approximate inverse systems) dynamical systems with arbitrarily relative degrees. Here the term "disturbance" is referred to the combination of the model uncertainties, the nonlinear parts of the system and the external disturbances. Only the upper and lower bounds of the disturbances is assumed as the *a priori* information.

By choosing small design parameters $\delta_i$ $(1 \le i \le r)$ and $\lambda$, the performance of the new observer may become better even for the high frequency disturbances. But there is a limitation about the choice of these parameters when the measurement noises are present. When it is implemented by a digital computer, the design parameters are also limited by the sampling period. For the class of nonminimum phase systems, the parameter $\gamma$ should be chosen large in order to get a good estimation.

The estimated disturbance is employed to formulate a state observer and to synthesize a pole assignment controller. Simulation results show that the proposed observer is of high robustness to the types of the disturbances and the model uncertainties. Experimental results show the new observer is of high practicality. Moreover, when the impulse disturbances exist, the experiment results show that the new observer is superior to the traditional disturbance observer.

For the systems with relatively large stochastic disturbances or measurement noises, the proposed disturbance observers should be modified. The proposed formulation is expected to be extended to the general nonminimum phase systems as well as much more complicated systems, such as the uncertain systems with delays, descriptor systems, affine parameter-dependent models, etc.

# Chapter 5

# VSS Theory-Based Disturbance Estimation Scheme and Its Applications for MIMO Systems

## 5.1 Introduction

Variable structure control systems are an established method of controlling uncertain dynamical systems and have been the focus of much research. Especially, their robust properties are well documented in [9,26,57]. But the main developments in this area are under the assumption that all the internal states are available to the control law. Of course, this is often not the case in practice. One way of overcoming this difficulty is to generate an estimate of the unavailable internal state. VSS type approaches are put forward by several authors [11,59-61,68]. The proposed observer systems are designed to provide asymptotic error decay in the presence of matched uncertainties. But all of the results are subjected to the minimum phase systems (with respect to the relation between the disturbance and the output) with the assumption that, from the point of view of state space, the partial states directly affected by the disturbances must directly appear in the outputs (For SISO system, this assumption is equivalent to that the system is minimum phase with relative degree one. From now on, we will also call this kind of MIMO systems "MIMO minimum phase dynamical systems with relative degree one".). An alternative approach is to design a controller in which only the input and output information is required. VSS type formulations are to restrict the class of sliding surfaces and control laws to those which require only output information. It has been explored in [12,13,15,25,50] recently. They are also subjected to the minimum phase dynamical systems with relative degree one.

It might be argued that if the disturbances can be estimated, the control problems of the systems with disturbances can be greatly put forward. For example, the state observer can be easily constructed by using the estimates of the disturbances. Further, a pole placement controller which also has the function to cancel the disturbances can be established. It is well known that the disturbance can be estimated by using the VSS equivalent control theory for SISO minimum phase dynamical systems with relative degree one [56,57]. But for the systems with higher relative degrees, no result has been reported. Usually, it is regarded impossible. This is because that the VSS type estimation is constructed by discontinuous functions. A further step can not be done to generate new useful signals.

Based on the transfer function method, this chapter presents a disturbance estimation method for the MIMO minimum phase dynamical systems with arbitrary relative degrees. From the point of view of sliding mode control, one of the most important benefits of VSS, the equivalent control method, is sufficiently employed in this chapter.

The organization of this chapter is as follows. Section 5.2 gives the problem formulation. In section 5.3, the disturbances are estimated independently based on the obtained important equations. In section 5.4, a VSS type state observer is constructed by applying the estimates of the disturbances. In section 5.5, a pole placement controller is synthesized to place desired poles and to cancel the disturbances. Section 5.6 gives a design example and the simulation results.

## 5.2 Problem Statement

Consider the uncertain dynamical system described by



$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Kv(x,u,t) \\ y(t) = C^T x(t) \end{cases} \tag{5.1}$$

where $x(t) \in R^n$ is the unknown state vector, $u(t) \in R^r$ is the input and $y(t) \in R^p$ is the output, $v(x,u,t) \in R^m$ is the disturbance (or system nonlinearitis and any model uncertainties). The quantities $n$, $r$, $p$ and $m$ are known constants satisfying $n > p \ge r$ and $n > p \ge m$. $A$, $B$, $K$ and $C$ are known matrices with appropriate dimensions and are described in the observable canonical form

$$A = \left[\begin{array}{cc|cc|c|cc}
-a_{11}^{(1)} & 1 \ddots & -a_{12}^{(1)} & & & -a_{1p}^{(1)} & \\
\vdots & 1 & \vdots & 0 & \cdots & \vdots & 0 \\
-a_{n_1 1}^{(1)} & 0\cdots0 & -a_{n_1 2}^{(1)} & & & -a_{n_1 p}^{(1)} & \\
\hline
-a_{11}^{(2)} & & -a_{12}^{(2)} & 1 \ddots & & -a_{1p}^{(2)} & \\
\vdots & 0 & \vdots & 1 & \cdots & \vdots & 0 \\
-a_{n_2 1}^{(2)} & & -a_{n_2 2}^{(2)} & 0\cdots0 & & -a_{n_2 p}^{(2)} & \\
\hline
\cdots & & \cdots & & \ddots & \cdots & \\
\hline
-a_{11}^{(p)} & & -a_{12}^{(p)} & & & -a_{1p}^{(p)} & 1 \ddots \\
\vdots & 0 & \vdots & 0 & \cdots & \vdots & 1 \\
-a_{n_p 1}^{(p)} & & -a_{n_p 2}^{(p)} & & & -a_{n_p p}^{(p)} & 0\cdots0
\end{array}\right] \tag{5.2}$$

$$B = \left[\begin{array}{ccc}
b_{11}^{(1)} & \cdots & b_{1r}^{(1)} \\
\vdots & \vdots & \vdots \\
b_{n_1 1}^{(1)} & \cdots & b_{n_1 r}^{(1)} \\
\hline
b_{11}^{(2)} & \cdots & b_{1r}^{(2)} \\
\vdots & \vdots & \vdots \\
b_{n_2 1}^{(2)} & \cdots & b_{n_2 r}^{(2)} \\
\hline
\vdots & \vdots & \vdots \\
\hline
b_{11}^{(p)} & \cdots & b_{1r}^{(p)} \\
\vdots & \vdots & \vdots \\
b_{n_p 1}^{(p)} & \cdots & b_{n_p r}^{(p)}
\end{array}\right], \quad
K = \left[\begin{array}{ccc}
k_{11}^{(1)} & \cdots & k_{1m}^{(1)} \\
\vdots & \vdots & \vdots \\
k_{n_1 1}^{(1)} & \cdots & k_{n_1 m}^{(1)} \\
\hline
k_{11}^{(2)} & \cdots & k_{1m}^{(2)} \\
\vdots & \vdots & \vdots \\
k_{n_2 1}^{(2)} & \cdots & k_{n_2 m}^{(2)} \\
\hline
\vdots & \vdots & \vdots \\
\hline
k_{11}^{(p)} & \cdots & k_{1m}^{(p)} \\
\vdots & \vdots & \vdots \\
k_{n_p 1}^{(p)} & \cdots & k_{n_p m}^{(p)}
\end{array}\right], \quad
C = \left[\begin{array}{c|c|c|c}
1 & & & \\
0 & 0 & \cdots & 0 \\
\vdots & & & \\
0 & & & \\
\hline
 & 1 & & \\
0 & 0 & \cdots & 0 \\
 & \vdots & & \\
 & 0 & & \\
\hline
\vdots & \vdots & \ddots & \vdots \\
\hline
 & & & 1 \\
0 & 0 & \cdots & 0 \\
 & & & \vdots \\
 & & & 0
\end{array}\right] \tag{5.3}$$

It is obvious that $n = n_1 + n_2 + \cdots + n_p$.

In this chapter, we make the following assumptions.

**Assumption 5.1** The matrix $K$ is of full rank.

**Assumption 5.2** The system is in minimum phase (with respect to the relation between the disturbance and the output), i.e.

$$\begin{bmatrix} A-sI & K \\ C^T & 0 \end{bmatrix} \tag{5.4}$$

has full rank for all the $s$ satisfying $\mathrm{Re}(s)>0$.

**Assumption 5.3** The disturbance is bounded by

$$\|v(x,u,t)\|_2 \le \rho(y,u,t) \tag{5.5}$$

where $\|\cdot\|_2$ denotes the Euclidean norm, $\rho(y,u,t)\ge 0$ is a known scalar function.

It should be pointed out that the VSS type observers and controllers are discussed only under the assumption that $C^T K$ has full rank [11-13,25,59-61,68].

This chapter deals with the MIMO minimum phase dynamical systems even though $C^T K$ is not of full rank. The aim of this research is to estimate the disturbances and to construct a robust state observer and a pole placement controller based on the obtained estimated disturbances.

For simplicity, we call the term $v(x,u,t)$ "disturbance" of the system, and denote it by $v(t)$ in the following of this chapter.

## 5.3 Estimation of the Disturbance

### 5.3.1 Some Preparations

Without loss of generality, we also assume that $m=p$, i.e. the number of outputs is equal to the number of disturbances.

Now, we define the polynomials

$$a_{ij}(s)=a_{1j}^{(i)}s^{n_i-1}+a_{2j}^{(i)}s^{n_i-2}+\cdots+a_{n_j j}^{(i)} \tag{5.6}$$



$$b_{ij}(s)=b_{1j}^{(i)}s^{n_i-1}+b_{2j}^{(i)}s^{n_i-2}+\cdots+b_{n_j j}^{(i)} \tag{5.7}$$

$$k_{ij}(s)=k_{1j}^{(i)}s^{n_i-1}+k_{2j}^{(i)}s^{n_i-2}+\cdots+k_{n_j j}^{(i)} \tag{5.8}$$

$$K(s)=\begin{bmatrix} k_{11}(s) & k_{12}(s) & \cdots & k_{1p}(s) \\ k_{21}(s) & k_{22}(s) & \cdots & k_{2p}(s) \\ . & . & \cdots & . \\ k_{p1}(s) & k_{p2}(s) & \cdots & k_{pp}(s) \end{bmatrix} \tag{5.9}$$

It is obvious that

$$\deg(a_{ij}(s))<n_i,\ \ \deg(b_{ij}(s))<n_i,\ \ \deg(k_{ij}(s))<n_i \tag{5.10}$$

**Lemma 5.1** The matrix $\begin{bmatrix} A-sI & K \\ C^T & 0 \end{bmatrix}$ has full rank for all the $s$ satisfying $\mathrm{Re}(s)>0$ if and only if $\det(K(s))$ is a Hurwitz polynomial.

**Proof**: By some elementary row operations, the result can be easily proved to be valid.

If we denote the differential operator by $s$, then Equation (5.1) can be rewritten in the following compact form

$$\begin{cases} (s^{n_1}+a_{11}(s))y_1(t)+\cdots+a_{1p}(s)y_p(t)= \\ \qquad b_{11}(s)u_1(t)+\cdots+b_{1r}(s)u_r(t)+k_{11}(s)v_1(t)+\cdots+k_{1p}(s)v_p(t) \\ \qquad \vdots \\ a_{p1}(s)y_1(t)+\cdots+(s^{n_p}+a_{pp}(s))y_p(t)= \\ \qquad b_{p1}(s)u_1(t)+\cdots+b_{pr}(s)u_r(t)+k_{p1}(s)v_1(t)+\cdots+k_{pp}(s)v_p(t) \end{cases} \tag{5.11}$$

i.e.

$$\begin{bmatrix} s^{n_1}+a_{11}(s) & \cdots & a_{1p}(s) \\ \vdots & \vdots & \vdots \\ a_{p1}(s) & \cdots & s^{n_p}+a_{pp}(s) \end{bmatrix}\begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

$$=\begin{bmatrix} b_{11}(s) & \cdots & b_{1r}(s) \\ \vdots & \vdots & \vdots \\ b_{p1}(s) & \cdots & b_{pr}(s) \end{bmatrix}\begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{bmatrix}+\begin{bmatrix} k_{11}(s) & \cdots & k_{1p}(s) \\ \vdots & \vdots & \vdots \\ k_{p1}(s) & \cdots & k_{pp}(s) \end{bmatrix}\begin{bmatrix} v_1(t) \\ \vdots \\ v_p(t) \end{bmatrix} \tag{5.12}$$

Let $k(s)=\det(K(s))$ and $q_0=\deg(k(s))$. Suppose

$$k(s)=k_0 s^{q_0}+\cdots+k_{q_0} \tag{5.13}$$

where $k_0 \neq 0$. By Assumption 5.2 and Lemma 5.1, it is obvious that $k(s)$ is a Hurwitz polynomial. From the determination calculation method of a matrix, we can easily concluded that $q_0 \le n_1-1+n_2-1+\cdots+n_p-1=n-p$.

Now, multiplying (5.12) by $adj(K(s))$ yields

$$adj(K(s))\begin{bmatrix} s^{n_1}+a_{11}(s) & \cdots & a_{1p}(s) \\ \vdots & \vdots & \vdots \\ a_{p1}(s) & \cdots & s^{n_p}+a_{pp}(s)\end{bmatrix}\begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t)\end{bmatrix}$$

$$=adj(K(s))\begin{bmatrix} b_{11}(s) & \cdots & b_{1r}(s) \\ \vdots & \vdots & \vdots \\ b_{p1}(s) & \cdots & b_{pr}(s)\end{bmatrix}\begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t)\end{bmatrix}+\det(K(s))\begin{bmatrix} v_1(t) \\ \vdots \\ v_p(t)\end{bmatrix} \tag{5.14}$$

where $\det\{adj(K(s))\}=\{\det(K(s))\}^{p-1}=\{k(s)\}^{p-1}$ is also a Hurwitz polynomial.

Let $l_i$ be the highest order of $s$ of all the entries in the $i$-th row of $adj(K(s))\begin{bmatrix} s^{n_1}+a_{11}(s) & \cdots & a_{1p}(s) \\ \vdots & \vdots & \vdots \\ a_{p1}(s) & \cdots & s^{n_p}+a_{pp}(s)\end{bmatrix}$. So, there exists a vector $\alpha_i$ such that the $i$-th row of the left hand side of (5.14) can be expressed as $s^{l_i}(\alpha_i y(t))-\Phi_i(s)y(t)$, where $\Phi_i(s)$ is a row vector whose entries are at most $(l_i-1)$-th order polynomials of $s$. Let $h_i$ be the highest order of $s$ of all the entries in the $i$-th row of $adj(K(s))\begin{bmatrix} b_{11}(s) & \cdots & b_{1r}(s) \\ \vdots & \vdots & \vdots \\ b_{p1}(s) & \cdots & b_{pr}(s)\end{bmatrix}$. So the $i$-th row of $adj(K(s))\begin{bmatrix} b_{11}(s) & \cdots & b_{1r}(s) \\ \vdots & \vdots & \vdots \\ b_{p1}(s) & \cdots & b_{pr}(s)\end{bmatrix}\begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t)\end{bmatrix}$ can be expressed as $\Psi_i(s)u(t)$, where



$\Psi_i(s)$ is a row vector whose entries are at most $h_i$-th order polynomials of $s$.

It is easy to see that $l_i>q_0$ and $l_i>h_i$, otherwise, Equation (5.14) will contradict with the original Differential Equation (5.1). Then, $l_i-q_0$ can be regarded as the "relative degree" (with respect to the relation between the disturbance and the output) of the $i$-th equation in (5.14).

Thus, Equation (5.14) can be rewritten as

$$\begin{cases} s^{l_1}(\alpha_1 y(t))=\Phi_1(s)y(t)+\Psi_1(s)u(t)+k(s)v_1(t) \\ \vdots \\ s^{l_p}(\alpha_p y(t))=\Phi_p(s)y(t)+\Psi_p(s)u(t)+k(s)v_p(t)\end{cases} \tag{5.15}$$

Because $A$, $B$, $C$ and $K$ are known matrices, $\alpha_i$, $\Phi_i(s)$, $\Psi_i(s)$ and $k(s)$ can be calculated.

Now, the equations in (5.15) are rearranged. For the $i$-th equation in (5.15), we introduce an $n$-th order Hurwitz polynomial

$$f_i(s)=\frac{1}{k_0}k(s)\cdot(s+\lambda)^{l_i-q_0} \tag{5.16}$$

Then, the $i$-th equation in (5.15) can be rewritten as

$$\alpha_i y(t)=\frac{f_i(s)-s^{l_i}}{f_i(s)}(\alpha_i y(t))+\frac{\Phi_i(s)}{f_i(s)}y(t)+\frac{\Psi_i(s)}{f_i(s)}u(t)+\frac{k_0}{(s+\lambda)^{l_i-q_0}}v_i(t) \tag{5.17}$$

Multiplying (5.17) by $s+\lambda$ yields

$$\begin{aligned}(s+\lambda)(\alpha_i y(t)) &= (s+\lambda)\left\{\frac{f_i(s)-s^{l_i}}{f_i(s)}(\alpha_{i1}y_{i1}+\cdots+\alpha_{i\tau_i}y_{i\tau_i})+\frac{\Phi_i(s)}{f_i(s)}y(t)+\frac{\Psi_i(s)}{f_i(s)}u(t)\right\} \\ &\quad +\frac{k_0}{(s+\lambda)^{l_i-q_0-1}}v_i(t)\end{aligned} \tag{5.18}$$

If we define

$$l_i-q_0=\eta_i \tag{5.19}$$

$$z_i(t)=\alpha_i y(t) \tag{5.20}$$

$$L_i(y(t),u(t)) = (s+\lambda)\left\{\frac{f_i(s)-s^{l_i}}{f_i(s)}(\alpha_i y(t)) + \frac{\Phi_i(s)}{f_i(s)}y(t) + \frac{\Psi_i(s)}{f_i(s)}u(t)\right\} \tag{5.21}$$

then, (5.18) can be simplified as

$$\dot{z}_i(t) + \lambda z_i(t) = L_i(y(t),u(t)) + \frac{k_0}{(s+\lambda)^{\eta_i-1}}v_i(t) \tag{5.22}$$

It should be pointed out that $z_i(t)$ and $L_i(y(t),u(t))$ are available signals. Equation (5.22) will be employed to estimate the disturbance $v_i(t)$.

### 5.3.2 Disturbance Estimation

It should be mentioned that the discussion in this section inherently assumes zero initial conditions for all internal sates of the system. Fortunately, this treatment does not lose any generality since, for a stable closed-loop linear system, non-zero initial conditions only contribute to the solution of the state (or the system output) an additive term which decays to zero exponentially. Thus, all of the initial conditions the filtered inputs, filtered outputs and filtered disturbances can be assumed to be zeros (suppose $t_0$ is the starting time).

In this section, based on the Differential Equation (5.22), the disturbance $v_i(t)$ is estimated by a procedure similar to that of Theorem 4.2. As the VSS method is employed, the bounds of the filters of the disturbance $v_i(t)$ must be estimated.

For positive constant $\lambda$, similar to Lemma 4.1, the next lemma is obtained.

**Lemma 5.2** The upper bound of $\left|\frac{1}{(s+\lambda)^j}v_i(t)\right|$ can be estimated as

$$\left|\frac{1}{(s+\lambda)^j}v_i(t)\right| \le \frac{1}{(s+\lambda)^j}\rho(y(t),u(t),t) \triangleq \omega_j(t), \text{ for } j \ge 0 \tag{5.25}$$

Similar to Theorem 4.2, based on Equation (5.22), the disturbance $v_i(t)$ can be estimated by the next theorem.



**Theorem 5.1** Construct the differential equations

$$\dot{\hat{z}}_i(t) + \lambda\hat{z}_i(t) = L_i(y(t),u(t)) + k_0 w_{i1}(t),\ \hat{z}_i(t_0) = 0 \tag{5.24}$$

$$\dot{\hat{w}}_{i,\iota_i-1}(t) + \lambda\hat{w}_{i,\iota_i-1}(t) = w_{i,\iota_i}(t),\ \hat{w}_{i,\iota_i-1}(t_0) = 0 \tag{5.25}$$

where $\hat{z}_i(t)$ and $\hat{w}_{i,\iota_i-1}(t)$ $(1 < \iota_i \le \eta_i)$ are the signals which can be obtained by solving the Differential Equations (5.24) and (5.25), $w_{i1}(t)$ and $w_{i,\iota_i}(t)$ are the inputs described by

$$w_{i1}(t) = \frac{k_0\{z_i(t)-\hat{z}_i(t)\}\omega^2_{\eta_i-1}(t)}{\left|k_0\{z_i(t)-\hat{z}_i(t)\}\right|\omega_{\eta_i-1}(t)+\delta_{i1}},\ \delta_{i1} > 0 \tag{5.26}$$

and

$$w_{i\iota_i}(t) = \frac{\{w_{i,\iota_i-1}(t)-\hat{w}_{i,\iota_i-1}(t)\}\omega^2_{\eta_i-\iota_i}(t)}{\left|w_{i,\iota_i-1}(t)-\hat{w}_{i,\iota_i-1}(t)\right|\omega_{\eta_i-\iota_i}(t)+\delta_{i\iota_i}},\ \delta_{i\iota_i} > 0 \text{ (for } 1<\iota_i \le \eta_i) \tag{5.27}$$

respectively. It can be concluded that $w_{i,\iota_i}(t)$ are the corresponding approximate estimates of $\frac{1}{(s+\lambda)^{\eta_i-\iota_i}}v_i(t)$ for $1 \le \iota_i \le \eta_i$ (Particularly, $w_{i,\eta_i}(t)$ is the estimate of $v_i(t)$). Therefore, there exist $T_i > t_0$ and $\varepsilon_i(\delta_{i1},\cdots,\delta_{i,\eta_i}) > 0$ such that

$$\left|\frac{1}{(s+\lambda)^{\eta_i-\iota_i}}v_i(t) - w_{i,\iota_i}(t)\right| \le \varepsilon_i(\delta_{i1},\cdots,\delta_{i,\eta_i}) \tag{5.28}$$

where $\varepsilon_i(\delta_{i1},\cdots,\delta_{i,\eta_i}) \to 0$ as $\sum_{j=1}^{\eta_i}\delta_{ij} \to 0$.

**Proof**: For proof, see Theorem 4.2.

The block diagram of the disturbances estimator for MIMO minimum phase systems with arbitrarily relative degrees is shown in Figure 5.1, where the functions $f_{i1}(\cdot)$ and $f_{i,\iota_i}(\cdot)$ (for $1 \le i \le p$, $2 \le \iota_i \le \eta_i$) are defined as

$f_{i1}(\beta) = \frac{k_0\beta\omega^2_{\eta_i-1}(t)}{|k_0\beta|\omega_{\eta_i-1}(t)+\delta_{i1}}$ and $f_{i,\iota_i}(\beta) = \frac{\beta\omega^2_{\eta_i-\iota_i}(t)}{|\beta|\omega_{\eta_i-\iota_i}(t)+\delta_{i\iota_i}}$ for an argument $\beta$.

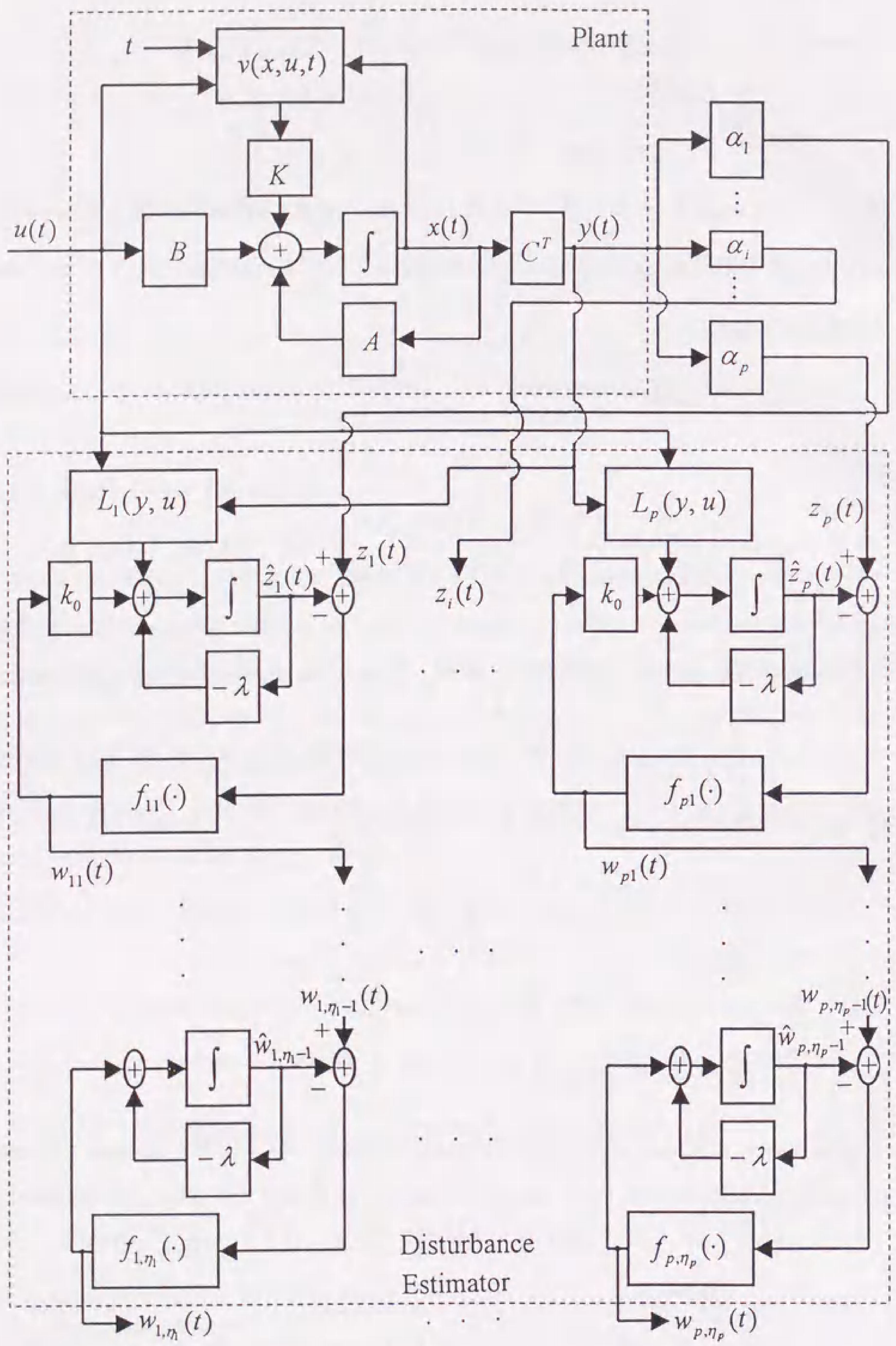



Fig. 5.1 The block diagram of the disturbance estimator for MIMO systems.



## 5.4 Construction of the State Observer

In this section, a robust state observer will be formulated by using the obtained estimates of the disturbances.

