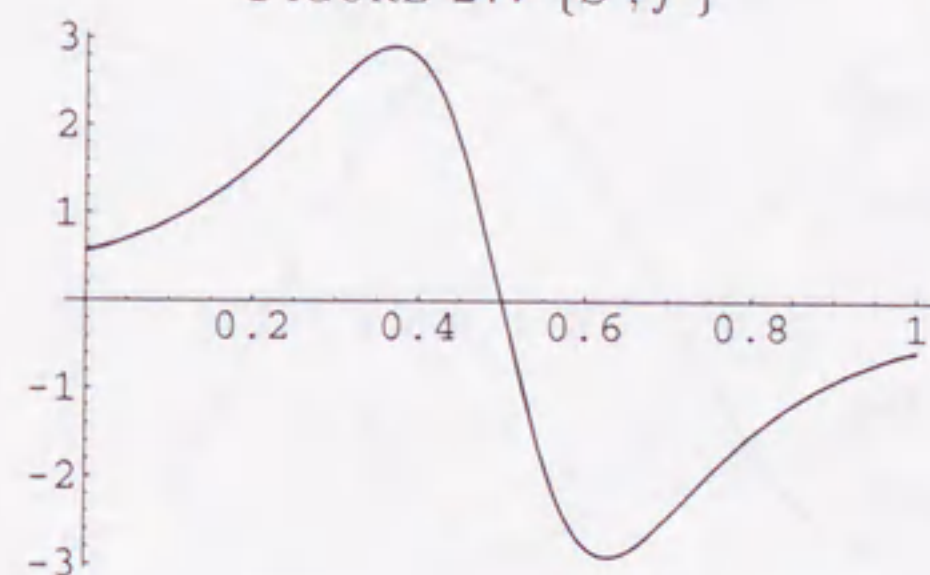
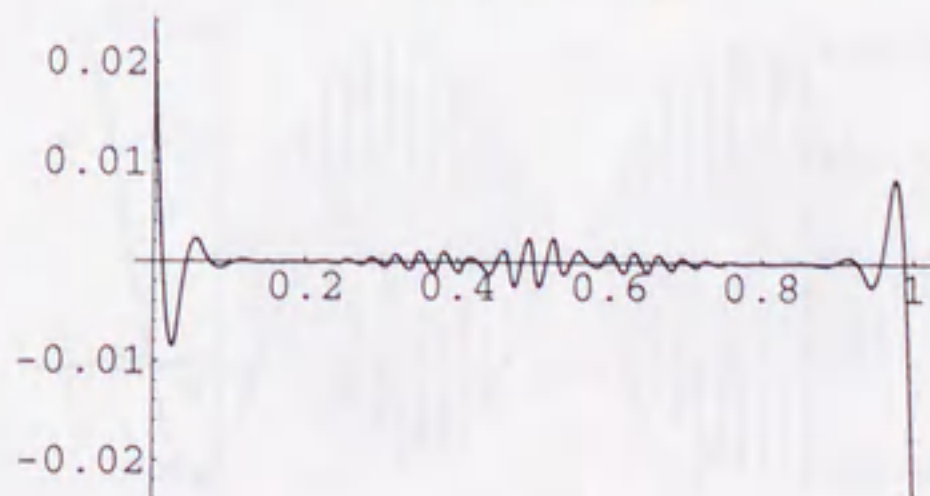
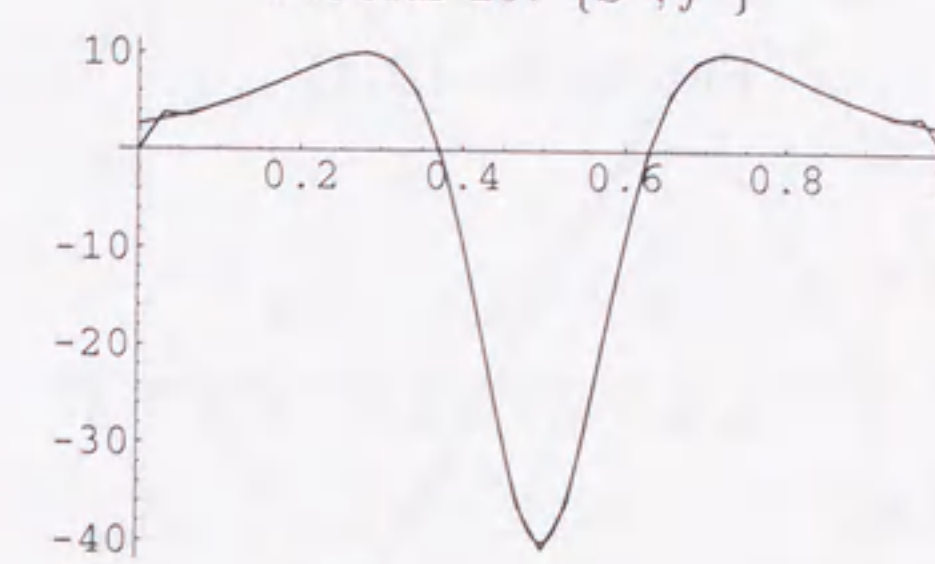
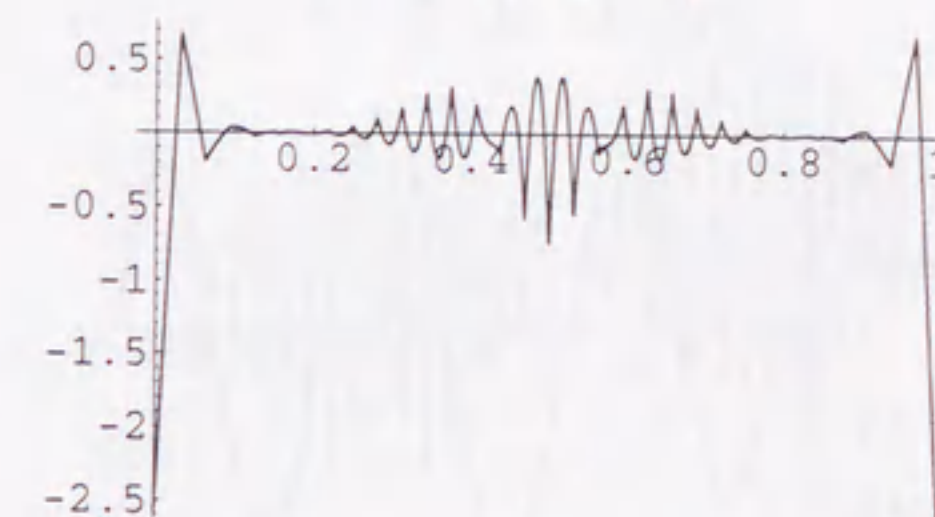
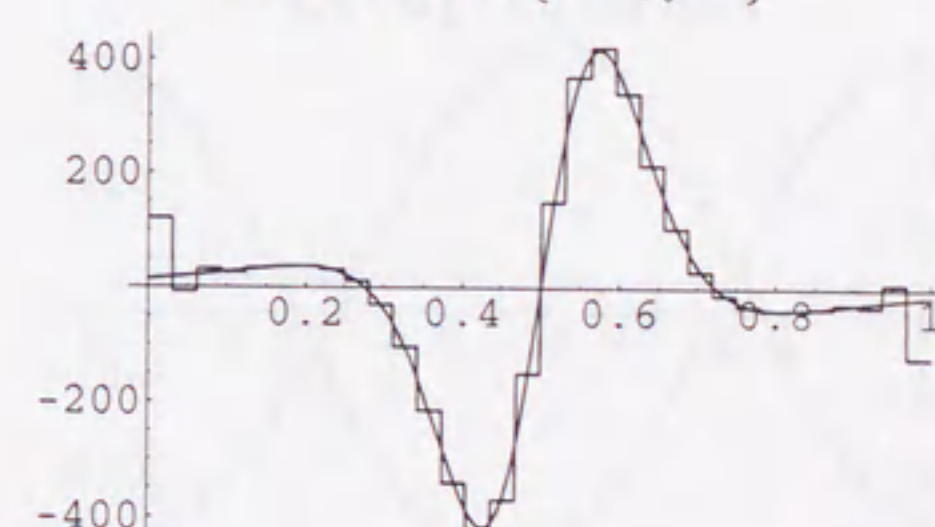
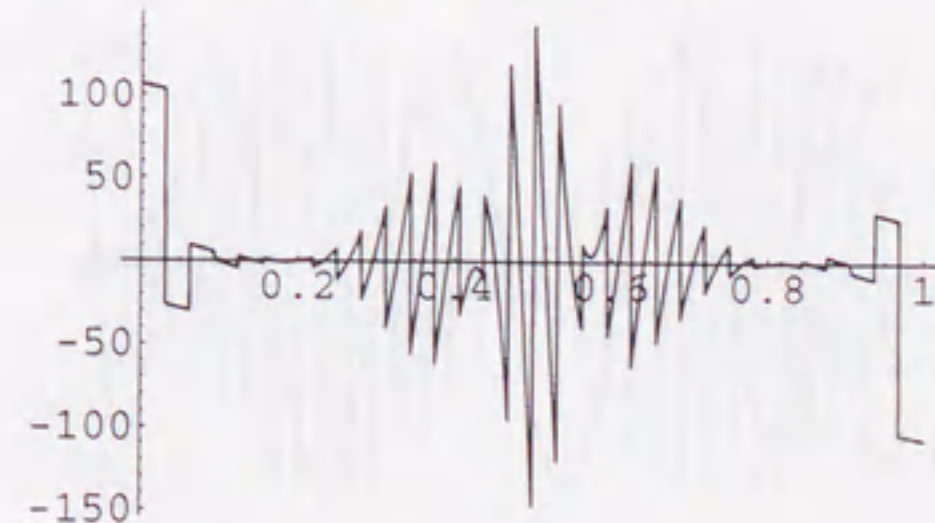
As we mentioned in Chapter 1, the first function is a typical example that causes Runge's phenomenon. Compare Figures 1-6 and 9-14. We can see that the spline function is more accurate than the modified Bernstein polynomial in this case. We can consider that this fact is due to the flexibility of spline functions. It is guaranteed that the modified Bernstein polynomial is more precise than the spline function for every sufficiently large n , but not for a fixed n that is not large enough; that is, $n = 32$ is too small to exhibit the advantage of the modified Bernstein polynomial in this case.

