Coding Theory for Constrained Channels

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Summary

Codes for constrained channels (also called "*modulation codes*" or "*line codes*") are needed in digital storage devices and digital data transmission. Although many codes for the constrained channels were invented and have been used in actual systems, there remain many problems which are both interesting and important. Three problems which are fundamental in constructing the codes for the constrained channels are addressed in this thesis.

First, we shall investigate a general code construction procedure due to Adler, Coppersmith and Hassner, for a class of constrained channels represented by labeled graphs. The scope of the decoder of a code constructed by the procedure is one of the measures of the complexity of the decoder and is bounded from above by a linear function of the scope of a conjugacy constructed by the procedure. We shall investigate the problem of searching the minimum scope of the conjugacy constructed by the procedure. We shall establish some properties of the conjugacies constructed by the procedure and give an algorithm searching the conjugacy with the minimum scope. We shall apply the algorithm to (d, k)-constrained channels.

Second, we shall consider an input-constraint which requires that the input sequences to the channel should have a given frequency f component, independent of the source statistics. Codes for this input-constraint are of practical interest in some digital recording systems. It can be considered that sequences with a spectral line at f contain the f component. We shall give several necessary and sufficient conditions for an encoder to generate sequences with a spectral line of amplitude at least a given constant at a given frequency $f = f_S k/n$, independent of the source statistics, where f_S is the symbol frequency, and $k(\geq 0)$ and $n(> 0)$ are integers with $\gcd(k, n) = 1$. Some of the conditions are described in terms of labeled graphs, so that we can apply the ACH procedure to construct codes satisfying the conditions. We shall define a biased coboundary condition at the frequency f and prove that this condition is necessary and sufficient for the sequences encoded by an encoder to have a spectral density null at f and that if the sequences generated by the encoder have a spectral density null at f, then the amplitude

of the spectral line at f is independent of the source statistics. We shall give several other related results about spectral lines. We shall also define the canonical graph for a spectral density null constraint with a nonzero spectral line at a given frequency f.

We shall define an order-K biased coboundary condition at the frequency f as an extension of the biased coboundary condition at f. We shall show that the condition is necessary and sufficient for the sequences encoded by an encoder to have an order-K spectral density null at f, i.e., for the spectral density function of the encoded sequences and its first $2K-1$ derivatives to vanish at f. We shall derive the lower bound on the minimum Euclidean distance of a code for an order-K spectral density null constraint. We shall also define the canonical graph for an order-K spectral density null constraint with a nonzero spectral line at f.

Finally, we shall investigate the structure of the canonical graphs for spectral null constraints and for spectral density null constraints. Each irreducible component of the canonical graph represents a class of spectral null codes or of spectral density null codes, and can be used to construct the spectral null or the spectral density null codes by some code construction schemes, for example, the ACH procedure. We shall consider the problem of identifying all irreducible components of the canonical graph for the spectral null constraint or for the spectral density null constraint at a given frequency $f = f_S k/n$. We shall show that for the canonical graph for the spectral null constraint at f, our problem is solvable. We shall prove that there is only one irreducible component for the canonical graph for the spectral density null constraint with a spectral line at f. We shall identify all irreducible components of the canonical graph for the spectral null constraint at $f = f_S k/n$ when n is a prime number or the double of a prime number. Using computer, we shall identify all irreducible components of the canonical graph for the spectral null constraint at $f = f_S k/n$ for $n \leq 20$. The irreducible components of the canonical graph for a second order the spectral null at dc (i.e., $f = 0$) shall also be identified.

Contents

Notation

Throughout this thesis we assume that k is a nonnegative integer and n a positive integer with $\gcd(k, n) = 1$. We also put $f = f_S k/n$ where f_S is the symbol frequency.

Z : the set of integers.

i : the imaginary unit.

ω : $\omega = \exp(-\mathrm{i}2\pi k f_S/n)$.

$\cong$: $G \cong H$ means that graph G is label-preserving graph isomorphic to graph H.

$\mathcal{S}(G)$: the set of states of a graph G.

$\mathcal{E}(G)$: the set of edges of a graph G.

Σ_A : a full shift with alphabet A.

Σ_N : a full shift with alphabet A and $N = \#A$.

σ : a shift map.

$h(X)$: the topological entropy of X.

Λ_G : a topological dynamical system defined by a directed graph G.

$i_G(e)$: the initial end state of an edge e in a directed graph G.

$t_G(e)$: the terminal end state of an edge e in a directed graph G.

$L_G(n)$: all n-blocks in Λ_G.

L_G : $L_G = \bigcap_{n \geq 1} L_G(n)$.

M_G : the adjacency matrix of G.

exp : the exponential function.

$\sum_{\eta} d$: $\sum_{\eta} d = \sum_{i=0}^{L-1} d(a_i, a_{i+1})$ for block $\eta = a_0 \cdots a_{L-1}$.

$\sum_{\zeta}^{[H]} d$: $\sum_{\zeta}^{[H]} d = \sum_{\substack{m=0 \\ (a_m, a_{m+1}) \in E(H)}}^{L-1} d(a_m, a_{m+1})$ where H is a subgraph, $\zeta = a_0 a_1 \cdots a_{L_1}$ and $d(a_m, a_{m+1})$ is the label of the edge (a_m, a_{m+1}).

$\langle x, y \rangle$: for complex numbers x and y, $\langle x, y \rangle = \mathrm{Re}x\mathrm{Re}y + \mathrm{Im}x\mathrm{Im}y$.

$\mathrm{RDS}_f(\boldsymbol{a})$: $\mathrm{RDS}_f(\boldsymbol{a}) = \sum_{m=0}^{L-1} a_m \exp(-\mathrm{i}2\pi mk/n)$ for block $\boldsymbol{a} = a_0 a_1 \cdots a_{L-1}$.

$\ell(\sigma)$: the index of the periodic component which σ belongs to, i.e., $\sigma \in B_{\ell(\sigma)}$.

$rds_{f,\gamma}(\boldsymbol{a})$: $rds_{f,\gamma}(\boldsymbol{a}) = \omega^{\ell(i_G(\eta))} \mathrm{RDS}_f(\gamma(\eta))$ where η is the cycle generating block $\boldsymbol{a}$.

$\mathrm{RDS}_f^{(j)}(\boldsymbol{a})$: $\mathrm{RDS}_f^{(j)}(\boldsymbol{a}) = \sum_{m=0}^{L-1} \mathrm{RDS}_f^{(j-1)}(a_0 \cdots a_m)$ for block $\boldsymbol{a} = a_0 \cdots a_{L-1}$.

$M_f^{(k)}(\boldsymbol{a})$: $M_f^{(k)}(\boldsymbol{a}) = \sum_{i=0}^{L-1} i^k \omega^i a_i$ for $\boldsymbol{a} = a_0 \cdots a_{L-1}$.

$\boldsymbol{a}^{[i]}$: $a_i a_{i+1} \cdots a_{L-1} a_0 \cdots a_{i-1}$ for block $\boldsymbol{a} = a_0 \cdots a_{L-1}$.

$\boldsymbol{a}^i$: $\underbrace{\boldsymbol{a} \cdot \boldsymbol{a} \cdots \boldsymbol{a}}_{i \text{ times}}$ for block $\boldsymbol{a} = a_0 \cdots a_{L-1}$.

$\mathcal{N}$: the set of all blocks $\boldsymbol{a}$ with $n \not| \lg(\boldsymbol{a})$.

$C(G)$: $C(G) = \{\boldsymbol{s} : \boldsymbol{s}$ is a block generated by a cycle in $G\}$.

G_f : the canonical graph for a first-order spectral null at f.

$I_f(\boldsymbol{a})$: an irreducible component of G_f in which a cycle generates $\boldsymbol{a}$.

$C_f(\boldsymbol{a})$: $C_f(\boldsymbol{a}) = C(I_f(\boldsymbol{a}))$.

Chapter 1

Introduction

In digital data transmissions and digital data recording devices we need codes (or encoders) which encode input data sequences into sequences consistent with given input-constraints. Such codes (called codes for constrained channels) have been investigated by many researchers for several decades and many code construction schemes for such codes were presented. In 1980's and 1990's a number of theoretical results on codes for constrained channels have been obtained, e.g., existence theorems of efficient codes for constrained channels, and theoretical explanations of heuristic methods of constructing codes for several kinds of constrainted channels have been given. These results have contributed to the increases of the recording density and of the reliability of recording devices. Information theory and symbolic dynamics are used to develop these results. Conversely, the coding theory for constrained channels has given stimulation to symbolic dynamics.

Though the coding theory for constrained channels are growing rapidly because of theoretical interest and strong requirements from practical applications, many fundamental problems remain unsolved. In this thesis we consider the problem of searching the optimal codes, of characterizing several kinds of input-constraints, and of listing up all the possible codes for spectral null constraints. These problems arise in different steps of constructing codes for constrained channels. However, all of them are strongly connected with (not necessarily finite) directed graphs with labeled edges.

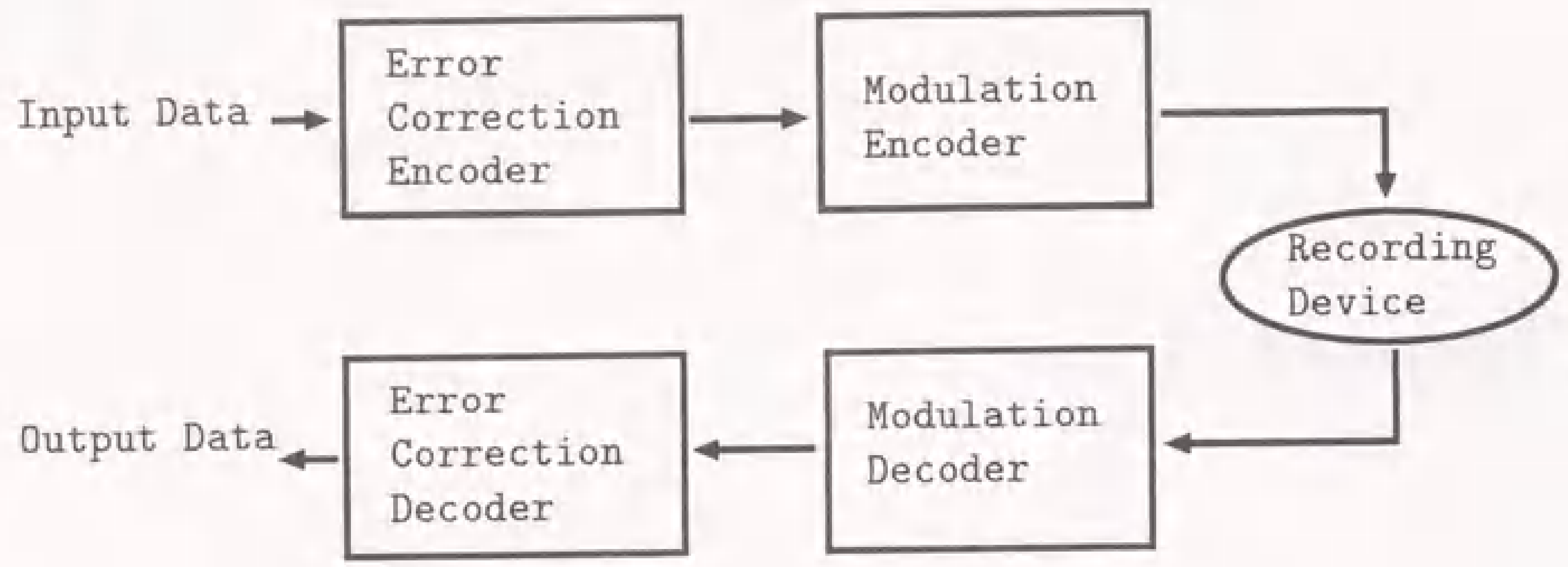


Figure 1.1: Coding scheme for storage device

1.1 Constrained channels and codes for them

A digital datum is recorded on and retrieved from a storage device through a number of steps shown in Fig.1.1. In the first step data sequences are encoded by an error correction encoder. In the next step the output sequences from the error correction encoder are encoded by a modulation encoder. This is our main concern in this thesis. Though both error correction codes and modulation codes are used to protect data from errors occurring in the digital storage devices, they are designed for different kinds of channels. Error correction encoders are designed for channels which are affected by noises. Modulation encoders are designed for channels which can hardly transmit sequences which do not satisfy some special requirements. Such channels are called "*(input) constrained channels*", and the special requirements which input sequences transmitted by the channels have to satisfy are called "*input-constraints.*" There are two kinds of 'constrained channels.' One is a channel which has a 'physical' input-constraint, such as a spectral null constraint at dc: an input sequence which does not satisfy such a constraint can not physically be transmitted through the channel. The other is a channel with a rather artificial input-constraint, such as a run-length limited constraint: a code which generates sequences consistent with such a constraint is expected to increase the performance of a storage system. In this thesis we assume that input-constraints are given by other persons, e.g., the designers of (physical) devices, and our main concerns are the characterization of codes for the constraints and the construction of them.

The following input-constraints are mainly investigated in this thesis:

- "*(d, k)-constraints,*"
- "*spectral null*", "*spectral line*" and "*spectral density null*" constraints.

The (d, k)-constraints and the spectral null constraints have been studied by many researchers since 1960's, and used in the design of digital storage devices and digital data transmissions.

Here we explain (d, k)-constraints. Constraints on the spectrum are explained in Section 1.3. Let d and k be nonnegative integers with $d < k$. A (d, k)-sequences are binary sequences where the run of '0' symbols between consecutive '1' symbols must have length at least d and at most k. A (d, k)-constraint is a constraint which requires that input sequences to the channel should be (d, k)-sequences. (d, k)-sequences are typically used in magnetic recording devices. The parameter d controls the highest transition frequency and the frequency is related to intersymbol interference. The length of the longest run of '0' symbols is bounded by k so that the retrieved signal contain timing information. In digital data recording it is desirable that the retrieved signal should have timing information.

When we design codes for digital recording devices there are many factors to be taken into account for, e.g., recording density, error probability, cost of implementation, and so on. The parameters d and k are determined based on these factors by the designers of the storage device, and code designers have to design codes for the device on the assumption that the digital recording channel accepts only sequences satisfying the (d, k)-constraint. Therefore we need a coding scheme according to which (arbitrary binary) data sequences can be encoded into (d, k)-constraint sequences. See [1], [2] and [3] for more details.

The set of all (bi-infinite) (d, k)-sequences is represented by a directed graph with labeled edges. For example, the set of all $(0, 3)$-sequences are described by reading the labels along paths in the graph given in Fig. 1.2, e.g.,

$$\cdots 1\,0\,1\,0\,0\,1\,1\,1\,0\,0\,0\,1\,0\,0\,1\,0\,1\,1\,0 \cdots$$

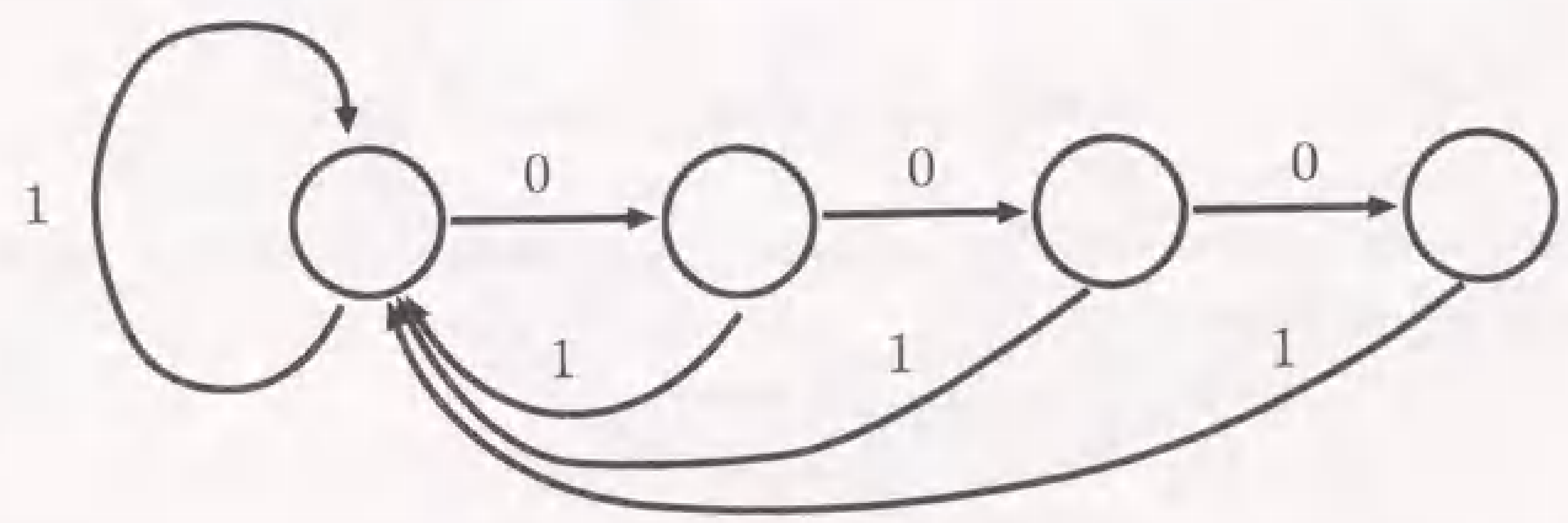


Figure 1.2: $(0, 3)$-constrained system.

There are many other constraints which are represented by graphs with labeled edges. The run-length limited (RLL) constraint (which is closely related to the (d, k)-constraint) is another example of such a constraint. General *constrained channels represented by directed graphs with labeled edges* were first investigated by Shannon[4].

When we are given an input-constraint, we must transmit the user information through a channel with the constraint or record it into a storage device with the constraint. Therefore we need a code which encodes any sequences of input symbols to sequences (of channel symbols) consistent with the input constraint. A code construction method, which will be called *the ACH procedure*, for constrained channels was proposed by Adler, Coppersmith and Hassner[5]. In Chapter 3 we shall consider the problem of finding an optimal code generated by the procedure. In Chapters 4 and 5 we represent three kinds of constraints on the frequency spectrum in terms of graphs so that we can apply the ACH procedure to the constraints. *Canonical graphs* are infinite state graphs which characterize constraints on the frequency spectrum. It can be considered that each irreducible component of the graphs represents a class of codes for the constraint on the frequency spectrum. In Chapter 6 we shall consider the problem of identifying all irreducible components of canonical graphs in order to obtain optimal codes. In the following sections we shall explain background, our problems and our results on the problems.

1.2 Sliding block decoders

In this section we explain encoders having sliding block decoders for constrained channels. First we describe a simple code construction scheme for constrained channels and then a much better code construction scheme due to Adler, Coppersmith and Hassner[5], which is our main concern of Chapter 3.

Shannon defined the (noiseless) channel capacity $h(G)$ of the channel represented by a graph G by

$$h(G) = \lim_{n\to\infty} \frac{1}{n} \log_2 N(n),$$

where $N(n)$ is the number of blocks(or words) generated by paths of G of length n[4]. For large n, we can write

$$N(n) \cong 2^{h(G)n}. \tag{1.1}$$

We define the jth "*power graph*" G^j of G: the state set of G^j is that of G; each edge of G^j corresponds to a sequence of state transitions of length j in G. The second power

graph of the graph in Fig. 1.2 is shown in Fig. 1.3. We have

$$h(G^j) = jh(G) \tag{1.2}$$

for every positive integer j (Proposition 2.1).