We introduce a stable block diagonal matrix defined by

$$G=\left[\begin{array}{ccc|ccc|c|ccc}
-g_{11}^{(1)} & 1 & & & & & & & & \\
\vdots & & \ddots\ 1 & & 0 & & \cdots & & 0 & \\
-g_{n_1 1}^{(1)} & 0\cdots 0 & & & & & & & & \\
\hline
 & & & -g_{12}^{(2)} & 1 & & & & & \\
 & 0 & & \vdots & & \ddots\ 1 & \cdots & & 0 & \\
 & & & -g_{n_2 2}^{(2)} & 0\cdots 0 & & & & & \\
\hline
 & \cdots & & & \cdots & & \ddots & & \cdots & \\
\hline
 & & & & & & & -g_{1p}^{(p)} & 1 & \\
 & 0 & & & 0 & & \cdots & \vdots & & \ddots\ 1 \\
 & & & & & & & -g_{n_p p}^{(p)} & 0\cdots 0 & 
\end{array}\right] \tag{5.29}$$

Now, consider the robust state observer of system (5.1) described by

$$\begin{cases}\dot{\hat{x}}(t)=A\hat{x}(t)+Bu(t)+Kw(t)+(A-G)C\{y(t)-\hat{y}(t)\}\\ \hat{y}(t)=C^T\hat{x}(t)\end{cases} \tag{5.30}$$

where $\hat{x}(t)$ is the estimated state with the initial condition $\hat{x}(t_0)=0$; $w(t)$ is defined as

$$w(t)=\lfloor w_{1\eta_1},\ \cdots,\ w_{p\eta_p}\rfloor^T \tag{5.31}$$

in which $w_{i\eta_i}$ are defined in Theorem 5.1 for $1\le i\le p$.

Let

$$\bar{x}(t)=x(t)-\hat{x}(t) \tag{5.32}$$

be the state error. From (1) and (5.30),

$$\dot{\bar{x}}(t)=G\bar{x}(t)+K\{v(t)-w(t)\} \tag{5.33}$$

where the fact $A-(A-G)CC^T=G$ is employed.

As $w_{i,\eta_i}(t)$ are the corresponding approximate estimates of $v_i(t)$ for $1\le i\le p$, by employing the stability of matrix $G$, it can be concluded that there exist $T > t_0$ and $\varepsilon(\delta_{11},\cdots,\delta_{1,\eta_1},\cdots,\delta_{p1},\cdots,\delta_{p,\eta_p}) > 0$ such that

$$|\bar{x}(t)| \le \varepsilon(\delta_{11},\cdots,\delta_{1,\eta_1},\cdots,\delta_{p1},\cdots,\delta_{p,\eta_p}) \tag{5.34}$$

where $\varepsilon(\delta_{11},\cdots,\delta_{1,\eta_1},\cdots,\delta_{p1},\cdots,\delta_{p,\eta_p}) \to 0$ as $\sum_{i=1}^{p}\sum_{j=1}^{\eta_i}\delta_{ij} \to 0$. Thus, a state observer is approximately formulated.

**Theorem 5.2** The VSS observer of system (5.1) can be approximately constructed by Equation (5.30), where the entries of $w(t)$ are characterized by Theorem 5.1 .

## 5.5 Design of the Pole Assignment Controller

In this section, we assume that the disturbance is matched, i.e. $B = K$. For simplicity, we also assume that the upper bound $\rho(u, y, t)$ of the disturbance has no relation with the input $u(t)$.

Let the desired closed-loop system poles are determined by the roots of the Hurwitz polynomial

$$d(s) = s^n + d_1 s^{n-1} + \cdots + d_n \tag{5.35}$$

Consider a linear state feedback control law defined by

$$u(t) = -Q^T \hat{x}(t) - w(t) + \gamma(t) \tag{5.36}$$

where $Q$ is an $n \times p$ feedback gain matrix, $\hat{x}(t)$ is the estimated state obtained in Theorem 5.2, $\gamma(t)$ is a $p \times 1$ uniformly bounded external input and $w(t)$ is employed to cancel the disturbance $v(t)$. With an appropriate choice of the feedback gain matrix $Q$, the characteristic equation of the closed-loop system becomes

$$\det(sI - A + BQ^T) = d(s) \tag{5.37}$$

By using the control law (6.36), the closed-loop system is described by



$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = \begin{bmatrix} A & -BQ^T \\ (A-G)C^T C & A-(A-G)C^T C - BQ^T \end{bmatrix}\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}(v(t) - w(t)) + \begin{bmatrix} B \\ B \end{bmatrix}\gamma(t) \tag{5.38}$$

Since

$$\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \tag{5.39}$$

by using the fact

$$A - (A-G)C^T C = G \tag{5.40}$$

(5.38) gives

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix} = \begin{bmatrix} A - BQ^T & BQ^T \\ 0 & G \end{bmatrix}\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}(v(t) - w(t)) + \begin{bmatrix} B \\ 0 \end{bmatrix}\gamma(t) \tag{5.41}$$

where $\begin{bmatrix} A - BQ^T & BQ^T \\ 0 & G \end{bmatrix}$ is a stable matrix. Since $\gamma(t)$ is a uniformly bounded signal and $v(t) - w(t)$ is bounded as $t \to \infty$ (see (5.28)), it can be easily concluded that $\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix}$ is uniformly bounded on $[t_0, \infty)$. Further, the input determined in (6.36) is uniformly bounded.

Let

$$y_d(t) \triangleq C(sI - A + BQ^T)^{-1} B\gamma(t) \tag{5.42}$$

Then, by applying Theorems 5.1 and 5.2 to (5.41), it can be concluded that there exist $\bar{T} > t_0$ and $\bar{\varepsilon}(\delta_{11},\cdots,\delta_{1,\eta_1},\cdots,\delta_{p1},\cdots,\delta_{p,\eta_p}) > 0$ such that

$$|y(t) - y_d(t)| \le \bar{\varepsilon}(\delta_{11},\cdots,\delta_{1,\eta_1},\cdots,\delta_{p1},\cdots,\delta_{p,\eta_p}) \tag{5.43}$$

where $\bar{\varepsilon}(\delta_{11},\cdots,\delta_{1,\eta_1},\cdots,\delta_{p1},\cdots,\delta_{p,\eta_p}) \to 0$ as $\sum_{i=1}^{p}\sum_{j=1}^{\eta_i}\delta_{ij} \to 0$, i.e. the pole assignment can be approximately achieved as $t \to \infty$. Therefore, we have the next theorem.

**Theorem 5.3** Consider the overall control system consisting of the controlled system (5.1), the disturbance estimator described in Theorem 5.1, the state observer

(5.30) and the control input (5.36). Then, the state of the overall control system is uniformly bounded and the transfer function of the closed-loop system approaches $\Omega_d(t)$.

The block diagram of the pole assignment controller for the MIMO minimum phase systems with arbitrary relative degrees is shown in Figure 5.2.

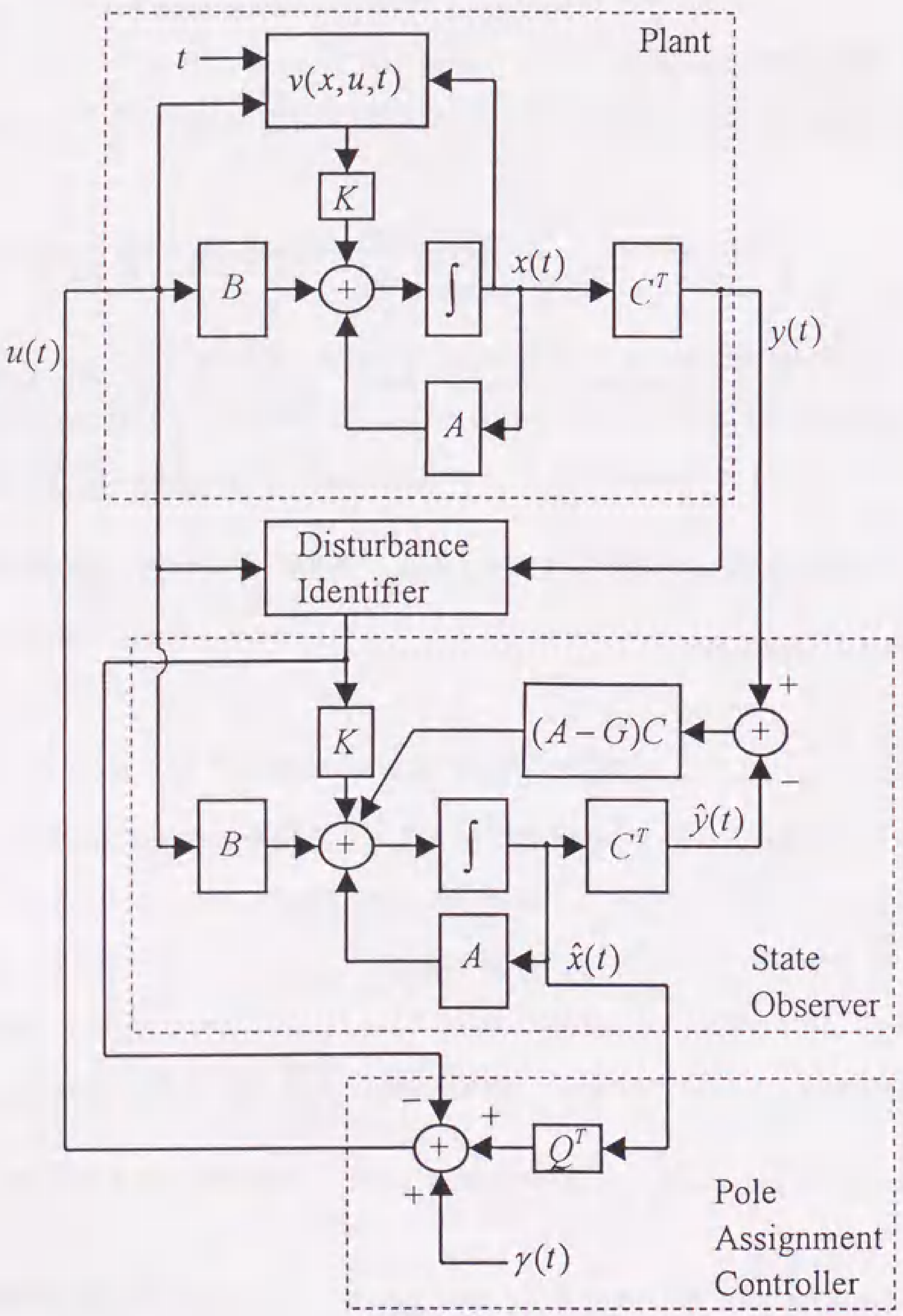



Fig. 5.2 The block diagram of the pole assignment controller for MIMO systems.



The proposed algorithm of the disturbance estimation method and its applications for MIMO minimum phase systems with arbitrarily relative degrees is illustrated in Table 5.1.

Table 5.1 The algorithm of disturbance estimation and its applications for MIMO systems

| | |
|---|---|
| Plant | $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Kv(x,u,t) \\ y(t) = C^T x(t) \end{cases}$ |
| *A priori* information | $\lVert v(x,u,t) \rVert_2 \le \rho(y,u,t)$ |
| Upper bound of $\left\lvert \frac{1}{(s+\lambda)^j} v_i(t) \right\rvert$ | $\left\lvert \frac{1}{(s+\lambda)^j} v_i(t) \right\rvert \le \frac{1}{(s+\lambda)^j} \rho(y(t),u(t),t) \triangleq \omega_j(t)$ |
| Transformed plant | $\begin{cases} s^{l_1}(\alpha_1 y(t)) = \Phi_1(s)y(t) + \Psi_1(s)u(t) + k(s)v_1(t) \\ \vdots \\ s^{l_p}(\alpha_p y(t)) = \Phi_p(s)y(t) + \Psi_p(s)u(t) + k(s)v_p(t) \end{cases}$ |
| Estimate of $v_i(t)$ | $w_{i,\eta_i}(t)$ (Based on the i-th equation of the transformed plant, the algorithm in Table 4.1 is used.) |
| Estimate of $v(t)$ | $w(t) = \left[ w_{1\eta_1}, \ \cdots, \ w_{p\eta_p} \right]^T$ |
| State observer | $\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + Kw(t) + (A-G)C\{y(t) - \hat{y}(t)\} \\ \hat{y}(t) = C^T \hat{x}(t) \end{cases}$ |
| Desired poles | Zeros of $d(s) = s^n + d_1 s^{n-1} + \cdots + d_n$ |
| Feedback gain | $Q$ ($n \times p$ matrix) |
| External input | $\gamma(t)$ ($p \times 1$ uniformly bounded vector) |
| Pole assignment control input | $u(t) = -Q^T \hat{x}(t) - w(t) + \gamma(t)$ |

### 5.6 Design Example and Simulation Results

In this section, an unstable MIMO minimum phase system is presented to show the design procedure and the computer simulation results.

**Example**: Consider an unstable minimum phase system described by

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 2 & -1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} (u(t) + v(t)) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix} \end{cases} \tag{5.44}$$

where the disturbance vector $v(t)$ is governed by

$$v(t) = \begin{bmatrix} (0.5\sin\frac{t}{2} + 0.2\sin t)\dfrac{0.4\|y(t)\|_2^2}{\|y(t)\|_2 + 1} \\ (0.3\sin 3t + 0.7\sin 2t)\exp(-0.5\|y(t)\|_2) \end{bmatrix} \tag{5.45}$$

its upper bound is known as $\rho(y(t),t) = 1 + 0.5\|y(t)\|_2$. Suppose the starting time is $t_0 = 0$. The unknown initial state is assumed to be $x(0) = [1, -2, 1, -2, -3]^T$. The desired closed-loop poles are determined by the roots of the polynomial

$$\begin{aligned} d(s) &= (s+1)(s+2)(s^3 + 6s^2 + 9s + 12) \\ &= s^5 + 9s^4 + 29s^3 + 51s^2 + 54s + 24 \end{aligned} \tag{5.46}$$

The external reference input is described by

$$\gamma(t) = \begin{bmatrix} 2\sin t + 3\sin\frac{t}{2} \\ 2\sin(t + \frac{\pi}{4}) \end{bmatrix} \tag{5.47}$$

Then, the feedback gain $Q$ can be calculated as

$$Q = \begin{bmatrix} 3 & 2 & 2 & 2 & 1 \\ -2 & 1 & -1 & 1 & 1 \end{bmatrix}^T \tag{5.48}$$

Now, let us estimate the disturbances. The given system can be written in the following compact form



$$\begin{bmatrix} s^2 + s - 1 & -(2s+3) \\ -(2s^2 + s + 2) & s^3 + s^2 + 2s - 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ s+1 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) + v_1(t) \\ u_2(t) + v_2(t) \end{bmatrix} \tag{5.49}$$

Thus, multiplying the above equation by $adj(\begin{bmatrix} 2 & -1 \\ s+1 & 1 \end{bmatrix}) = \begin{bmatrix} 1 & 1 \\ -(s+1) & 2 \end{bmatrix}$ yields

$$\begin{bmatrix} -(s^2 + 3) & s^3 + s^2 - 4 \\ -(s^3 + 6s^2 + 2s + 3) & 2s^3 + 4s^2 + 9s + 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = (s+3) \begin{bmatrix} u_1(t) + v_1(t) \\ u_2(t) + v_2(t) \end{bmatrix} \tag{5.50}$$

Choose the Hurwitz polynomial $f(s)$ in (5.16) as $f(s) = (s+1)^2(s+3)$. Then, dividing the above equation by $f(s)$ and rearranging it, we have

$$\begin{aligned} & y_2(t) + \frac{s^3 + s^2 - 4 - (s+1)^2(s+3)}{(s+1)^2(s+3)} y_2(t) - \frac{s^2 + 3}{(s+1)^2(s+3)} y_1(t) \\ &= \frac{1}{(s+1)^2}(u_1(t) + v_1(t)) \end{aligned} \tag{5.51}$$

and

$$\begin{aligned} & 2y_2(t) - y_1(t) + \frac{s^3 - (s+1)^2(s+3)}{(s+1)^2(s+3)}(2y_2(t) - y_1(t)) \\ & + \frac{(4s^2 + 9s + 1)y_2(t) - (6s^2 + 2s + 3)y_1(t)}{(s+1)^2(s+3)} = \frac{1}{(s+1)^2}(u_2(t) + v_2(t)) \end{aligned} \tag{5.52}$$

Multiplying the both sides of the above two equations by $(s+1)$ yields

$$\begin{aligned} \dot{y}_2(t) + y_2(t) &= \frac{s^2 + 3}{(s+1)(s+3)} y_1(t) + \frac{4s^2 + 7s + 7}{(s+1)(s+3)} y_2(t) \\ & + \frac{1}{s+1} u_1(t) + \frac{1}{s+1} v_1(t) \end{aligned} \tag{5.53}$$

and

$$\begin{aligned} \frac{d}{dt}(2y_2(t) - y_1(t)) + (2y_2(t) - y_1(t)) &= \frac{s^2 - 5s}{(s+1)(s+3)} y_1(t) \\ & + \frac{6s^2 + 5s + 5}{(s+1)(s+3)} y_2(t) + \frac{1}{s+1} u_2(t) + \frac{1}{s+1} v_2(t) \end{aligned} \tag{5.54}$$

Corresponding to the above obtained equations, the estimated differential equations are constructed as

$$\dot{\hat{z}}_1(t) + \hat{z}_1(t) = \frac{s^2+3}{(s+1)(s+3)} y_1(t) + \frac{4s^2+7s+7}{(s+1)(s+3)} y_2(t)$$

$$+ \frac{1}{s+1} u_1(t) + w_{11}(t), \quad \hat{z}_1(0) = 0 \tag{5.55}$$

and

$$\dot{\hat{z}}_2(t) + \hat{z}_2(t) = \frac{s^2-5s}{(s+1)(s+3)} y_1(t) + \frac{6s^2+5s+5}{(s+1)(s+3)} y_2(t)$$

$$+ \frac{1}{s+1} u_2(t) + w_{21}(t), \quad \hat{z}_2(0) = 0 \tag{5.56}$$

where $w_{11}(t)$ and $w_{21}(t)$ are given as

$$w_{11}(t) = \frac{\left\{\frac{1}{s+1}(1+0.5\|y(t)\|_2)\right\}^2 \{y_2(t) - \hat{z}_1(t)\}}{\left\{\frac{1}{s+1}(1+0.5\|y(t)\|_2)\right\} |y_2(t) - \hat{z}_1(t)| + \delta_{11}} \tag{5.57}$$

and

$$w_{21}(t) = \frac{\left\{\frac{1}{s+1}(1+0.5\|y(t)\|_2)\right\}^2 \{2y_2(t) - y_1(t) - \hat{z}_2(t)\}}{\left\{\frac{1}{s+1}(1+0.5\|y(t)\|_2)\right\} |2y_2(t) - y_1(t) - \hat{z}_2(t)| + \delta_{21}} \tag{5.58}$$

respectively. Thus, $w_{11}(t)$ and $w_{21}(t)$ can be regarded as the approximate estimates of $\frac{1}{s+1} v_1(t)$ and $\frac{1}{s+1} v_2(t)$ respectively.

Now, based on the estimates $w_{11}(t)$ and $w_{21}(t)$ of the first order filters of the disturbances, let us estimate the disturbances $v_1(t)$ and $v_2(t)$. The estimated equations are constructed as

$$\dot{\hat{w}}_{11}(t) + \hat{w}_{11}(t) = w_{12}(t), \qquad \hat{w}_{11}(0) = 0 \tag{5.59}$$

and

$$\dot{\hat{w}}_{21}(t) + \hat{w}_{21}(t) = w_{22}(t), \qquad \hat{w}_{21}(0) = 0 \tag{5.60}$$



where $w_{12}(t)$ and $w_{22}(t)$ are chosen as

$$w_{12}(t) = \frac{\{1+0.5\|y(t)\|_2\}^2 \{w_{11}(t) - \hat{w}_{11}(t)\}}{\{1+0.5\|y(t)\|_2\} |w_{11}(t) - \hat{w}_{11}(t)| + \delta_{12}} \tag{5.61}$$

and

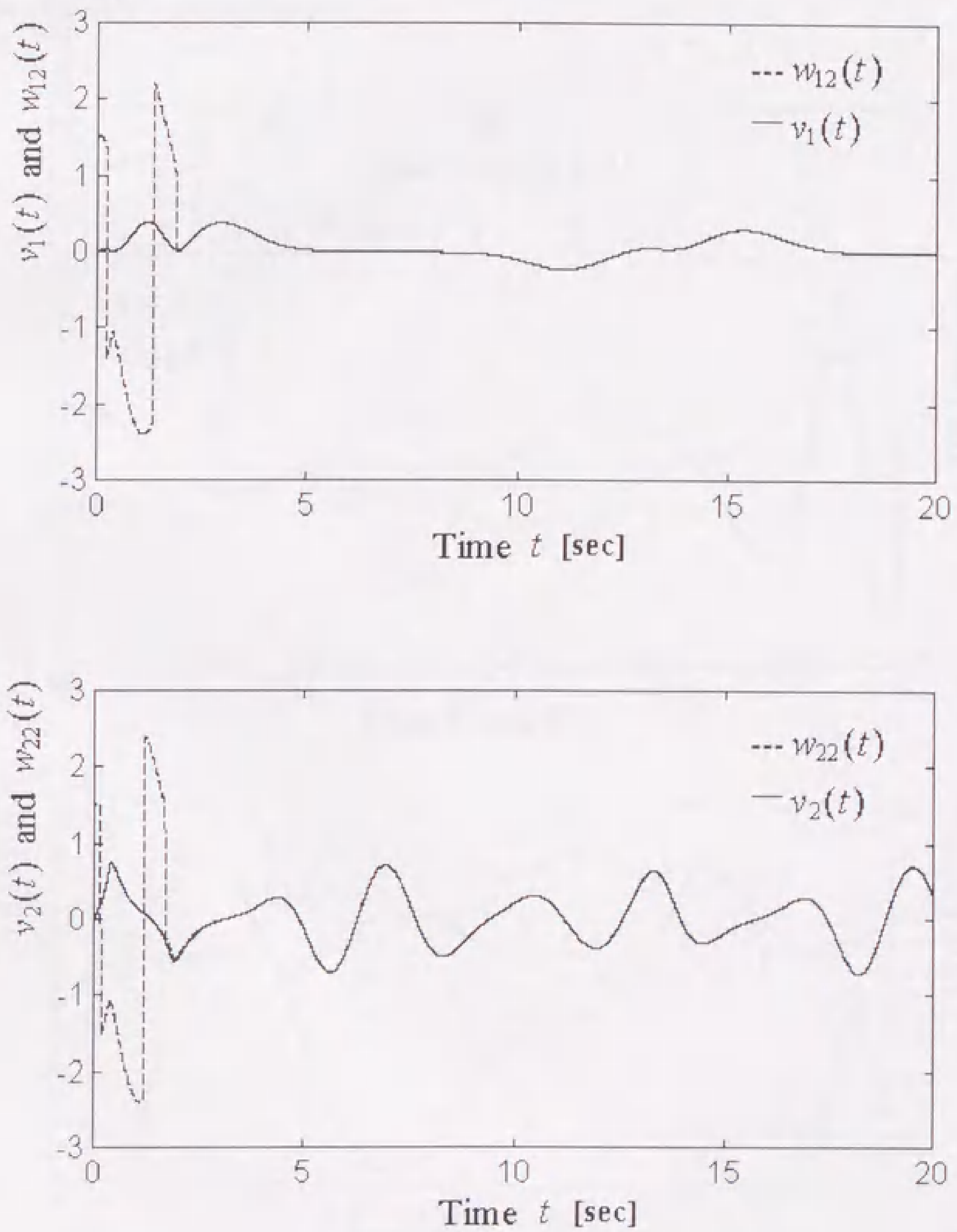



Fig. 5.3 The disturbances $v_1(t)$ and $v_2(t)$ and their corresponding estimates $w_{12}(t)$ and $w_{22}(t)$.

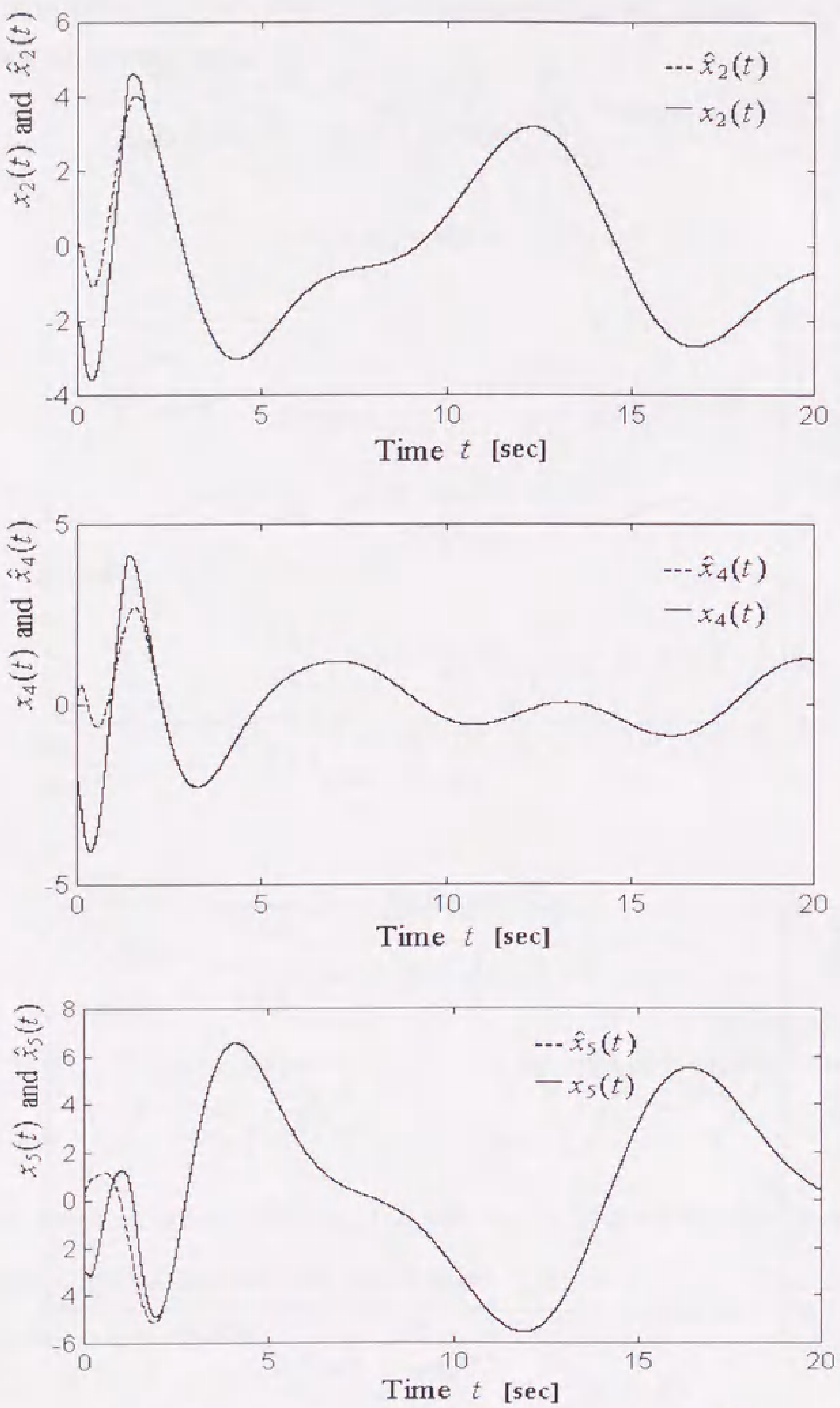



Fig. 5.4 The genuine states $x_2(t)$, $x_4(t)$ and $x_5(t)$ and their corresponding estimates $\hat{x}_2(t)$, $\hat{x}_4(t)$ and $\hat{x}_5(t)$.



$$w_{22}(t) = \frac{\{1+0.5\|y(t)\|_2\}^2\{w_{21}(t)-\hat{w}_{21}(t)\}}{\{1+0.5\|y(t)\|_2\}|w_{21}(t)-\hat{w}_{21}(t)|+\delta_{22}} \tag{5.62}$$

respectively. From the results of Theorem 5.1, $w_{12}(t)$ and $w_{22}(t)$ can be regarded as the estimates of the disturbances $v_1(t)$ and $v_2(t)$ respectively.