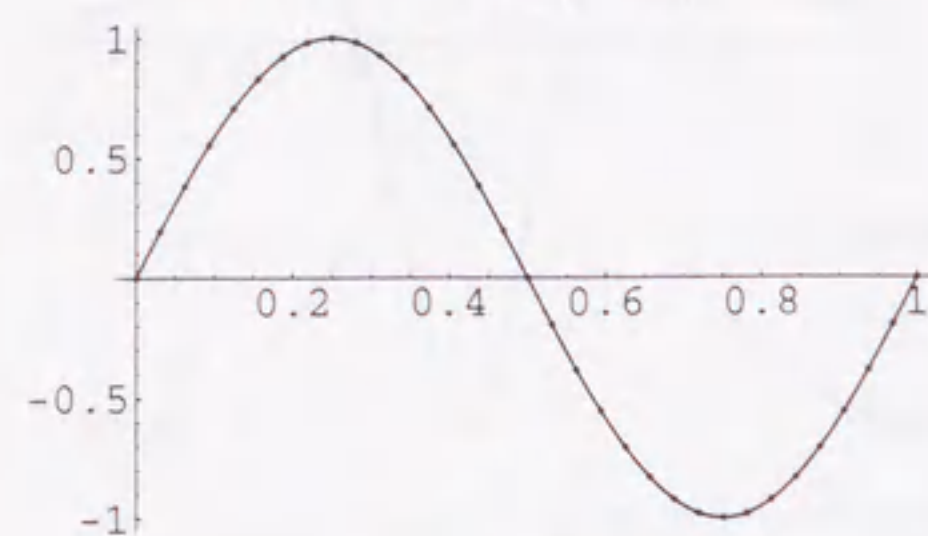
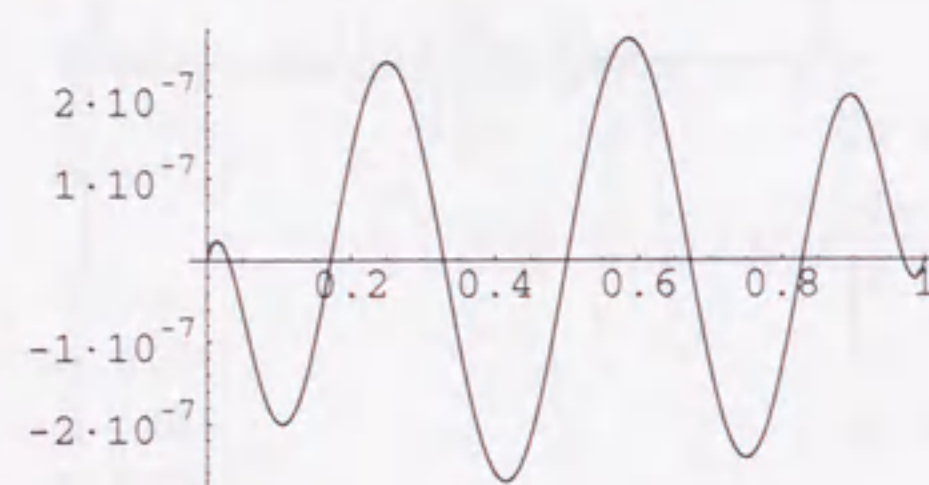
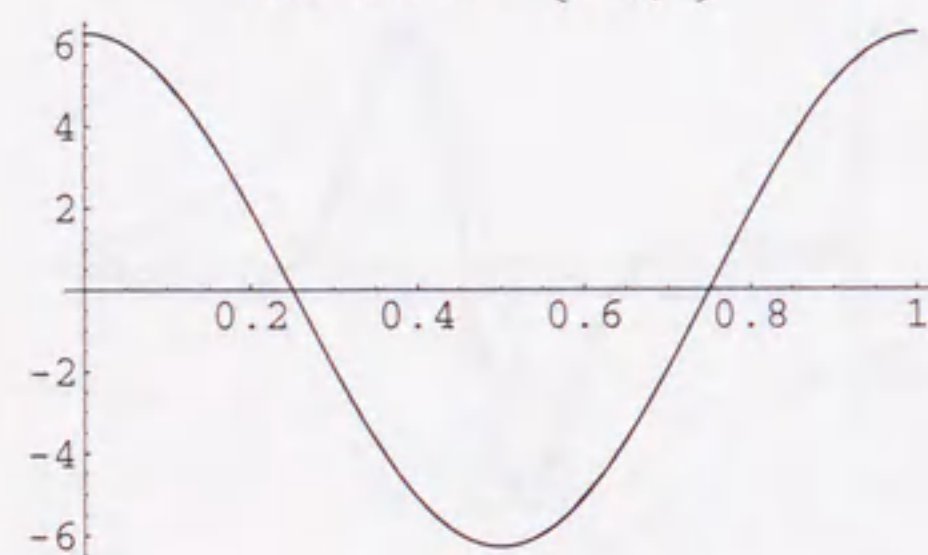
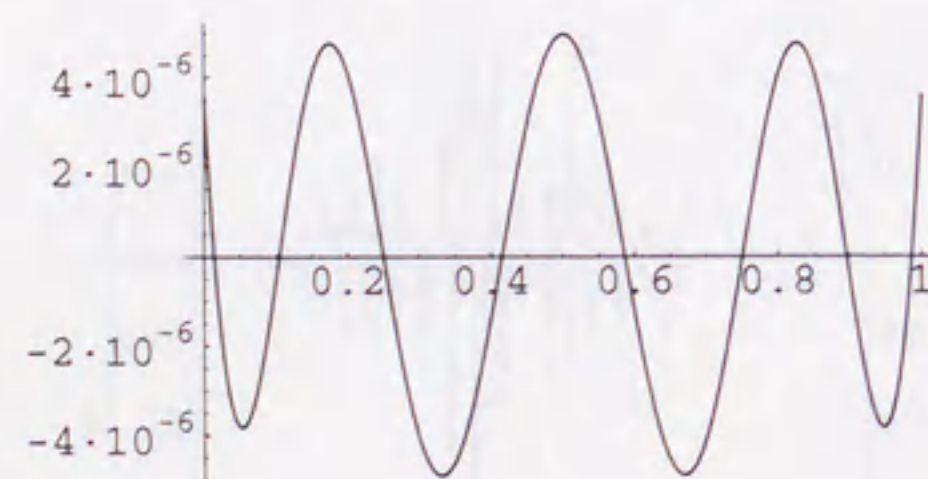
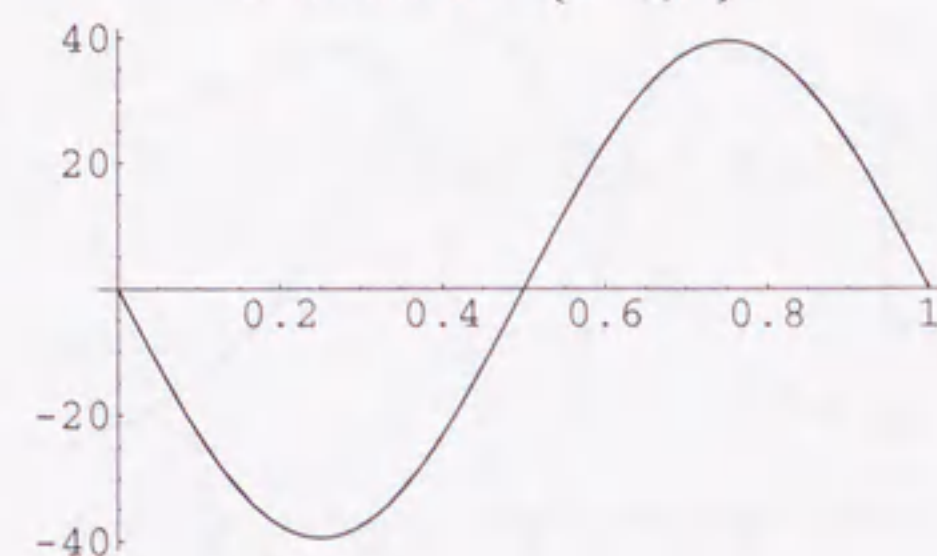
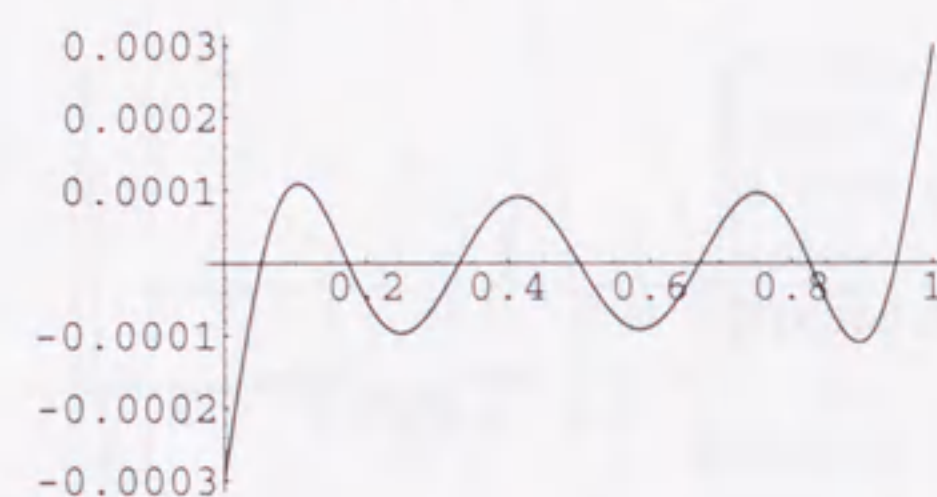
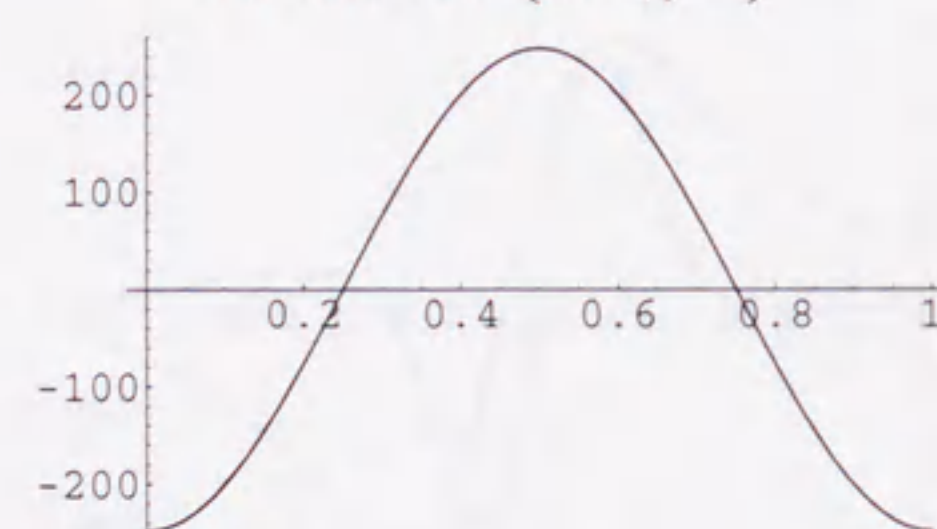
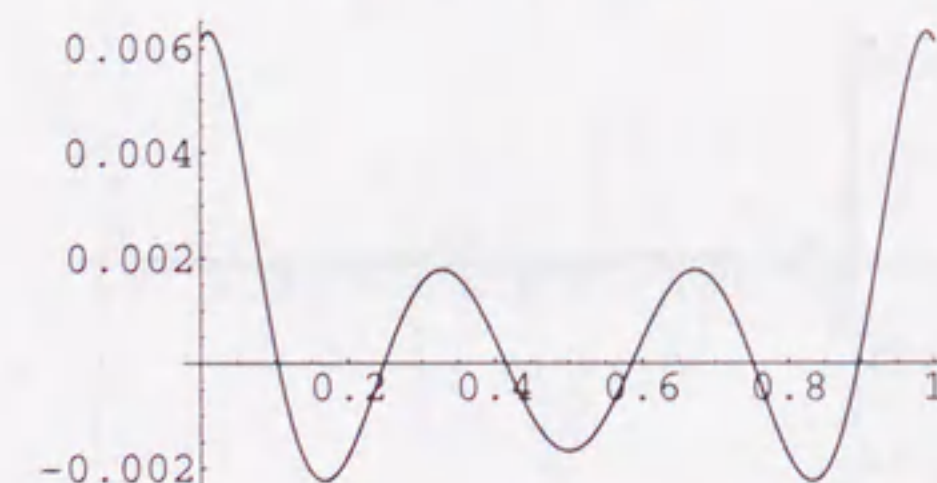
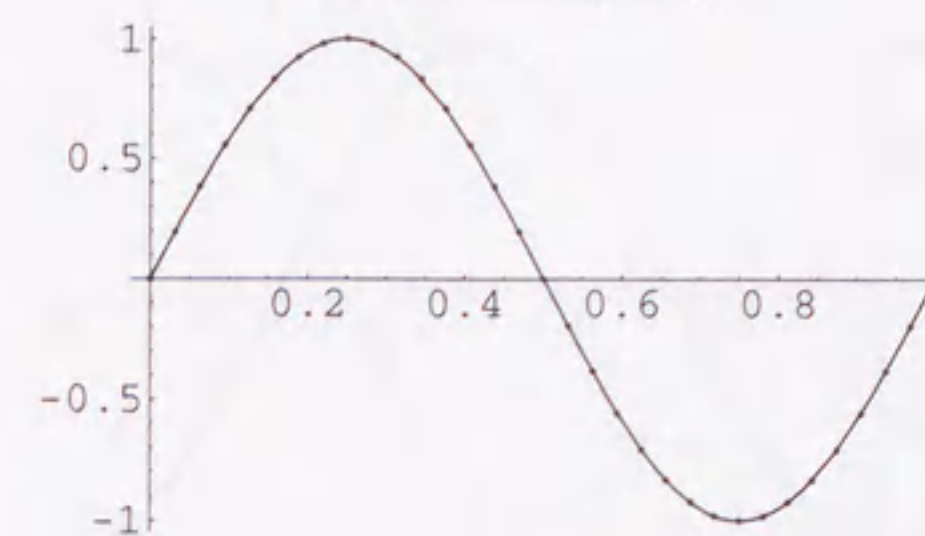