Using (1.1) and (1.2), we can construct a code which encodes input sequences (i.e., any binary sequences) into sequences consistent with a given constraint represented by a graph G. Let p and q be integers such that

$$ph(G) \geq q. \tag{1.3}$$

The ratio $(q/p)/h(G)$ is the efficiency of the code. By (1.1) and (1.2) we know that for every positive integer j, each state of G^j has about $2^{h(G)j}$ outgoing edges. From (1.3) we may expect that there is an integer ℓ such that for each state of $G^{p\ell}$ there are more than $2^{q\ell}$ edges. Therefore, for each state in $G^{p\ell}$ we can define a one-to-one mapping from the set of binary sequences of length $q\ell$ to the set of outgoing edges from the state. An example of such an assignment for the graph in Fig. 1.3 is given in Fig. 1.4 where we consider the case $p = 2$ and $\ell = q = 1$. The redundant edges (i.e., the edges to which no input symbol is assigned) are omitted in the figure. Each label xy/a means that input symbol a is assigned to the edge having label xy. We can encode input sequences into sequences consistent with the input constraint as follows. We fix a state, which is called "*initial state.*" In Fig. 1.4 let A be the initial state. Assume that an input sequence $\alpha = 010010001$ is given. There is a path, say x, of G^2 such that the sequence of input

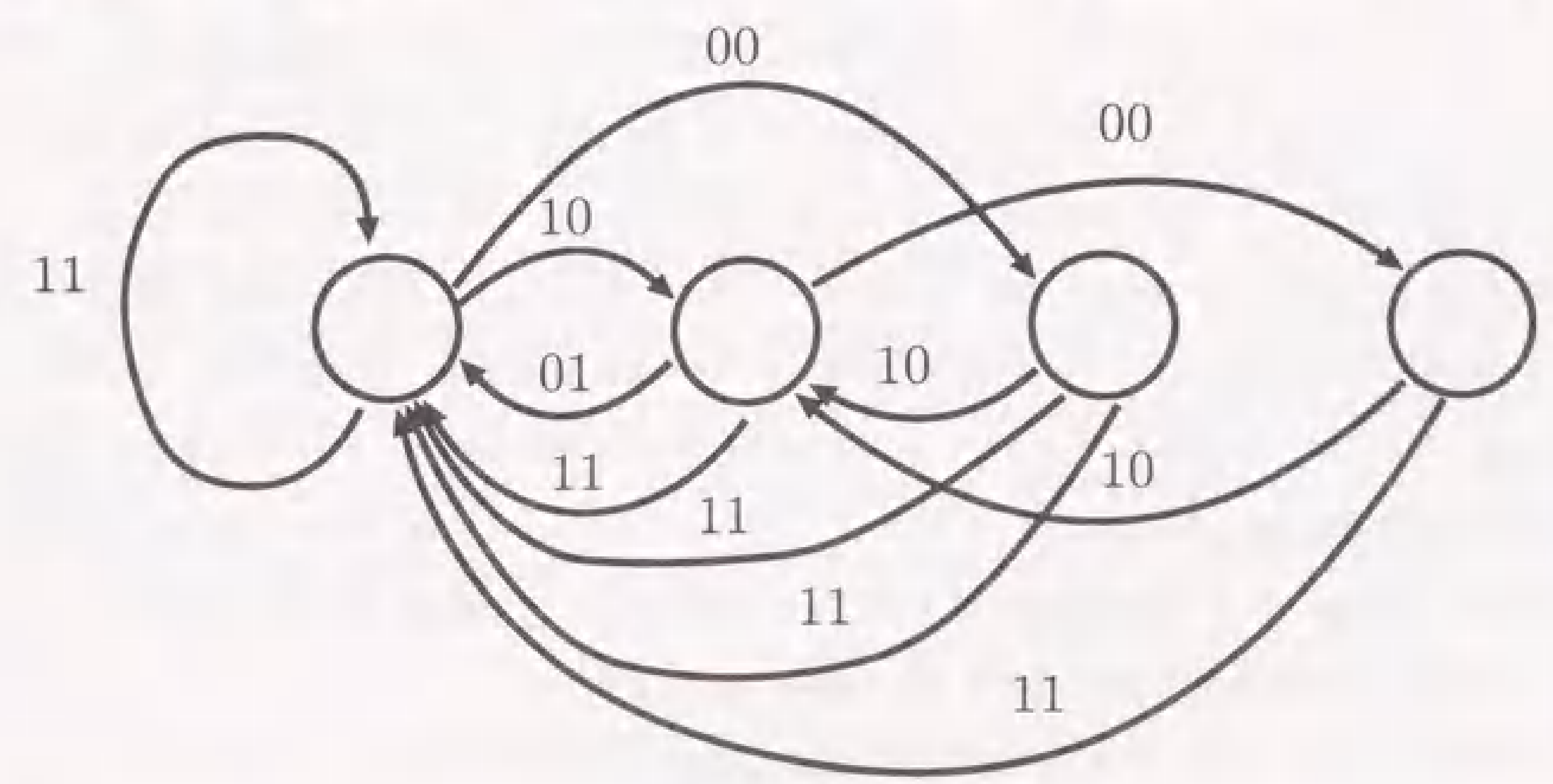


Figure 1.3: Second power graph of $(0, 3)$-constrained system.

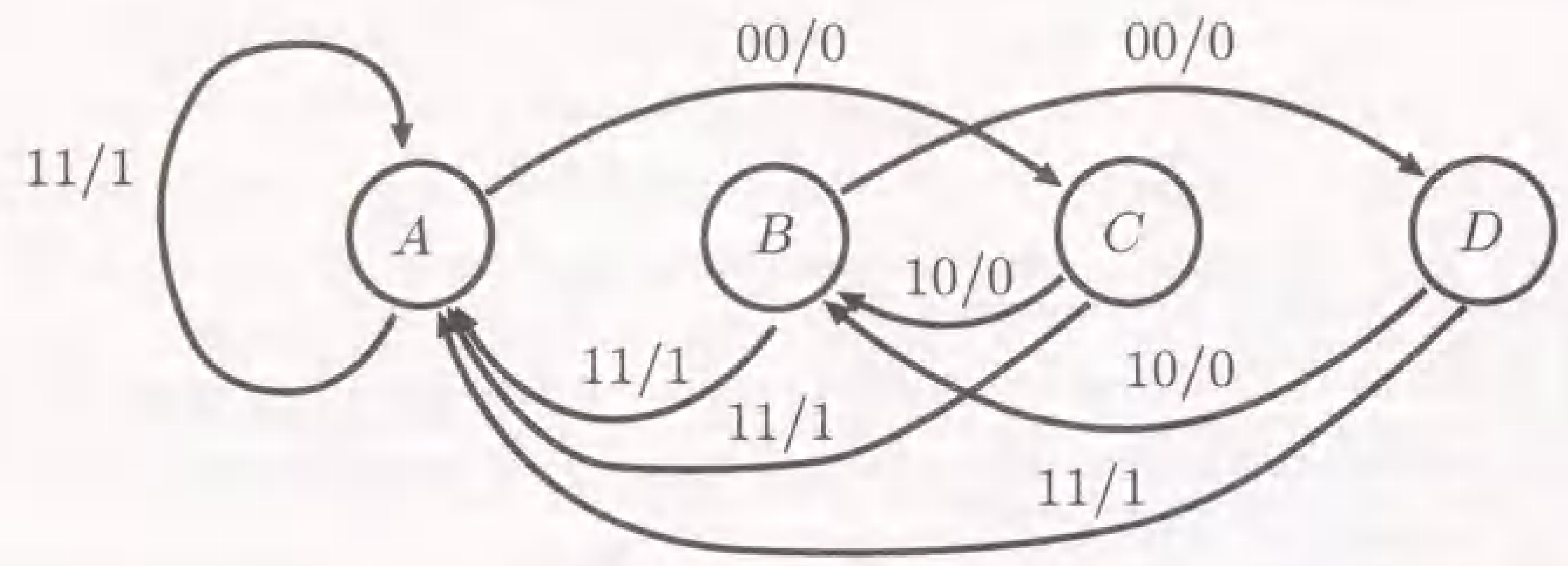


Figure 1.4: One-to-one assignment

channel symbols	decoded symbol
00	0
10	0
11	1

Table 1.1: Decoding rule for Fig. 1.4

symbols along $\boldsymbol{x}$ is equal to α and the initial state of $\boldsymbol{x}$ is A. Then the encoded sequence is the sequence of output symbols along $\boldsymbol{x}$, in this case, 00 11 00 10 11 00 10 00 11. The decoding procedure is also given by the simple mapping given by Table. 1.1.

The simple code construction scheme described above has some disadvantages. For our example the code construction scheme gives simple encoding and decoding rules because each state of the second power graph of the $(0, 3)$-constraint has at least 2 outgoing edges. However, for almost all constraints G of practical interest, the integer ℓ such that each state of $G^{p\ell}$ has at least $2^{q\ell}$ is large if we want to construct a code for G with high efficiency. For example we consider the $(2, 7)$ constraint with $p = 2$ and $q = 1$. The parameter ℓ is 14 by the above construction. If we use the code construction scheme given in the following (which is also explained in Section 3.2) then we get a code such that the parameter corresponding to ℓ is 4. Therefore, the above code construction scheme is hardly used in practical systems. The efficiency of the code given by the scheme is always less than 100% if $h(G)$ is an irrational number. Moreover, even if $h(G)$ is a rational number, it cannot give a code with 100% efficiency. For example consider the constraint given by the graph in Fig. 1.5 (the edge labeling does not matter here). Although the capacity of the channel is 1, for any ℓ the number of blocks generated by paths of length ℓ starting from state B is less than 2^m.

As described above the problem of constructing codes for constrained channels can be solved in principle but it is not easy to construct codes for constrained channels arising in practical systems. Therefore, many works have been done on the problem. For

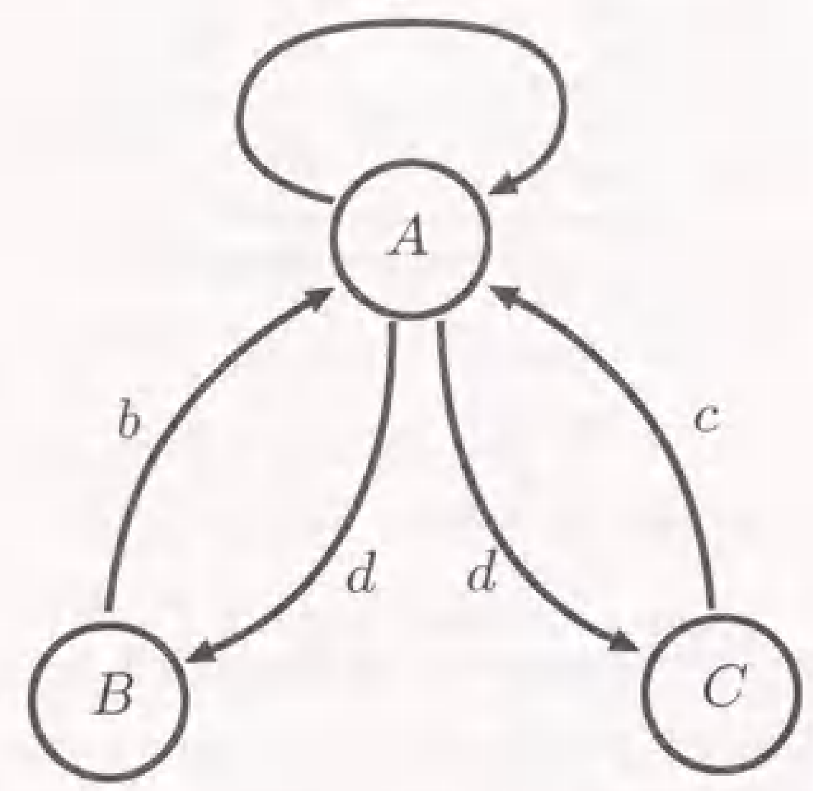


Figure 1.5: A constrained channel: G_0

example, Franaszek [6,7,8], and Lempel and Cohn[9] studied code construction methods for codes suitable for implementation by electronic circuits. However, many theoretical problems which practical code designers were also interested in, had been left unsolved for several decades. In 1980's a lot of nice results in coding theory for constrained channels were presented by many researchers. The first work among them was done by Adler, Coppersmith and Hassner [5]. For a class of constrained channels (called "*subshifts of finite type (SFT)*") they presented a concrete code construction procedure (the ACH procedure) by which we can obtain codes which are not only efficient but also easily implemented by electronic circuits. Since a set of (d, k)-sequences is of finite type, they were able to successfully apply the ACH procedure to $(2, 7)$-constraint. One of advantages of the procedure is that the decoding rule of the code constructed by the procedure is a sliding block mapping and hence the rule has the finite error propagation property. Next we explain sliding block decoders.

Let G be the directed graph representing an input-constraint. Let C be the capacity of the channel and let p and q be positive integers such that $p/q \leq C$. Let n be the size of the source alphabet. We assume that the size of the channel alphabet is also n. The sliding block decoder implements the "*sliding block decoder mapping*," which is a block map, say π, of the set of bi-infinite sequences of the qth power graph G^q of G onto the set of free n^p-ary bi-infinite sequences. This is given by a mapping h of the set of allowable blocks of length $m + a + 1$ of edges of G^q onto the set of n^p-ary symbols, where m and a are nonnegative integers defined as follows: for any bi-infinite sequence $y = (y_i)_{i \in \mathbb{Z}}$ of G^q

$$\pi(y) = (x_i)_{i \in \mathbb{Z}},$$

where

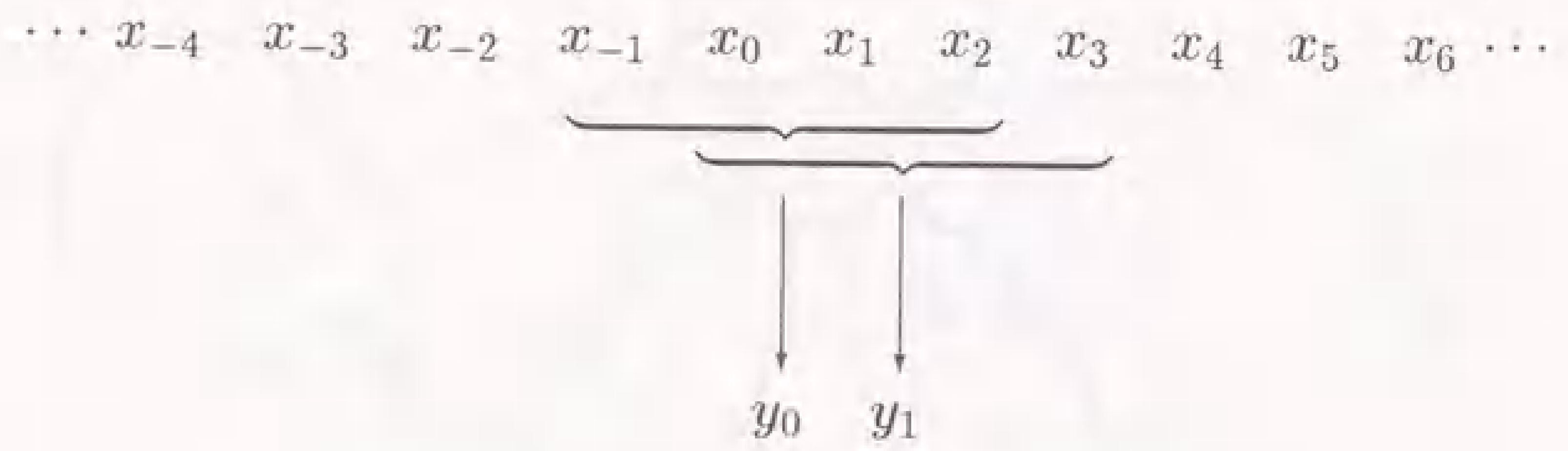


Figure 1.6: Example of sliding block decoder

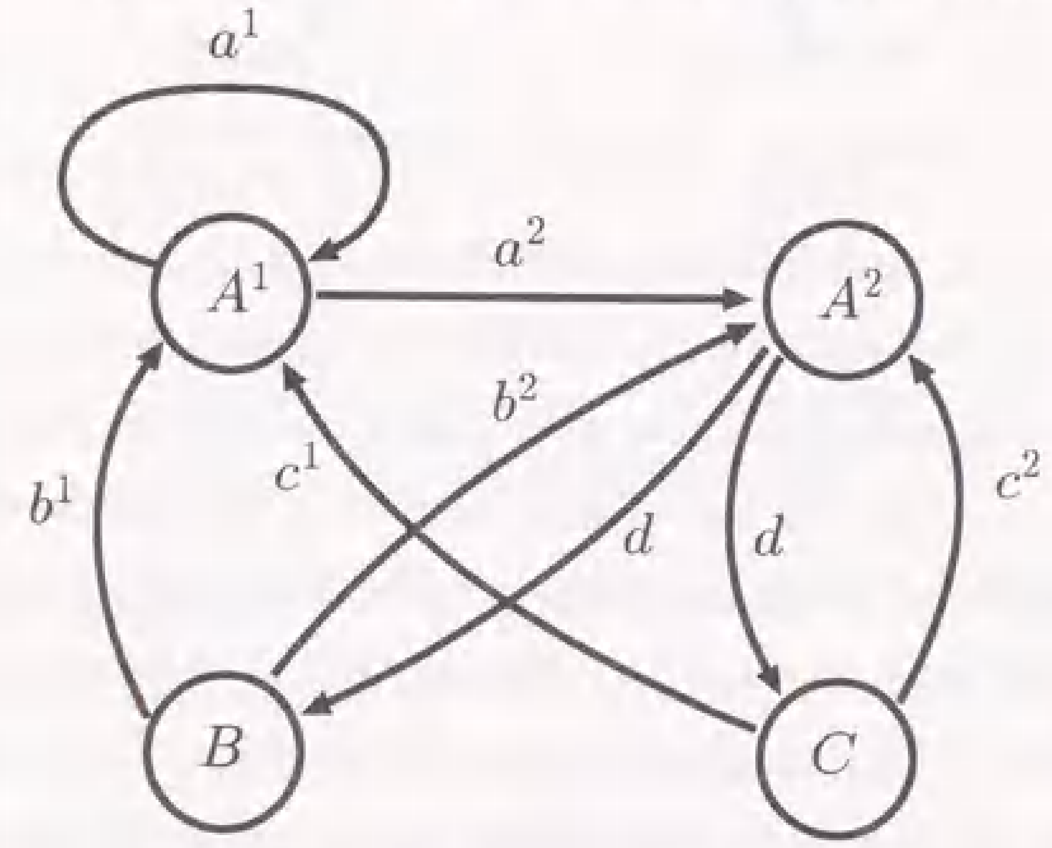


Figure 1.7: G_1

$$x_i = h(y_{i-m}y_{i-m+1}\cdots y_{i+a}), \qquad i \in Z.$$

We may assume that $s = m + a + 1$ equals the scope of π, i.e., the smallest value of $m + a + 1$ for h giving π. At each time i, the decoder sees the block $y_{i-m}\cdots y_{i-a}$ of length sq, retained in its buffer, of a sequence y from the constrained system and outputs a source block $h(y_{i-m}\cdots y_{i+a})$ of length p. Then the subsequent block y_{i+a+1} of length q is shifted into the buffer (the block y_{i-m} is shifted out) and the next source block $h(y_{i-m+1}\cdots y_{i+a+1})$ of length p is output (see Fig. 1.6, where $a = 2$ and $m = 2$ are assumed). It is clear that a single error occurring in the channel causes at most sp errors in decoding. This property is called the limited error propagation property and is desirable in practical applications. The size of the buffer memory of the sliding block decoder is related to the complexity of the circuit implementing the block map and is bounded by sq. Thus it is desirable to find π with the smallest scope.

The ACH procedure[5] transforms G^q into a graph H with each row sum at least

h
$h(aa) = a^1$
$h(ad) = a^2$
$h(ba) = b^1$
$h(bd) = b^2$
$h(ca) = c^1$
$h(cd) = c^2$
$h(dc) = d$
$h(db) = d$

h^{-1}
$h^{-1}(a^1) = a$
$h^{-1}(a^2) = a$
$h^{-1}(b^1) = b$
$h^{-1}(b^2) = b$
$h^{-1}(c^1) = c$
$h^{-1}(c^2) = c$
$h^{-1}(d) = d$

Table 1.2: local maps of ϕ and ϕ^{-1}

n^p (such a graph is said to be *nice*). The transformation induces a conjugacy (i.e. 1-1 and onto block map) from the set of bi-infinite sequences of G^q to the set of bi-infinite sequences of H. They proved it by using the technique, which is called the "*state-splitting*," that there exists such a transformation for every input-constraint of finite type. The technique was given by generalizing the technique introduced by Marcus[10] in order to prove a theorem related to the classification problem in symbolic dynamics, and his proof suggests that we can construct a code with 100% efficiency in the case where the channel capacity is a rational number p/q. There is a natural block map of scope 1 which transforms a bi-infinite sequence of H to a free n^p-ary bi-infinite sequence, and the composition of that conjugacy induced by the state-splittings and this block map gives the sliding block decoder mapping.

For example if we apply the technique once to the graph G_0 in Fig. 1.6 then we can obtain the graph G_1 in Fig. 1.7 (the precise description of the procedure will be given in Chapter. 3). The transformation induces the block map ϕ of the set of sequences of G_0 to the set of sequences of G_1 with its local map h given in Table. 1.2. The scope of the block map is 2 with $\ell = 0$ and $a = 1$. It is easily seen that the block map is bijective (see Fig. 1.8). The local map of the inverse of the map is also given in Table. 1.2. Note that each state in G_1 has 2 outgoing edges (so it is nice in this case) and hence for each state we can define a one-to-one mapping of the binary alphabet to the set of outgoing edges from the state. Therefore, we can construct a finite state encoder which encodes every binary sequence into a sequence consistent with the constraint represented by G_1. Then by applying ϕ^{-1} we can transform the sequence of G_1 into a sequence consistent with the constraint represented by G_0. Conversely, we can get the sequences of G_1 from the sequences of G_0 by applying ϕ (See Section 3.2).

As described in this example a state-splitting of 1-round can give a nice graph. However, in general a state-splitting of 'r *rounds*' is needed in order to obtain a nice graph and the state-splitting of r rounds induces the block map with scope $r + 1$. It is desirable that the parameter r be as small as possible because of the following reasons: (1)

one channel symbol error affects a maximum of rq symbols in a decoded sequence; (2) the complexity of an encoder and a decoder obtained by the ACH procedure depends on r. Thus we shall consider the problem of searching the minimum scope conjugacy induced by the ACH procedure. In Chapter 3 we shall establish several properties of the ACH procedure, and using them we shall present an algorithm searching the sliding block decoder with the minimum scope. From Theorem 3.1 we know that finding the sliding block decoder with the minimum scope constructed by the ACH procedure is equivalent to finding the minimum scope conjugacy which is of $(0, r)$ type and whose inverse is of $(0, 0)$ type. The ACH procedure is based on integral vectors which are called *approximate eigenvectors* for the adjacency matrix of the graph representing the given constraint. An *r-round vector* is an approximate eigenvector such that a state-splitting of r rounds based on the vector generates a nice graph. Therefore our problem of finding the conjugacy with the minimum scope is the problem of finding the minimum r^* such that there is an r^*-round vector. By Proposition 3.1 we need not take into account 'empty splittings' in searching the minimum r^*. Theorem 3.2, Corollary 3.2 and Corollary 3.3 give necessary conditions for an approximate eigenvector to be an r-round vector. From Corollary 3.2 and Corollary 3.3, we know a finite region where r-round vectors should exist. Since for every approximate eigenvector $\boldsymbol{v}$ in the region we can check in a finite number of steps whether $\boldsymbol{v}$ is an r-round vector or not, our problem is solvable. An efficient algorithm is presented using the properties of r-round vectors given in Theorem 3.2 and Corollary 3.1. Based on many examples it was conjectured that the approximate eigenvector $\boldsymbol{v}^*$ such that the maximum element of $\boldsymbol{v}^*$ is minimum over all approximate eigenvectors would give the minimum scope conjugacy[5, Appendix]. However we give a counter example to this in Section 3.4. Furthermore examples presented in the section suggest us that there seems to be no simple algorithm searching the sliding block decoders with the minimum scope.