Based on the obtained estimates of the disturbances, from the results of Theorem 5.2, the observer is formulated as

$$\begin{cases} \dot{\hat{x}}(t) = \begin{bmatrix} -1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 & 0 \\ 2 & -1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} (u(t) + \begin{bmatrix} w_{12}(t) \\ w_{22}(t) \end{bmatrix}) \\ \quad + (\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 6 \\ 0 & 12 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ -1 & -3 \\ -2 & 1 \\ -1 & 2 \\ -2 & -1 \end{bmatrix})(y(t) - \hat{y}(t)), \qquad \hat{x}(0) = 0 \\ \hat{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \hat{x}(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_3(t) \end{bmatrix} \end{cases} \tag{5.63}$$

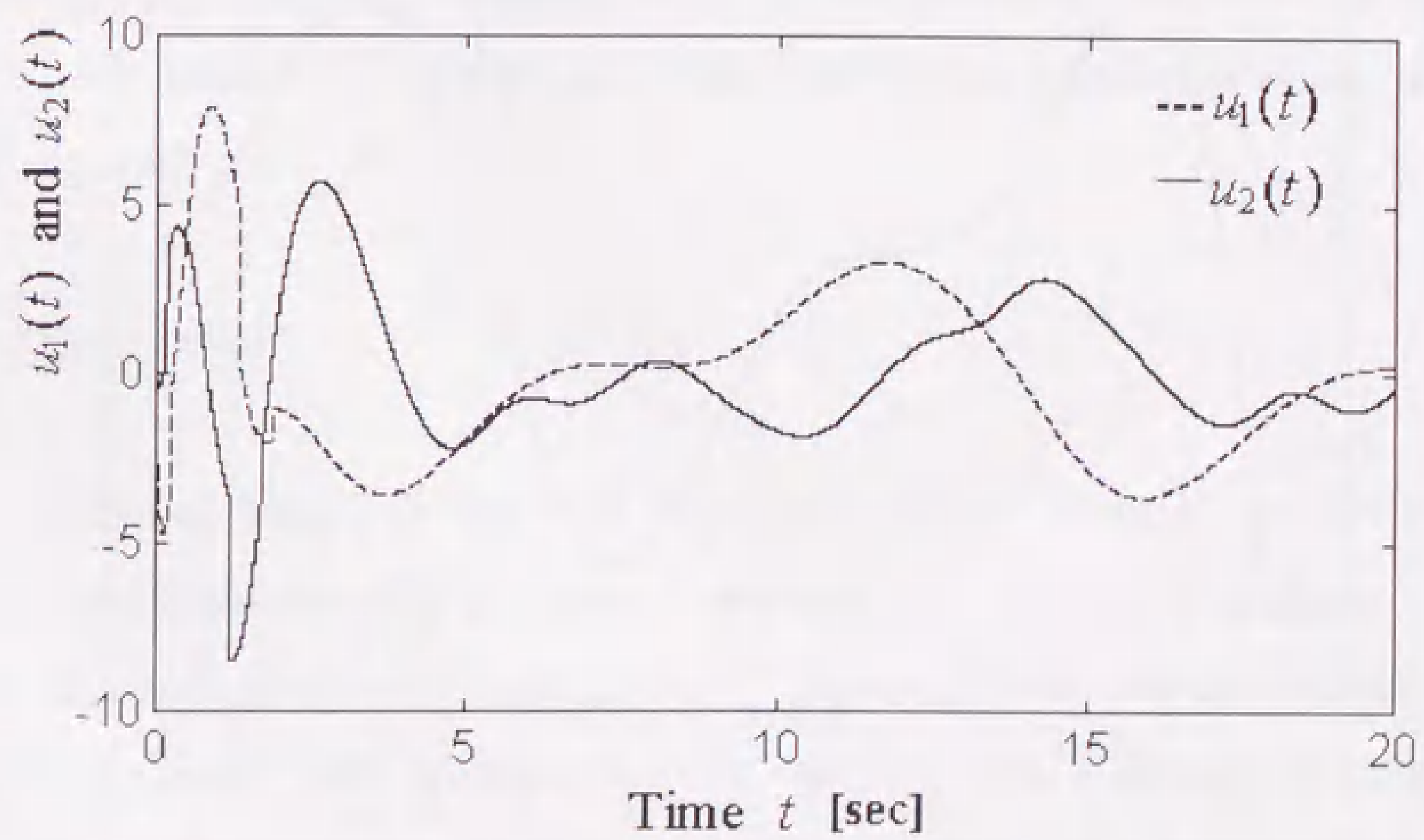
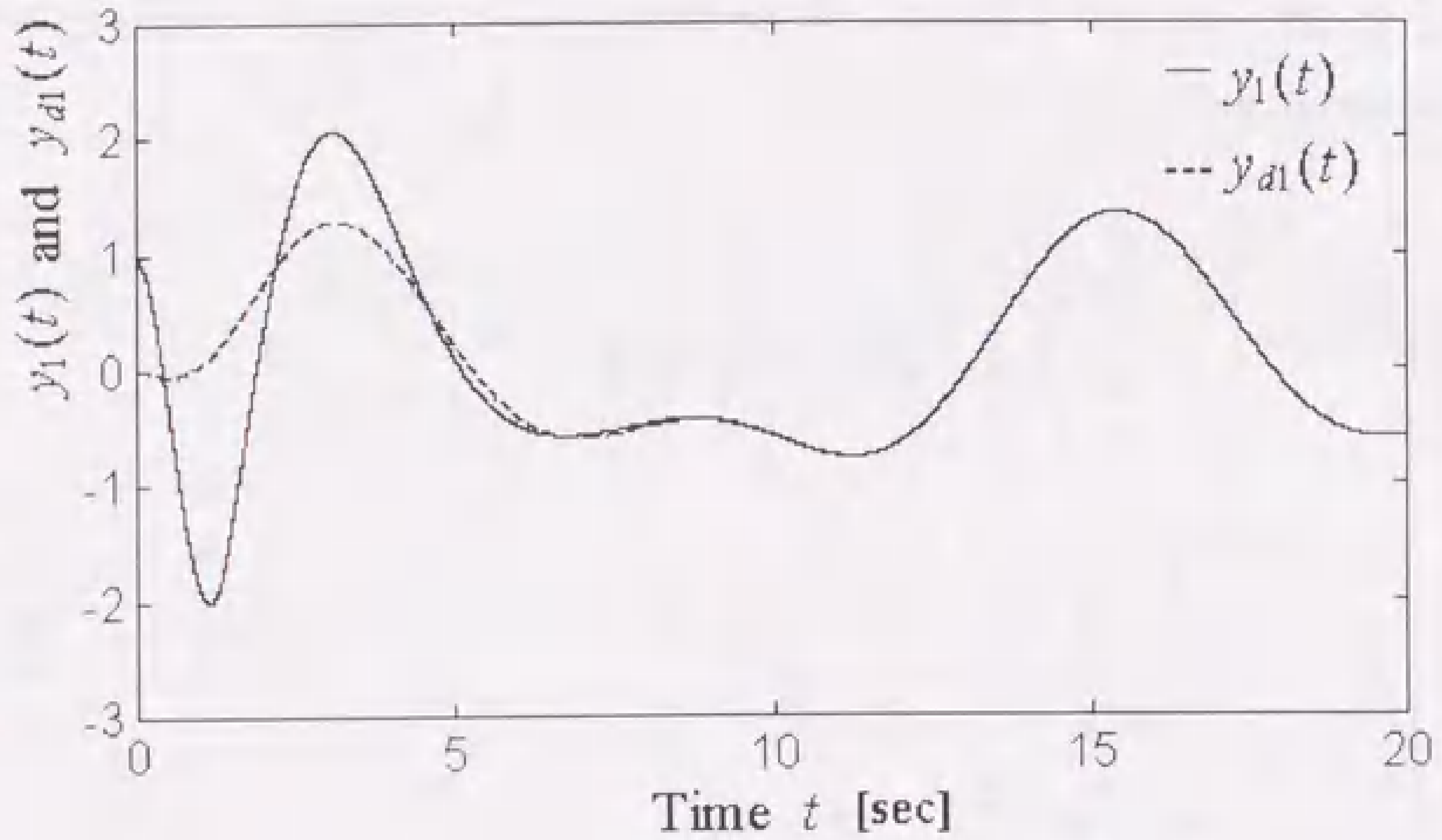



Fig. 5.5 The pole placement control inputs $u_1(t)$ and $u_2(t)$.

where the stable matrix $G$ is chosen as $G = \left[\begin{array}{cc|ccc} -2 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -6 & 1 & 0 \\ 0 & 0 & -12 & 0 & 1 \\ 0 & 0 & -8 & 0 & 0 \end{array}\right]$.



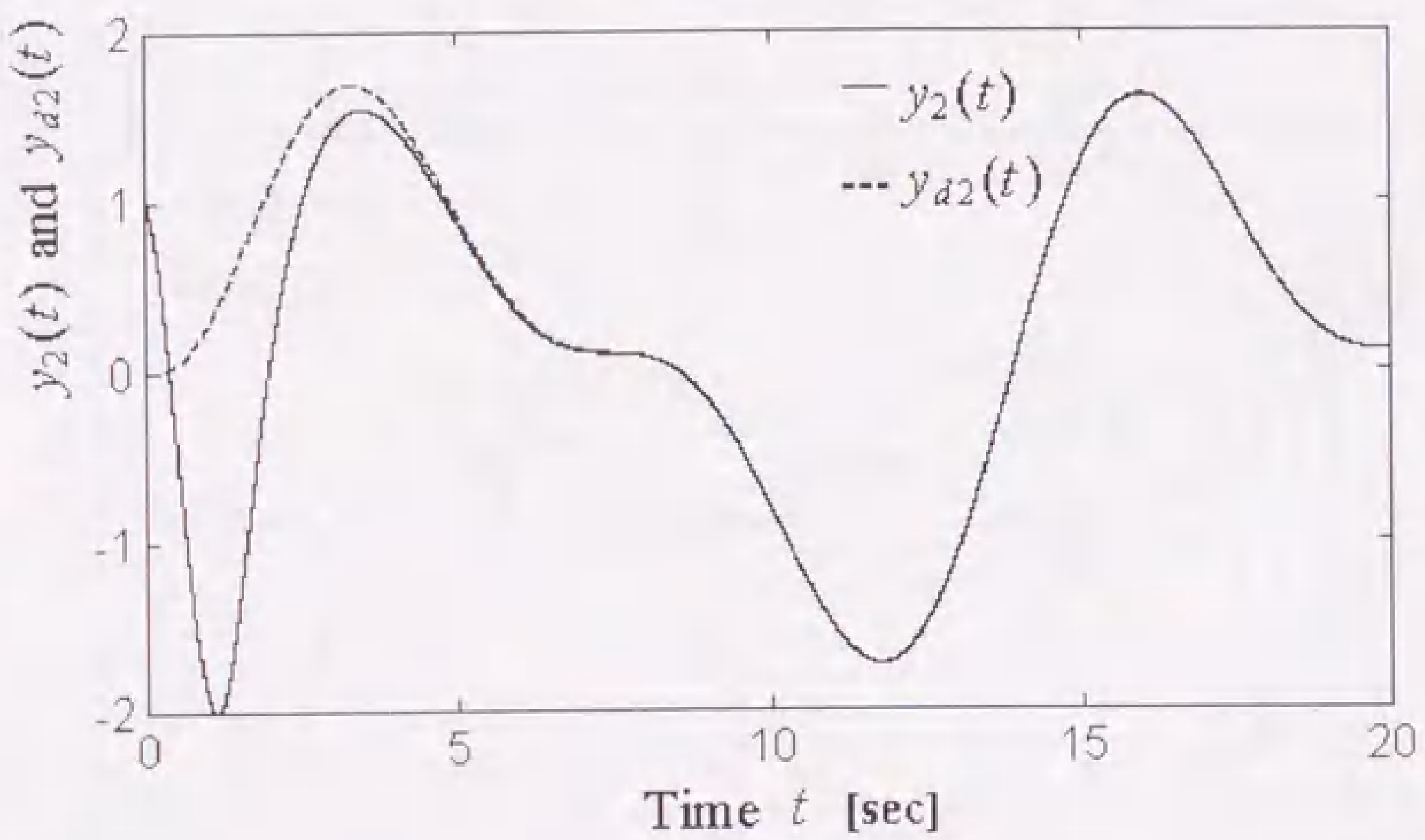




Fig. 5.6 The outputs $y_1(t)$ and $y_2(t)$ of the controlled system and the comparisons to their corresponding desired outputs $y_{d1}(t)$ and $y_{d2}(t)$.



Therefore, the pole placement controller can be constructed as

$$u(t) = -\begin{bmatrix} 3 & 2 & 2 & 2 & 1 \\ -2 & 1 & -1 & 1 & 1 \end{bmatrix}\hat{x}(t) + \gamma(t) - \begin{bmatrix} w_{12}(t) \\ w_{22}(t) \end{bmatrix} \tag{5.64}$$

The desired outputs are governed by $y_d(t) \triangleq C(sI - A + BQ^T)^{-1}B\gamma(t)$, where the numerical matrices $A$, $B$ and $C$ are described in the given system, and $Q$ are calculated before.

In the presented computer simulation process, the parameters $\delta_{ij}$ are all chosen as $0.001$, the sampling period is chosen as $5\times10^{-4}$ second. Figure 5.3 shows the real disturbances and their corresponding estimates. It is obvious that after 2.5s almost perfect identifications are made. Figure 5.4 shows the states $x_2(t)$, $x_4(t)$ and $x_5(t)$ plotted for comparison against their corresponding estimates $\hat{x}_2(t)$, $\hat{x}_4(t)$ and $\hat{x}_5(t)$ (We need not to investigate $x_1(t)$ and $x_3(t)$ as they are the measurable signals $y_1(t)$ and $y_2(t)$, respectively). It can be seen that after 3s almost perfect estimations are obtained. Figure 5.5 shows the pole placement control inputs $u_1(t)$ and $u_2(t)$. Figure 5.6 shows the outputs of the controlled system and the desired outputs. It can be seen that after 7s the tracking problems are almost perfectly achieved.

### 5.7 Conclusions

In this chapter, based on the VSS equivalent control method, the disturbance (or system nonlinearities plus any model uncertainties) estimation problem which has been regarded unsolvable until now is overcome for MIMO minimum phase dynamical systems with arbitrarily relative degrees. By estimating the higher order filters of the disturbances, the lower order filters of the disturbances, eventually the disturbances, are inductively estimated. The estimated disturbances are then used to generate a robust state observer and a pole placement controller which also has the function to cancel the disturbances. Design example and simulation results show that the proposed algorithm is effective for practical applications. In order to be implement by a digital computer, the proposed algorithms are approximately formulated by differentiable functions.

The obtained results are expected to be extended to MIMO nonminimum phase uncertain systems with arbitrarily relative degrees. Further, for much more complicated systems, such as the uncertain systems with delays, descriptor systems, affine parameter-dependent models, a similar formulation is expected to be derived. Also, for the systems with relatively large stochastic disturbances or measurement noises, the proposed disturbance observer should be modified.

# Chapter 6

# Implicit State Estimators and Their Application to Pole Assignment Controllers for Systems with Uncertainties

## 6.1 Introduction

The problem of controlling uncertain dynamical systems subject to external disturbances has been one of the interesting topics recently. Many of the proponents of the associated theoretical developments have found it convenient to assume that the system state vector is available for use by the control scheme [55,57]. In practice, it is not always possible to measure the state vector. In such cases, either a design method based solely upon the input and output information is required, or a suitable estimate of the state vector has to be constructed for use in the original control law. This chapter considers the latter approach.

As for the state estimation problem for the systems with uncertainties, relatively few authors have considered it. It is known that the VSS theory has many advantages in solving the problems with uncertainties [9,55,57,71]. But about its application to the state and disturbance estimation problems, very few theoretical works have been reported.

Utkin presents a discontinuous observer by forcing the error between the estimated and measured outputs to exhibit a sliding mode [56]. And it is pointed out that the proposed method finds difficulty in selecting the switched gain owing to the uncertainty of the initial condition. Walcott *et al* use a Lyapunov-based approach to formulate an observer in the presence of bounded disturbances [59-61]. Edwards and Spurgeon [11] effectively consider the problem first proposed in [60]. However, all these results are subject to MIMO minimum phase systems with relative degree one (with respect to the relation of disturbance-output). For the uncertain systems (even for SISO uncertain systems) with higher relative degrees, no reports has been found about the design problems of the state observers.

Based on the results of Chapters 4 and 5, this chapter deals with the robust observer design problems for minimum phase dynamical systems with arbitrarily relative degrees (with respect to the relation between the disturbance and the output). In section 6.2, the problem is formulated. In section 6.3, by introducing a filter with $n$ different poles, a new implicit observer is formulated by employing the first order filters of the input, output and disturbance. In section 6.4, the robust state observers are formulated for the uncertain systems with arbitrarily relative degrees. For plants with relative degree one, the observer is constructed without any *a priori* knowledge of the disturbance. For plants with higher relative degrees, the observer is constructed by the estimates of the first order filters of the disturbance and the filters of the input and the output. In section 6.5, the obtained observer and the estimated disturbance are applied to a pole assignment controller which also has the function to cancel the disturbance. In section 6.6, numerical examples are given to illustrate the proposed algorithms. Finally, the implicit state observer techniques are extended to MIMO minimum phase dynamical systems with uncertainties.

## 6.2 Problem Formulation

Consider the system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) + kv(x,u,t), \; x(t_0) = x_0 \\ y(t) = c^T x(t) \end{cases} \tag{6.1}$$



where $x(t)$ is an unknown state vector with known dimension $n$; $t_0$ is the starting time, $x_0$ is the unknown initial state; $u(t)$ and $y(t)$ are scalar input and output, respectively; $v(x,u,t)$ is a signal composed of the model uncertainties, the nonlinear parts of the system and the disturbances; and $A$, $b$, $k$, $c$ are known matrices and are described in the observable canonical form

$$A = \begin{bmatrix} -a_1 & I \\ \vdots & \cdots \\ -a_n & 0 \end{bmatrix} \triangleq \begin{bmatrix} -a & \begin{matrix} I \\ 0 \end{matrix} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \; k = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6.2}$$

**Assumption 6.1** The signal $v(x,u,t)$ is bounded by a known function of $y(t)$, $u(t)$ and $t$

$$|v(x,u,t)| \le \rho(y,u,t) \tag{6.3}$$

where $\rho(y,u,t)$ is a known function.

**Assumption 6.2** The polynomial

$$k(s) = k_1 s^{n-1} + \cdots + k_{n-1}s + k_n \tag{6.4}$$

is a Hurwitz polynomial.

This chapter attempts to construct a robust state observer and a state feedback pole-assignment controller for system (6.1).

In the following of this chapter, for simplicity, the signal $v(x,u,t)$ will be called the disturbance of the system, and it is denoted by $v(t)$.

## 6.3 The Implicit Observers

### 6.3.1 The Traditional Implicit State Observer

To begin with, define a stable $n \times n$ matrix $F$ by

$$F = \begin{bmatrix} -f & \begin{matrix} I \\ 0 \end{matrix} \end{bmatrix} \tag{6.5}$$

Then, Equation (6.1) can be rewritten as

$$\dot{x}(t) = Fx(t) + (f - a)y(t) + bu(t) + kv(t)$$

$$\triangleq Fx(t) + h_a y(t) + h_b u(t) + h_c v(t), \quad x(t_0) = x_0 \tag{6.6}$$

Now, let us define the following three matrices.

$$L(h) = \begin{bmatrix} h_1 & h_2 & \cdots & h_{n-1} & h_n \\ h_2 & h_3 & \cdots & h_n & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ h_{n-1} & h_n & \cdots & 0 & 0 \\ h_n & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{6.7}$$

$$U(f) = \begin{bmatrix} 0 & f_1 & \cdots & f_{n-2} & f_{n-1} \\ 0 & 0 & \cdots & f_{n-3} & f_{n-2} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & f_1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{6.8}$$

$$H(f,h) = L(h)(I + U(f)) - L(f)U(h) \tag{6.9}$$

Some useful properties of the matrix $H(f,h)$ are stated in the next lemma.

**Lemma 6.1** $H(f,h)$ is a symmetric matrix satisfying

$$H(f,h)\frac{\xi(s)}{\det(sI - F)} = (sI - F)^{-1}h \tag{6.10}$$

where $\xi(s) = \left[s^{n-1}, \cdots, s, 1\right]^T$. Further, if the polynomials

$$f(s) \triangleq \det(sI - F) = s^n + f_1 s^{n-1} + \cdots + f_{n-1}s + f_n \tag{6.11}$$

and

$$h(s) = h_1 s^{n-1} + \cdots + h_{n-1}s + h_n \tag{6.12}$$

are coprime, then $H(f,h)$ is nonsingular.

**Proof**: For proof, see [36].

It is worth mentioning that the initial conditions for all the filters of the input, output and disturbance are assumed to be zero in this chapter. Fortunately, this treatment does not lose any generality since non-zero initial conditions only contribute to the state some additive terms which decay to zero exponentially.

In this chapter, $s$ denotes, as the case may be, the Laplace transform variable or



the differential operator $\frac{d}{dt}$. Taking the Laplace transform of Equation (6.6) gives

$$X(s) = (sI - F)^{-1}\left\{h_a Y(s) + h_b U(s) + h_c V(s) + x_0\right\} \tag{6.13}$$

By Lemma 1, from (6.13), the state vector can be reconstructed as

$$x(t) = H(f,h_a)\frac{\xi(s)}{f(s)}y(t) + H(f,h_b)\frac{\xi(s)}{f(s)}u(t)$$

$$+H(f,h_c)\frac{\xi(s)}{f(s)}v(t) + H(f, x_0)z(t) \tag{6.14}$$

where $z(t)$ is an exponentially decreasing vector which is defined as

$$\dot{z}(t) = F^T z(t), \quad z(t_0) = [1, 0, \cdots, 0]^T \tag{6.15}$$

**Remark 6.1** It should be pointed out that $s$ denotes the differential operator and $\frac{\xi(s)}{f(s)}v(t)$ is not available in Equation (6.14).

As the state can be reconstructed by Equation (6.14), the traditional implicit state observer is constructed by the next theorem.

**Theorem 6.1** The implicit state observer $\hat{x}(t)$ can be formulated as

$$\hat{x}(t) = H(f,h_a)\frac{\xi(s)}{f(s)}y(t) + H(f,h_b)\frac{\xi(s)}{f(s)}u(t) + H(f,h_c)\frac{\xi(s)}{f(s)}v(t) \tag{6.16}$$

**Proof**: Since $z(t) \to 0$, it is obvious that $x(t) - \hat{x}(t) \to 0$ as $t \to \infty$, where the roots of $f(s)$ determine the converging speed.

### 6.3.2 A New Implicit State Observer

In this section, some operations will be carried out on the state observer described in Equation (6.16). Now, let us consider the Hurwitz polynomial $f(s)$ in (6.16) defined as

$$f(s) = s^n + f_1 s^{n-1} + \cdots + f_n$$

$$= (s + \lambda_1)(s + \lambda_2)\cdots(s + \lambda_n) \tag{6.17}$$

where $\lambda_i \neq \lambda_j$ as $i \neq j$, for $i, j = 1, \cdots, n$.

Pre-multiplying (6.16) by the vector $\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \cdots, (-1)^{n-1}\right]$ yields

$$\begin{aligned}&\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \cdots, (-1)^{n-1}\right]\hat{x}(t)\\ &= \left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \cdots, (-1)^{n-1}\right]H(f, h_a)\frac{\xi(s)}{f(s)}y(t)\\ &+\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \cdots, (-1)^{n-1}\right]H(f, h_b)\frac{\xi(s)}{f(s)}u(t)\\ &+\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \cdots, (-1)^{n-1}\right]H(f, h_c)\frac{\xi(s)}{f(s)}v(t)\end{aligned} \tag{6.18}$$

**Lemma 6.2** For the matrix $H(f, h)$ defined in (6.7)-(6.9), the following equation is valid.

$$\left[\lambda^{n-1}, -\lambda^{n-2}, \cdots, (-1)^{n-1}\right]H(f, h) = \chi\left[1 \ g_1 \ \cdots \ g_{n-1}\right] \tag{6.19}$$

where $g_1, \cdots, g_{n-1}$ and $\chi$ are described by

$$f(s) = (s^{n-1} + g_1 s^{n-2} + \cdots + g_{n-1})(s + \lambda) \tag{6.20}$$

and

$$\chi = h_1\lambda^{n-1} - h_2\lambda^{n-2} + \cdots + (-1)^{n-1}h_n \tag{6.21}$$

respectively.