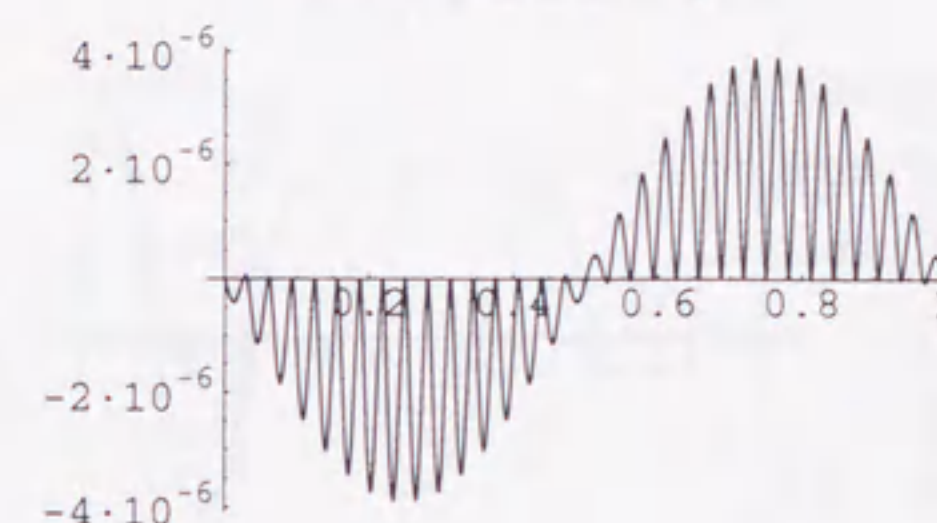
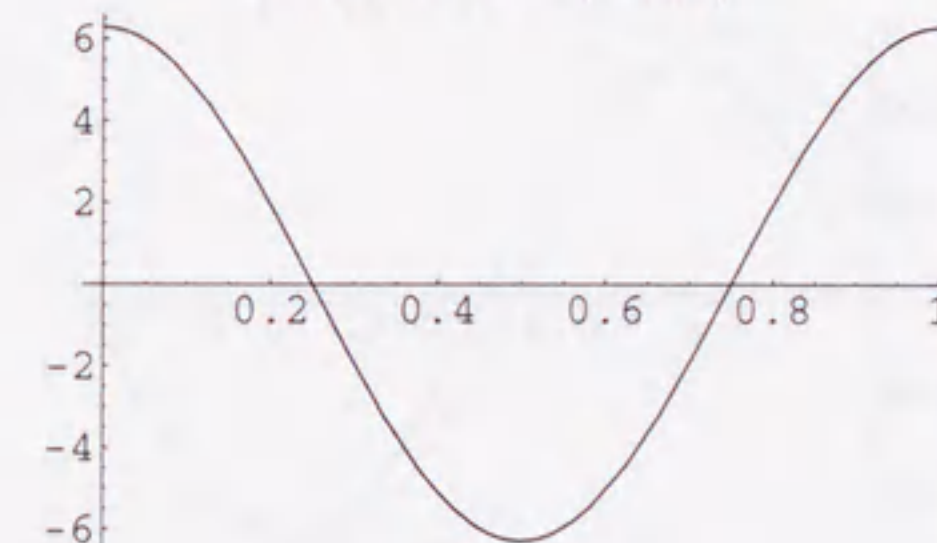
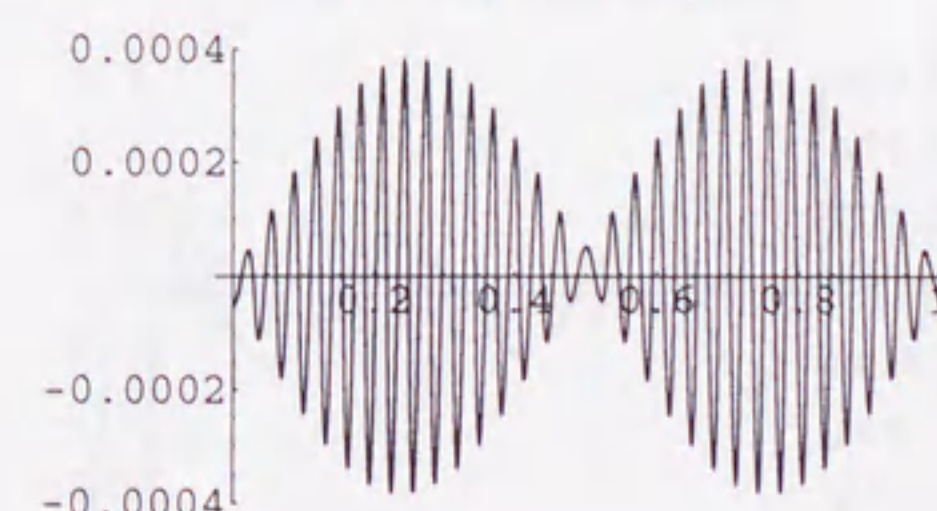
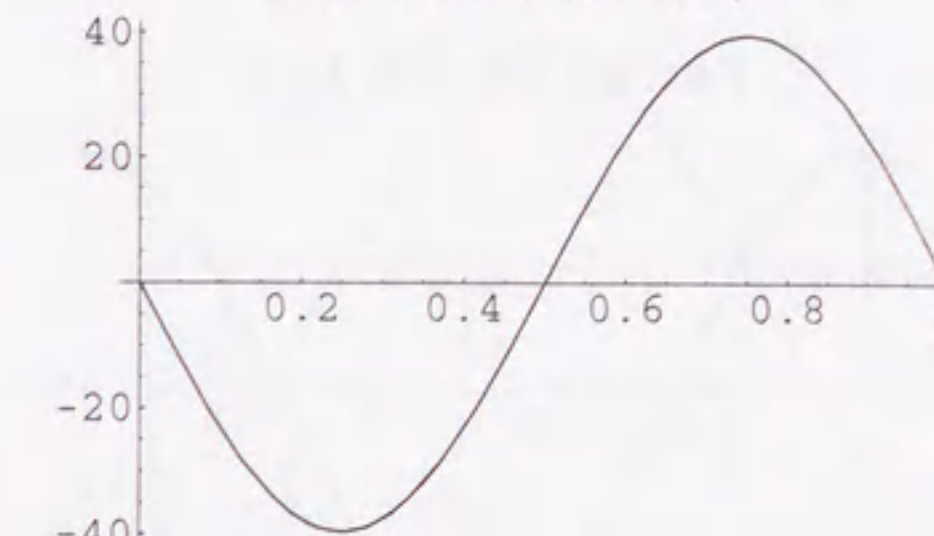
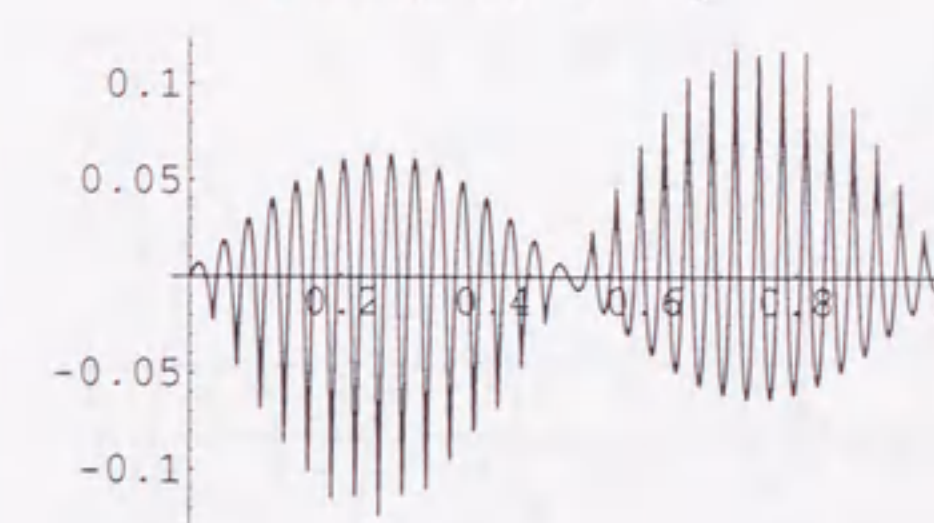
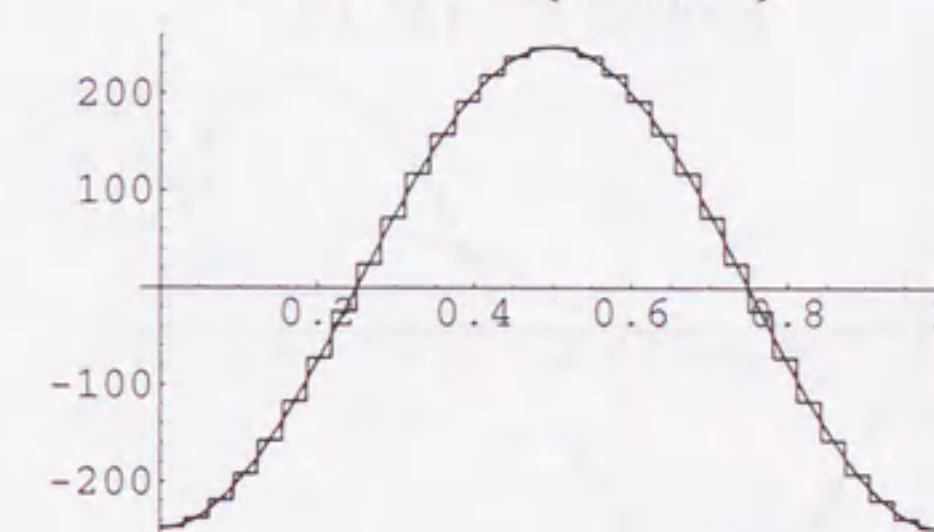
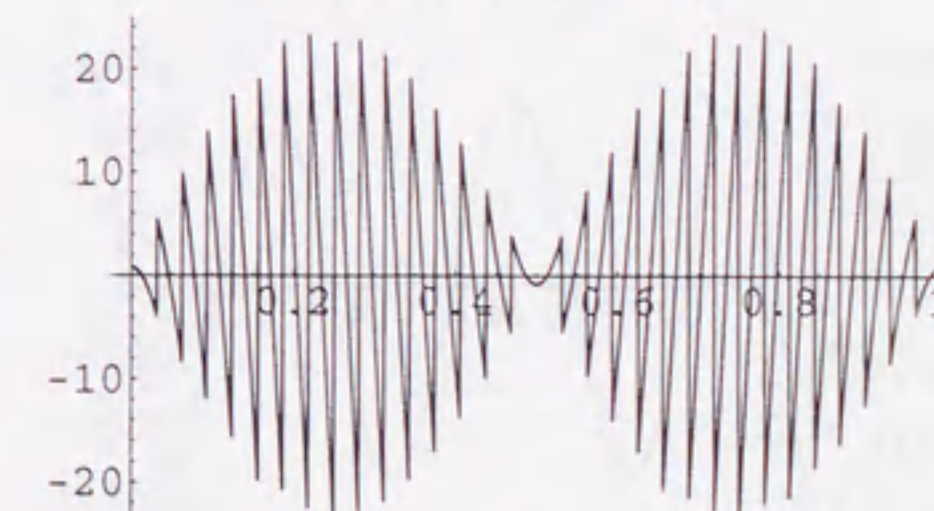
(The sequel of this explanation jumps to the page after all the figures for the convenience of layout.)