Marcus[11] showed that code construction techniques which are invented for constraints of finite type also can be applied to more general constrained channels, which is called "*sofic systems.*" His method is to approximate a sofic constrained system by a constrained system of finite type from the inside in terms of capacity and to apply the code construction method for constraints of finite type to the approximated system.

$$\begin{array}{rccccccccccccc} x = & \cdots & a & a & d & c & d & b & a & d & c & a & a & \cdots \\ \phi(x) = & \cdots & a^1 & a^2 & d & c^2 & d & b^1 & a^2 & d & c^1 & a^1 & & \cdots \end{array}$$

Figure 1.8: Example of $\phi(x)$

Almost all constraints of practical importance can be represented by sofic systems. Therefore, the ACH procedure and the algorithm given in Chapter 3 can be applied to almost all constrained systems of practical importance.

1.3 Spectral nulls and spectral lines

In Chapters 4 and 5, we shall introduce several kinds of constraints with respect to the power spectrums of the input sequences of channels. As described in the previous section, once we get a representation of a constrained system by a directed labeled graph, we can construct an efficient code for the constraint by the ACH procedure. Therefore we shall characterize our constraints in terms of graphs.

Let f be a given frequency. There are channels such that input sequences to them should have no f component. The following are examples in which such channels are used.

a) In the baseband transmission system, an electronic circuit which transmits the dc component of signals correctly is expensive and if a circuit does not transmit the dc component of signals correctly then the signals containing the dc component decrease the performance of the system containing the circuit.

b) In magnetic recording systems, the servo position information must also be retrieved from reproducing signals (see Fig. 1.9). To attain a high recording density and to position the head precisely, it is desirable that the user information and the servo position information be recorded on the same track. The system, which is called "*buried servo*," records both of the user information and the servo position information on the same track but partitions them into separate frequency bands [12]. Therefore, in such systems it is required that the data signal should have no f_p component, where f_p is the frequency of the servo signal (Fig. 1.10).

The constraints which such channels have, are called "*spectral null constraints*" at f and codes which produce sequences consistent with a spectral null constraint at f are called "*spectral null codes*" at f. Many spectral null codes have been presented and used in practical systems.

On the other hand, in a digital recording or communication systems the existence of a specified frequency component might be desirable when we need to extract information about clock only from reproduced or received signals. In fact, we require a spectral null constraint at f in order to insert the servo signal of frequency f into an encoded

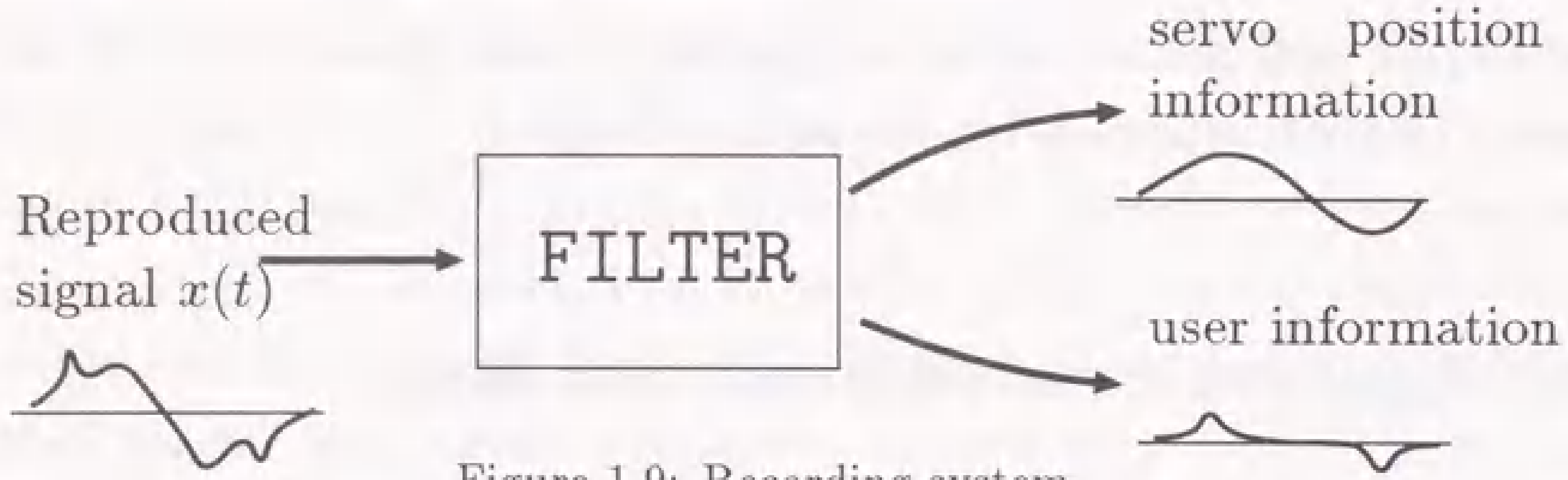


Figure 1.9: Recording system

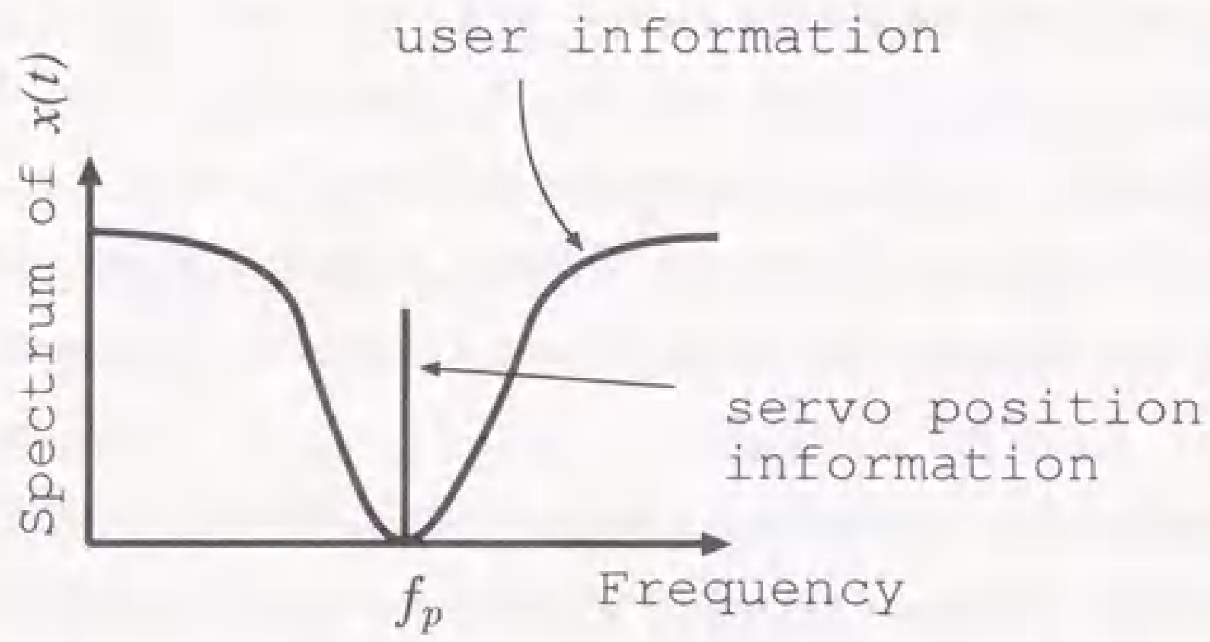


Figure 1.10: Spectrum of reproduced signal $x(t)$

sequence. If we want to use a system like a buried servo in an optical storage device then we need such a code, because the channel given by the optical storage device is intrinsically digital and, therefore we can not add any analog servo signal to any data sequence[13]. Another example of such a channel is the digital recording system produced by Borgers et al.[14]. In the system the (analog) servo signal can not be recorded on the medium. Hence they used spectral null codes which generate sequences having spectral lines at low frequencies for the equiprobable information source (Immink

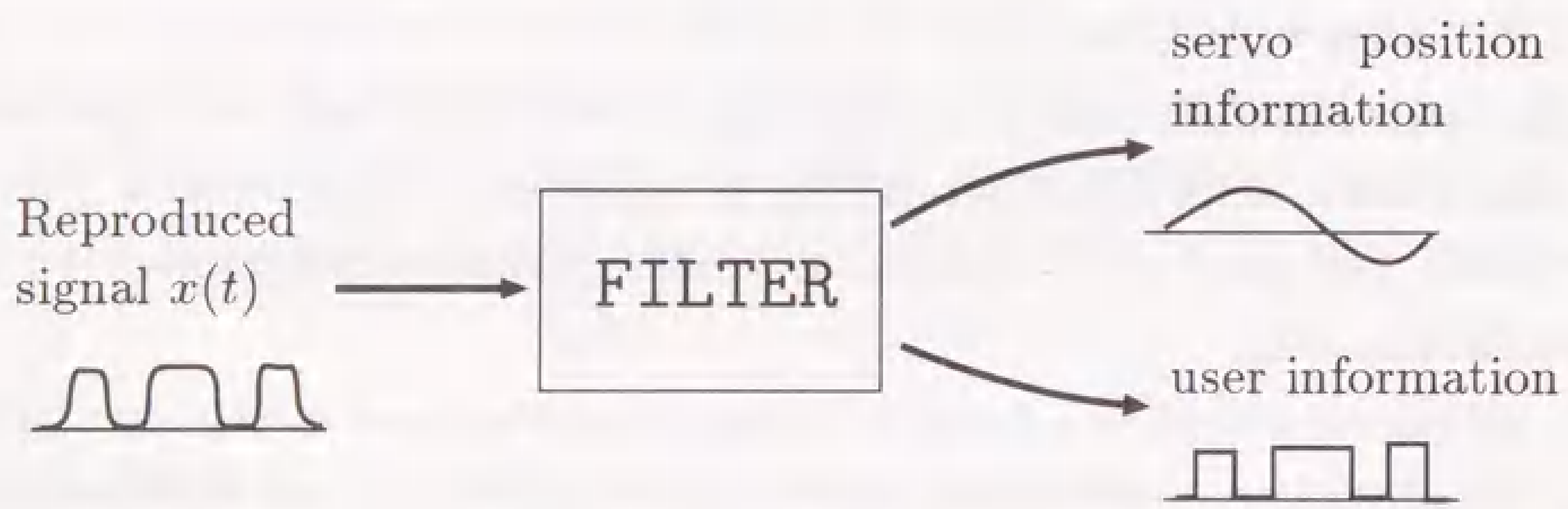


Figure 1.11: Generating pilot tone from reproducing signal

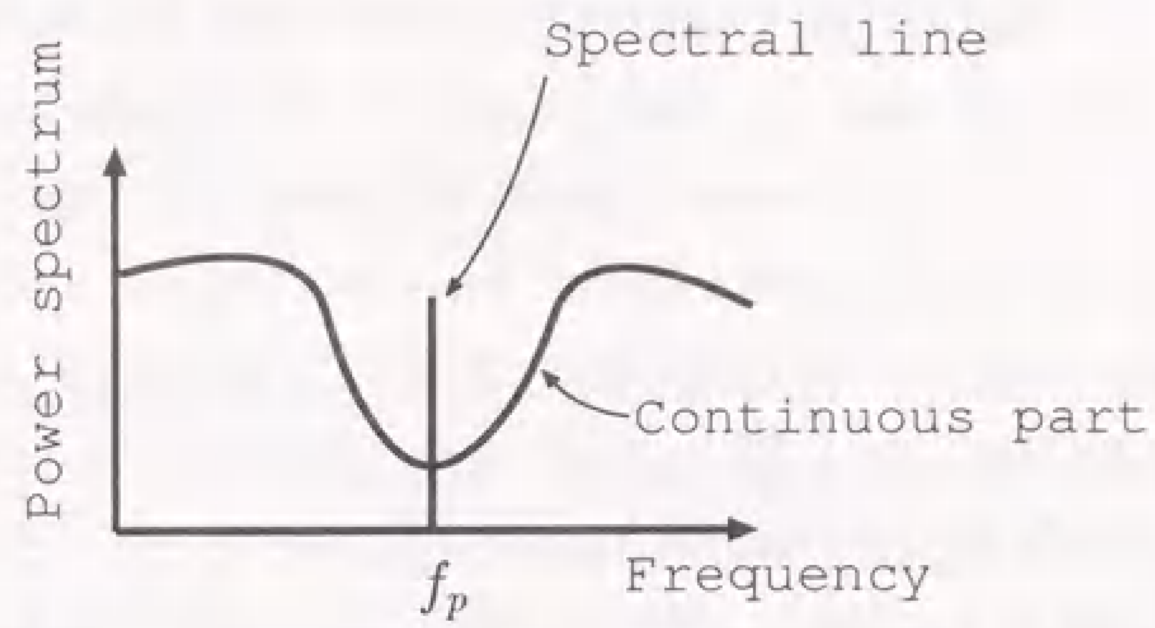


Figure 1.12: Power spectrum having spectral line at f_p

supported theoretically the general existence of such spectral null codes at dc[15, Chapter 11],[16]) so that the servo information may be extracted only from the reproduced signal. Therefore a code which produces sequences having the specified frequency f component for every information source is of practical interest (see Fig. 1.11). Given an information source, the power spectrum of the sequence generated by a code consists of the continuous part and the discrete part (the spectral line). In this thesis we shall concentrate on conditions for the sequences generated by the code to have spectral lines of amplitude not less than given values (Fig. 1.12) because it can be considered that the spectral line corresponds to the pilot tone, which can be used as the servo signal. Therefore our input constraint is described as follows: there should exist a spectral line of amplitude not less than the given value at the specified frequency, independent of the source statistics. In Chapter 4 we shall study characterizations and representations of the codes satisfying this constraint.

Our results on codes satisfying our constraint are related to the known results on spectral null codes. Therefore we shall first mention the results on spectral null codes. We consider the encoder to be a directed graph with labeled edges. The labeling determines a function of the Markov chain given by a transition probability matrix compatible with the graph. It was pointed out that the "*running digital sum*" at dc (denoted RDS_0) of the encoded sequence plays an important role in constructing and analyzing encoders whose encoded message has only small amounts of low frequency components, where for a sequence $\boldsymbol{a} = a_0 a_1 \cdots a_L$ with $a_i \in C$, RDS_0 of $\boldsymbol{a}$ is defined by

$$\mathrm{RDS}_0(\boldsymbol{a}) = \sum_{m=0}^{L} a_m.$$

The RDS_0 is also called the digital sum variation(DSV). Yasuda and Inose [17], [18] proved that for an encoder the following three conditions are equivalent: (1) the encoder satisfies the "*finite* RDS_0 *condition*" (i.e., for every encoded sequence from the encoder, RDS_0 takes its value in a finite range); (2) the encoder satisfies a "*loop-sum-zero condition*" (i.e., for every encoded sequence generated by a cycle of the encoder, its RDS_0 value is 0); (3) the encoded sequence has a spectral null at dc (i.e., the encoded sequence has zero mean and the power spectral density vanishes at dc). This was rediscovered in [19], [20]. Let k be a nonnegative integer and let n be a positive integer with $\gcd(k,n)=1$. Let f_S be the "*symbol frequency.*" Yoshida and Yajima[21] defined the running digital sum at $f = kf_S/n$ (denoted RDS_f) of a sequence $\boldsymbol{a} = a_0a_1\cdots a_L$ to be

$$\mathrm{RDS}_f(\boldsymbol{a}) = \sum_{m=0}^{L} a_m \exp\left(-i2\pi\frac{k}{n}m\right), \tag{1.4}$$

where i is the imaginary unit. (They call $\mathrm{RDS}_f(\boldsymbol{a})$ a weighted digital sum variation of $\boldsymbol{a}$.) Extending the above result, Yoshida and Yajima [22], and Marcus and Siegel [23] proved that the following three conditions are equivalent: (1) the encoder satisfies the finite RDS_f condition; (2) for every encoded sequence generated by a cycle of length a multiple of n in the labeled graph representing the encoder, its RDS_f value is 0; (3) the encoded message has a spectral null at f (i.e., the encoded message has a zero spectral line and the power spectral density vanishes at f). Moreover, Marcus and Siegel [23] showed that a "*coboundary condition at f*" is equivalent to the above three conditions and that the coboundary condition is useful in constructing encoders for a spectral null at f. The conditions (1) and (2) above and the coboundary condition are stated in terms of graphs and it can be seen that the graph transformation used in the ACH procedure preserves these conditions, so that we can apply the ACH procedure to a graph satisfying the conditions in order to construct an encoder for a spectral null at f[23], [24].

Our input-constraint is stated as follows: the input sequences to the channel should have a spectral line of amplitude at least a given value at f. Therefore we need a code by which data sequences are encoded into sequences having a spectral line of amplitude at least the given value at f, independent of the source statistics. In Chapter 4 we shall study the problem of characterizing the the encoders for such constraints. For every encoder and for every positive constant c we shall give several necessary and sufficient conditions for the sequences generated by the encoder to have a spectral line of amplitude not less than c at f, independent of transition probability matrices compatible with the graph representing the encoder (Theorem 4.4). One is the following: for every block η

generated by a cycle in the labeled graph representing the encoder

$$\left|\frac{\mathrm{RDS}_f(\eta)}{\lg(\eta)}\right| \geq c.$$

Another condition is similar to the coboundary condition for a spectral null constraint at f. We can easily determine whether a given graph (not necessarily representing an encoder) satisfies these conditions. Moreover, the ACH procedure preserves them as above. Hence using the ACH procedure we can construct an encoder which generates sequences having a spectral line of amplitude not less than the given value at f independent of the source statistics. The characterizations of the encoders for our input-constraints given by Theorem 4.4 are similar in form to those of the encoders for spectral null constraints. However the proof of Theorem 4.4 is very different from that of the characterization theorem for spectral null codes.

Theorem 4.6 characterizes an encoder which generates sequences having a spectral line of amplitude a given value at f. It states that for every encoder and every positive number c, the following conditions are equivalent: (A1) for every block η generated by a cycle in the labeled graph representing the encoder

$$\left|\frac{\mathrm{RDS}_f(\eta)}{\lg(\eta)}\right| = c;$$

(A2) the labeled graph representing the encoder satisfies a *biased coboundary condition* with respect to a constant d with $|d| = c$ at f; (A3) for every information source the sequences generated by the encoder has a spectral line of amplitude c; (A4) for an information source the sequences generated by the encoder have a spectral line at f and the continuous part of the spectrum of the sequences vanishes at f. In Fig. 1.13 we show an example of the power spectrum of an encoder satisfying (A2) at f_p. By (A3) the amplitude of the spectral line does not depend on information sources, that is, it is equal to c for every information source. Thus such an encoder is ideal in the sense of SNR (signal-to-noise ratio), where SNR is defined to be the amplitude of the spectral line divided by the power spectral density[15, Chapter 11], [16]. The condition (A4) says that if an encoder satisfies one of the above conditions then the sequences generated by the encoder have a frequency region of suppressed components around f except f (see Fig. 1.12). This suggests that the servo signal of frequency f is easily extracted from the reproduced signal by using filters. These characterizations of encoders for our input-constraints are similar in form to those of encoders for the

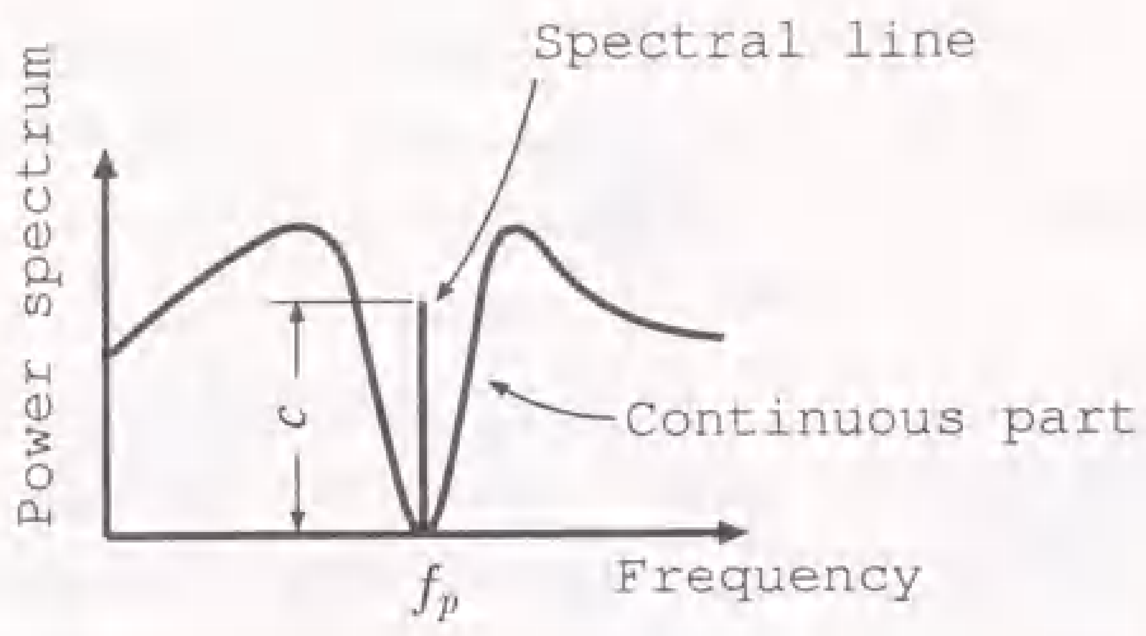


Figure 1.13: Power spectrum of code satisfying biased coboundary condition

spectral null constraints. However, in an engineering viewpoint the meaning of our input-constraints is very different from that of spectral null constraints: although the sequences generated by a spectral null code contain no frequency f component(see Fig. 1.10; the spectral line at f_p in the figure is due to the servo signal which is always pre-recorded in the medium), the encoded sequences generated by a code for our input-constraint has a spectral line at f, that is, an f component, independent of the source statistics (see Fig. 1.12 and Fig. 1.13).