**Proof**: The proof is given in A.9 of the Appendix.

Therefore, by applying Lemma 6.2, we have

$$\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \cdots, (-1)^{n-1}\right]H(f, h_a) = \chi_{ia}\left[1 \ g_{i,1} \ \cdots \ g_{i,n-1}\right] \tag{6.22}$$

$$\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \cdots, (-1)^{n-1}\right]H(f, h_b) = \chi_{ib}\left[1 \ g_{i,1} \ \cdots \ g_{i,n-1}\right] \tag{6.23}$$

$$\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \cdots, (-1)^{n-1}\right]H(f, h_c) = \chi_{ic}\left[1 \ g_{i,1} \ \cdots \ g_{i,n-1}\right] \tag{6.24}$$

where

$$\chi_{ia} = (f_1 - a_1)\lambda_i^{n-1} - (f_2 - a_2)\lambda_i^{n-2} + \cdots + (-1)^{n-1}(f_n - a_n) \tag{6.25}$$



$$\chi_{ib} = b_1\lambda_i^{n-1} - b_2\lambda_i^{n-2} + \cdots + (-1)^{n-1}b_n \tag{6.26}$$

$$\chi_{ic} = k_1\lambda_i^{n-1} - k_2\lambda_i^{n-2} + \cdots + (-1)^{n-1}k_n \tag{6.27}$$

and $g_{i,1}, \cdots, g_{i,n-1}$ are determined by

$$f(s) = (s^{n-1} + g_{i,1}s^{n-2} + \cdots + g_{i,n-1})(s + \lambda_i) \tag{6.28}$$

Thus, by (6.22)-(6.28), Equation (6.18) can be simplified as

$$\begin{aligned}&\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \cdots, (-1)^{n-1}\right]\hat{x}(t)\\ &= \chi_{ic}\frac{v(t)}{s+\lambda_i} + \chi_{ia}\frac{y(t)}{s+\lambda_i} + \chi_{ib}\frac{u(t)}{s+\lambda_i}\end{aligned} \tag{6.29}$$

Now, for $i = 1, 2, \cdots, n$, writing the $n$ equations in (6.29) in a compact form yields

$$\begin{aligned}&\begin{bmatrix}\lambda_1^{n-1} & -\lambda_1^{n-2} & \cdots & (-1)^{n-2}\lambda_1 & (-1)^{n-1}\\ \lambda_2^{n-1} & -\lambda_2^{n-2} & \cdots & (-1)^{n-2}\lambda_2 & (-1)^{n-1}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ \lambda_n^{n-1} & -\lambda_n^{n-2} & \cdots & (-1)^{n-2}\lambda_n & (-1)^{n-1}\end{bmatrix}\hat{x}(t)\\ &= \begin{bmatrix}\frac{\chi_{1c}}{s+\lambda_1}v(t)\\ \vdots\\ \frac{\chi_{nc}}{s+\lambda_n}v(t)\end{bmatrix} + \begin{bmatrix}\frac{\chi_{1a}}{s+\lambda_1}y(t)\\ \vdots\\ \frac{\chi_{na}}{s+\lambda_n}y(t)\end{bmatrix} + \begin{bmatrix}\frac{\chi_{1b}}{s+\lambda_1}u(t)\\ \vdots\\ \frac{\chi_{nb}}{s+\lambda_n}u(t)\end{bmatrix}\end{aligned} \tag{6.30}$$

On the other hand, it is well known that the Vandermonde matrix

$$\Lambda \triangleq \begin{bmatrix}\lambda_1^{n-1} & -\lambda_1^{n-2} & \cdots & (-1)^{n-2}\lambda_1 & (-1)^{n-1}\\ \lambda_2^{n-1} & -\lambda_2^{n-2} & \cdots & (-1)^{n-2}\lambda_2 & (-1)^{n-1}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ \lambda_n^{n-1} & -\lambda_n^{n-2} & \cdots & (-1)^{n-2}\lambda_n & (-1)^{n-1}\end{bmatrix} \tag{6.31}$$

is nonsingular when $\lambda_i \neq \lambda_j$ for $i \neq j \ (i, j = 1, \cdots, n)$. Therefore, by pre-multiplying (6.30) with $\Lambda^{-1}$, the state observer can be reconstructed by the first order filters of $v(t)$, $y(t)$ and $u(t)$.

**Theorem 6.2** A new implicit observer of the state $x(t)$ can be formulated as

$$\hat{x}(t) = \Lambda^{-1}\left\{\begin{bmatrix} \frac{\chi_{1c}}{s+\lambda_1}v(t) \\ \vdots \\ \frac{\chi_{nc}}{s+\lambda_n}v(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1a}}{s+\lambda_1}y(t) \\ \vdots \\ \frac{\chi_{na}}{s+\lambda_n}y(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1b}}{s+\lambda_1}u(t) \\ \vdots \\ \frac{\chi_{nb}}{s+\lambda_n}u(t) \end{bmatrix}\right\} \tag{6.32}$$

**Proof**: As the expression of $\hat{x}(t)$ in (6.32) is just an algebraic transform of Equation (6.16), the proof is same as that of Theorem 6.1.

**Remark 6.2** In the above implicit observer, the first order filters of the disturbance are not available. They will be estimated in the next section.

## 6.4 Description of the Robust State Observers

In this chapter, the system is divided into the following two cases:

**Case 1** $k_1 \neq 0$

**Case 2** $k_i = 0 (i = 1,2,\cdots,r-1)$, but $k_r \neq 0$ $(r > 1)$

By defining

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \tag{6.33}$$

$$b(s) = b_1 s^{n-1} + \cdots + b_{n-1}s + b_n \tag{6.34}$$

$$k(s) = k_1 s^{n-1} + \cdots + k_{n-1}s + k_n \tag{6.35}$$

the Differential Equation (6.1) can be rewritten as

$$a(s)y(t) = b(s)u(t) + k(s)v(t) \tag{6.36}$$

Now, the robust state observer will be given for each case.

**Case 1** $k_1 \neq 0$

In this case, choose $n$ different Hurwitz polynomials as

$$\hat{f}_i(s) = \frac{1}{k_1}k(s)\cdot(s+\lambda_i) \tag{6.37}$$



where $\lambda_i$ $(i = 1,\cdots,n)$ are defined in (6.17).

Then, dividing Equation (6.36) by $\hat{f}_i(s)$ yields

$$\frac{1}{s+\lambda_i}v(t) = \frac{1}{k_1}\left\{\frac{a(s)}{\hat{f}_i(s)}y(t) - \frac{b(s)}{\hat{f}_i(s)}u(t)\right\} \tag{6.38}$$

So, $\frac{1}{s+\lambda_i}v(t)$ can be expressed by available signals. Therefore, by using Theorem 6.2, the state observer $\hat{x}(t)$ can be constructed by the known signals composed of $y(t)$ and the filters of $y(t)$ and $u(t)$.

**Theorem 6.3** In the case $k_1 \neq 0$, the robust observer can be formulated as

$$\hat{x}(t) = \Lambda^{-1}\left\{\frac{1}{k_1}\begin{bmatrix} \chi_{1c}(\frac{a(s)}{\hat{f}_1(s)}y(t) - \frac{b(s)}{\hat{f}_1(s)}u(t)) \\ \vdots \\ \chi_{nc}(\frac{a(s)}{\hat{f}_n(s)}y(t) - \frac{b(s)}{\hat{f}_n(s)}u(t)) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1a}}{s+\lambda_1}y(t) \\ \vdots \\ \frac{\chi_{na}}{s+\lambda_n}y(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1b}}{s+\lambda_1}u(t) \\ \vdots \\ \frac{\chi_{nb}}{s+\lambda_n}u(t) \end{bmatrix}\right\} \tag{6.39}$$

**Proof**: The theorem is obvious by replacing the terms $\frac{1}{s+\lambda_i}v(t)$ in Equation (6.32) by their available expressions described in (6.38).

**Remark 6.3** It is should be noted that no *a priori* information of the disturbance is needed in this case, and there is no necessity to estimate the disturbance. The state observer is formulated by the filters of the input and the output. Furthermore, discontinuous formulations as in [11] can be avoided.

The block diagram of the implicit state estimator for the minimum phase systems with relative degree one is shown in Figure 6.1.

The algorithm of the implicit state estimator formulation for the minimum phase systems with relative degree one is illustrated in Table 6.1.

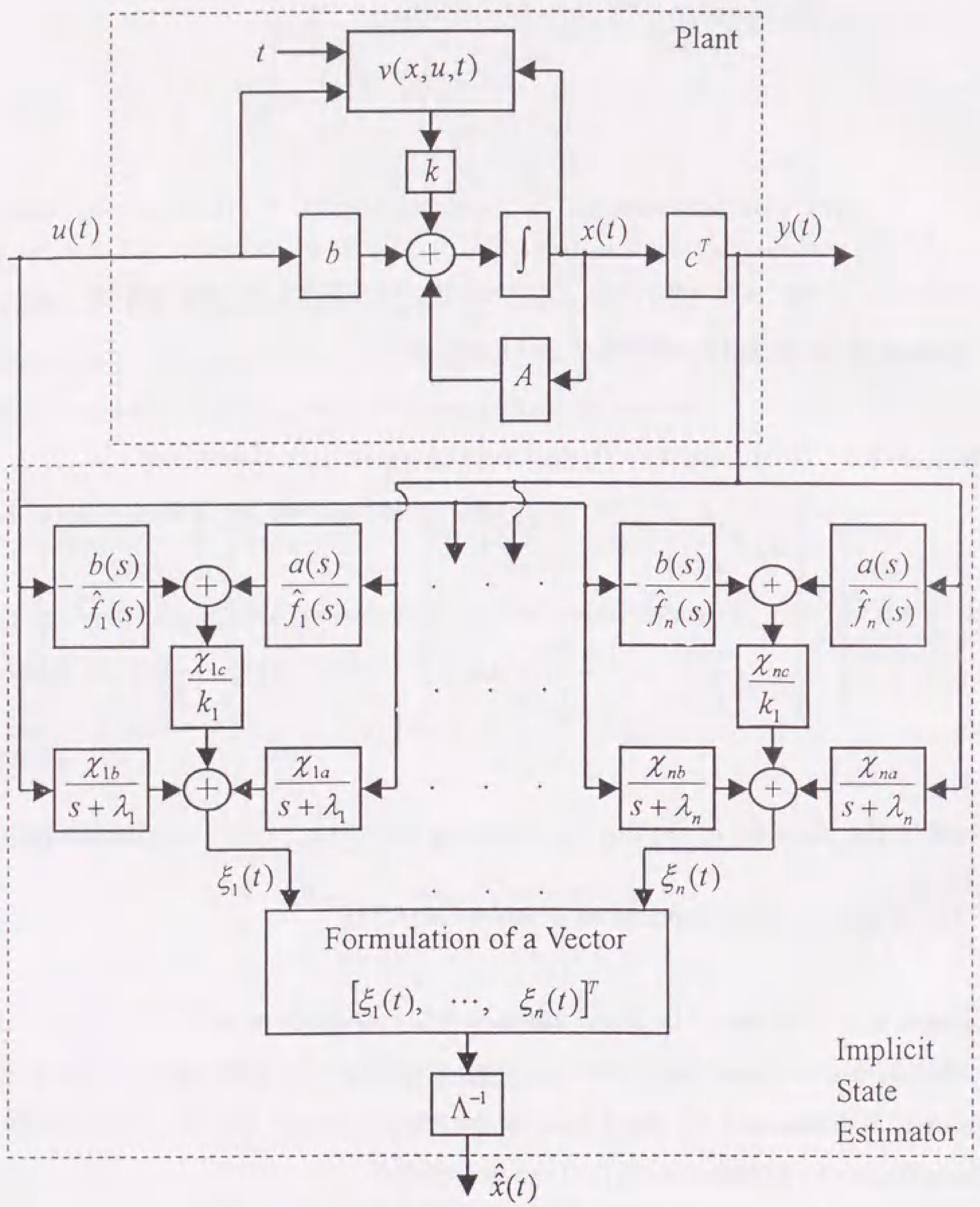



Fig. 6.1 The block diagram of the implicit state estimator for the minimum phase systems with relative degree one.



Table 6.1 The algorithm of the implicit state estimator formulation for the minimum phase systems with relative degree one

| | |
|---|---|
| Plant | $\begin{cases} \dot{x}(t) = Ax(t) + bu(t) + kv(t), \ x(t_0) = x_0 \\ y(t) = c^T x(t) \end{cases} \quad (k_1 \neq 0)$ |
| Introduction of parameters | $\lambda_i \ ( \ i = 1, \cdots, n)$ where $\lambda_i \neq \lambda_j$ for $i \neq j \ (i, j = 1, \cdots, n)$ |
| Formulation of a matrix | $\Lambda \triangleq \begin{bmatrix} \lambda_1^{n-1} & -\lambda_1^{n-2} & \cdots & (-1)^{n-2}\lambda_1 & (-1)^{n-1} \\ \lambda_2^{n-1} & -\lambda_2^{n-2} & \cdots & (-1)^{n-2}\lambda_2 & (-1)^{n-1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \lambda_n^{n-1} & -\lambda_n^{n-2} & \cdots & (-1)^{n-2}\lambda_n & (-1)^{n-1} \end{bmatrix}$ |
| Implicit state | $\hat{x}(t) = \Lambda^{-1} \left\{ \begin{bmatrix} \frac{\chi_{1c}}{s+\lambda_1} v(t) \\ \vdots \\ \frac{\chi_{nc}}{s+\lambda_n} v(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1a}}{s+\lambda_1} y(t) \\ \vdots \\ \frac{\chi_{na}}{s+\lambda_n} y(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1b}}{s+\lambda_1} u(t) \\ \vdots \\ \frac{\chi_{nb}}{s+\lambda_n} u(t) \end{bmatrix} \right\}$ |
| Estimate of $\frac{1}{s+\lambda_i} v(t)$ | $\frac{1}{s+\lambda_i} v(t) = \frac{1}{k_1} \left\{ \frac{a(s)}{\hat{f}_i(s)} y(t) - \frac{b(s)}{\hat{f}_i(s)} u(t) \right\}$ for $i = 1, \cdots, n$ |
| Implicit state estimator | $\hat{\hat{x}}(t) = \Lambda^{-1} \left\{ \frac{1}{k_1} \begin{bmatrix} \chi_{1c}(\frac{a(s)}{\hat{f}_1(s)} y(t) - \frac{b(s)}{\hat{f}_1(s)} u(t)) \\ \vdots \\ \chi_{nc}(\frac{a(s)}{\hat{f}_n(s)} y(t) - \frac{b(s)}{\hat{f}_n(s)} u(t)) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1a}}{s+\lambda_1} y(t) \\ \vdots \\ \frac{\chi_{na}}{s+\lambda_n} y(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1b}}{s+\lambda_1} u(t) \\ \vdots \\ \frac{\chi_{nb}}{s+\lambda_n} u(t) \end{bmatrix} \right\}$ |

**Case 2** $k_i = 0 (i = 1,2,\cdots,r-1)$, but $(r > 1)$

In the following, the disturbance will be estimated by using the VSS theory. First of all, the upper bounds of the filters of the disturbance must be estimated. For positive constant $\lambda$, similar to Lemma 4.1, the next lemma is obvious.

**Lemma 6.3** The upper bound of $\left|\frac{1}{(s+\lambda)^i} v(t)\right|$ can be estimated as

$$\left|\frac{1}{(s+\lambda)^i} v(t)\right| \le \frac{1}{(s+\lambda)^i} \rho(y(t), u(t), t) \triangleq \omega_i(t) \tag{6.40}$$

**Proof**: The proof is omitted.

Now, we introduce a Hurwitz polynomial $l(s)$ as

$$l(s) = s^n + l_1 s^{n-1} + \cdots + l_n = \frac{1}{k_r} k(s)(s+\lambda)^r \tag{6.41}$$

Dividing the both sides of (6.36) by $l(s)$ yields

$$y(t) = k_r \left\{ \frac{l(s) - a(s)}{k(s)(s+\lambda)^r} y(t) + \frac{b(s)}{k(s)(s+\lambda)^r} u(t) \right\} + \frac{k_r}{(s+\lambda)^r} v(t) \tag{6.42}$$

Then, multiplying the both sides of (6.42) with $s+\lambda$ gives

$$\dot{y}(t) + \lambda y(t) = k_r \left\{ \frac{l(s) - a(s)}{k(s)(s+\lambda)^{r-1}} y(t) + \frac{b(s)}{k(s)(s+\lambda)^{r-1}} u(t) \right\} + \frac{k_r}{(s+\lambda)^{r-1}} v(t) \tag{6.43}$$

Based on Equation (6.43), the disturbance is estimated by the next proposition.

**Proposition 6.1** Construct the differential equations

$$\dot{\hat{y}}(t) + \lambda \hat{y}(t) = k_r \left\{ \frac{f(s) - a(s)}{k(s)(s+\lambda)^{r-1}} y(t) + \frac{b(s)}{k(s)(s+\lambda)^{r-1}} u(t) \right\} + k_r w_1(t), \quad \hat{y}(t_0) = 0 \tag{6.44}$$

$$\dot{\hat{w}}_{i-1}(t) + \lambda \hat{w}_{i-1}(t) = w_i(t), \quad \hat{w}_{i-1}(t_0) = 0 \ (\text{for } 2 \le i \le r) \tag{6.45}$$

where $w_1(t)$ and $w_i(t)$ $(2 \le i \le r)$ are given by



$$w_1(t) = \frac{k_r \{y(t) - \hat{y}(t)\} \omega_{r-1}^2(t)}{|k_r \{y(t) - \hat{y}(t)\}| \omega_{r-1}(t) + \delta_1}, \quad \delta_1 > 0 \tag{6.46}$$

and

$$w_i(t) = \frac{\{w_{i-1}(t) - \hat{w}_{i-1}(t)\} \omega_{r-i}^2(t)}{|w_{i-1}(t) - \hat{w}_{i-1}(t)| \omega_{r-i}(t) + \delta_i}, \quad \delta_i > 0 \ (\text{for } 2 \le i \le r) \tag{6.47}$$

respectively; $\hat{y}(t)$ and $\hat{w}_{i-1}(t)$ $(2 \le i \le r)$ are signals generated by Equations (6.44) and (6.45), respectively. It can be concluded that, when $\sum_{i=1}^{r} \delta_i$ is very small, $w_i(t)$ can be approximately regarded as the corresponding estimates of $\frac{1}{(s+\lambda)^{r-i}} v(t)$ for $1 \le i \le r$ as $t$ is large enough. Therefore, there exist $T_i > t_0$ and $\varepsilon_i(\delta_1,\cdots,\delta_i) > 0$ such that

$$\left|\frac{1}{(s+\lambda)^{r-i}} v(t) - w_i(t)\right| \le \varepsilon_i(\delta_1,\cdots,\delta_i) \tag{6.48}$$

where $\varepsilon_i(\delta_1,\cdots,\delta_i) \to 0$ as $\sum_{j=1}^{i} \delta_j \to 0$, for $1 \le i \le r$.

**Proof**: For the proof, see Theorem 4.2.

**Remark 6.4** Theorem 6.4 is also valid for Case 1, in which $w_1(t)$ can be regarded as the estimate of the disturbance $v(t)$.

From Theorem 6.2 and Proposition 6.1, the observer for Case 2 can be approximately constructed in the following theorem.

**Theorem 6.4** For Case 2, the robust state observer of system (6.1) can be approximately constructed as

$$\hat{\hat{x}}(t) = \Lambda^{-1} \left\{ \begin{bmatrix} \chi_{c1} \cdot w_{1,r-1}(t) \\ \vdots \\ \chi_{cn} \cdot w_{n,r-1}(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1a}}{s+\lambda_1} y(t) \\ \vdots \\ \frac{\chi_{na}}{s+\lambda_n} y(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1b}}{s+\lambda_1} u(t) \\ \vdots \\ \frac{\chi_{nb}}{s+\lambda_n} u(t) \end{bmatrix} \right\} \tag{6.49}$$

where $w_{i,r-1}(t)$ are the corresponding estimates of $\frac{1}{s+\lambda_i}v(t)$ obtained in Proposition 6.1 for $i = 1, 2, \cdots, n$.

**Proof**: From (6.14), (6.32) and (6.49), we have

$$x(t) - \hat{\hat{x}}(t) = \{x(t) - \hat{x}(t)\} + \{\hat{x}(t) - \hat{\hat{x}}(t)\}$$

$$= H(f, x_0)z(t) + \Lambda^{-1}\begin{bmatrix} \chi_{c1}\left\{\frac{1}{s+\lambda_1}v(t) - w_{1,r-1}(t)\right\} \\ \vdots \\ \chi_{cn}\left\{\frac{1}{s+\lambda_n}v(t) - w_{n,r-1}(t)\right\} \end{bmatrix} \quad (6.50)$$

From (6.15) and Proposition 6.1, it can be easily concluded that there exist $\overline{T} > t_0$ and $\overline{\varepsilon}(\delta_1, \cdots, \delta_r) > 0$ such that

$$\left|x(t) - \hat{\hat{x}}(t)\right| \le \overline{\varepsilon}(\delta_1, \cdots, \delta_r) \quad (6.51)$$

where $\overline{\varepsilon}(\delta_1, \cdots, \delta_r) \to 0$ as $\sum_{i=1}^{r}\delta_i \to 0$. Thus, $\hat{\hat{x}}(t)$ defined in (6.49) is an approximate estimate of the state $x(t)$ if $\sum_{i=1}^{r}\delta_i$ is very small.

**Remark 6.5** The signals $w_{i,r-1}(t)$ (for $i = 1, 2, \cdots, n$) can be either individually generated by a procedure similar to that of Proposition 6.1, or calculated by $\frac{s+\lambda}{s+\lambda_i}w_{r-1}(t)$, where $w_{r-1}(t)$ is generated in Proposition 6.1.

**Remark 6.6** From Theorem 6.2, it can be seen that the state can be asymptotically expressed by the first filters of the input, output and disturbance. This is a reason why we do not employ the estimate $w_r(t)$ of the disturbance to generate the state observer directly by a differential equation.

The block diagram of the implicit state estimator for the minimum phase systems with higher relative degrees is shown in Figure 6.2.



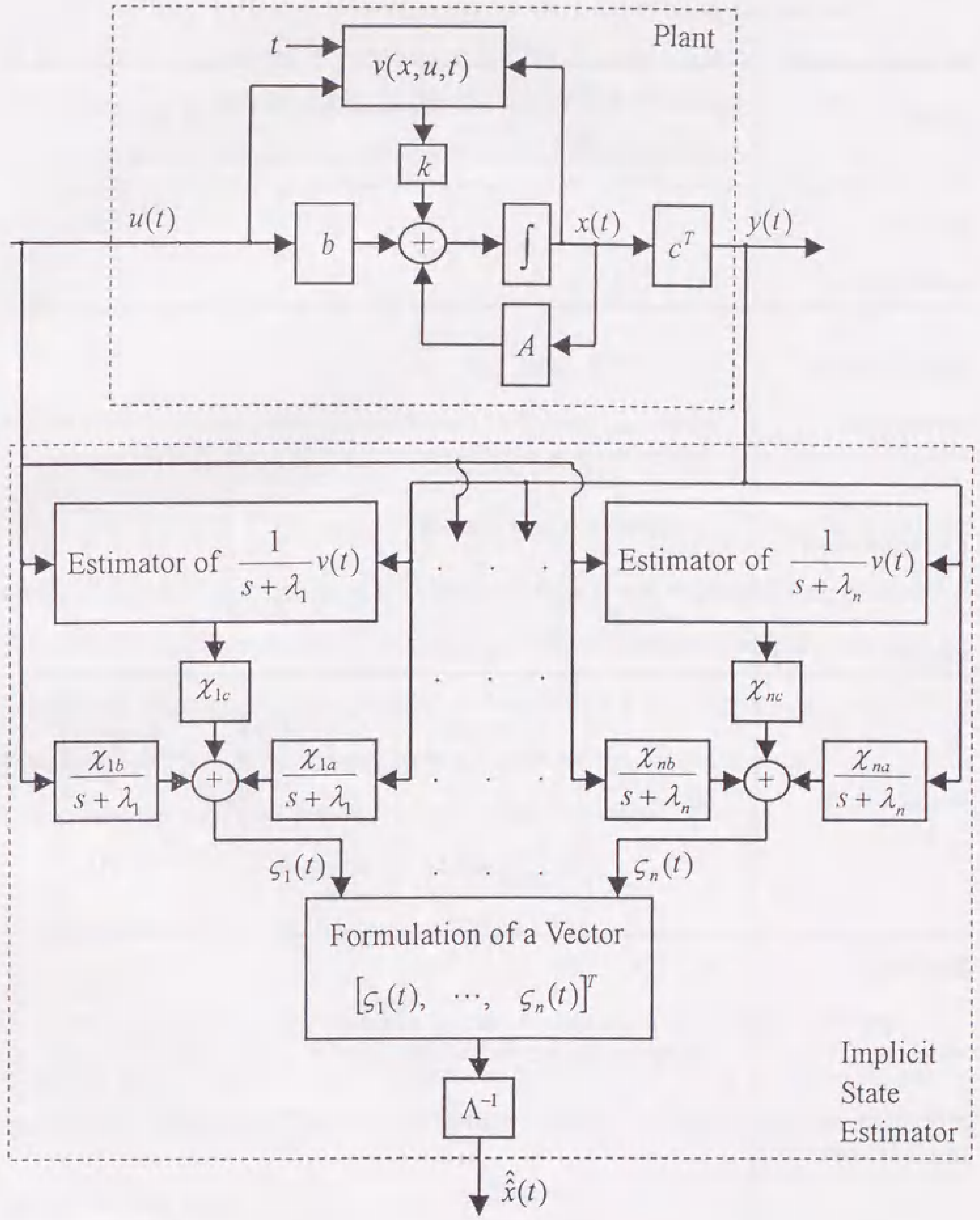



Fig. 6.2 The block diagram of the implicit state estimator for the minimum phase systems with higher relative degrees.

Table 6.1 The algorithm of the implicit state estimator formulation for the minimum phase systems with relative degree $r$ ($r > 1$)

| | |
|---|---|
| Plant | $\begin{cases} \dot{x}(t) = Ax(t) + bu(t) + kv(t), \; x(t_0) = x_0 \\ y(t) = c^T x(t) \end{cases} \quad (k_r \neq 0)$ |
| *A priori* information | $\lvert v(t) \rvert \le \rho(y, u, t)$ |
| Introduction of parameters | $\lambda_i \; (i = 1, \cdots, n)$ where $\lambda_i \neq \lambda_j$ for $i \neq j \; (i, j = 1, \cdots, n)$ |
| Formulation of a matrix | $\Lambda \triangleq \begin{bmatrix} \lambda_1^{n-1} & -\lambda_1^{n-2} & \cdots & (-1)^{n-2}\lambda_1 & (-1)^{n-1} \\ \lambda_2^{n-1} & -\lambda_2^{n-2} & \cdots & (-1)^{n-2}\lambda_2 & (-1)^{n-1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \lambda_n^{n-1} & -\lambda_n^{n-2} & \cdots & (-1)^{n-2}\lambda_n & (-1)^{n-1} \end{bmatrix}$ |
| Implicit state | $\hat{x}(t) = \Lambda^{-1} \left\{ \begin{bmatrix} \frac{\chi_{1c}}{s+\lambda_1} v(t) \\ \vdots \\ \frac{\chi_{nc}}{s+\lambda_n} v(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1a}}{s+\lambda_1} y(t) \\ \vdots \\ \frac{\chi_{na}}{s+\lambda_n} y(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1b}}{s+\lambda_1} u(t) \\ \vdots \\ \frac{\chi_{nb}}{s+\lambda_n} u(t) \end{bmatrix} \right\}$ |
| Estimate of $\frac{1}{s+\lambda_i} v(t)$ | $w_{i,r-1}(t)$, for $i = 1, \cdots, n$ (By using the algorithm illustrated in Table 4.2) |
| Implicit state estimator | $\hat{\hat{x}}(t) = \Lambda^{-1} \left\{ \begin{bmatrix} \chi_{c1} \cdot w_{1,r-1}(t) \\ \vdots \\ \chi_{cn} \cdot w_{n,r-1}(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1a}}{s+\lambda_1} y(t) \\ \vdots \\ \frac{\chi_{na}}{s+\lambda_n} y(t) \end{bmatrix} + \begin{bmatrix} \frac{\chi_{1b}}{s+\lambda_1} u(t) \\ \vdots \\ \frac{\chi_{nb}}{s+\lambda_n} u(t) \end{bmatrix} \right\}$ |



## 6.5 A Pole Assignment Controller

In this section, for simplicity, we assume that the disturbance is not directly related to the control input. We also assume that the disturbance is matched, i.e. $b = k$.