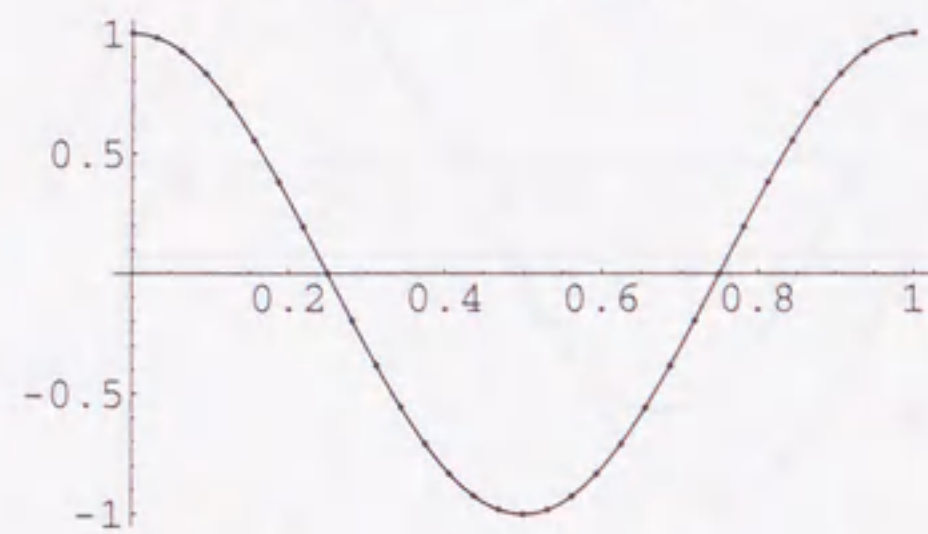
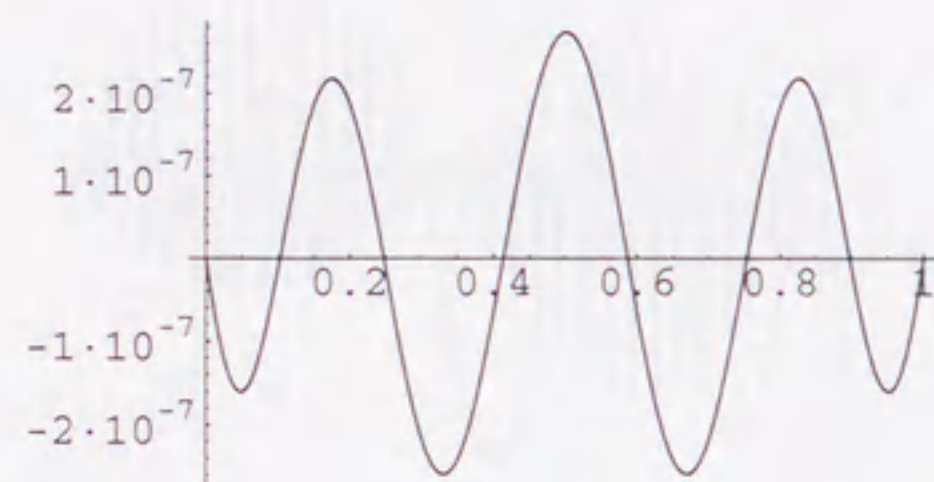
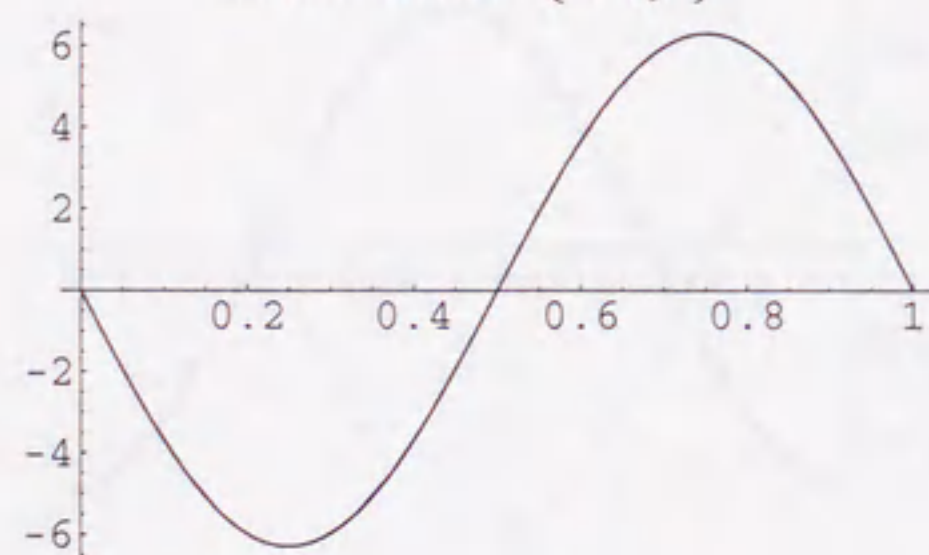
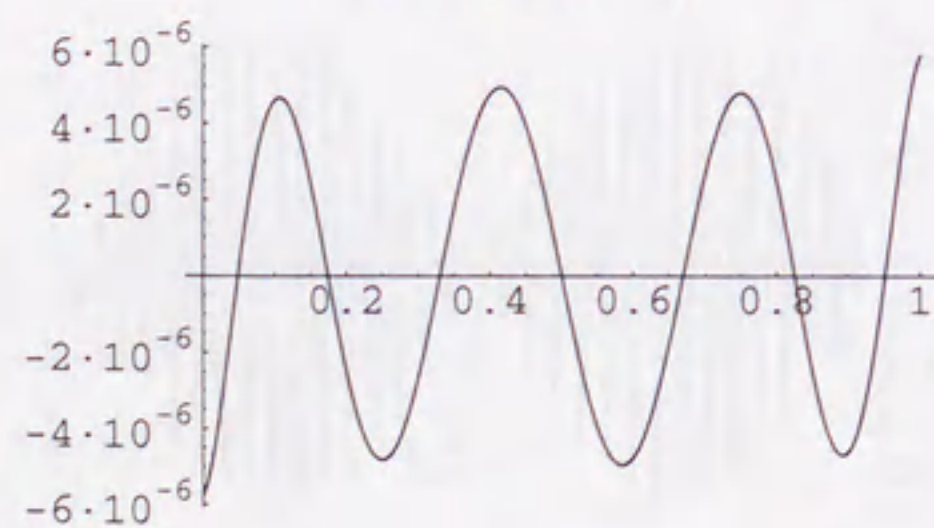
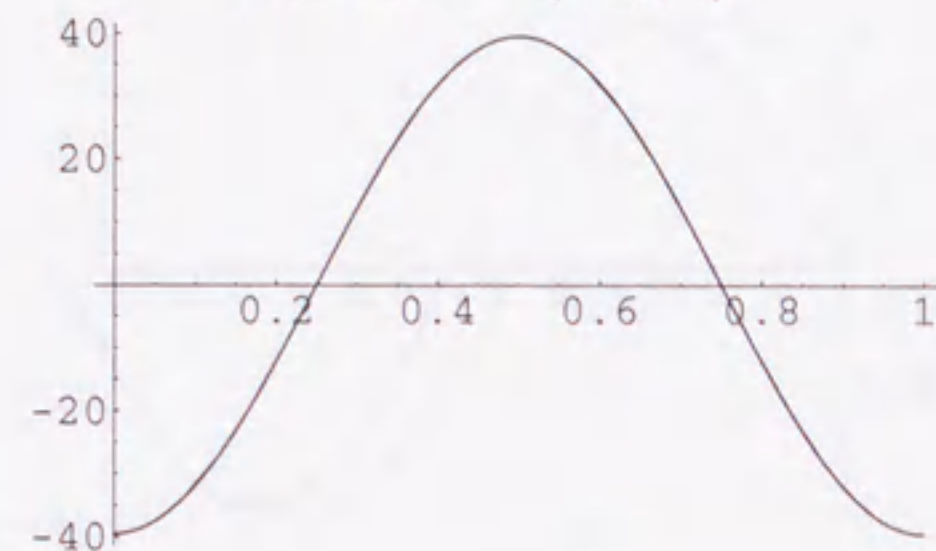
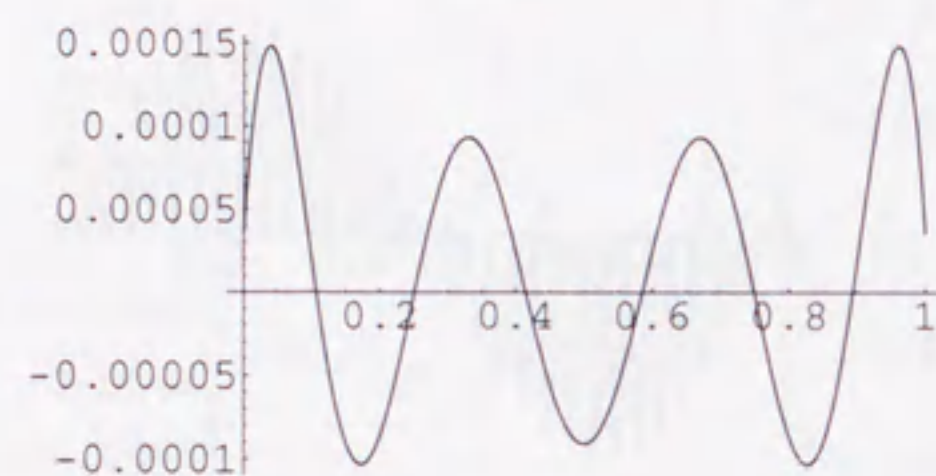
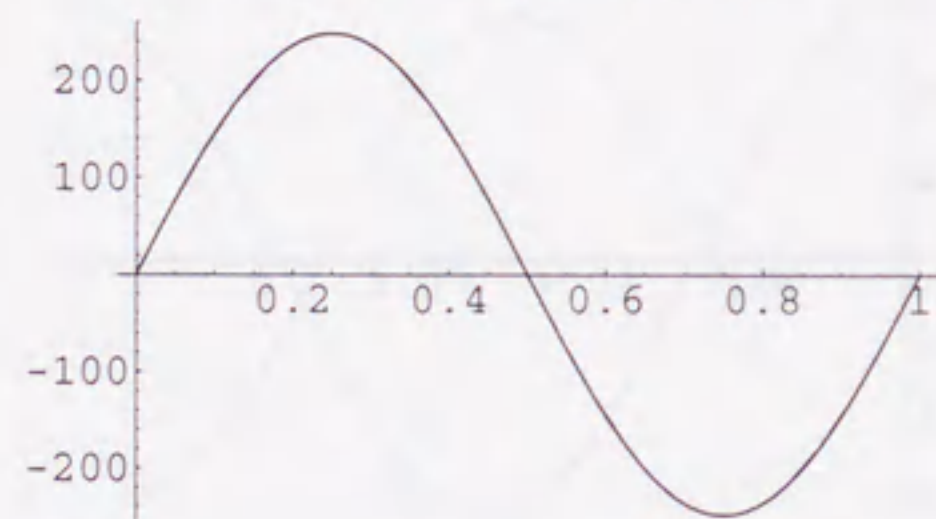
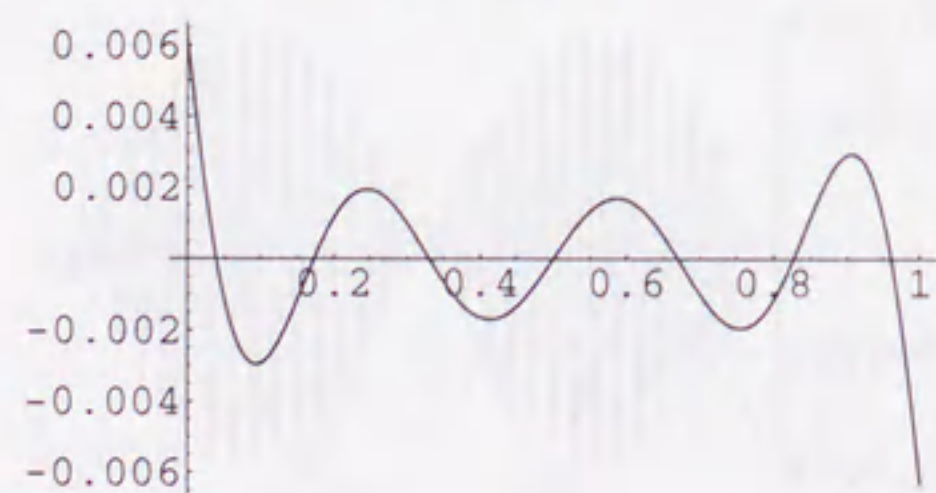
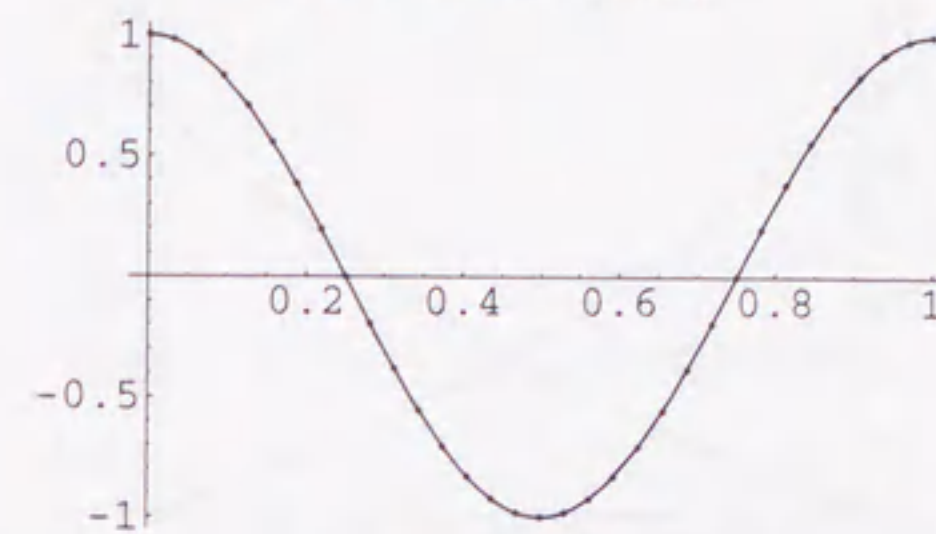
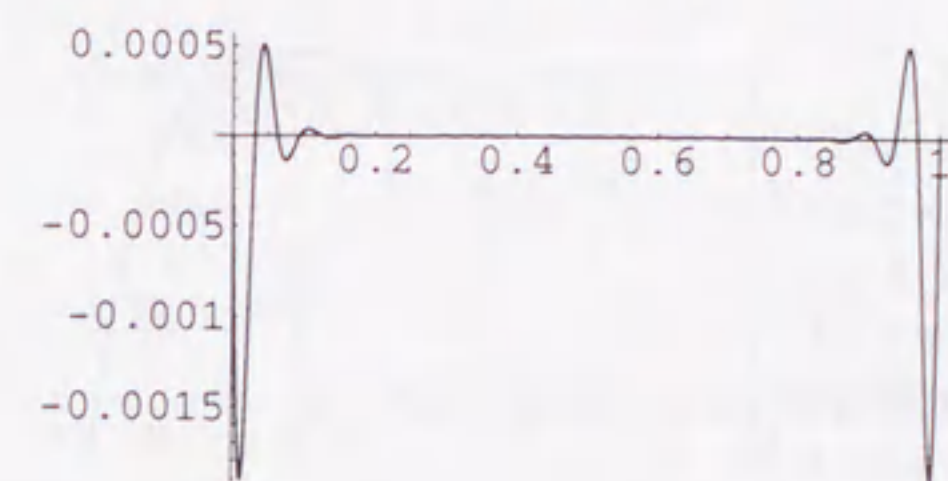
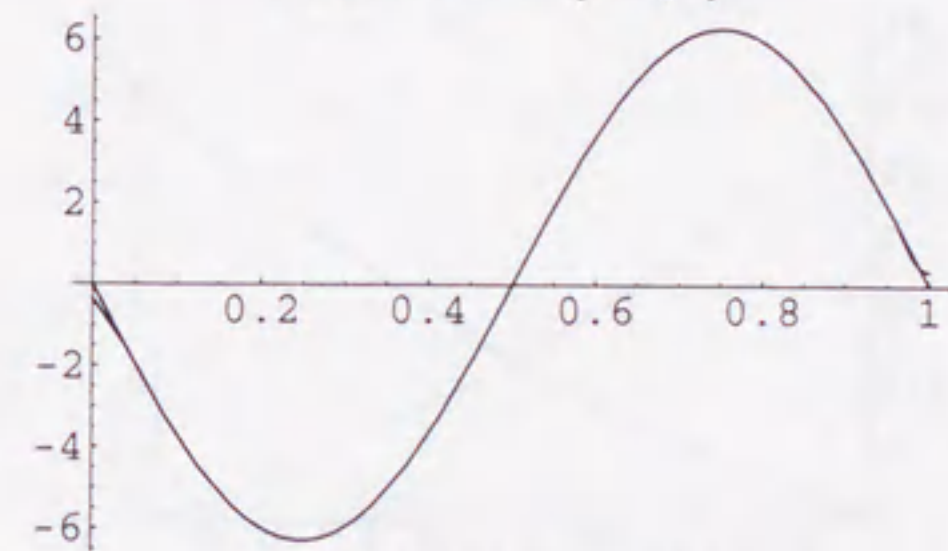
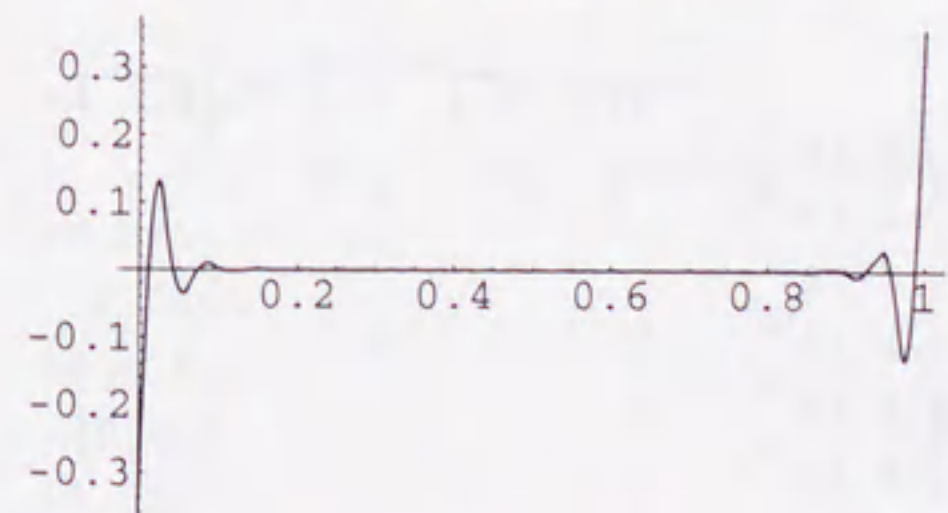
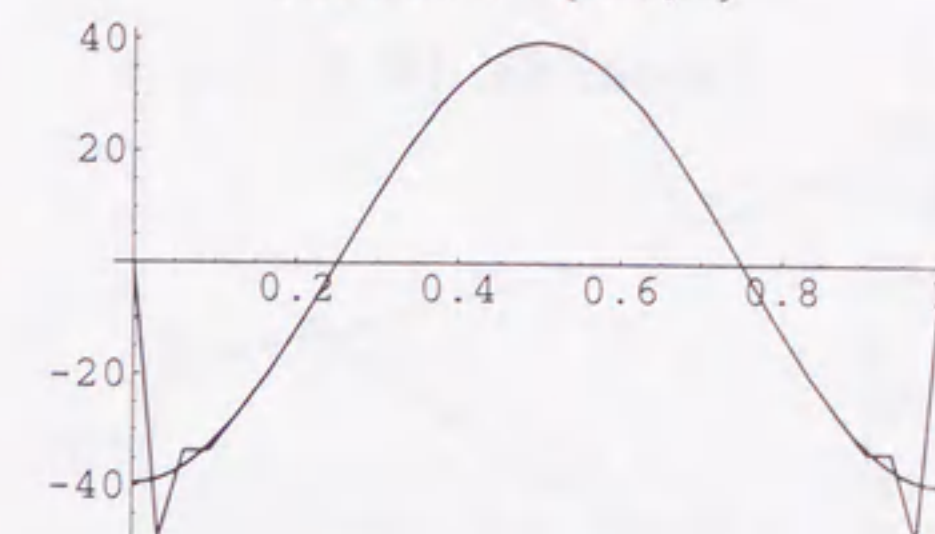
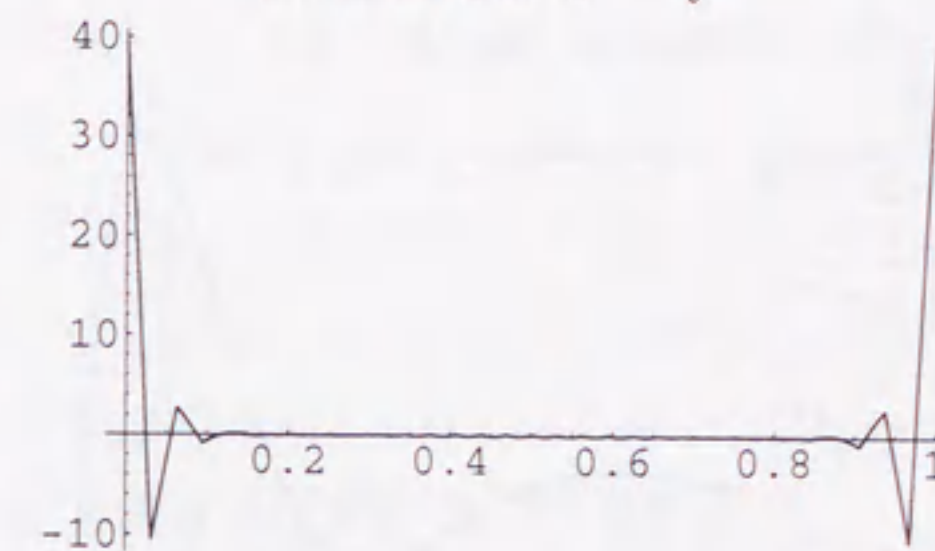