If the continuous part of the spectrum of the sequences generated by an encoder vanishes at f then we say that the sequences have a *spectral density null* at f. Necessary and sufficient conditions for an encoder to generate sequences with a spectral density null at f are also given in Theorem 4.5. They are similar to those in Theorem 4.6.

Some conditions in Theorem 4.6 are stated in terms of graphs and the graph transformation used in the ACH procedure preserves them. From these facts it follows that using the ACH procedure and Marcus' approximation theorem, we can construct an encoder which generates the sequences having a pilot tone at f and a frequency region of suppressed components around f. Therefore we can record the servo information and the user information on the same track even if the medium is intrinsically digital. The pilot tone can be obtained easily only by filtering the reproduced signal.

Marcus and Siegel introduced "*canonical graphs*" for spectral null constraints which are infinite state graphs[23]. These graphs characterize spectral null codes and also can be used in the design of spectral null codes. We shall also define a canonical graph for a spectral density null constraint with a nonzero spectral line of amplitude c at f, for $c > 0$. The graph has the following properties: (P1) every finite subgraph G of the graph and every transition probability matrix compatible with G define a stochastic process which has a spectral density null and a spectral line of amplitude c at f; (P2) if a stochastic process defined by a finite graph G and a transition probability matrix

compatible with G has a spectral density null and a spectral line of amplitude c at f then G is collapsed (i.e., label-preserving graph homomorphic) to a subgraph of the canonical graph(Proposition 4.3). The canonical graphs are our main concern in Chapter 6.

1.4 Higher order spectral density nulls

Recently, a spectral null has been extended to a higher order spectral null by several authors. Immink [25] proposed a bipolar code (with symbols ± 1), called a DC^2-constrained code, which has not only a spectral null but also a second order spectral null at dc (that is, $f = 0$, or $k = 0$ and $n = 1$). Immink and Beenker [26] proposed a code such that the power spectrum and its derivatives through order $2K - 1$ vanish at dc(a "*spectral null of order K*" at f). Such a code achieves a large rejection of low frequency components. A lower bound on the minimum distance of a code generating sequences with a higher order spectral null at dc was also derived [26], [27]. Monti and Pierobon [28] presented a necessary and sufficient condition for a code with complex channel symbols to generate sequences having an order-K spectral null at dc and defined canonical graphs for the second order spectral null at dc. Eleftheriou and Cideciyan [29], Karabed and Siegel [24] extended these results to the frequencies f of rational submultiples of the symbol frequency. They have shown that a code has an order-K spectral null at f if and only if the code satisfies an order-K coboundary condition at f. Karabed and Siegel [24] derived lower bounds on the minimum Euclidean distance of a code with an integer alphabet generating sequences with a higher order spectral null at f. This result gives a theoretical explanation of a heuristic criterion of constructing codes for "*partial response channels.*" [24]

A spectral density null constraint can also be extended to a higher order spectral density null constraint. The purpose of Chapter 5 is to characterize the codes for the higher order spectral density null constraint at f. We shall define an *order-K biased coboundary condition* at f, which can be considered as an extension of the order-K coboundary condition of [24] or the biased coboundary condition of [30]. We shall prove that for every positive integer K and for every encoder represented by a labeled graph G, the following conditions are equivalent(Theorem 5.1): (D1) the sequences generated by the encoder has a spectral density null of order K at f, i.e., the continuous part of the spectrum of the sequences and its derivatives through order $2K - 1$ vanish at f; (D2) G satisfies an order-K biased coboundary condition at f; (D3) for every j with $1 \leq j \leq K$ and for every pair $\boldsymbol{x}, \boldsymbol{y}$ of blocks generated by cycles in g with the same initial state, the same terminal state and the same length, the *order-j running digital*

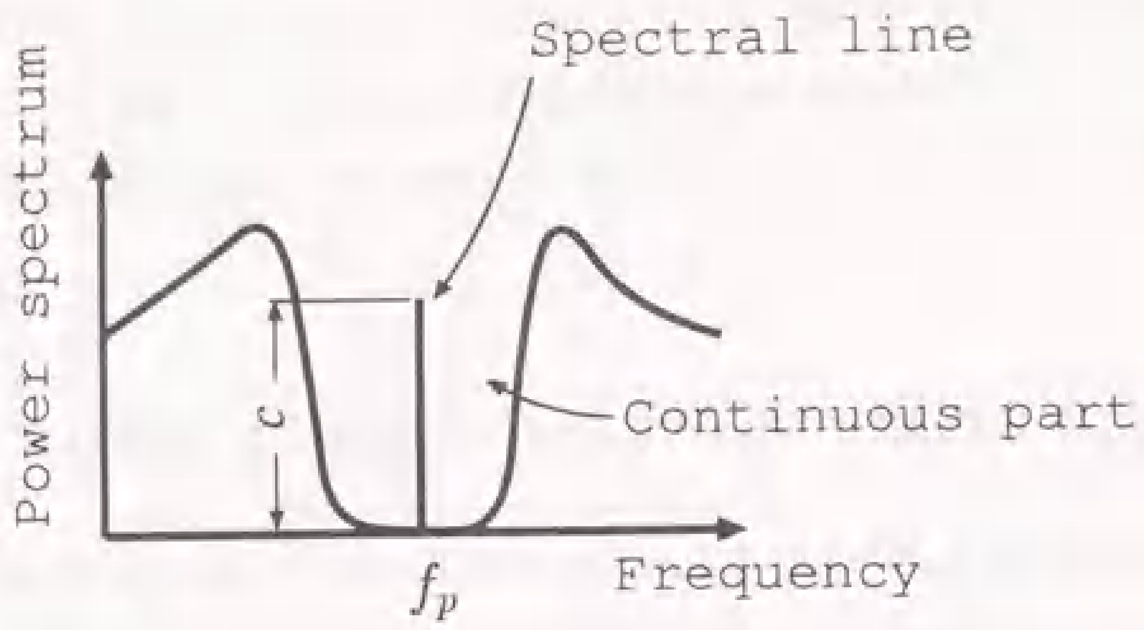


Figure 1.14: Power spectrum of code having higher order spectral density null

sum $\mathrm{RDS}_f^{(j)}(\boldsymbol{x}-\boldsymbol{y})$ (see Definition 5.1) of $\boldsymbol{x}-\boldsymbol{y}$ is 0; (D4) for every j with $0 \le j \le K-1$ and for every pair $\boldsymbol{x}$, $\boldsymbol{y}$ of blocks generated by cycles in G with the same initial state, the same terminal state and the same length, the *order-j moment* $M_f^{(j)}(\boldsymbol{x}-\boldsymbol{y})$ (see Definition 5.1) of $\boldsymbol{x}-\boldsymbol{y}$ is 0. By each of the conditions (D2), (D3) and (D4), the higher order spectral density null is also characterized in terms of graphs.

We can construct a finite state graph H (not necessarily representing an encoder) which satisfies the condition (D2). It can be seen that the graph transformation used in the ACH procedure preserves (D2). Therefore we can obtain a higher order spectral density null code from H by the ACH procedure. In Fig. 1.14 we illustrate the frequency spectrum of the sequences generated by a higher order spectral density null code with a spectral line at f_p. When we use a higher order spectral density null code with a spectral line at f_p, we may expect that a larger rejection of frequency components around f_p can be realized and that we can extract the servo signal from the signal encoded by the code very easily(cf. Fig. 1.13).

We shall give a lower bound on the minimum Euclidean distance of a code for an order-K spectral density null constraint(Proposition 5.1). We shall define the canonical graphs for higher order spectral density null constraints with nonzero spectral lines. These graphs have the properties which are similar to the properties of canonical graphs for spectral null constraints and those for spectral density null constraints with nonzero spectral lines. The canonical graphs for higher order spectral density null constraints can also be used to construct higher order spectral density null codes.

1.5 Irreducible components of canonical graphs

Canonical graphs for spectral null constraints and those for spectral density null constraints do not only characterize these constraints but also can be used to construct

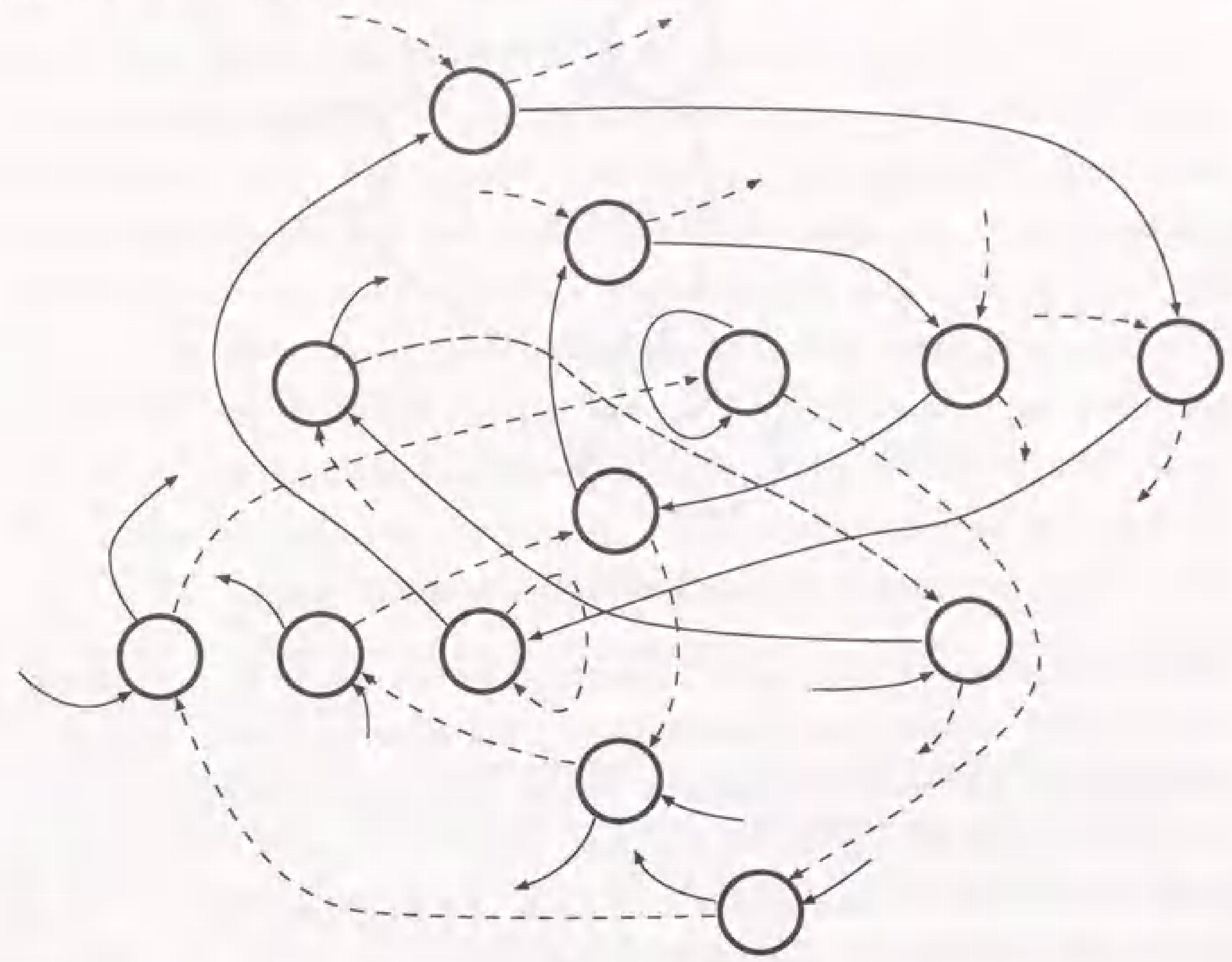


Figure 1.15: Canonical graph for spectral null at $f_S/3$

codes for the constraints. For example suppose that we want to construct a code for the spectral null constraint at $f_S/3$. A part of the canonical graph for the spectral null constraint at $f_S/3$ is illustrated in Fig. 1.15 where solid arrows mean edges with label 1 and dotted arrows mean edges with label -1. Let G be the graph shown in Fig. 1.16 which is a finite subgraph of the canonical graph for the spectral null constraint at $f_S/3$. Then by one of the properties of the canonical graphs, G has a spectral null at $f_S/3$. Since the capacity $h(G)$ of G is 0.551463, by (1.2) we know that $h(G^2) > 1$. We can find a sub-constrained system H of G which is of finite type with $h(H) > 1$ (e.g., by using Marcus' approximation theorem). Therefore by applying the ACH procedure to H we can obtain a spectral null encoder.

When we want to get an 'optimal' code in this way, many problems arise. One of them is 'how do we choose graph G from the canonical graph?' To find a subgraph which induces the optimal code, we must at least be able to know all "*irreducible components*" of the canonical graph in order to get useful information about the optimal code when we construct spectral null codes in the way above. It will be useful to identify all irreducible components of the canonical graphs even if we use other code construction schemes for spectral null codes than the ACH procedure. Canonical graphs for first order spectral null constraints are infinite graphs, and consist of disconnected irreducible

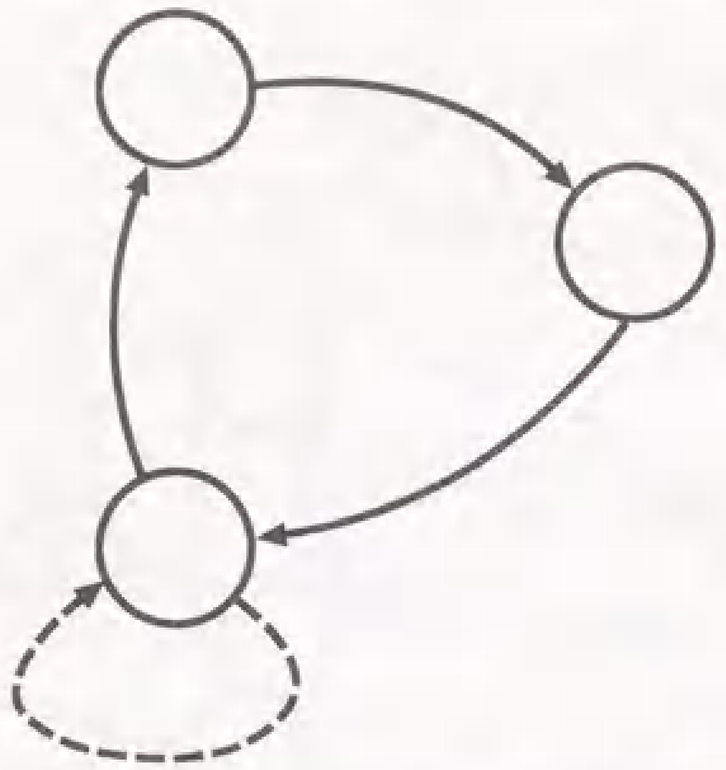


Figure 1.16: Subgraph of canonical graphs

components (Proposition 6.2 and Proposition 6.4). It seems to be impossible to identify all irreducible compoentns by observation. Therefore we need a systematic way of identifying all irreducible components. A code for a spectral null constraint at $f_S/2$ is used in an actual system[12] and two codes which have spectral lines at $f_S/60$ and at $f_S/80$, respectively, are also used in a practical storage system[14]. So we want to solve our identification problem as generally as possible.

In Chapter 6 we shall give results on our problem of identifying irreducible components of canonical graphs. Assume that $f = f_S k/n$ where k and n are relatively prime positive integers. For every canonical graph, the period of the graph is a divisor of n(Remark 6.1). We shall show that there is only one irreducible component of the canonical graph for a spectral density null constraint with a nonzero spectral line at f and that there is only one irreducible component of the canonical graph for a spectral null constraint at f with period n (Proposition 6.6). Therefore we shall concentrate on the problem of identifying irreducible components with period less than n for canonical graphs for spectral null constraints at f. We shall give an algorithm solving our problem for canonical graphs for first order spectral null constraints at f (Proposition 6.12). In Subsection 6.3.4 we shall show results obtained by applying it for n with $n \leq 20$. But it is impractical for large n because the size of a system of equations to be solved in the algorithm is of size $2^n \times \phi(n)$, where $\phi(n)$ is the euler number of n. Therefore we need a systematic way of identifying them without solving the system of equations.

We consider the equivalence relation in the set $\mathcal{N}$ of blocks of length not a multiple of n such that for every pair $\boldsymbol{a}$, $\boldsymbol{b}$ of blocks in $\mathcal{N}$, $\boldsymbol{a}$ is equivalent to $\boldsymbol{b}$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ are generated by cycles in the same irreducible component. This equivalence relation partitions the set $\mathcal{N}$ into the equivalence classes each of which corresponds to exactly one irreducible component with period less than n (Corollary 6.3). Since for every block $\boldsymbol{a}$ in $\mathcal{N}$ we can generate the irreducible component in which $\boldsymbol{a}$ is generated

by a cycle, we can identify the irreducible components by giving the representative blocks of those equivalence classes. For the cases where n is a prime number or n is the double of an odd prime number, we shall explicitly give all representative blocks of those equivalence classes, and hence all irreducible components. If n is a positive prime integer, then the canonical graph for a spectral null constraint at $f = f_S k/n$ consists of $n+1$ irreducible components (Theorem 6.2). For example, the canonical graph for a spectral null constraint at $f_S/3$ shown in Fig. 1.15 contains four irreducible components, which are shown in Fig. 6.3. If n is the double of an odd prime integer d then the number of irreducible components for the spectral null at $f = f_S k/n$ is $1 + (d+1)/2 + (2^{d-1} - 1)/2$ (Theorem 6.2). We shall also give all the representative blocks of irreducible components in the canonical graph for a second order spectral null constraint at dc (Theorem 6.4).

The general solution for the problem of determining the number of irreducible components of canonical graphs for first order spectral null constraints has not been obtained yet. Few results are known on the problem of identifying irreducible components of canonical graphs for higher order spectral null constraints and higher order spectral density null constraints at f for $f > 0$.

Chapter 2

Symbolic Dynamics and Coding Theory for Constrained Channels

It was shown that symbolic dynamics and automata theory have close connections with coding theory for constrained channels or storage devices[5], [31]. Since we state our results in terms of symbolic dynamics frequently, we give necessary background on symbolic dynamics here. Symbolic dynamics is a branch of dynamical systems and ergodic theory. Almost all definitions in this chapter can be defined more generally in dynamical systems or ergodic theory[5], [32].

2.1 Symbolic dynamics

Let A be a finite set of symbols (an alphabet). We define Σ_A by

$$\Sigma_A = \{(x_i)_{i \in Z} : x_i \in A\}.$$

We call Σ_A a full n-shift or a full shift over the alphabet A, where $n = \#A$. We also write Σ_n to mean Σ_A because only the number of elements in A is essentially a matter of concern. We define a metric d on Σ_A by

$$d(x, y) = \sum_{i=-\infty}^{\infty} \frac{\overline{d}(x_i, y_i)}{2^{|i|}}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \Sigma_A$, where $\bar{d}$ is a function such that if $a \neq b$ then $\bar{d}(a,b) = 1$ otherwise $\bar{d}(a,b) = 0$. With respect to the topology induced by the metric d, Σ_A is compact. Let X be a closed subset of Σ_A and let T be an homeomorphism (that is, T is continuous and bijective, and T^{-1} is also continuous) of X. Assume that X is T-invariant, i.e., $X = T(X)$. Then (X, T) is a "*topological dynamical system*." The shift map σ_A of Σ_A is the map defined by

$$(\sigma_A(\boldsymbol{x}))_i = x_{i+1} \qquad \text{for } \boldsymbol{x} \in \Sigma_A \text{ and } i \in Z.$$

We call the dynamical system $(X, \sigma_A|_X)$ a "*subshift*" of Σ_A, where $\sigma_A|_X$ is the restriction of σ_A to X. In this thesis, we mean the shift map of any subshift by σ because this abbreviation will not cause any confusion. Since we only consider dynamical systems having σ as their transformation, we regard closed σ-invariant subsets as subshifts. We call a finite sequence of symbols a "*block*". Let $\boldsymbol{x} \in X$. Let $n, i \in Z$ and assume that $n > 0$. We call $x_i x_{i+1} \cdots x_{i+n-1}$ an "*n-block*" in X or simply a "*block in*" X. The empty block is also considered as a block. For a block $\boldsymbol{s}$, we mean the length of $\boldsymbol{s}$ by $\lg(\boldsymbol{s})$. Let H be a set of blocks in Σ_A. Let U_H be a set of all bi-infinite sequences in which any block in H does not appear. Then we can show that U_H is a subshift of Σ_A. We call U_H the subshift induced by H. Conversely, for every subshift U there is a set H' of blocks such that U is equal to the subshift induced by H'.

Definition 2.1 For a subshift X if there is a finite set H of blocks such that X is equal to the subshift induced by H then X is said to be a "*subshift of finite type (SFT)*".

Definition 2.2 Let U be a subshift, let B be a finite alphabet and let $Y \subset \Sigma_B$ be an SFT. If there is a continuous surjective map $f : Y \to U$ which commutes with σ (i.e., $\sigma \circ f = f \circ \sigma$), then we call U a "*sofic system*."