Let the desired closed-loop transfer function be represented by

$$G_d(s) = \frac{b(s)}{d(s)} \tag{6.52}$$

where the zeros of the Hurwitz polynomial

$$d(s) = s^n + d_1 s^{n-1} + \cdots + d_n \tag{6.53}$$

determine the desired closed-loop poles.

Consider a state feedback control law defined by

$$u(t) = -\kappa^T \hat{\hat{x}}(t) - w_r(t) + \gamma(t) \tag{6.54}$$

where $\kappa$ is an $n \times 1$ feedback gain vector, $\hat{\hat{x}}(t)$ is the estimated state obtained in Theorem 6.3 or Theorem 6.4, $\gamma(t)$ is a uniformly bounded external input and the disturbance estimate $w_r(t)$ obtained in Proposition 6.1 is employed to cancel the disturbance $v(t)$. With an appropriate choice of the feedback gain vector $\kappa$, the characteristic equation of the closed-loop system becomes

$$\det(sI - A + b\kappa^T) = d(s) \tag{6.55}$$

The calculation method of $\kappa$ can be found in [36].

For the system (6.1) controlled by (6.54), we obtain the following theorem.

**Theorem 6.5** With the pole assignment controller (6.54), the global system is uniformly bounded, and the overall system output $y(t)$ approximately tracks the desired output $y_d(t) = \frac{b(s)}{d(s)} \gamma(t)$ as $t \to \infty$.

**Proof**: By using the control law described in (6.54), the system (6.1) will be described by

$$\dot{x}(t) = (A - b\kappa^T)x(t) + b\gamma(t) + b\kappa^T \left\{ x(t) - \hat{\hat{x}}(t) \right\} + b(v(t) - w_r(t)) \tag{6.56}$$

Since $\gamma(t)$ is a uniformly bounded signal and $A - b\kappa^T$ is a stable matrix, by applying the results that $\{x(t) - \hat{x}(t)\}$ and $\{v(t) - w_r(t)\}$ is very small as $t \to \infty$, it can be easily concluded that the state $x(t)$ is uniformly bounded. Then, by Theorem 6.3 or Theorem 6.4, the estimated state $\hat{x}(t)$ is also uniformly bounded. So, the input determined in (6.54) is uniformly bounded. Therefore, from (6.56), we have

$$y(t) = \frac{b(s)}{d(s)}\gamma(t) + \frac{b(s)}{d(s)}\left\{\kappa^T\left\{x(t) - \hat{x}(t)\right\} + \left\{v(t) - w_r(t)\right\}\right\} \tag{6.57}$$

Let

$$y_d(t) \triangleq \frac{b(s)}{d(s)}\gamma(t) \tag{6.58}$$

Thus, (6.57) gives

$$y(t) - y_d(t) = \frac{b(s)}{d(s)}\left\{\kappa^T\left\{x(t) - \hat{x}(t)\right\} + \left\{v(t) - w_r(t)\right\}\right\} \tag{6.59}$$

By Proposition 6.1 and Theorem 6.3 or 6.4 and the fact that $d(s)$ is a Hurwitz polynomial, it can be concluded that there exist $T' > t_0$ and $\varepsilon'(\delta_1, \cdots, \delta_r) > 0$ such that

$$|y(t) - y_d(t)| \le \varepsilon'(\delta_1, \cdots, \delta_r) \tag{6.60}$$

where $\varepsilon'(\delta_1, \cdots, \delta_r) \to 0$ as $\sum_{i=1}^{r}\delta_i \to 0$. Thus, $y(t) - y_d(t)$ can be made very small as $t \to \infty$, i.e. the pole assignment can be approximately achieved as $t \to \infty$.

The algorithm of the proposed pole assignment control formulation is summarized in Table 6.3.

## 6.6 Design Examples

In this section, for the two possible cases discussed in section 6.4, examples will be presented to show the design procedure and the simulation results.



Table 6.3 The algorithm of the proposed pole assignment control

| | |
|---|---|
| Plant | $a(s)y(t) = b(s)u(t) + k(s)v(x,u,t)$, |
| *A priori* information | $\lvert v(x,u,t)\rvert \le \rho(y,u,t)$ |
| Desired poles | The roots of the Hurwitz polynomial $d(s) = s^n + d_1 s^{n-1} + \cdots + d_n$ |
| Identified disturbance | $w_r(t)$ (By using the algorithm presented in Table 4.1) |
| State observer | $\hat{x}(t)$ (By employing the algorithm presented in Table 6.1 or 6.2) |
| Feedback gain | $\kappa$ |
| External input | $\gamma(t)$ (a uniformly bounded signal) |
| Pole assignment controller | $u(t) = -\kappa^T\hat{x}(t) - w(t) + \psi(t)$ |

**Example 6.1** Consider a stable system with relative degree one described by

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} v(t) \\ y(t) = [1,\ 0]x(t) = x_1(t) \end{cases} \tag{6.61}$$

where the disturbance is governed by $v(t) = (\sin 2t)\dfrac{0.5y(t)\{y(t) + 2u(t)\}}{|y(t)| + 0.5}$, the input is assumed to be $u(t) = \sin t$. Suppose the starting time is $t_0 = 0$. The unknown initial condition is assumed to be $x_0 = [-1,\ 2]^T$. The purpose of this example is to estimate the state $x(t)$. As the state $x_1(t)$ is the output, we only need to estimate the state $x_2(t)$.

We choose the parameters $\lambda_1$ and $\lambda_2$ as $\lambda_1 = 1$ and $\lambda_2 = 2$, i.e. the Hurwitz polynomial in (6.17) is chosen as $f(s) = s^2 + 3s + 2$. Then, we have

$$\chi_{1a} = 1,\ \chi_{2a} = 3,\ \chi_{1b} = 1,\ \chi_{2b} = 2,\ \chi_{1c} = -1,\ \chi_{2c} = 0 \tag{6.62}$$

As $k(s) = s + 2$, the Hurwitz polynomials defined in (6.37) is chosen as

$$\hat{f}_1(s) = (s+2)(s+1),\ \hat{f}_2(s) = (s+2)(s+2) \tag{6.63}$$

From Theorem 6.3, the implicit observer can be constructed as

$$\hat{x}(t) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\left\{ \dfrac{s^2+s+1}{(s+2)(s+1)} y(t) - \dfrac{s}{(s+2)(s+1)} u(t) \right\} \\ 0 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{s+1} y(t) \\ \dfrac{3}{s+2} y(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{s+1} u(t) \\ \dfrac{2}{s+2} u(t) \end{bmatrix} \right\} \tag{6.64}$$

The computer simulation results of $x_2(t)$ and $\hat{x}_2(t)$ are given in Figure 6.3, in which the sampling period is set to 0.001 second. The difference at the beginning is due to the initial conditions.

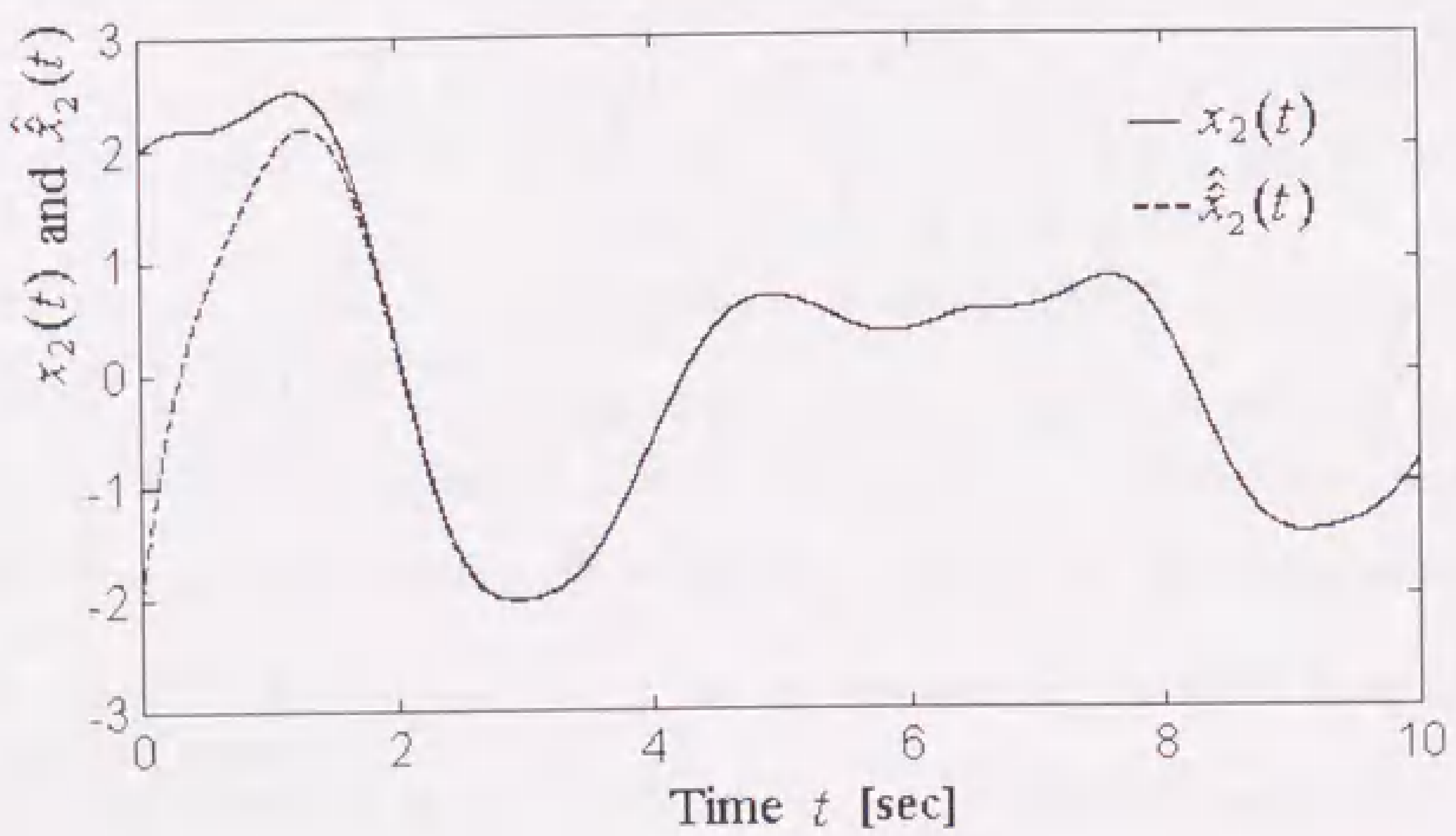



Fig. 6.3 The genuine state $x_2(t)$ and its estimate $\hat{x}_2(t)$ for Example 6.1.



**Example 6.2** Consider an unstable system with relative degree 2 described by

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(t) + v(t)) \\ y(t) = [1, 0] x(t) = x_1(t) \end{cases} \tag{6.65}$$

The disturbance $v(t)$ is governed by $v(t) = (0.5\cos t + 0.25\sin 2t)\dfrac{0.5y(t)x_2(t)}{|x_2(t)| + 0.5}$, its upper bound is known as $\rho(y(t), t) = 0.5|y(t)|$. Suppose the starting time is $t_0 = 0$. The unknown initial state $x_0$ is assumed to be $x_0 = [1,\ 1]^T$. The external reference input is assumed to be

$$\gamma(t) = 3\sin t \tag{6.66}$$

The desired closed-loop poles are supposed to be the roots of the polynomial

$$d(s) = s^2 + 6s + 9 \tag{6.67}$$

The purpose of this example is to estimate the state $x(t)$ and to synthesize a pole assignment controller to achieve the above goal.

From (6.55), the feedback gain $\kappa$ can be calculated as

$$\kappa = [15,\ 7]^T \tag{6.68}$$

Choose the parameters $\lambda_1$ and $\lambda_2$ as $\lambda_1 = 1$ and $\lambda_2 = 2$, i.e. the Hurwitz polynomial in (6.17) is considered as

$$f(s) = (s+1)(s+2) \tag{6.69}$$

Then, we have $\chi_{1a} = 3$, $\chi_{2a} = 7$, $\chi_{1b} = \chi_{1c} = -1$, $\chi_{2b} = \chi_{2c} = -1$. From Theorem 6.2, the implicit state observer can be constructed as

$$\hat{x}(t) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \dfrac{-1}{s+1} v(t) \\ \dfrac{-1}{s+2} v(t) \end{bmatrix} + \begin{bmatrix} \dfrac{3}{s+1} y(t) \\ \dfrac{7}{s+2} y(t) \end{bmatrix} + \begin{bmatrix} \dfrac{-1}{s+1} u(t) \\ \dfrac{-1}{s+2} u(t) \end{bmatrix} \right\} \tag{6.70}$$

where $\dfrac{1}{s+1} v(t)$ and $\dfrac{1}{s+2} v(t)$ are unknown.

Now, let us estimate the first order filters of the disturbance and the disturbance. As $k(s)=1$, choose the Hurwitz polynomial $l(s)$ in (6.42) as $l(s)=(s+1)^2$. From (6.44), we have

$$\dot{y}(t)+y(t)=\frac{3s}{s+1}y(t)+\frac{1}{s+1}u(t)+\frac{1}{s+1}v(t) \tag{6.71}$$

By Theorem 6.4, the following differential equations are constructed.

$$\dot{\hat{y}}(t)+\hat{y}(t)=\frac{3s}{s+1}y(t)+\frac{1}{s+1}u(t)+w_1(t),\ \ \hat{y}(0)=0 \tag{6.72}$$

$$\dot{\hat{w}}_1(t)+\hat{w}_1(t)=w_2(t),\ \ \hat{w}_1(0)=0 \tag{6.73}$$

where $w_1(t)$ and $w_2(t)$ are given as

$$w_1(t)=\frac{\left\{\frac{0.5}{s+1}|y(t)|\right\}^2\{y(t)-\hat{y}(t)\}}{\left\{\frac{0.5}{s+1}|y(t)|\right\}\left|\{y(t)-\hat{y}(t)\}\right|+\delta_1} \tag{6.74}$$

and

$$w_2(t)=\frac{\{0.5|y(t)|\}^2\{w_1(t)-\hat{w}_1(t)\}}{0.5|y(t)|\left|\{w_1(t)-\hat{w}_1(t)\}\right|+\delta_2}. \tag{6.75}$$

Therefore, $w_1(t)$ and $w_2(t)$ can be regarded as the estimates of $\frac{1}{s+1}v(t)$ and $v(t)$, respectively.

On the above preparations, from Theorem 6.4, the state observer is formulated as

$$\hat{x}(t)=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}^{-1}\left\{\begin{bmatrix}-w_1(t)\\ -\frac{s+1}{s+2}w_1(t)\end{bmatrix}+\begin{bmatrix}\frac{3}{s+1}y(t)\\ \frac{7}{s+2}y(t)\end{bmatrix}+\begin{bmatrix}\frac{-1}{s+1}u(t)\\ \frac{-1}{s+2}u(t)\end{bmatrix}\right\} \tag{6.76}$$

Therefore, the state feedback pole assignment controller can be constructed as

$$u(t)=-[15,\ 7]\hat{x}(t)+\gamma(t)-w_2(t) \tag{6.77}$$



In the computer simulation process, $\delta_i$ are chosen as $\delta_1=\delta_2=2.5\times10^{-4}$, the sampling period is set to $5\times10^{-4}$ second. The simulation results are shown in Figures 6.3-6.7.

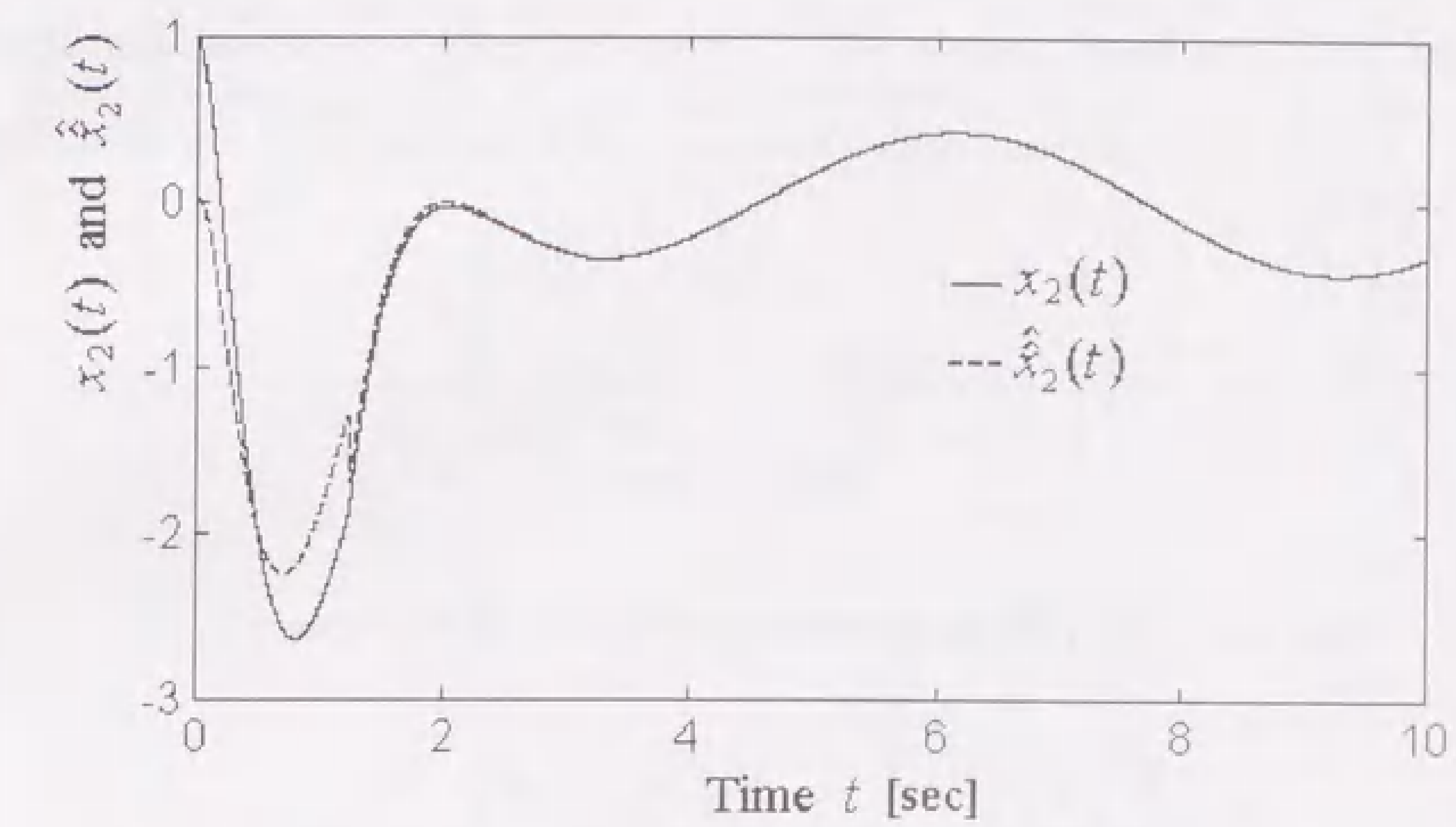



Fig. 6.4 The genuine state $x_2(t)$ and its estimate $\hat{x}_2(t)$ of Example 6.2.

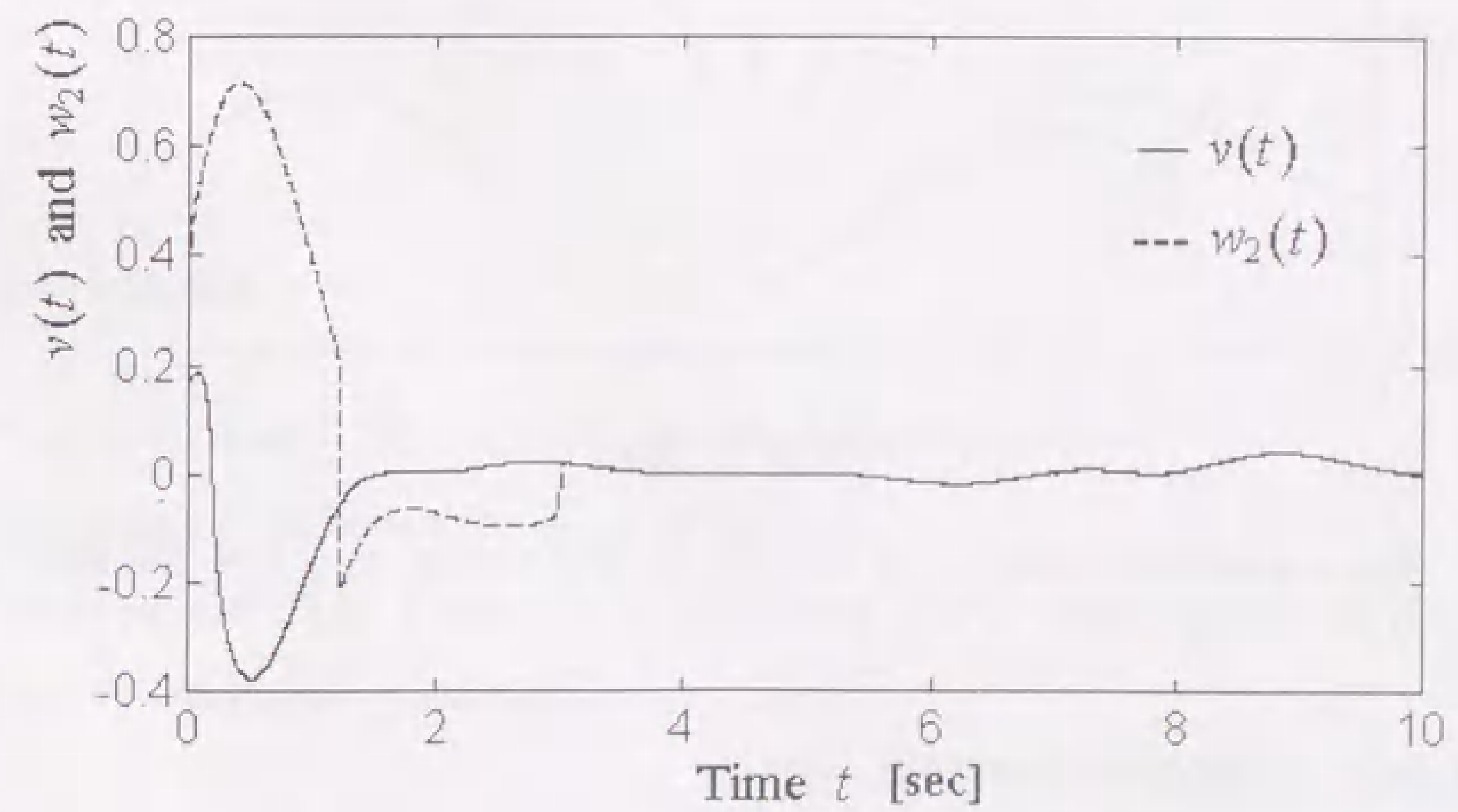



Fig. 6.5 The disturbance $v(t)$ and its estimate $w_2(t)$ of Example 6.2.

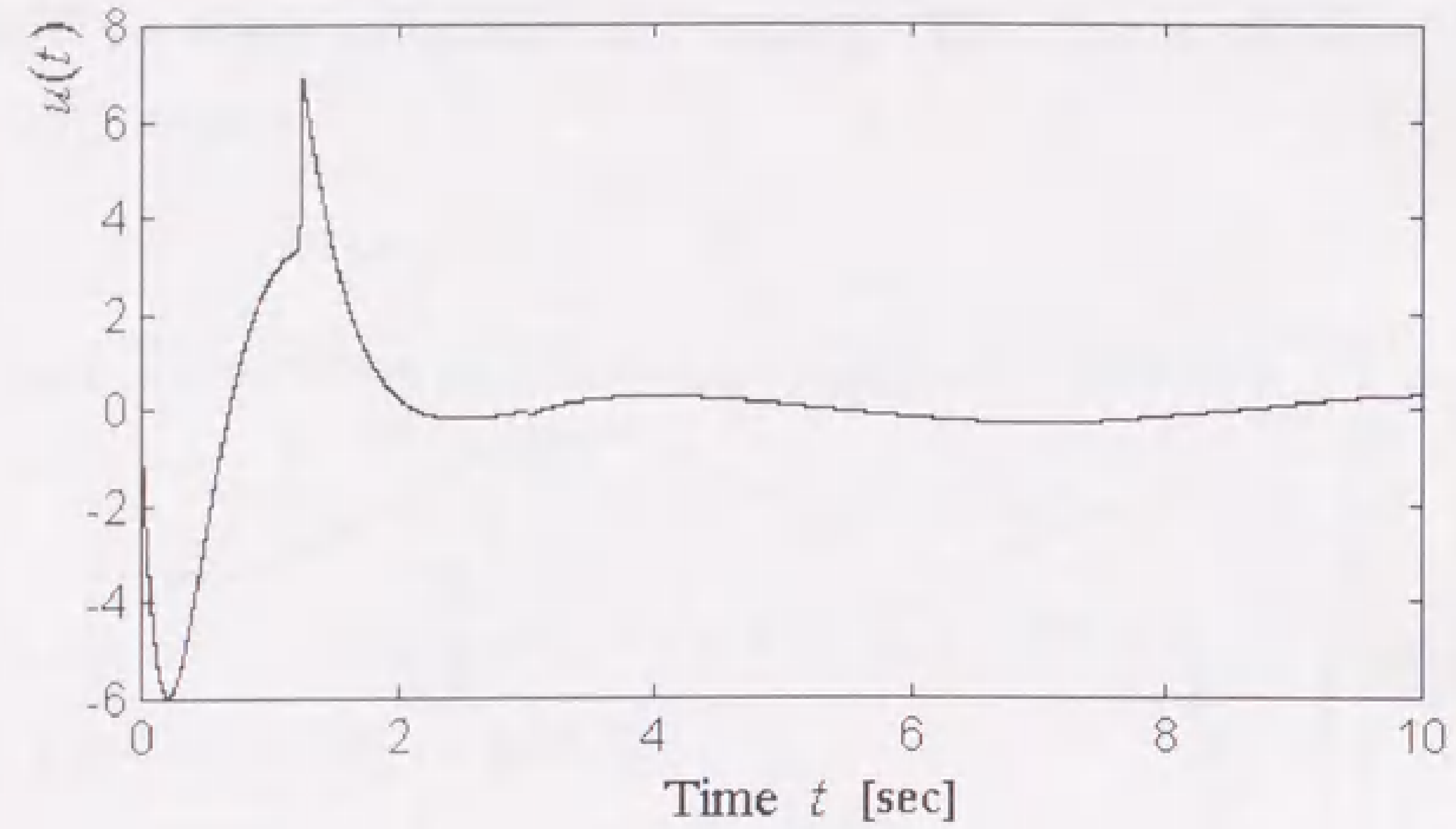



Fig. 6.6 The pole assignment control $u(t)$ of Example 6.2.