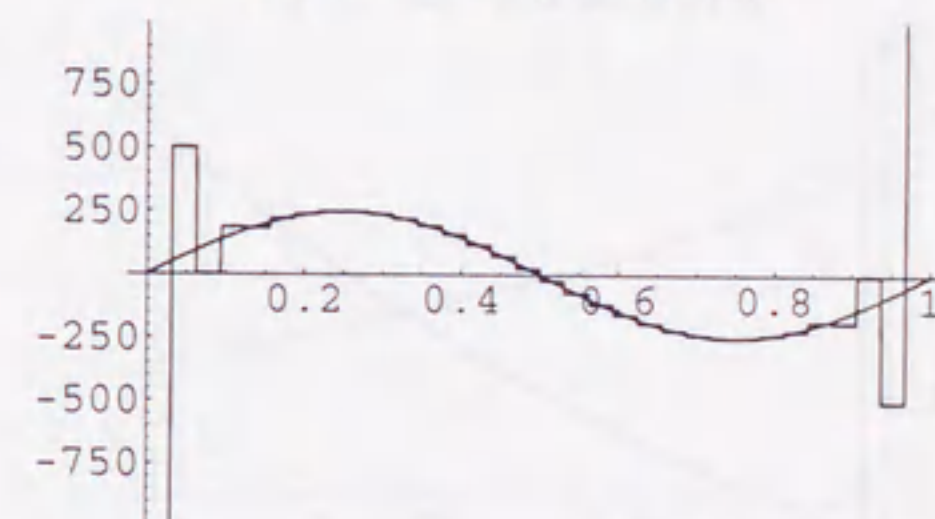
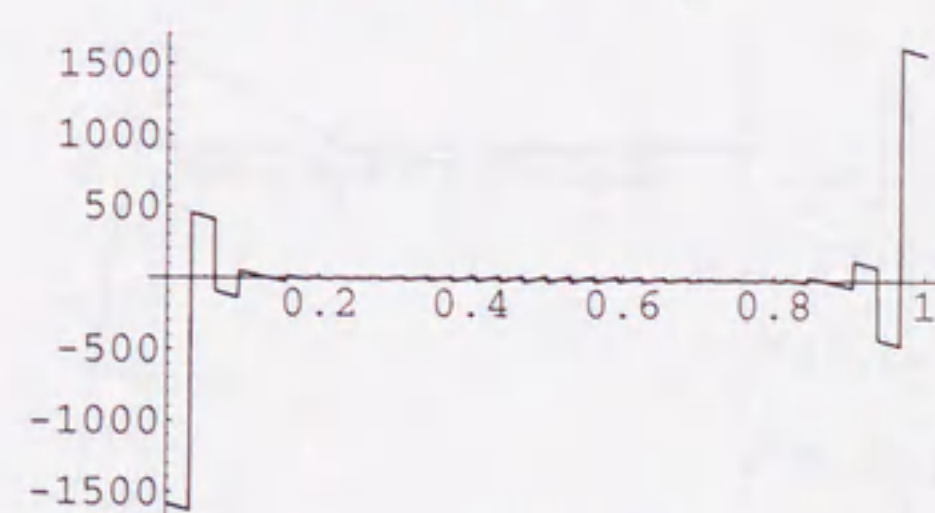
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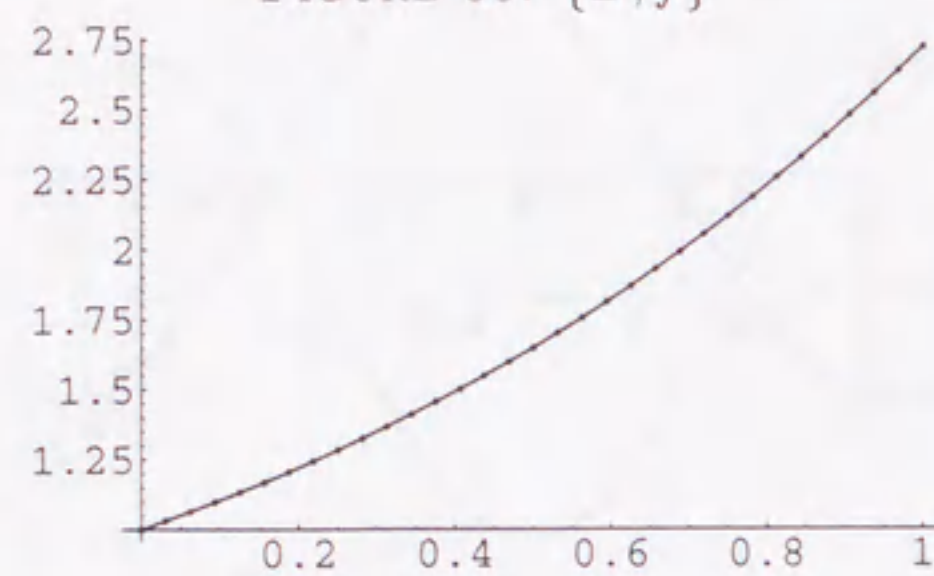
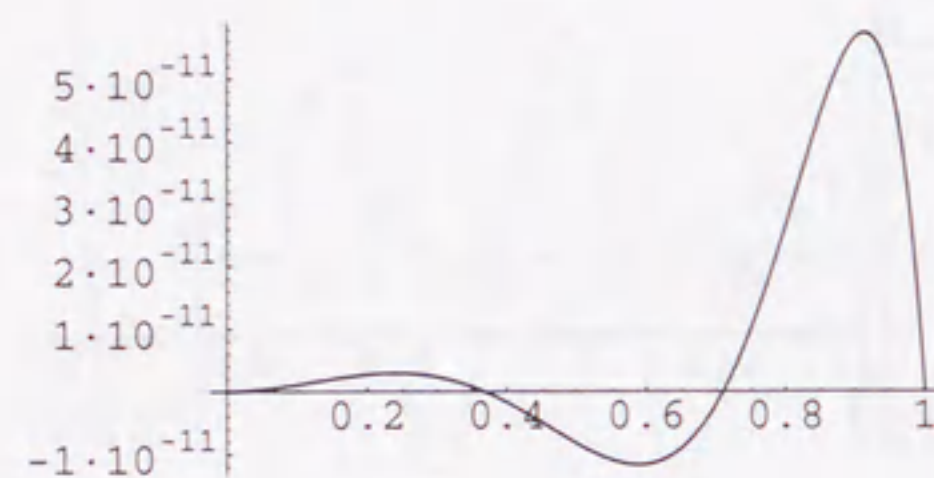
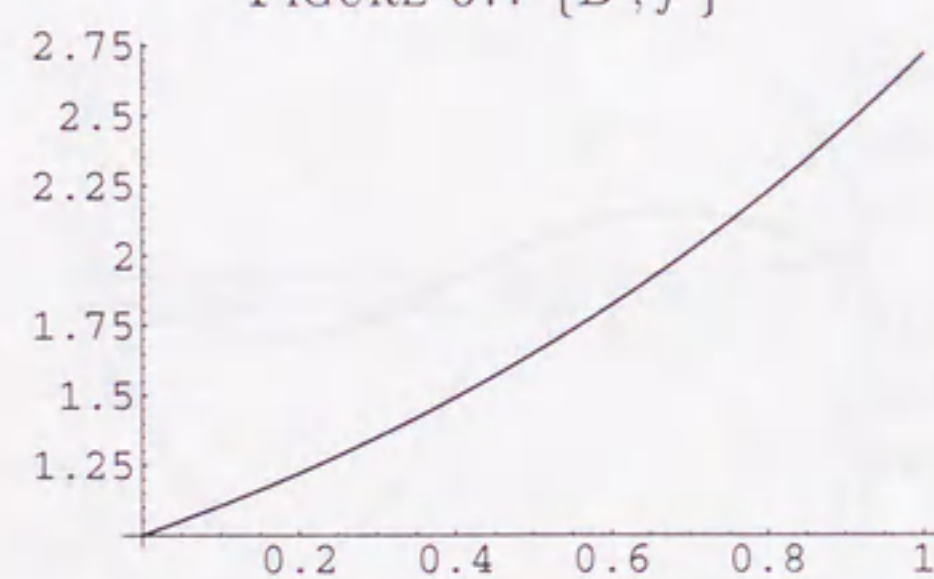
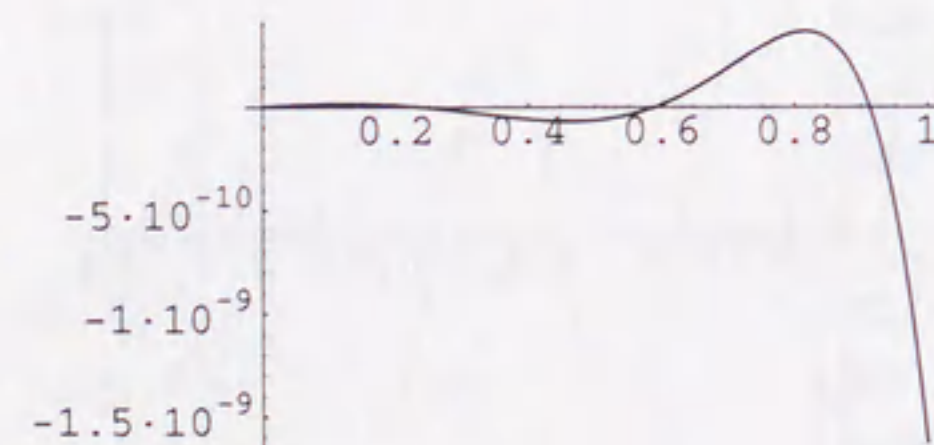
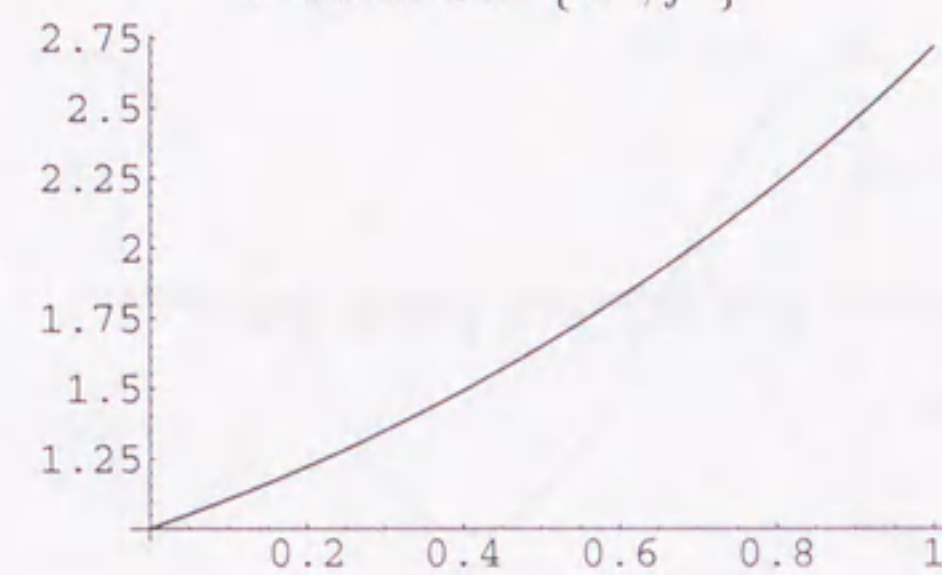
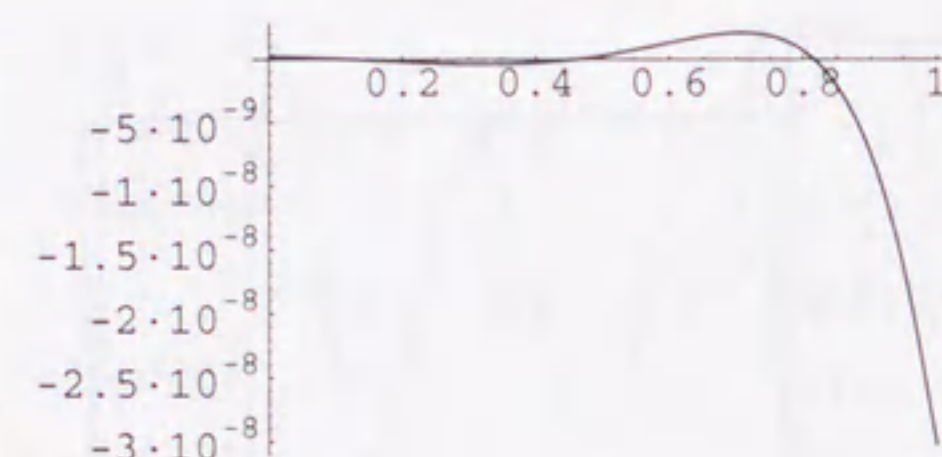
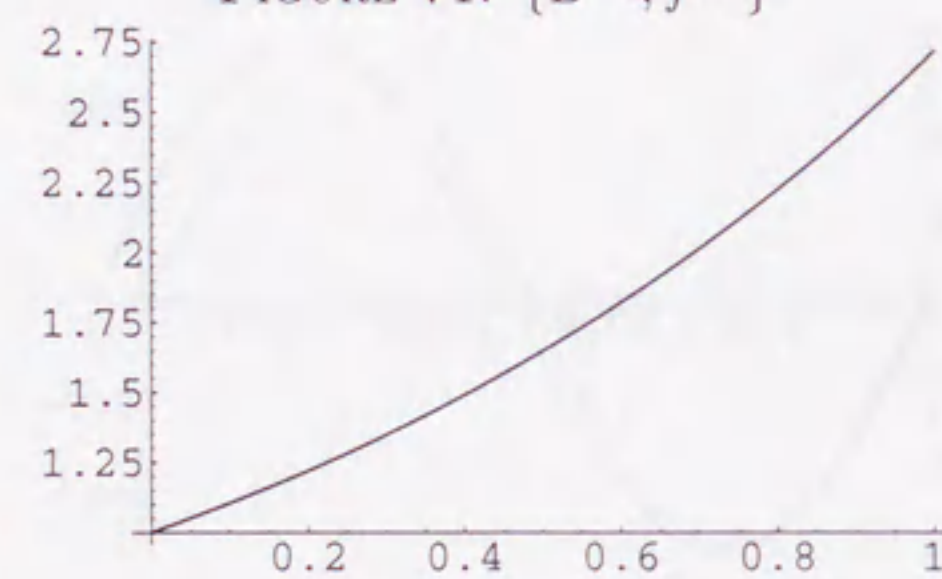
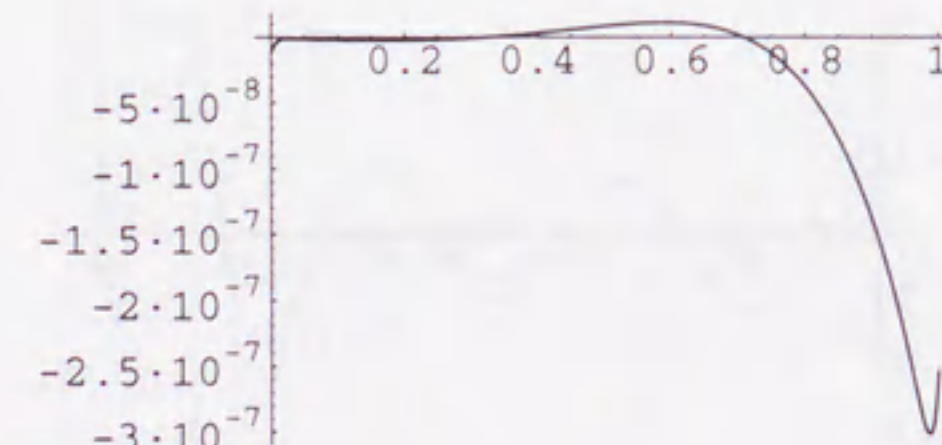
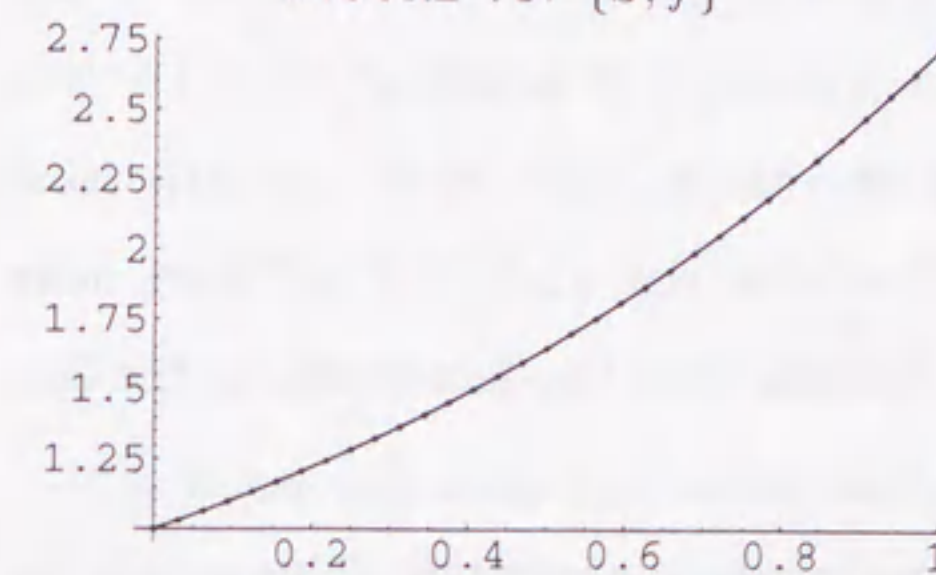
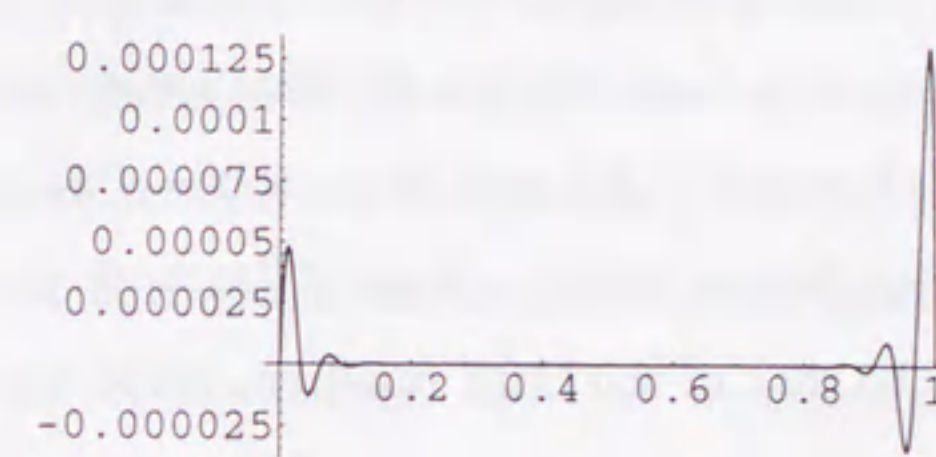
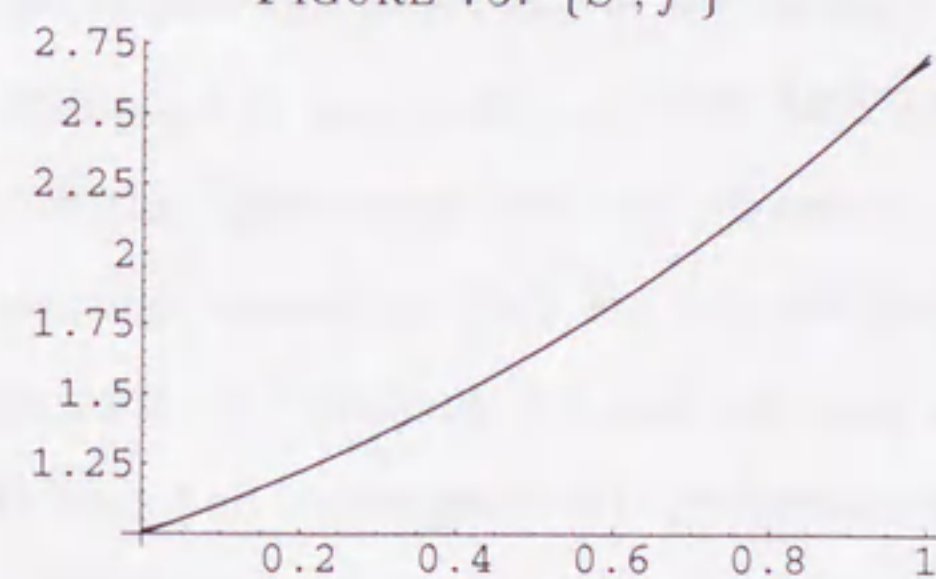