Sofic systems and SFT's are important classes of subshifts in both symbolic dynamics and coding theory for constrained channels.

Let $X \subset \Sigma_A$ and $Y \subset \Sigma_B$ be subshifts. Let $m \geq 1$, and let π be a function of the set of m-blocks in X to B. Let $f : X \to Y$ be a function, and assume that there is an integer n such that

$$(f(\boldsymbol{x}))_i = \pi(x_{i-n} x_{i-n+1} \cdots x_{i-n+m-1}), \qquad \text{for every } \boldsymbol{x} \in X \text{ and } i \in Z.$$

Then f is said to be an "*m-block map of* $(n, m-n-1)$ *type*." The map π is said to be a "*local map*" of f. We shall use the same symbol π to refer f, and for a positive

integer p with $p \geq m$ we shall use the following convention:

$$\pi(x_1 x_2 \cdots x_p) = \pi(x_1 x_2 \cdots x_m)\pi(x_2 x_3 \cdots x_{m+1}) \cdots \pi(x_{p-m+1} x_{p-m+2} \cdots x_p),$$

for every p-block. It is easy to see that π is continuous and commutes with σ. Conversely, every continuous map which commutes with σ is a block map[33]. Therefore, for every sofic system U there are an SFT and a block map $\pi : X \to U$ such that $U = \pi(X)$. For a block map π the "*scope*" of π means the minimum number m such that π is an m-block map. The scope of π is also referred as the "*window size*" of π. If there is a bijective block map π of U to V, then we say that U is conjugate to V and π is called a conjugacy. We note that π^{-1} is also a block map.

Let X be a noiseless channel with an input-constraint. Shannon defined the noiseless *channel capacity* of X as follows[4]:

$$h(X) = \lim_{n \to \infty} \frac{\log_2(\#\{\text{all } n\text{-blocks in } X\})}{n}. \tag{2.1}$$

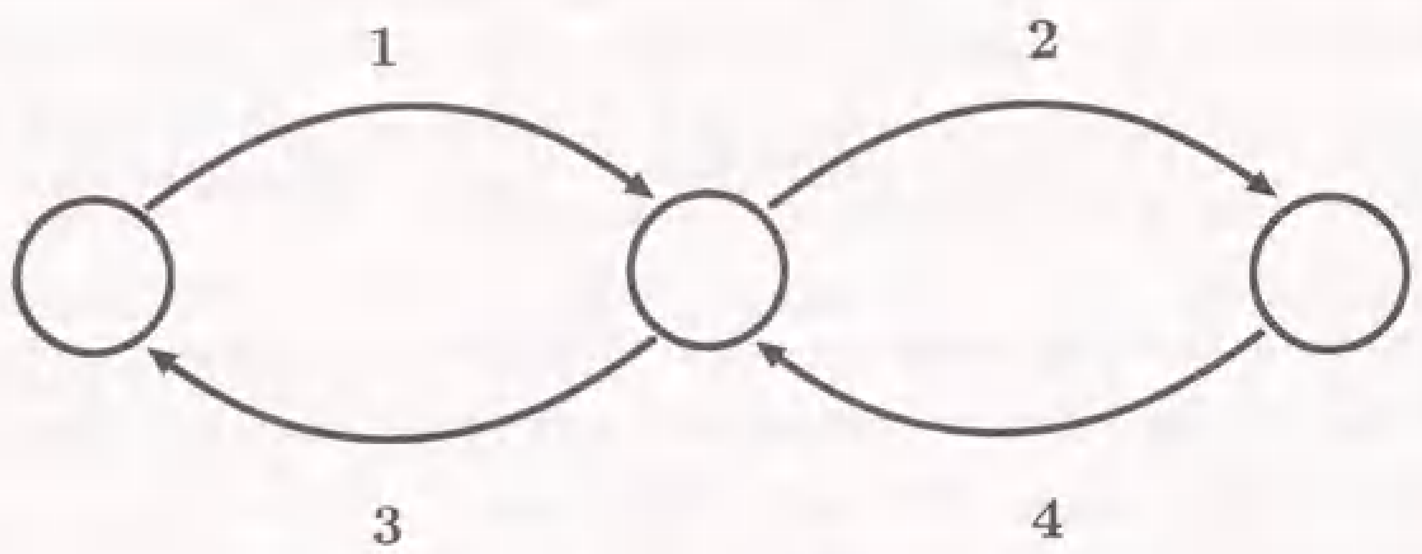
In the theory of dynamical systems the "*(topological) entropy*" is defined for a topological dynamical system and the definition reduces to (2.1) for a symbolic dynamical system[32, Chapter 16]. Shannon also defined the entropy of a Markov chain[4] (this corresponds to the measure theoretical entropy of a measure theoretical dynamical system in symbolic dynamics[32, Chapter 10, 12]) and showed a relation between these two entropies. An exposition on this subject in terms of ergodic theory can be found in [34, Appendix].

Let $G = (S, E)$ be a directed graph which consists of a set S of states and a set E of edges. We write $\mathcal{S}(G) = S$ and $\mathcal{E}(G) = E$. For each edge $e \in E$, $i_G(e)$ and $t_G(e)$ denote its initial and terminal states of e, respectively. We define Λ_G as the set of all bi-infinite sequences of edges over G, i.e.,

$$\Lambda_G = \{(e_i)_{i \in Z} : e_i \in E, t_G(e_i) = i_G(e_{i+1}) \text{ for all } i \in Z\}.$$

We note that Λ_G is an SFT. For a path $\boldsymbol{e} = e_1 e_2 \cdots e_m$ we put $i_G(\boldsymbol{e}) = i_G(e_1)$ and $t_G(\boldsymbol{e}) = t_G(e_m)$. A "*cycle*" is a block $\boldsymbol{e}$ in Λ_G such that $i_G(\boldsymbol{e}) = t_G(\boldsymbol{e})$. For a cycle $s_0 s_1 \cdots s_{L-1}$, we use a convention: for $i \geq L$, $s_i \equiv s_{i \bmod L}$. For $n \geq 1$, $L_G(n)$ denotes the set of all n-blocks in Λ_G and we put $L_G = \bigcup_{n \geq 1} L_G(n)$. We call Λ_G a "*graph shift*" and a block in Λ_G a "*path*." We regard a directed graph with labeled edges as a model of a constrained channel or a constraint itself.

Example 2.1 Let G be a graph given in Fig. 2.1. We distinguish each edge by number. A sequence given in Fig. 2.1 is a part of an element in $L(G)$.



$$\cdots 1\ 3\ 1\ 2\ 4\ 2\ 4\ 2\ 3\ 1 \cdots$$

Figure 2.1: Example of graph shift

For a graph $G = (S, E)$ the "*adjacency matrix*" $M_G = [s_{\alpha\beta}]$ of G is an integral matrix defined as follows: for $\alpha, \beta \in S$, $s_{\alpha\beta} = \#\{e \in E : i_G(e) = \alpha, t_G(e) = \beta\}$. It is well known that $h(\Lambda_G) = \log \lambda$, where λ is the largest real eigenvalue of M_G [4]. For $\alpha \in S$ we define $\mathcal{F}_G(\alpha)$ by $\mathcal{F}_G(\alpha) = \{e \in E : i_G(e) = \alpha\}$. If for every (ordered) pair α, β of states in S there exists an m-block $e_1 ... e_m \in L_G$ such that $i_G(e_1) = \alpha$ and $t_G(e_m) = \beta$, then we say that G (and M_G) is "*irreducible.*" For a positive integer k, we define the graph $G^k = (S, L_G(k))$ as follows: for $w = e_1 \cdots e_k \in L_G(k)$, $i_{G^k}(w) = i_G(e_1)$ and $t_{G^k}(w) = t_G(e_k)$. We call G^k the "*k-th power graph*" of G.

Proposition 2.1 Let G be a directed graph. For every positive integer k, we have

$$h(\Lambda_{G^k}) = kh(\Lambda_G).$$

Proposition 2.2 *[35, p. 9]* Let Λ_1 and Λ_2 be subshifts and let $\pi : \Lambda_1 \to \Lambda_2$ be an onto map. If π is finite-to-one or 1-1 almost everywhere, then $h(\Lambda_1) = h(\Lambda_2)$.

Definition 2.3 Let π be a factor map from Λ_1 to Λ_2. Suppose that if $\boldsymbol{x}, \boldsymbol{y} \in \Lambda_1$, $\boldsymbol{x} \neq \boldsymbol{y}$, and there exists n such that for all $i \leq n$ $x_i = y_i$ then $\pi(\boldsymbol{x}) \neq \pi(\boldsymbol{y})$. Then π is said to be "*right closing.*" Similarly we can define the notion of "*left closing.*" If π is a 1-block map and for every distinct edges e, e' with $i_G(e) = i_G(e')$, $\pi(e) \neq \pi(e')$, then π is said to be "*right resolving.*" Similarly, we define a "*left resolving*" map. If π is left closing and right closing then π is said to be "*bi-closing.*"

A right resolving block map is also said to be "*deterministic*" and a directed graph with a deterministic block map is also said to be deterministic.

We can prove the following

Proposition 2.3 Every right or left closing map is finite-to-one.

From Proposition 2.2 and Proposition 2.3 we note that if S is a sofic system given by $S = \pi(X)$ then $h(S) = h(X)$ where X is an SFT and π a right closing block map.

A sofic system S is said to be "*irreducible*" if for every pair of blocks $\boldsymbol{w}, \boldsymbol{z}$ in S, there is a word $\boldsymbol{t}$ such that $\boldsymbol{wtz}$ is in S.

Proposition 2.4 *[36, p. 744]* Let S be a constrained system. The following are equivalent:

1) S is irreducible;

2) S is presented by some irreducible graph, that is, S is an image of an irreducible SFT by a block map.

Lemma 2.1 Let S be an irreducible sofic system, let G be a directed graph and let π be a block map such that $S \subseteq \pi(\Lambda_G)$. Then for some irreducible component G' of G, $S \subseteq \pi(\Lambda_{G'})$.

2.2 Codes for constrained channels

We describe one of the main problems of coding theory for constrained channels in terms of symbolic dynamics.

Let S be an irreducible input-constraint. We assume that S is a sofic system. By the modified subset construction, which is well-known in automata theory (e.g., see[36, Proof of Theorem 3]), we may assume that S is an image of an SFT by a right resolving block map. From Proposition 2.4 and Lemma 2.1, we may assume that S is represented by a deterministic irreducible graph G. Let π be the labeling of G. Let λ be the largest eigenvalue of G. For a state s of G, let $N(s, n)$ denote the number of distinct blocks with initial state s. Since G is irreducible, from (2.1) there is a positive constant a such that $a\lambda^n \leq N(s, n)$ for all sufficiently large n and for all states s. Since π is right resolving, the number of n-blocks generated by paths starting from s is $N(s, n)$. Therefore, if $m/n < \lambda$ then we can construct a finite state code with rate m/n. However, if we construct a code such that $\lambda - m/n$ is very small, then the code word length of the resulting code may be very long and it may be impractical to implement such a code by logic circuits. Thus it is one of the main problems in coding theory for constrained channels to find general code construction algorithms which produce efficient and simple codes for given constraints.

Adler, Coppersmith and Hassner[5] proposed a code construction procedure by which we can always construct a code for every SFT channel. We shall call it "*the ACH procedure*", which we shall describe in Section 3.2. They proved the following:

Theorem 2.1 Let X be an SFT and let N be a positive integer. Assume that $h(X) \geq \log N$. Then there are a graph G and a conjugacy π such that 1) $\pi(\Lambda_G) \subset X$; 2) every row sum of M_G is at least N; 3) π is $(0,0)$ type and π^{-1} $(0,m)$ type for some nonnegative integer m. We can construct G and π by the ACH procedure.

We can construct a right closing factor map $\eta : \Lambda_G \to \Sigma_N$ because of the property 2) above. The triple $(\Lambda_G, \eta, \pi^{-1})$ defines a finite state encoder, and the composition of η^{-1} and π a sliding block decoder. These constructions are explained in the next chapter. The scope of the sliding block decoder is a complexity measure of a code obtained by the ACH procedure. This is the main concern of Chapter 3.

The following results tell us that the ACH procedure can be applied to sofic systems.

Proposition 2.5 *[11, Proposition 3]* Let S be an irreducible sofic system. Then there are SFT's $\Lambda_1, \Lambda_2, \cdots$, such that

1) $\Lambda_l \subset S$ for every l;

2) $\sup_l h(\Lambda_l) = h(S)$.

Therefore, if $r < h(S)$ for some rational number r then we can construct a code for a constrained channel represented by S with coding rate r by using the ACH procedure.

Definition 2.4 A sofic system S is "*almost of finite type*"(AFT) if there is an irreducible SFT X and an onto factor map $\pi : X \to S$ that is one-to-one on an open set.

Definition 2.5 *[11, Definition 2]* Let Λ_1 and Λ_2 be subshifts, and let $\pi : \Lambda_1 \to \Lambda_2$ be a factor map. A "*resolving block*" is a block in Λ_2 $s = s_1 \cdots s_t$ for which there exists an $i \in [1,t]$ such that if $\boldsymbol{u} = u_1 \cdots u_w$ and $\boldsymbol{v} = v_1 \cdots v_w$, $t \leq w$ are blocks in Λ_1 with $\pi(\boldsymbol{u}) = \boldsymbol{s} = \pi(\boldsymbol{v})$, then $u_i = v_i$.

Proposition 2.6 *[11, p.374]* Let $\pi : \Lambda \to S$ be an onto factor map from an irreducible SFT to a sofic system. The following are equivalent:

1) π is 1-1 on an open set;

2) π is 1-1 on an open dense set of full measure;

3) π is bi-closing and has a resolving block.

From these results, we note that irreducible finite subgraphs of canonical graphs (defined in [23]) with period less than n are AFT. Therefore, we can apply code construction procedure given in [5], [11] and [37] to those subgraphs in order to obtain spectral null codes.

2.3 Representation of constrained channels

In this thesis we will use several ways to represent constrained systems because of the convenience of analyzing our problems and of stating our results.

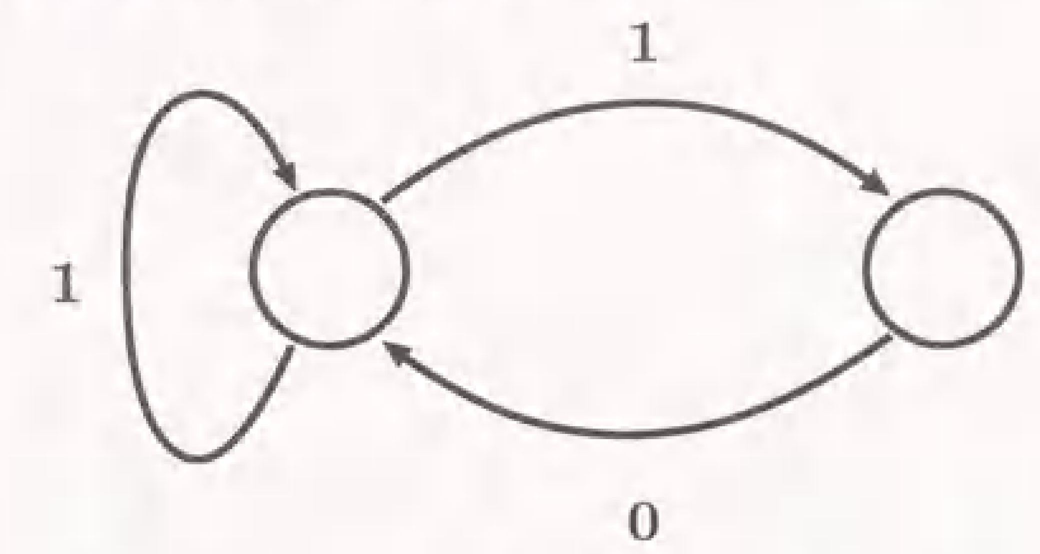


Figure 2.2: Example of Mealy type FSTD

Let G be a directed graph. Suppose that every entry of M_G is 0 or 1. Then the following set is an SFT:

$$\tilde{\Lambda}_G = \{(x_i)_{i \in \mathbf{Z}} : x_i \in \mathcal{S}(G), x_{i+1} \in \mathcal{F}_G(x_i)\}.$$

A path in $\tilde{\Lambda}_G$ is a finite sequence of states in G. Such a model for a constrained channel is used in Chapter 4 and Chapter 5.

A "*Mealy finite sequential state machine(FSSM)*" is a directed graph with labeled edges. A "*Moore FSSM*" is a directed graph with labeled states. We also call an FSSM a "*finite state transition diagram (FSTD)*" or a "*finite state transition graph (FSTG).*"

Let S be a sofic system and assume that S is represented by a directed graph G and a block map π. Let m be the scope of π. Consider a directed graph G' with edge set $L_G(m)$ and state set $L_G(m-1)$ such that for any pair of states $x_1 \cdots x_{m-1}$ and $y_1 \cdots y_{m-1}$ if $x_2 \cdots x_{m-1} = y_1 \cdots y_{m-2}$ then there is an edge $x_1 x_2 \cdots x_{m-1} y_{m-1}$ from $x_1 \cdots x_{m-1}$ to $y_1 \cdots y_{m-1}$ with label $\pi(x_1 x_2 \cdots x_{m-1} y_{m-1})$. We call $\Lambda_{G'}$ a higher block representation or a higher block shift of Λ_G. It is easy to see that $\Lambda_{G'}$ and Λ_G are conjugate. Let π' be the labeling of edges in G'. Then we also note that $\pi'(\Lambda_{G'}) = \pi(\Lambda_G)$. This construction shows that every constrained channel represented by a Mealy FSSM can be also represented by a Moore FSSM, and vice versa. Since constrained channels which we are concerned with are sofic systems, FSSM is general enough to represent any constrained channel. As models of constrained channels or constraints themselves, we use directed graphs with labeled edges in Chapter 6. But in Chapter 4 and 5 we use directed graphs with labeled states because they are suitable for spectral analysis of codes given as functions of finite Markov chains. In Chapter 3 we consider subshifts of finite type which are represented by Mealy FSSM's.

Example 2.2 In Fig. 2.2 we present an example of Mealy type FSTD. The Moore-type representation of the FSTD is given in Fig. 2.3.

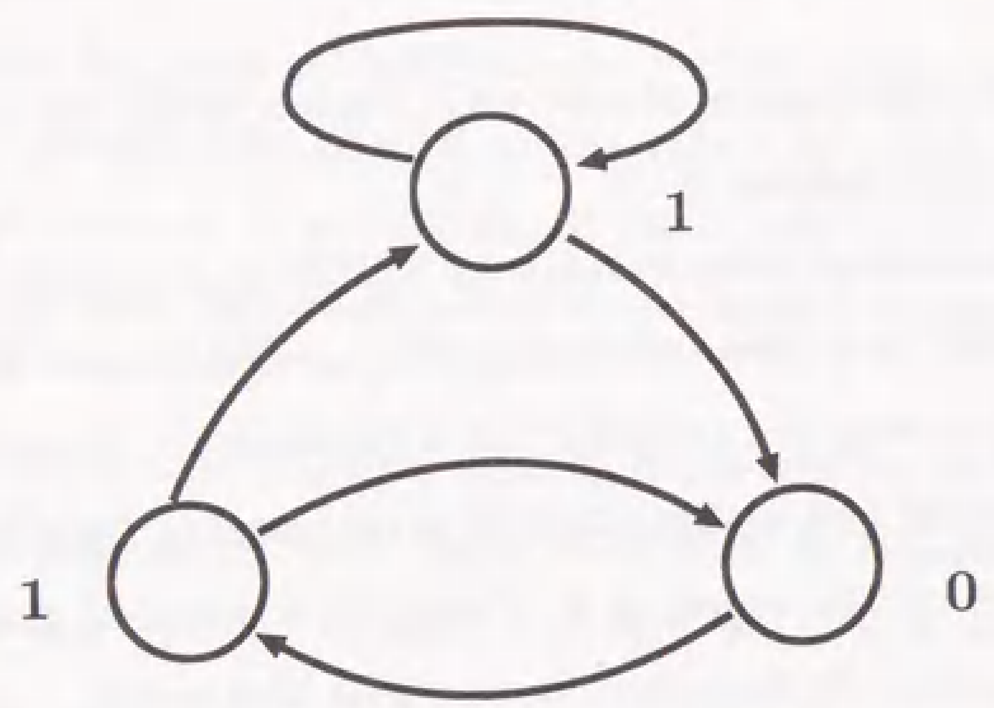


Figure 2.3: Example of Mealy type FSTD

Chapter 3

Minimum Scope for Sliding Block Decoder Mappings

3.1 Introduction

In this chapter we shall investigate the scope of the conjugacy and consider the problem of searching the conjugacy induced by the state-splitting with the minimum scope.

Adler, Coppersmith, and Hassner presented a code construction procedure for constrained channels of finite type[5]. We shall call it the ACH procedure. A state-splitting algorithm is the key idea of the ACH procedure, and it induces a finite-state encoder and a conjugacy of a certain type (a sliding block decoder mapping). The number of states of the encoder and the scope of the conjugacy may be considered as appropriate complexity measures of the code. Here we investigate the scope of the conjugacy.

In Section 3.3 we establish some properties of the state-splitting. First we shall establish a relation between a conjugacy of $(0, m)$ type and a state-splitting of m rounds. We shall show that we need not take into account "empty splittings" (Proposition 3.1) and establish a property of approximate eigenvectors of m rounds (Theorem 3.2). Using this we shall bound m-round approximate eigenvectors from above and show a fact which is useful to reduce the computation time for searching the conjugacy with the minimum scope. We present a searching algorithm and give the minimum scopes of the conjugacies for (d, k)-constraints in Section 3.4 by applying the searching algorithm. We also show that apparently good strategies for this searching problem do not work for all constrained channels.