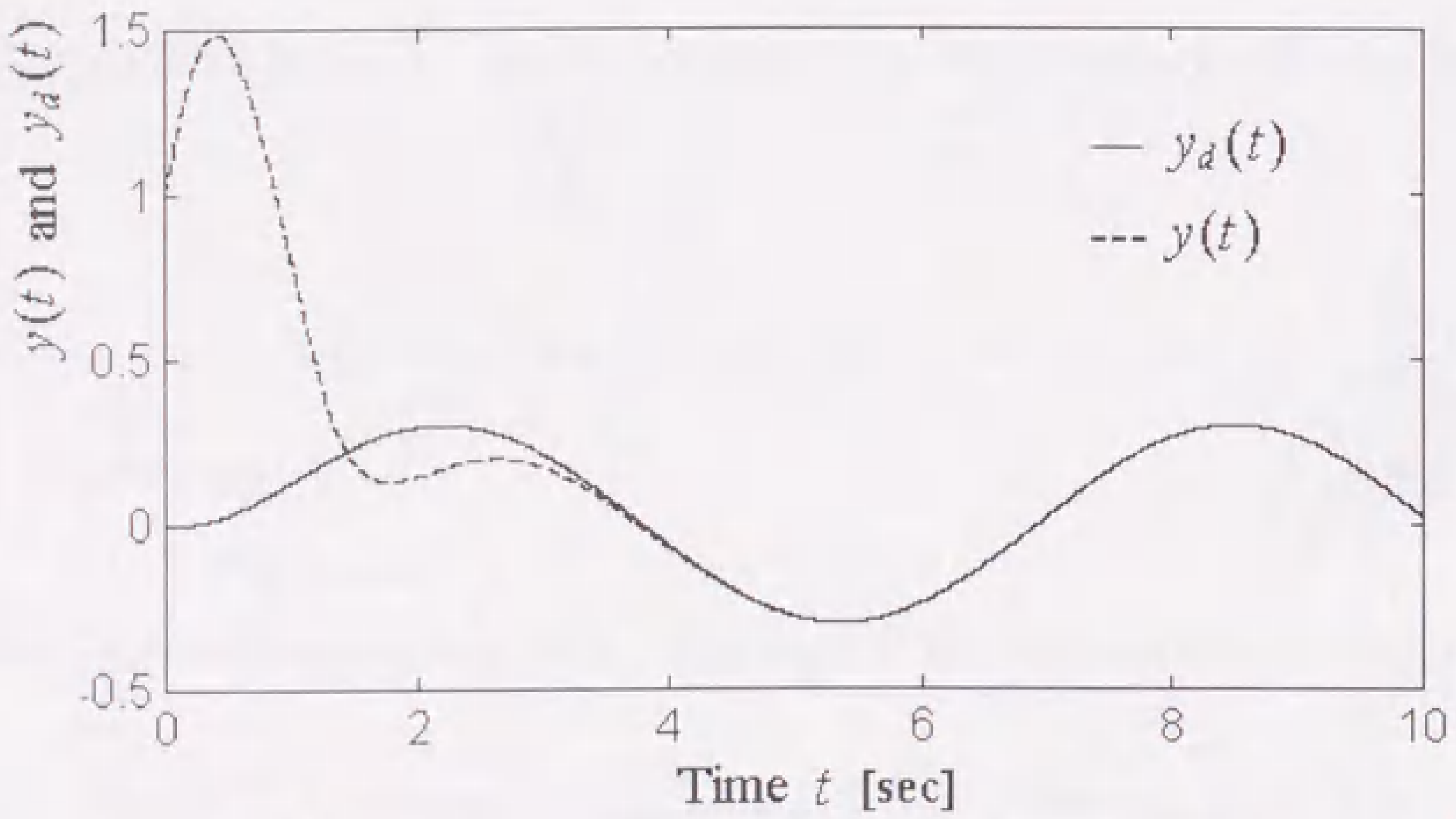



Fig. 6.7 The controlled output $y(t)$ and the desired output $y_d(t)$ of Example 6.2.

## 6.7 Extension to MIMO Uncertain Systems

Consider the uncertain dynamical system described by



$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Kv(x,u,t) \\ y(t) = C^T x(t) \end{cases} \tag{6.78}$$

where $x(t) \in R^n$ is the unknown state vector, $u(t) \in R^r$ is the input and $y(t) \in R^p$ is the output, $v(x,u,t) \in R^m$ is the disturbance (or system nonlinearities and any model uncertainties). $A$, $B$, $K$ and $C$ are known matrices with appropriate dimensions and are described in the observable canonical form

$$A = \left[\begin{array}{cc|cc|c|cc} -a_1^{(1)} & \begin{matrix} I \\ 0 \end{matrix} & -a_2^{(1)} & 0 & \cdots & -a_p^{(1)} & 0 \\ \hline -a_1^{(2)} & 0 & -a_2^{(2)} & \begin{matrix} I \\ 0 \end{matrix} & \cdots & -a_p^{(2)} & 0 \\ \hline \cdots & & \cdots & & \ddots & \cdots & \\ \hline -a_1^{(p)} & 0 & -a_2^{(p)} & 0 & \cdots & -a_p^{(p)} & \begin{matrix} I \\ 0 \end{matrix} \end{array}\right] \tag{6.79}$$

$$B = \left[\begin{array}{c} b_1^{(1)} \cdots b_r^{(1)} \\ \hline b_1^{(2)} \cdots b_r^{(2)} \\ \hline \vdots \\ \hline b_1^{(p)} \cdots b_r^{(p)} \end{array}\right], \quad K = \left[\begin{array}{c} k_1^{(1)} \cdots k_m^{(1)} \\ \hline k_1^{(2)} \cdots k_m^{(2)} \\ \hline \vdots \\ \hline k_1^{(p)} \cdots k_m^{(p)} \end{array}\right] \tag{6.80}$$

$$C^T = \left[\begin{array}{c|c|c|c} 1\ 0 \cdots 0 & 0 & \cdots & 0 \\ \hline 0 & 1\ 0 \cdots 0 & \cdots & 0 \\ \hline \cdots & \cdots & \ddots & \cdots \\ \hline 0 & 0 & \cdots & 1\ 0 \cdots 0 \end{array}\right] \tag{6.81}$$

It is obvious that $n = n_1 + n_2 + \cdots + n_p$.

For the system (6.78), we make the following assumptions.

**Assumption 6.3** The matrix $K$ is of full rank.

**Assumption 6.4** The system is in minimum phase (with respect to the relation between the disturbance and the output), i.e.

$$\left[\begin{array}{c|c} A - sI & K \\ \hline C^T & 0 \end{array}\right] \tag{6.82}$$

has full rank for all the $s$ satisfying $\mathrm{Re}(s) > 0$.

**Assumption 6.5** The disturbance is bounded by

$$\|v(x,u,t)\|_2 \leq \rho(y,u,t) \tag{6.83}$$

where $\|\cdot\|_2$ denotes the Euclidean norm, $\rho(y,u,t) \geq 0$ is a known scalar function.

**Assumption 6.6** The number of outputs is not smaller than that of the disturbances.

Without loss of generality, we also assume that $m = p$, i.e. the number of outputs is equal to the number of disturbances.

To begin with, we introduce a stable $n \times n$ matrix $F$ by

$$F = \left[\begin{array}{cc|c|c|cc} -f^{(1)} & \begin{matrix} I \\ 0 \end{matrix} & 0 & \cdots & 0 & \\ \hline 0 & & -f^{(2)} \ \begin{matrix} I \\ 0 \end{matrix} & \cdots & 0 & \\ \hline \cdots & & \cdots & \ddots & \cdots & \\ \hline 0 & & 0 & \cdots & -f^{(p)} & \begin{matrix} I \\ 0 \end{matrix} \end{array}\right] \triangleq \left[\begin{array}{c|c|c|c} F^{(1)} & 0 & \cdots & 0 \\ \hline 0 & F^{(2)} & \cdots & 0 \\ \hline \cdots & \cdots & \ddots & \cdots \\ \hline 0 & 0 & \cdots & F^{(3)} \end{array}\right] \tag{6.84}$$

Define

$$x(t) = \left[x_1^{(1)}(t), \cdots, x_{n_1}^{(1)}(t), \ \cdots, \ x_1^{(p)}(t), \cdots, x_{n_p}^{(p)}(t)\right]^T$$

$$\triangleq \left[x^{(1)}(t) \vdots \cdots \vdots x^{(p)}(t)\right]^T \tag{6.85}$$

$$y(t) = \left[y^{(1)}(t), \ \cdots, \ y^{(p)}(t)\right]^T \tag{6.86}$$

$$u(t) = \left[u^{(1)}(t), \ \cdots, \ u^{(r)}(t)\right]^T \tag{6.87}$$

$$v(t) = \left[v^{(1)}(t), \ \cdots, \ v^{(p)}(t)\right]^T \tag{6.88}$$

Then, from (6.78)-(6.81) and (6.84)-(6.88), for $1 \leq i \leq p$, we have

$$\dot{x}^{(i)}(t) = F^{(i)}x^{(i)}(t) + (f^{(i)} - a_i^{(i)})y^{(i)}(t)$$

$$-\sum_{\substack{j=1 \\ j \neq i}}^{p} a_j^{(i)} y^{(j)}(t) + \sum_{j=1}^{r} b_j^{(i)} u^{(j)}(t) + \sum_{j=1}^{p} k_j^{(i)} v^{(j)}(t)$$



$$\triangleq F^{(i)}x^{(i)}(t) + \sum_{j=1}^{p} h_{aj}^{(i)} y^{(j)}(t) + \sum_{j=1}^{r} h_{bj}^{(i)} u^{(j)}(t) + \sum_{j=1}^{p} h_{kj}^{(i)} v^{(j)}(t) \tag{6.89}$$

Then, for $1 \leq i \leq p$, similar to Theorem 6.1, (6.89) gives

$$x^{(i)}(t) = \sum_{j=1}^{p} H(f^{(i)}, h_{aj}^{(i)}) \begin{bmatrix} s^{n_i-1} \\ \vdots \\ s \\ 1 \end{bmatrix} \frac{y^{(j)}(t)}{f^{(i)}(s)}$$

$$+ \sum_{j=1}^{r} H(f^{(i)}, h_{bj}^{(i)}) \begin{bmatrix} s^{n_i-1} \\ \vdots \\ s \\ 1 \end{bmatrix} \frac{u^{(j)}(t)}{f^{(i)}(s)}$$

$$+ \sum_{j=1}^{p} H(f^{(i)}, h_{kj}^{(i)}) \begin{bmatrix} s^{n_i-1} \\ \vdots \\ s \\ 1 \end{bmatrix} \frac{v^{(j)}(t)}{f^{(i)}(s)} \tag{6.90}$$

where $f^{(i)}(s) = s^{n_i} + f_1^{(i)} s^{n_i-1} + \cdots + f_{n_i}^{(i)}$.

**Remark 6.7** The above expression can be regarded as an implicit observer of the state. It should be pointed out that the signals in the vector $\left[s^{n_i-1} \ \cdots \ s \ 1\right] \frac{v^{(j)}(t)}{f^{(i)}(s)}$ (for $i, j = 1, 2, \cdots, p$) are not available.

Now, some operations will be carried out on the state observer (6.90). Here, the Hurwitz polynomials $f^{(i)}(s)$, for $1 \leq i \leq p$, in (6.90) are considered as

$$f^{(i)}(s) = s^{n_i} + f_1^{(i)} s^{n_i-1} + \cdots + f_{n_i}^{(i)}$$

$$= (s + \lambda_1^{(i)})(s + \lambda_2^{(i)}) \cdots (s + \lambda_{n_i}^{(i)}) \tag{6.91}$$

where $\lambda_l^{(i)} \neq \lambda_j^{(i)}$ as $l \neq j$, for $l, j = 1, \cdots, n_i$.

For $1 \leq i \leq p$, pre-multiplying the $i$-th equation in (6.90) by the vector $\left[\left(\lambda_l^{(i)}\right)^{n_i-1}, -\left(\lambda_l^{(i)}\right)^{n_i-2}, \cdots, (-1)^{n_i-1}\right]$ respectively yields

$$\left[\left(\lambda_l^{(i)}\right)^{n_i-1},-\left(\lambda_l^{(i)}\right)^{n_i-2},\cdots,(-1)^{n_i-1}\right]x^{(i)}(t)$$

$$=\sum_{j=1}^{p}\left[\left(\lambda_l^{(i)}\right)^{n_i-1},-\left(\lambda_l^{(i)}\right)^{n_i-2},\cdots,(-1)^{n_i-1}\right]H(f^{(i)},h_{aj}^{(i)})\begin{bmatrix}s^{n_i-1}\\ \vdots\\ s\\ 1\end{bmatrix}\frac{y^{(j)}(t)}{f^{(i)}(s)}$$

$$+\sum_{j=1}^{r}\left[\left(\lambda_l^{(i)}\right)^{n_i-1},-\left(\lambda_l^{(i)}\right)^{n_i-2},\cdots,(-1)^{n_i-1}\right]H(f^{(i)},h_{bj}^{(i)})\begin{bmatrix}s^{n_i-1}\\ \vdots\\ s\\ 1\end{bmatrix}\frac{u^{(j)}(t)}{f^{(i)}(s)}$$

$$+\sum_{j=1}^{p}\left[\left(\lambda_l^{(i)}\right)^{n_i-1},-\left(\lambda_l^{(i)}\right)^{n_i-2},\cdots,(-1)^{n_i-1}\right]H(f^{(i)},h_{kj}^{(i)})\begin{bmatrix}s^{n_i-1}\\ \vdots\\ s\\ 1\end{bmatrix}\frac{v^{(j)}(t)}{f^{(i)}(s)} \quad (6.92)$$

Thus, by applying Lemma 6.2, Equation (6.92) can be simplified as

$$\left[\left(\lambda_l^{(i)}\right)^{n_i-1},-\left(\lambda_l^{(i)}\right)^{n_i-2},\cdots,(-1)^{n_i-1}\right]x^{(i)}(t)$$

$$=\sum_{j=1}^{p}\frac{\chi_{a\,j,l}^{(i)}}{s+\lambda_l^{(i)}}y^{(j)}(t)+\sum_{j=1}^{r}\frac{\chi_{b\,j,l}^{(i)}}{s+\lambda_l^{(i)}}u^{(j)}(t)+\sum_{j=1}^{p}\frac{\chi_{k\,j,l}^{(i)}}{s+\lambda_l^{(i)}}v^{(j)}(t) \quad (6.93)$$

where

$$\chi_{a\,j,l}^{(i)}=h_{aj1}^{(i)}\lambda_l^{n_i-1}-h_{aj2}^{(i)}\lambda_l^{n_i-2}+\cdots+(-1)^{n_i-1}h_{ajn_i}^{(i)} \quad (6.94)$$

$$\chi_{b\,j,l}^{(i)}=h_{bj1}^{(i)}\lambda_l^{n_i-1}-h_{bj2}^{(i)}\lambda_l^{n_i-2}+\cdots+(-1)^{n_i-1}h_{bjn_i}^{(i)} \quad (6.95)$$

$$\chi_{k\,j,l}^{(i)}=h_{k\,j1}^{(i)}\lambda_l^{n_i-1}-h_{k\,j2}^{(i)}\lambda_l^{n_i-2}+\cdots+(-1)^{n_i-1}h_{k\,jn_i}^{(i)} \quad (6.96)$$

Then, for $l=1,2,\cdots,n_i$, Equation (6.93) can be written in the next compact form

$$\begin{bmatrix}\left(\lambda_1^{(i)}\right)^{n_i-1} & -\left(\lambda_1^{(i)}\right)^{n_i-2} & \cdots & (-1)^{n_i-2}\lambda_1^{(i)} & (-1)^{n_i-1}\\ \left(\lambda_2^{(i)}\right)^{n_i-1} & -\left(\lambda_2^{(i)}\right)^{n_i-2} & \cdots & (-1)^{n_i-2}\lambda_2^{(i)} & (-1)^{n_i-1}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ \left(\lambda_{n_i}^{(i)}\right)^{n_i-1} & -\left(\lambda_{n_i}^{(i)}\right)^{n_i-2} & \cdots & (-1)^{n_i-2}\lambda_{n_i}^{(i)} & (-1)^{n_i-1}\end{bmatrix}\begin{bmatrix}x_1^{(i)}(t)\\ \vdots\\ x_{n_i}^{(i)}(t)\end{bmatrix}$$



$$=\sum_{j=1}^{p}\begin{bmatrix}\frac{\chi_{a\,j,1}^{(i)}}{s+\lambda_1^{(i)}}y^{(j)}(t)\\ \vdots\\ \frac{\chi_{a\,j,n_i}^{(i)}}{s+\lambda_{n_i}^{(i)}}y^{(j)}(t)\end{bmatrix}+\sum_{j=1}^{r}\begin{bmatrix}\frac{\chi_{b\,j,1}^{(i)}}{s+\lambda_1^{(i)}}u^{(j)}(t)\\ \vdots\\ \frac{\chi_{b\,j,n_i}^{(i)}}{s+\lambda_{n_i}^{(i)}}u^{(j)}(t)\end{bmatrix}+\sum_{j=1}^{p}\begin{bmatrix}\frac{\chi_{k\,j,1}^{(i)}}{s+\lambda_1^{(i)}}v^{(j)}(t)\\ \vdots\\ \frac{\chi_{k\,j,n_i}^{(i)}}{s+\lambda_{n_i}^{(i)}}v^{(j)}(t)\end{bmatrix} \quad (6.97)$$

On the other hand, it is well known that the Vandermonde matrix

$$\Lambda^{(i)}\triangleq\begin{bmatrix}\left(\lambda_1^{(i)}\right)^{n_i-1} & \left(\lambda_1^{(i)}\right)^{n_i-2} & \cdots & (-1)^{n_i-2}\lambda_1^{(i)} & (-1)^{n_i-1}\\ \left(\lambda_2^{(i)}\right)^{n_i-1} & \left(\lambda_2^{(i)}\right)^{n_i-2} & \cdots & (-1)^{n_i-2}\lambda_2^{(i)} & (-1)^{n_i-1}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ \left(\lambda_{n_i}^{(i)}\right)^{n_i-1} & \left(\lambda_{n_i}^{(i)}\right)^{n_i-2} & \cdots & (-1)^{n_i-2}\lambda_{n_i}^{(i)} & (-1)^{n_i-1}\end{bmatrix} \quad (6.98)$$

is nonsingular when $\lambda_l^{(i)}\neq\lambda_j^{(i)}$ as $l\neq j$, for $l,j=1,\cdots,n_i-1$.

Therefore, by pre-multiplying (6.97) with $\left(\Lambda^{(i)}\right)^{-1}$, the partial states $\left[x_1^{(i)}(t),x_2^{(i)}(t),\cdots,x_{n_i}^{(i)}\right]^T$ can be expressed by the first order filters of the inputs, outputs and disturbances. Consequently, we have the following theorem.

**Theorem 6.6** A new implicit observer of the partial states $\left[x_1^{(i)}(t),x_2^{(i)}(t),\cdots,x_{n_i}^{(i)}\right]^T$ can be formulated as

$$\begin{bmatrix}\hat{x}_1^{(i)}(t)\\ \vdots\\ \hat{x}_{n_i}^{(i)}(t)\end{bmatrix}=\left(\Lambda^{(i)}\right)^{-1}\left\{\sum_{j=1}^{p}\begin{bmatrix}\frac{\chi_{a\,j,1}^{(i)}}{s+\lambda_1^{(i)}}y^{(j)}(t)\\ \vdots\\ \frac{\chi_{a\,j,n_i}^{(i)}}{s+\lambda_{n_i}^{(i)}}y^{(j)}(t)\end{bmatrix}+\sum_{j=1}^{r}\begin{bmatrix}\frac{\chi_{b\,j,1}^{(i)}}{s+\lambda_1^{(i)}}u^{(j)}(t)\\ \vdots\\ \frac{\chi_{b\,j,n_i}^{(i)}}{s+\lambda_{n_i}^{(i)}}u^{(j)}(t)\end{bmatrix}+\sum_{j=1}^{p}\begin{bmatrix}\frac{\chi_{k\,j,1}^{(i)}}{s+\lambda_1^{(i)}}v^{(j)}(t)\\ \vdots\\ \frac{\chi_{k\,j,n_i}^{(i)}}{s+\lambda_{n_i}^{(i)}}v^{(j)}(t)\end{bmatrix}\right\} \quad (6.99)$$

for $i=1,2,\cdots,p$.

**Proof**: The proof is similar to that of Theorem 6.1.

**Remark 6.8** In the state expression (6.99), only the first order filters of the disturbances are unknown.

Similar to SISO systems, for the systems with relative degree one, the first order filters of the disturbances can be expressed by available signals; for the systems with higher relative degrees, the first order filters of the disturbances can be estimated by Theorem 5.1. Thus, an implicit state observer can be formulated based on (6.99) for MIMO minimum phase uncertain systems.

As the disturbances can be estimated by Theorem 5.1, the obtained implicit state observer and the estimated disturbance can be applied to a pole assignment controller similar to that in SISO systems.

### 6.8 Conclusions

In this chapter, the state is mathematically expressed by the first order filters of the input, output and disturbance for the systems with uncertainties. For the systems with relative degree one, the state is expressed by the filters of the in put and the output, where no *a priori* information about the disturbance is required. For the systems with higher relative degrees, the state is expressed by the estimates of the first order filters of the disturbance, and the filters of the input and output. Then, the estimated disturbance and generated state observer are employed to construct a state feedback controller to place the desired poles and to cancel the disturbance. Examples and simulation results show that the proposed algorithms are effective for practical applications. Finally, the results are extended to MIMO minimum phase uncertain systems.

For the SISO nonminimum phase dynamical systems, a similar result can be obtained by applying Theorem 4.3.

The obtained results are expected to be extended to much more complicated uncertain systems, for example, MIMO nonminimum phase uncertain systems, the uncertain systems with delays, or the uncertain descriptor systems .

# Chapter 7

# Conclusions

In this thesis, for the state space described uncertain systems, the digital implementation of well-designed continuous-time VSS control systems, and the discrete-time sliding mode control techniques are studied; for the input-output represented uncertain models in minimum phases as well as nonminimum phases, the disturbance identification, state observer formulation and the robust control design problems are studied by employing the VSS theory. In the following, we shall conclude this thesis and point out the future research subjects.

### 7.1 Conclusion of the Thesis

For solving the problem associated with implementation by computer of a well designed continuous-time variable structure control system, Chapter 2 proposes a more general definition, which is called "discrete-time weak-pseudo-sliding mode", to describe the zigzag behavior along the hyperplane for the corresponding discrete-time system. An estimation of the upper bound of the sampling period is presented to ensure that the weak-pseudo-sliding mode can take place along the prescribed hyperplane. If the chosen sampling period is not greater than this upper bound, the zero-order-hold of the continuous-time VSC structure can still be used without any modification, and the stability of the sampled data system can also be guaranteed.

The obtained results are also confirmed by an example.

In Chapter 3, by applying the adaptive algorithms with dead-zone, the unknown parameters and the upper bound of the disturbances are estimated for the system with model uncertainties, unmodeled dynamics and bounded disturbances. Based on these estimates, a discrete robust quasi-sliding mode controller is proposed to guarantee the system to be stable in the sense that all signals in the loop remain bounded. The new controller makes the best use of the advantages of the adaptive control and sliding mode control. Simulation results show the effectiveness of the proposed algorithm. The method of estimating the parameters and the upper bound of the absolute values of the perturbations is also an important contribution to the adaptive control theory. If the unknown term $f(k, u(k), x(k))$ is matched and slow varying with respect to $k$, the proposed controller can be modified by using the techniques proposed in [2,14].

In Chapter 4, based on the VSS equivalent control method, the disturbance observer which was regarded impossible until now is formulated for minimum phase dynamical systems with arbitrarily relative degrees. Here the term "disturbance" is referred to the combination of the model uncertainties, the nonlinear parts of the system and the external disturbances. Only the upper bound of the disturbances is assumed as the *a priori* information. By choosing small design parameters $\delta_i$ ($1 \le i \le r$) and $\lambda$, the performance of the new observer may become better even for the high frequency disturbances But there is a limitation about the choice of these parameters when the measurement noises are present. When it is implemented by a digital computer, the design parameters are also limited by the sampling period. Also, based on the approximate inverse systems techniques, the proposed formulation is extended to a class of nonminimum phase systems. The estimated disturbance is used to construct a state observer and a pole assignment controller. Design example is presented to show the effectiveness of the proposed algorithm. The new disturbance observer is also applied to control a linear motor which shows the new observer is of practicality. Moreover, experiment results show that the new observer is superior to the traditional disturbance observer when the impulse disturbances exist.



In Chapter 5, based on the VSS equivalent control method, the disturbance (or system nonlinearities plus any model uncertainties) estimation problem which has been regarded unsolvable until now is overcome for MIMO minimum phase dynamical systems with arbitrary relative degrees. The estimated disturbances are then used to generate a robust state observer and a pole placement controller which also has the function to cancel the disturbances. An example is given to show the design procedure and the effectiveness of the proposed formulation.

In Chapter 6, the state is mathematically expressed for (SISO or MIMO) minimum phase dynamical systems. For the system with relative degree one, the state is expressed by the filters of the input and output. For the system with higher relative degrees, the state is expressed by the filters of the input and output, and the estimates of the first order filters of the disturbance. Then, the estimated disturbance and generated state observer are employed to construct a state feedback controller to place the desired poles and to cancel the disturbance. Examples are given for each case to show the effectiveness of the implicit state observers.

## 7.2 The Future Research Subjects

In this thesis, there are several unsolved problems which may be the future research subjects.

1) In Chapter 2, for the multivariable uncertain systems, the upper bound of the sampling period to assure the stability of the sampled-data systems is expected to be derived.

2) In Chapter 3, the affine parameter-dependent models may be considered as the future research. Also, the proposed method is expected to be extended to input-output models with parameter uncertainties, unmodeled dynamics and bounded disturbances.

3) The new disturbance identification method formulated in Chapters 4 and 5 should be extended to MIMO nonminimum phase uncertain systems. Further, for much more complicated systems, such as the uncertain systems with delays, descriptor

systems, affine parameter-dependent models, a similar formulation is expected to be derived. Also, for the systems with relatively large stochastic disturbances or measurement noises, the proposed disturbance observer should be modified.