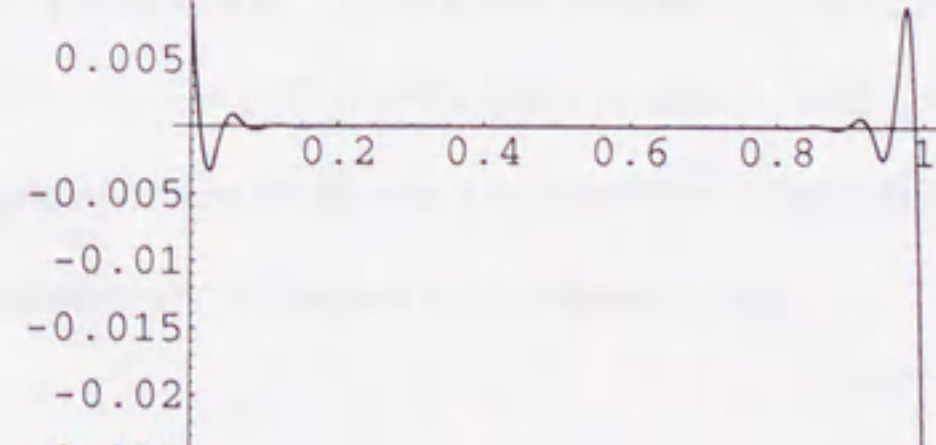
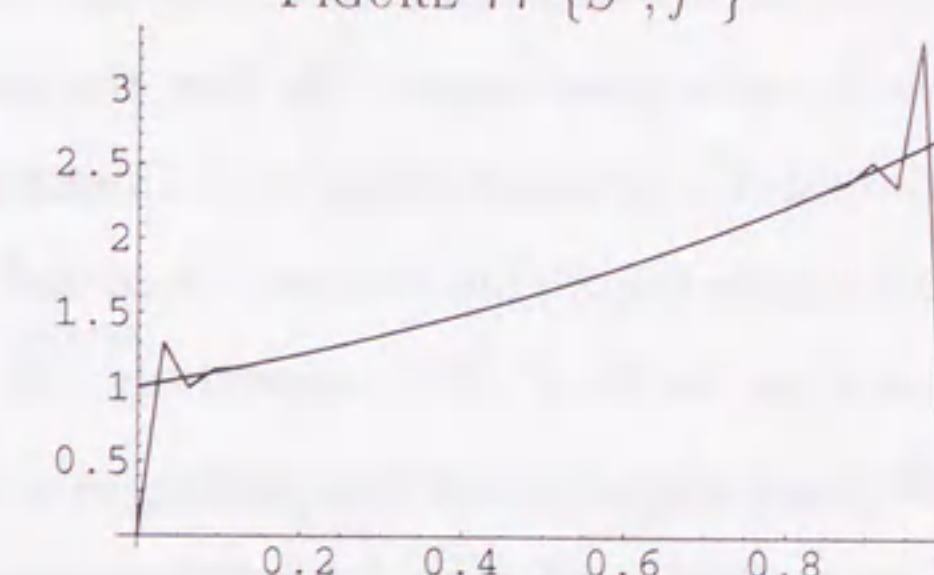
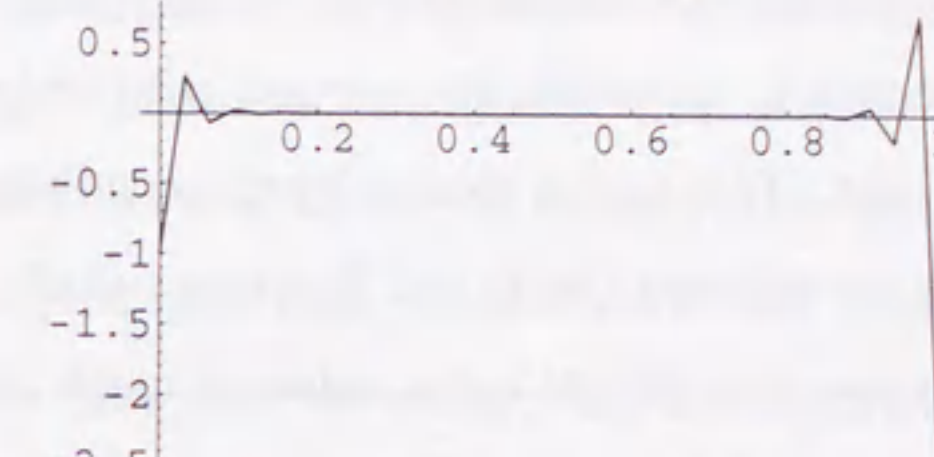
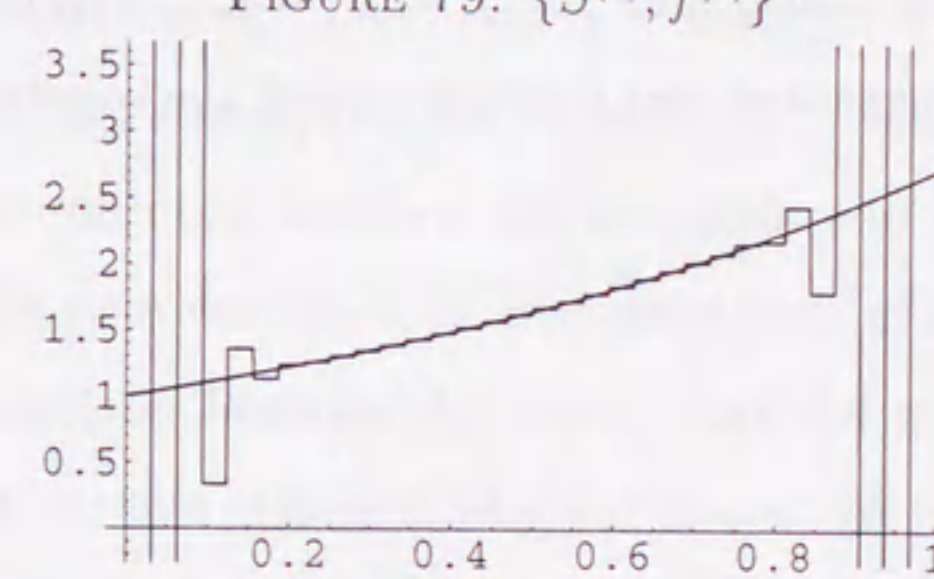
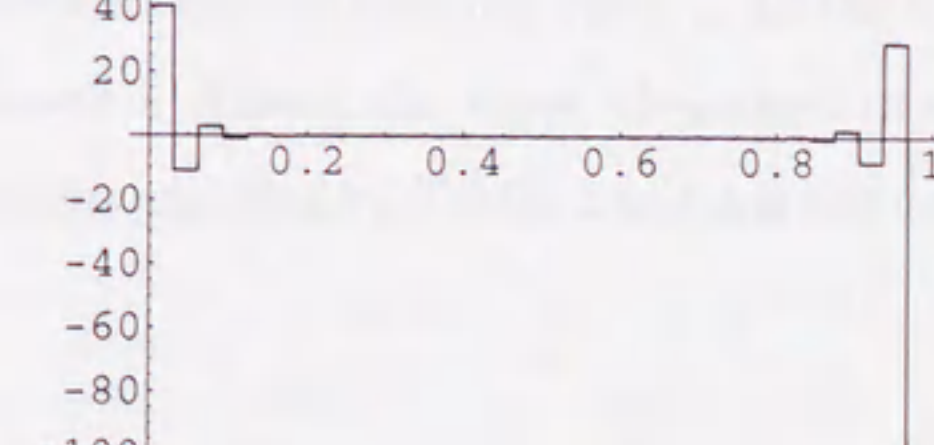
FIGURE 1. $\{B, f\}$ FIGURE 2. $B - f$ FIGURE 3. $\{B', f'\}$ FIGURE 4. $B' - f'$ FIGURE 5. $\{B'', f''\}$ FIGURE 6. $B'' - f''$ FIGURE 7. $\{B^{(3)}, f^{(3)}\}$ FIGURE 8. $B^{(3)} - f^{(3)}$ FIGURE 9. $\{S, f\}$ FIGURE 10. $S - f$ FIGURE 11. $\{S', f'\}$ FIGURE 12. $S' - f'$ FIGURE 13. $\{S'', f''\}$ FIGURE 14. $S'' - f''$ FIGURE 15. $\{S^{(3)}, f^{(3)}\}$ FIGURE 16. $S^{(3)} - f^{(3)}$ 