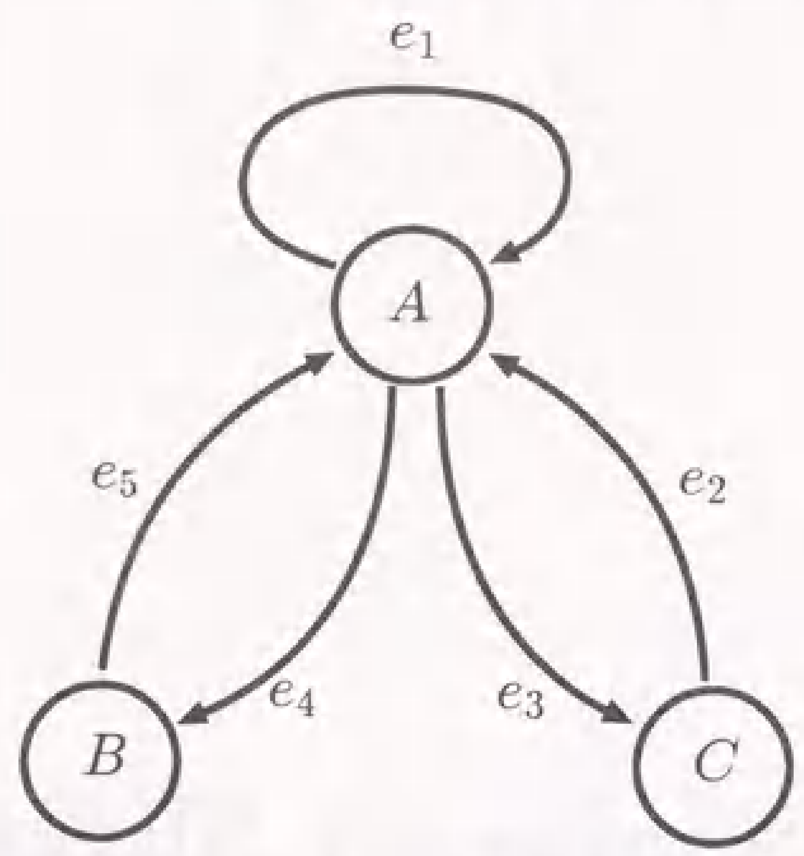


Figure 3.1: A constrained channel

3.2 State-splitting

The ACH procedure mentioned above was introduced by Marcus [10] and was extended and used to construct codes for constrained channels by Adler, Coppersmith and Hassner [5]. Since our results depend strongly on this ACH procedure, we explain it here. Expositions of the ACH procedure can be found in [38], [39] and [15].

Let N be a positive integer and let $G = (S, E)$ be a directed graph with the adjacency matrix M_G. Graph G is said to be *nice* with respect to N if there exists some irreducible subgraph H of G such that each state of H has at least N outgoing edges. Suppose that $\log N \leq h(\Lambda_G)$. Then there exists a nonnegative integral vector v with $Nv \leq M_G v$ [9] and such v is called an *approximate eigenvector* for M_G (or G) with respect to N, where inequality between two vectors means componentwise inequality. Assume that for each $\alpha \in S$ we are given a partition $\{B(\alpha,1),\ldots,B(\alpha,k(\alpha))\}$ of $\mathcal{F}_G(\alpha)$. We make a graph G' as follows: the set of states of G' is $S' = \{(\alpha,m) : \alpha \in V, 1 \leq m \leq k(\alpha)\}$ and the set of edges of G' is $E' = \{(e,m) : e \in E, 1 \leq m \leq k(t_G(e))\}$. The initial and terminal state mappings of G' are defined by $i_{G'}((e,m)) = (i_G(e), n)$ and $t_{G'}((e,m)) = (t_G(e), m)$, where n is the number such that $e \in B(i_G(e), n)$. We call this procedure transforming G into G' a *state-splitting of one round* (see [5]) and we say that G' is obtained from G by the state-splitting of one round. Let $\rho : \Lambda_G \to \Lambda_{G'}$ be a 2-block map of (0,1) type defined by that $\rho(e_1, e_2) = (e_1, n)$ for $e_1 e_2 \in L_G$, where n is the number such that $e_2 \in B(t_G(e_1), n)$. Then ρ is a conjugacy and its inverse is given by that $\rho^{-1}((e,k)) = e$ for $(e,k) \in E'$.

Example 3.1 Let G be the graph in Fig. 3.1. We partition $\mathcal{F}_G(A) = \{e_1, e_2, e_3\}$ into

$$B(A,1) = \{e_1\} \quad \text{and} \quad B(A,2) = \{e_3, e_4\}.$$

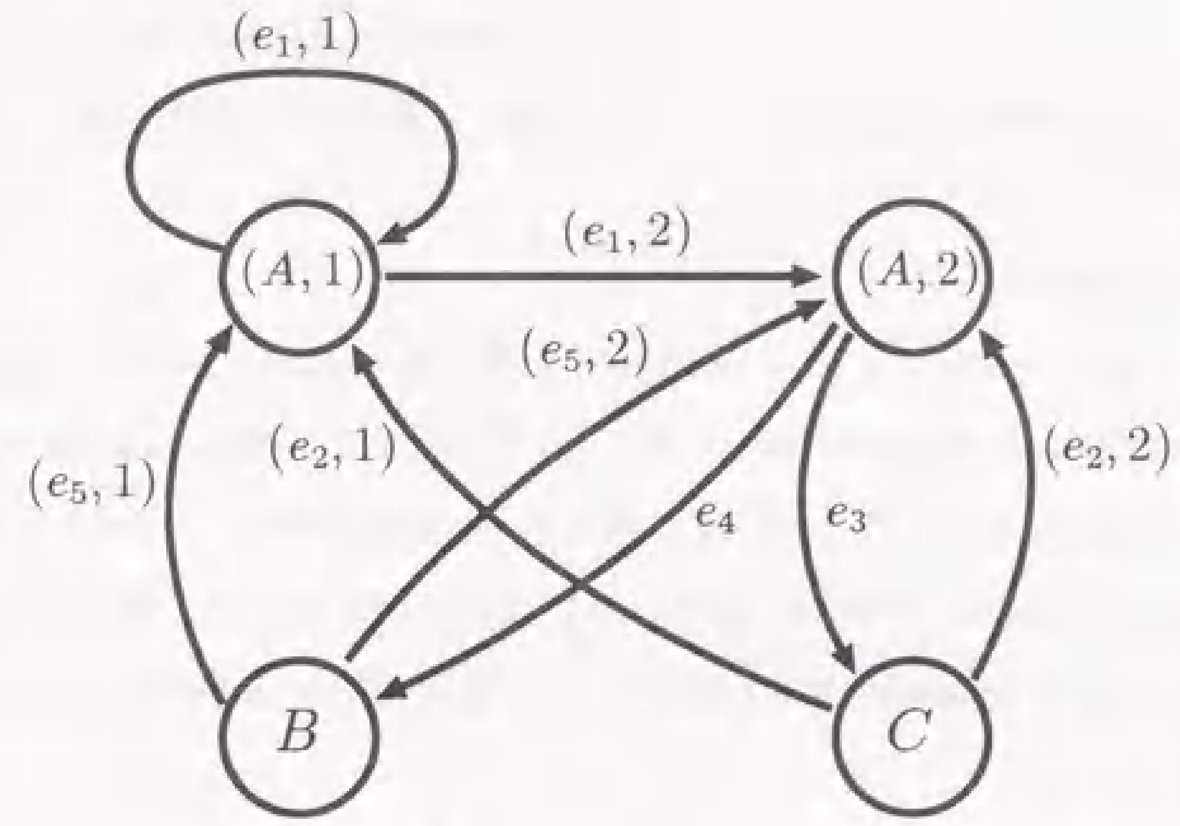


Figure 3.2: G_1

The state-splitting induced by this partition gives the graph in Fig. 3.2 (these graphs correspond to the graphs in Fi.g 1.5 and in Fig. 1.7). The local map ρ of $(0,1)$ type induced by this state-splitting is given as follows:

$$\rho(e, e') = \begin{cases} (e, i) & \text{if } e = e_1, e = e_2 \text{ or } e = e_5; \\ e & \text{otherwise,} \end{cases}$$

where i is the number such that $e' \in B(A, i)$.

Let v be an approximate eigenvector for G with respect to N. Let v' be a nonnegative integral vector with index set S' and suppose that v' satisfies the following conditions: for each $\alpha \in S$ $v(\alpha) = \sum_{j=1}^{k(\alpha)} v'((\alpha, j))$ and $Nv'((\alpha,j)) \leq \sum_{e \in B(\alpha,j)} v(t_G(e)), 1 \leq j \leq k(\alpha)$. Then v' is an approximate eigenvector for G' with respect to N. We say that the state-splitting is *compatible* with v with respect to N and *induces* v' from v. If v and v' have the same number of $0's$, then we say that it is strongly compatible with v with respect to N. Let $\hat{G}$ be a directed graph and let k be a nonnegative integer. If there is a sequence of graphs $G_0, G_1, \ldots, G_k$ such that $G_0 = G, G_k = \hat{G}$ and for each $i \leq k-1$, G_{i+1} is obtained from G_i by a state-splitting of one round, then we say that $\hat{G}$ is obtained from G by a *state-splitting of k rounds*. The state-splitting of k rounds is said to be *compatible with* v and induces v' from v [strongly compatible with v] with respect to N if besides the sequence of graphs $G_0, \ldots, G_k$ as above, there is a sequence of vectors $v_0, v_1, \ldots, v_k$ with $v_0 = v$ and $v_k = v'$ such that

(2-1) For each $i \leq k-1$, the state-splitting of one round obtaining G_{i+1} from G_i is compatible with v_i [strongly compatible with v_i] with respect to N, and v_{i+1} is an approximate eigenvector for G_{i+1} induced from v_i by the state-splitting.

If in addition the following condition is satisfied, v is said to be a *k-round vector* for G with respect to N:

(2-2) Each component of v_k is 0 or 1.

If v is a k-round vector then G_k is a nice graph. It has been proved in [5, Theorem 6.1] that for any approximate eigenvector v for G, G is transformed into a nice graph with respect to N by a state-splitting of m rounds compatible with v for some integer m, and then it is implicit that there is an m-round vector for G. Thus G has an m-round vector if and only if G is transformed into a nice graph by a state-splitting of m rounds.

Example 3.2 Put

$$T = \begin{pmatrix} 1 & 1 & \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and, then, T is the adjacency matrix of the graph G in Fig. 3.1. We note that the maximum eigenvalue of T is 2 and that an eigenvector corresponding to 2 $v = (2\ 1\ 1)^t$. Put

$$T' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

which is the adjacency matrix of the graph in Fig. 3.2 and let $v' = (1\ 1\ 1\ 1)^t$. We note that the maximum eigenvalue of T' is 2 and v' is an eigenvector corresponding to 2. We have

$$\begin{aligned} v(A) &= v'((A,1)) + v'((A,2)) \\ 2v'((A,1)) &= v'((A,1)) + v'((A,2)) \\ 2v'((A,1)) &= v'(B) + v'(C). \end{aligned}$$

Therefore the state-splitting given in Example 3.1 is compatible with v with respect to 2 and induces v' from v. Thus v is a 1-round vector of G(or T).

Here we describe the method of constructing codes due to Adler, Coppersmith, and Hassner [5] for a source $\{1, \ldots, N\}^{\mathbb{N}}$ and a constrained system which is given by Λ_G, where G is a directed graph. We may assume that G is irreducible. In fact if G is not irreducible then we can take an irreducible subgraph G' with $h(\Lambda_G) = h(\Lambda_{G'})$ and

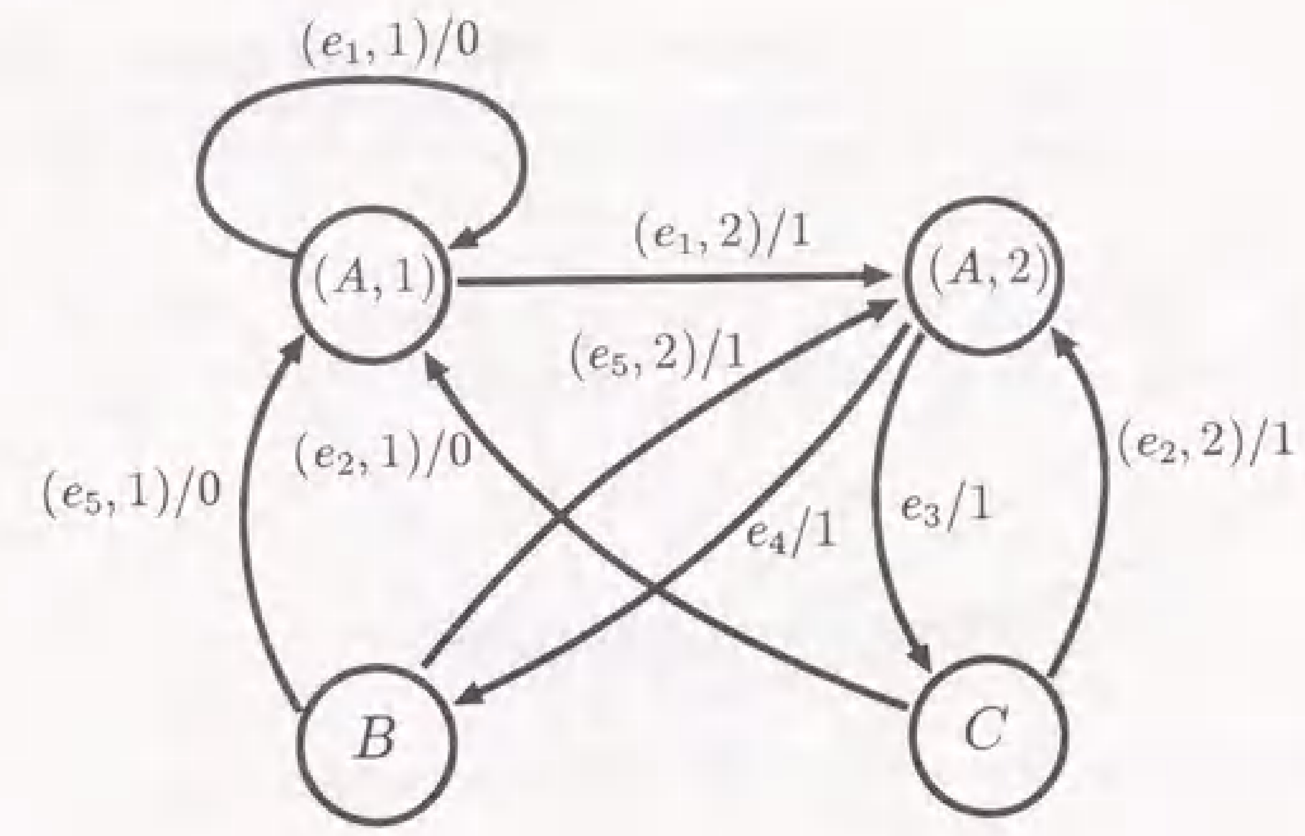


Figure 3.3: Encoder for graph in Fig. 3.1

replace G by G'. Let p and q be positive integers such that $p \log N \leq qh(\Lambda_G)$. Put $\overline{G} = G^q$ and let $M_{\overline{G}}$ be the adjacency matrix of $\overline{G}$. Let v be an approximate eigenvector for $\overline{S}$ with respect to N^p. By the method of [5, Theorem 6.1] we obtain a nice graph H from $\overline{G}$ by a state-splitting of k rounds compatible with v with respect to N^p for some nonnegative integer k. Let $\rho : \Lambda_{\overline{G}} \to \Lambda_H$ be the conjugacy corresponding to the state-splitting, where ρ is of $(0, k)$ type and ρ^{-1} is of $(0, 0)$ type. Let $H' = (S', E')$ be an irreducible subgraph of H such that each state in S' has N^p outgoing edges. Let $\phi : \Lambda_{H'} \to \{1, \ldots, N^p\}^{\mathbb{Z}}$ be a 1-block onto map such that for $e, d \in E'$ if $i_{H'}(e) = i_{H'}(d)$ and $\phi(e) = \phi(d)$ then $e = d$ (such block maps are said to be right resolving or unifilar, see [5]). We fix a state $\overline{\alpha} \in S'$, which is the "initial state" for encoding. For any $x \in \{1, \ldots, N^p\}^{\mathbb{N}}$, we can obtain a half infinite sequence $y = (y_i)_{i \geq 0}$ of edges over H such that $\phi(y_0)\phi(y_1)\ldots = x$ and $i_H(y_0) = \overline{\alpha}$. Then we obtain $\rho^{-1}(y) = z$ which is the half infinite sequence in $\Lambda_{\overline{G}}$ into which x is encoded. Since $\rho(z) = y$ we can decode z into the original data x by $\phi(y) = \phi(\rho(z))$. This coding scheme is implemented a logical circuit with a buffer memory. The code construction procedure described above is called "*ACH procedure.*"

Example 3.3 We use the graph in Fig. 3.2 again. We define the right resolving 1-block map ϕ as shown in Fig. 3.3 where e/x means $\phi(e) = x$. Let $(A, 1)$ be the initial state. Suppose that an input sequence $x = 011100101\cdots$ is given. Then we have

$$\phi((e_1,1)(e_1,2)e_3(e_2,2)e_4(e_5,1)(e_1,2)e_4(e_2,1)\cdots) = x.$$

Let ρ be the 2-block map given in Example 3.2. The 1-block map ρ^{-1} is given by

$$\rho^{-1}(e) = \begin{cases} e' & \text{if } e = (e', i); \\ e & \text{otherwise.} \end{cases}$$

So, we get

$$\rho^{-1}(x) = e_1 e_1 e_3 e_2 e_4 e_5 e_1 e_4 e_2 \cdots.$$

This is the encoding step. Put $y = \rho^{-1}(x)$. The decoding step is given by

$$\begin{aligned} \phi(\rho(y)) &= \phi((e_1,1)(e_1,2)e_3(e_2,2)e_4(e_5,1)(e_1,2)e_4\cdots) \\ &= 011100101\cdots. \end{aligned}$$

3.3 Properties of state-splitting

As stated in the previous section, an approximate eigenvector plays an important role in the ACH procedure. In this section we shall give some properties on state-splitting and approximate eigenvectors which will be used in our searching algorithm.

Conjugacies induced by the ACH procedure are characterized in terms of the state-splitting as follows.

Theorem 3.1 Let G and H be irreducible directed graphs and let m be a nonnegative integer. Then there exists a conjugacy $\pi : \Lambda_G \to \Lambda_H$ which is of $(0, m)$ type and whose inverse is of $(0, 0)$ type if and only if H is obtained from G by a state-splitting of m rounds.

Proof: See Appendix A.1. □

Therefore our problem changes into the problem of finding the smallest integer m such that there is an m-round vector.

In [5] Adler, Coppersmith, and Hassner have proved that for any positive integer N, for any irreducible directed graph G with $\log N \leq h(\Lambda_G)$ is transformed into a nice graph with respect to N by a state-splitting of k rounds compatible with an approximate eigenvector for G for some positive integer k. Moreover it is implicit in their proof that if G is transformed into a nice graph by a state-splitting of k rounds, then there exists a k-round vector for G. From the following proposition we note that G is transformed into a nice graph by a state-splitting of k-round strongly compatible with the vector.

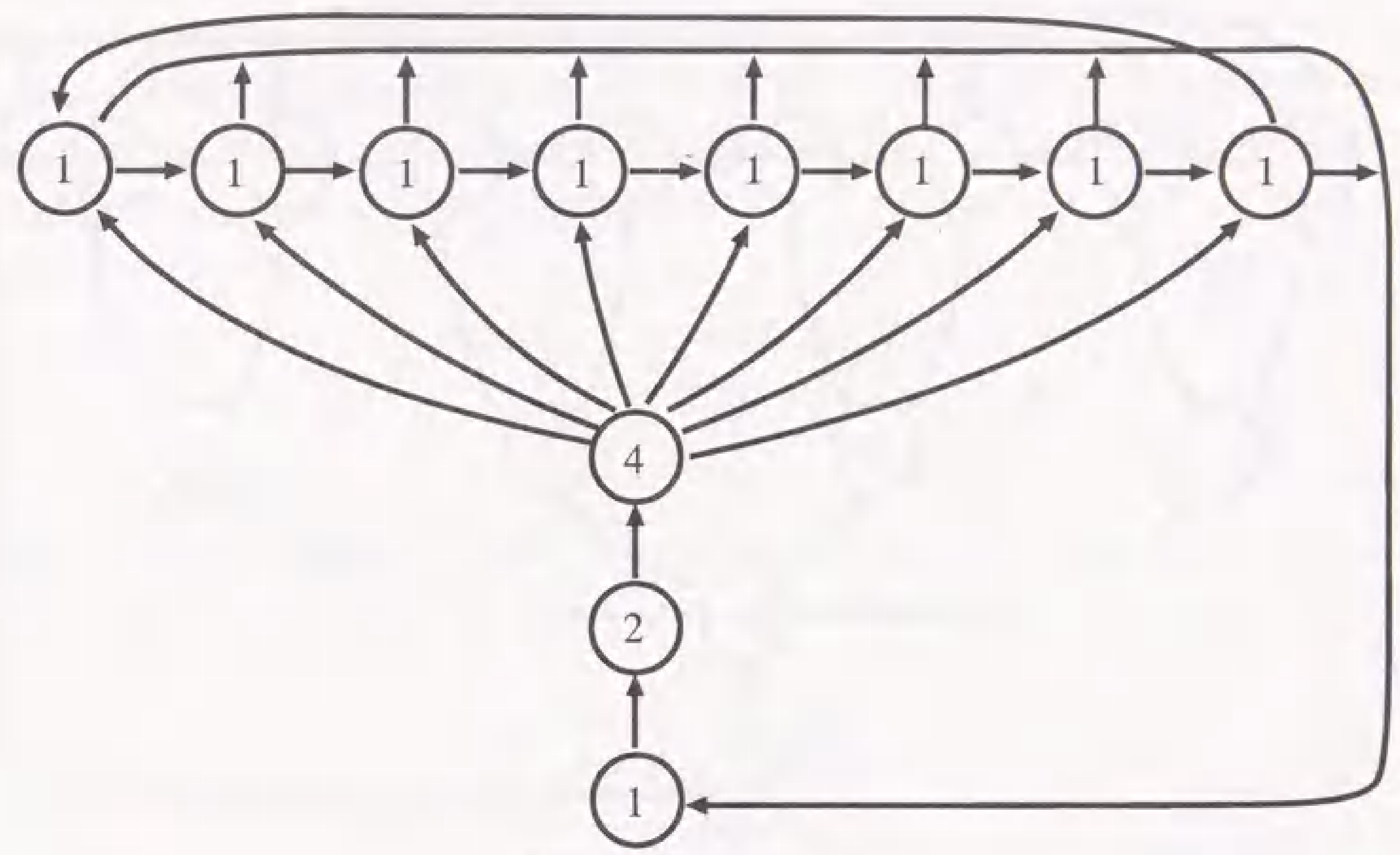


Figure 3.4: Graph G

Proposition 3.1 Let m be a nonnegative integer and let N be a positive integer. Let G be a directed graph such that $\log N \leq h(\Lambda_G)$, and let v be an m-round vector for G with respect to N. Then there is a state-splitting of m-round which is strongly compatible with v with respect to N and induces a vector with all components 0 or 1.