4) The implicit state observer techniques proposed in Chapter 6 should be extended to MIMO nonminimum phase uncertain systems with arbitrarily relative degrees. For descriptor systems with disturbances or the uncertain systems with delays, the implicit state observer formulations may be considered.

Apart from the above remained problems, the VSS control should be combined with the intelligent control methods (Fuzzy Control, Neural Network Control, etc.) to improve the performance of the controlled systems. Also, the control problems of nonlinear systems should be studied. Because very high level mathematical knowledge is required for controlling nonlinear systems, very few good results have been reported until now. From now on, by combining the VSS theory and Lyapunov method with the differential geometry theory, the control problems for nonlinear systems are expected to be studied.

# Appendix

## A.1 Proof of Theorem 2.2

From the relation (2.46), we get

$$\begin{aligned} |s(k+1)| &= |s(k)) + hF(h,x(k)) - hG(x(k))\,\mathrm{sgn}(s(k))| \\ &= \left|\left\{s(k) + hF(h,x(k)) - hG(x(k))\,\mathrm{sgn}(s(k))\right\}\mathrm{sgn}(s(k))\right| \\ &= \left||s(k)| - h\left\{G(x(k)) - F(h,x(k))\,\mathrm{sgn}(s(k))\right\}\right| \end{aligned} \quad \text{(A1)}$$

On the other hand, by the result (2.53) of Lemma 2.7,

$$\begin{aligned} & h\left\{G(x(k)) - F(h,x(k))\,\mathrm{sgn}(s(k))\right\} \\ & \le h\left\{G(x(k)) + |F(h,x(k))|\right\} \\ & < \varepsilon \|x(k)\|_1 \end{aligned} \quad \text{(A2)}$$

and

$$\begin{aligned} & h\left\{G(x(k)) - F(h,x(k))\,\mathrm{sgn}(s(k))\right\} \\ & \ge h\{G(x(k)) - |F(h,x(k))|\} \\ & > \frac{h\delta\|x(k)\|_1}{8} \end{aligned} \quad \text{(A3)}$$

Summing up (A2) and (A3) yields

$$\frac{h\delta\|x(k)\|_1}{8} < h\left\{G(x(k)) - F(h,x(k))\,\mathrm{sgn}(s(k))\right\} < \varepsilon\|x(k)\|_1 \quad \text{(A4)}$$

About the relation between $|s(k)|$ and $\varepsilon\|x(k)\|_1$, it can be divided into the next two cases:

**Case 1** There exist infinite integers, say a sequence $\{k_i : i = 1,2,\cdots\}$, such that

$$|s(k_i)| < \varepsilon\|x(k_i)\|_1 \tag{A5}$$

**Case 2** There exist finite integers $k_j$ $(j = 1,2,\cdots,N)$ such that $|s(k_j)| < \varepsilon\|x(k_j)\|_1$, then there exist an integer $K$ such that for all $k \geq K$

$$|s(k)| \geq \varepsilon\|x(k)\|_1 \tag{A6}$$

Now, detailed analyses are given for each case.

**Case 1** From (A1), (A4) and the assumption (A5), we can easily know that

$$|s(k_i+1)| < \varepsilon\|x(k_i)\|_1 \tag{A7}$$

If $|s(k_i+1)| < \varepsilon\|x(k_i+1)\|_1$, then $k_i+1 = k_{i+1}$ holds.

If $|s(k_i+1)| \geq \varepsilon\|x(k_i+1)\|_1$, by using the inequality (A4) for $k = k_i+1$, $|s(k_i+2)|$ can be calculated as

$$\begin{aligned}|s(k_i+2)| &= \big||s(k_i+1)| - h\{G(x(k_i+1)) - F(h,x(k_i+1))\,\mathrm{sgn}(s(k_i+1))\}\big| \\ &= |s(k_i+1)| - h\{G(x(k_i+1)) - F(h,x(k_i+1))\,sign(s(k_i+1))\} \\ &< |s(k_i+1)| - \frac{h\delta\|x(k_i+1)\|_1}{8}\end{aligned} \tag{A8}$$

Therefore,

$$|s(k_i+2)| < |s(k_i+1)| < \varepsilon\|x(k_i)\|_1 \tag{A9}$$

If $|s(k_i+2)| < \varepsilon\|x(k_i+2)\|_1$, then $k_i+2 = k_{i+1}$ holds.

If $|s(k_i+2)| \geq \varepsilon\|x(k_i+2)\|_1$, then similar to (A8), we have

$$|s(k_i+3)| < |s(k_i+2)| - \frac{h\delta\|x(k_i+2)\|_1}{8} \tag{A10}$$

Thus,

$$|s(k_i+3)| < |s(k_i+2)| < |s(k_i+1)| < \varepsilon\|x(k_i)\|_1 \tag{A11}$$



By mathematical induction, we have

$$|s(k)| \geq \varepsilon\|x(k)\|_1 \text{ and } |s(k+1)| < |s(k)| < \varepsilon\|x(k_i)\|_1 \tag{A12}$$

for $k_{i+1} > k > k_i$.

By repeating this procedure infinitely, we can prove that the weak-pseudo-sliding mode takes place along the hyperplane.

In the second case, from (A1) and (A4), the next relation can be proved for all $k \geq K$.

$$\begin{aligned}|s(k+1)| &= \big||s(k)| - h\{G(x(k)) - F(h,x(k))\,\mathrm{sgn}(s(k))\}\big| \\ &= |s(k)| - h\{G(x(k)) - F(h,x(k))\,\mathrm{sgn}(s(k))\} \\ &< |s(k)| - \frac{h\delta\|x(k)\|_1}{8}\end{aligned} \tag{A13}$$

So, $s(k)$ is monotonically decreasing.

Now, by summing both sides of inequalities (A13) for all $k \geq K$,

$$|s(k+1)| < |s(K)| - \frac{h\delta}{8}\sum_{j=K}^{k}\|x(j)\|_1 \tag{A14}$$

From (A14), it can be easily concluded that $\sum_{j=K}^{k}\|x(j)\|_1$ converges as $k \to \infty$.

Therefore, $x(k) \to 0$ and $s(k) \to 0$, as $k \to \infty$. The theorem has been proved.

## A.2 Proof of Lemma 2.8

For any $(k+1)h \geq t \geq kh$, we write $t$ as $t = kh + \eta h\ (1 \geq \eta \geq 0)$. From (2.33), we have

$$x(t) = \exp(A\eta h)x(k) + \int_{kh}^{t}\exp\{A(t-\tau)\}d\tau \cdot b \cdot u(k) \tag{A15}$$

By pre-multiplying (A15) with the vector $c$, a relation similar to (2.38) can be obtained.

$$s(t) = s(k) + c\int_0^{\eta h}\exp\{A\tau\}d\tau \cdot \{x(k) + b \cdot u(k)\} \tag{A16}$$

Here, (A16) can be regarded as replacing $h$ by $h\eta$ in (2.38). Therefore, by some similar operations in deriving (2.46), (A16) can be written as

$$s(t) = s(k) + \eta h F(\eta h, x(k)) - \eta h G(x(k))\,\mathrm{sgn}(s(k)) \tag{A17}$$

where $F(\eta h, x(k))$ is obtained by replacing $h$ with $\eta h$ in the definition of $F(h, x(k))$.

If $h$ meets the condition $h \le h_m$, then $\eta h \le h_m$ as $1 \ge \eta \ge 0$. Thus, by Lemma 2.7, it can also be concluded that

$$\frac{\delta\|x(k)\|_1}{8} + |F(\eta h, x(k))| < G(x(k)) \le \frac{\varepsilon\|x(k)\|_1}{2h} \tag{A18}$$

Therefore, we have

$$\frac{h\delta\|x(k)\|_1}{8} < h\{G(x(k)) - F(\eta h, x(k))\,\mathrm{sgn}(s(k))\} < \varepsilon\|x(k)\|_1 \tag{A19}$$

For $(k_i + 1)h \ge t > k_i h$, from (A17) and (A19), we can easily get

$$|s(t)| < \varepsilon\|x(k_i)\|_1 \tag{A20}$$

where the fact $|s(k_i)| < \varepsilon\|x(k_i)\|_1$ is used. Making one more step, for $(k_i + 2)h \ge t > (k_i + 1)h$, from (A17) and (A19), we have

$$\begin{aligned}
|s(t)| &= |s(k_i + 1) + \eta h F(\eta h, x(k_i + 1)) - \eta h G(x(k_i + 1))\,\mathrm{sgn}(s(k_i + 1))| \\
&= |s(k_i + 1)| - \eta h\{G(x(k_i + 1)) - F(\eta h, x(k_i + 1))\,\mathrm{sgn}(s(k_i + 1))\} \\
&< |s(k_i + 1)| - \frac{\eta h \delta\|x(k_i + 1)\|_1}{8} \\
&\le |s(k_i + 1)| < \varepsilon\|x(k_i)\|_1
\end{aligned} \tag{A21}$$

where the facts $|s(k_i + 1)| \ge \varepsilon\|x(k_i + 1)\|_1$ and $|s(k_i + 1)| < \varepsilon\|x(k_i)\|_1$ are used. By mathematical induction, for $k_{i+1}h \ge t > (k_{i+1} - 1)h$, we have

$$\begin{aligned}
|s(t)| &= |s(k_{i+1} - 1) + \eta h F(\eta h, x(k_{i+1} - 1)) - \eta h G(x(k_{i+1} - 1))\,\mathrm{sgn}(s(k_{i+1} - 1))| \\
&= |s(k_{i+1} - 1)| - \eta h\{G(x(k_{i+1} - 1)) - F(\eta h, x(k_{i+1} - 1))\,\mathrm{sgn}(s(k_{i+1} - 1))\} \\
&< |s(k_{i+1} - 1)| - \frac{\eta h \delta\|x(k_{i+1} - 1)\|_1}{8}
\end{aligned}$$



$$\le |s(k_{i+1} - 1)| < |s(k_i + 1)| < \varepsilon\|x(k_i)\|_1 \tag{A22}$$

where $|s(k_{i+1} - 1)| \ge \varepsilon\|x(k_{i+1} - 1)\|_1$ and $|s(k_{i+1} - 1)| < \cdots < |s(k_i + 1)| < \varepsilon\|x(k_i)\|_1$ are employed.

Therefore, from the above analysis, the conclusion of this lemma is obvious.

## A.3 Proof of Lemma 2.9

By the definition of $\underline{x}(t)$ and the fact $x_n(t) = s(t) - \underline{c} \cdot \underline{x}(t)$,

$$\begin{aligned}
\|x(k)\|_1 &= \|\underline{x}(k)\|_1 + |x_n(k)| \\
&\le \|\underline{x}(k)\|_1 + \|\underline{c}\|_\infty \|\underline{x}(k)\|_1 + |s(k)| \\
&< (1 + \|\underline{c}\|_\infty)\|\underline{x}(k)\|_1 + \varepsilon\|x(k)\|_1
\end{aligned} \tag{A23}$$

By solving $\|x(k)\|_1$ in (A23),

$$\|x(k)\|_1 < \frac{1}{1-\varepsilon}(1 + \|\underline{c}\|_\infty)\|\underline{x}(k)\|_1 \tag{A24}$$

Now, we start with $\underline{x}(k_1)$ and define $k_{i_1} = k_1$. Let $M_1 = \left[\frac{k_2 - k_1}{L}\right]$ represent the integer part of $\frac{k_2 - k_1}{L}$. For the defined $M_1$, there are two cases.

**Case 1** $M_1 \ge 1$**:**

From (2.65), by employing Lemma 2.8, we have

$$\begin{aligned}
\|\underline{x}(k_1 + mL + l)\|_1 &\le \|\exp(\Gamma Lh)\|_1 \|\underline{x}(k_1 + (m-1)L + l)\|_1 \\
&\quad + \int_0^{Lh} \|\exp(\Gamma(Lh - \tau))\|_1 |s((k_1 + (m-1)L + l)h + \tau)| d\tau \\
&\le \frac{1}{8}\|\underline{x}(k_1 + (m-1)L + l)\|_1 + \varepsilon\|x(k_1)\|\int_0^{Lh} d_2 e^{-d_1(Lh-\tau)} d\tau
\end{aligned}$$

$$\leq \frac{1}{8}\|\underline{x}(k_1+(m-1)L+l)\|_1 + \frac{d_2\varepsilon(1+\|\underline{c}\|_\infty)}{d_1(1-\varepsilon)}\|\underline{x}(k_1)\|_1 \tag{A25}$$

where $1 \leq m \leq M_1$, $0 \leq l \leq L-1$, but $m$ and $l$ meet the condition $k_1 + mL + l \leq k_2$.

Now, we choose $\varepsilon$ as

$$\varepsilon \leq \varepsilon_m \tag{A26}$$

i.e.

$$\frac{d_2\varepsilon(1+\|\underline{c}\|_\infty)}{d_1(1-\varepsilon)} \leq \frac{3}{4} \tag{A27}$$

Thus, by (A25), it can be proved that

$$\|\underline{x}(k_1+L+i)\|_1 \leq \frac{7}{8}\max_{l=0}^{L-1}\|\underline{x}(k_1+l)\|_1 \tag{A28}$$

for all $0 \leq i \leq k_2 - k_1 - L$. In fact, from (A25), for $0 \leq l \leq L-1$, letting $m=1$ gives

$$\|\underline{x}(k_1+L+l)\|_1 \leq \frac{1}{8}\|\underline{x}(k_1+l)\|_1 + \frac{3}{4}\|\underline{x}(k_1)\|_1$$

$$\leq \frac{7}{8}\max_{l=0}^{L-1}\|\underline{x}(k_1+l)\|_1 \tag{A29}$$

From (A25), for $0 \leq l \leq L-1$, letting $m=2$ yields

$$\|\underline{x}(k_1+2L+l)\|_1 \leq \frac{1}{8}\|\underline{x}(k_1+L+l)\|_1 + \frac{3}{4}\|\underline{x}(k_1)\|_1$$

$$\leq \frac{7}{8}\max_{l=0}^{L-1}\|\underline{x}(k_1+l)\|_1 \tag{A30}$$

where (A29) is applied. By mathematical induction, the inequality (A28) can be proved to be valid for $0 \leq i \leq k_2 - k_1 - L$.

In the following, we will show that (A28) is also valid for $k_2 - k_1 - L < i \leq k_2 - k_1 - 1$. It is known that there must exist $k_\gamma$ such that $k_\gamma \leq k_2 + L - 1 < k_{\gamma+1}$. For $1 \leq \omega \leq k_3 - k_2$, from (2.65), applying Lemma 8 and



(A27) yields

$$\|\underline{x}(k_2+\omega)\|_1 \leq \frac{1}{8}\|\underline{x}(k_2-L+\omega)\|_1 + \frac{3}{4}\max\{\|\underline{x}(k_2)\|_1, \|\underline{x}(k_1)\|_1\} \tag{A31}$$

By using the relations

$$\|\underline{x}(k_2)\|_1 \leq \frac{7}{8}\max_{l=0}^{L-1}\|\underline{x}(k_1+l)\|_1 \quad \text{and} \quad \|\underline{x}(k_2-L+\omega)\|_1 \leq \frac{7}{8}\max_{l=0}^{L-1}\|\underline{x}(k_1+l)\|_1$$

for $1 \leq \omega \leq k_3 - k_2$ implied by (A28), (A31) can be estimated as

$$\|\underline{x}(k_2+\omega)\|_1 \leq \frac{7}{8}\max_{l=0}^{L-1}\|\underline{x}(k_1+l)\|_1 \tag{A32}$$

For $k_3 - k_2 < \omega \leq k_4 - k_2$, from (3.39), it gives

$$\|\underline{x}(k_2+\omega)\|_1 \leq \frac{1}{8}\|\underline{x}(k_2-L+\omega)\|_1 + \frac{3}{4}\max_{t=1}^{3}\{\|\underline{x}(k_t)\|_1\} \tag{A33}$$

where Lemma 2.8 and (A27) are used. Similarly, by using the results implied by (A28) and (A32), (A33) gives

$$\|\underline{x}(k_2+\omega)\|_1 \leq \frac{7}{8}\max_{l=0}^{L-1}\|\underline{x}(k_1+l)\|_1 \tag{A34}$$

for $k_3 - k_2 < \omega \leq k_4 - k_2$. By mathematical induction, for $k_r - k_2 < \omega \leq L-1$, we get

$$\|\underline{x}(k_2+\omega)\|_1 \leq \frac{1}{8}\|\underline{x}(k_2-L+\omega)\|_1 + \frac{3}{4}\max_{t=1}^{r}\{\|\underline{x}(k_t)\|_1\} \tag{A35}$$

Thus, by applying the obtained results before this step, (A35) yields

$$\|\underline{x}(k_2+\omega)\|_1 \leq \frac{7}{8}\max_{l=0}^{L-1}\|\underline{x}(k_1+l)\|_1 \tag{A36}$$

for $k_r - k_2 < \omega \leq L-1$.

Therefore, for all $0 \leq i \leq k_2 - k_1 - 1$, we get

$$\|\underline{x}(k_1+L+i)\|_1 \leq \frac{7}{8}\max_{l=0}^{L-1}\|\underline{x}(k_1+l)\|_1 \tag{A37}$$

We denote $k_{l_2} = k_2$.

**Case 2** $M_1 = 0$:

In this circumstances, we can conclude that there exists $k_r$, such that $k_r \le k_1 + L < k_{r+1}$. Define $M_1^{'} = \left[\frac{k_{r+1} - k_1}{L}\right]$ as the integer part of $\frac{k_{r+1} - k_1}{L}$. Based on the inequality (2.65), similar to (A25), we have

$$\|\underline{x}(k_1 + m^{'}L + l)\|_1 \le \|\exp(\Gamma Lh)\|_1 \|\underline{x}(k_1 + (m^{'} - 1)L + l)\|_1$$

$$+ \int_0^{Lh} \|\exp(\Gamma(Lh - \tau))\|_1 |s((k_1 + (m^{'} - 1)L + l)h + \tau)| d\tau$$

$$\le \frac{1}{8}\|\underline{x}(k_1 + (m^{'} - 1)L + l)\|_1 + \varepsilon \cdot \max_{\iota=1}^{r} \|x(k_\iota)\| \int_0^{Lh} d_2 e^{-d_1(Lh-\tau)} d\tau$$

$$\le \frac{1}{8}\|\underline{x}(k_1 + (m^{'} - 1)L + l)\|_1 + \frac{3}{4}\max_{\iota=1}^{r} \|\underline{x}(k_\iota)\|_1 \tag{A38}$$

where (A27) is used, $1 \le m^{'} \le M_1^{'}$, $0 \le l \le L - 1$, but $m^{'}$ and $l$ meet the condition $k_1 + m^{'}L + l \le k_{r+1}$. Similar to the analysis in (A29) and (A30), by employing the assumed condition $k_r \le k_1 + L$, we can also get

$$\|\underline{x}(k_1 + L + i)\|_1 \le \frac{7}{8}\max_{l=0}^{L-1} \|\underline{x}(k_1 + l)\|_1 \tag{A39}$$

for all $0 \le i \le k_{r+1} - k_1 - L$.

In the following, we show that (A39) is also valid for $k_{r+1} - k_1 - L < i \le k_{r+1} - k_1 - 1$. It is known that there must exist $k_{r'}$, such that $k_{r'} \ge k_{r+1}$ and $k_{r'} \le k_{r+1} + L - 1 < k_{r'+1}$. Similar to the corresponding analysis in case 1, for $1 \le \upsilon \le k_{r+2} - k_{r+1}$, from (2.65), we obtain

$$\|\underline{x}(k_{r+1} + \upsilon)\|_1 \le \frac{1}{8}\|\underline{x}(k_{r+1} - L + \upsilon)\|_1 + \frac{3}{4}\max_{\iota=1}^{r+1}\{\|\underline{x}(k_\iota)\|_1\} \tag{A40}$$

By using (A39), (A40) is estimated as

$$\|\underline{x}(k_{r+1} + \upsilon)\|_1 \le \frac{7}{8}\max_{l=0}^{L-1} \|\underline{x}(k_1 + l)\|_1 \tag{A41}$$



For $k_{r+2} - k_{r+1} < \upsilon \le k_{r+3} - k_{r+1}$, from (A38), it gives

$$\|\underline{x}(k_{r+1} + \upsilon)\|_1 \le \frac{1}{8}\|\underline{x}(k_{r+1} - L + \upsilon)\|_1 + \frac{3}{4}\max_{\iota=1}^{r+2}\{\|\underline{x}(k_\iota)\|_1\} \tag{A42}$$

By using the results (A39) and (A41), (A42) yields

$$\|\underline{x}(k_{r+1} + \upsilon)\|_1 \le \frac{7}{8}\max_{l=0}^{L-1} \|\underline{x}(k_1 + l)\|_1 \tag{A43}$$

for $k_{r+2} - k_{r+1} < \upsilon \le k_{r+3} - k_{r+1}$. By mathematical induction, for $k_{r'} - k_{r+1} < \upsilon \le L - 1$, we can similarly conclude that

$$\|\underline{x}(k_{r+1} + \upsilon)\|_1 \le \frac{1}{8}\|\underline{x}(k_{r+1} - L + \upsilon)\|_1 + \frac{3}{4}\max_{\iota=1}^{r'}\{\|\underline{x}(k_\iota)\|_1\} \tag{A44}$$

By applying the obtained results before this step, (A44) yields

$$\|\underline{x}(k_{r+1} + \upsilon)\|_1 \le \frac{7}{8}\max_{l=0}^{L-1} \|\underline{x}(k_1 + l)\|_1 \tag{A45}$$

for $k_{r'} - k_{r+1} < \upsilon \le L - 1$.

Therefore, for all $0 \le i \le k_{r+1} - k_1 - 1$, we have

$$\|\underline{x}(k_1 + L + i)\|_1 \le \frac{7}{8}\max_{l=0}^{L-1} \|\underline{x}(k_1 + l)\|_1 \tag{A46}$$

We denote $k_{i_2} = k_{r+1}$.

Summing up the analysis in cases 1 and 2, we have

$$\|\underline{x}(k_{i_1} + L + i)\|_1 \le \frac{7}{8}\max_{l=0}^{L-1} \|\underline{x}(k_{i_1} + l)\|_1 \tag{A47}$$

for all $0 \le i \le k_{i_2} - k_{i_1} - 1$.

Now, we start with $\underline{x}(k_{i_2})$. By a similar analysis, we can define $k_{i_3}$, and we can get

$$\|\underline{x}(k_{i_2} + L + i)\|_1 \le \frac{7}{8}\max_{l=0}^{L-1} \|\underline{x}(k_{i_2} + l)\|_1 \tag{A48}$$

for all $0 \le i \le k_{i_3} - k_{i_2} - 1$.

By mathematical induction, a subsequence $\{k_{l_j}\}$, $j = 1,2,\cdots$, can be defined such that $k_{l_{j+1}} \geq k_{l_j} + L$ and

$$\left\|\underline{x}(k_{l_j} + L + \ell)\right\|_1 \leq \frac{7}{8} \max_{l=0}^{L-1}\{\left\|\underline{x}(k_{l_j} + l)\right\|_1\} \tag{A49}$$

for $0 \leq \ell \leq k_{l_{j+1}} - k_{l_j} - 1$. The lemma is proved.

## A.4 Proof of Lemma 4.2

From (4.26), it yields

$$\begin{aligned} \frac{d}{dt}\overline{w}_1^2(t) &= -2\lambda\overline{w}_1^2(t) + 2\overline{w}_1(t)\left\{\frac{1}{(s+\lambda)^{r-2}}v(t) - w_2(t)\right\} \\ &= -2\lambda\overline{w}_1^2(t) + 2\overline{w}_1(t)\frac{1}{(s+\lambda)^{r-2}}v(t) \\ &\quad -2\overline{w}_1(t)\omega_{r-2}(t)\cdot \mathrm{sgn}\left\{\overline{w}_1(t) + w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t)\right\} \end{aligned} \tag{A50}$$

Concerning the relation between the functions $\overline{w}_1(t)$ and $w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t)$, we will divide it into three cases.

**Case 1** There exists a positive constant $T_1$, such that

$$|\overline{w}_1(t)| \geq \left|w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t)\right| \tag{A51}$$

for all $t > T_1$.

**Case 2** There exists a positive constant $T_2$, such that

$$|\overline{w}_1(t)| \leq \left|w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t)\right| \tag{A52}$$

for all $t > T_2$.

**Case 3** Neither Case 1 nor Case 2.



Now, a detailed analysis is given for each case.

**Case 1** In this case, (A50) gives

$$\begin{aligned} \frac{d}{dt}\overline{w}_1^2(t) &= -2\lambda\overline{w}_1^2(t) \\ &\quad +2\overline{w}_1(t)\frac{1}{(s+\lambda)^{r-2}}v(t) - 2\overline{w}_1(t)\omega_{r-2}(t)\cdot \mathrm{sgn}\{\overline{w}_1(t)\} \\ &\leq -2\lambda\overline{w}_1^2(t) \end{aligned} \tag{A53}$$

It can be concluded that $\overline{w}_1(t)$ approaches zero exponentially as $t \to \infty$.

**Case 2** Since $w_1(t)$ is an estimate of $\frac{1}{(s+\lambda)^{r-1}}v(t)$, i.e.

$$w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t) \to 0 \tag{A54}$$

then from (A52), it can be easily concluded that $\overline{w}_1(t) \to 0$ (as $t \to \infty$).

**Case 3** If the next relation holds for instant $t_0$

$$|\overline{w}_1(t_0)| \geq \left|w_1(t_0) - \frac{1}{(s+\lambda)^{r-1}}v(t_0)\right| \tag{A55}$$

then, from (A50), it gives

$$\frac{d}{dt}\overline{w}_1^2(t_0) \leq -2\lambda\overline{w}_1^2(t_0) \tag{A56}$$

i.e. as $t$ increases from $t_0$, $\overline{w}_1^2(t)$ decreases until the following relation (A57) holds, otherwise

$$|\overline{w}_1(t)| \leq \left|w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t)\right| \tag{A57}$$

will contradict the assumption of Case 3.

Thus, when (A55) holds for some instant, sooner or later (A57) will hold as $t$ increases from this instant. From the assumption of Case 3, it can be seen that this kind of instants exist infinitely (at least countably infinite), and their values can approach infinity. Therefore, by employing the fact

$$w_1(t) - \frac{1}{(s+\lambda)^{r-1}} v(t) \to 0 \tag{A58}$$

we can concluded that

$$\overline{w}_1(t) \to 0 \quad (t \to \infty) \tag{A59}$$

Therefore, by combining the above three cases, Lemma 4.2 is proved.

## A.5 Proof of Theorem 4.2

Mathematical induction principle will be employed to prove this theorem.