2. The case $f(x) = 1/(1 + 5(2x - 1)^2)$.

FIGURE 17. $\{B, f\}$ FIGURE 18. $B - f$ FIGURE 19. $\{B', f'\}$ FIGURE 20. $B' - f'$ FIGURE 21. $\{B'', f''\}$ FIGURE 22. $B'' - f''$ FIGURE 23. $\{B^{(3)}, f^{(3)}\}$ FIGURE 24. $B^{(3)} - f^{(3)}$ FIGURE 25. $\{S, f\}$ FIGURE 26. $S - f$ FIGURE 27. $\{S', f'\}$ FIGURE 28. $S' - f'$ FIGURE 29. $\{S'', f''\}$ FIGURE 30. $S'' - f''$ FIGURE 31. $\{S^{(3)}, f^{(3)}\}$ FIGURE 32. $S^{(3)} - f^{(3)}$ 

3. The case $f(x) = \sin 2\pi x$.FIGURE 33. $\{B, f\}$ FIGURE 34. $B - f$ FIGURE 35. $\{B', f'\}$ FIGURE 36. $B' - f'$ FIGURE 37. $\{B'', f''\}$ FIGURE 38. $B'' - f''$ FIGURE 39. $\{B^{(3)}, f^{(3)}\}$ FIGURE 40. $B^{(3)} - f^{(3)}$ FIGURE 41. $\{S, f\}$ FIGURE 42. $S - f$ FIGURE 43. $\{S', f'\}$ FIGURE 44. $S' - f'$ FIGURE 45. $\{S'', f''\}$ FIGURE 46. $S'' - f''$ FIGURE 47. $\{S^{(3)}, f^{(3)}\}$ FIGURE 48. $S^{(3)} - f^{(3)}$ 

4. The case $f(x) = \cos 2\pi x$.FIGURE 49. $\{B, f\}$ FIGURE 50. $B - f$ FIGURE 51. $\{B', f'\}$ FIGURE 52. $B' - f'$ FIGURE 53. $\{B'', f''\}$ FIGURE 54. $B'' - f''$ FIGURE 55. $\{B^{(3)}, f^{(3)}\}$ FIGURE 56. $B^{(3)} - f^{(3)}$ FIGURE 57. $\{S, f\}$ FIGURE 58. $S - f$ FIGURE 59. $\{S', f'\}$ FIGURE 60. $S' - f'$ FIGURE 61. $\{S'', f''\}$ FIGURE 62. $S'' - f''$ FIGURE 63. $\{S^{(3)}, f^{(3)}\}$ FIGURE 64. $S^{(3)} - f^{(3)}$ 

4. The case $f(x) = e^x$.FIGURE 65. $\{B, f\}$ FIGURE 66. $B - f$ FIGURE 67. $\{B', f'\}$ FIGURE 68. $B' - f'$ FIGURE 69. $\{B'', f''\}$ FIGURE 70. $B'' - f''$ FIGURE 71. $\{B^{(3)}, f^{(3)}\}$ FIGURE 72. $B^{(3)} - f^{(3)}$ FIGURE 73. $\{S, f\}$ FIGURE 74. $S - f$ FIGURE 75. $\{S', f'\}$ FIGURE 76. $S' - f'$ FIGURE 77. $\{S'', f''\}$ FIGURE 78. $S'' - f''$ FIGURE 79. $\{S^{(3)}, f^{(3)}\}$ FIGURE 80. $S^{(3)} - f^{(3)}$ 

Next, we selected the second function because it is gentler than the first but it also causes Runge's phenomenon. (In fact, the poles $\zeta = 1/2 \pm \sqrt{5}/10$ satisfy $|\zeta^\zeta/(\zeta-1)^{\zeta-1}| = 0.916\cdots < 1$.) Through Figures 17,18 and 25,26, we can see that the S is a little more accurate than the B , but Figures 19-24 and 27-32 exhibit that the $B', B'', B^{(3)}$ are more precise than the $S', S'', S^{(3)}$, respectively. We can consider that this fact is due to the lack of differentiability of spline functions.

Our method is effective for functions having good analytic properties. This is why we selected the remained three functions, all of which are entire functions.