Proof: See Appendix A.2. □

Our algorithm for finding the minimum integer m such that there is an m-round vector, which will be described later, is essentially an exhaustive search over approximate eigenvectors. The algorithm searches an m-round vector by trying all possible state-splitting of m rounds. Therefore it is worth giving necessary conditions for an approximate eigenvector to be an m-round vector in order to reduce the number of vectors to be check.

We begin with the following example.

Example 3.4 We consider the graph G given in Fig. 3.4. The maximum eigenvalue of the graph is 2 and numbers in circles of the graph mean elements of an eigenvector with respect to 2. Fig. 3.5 and Fig. 3.6 show that the vector is a 2-rounds vector. From these figures we note that the vector is 1-round vector for G^2 with respect to 2^2.

This observation can be generalized as follows.

Theorem 3.2 Let m and N be positive integers and let $G = (V, E)$ be a directed graph with $\log N \leq h(\Lambda_G)$. Let v be an m-round vector for G with respect to N. Then v is a

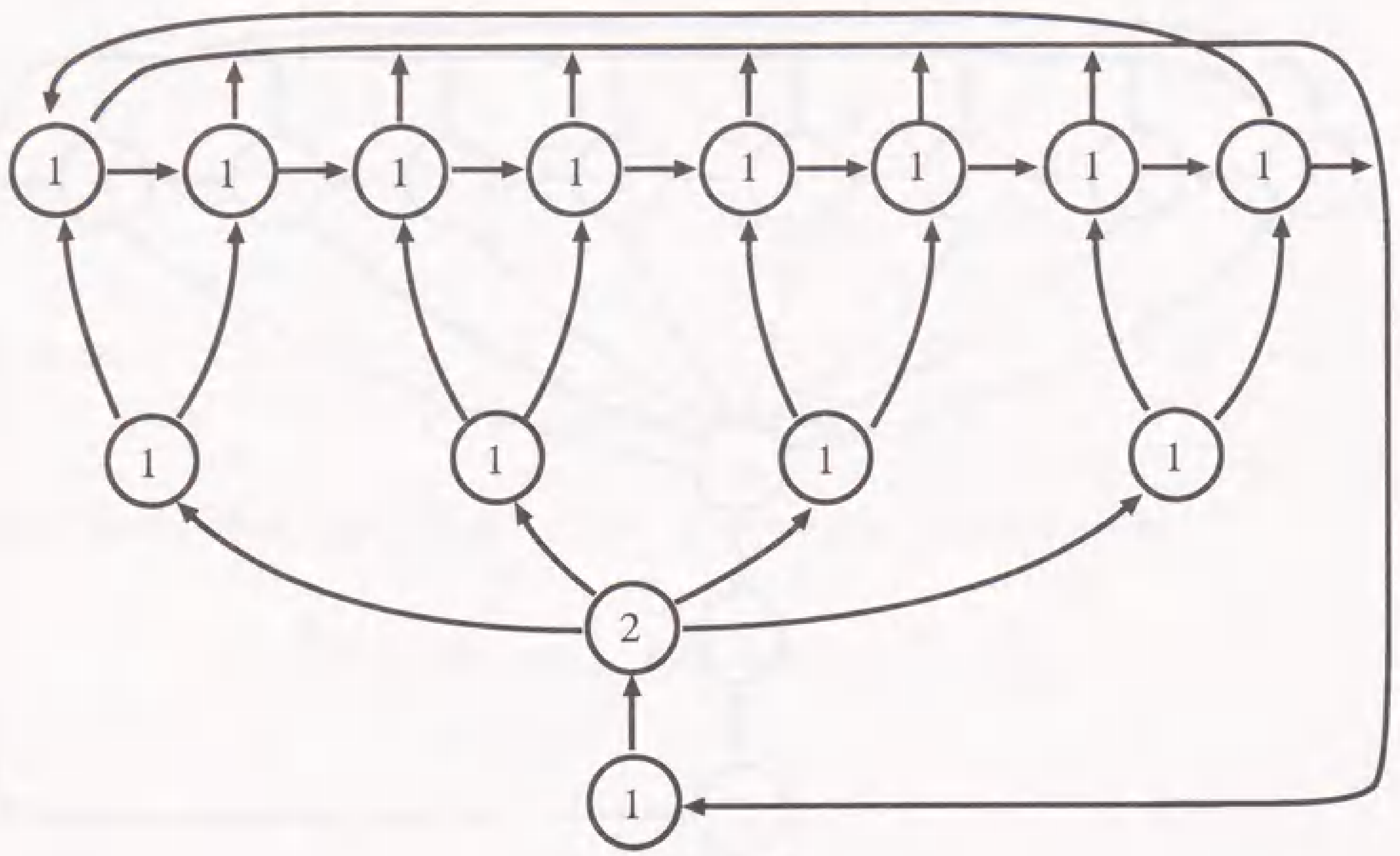


Figure 3.5: G_1

1-round vector for G^m with respect to N^m.

Proof: See Appendix A.3. □

Let v be an approximate eigenvector. If we find that v is not a 1-round vector with respect to N^m then we can reduce the region where m-round vectors exist.

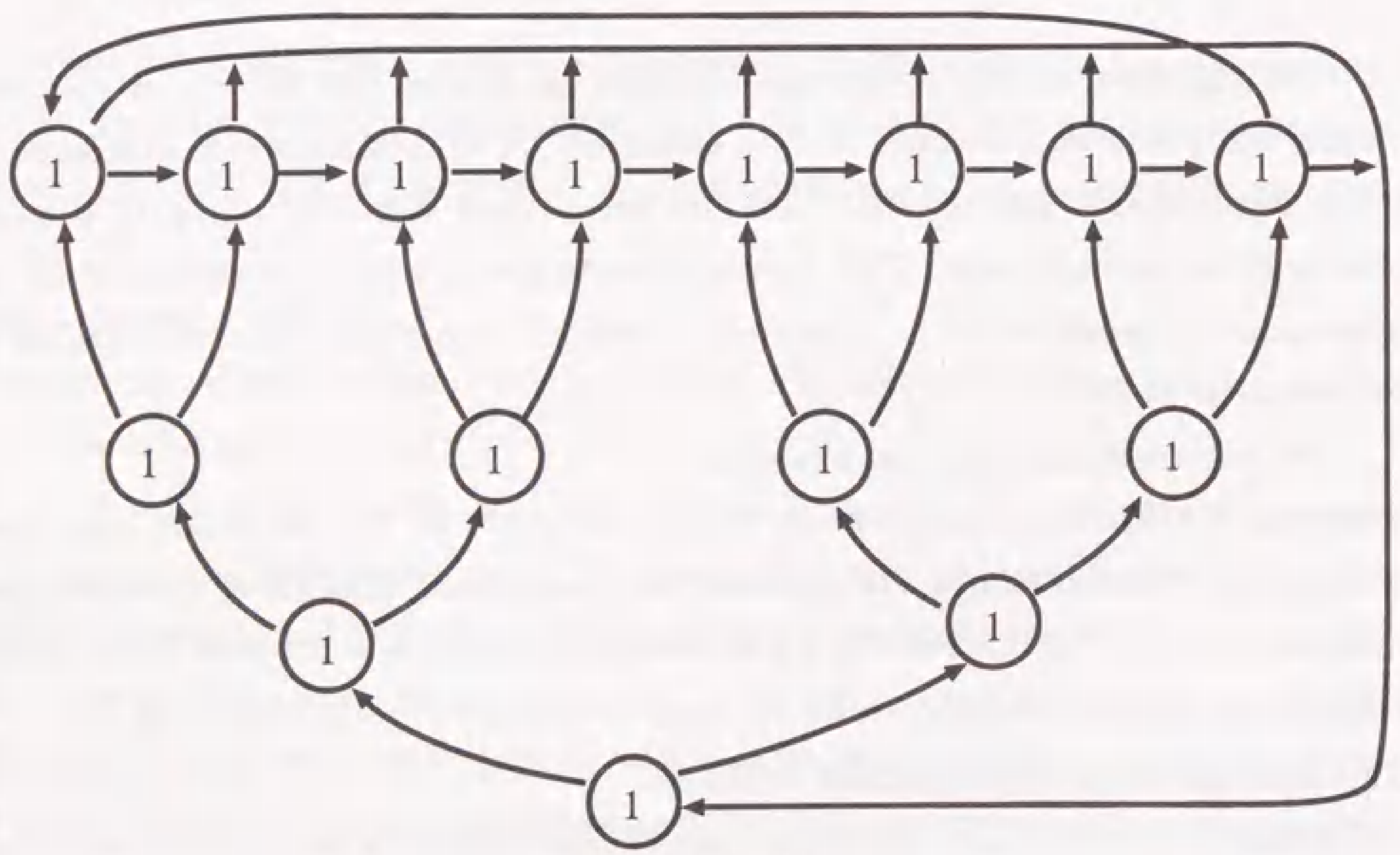


Figure 3.6: G_2

Corollary 3.1 Let v be an m-round vector for G with respect to N and let u be an approximate eigenvector for G with respect to N. Assume that $v \leq u$ and u is not a 1-round vector for G^m with respect to N^m, so that there is a state $\alpha \in V$ such that $u(\alpha) > 1$ and for any partition with $u(\alpha)$ elements $\{B_1, ..., B_{u(\alpha)}\}$ of $\{e_1...e_m \in L_G : i_G(e_1) = \alpha\}$, there is an i with

$$N^m > \sum_{e_1...e_m \in B_i} u(t_G(e_m)). \tag{3.1}$$

Then $v(\alpha) \leq u(\alpha) - 1$.

Proof: Suppose that $v(\alpha) = u(\alpha)$. By the proof of Theorem 3.2, there is a partition $\{B_1, ..., B_{u(\alpha)}\}$ of $\{e_1...e_m \in L_G : i_G(e_1) = \alpha\}$ such that for each $1 \leq k \leq v(\alpha)$,

$$N^m \leq \sum_{e_1...e_m \in B_k} v(t_G(e_m)),$$

and hence by the assumption

$$N^m \leq \sum_{e_1...e_m \in C_k} u(t_G(e_m)),$$

for each $1 \leq k \leq u(\alpha)$, which contradicts (3.1). □

The next two results confine the region where m-round vectors exist.

Corollary 3.2 Let G be a directed graph with the adjacency matrix S and let m be a nonnegative integer. If v is an m-round vector for G, then we have $v \leq S^m \underline{e}$, where $\underline{e}$ is the vector with all components 1.

Proof: If $m = 0$, the corollary is trivial. Assume that $m \geq 1$. By Theorem 3.2 v is a 1-round vector for G^m with respect to N^m. Since each state α in G^m has $(S^m \underline{e})(\alpha)$ outgoing edges, α can be split into at most $(S^m \underline{e})(\alpha)$ states by any state-splitting of one round. Thus we get $v(\alpha) \leq (S^m \underline{e})(\alpha)$. □

Corollary 3.3 Let m be a nonnegative integer and let N be a positive integer. Let G be an irreducible directed graph with $\log N = h(\Lambda_G)$. If v is an m-round vector for G with respect to N, then $v \leq N^m \underline{e}$.

Proof: Let S be the adjacency matrix of G. Assume that $v(\alpha) = 0$ for some $\alpha \in V$. We have $Nv = Sv$ since the maximal real eigenvalue of S is N. Therefore it follows that $v(t_G(d)) = 0$ for all $d \in \mathcal{F}_G(\alpha)$. But this means $v = 0 \leq N^m \underline{e}$ since G is irreducible.

Now we assume that $v > 0$. Any graph obtained by any state-splitting of any irreducible graph is also irreducible and G^m is irreducible. S^m is the adjacency matrix of G^m and $N^m v = S^m v$. Therefore for any state-splitting of one round compatible with v, any induced vector by it must be positive by the same argument above. By Theorem 3.2 v is a 1-round vector for G^m. Let $H = (V', E')$ be a nice graph obtained by a state-splitting of one round which is compatible with v and induces the vector $\underline{e}'$ with all components 1. Suppose that for some $\alpha \in V$ $v(\alpha) > N^m$. This implies that there exists $a \in V'$ with $|\mathcal{F}_H(a)| > N^m$. On the other hand by the definition of state-splitting it follows that $N^m \underline{e}' = T\underline{e}'$, where T is the adjacency matrix of H. This implies that each state of H has just N^m outgoing edges. □

Let N be the number of information symbols, let G be a directed graph which represents a constrained system with $\log N \le h(\Lambda_G)$, and let M_G be the adjacency matrix of G. We want to know the minimum scope of sliding block codes for G obtained by the state-splitting due to Adler, Coppersmith and Hassner [5]. By Theorem 3.1 and the paragraph after the proof of the theorem, our problem changes into the problem of finding the smallest integer m such that there is an m-round vector for G with respect to N. Assume that we know that there exists a k-round vector for G with respect to N for some positive integer k. If there exists an m-round vector v for G with $m < k$, then by Theorem 3.4 v is a 1-round vector for G and by Corollary 3.2 $v \le M_G^m \underline{e} \le M_G^{k-1}\underline{e}$. Thus we can get the answer to the above question by trying all state-splittings for all 1-round vectors v for G^m with respect to N^m.

An algorithm which determines the smallest integer m such that there is an m-round vector for G with respect to N is given in Fig. 3.7 In the figure f_a is the mapping from the set V_I of nonnegative integral vectors for G into itself having the following two properties:

1. for any $u \in V_I$, $Nf_a(u) \le Sf_a(u)$ and $f_a(u) \le u$;

2. for $u, v \in V_I$, if $v \le u$ and $Nv \le Sv$, then $v \le f_a(u)$.

The computing procedure of this mapping have been given by Adler, Coppersmith and Hassner [5], using the method due to Franaszek [8]. Let m be a positive integer, let v be an m-round vector for G and let u be an approximate eigenvector for G^m. Suppose that $v \le u$ and u is not a 1-round vector for G^m. Let u' be the vector defined in lines 8-9. Then, by Corollary 3.1 and the properties of f_a we know that $v \le f_a(u')$. Thus the procedure defined by lines 5-11 finds the largest vector over the vectors which are approximate eigenvectors for G and 1-round vectors for G^m. For an approximate eigenvector v, we do not need to check state-splittings by which the number of 0 components of v increases in lines 15-16 (Proposition 3.1).

3.4 Examples

Here we give examples. The (d,k)-constraint is given by the set of bi-infinite sequences of binary labels along the bi-infinite paths of the directed graph $G_{d,k}$ with labeled edges given in Fig. 3.8 or Fig. 3.9. For each edge e of $G_{d,k}$ let $\eta_{d,k}(e)$ be the symbol labeled on e. Since we consider codes with rate $1/q$ here, let $q = q(d,k)$ be the smallest integer such that $h(\Lambda_{(G_{d,k})^{q(d,k)}}) \ge 1$. Let $m(d,k)$ be the smallest integer m such that there is an m-round vector for $(G_{d,k})^{q(d,k)}$ with respect to 2. Let $r(d,k) = k$ if $k < \infty$ and let $r(d,k) = d$ if $k = \infty$. Given d and k, we consider a state-splitting of $m(\ge m(d,k))$ rounds which transform $G = (G_{d,k})^q$ into a nice graph, where $q = q(d,k)$. Each bi-infinite sequence of Λ_G is represented as

$$...(x_{-q}x_{-q+1}...x_{-1})(x_0...x_{q-1})(x_q...x_{2q-1})...,$$

for some $(x_i)_{i\in\mathbb{Z}} \in \Lambda_{G_{d,k}}$. We can consider $\eta_{d,k}$ a block map of (0,0) type. Let $\boldsymbol{x} = (x_i)_{i\in\mathbb{Z}} \in \Lambda_{G_{d,k}}$ and let $\boldsymbol{z} = (z_i)_{i\in\mathbb{Z}} = \eta(\boldsymbol{x})$. We note that the block $z_{-r}z_{-r+1}...z_{-1}$ determines $i_{G_{d,k}}(x_0)$, where $r = r(d,k)$. Since $\eta_{d,k}$ is right resolving, from the block $z_{-r}z_{-r+1}...z_{(m+1)q-1}$ we can determine the block $x_0x_1...x_{(m+1)q-1}$ and hence the $(m+1)$-block $(x_0...x_{q-1})(x_q...x_{2q-1})...(x_{mq}...x_{(m+1)q-1})$ of Λ_G. As described in Section 3.2 we can decode one information symbol from this block. Thus we can construct the decoder whose buffer memory size is at most $(m+1)q+r$. Since the decoder is sliding block mapping, one error of channel symbol causes at most $(m+1)+\lceil r/q \rceil$ errors of information symbol.

Let $G = (V,E)$ be a directed graph. Suppose that $H = (U,F)$ is a nice graph obtained by a state-splitting of m round from G for some positive integer m. Let $\pi : \Lambda_G \to \Lambda_H$ be the conjugacy corresponding to the state-splitting. Assume that there is a numbering $\chi : U \to \{0,1\}$ such that

$$|\{\chi(t_H(e)) : e \in \mathcal{F}_H(\alpha)\}| \ge 2, \qquad \text{for any } \alpha \in U. \tag{3.2}$$

Define $\phi : F \to \{0,1\}$ by $\phi(e) = \chi(t_H(e))$ for each $e \in F$ and define $\zeta : \Lambda_G \to \{0,1\}$ by $\zeta = \phi \circ \pi$. If $m = 1$ then ζ is a 2-block map. But by construction, we note that for any $e_1, e_1', e_2 \in E$ with $t_G(e_1) = t_G(e_1') = i_G(e_2)$, $\zeta(e_1e_2) = \zeta(e_1'e_2)$. This means that ζ is a 1-block map of Λ_G, and thus we can construct the decoder whose buffer memory size is at most $q+r$. For $m > 2$ we also find that we can construct the decoder whose buffer memory size is at most $mq+r$ by applying the above construction to a nice graph obtained by a state-splitting(see [5, pp19]), if there is a numbering χ which satisfies (3.2) for the nice graph.

We applied the algorithm in Fig. 3.7 in order to determine $m(d,k)$ for (d,k)-constraints with d =2,...,9 and $k = d+1$, $d+2$, ...,18, 19, ∞. Results are presented in Table I. In the table, to save space we use the following convention: if (d,k) appears on the table and $(d,k+1)$,...,$(d,k+i)$ do not appear, then $q(d,k) = q(d,k+1) = ... = q(d,k+i)$ and $m(d,k) = m(d,k+1) = ... = m(d,k+i)$. We have not yet known the exact value of $m(8,17)$ because our program implemented the algorithm needs too much time to compute it by the computer we used (Sun-3/60). For each (d,k) in Table I except for (8,17), there is a numbering of states which satisfies (3) for the nice graph obtained from $(G_{d,k})^{q(d,k)}$ by state-splitting of $m(d,k)$ rounds we have found. Theorem 3.2 is actually useful to reduce the computation time for constraints in Table I. We note, however, that the converse of Theorem 3.2 does not hold, since there is a 1-round vector for $\left((G_{7,13})^4\right)^4$ with respect to 2^4 but there is no 4-round vector for $(G_{7,13})^4$ with respect to 2. We also note that for our search the strategy which checks only state-splittings that split every state into as many states as possible in every round is not successful. In fact, by that strategy we cannot get $m(9,12)$ for the approximate eigenvector

$$(2\ 2\ 2\ 3\ 3\ 4\ 4\ 5\ 7\ 8\ 6\ 6\ 5\ 4\ 3\ 1).$$

In the first round the 9th state can be split into 5 states. By any state-splittings of 4 rounds which split the 9th state into 5 states in the first round, the graph can not be transformed into any nice graph. But by a state-splitting of 4 rounds which split the 9th state into 4 states in the first round the graph is transformed into a nice graph.