**Step 1** Based on Equation (4.18), $\frac{1}{(s+\lambda)^{r-1}} v(t)$ will be estimated. For this purpose, let us consider the dynamical system described by

$$\dot{\hat{y}}(t) + \lambda \hat{y}(t) = k_r \left\{ \frac{f(s) - a(s)}{k(s)(s+\lambda)^{r-1}} y(t) + \frac{b(s)}{k(s)(s+\lambda)^{r-1}} u(t) \right\}$$

$$+ k_r w_1(t), \quad \hat{y}(t_0) = 0 \tag{4.34}$$

where $w_1(t)$ is the input determined as

$$w_1(t) = \frac{k_r \{y(t) - \hat{y}(t)\} \omega_{r-1}^2(t)}{\left|k_r \{y(t) - \hat{y}(t)\}\right| \omega_{r-1}(t) + \delta_1} \tag{4.36}$$

$\hat{y}(t)$ which can be computed by solving Equation (4.34) is the estimate of $y(t)$.

Let

$$\overline{y}(t) \triangleq y(t) - \hat{y}(t) \tag{A60}$$

Then, combining (4.18) and (4.34) yields

$$\dot{\overline{y}}(t) + \lambda \overline{y}(t) = k_r \left\{ \frac{1}{(s+\lambda)^{r-1}} v(t) - w_1(t) \right\} \tag{A61}$$

It can be proved that $\overline{y}(t)$ is uniformly bounded and

$$|\overline{y}(t)| \le \sqrt{\frac{\delta_1}{\lambda}} \ \text{(as } t \to \infty) \tag{A62}$$

where $\delta_1 > 0$ is a small constant. The proof of relation (A62) is given in Appendix A.6.



As $\delta_1 \to 0$, we can see that $\overline{y}(t) \to 0$ as $t \to \infty$. By the equivalent control method, from (A61), it can be concluded that $\lim_{\delta_1 \to 0} w_1(t) = \omega_{r-1}(t) \cdot sign(k_r \overline{y}(t))$ can be regarded as the estimate of $\frac{1}{(s+\lambda)^{r-1}} v(t)$ as $t$ is sufficiently large.

So, for a very small positive constant $\delta_1$, $w_1(t)$ can be approximately regarded as the estimate of $\frac{1}{(s+\lambda)^{r-1}} v(t)$ as $t$ is sufficiently large. Thus, there exist $T_1 > t_0$ and a quantity $\varepsilon_1(\delta_1) > 0$ such that

$$\left| \frac{1}{(s+\lambda)^{r-1}} v(t) - w_1(t) \right| \le \varepsilon_1(\delta_1) \tag{A63}$$

for all $t > T_1$, where $\varepsilon_1(\delta_1) \to 0$ as $\delta_1 \to 0$.

**Step 2** Now, we use $w_1(t)$ to approximately estimate $\frac{1}{(s+\lambda)^{r-2}} v(t)$ by appealing to the following identical differential equation.

$$\frac{d}{dt} \left\{ \frac{1}{(s+\lambda)^{r-1}} v(t) \right\} + \frac{\lambda}{(s+\lambda)^{r-1}} v(t) = \frac{1}{(s+\lambda)^{r-2}} v(t) \tag{A64}$$

Consider the dynamical system described by

$$\dot{\hat{w}}_1(t) + \lambda \hat{w}_1(t) = w_2(t), \quad \hat{w}_1(t_0) = 0 \tag{A65}$$

where $w_2(t)$ is determined as

$$w_2(t) = \frac{\{w_1(t) - \hat{w}_1(t)\} \omega_{r-2}^2(t)}{|w_1(t) - \hat{w}_1(t)| \omega_{r-2}(t) + \delta_2} \tag{A66}$$

$\hat{w}_1(t)$ is a signal which can be obtained by solving Equation (A65).

Let

$$\overline{w}_1(t) \triangleq \frac{1}{(s+\lambda)^{r-1}} v(t) - \hat{w}_1(t) \tag{A67}$$

Then, from (A64) and (A65), we have

$$\dot{\overline{w}}_1(t)+\lambda\overline{w}_1(t)=\frac{1}{(s+\lambda)^{r-2}}v(t)-w_2(t) \tag{A68}$$

It can be concluded that

$$|\overline{w}_1(t)|\le\sqrt{\frac{1}{\lambda}\{\delta_2+M_2\varepsilon_1(\delta_1)\}}\quad(\text{as } t\to\infty) \tag{A69}$$

where $\delta_2$ and $M_2$ are positive constants. The proof of (A69) is given in Appendix A.7.

Thus, as $\sum_{i=1}^{2}\delta_i\to 0$, we can see that $\overline{w}_1(t)\to 0$ as $t\to\infty$. By the equivalent control method, from (A68), it is obvious that $\lim_{\sum_{i=1}^{2}\delta_i\to 0} w_2(t)$ can be regarded as the estimate of $\frac{1}{(s+\lambda)^{r-2}}v(t)$ as $t$ is sufficiently large.

So, for very small positive constants $\delta_i$ $(i=1,2)$, $w_2(t)$ can be approximately regarded as the estimate of $\frac{1}{(s+\lambda)^{r-2}}v(t)$ as $t$ is sufficiently large. Thus, there exist $T_2>t_0$ and a quantity $\varepsilon_2(\delta_1,\delta_2)>0$ such that

$$\left|\frac{1}{(s+\lambda)^{r-2}}v(t)-w_2(t)\right|<\varepsilon_2(\delta_1,\delta_2) \tag{A70}$$

where $\varepsilon_2(\delta_1,\delta_2)\to 0$ as $\sum_{i=1}^{2}\delta_i\to 0$.

**Step i** $(3\le i\le r)$ Based on the identical differential equation

$$\frac{d}{dt}\left\{\frac{1}{(s+\lambda)^{r-i+1}}v(t)\right\}+\frac{\lambda}{(s+\lambda)^{r-i+1}}v(t)=\frac{1}{(s+\lambda)^{r-i}}v(t) \tag{A71}$$

we consider the dynamical system described by

$$\dot{\hat{w}}_{i-1}(t)+\lambda\hat{w}_{i-1}(t)=w_i(t),\quad \hat{w}_{i-1}(t_0)=0 \tag{A72}$$



where $w_i(t)$ is determined as

$$w_i(t)=\frac{\{w_{i-1}(t)-\hat{w}_{i-1}(t)\}\omega_{r-i}^2(t)}{|w_{i-1}(t)-\hat{w}_{i-1}(t)|\omega_{r-i}(t)+\delta_i} \tag{A73}$$

$\hat{w}_{i-1}(t)$ is a signal generated by Equation (A72). In (A73), $w_{i-1}(t)$ is the estimate of $\frac{1}{(s+\lambda)^{r-i+1}}v(t)$ obtained in the $(i-1)$-th step.

Let

$$\overline{w}_{i-1}(t)\triangleq\frac{1}{(s+\lambda)^{r-i+1}}v(t)-\hat{w}_{i-1}(t) \tag{A74}$$

Similar to Appendix A.7, we can prove that $\overline{w}_{i-1}(t)$ can be controlled very small by choosing small $\sum_{j=1}^{i}\delta_j$. Thus, $w_i(t)$ can be approximately regarded as the estimate of $\frac{1}{(s+\lambda)^{r-i}}v(t)$, i.e. there exist $T_i>t_0$ and $\varepsilon_i(\delta_1,\cdots,\delta_i)>0$ such that

$$\left|\frac{1}{(s+\lambda)^{r-i}}v(t)-w_i(t)\right|<\varepsilon_i(\delta_1,\cdots,\delta_i) \tag{A75}$$

for all $t>T_i$, where $\varepsilon_i(\delta_1,\cdots,\delta_i)\to 0$ as $\sum_{j=1}^{i}\delta_j\to 0$.

By combining the above analysis, Theorem 4.2 is proved.

## A.6 Proof of Relation (A62)

From (A61), differentiating $(\overline{y}(t))^2$ yields

$$\frac{d}{dt}(\overline{y}(t))^2=-2\lambda(\overline{y}(t))^2+2\overline{y}(t)k_r\left\{\frac{1}{(s+\lambda)^{r-1}}v(t)-w_1(t)\right\}$$

$$=-2\lambda(\overline{y}(t))^2+2\overline{y}(t)k_r\frac{1}{(s+\lambda)^{r-1}}v(t)$$

$$-2|\overline{y}(t)k_r|\omega_{r-1}(t)+2\delta_1\frac{|k_r\overline{y}(t)|\omega_{r-1}(t)}{|k_r\overline{y}(t)|\omega_{r-1}(t)+\delta_1}$$

$$\le -2\lambda(\overline{y}(t))^2+2\delta_1 \tag{A76}$$

Based on (A76), it can be easily obtained that

$$(\overline{y}(t))^2 \le e^{-2\lambda(t-t_0)}(\overline{y}(t_0))^2+2\delta_1\int_{t_0}^{t}e^{-2\lambda(t-\tau)}d\tau$$

$$\le e^{-2\lambda(t-t_0)}(\overline{y}(t_0))^2+\frac{\delta_1}{\lambda}(1-e^{-2\lambda(t-t_0)})$$

$$\to \frac{\delta_1}{\lambda} \quad (\text{as } t\to\infty) \tag{A77}$$

Thus, relation (A62) is proved.

## A.7 Proof of Relation (A69)

From (A68), differentiating $(\overline{w}_1(t))^2$ yields

$$\frac{d}{dt}(\overline{w}_1(t))^2=-2\lambda(\overline{w}_1(t))^2+2\overline{w}_1(t)\left\{\frac{1}{(s+\lambda)^{r-2}}v(t)-w_2(t)\right\}$$

$$=-2\lambda(\overline{w}_1(t))^2+2\left\{\{w_1(t)-\hat{w}_1(t)\}+\left\{\frac{1}{(s+\lambda)^{r-1}}v(t)-w_1(t)\right\}\right\}$$

$$\times\left\{\frac{1}{(s+\lambda)^{r-2}}v(t)-w_2(t)\right\}$$

$$=-2\lambda(\overline{w}_1(t))^2+2\{w_1(t)-\hat{w}_1(t)\}\left\{\frac{1}{(s+\lambda)^{r-2}}v(t)-w_2(t)\right\}$$

$$+2\left\{\frac{1}{(s+\lambda)^{r-1}}v(t)-w_1(t)\right\}\left\{\frac{1}{(s+\lambda)^{r-2}}v(t)-w_2(t)\right\} \tag{A78}$$

The second term of the right side of (A78) can be estimated as



$$\{w_1(t)-\hat{w}_1(t)\}\left\{\frac{1}{(s+\lambda)^{r-2}}v(t)-w_2(t)\right\}$$

$$=\{w_1(t)-\hat{w}_1(t)\}\left\{\frac{1}{(s+\lambda)^{r-2}}v(t)-\omega_{r-2}\cdot\operatorname{sgn}\{w_1(t)-\hat{w}_1(t)\}\right\}$$

$$+\frac{\delta_2|w_1(t)-\hat{w}_1(t)|\omega_{r-2}(t)}{|w_1(t)-\hat{w}_1(t)|\omega_{r-2}(t)+\delta_2}$$

$$\le\frac{\delta_2|w_1(t)-\hat{w}_1(t)|\omega_{r-2}(t)}{|w_1(t)-\hat{w}_1(t)|\omega_{r-2}(t)+\delta_2}$$

$$\le\delta_2 \tag{A79}$$

Let $M_2$ be the upper bound of $\left|\frac{1}{(s+\lambda)^{r-2}}v(t)-w_2(t)\right|$. Then, by using (A63) and (A79), (A78) can be estimated as

$$\frac{d}{dt}\overline{w}_1^2(t)\le-2\lambda(\overline{w}_1(t))^2+2\delta_2+2\varepsilon_1(\delta_1)\cdot M_2,\ \text{as } t>T_1 \tag{A80}$$

Now, let

$$s_2(t)=\frac{d}{dt}\overline{w}_1^2(t)+2\lambda(\overline{w}_1(t))^2-2\delta_2-2M_2\varepsilon_1(\delta_1) \tag{A81}$$

Thus, (A80) follows that $s_2(t)\le 0$ as $t>T_1$. Solving the Differential Equation (A81) yields

$$(\overline{w}_1(t))^2=e^{-2\lambda(t-T_1)}(\overline{w}_1(T_1))^2+\int_{T_1}^{t}e^{-2\lambda(t-\tau)}\{s_2(\tau)+2\delta_2+2M_2\varepsilon_1(\delta_1)\}d\tau$$

$$\le e^{-2\lambda(t-T_1)}(\overline{w}_1(T_1))^2+\frac{1}{\lambda}\{\delta_2+M_2\varepsilon_1(\delta_1)\}(1-e^{-2\lambda(t-T_1)})$$

$$\to\frac{1}{\lambda}\{\delta_2+M_2\varepsilon_1(\delta_1)\} \quad (\text{as } t\to\infty) \tag{A82}$$

Thus, relation (A69) is proved.

## A.8 Proof of Theorem 4.3

### A.8.1 Some Preliminaries

First, we consider the approximate inverse system of $s-\alpha$, where $\alpha \in C$ ($C$ denotes the set of complex numbers), $\mathrm{Re}(\alpha) \ge 0$. We consider

$$(s-\alpha)c(s) = (s+\beta)^{\gamma+1}, \tag{A83}$$

$$c(s) = s^{\gamma} + c_1 s^{\gamma-1} \cdots + c_{\gamma-1}s + c_{\gamma}, \tag{A84}$$

where $\beta \in C$, $\mathrm{Re}(\beta) > 0$, $\beta$ can be assigned in advance; $\gamma$ is a positive integer. The problem is finding $c(s)$ such that Equation (A83) holds. The order $\gamma$ is introduced so that the accuracy of the approximate inverse system becomes better.

Since Equation (A83) can not be satisfied exactly, we try to consider the approximate equation

$$(s-\alpha)c(s) \cong (s+\beta)^{\gamma+1}. \tag{A85}$$

Let

$$(s+\beta)^{\gamma+1} = s^{\gamma+1} + l_1 s^{\gamma} + \cdots + l_{\gamma}s + l_{\gamma+1}. \tag{A86}$$

It is easy to see that solving (A85) is equivalent to solving the following equation

$$Kc \cong l, \tag{A87}$$

where

$$K = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\alpha & 1 & \cdots & \vdots \\ 0 & -\alpha & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -\alpha \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ c_1 \\ \vdots \\ c_p \end{bmatrix}, \quad l = \begin{bmatrix} 1 \\ l_1 \\ \vdots \\ l_{p+1} \end{bmatrix}, \tag{A88}$$

$K$ is a $(\gamma+2)\times(\gamma+1)$ matrix, $c$ is a $(\gamma+1)\times 1$ vector, $l$ is a $(\gamma+2)\times 1$ vector.

Now, the solution of $c$ which may minimize the following criterion

$$J = (Kc-l)^*(Kc-l) \tag{A89}$$



will be derived [19], where $A^*$ denotes the complex conjugate of the transpose of $A$. It is well known that the least-square approximate solution is given by

$$c = (K^*K)^{-1}K^*l. \tag{90}$$

**Lemma A.8.1** If $\beta$ is chosen such that $0 < \mathrm{Re}(\beta) < 1 - \mathrm{Re}(\alpha)$ and $\mathrm{Im}(\beta) = -\mathrm{Im}(\alpha)$, then $s-\alpha$ can be approximated by $\dfrac{(s+\beta)^{\gamma+1}}{c(s)}$ as $\gamma \to \infty$.

**Proof:** It is well-known that there exists a unitary matrix $U \in C^{(\gamma+2)\times(\gamma+2)}$ such that [19]

$$U^*K = \begin{bmatrix} Q \\ 0 \end{bmatrix}, \text{ i.e. } K = U\begin{bmatrix} Q \\ 0 \end{bmatrix}, \tag{91}$$

where $Q \in C^{(\gamma+1)\times(\gamma+1)}$ is an upper triangular matrix. Thus, combining (89)-(91) yields

$$J = l^*U\begin{bmatrix} 0_{(\gamma+1)\times(\gamma+1)} & 0 \\ 0 & 1 \end{bmatrix}U^*l. \tag{92}$$

Now, express $U^*$ and $K$ as

$$U^* = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \tag{93}$$

where $U_{11} \in C^{(\gamma+1)\times(\gamma+1)}$, $U_{12} \in C^{1\times(\gamma+1)}$, $U_{21} \in C^{(\gamma+1)\times 1}$, $U_{22} \in C$; $K_1 \in C^{(\gamma+1)\times(\gamma+1)}$, $K_2 \in C^{1\times(\gamma+1)}$. From (91) and (93), we can also get $U_{21}K_1 + U_{22}K_2 = 0$, i.e.

$$U_{21} = -U_{22}K_2K_1^{-1} = -U_{22}\begin{bmatrix} 0 & \cdots & 0 & -\alpha \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha & 1 & 0 & \cdots & 0 \\ \alpha^2 & \alpha & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha^{\gamma} & \alpha^{\gamma-1} & \alpha^{\gamma-2} & \cdots & 1 \end{bmatrix} = \alpha U_{22}\begin{bmatrix} \alpha^{\gamma} \\ \vdots \\ \alpha \\ 1 \end{bmatrix} \tag{94}$$

Thus, from (92), (94) and (86), it gives

$$J = \left|[U_{21} \quad U_{22}]l\right|^2 = \left|U_{22}(\alpha^{\gamma+1} + l_1\alpha^{\gamma} + \cdots + l_\gamma \alpha + l_{\gamma+1})\right|^2 = |U_{22}|^2 |\alpha + \beta|^{2(\gamma+1)}. \quad (95)$$

It should be pointed out that $0 < |U_{22}| \le 1$. Since $\mathrm{Re}(\beta) > 0$, it can be seen that a necessary condition to make $J$ to be very small is that $\mathrm{Re}(\alpha) < 1$. This is why we make the assumption that the real parts of the roots of $k(s)$ is smaller than 1. Under this assumption, it is very clear that $J \to 0$ if $\gamma \to \infty$ and $\beta$ is chosen such that $0 < \mathrm{Re}(\beta) < 1 - \mathrm{Re}(\alpha)$ and $\mathrm{Im}(\beta) = -\mathrm{Im}(\alpha)$. Thus, Equation (83) can be approximately satisfied as $\gamma \to \infty$.

Lemma A.8.1 also tells us that $s - \bar{\alpha}$ can be approximated by $\dfrac{(s+\bar{\beta})^{\gamma+1}}{\bar{c}(s)}$ as $\gamma \to \infty$, where

$$\bar{c}(s) = [s^{\gamma} \quad \cdots \quad s \quad 1]\bar{c}, \quad 0 < \mathrm{Re}(\beta) < 1 - \mathrm{Re}(\alpha), \quad \mathrm{Im}(\beta) = -\mathrm{Im}(\alpha). \quad (96)$$

Therefore, $(s-\alpha)(s-\bar{\alpha})$ can be approximated by $\dfrac{\{(s+\beta)(s+\bar{\beta})\}^{\gamma+1}}{c(s)\bar{c}(s)}$ as $\gamma \to \infty$.

### A.8.2 Proof of Theorem 4.3

Consider the following approximate equations

$$(s - \eta_i)\theta_i(s) \cong (s + \chi_i)^{p+1}, \quad i = 1, \cdots, \tau, \quad (97)$$

$$(s - \alpha_j)\vartheta_j(s) \cong (s + \beta_j)^{p+1}, \quad j = 1, \cdots, \iota, \quad (98)$$



where $\chi_i$ are positive real numbers, $\beta_j$ are complex numbers. The solution which may minimize the following criterion is considered

$$H = \sum_{i=1}^{\tau}(N_i\theta_i - g_i)^T(N_i\theta_i - g_i) + \sum_{j=1}^{\iota}(K_j\vartheta_j - d_j)^*(K_j\vartheta_j - d_j), \quad (99)$$

By using the results of Lemma A.8.1, it can be seen that the following solutions can minimize $H$,

$$\theta_i = (N_i^T N_i)^{-1} N_i^T g_i, \quad \vartheta_j = (K_j^* K_j)^{-1} N_j^* d_j, \quad (100)$$

where $1 - \eta_i > \chi_i > 0$ for $i = 1, \cdots, \tau$; $0 < \mathrm{Re}(\beta_j) < 1 - \mathrm{Re}(\alpha_j)$ and $\mathrm{Im}(\beta_j) = -\mathrm{Im}(\alpha_j)$ for $j = 1, \cdots, \iota$. Under the above conditions, we can concluded that $H \to 0$ as $\gamma \to \infty$. Thus, Equations (97) and (98) can be approximately satisfied as $\gamma \to \infty$. Therefore, $\kappa_1(s)$ can be approximated by $\dfrac{\xi(s)}{\varsigma(s)}$ as $\gamma \to \infty$.

## A.9 Proof of Lemma 6.2

From the relation

$$f(s) = s^n + f_1 s^{n-1} + \cdots + f_n$$
$$= (s^{n-1} + g_1 s^{n-2} + \cdots + g_{n-1})(s + \lambda) \quad \text{(A101)}$$

the following two equations are obtained.

$$\begin{bmatrix} 1 & f_1 & f_2 & \cdots & f_{n-1} \\ 0 & 1 & f_1 & \cdots & f_{n-2} \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & f_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \lambda \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ 0 & 1 & g_1 & \cdots & g_{n-2} \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & g_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{(A102)}$$

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_2 & f_3 & f_4 & \cdots & 0 \\ . & . & . & \cdots & . \\ f_{n-1} & f_n & 0 & \cdots & 0 \\ f_n & 0 & 0 & \cdots & 0_1 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ g_1 & g_2 & g_3 & \cdots & 0 \\ . & . & . & \cdots & . \\ g_{n-2} & g_{n-1} & 0 & \cdots & 0 \\ g_{n-1} & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{(A103)}$$

Then,

$$[\lambda^{n-1},-\lambda^{n-2},\cdots,(-1)^{n-1}]H(f,h)$$

$$=[\lambda^{n-1},-\lambda^{n-2},\cdots,(-1)^{n-1}]\begin{bmatrix} h_1 & h_2 & \cdots & h_{n-1} & h_n\\ h_2 & h_3 & \cdots & h_n & 0\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ h_{n-1} & h_n & \cdots & 0 & 0\\ h_n & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix} 1 & f_1 & \cdots & f_{n-2} & f_{n-1}\\ 0 & 1 & \cdots & f_{n-3} & f_{n-2}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & \cdots & 1 & f_1\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix}$$

$$-[\lambda^{n-1},-\lambda^{n-2},\cdots,(-1)^{n-1}]\begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n\\ f_2 & f_3 & \cdots & f_n & 0\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ f_{n-1} & f_n & \cdots & 0 & 0\\ f_n & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix} 0 & h_1 & \cdots & h_{n-2} & h_{n-1}\\ 0 & 0 & \cdots & h_{n-3} & h_{n-2}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & \cdots & 0 & h_1\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}$$

$$=[h_1,\cdots,h_n]\begin{bmatrix} \lambda^{n-1} & 0 & 0 & \cdots & 0\\ -\lambda^{n-2} & \lambda^{n-1} & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ (-1)^{n-2}\lambda & (-1)^{n-3}\lambda^2 & (-1)^{n-4}\lambda^3 & \cdots & 0\\ (-1)^{n-1} & (-1)^{n-2}\lambda & (-1)^{n-3}\lambda^2 & \cdots & \lambda^{n-1}\end{bmatrix}$$

$$\times\begin{bmatrix} 1 & \lambda & 0 & \cdots & 0\\ 0 & 1 & \lambda & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & \lambda\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}\begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1}\\ 0 & 1 & g_1 & \cdots & g_{n-2}\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & g_1\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

$$-[\lambda^{n-1},-\lambda^{n-2},\cdots,(-1)^{n-1}]\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0\\ 0 & \lambda & 1 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & \lambda\end{bmatrix}$$

$$\times\begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1}\\ g_1 & g_2 & g_3 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ g_{n-2} & g_{n-1} & 0 & \cdots & 0\\ g_{n-1} & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} 0 & h_1 & \cdots & h_{n-2} & h_{n-1}\\ 0 & 0 & \cdots & h_{n-3} & h_{n-2}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & \cdots & 0 & h_1\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}$$

$$=[h_1,\cdots,h_n]\begin{bmatrix} \lambda^{n-1} & \lambda^n & 0 & \cdots & 0\\ -\lambda^{n-2} & 0 & \lambda^n & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ (-1)^{n-2}\lambda & 0 & 0 & \cdots & \lambda^n\\ (-1)^{n-1} & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1}\\ 0 & 1 & g_1 & \cdots & g_{n-2}\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & g_1\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$



$$-[\lambda^n,0,\cdots,0]\begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1}\\ g_1 & g_2 & g_3 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ g_{n-2} & g_{n-1} & 0 & \cdots & 0\\ g_{n-1} & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} 0 & h_1 & \cdots & h_{n-2} & h_{n-1}\\ 0 & 0 & \cdots & h_{n-3} & h_{n-2}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & \cdots & 0 & h_1\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}$$

$$=[\chi_c,h_1\lambda^n,\cdots,h_{n-1}\lambda^n]\begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1}\\ 0 & 1 & g_1 & \cdots & g_{n-2}\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & g_1\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

$$-\lambda^n[1,g_1,\cdots,g_{n-1}]\begin{bmatrix} 0 & h_1 & \cdots & h_{n-2} & h_{n-1}\\ 0 & 0 & \cdots & h_{n-3} & h_{n-2}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & \cdots & 0 & h_1\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}$$

$$=[\chi_c,h_1\lambda^n,\cdots,h_{n-1}\lambda^n]\begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1}\\ 0 & 1 & g_1 & \cdots & g_{n-2}\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & g_1\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

$$-\lambda^n[0,h_1,\cdots,h_{n-1}]\begin{bmatrix} 1 & g_1 & \cdots & g_{n-2} & g_{n-1}\\ 0 & 1 & \cdots & g_{n-3} & g_{n-2}\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & \cdots & 1 & g_1\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix}$$

$$=[\chi_c,0,\cdots,0]\begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1}\\ 0 & 1 & g_1 & \cdots & g_{n-2}\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & g_1\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

$$=\chi_c[1,g_1,\cdots,g_{n-1}] \tag{A104}$$

where

$$\chi_c=h_1\lambda^{n-1}-h_2\lambda^{n-2}+\cdots+(-1)^{n-1}h_n \tag{A105}$$

Therefore, Lemma 6.2 has been proved.

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## The Corresponding Papers of the Thesis

| Chapters | Corresponding Papers | Presentation Methods |
|---|---|---|
| Chapter 2 | Computer-controlled continuous-time variable structure systems with sliding modes | *International Journal of Control*, vol. 67, no. 4, pp. 619-639, 1997 |
| Chapter 3 | Robust quasi-sliding mode controller for discrete-time systems | *Systems and Control Letters* Vol. 35, (to appear), 1998 |
| Chapter 4 | Variable structure theory based disturbance identification and its applications | *International Journal of Control*, vol. 68, no. 2, pp. 373-384, 1997 |
| | Design of a nonlinear disturbance observer | Submitted to *IEEE Trans. Industrial Electronics* |
| Chapter 5 | VSS theory-based disturbance estimation scheme and its applications for MIMO systems | *International Journal of Control,* vol. 69, no. 6, pp. 733-752, 1998 |
| Chapter 6 | A robust estimator and its application for systems with uncertainties | *Applied Mathematics and Computer Science,* vol. 8, no. 1, pp101-122, 1998 |