In the case of the third function, all of the $B, B', B'', B^{(3)}$ are superior to the $S, S', S'', S^{(3)}$, respectively. However, the natural cubic spline function is also effective to some extent in this case. This fact is caused by the conditions $S''(0) = 0 = f''(0)$ and $S''(1) = 0 = f''(1)$. Thus we selected the fourth function, which satisfies neither $f''(0) = 0$ nor $f''(1) = 0$. In this case, the natural cubic spline is much inferior to that of the third function, while the errors of the $B, B', B'', B^{(3)}$ are almost invariant.

The final function is the simplest transcendental function and is also interesting because of the relationship $f = f' = f'' = f^{(3)}$. The effectiveness of our method is most conspicuous in the case of this function among the selected five functions.

CHAPTER 7

CONCLUSION

Finally, we give some concluding remarks and further aspects for the whole thesis.

In Chapter 2, we first introduced the left Bernstein quasi-interpolant operator due to P. Sablonnière and discussed its properties. We extracted the essence of the fact that Sablonnière's operator has good stability and convergence rate properties, and we clarified the structure of general operators that have similar properties. We derived more general and more detailed results than the preceding ones, as Theorems 2.1-2.4. It is remarkable that we treated the derivatives of the approximating polynomials, that we estimated their *uniform* convergence degree, and that we considered the case that we weaken the differentiability condition on approximated functions.

In Chapter 3, we defined the modified Bernstein operator ${}_{\alpha}B_n$ by using Stancu's operator $P_n^{(s)}$, where the parameter α was named *sharpness degree*. The cases $\alpha = 0$ and $\alpha = \infty$ correspond to the (ordinary) Bernstein and the Lagrange operators, respectively. Though both the Sablonnière and our operators are intermediate between the Bernstein and the Lagrange operators, they are not identical. We gave two kinds of representations of our operator as Theorems 3.1 and 3.2, and described as Theorem 3.3 that it has the good stability and convergence rate properties similar to those of Sablonnière's operator. We also considered the application of our operator to numerical quadrature, and we derived a kind of trapezoidal rule as Theorem 3.5. Furthermore, we investigated the positivity of this rule as Theorem 3.7. We demonstrated the effectiveness of this rule through Table 1. In the final section, we compared our operator with Sablonnière's. Among the three advantages of our operator listed there, the numerical stability property clarified by Tables 2 and 3 is the most significant for practical calculations.

In Chapter 4, we gave the Legendre expansion of the modified Bernstein polynomial as Theorem 4.1 and expressed a useful algorithm based on this theorem in the final section. Most of this chapter was spent to calculate the coefficients of the Legendre expansion.

In Chapter 5, we gave a criterion to select the sharpness degree α in the modified Bernstein operator ${}_{{}_\alpha}B_n$ by using the Peano kernel theorem. We presented a convenient formula as Theorem 5.1 to determine the *optimal* α . In the end of this chapter, we listed the optimal α for all $n \leq 64$ obtained from numerical experiments and we conjectured that it is $O(\sqrt{n})$.

In Chapter 6, we selected five functions and gave numerical examples through graphs to demonstrate our theory. We compared the modified Bernstein polynomial with the natural cubic spline functions, and we asserted that our method is superior to the interpolation by spline functions especially when the approximated function has a good analytic property. We clarified that our method is effective for numerical differentiation unlike the use of spline functions.

We consider that our research can be applied to the following directions at least.

- (1) Estimation of unknown functions from experimental data.

Though experimental data involve errors, our approximating polynomial is little affected by them. If we let ε be an error function, then we have

$$\|{}_{{}_\alpha}B_n(f + \varepsilon) - f\| \leq \|{}_{{}_\alpha}B_n f - f\| + \|{}_{{}_\alpha}B_n\| \|\varepsilon\|.$$

Since the value of $\|{}_{{}_\alpha}B_n\|$ is not very large, our method is of practical use.

- (2) Numerical integration and numerical differentiation.

In especial, the advantage that our operator can approximate higher derivatives of a function is effective for numerical differentiation. Furthermore, there is no problem concerning computational complexity if we obtain the Legendre expansion in advance, as we mentioned in Chapter 4.

- (3) Numerical solution of differential equations.

The Adams-type multistep methods for ODE use the Lagrange polynomial, which is numerically unstable when its degree is high. Therefore, it is difficult to increase the number of steps while preserving stability. However, we can generalize the Adams-type

methods if we replace the Lagrange polynomial by the modified Bernstein polynomial because it includes the Lagrange polynomial as a special case. There is a possibility that we can establish better methods regarding both precision and stability by controlling the sharpness degree of the modified Bernstein polynomial in the generalized Adams-type method. (Sablonnière stated a similar suggestion in [40].)

- (4) CAGD (Computer Aided Graphic Design).

We can generalize the Bézier curve if we replace the functional values by vectors and regard the variable x as a parameter in the expression of the modified Bernstein polynomial. (This relationship is parallel for the ordinary Bernstein polynomial and Bézier curve.)

We intend to generalize our method to the case that the sampling points are not necessarily equidistant, and moreover, to the multidimensional case.

ACKNOWLEDGMENTS

The author would like to express his deep gratitude to Professor Jun-ichiro Toriwaki of Nagoya University, who is the chief examiner of this thesis.

The author is greatly indebted to Professor Taketomo Mitsui of Nagoya University, who gave him many significant and detailed comments about the presentation of this thesis, especially about the English expressions.

The author wishes to give his hearty thanks to Professor Masaaki Sugihara of Nagoya University, who informed him of many references around the theme of this thesis and broadened his horizon in this field.

The author gratefully acknowledges helpful and frequent discussions with Associate Professor Hiroshi Sugiura of Nagoya University.

The author would like to give his deep gratitude to Professor Tatsuo Torii of Nanzan University (formerly, Nagoya University), who used to be the supervisor of the author and guided him to the field of numerical analysis.

Finally, the author wishes to express his hearty thanks to Professor Paul Sablonnière, who is the pioneer of the theme of this thesis and privately communicated with the author.

REFERENCES

- [1] M. Abramowitz and I. A. Stegun, Eds., "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables," Dover, New York, 1965.
- [2] N. I. Achieser, "Theory of Approximation," Ungar, New York, 1956.
- [3] H. Berens and R. A. DeVore, A characterization of Bernstein polynomials, in "Approximation III" (E. W. Cheney Ed.), pp. 213-219, Academic Press, New York, 1980.
- [4] P. Borwein and T. Erdélyi, "Polynomials and Polynomial Inequalities," Springer, Berlin/Heidelberg/New York, 1995.
- [5] P. L. Butzer, Linear combinations of Bernstein polynomials, *Canad. J. Math.* **5** (1953), 559-567.
- [6] L. Carlitz, Note on Nörlund's polynomial $B_n^{(z)}$, *Proc. Amer. Math. Soc.* **11** (1960), 452-455.
- [7] E. W. Cheney "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
- [8] P. G. Ciarlet, "Introduction to Numerical Linear Algebra and Optimisation," Cambridge Univ. Press, Cambridge, UK, 1989.
- [9] P. J. Davis, "Interpolation and Approximation," Blaisdell, Waltham, Massachusetts, 1963.
- [10] P. J. Davis and P. Rabinowitz, "Methods of Numerical Integration," 2nd ed., Academic Press, New York, 1984.
- [11] R. A. DeVore and G. G. Lorentz, "Constructive Approximation," Springer, Berlin/Heidelberg/New York, 1993.
- [12] Z. Ditzian, A global inverse theorem for combinations of Bernstein polynomials, *J. Approx. Theory* **26** (1979), 277-292.
- [13] Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer Series in Computational Mathematics, Vol. 9, Springer, Berlin/Heidelberg/New York, 1987.

- [14] H. Engels, "Numerical Quadrature and Cubature," Academic Press, London, 1980.
- [15] G. Farin, "Curves and Surfaces for Computer Aided Geometric Design," 3rd ed., Academic Press, New York, 1993.
- [16] W. Gautschi, The condition of orthogonal polynomials, *Math. Comp.* **26** (1972), 923–924.
- [17] W. Gautschi, The condition of polynomials in power form, *Math. Comp.* **33** (1979), 343–352.
- [18] H. H. Gonska and J. Meier, A bibliography on approximation of functions by Bernstein type operators, in "Approximation Theory IV" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 739–785, Academic Press, San Diego, 1983.
- [19] H. W. Gould, Stirling number representation problems, *Proc. Amer. Math. Soc.* **11** (1960), 447–451.
- [20] E. Isaacson and H. B. Keller, "Analysis of Numerical Methods," Wiley, New York, 1966.
- [21] C. Jordan, "Calculus of Finite Differences," 3rd ed., Chelsea, New York, 1965.
- [22] Y. Kageyama, Generalization of the left Bernstein quasi-interpolants, *J. Approx. Theory* **94** (1998), 306–329.
- [23] Y. Kageyama, A new class of modified Bernstein operators, *J. Approx. Theory* **101** (1999), 121–147.
- [24] Y. Kageyama, Legendre expansion of the modified Bernstein polynomials, submitted to *J. Approx. Theory*.
- [25] D. R. Kincaid and E. W. Cheney, "Numerical Analysis: Mathematics of Scientific Computing," Brooks/Cole Publ., Pacific Grove, California, 1991.
- [26] G. G. Lorentz, "Bernstein Polynomials," 2nd ed., Chelsea, New York, 1986.
- [27] G. G. Lorentz, "Approximation of Functions," 2nd ed., Chelsea, New York, 1986.
- [28] G. Meinardus, "Approximation of Functions: Theory and Numerical Methods," Springer, Berlin, 1967.
- [29] L. M. Milne-Thomson, "The Calculus of Finite Differences," Chelsea, New York, 1981.
- [30] I. P. Natanson, "Constructive Function Theory," Vol. III, Ungar, New York, 1965.
- [31] N. E. Nörlund, "Vorlesungen über Differenzenrechnung," Springer, Berlin, 1924.

- [32] M. J. D. Powell, "Approximation Theory and Methods," Cambridge Univ. Press, New York, 1981.
- [33] A. Ralston and P. Rabinowitz, "A First Course in Numerical Analysis," 2nd ed., McGraw-Hill, New York, 1978.
- [34] J. Riordan, "An Introduction to Combinatorial Analysis," Wiley, New York, 1958.
- [35] J. Riordan, "Combinatorial Identities," Wiley, New York, 1968.
- [36] T. J. Rivlin, "An Introduction to the Approximation of Functions," Dover, New York, 1981.
- [37] T. J. Rivlin, "Chebyshev Polynomials," 2nd ed., Wiley, New York, 1990.
- [38] P. Sablonnière, Bernstein quasi-interpolants on $[0, 1]$, in "Multivariate Approximation Theory, Vol. IV" (C. K. Chui, W. Schempp, and K. Zeller, Eds.), pp. 287–294, Birkhäuser, Basel, 1989.
- [39] P. Sablonnière, A family of Bernstein quasi-interpolants on $[0, 1]$, *Approx. Theory Appl.* **8**, No. 3 (1992), 62–76.
- [40] P. Sablonnière, Bernstein-type quasi-interpolants, in "Curves and Surfaces" (P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, Eds.), pp. 421–426, Academic Press, Boston, 1991.
- [41] P. Sablonnière, Representation of quasi-interpolants as differential operators and applications, *Internat. Ser. Numer. Math.* **132** (1999), 233–253.
- [42] L. Schumaker, "Spline Functions: Basic Theory," Wiley, New York, 1981.
- [43] D. D. Stancu, On a new positive linear polynomial operator, *Proc. Japan Acad.* **44** (1968), 221–224.
- [44] D. D. Stancu, Approximation of functions by a new class of linear polynomial operators, *Rev. Roumaine Math. Pures Appl.* **13** (1968), 1173–1194.
- [45] D. D. Stancu, Use of probabilistic methods in the theory of uniform approximation of continuous functions, *Rev. Roumaine Math. Pures Appl.* **14** (1969), 673–691.
- [46] D. D. Stancu, Approximation properties of a class of linear positive operators, *Studia Univ. Babeş-Bolyai Ser. Math.-Mech.* **15**, No. 2 (1970), 33–38.

- [47] D. D. Stancu, On the remainder of approximation of functions by means of a parameter-dependent linear polynomial operator, *Studia Univ. Babeş-Bolyai Ser. Math.-Mech.* **16**, No. 2 (1971), 59–66.
- [48] G. Szegő, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc. Coll. Publ., Providence, Rhode Island, 1975.
- [49] J. V. Uspensky, Sur les valeurs asymptotiques des coefficients de Cotes, *Bull. Amer. Math. Soc.* **31** (1925), 145–156.
- [50] J. V. Uspensky, On the expansion of the remainder in the Newton–Cotes formula, *Trans. Amer. Math. Soc.* **37** (1935), 381–396.
- [51] J. H. Van Lint and R. M. Wilson, "A Course in Combinatorics," Cambridge Univ. Press, Cambridge, UK, 1992.
- [52] J. Wimp, "Computation with Recurrence Relations," Pitman, Boston/London/Melbourne, 1984.
- [53] Wu Zhengchang, Norm of the Bernstein left quasi-interpolant operator, *J. Approx. Theory* **66** (1991), 36–43.