Let N be a positive integer and let $G = (V,E)$ be a directed graph with $\log N \leq h(\Lambda_G)$. Let $\boldsymbol{V}(G,N)$ denote the set of all approximate eigenvectors for G with respect to N. In [5, Appendix] it was suggested beginning state-splitting with the initial vector v^* such that $\max_{\alpha\in V} v^*(\alpha) = \min_{v\in \boldsymbol{V}(G,N)} \max_{\alpha\in V} v(\alpha)$ since the scope of sliding block mapping obtained by state-splitting compatible with v is connected vaguely to the size of $\max_{\alpha\in V} v(\alpha)$. In fact for all (d,k)-constraints such that we have found $m(d,k)$, we checked that state-splittings along the strategy give $m(d,k)$. But we have the following example. For the graph $G_1 = (V_1, E_1)$ in Fig. 3.10 one can find that $\min_{v\in \boldsymbol{V}(G_1,3)} \max_{\alpha\in V_1} v(\alpha) = 5$. Although there is no 2-round vector v with $\max_{\alpha\in V_1} v(\alpha) = 5$, the approximate eigenvector

$$(3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 2\ 2\ 2\ 5\ 7\ 5\ 1\ 1\ 5\ 4\ 3)$$

is a 2-round vector for G_1.

3.5 Summary and recent works

We have developed some results on the ACH procedure. We have bounded approximate eigenvectors which induce conjugacy of $(0,m)$ type with $m \geq 0$. We have presented an algorithm for finding the sliding block decoder with the minimum scope for a given constraint and applied the procedure to (d,k)-constraints. Examples in Section 3.4 shows that apparently good strategies for finding the sliding block decoder with the minimum scope do not work well for all constraints.

We shall mention recent results on finite-state codes with sliding block decoders for constrained channels. A relation between the ACH procedure and the code construction procedures due to Franaszek was investigated by Beal[40]. Beal also presented a code construction method for a channel of finite type. Ashley[41] gave a linear bound on the scope of sliding block decoders. Khayrallah have also studied the scope of sliding block decoders[42]. He has introduced a notion of "floating window." Ashley and Marcus has studied the structure of 'canonical encoders' for sliding block decoders[43]. They have given an example that the ACH procedure do not find the sliding block decoder with the minimum scope of a constrained channel S with $h(S) = \log K$ for an integer K. Franaszek and Thomas have proposed an efficient search strategy of finding the sliding block decoder with the minimum scope[44].

The number of states of the finite-state encoder is another appropriate measure of complexity of finite-state codes. Marcus and Roth have given bounds on the number of states of finite-state encoder with sliding block encoders[36].

A variant of the state-splitting algorithm, a variable length state-splitting algorithm, was presented by Adler, Friedman, Kitchens and Marcus[45] and it was successfully applied to a certain constraint by Heegard, Marcus and Siegel[46]. There are some expositions of the ACH procedure; Blahut[38], Immink[15], and Marcus, Siegel and Wolf[39].

Marcus[11] proved that for every channel almost of finite type, we can construct a code with 100% efficiency for a power system of the channel on the assumption that the channel capacity is a rational number. Karabed and Marcus [37] characterized various properties of finite-state codes, e.g., bounded error propagation and non-catastrophic code, in terms of symbolic dynamics. They also proved that for every channel almost of finite type with a rational channel capacity we can construct a code for the channel with 100 % efficiency only by splitting states very carefully.

INPUTS :An irreducible 0-1 matrix S.
:A positive integer N with $h(\Lambda_S) \geq \log N$.
OUTPUT :The smallest integer m such that there is an m-round vector for S.

```
begin u0 ← An approximate eigenvector for S with respect to N.
  Let n be an integer such that u0 is an n-round vector for S with respect to N.
  m ← 0
  while (m < n) do
    begin u ← f_a(S^m e )
    while (0 ≤ u, u ≠ 0 and there is an α as in the assumption of
                Corollary 3.7 for u) do
      begin u' ← u
        u'(α) ← u(α) − 1
        u ← f_a(u'(α))
      end
    if (u ≠ 0) then
      begin
        V← {v : v is a 1-round vector for S^m, v ≤ u }
        if (there is an m-round vector for S in V (check by
              trying all state-splittings for all vectors in V)) then
          Output m and exit
      end
    m ← m + 1
    end
  Output n
end.
```

Figure 3.7: An algorithm for searching the smallest integer m such that there is an m-round vector for S with respect to N.

(d,k)	q(d,k)	m(d,k)	(d,k)	q(d,k)	m(d,k)
(2,3)	4	0	(7,8)	9	0
(2,4)	3	0	(7,9)	6	4
(2,7)	2	3	(7,10)	5	4
(2,∞)	2	1	(7,11)	5	1
(3,4)	5	0	(7,12)	4	6
(3,5)	4	0	(7,13)	4	4
(3,6)	3	2	(7,14)	4	3
(3,7)	3	1	(7,15)	4	2
(4,5)	6	0	(8,9)	10	0
(4,6)	4	3	(8,10)	7	3
(4,7)	4	1	(8,11)	6	1
(4,8)	3	4	(8,12)	5	3
(4,9)	3	2	(8,13)	5	1
(4,10)	3	1	(8,16)	4	6
(5,6)	7	0	(8,17)	4	5 or 6
(5,7)	5	3	(8,18)	4	5
(5,8)	4	3	(8,19)	4	4
(5,9)	4	1	(8,∞)	4	2
(5,12)	3	4	(9,10)	11	0
(5,14)	3	3	(9,11)	7	7
(5,∞)	3	2	(9,12)	6	4
(6,7)	8	0	(9,13)	6	1
(6,8)	6	3	(9,14)	5	4
(6,9)	5	1	(9,15)	5	3
(6,10)	4	4	(9,16)	5	2
(6,11)	4	1	(9,∞)	4	4

Table 3.1: $m(d,k)$s for (d,k)-constraints.

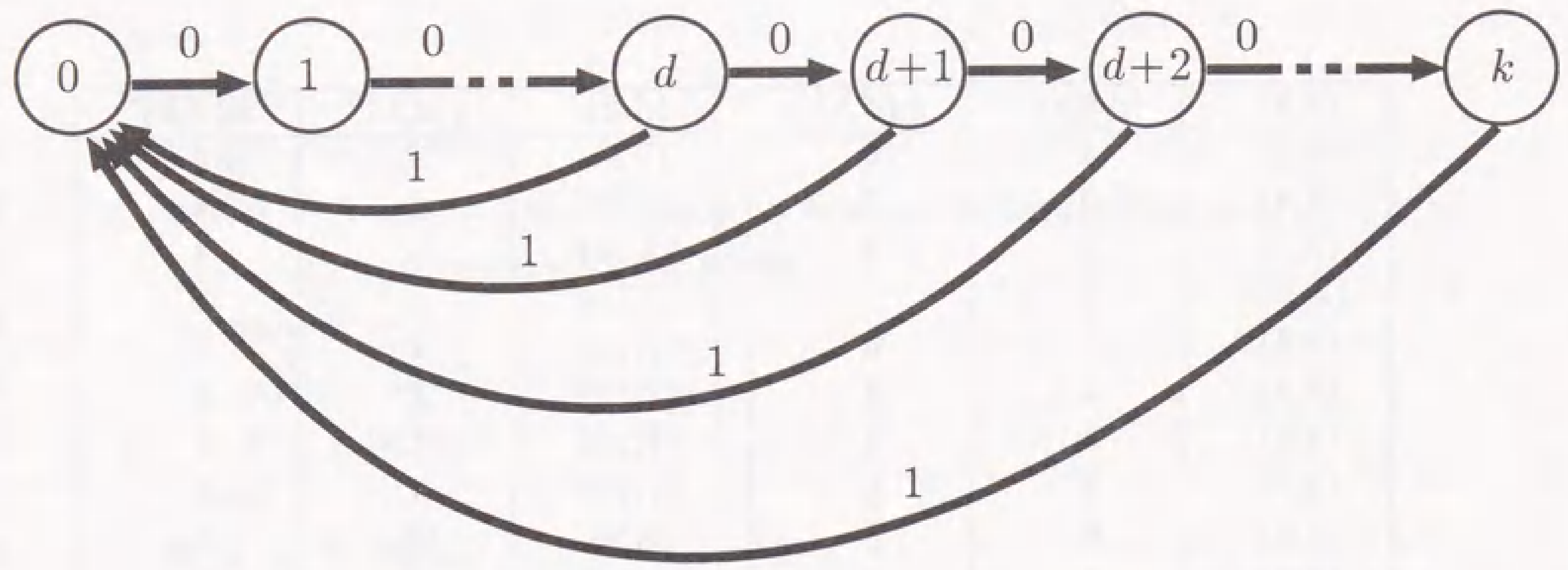


Figure 3.8: Directed graph with labeled edges of (d, k)-constraint $(k < \infty)$.

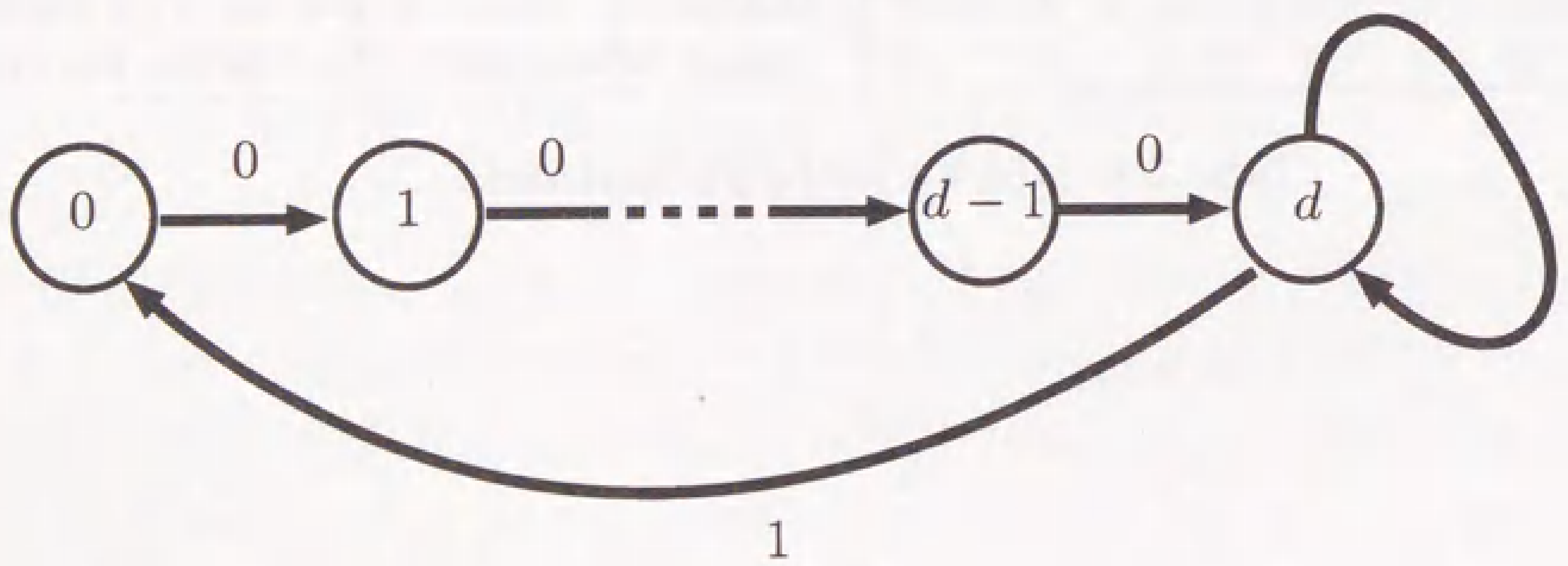


Figure 3.9: Directed graph with labeled edges of (d, ∞) constraint.

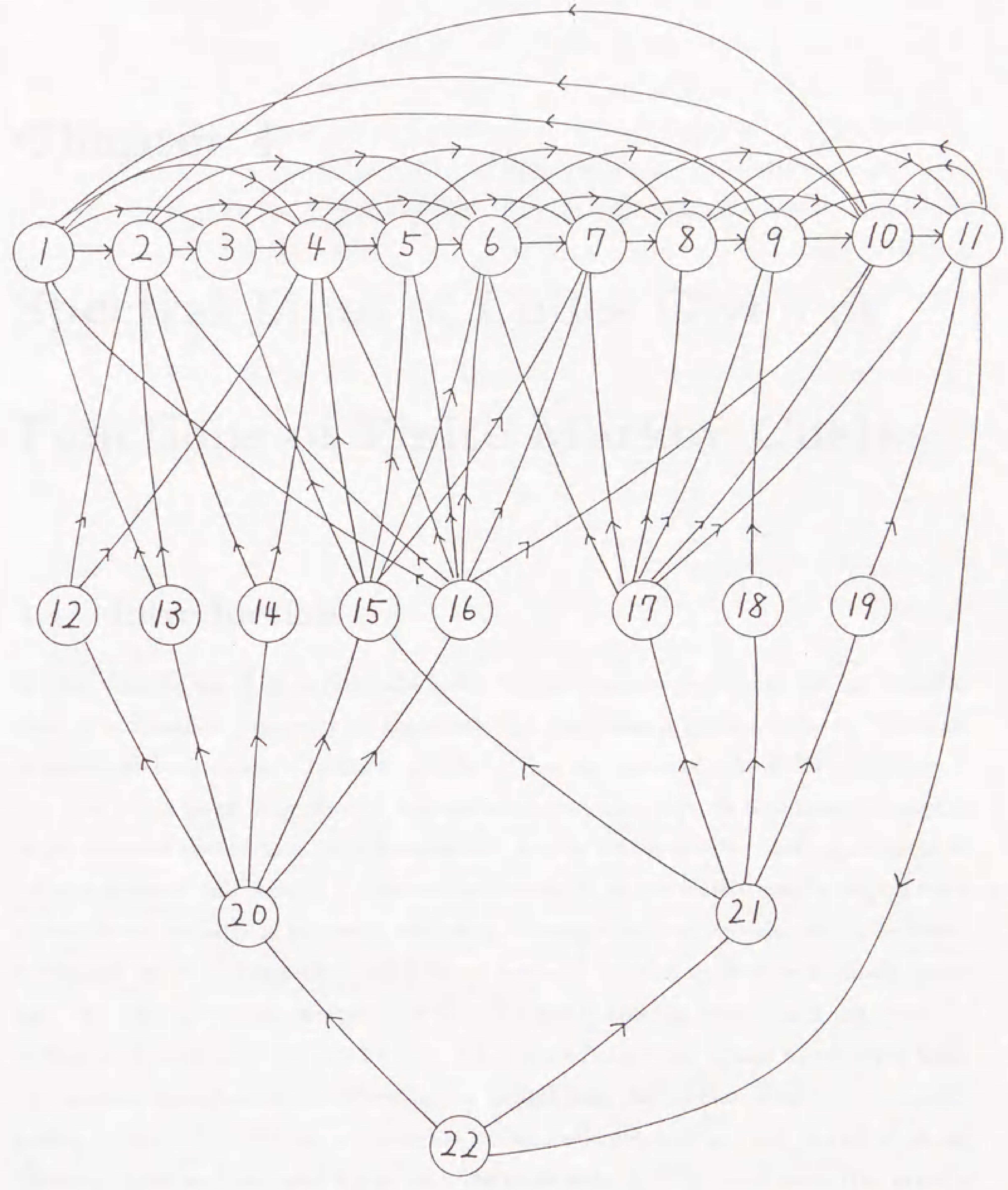


Figure 3.10: Directed graph $G_1 = (V_1, E_1)$.

Chapter 4

Spectral Lines of Codes Given as Functions of Finite Markov Chains

4.1 Introduction

In this chapter we shall investigate codes which generate sequences having spectral lines at a specified frequency of amplitude not less than a given constant. We shall characterize such codes in terms of graphs so that we can apply the ACH procedure.

Let f be a given frequency. A spectral null constraint requires that input sequences to the channel should have no f component. A code which satisfies this requirement is called a spectral null code at f. Spectral null codes at dc are widely used in digital data transmission and digital magnetic recording because electronic circuits which transmit or amplify the dc component of input signal correctly are very expensive or hardly available. In a digital storage system both the data signal and the servo signal are recorded on the same track in the system because it is required that they should be retrieved from the medium simultaneously. Therefore, a modulation code which produces sequences having a spectral null at the servo signal frequency is required so that we can separate the data signal and the servo signal from the retrieved signal by band filters (the system which uses such a code is called the buried servo system). Thus spectral null codes at frequencies other than dc have been studied[22], [23].

However, the buried servo system can not be used in intrinsically digital storage devices such as digital optical storage devices and magnetic storage devices using saturation recording with thin media. Borgers et al.[14] have proposed a recording system which uses a code producing spectral lines at low frequencies for equiprobable information sources, and Immink[15, Chapter 11], [16] has analyzed such a code. Thus a code having a spectral line at specified frequencies is of practical interest.

In Section 4.2 we shall give definitions and backgrounds. In Section 4.3, we shall prove our results on spectral lines at f, where f is a rational submultiples of the symbol frequency (Theorem 4.4). In Section 4.4 we shall prove the equivalence of the "*spectral density null*," i.e., the continuous part of the spectrum vanishes, and the "*biased coboundary condition*" at f (Theorem 4.5). Then we shall relate these conditions to the problem of constructing codes with nonzero spectral lines (Theorem 4.6). We shall give a sufficient condition for a code to have spectral lines simultaneously at several given frequencies, independent of the source statistics (Corollary 4.1). We shall give canonical graphs for spectral density nulls and nonzero spectral lines (Proposition 4.3).

In this chapter we use a Moore FSSM as the model of a constrained channel or of the channel itself. That is, we think of a directed graph with labeled states as the model of an encoder and we think of the sequence of labels (complex numbers) along a path in the directed graph as an encoded sequence generated by the encoder.

4.2 Spectral analysis of functions of finite Markov chains

In this section we present background for spectral analysis of functions of finite Markov chains and known results on spectral nulls of those processes.

In this and next chapters we consider (finite or infinite) directed graphs with labeled states. Let G be a finite directed graph. Without loss of generality we assume that $\#\mathcal{E}(G) \geq 1$. Let $\eta = a_0 a_1 \cdots a_{L-1}$ be a path in Λ_G (Since we consider Moore FSSM's as models of constrained channels, paths in Λ_G mean finite sequences of states in G). An *irreducible component* of G is a maximal irreducible subgraph of G. If $a_i \neq a_k$ for distinct i and k, then η is said to be *simple*. In this and next chapters, by (σ, τ) we mean an edge from state σ to τ. Since we assume that for every pair of states σ, τ there is at most one edges from σ to τ, this notation will not cause any confusion. If η is a cycle, then we set $a_L = a_0$ and we let $\mathrm{Eg}(\eta)$ denote the set of edges of η, that is, $\mathrm{Eg}(\eta) = \{(a_i, a_{i+1}) : 0 \leq i \leq L-1\}$. For two blocks η and ζ with $\big(t_G(\eta), i_G(\zeta)\big) \in \mathcal{E}(G)$, $\eta \cdot \zeta$ denotes the concatenation of η and ζ. For a cycle $\eta = a_0 a_1 \cdots a_{L-1}$ and a function $d : \mathcal{E}(G) \to \boldsymbol{C}$, we define $\sum_{\eta} d$ by $\sum_{\eta} d = \sum_{i=0}^{L-1} d(a_i, a_{i+1})$. Let P be a transition probability matrix of G. Then we have a Markov chain $\{s_n\}$ with a measure defined by P. By Π_G we mean the set of transition probability matrices P such that each edge has a positive transition probability (such matrices are said to be *compatible with* G). For $P \in \Pi_G$, a unique probability distribution p with $pP = p$ is called the *stationary distribution* of P. Let γ be a complex valued function of $\mathcal{S}(G)$. We put $\vec{\gamma} = \big(\gamma(b_1), \gamma(b_2), \ldots, \gamma(b_J)\big)^t$, where $\mathcal{S}(G) = \{b_1, b_2, \ldots, b_J\}$ and t means the transpose. We think of an FSTD (G, γ) as a model of an encoder. Almost all encoders in use (including block encoders) can be modeled by FSTD's. We have a function of the Markov chain $\{\gamma(s_n)\}$ determined by G and P, which is denoted (G, P, γ). The period of G is the greatest common divisor of cycle lengths. Let N be the period of G. Then $\mathcal{S}(G)$ can be divided into N sets $\{B_0, B_1, \ldots, B_{N-1}\}$ such that

$$(a, b) \in \mathcal{E}(G) \text{ and } a \in B_i \text{ imply } b \in B_{i+1 \bmod N}. \tag{4.1}$$

A partition $\{B_0, B_1, \ldots, B_{N-1}\}$ which satisfies (4.1) is called a *partition of* $\mathcal{S}(G)$ *by period*. The numbering $\ell : \mathcal{S}(G) \to \{0, 1, \ldots, N-1\}$ such that $a \in B_{\ell(a)}$ for $a \in \mathcal{S}(G)$ is called the *indexing* of $\mathcal{S}(G)$ *with respect to the partition*. By arranging rows and columns in order of an indexing of $\mathcal{S}(G)$ with respect to a partition by period, $P \in \Pi_G$ can be written in the following form

$$P = \begin{pmatrix} 0 & P_{0,1} & 0 & \cdots & 0 \\ 0 & 0 & P_{1,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_{N-2,N-1} \\ P_{N-1,0} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $P_{i,k}$ is a probability matrix of size $\#B_i \times \#B_k$. Throughout this chapter and the next chapter, we assume that every $P \in \Pi_G$ has this form. Corresponding to this, we write $p = (p_0, p_1, \ldots, p_{N-1})$, where p is the stationary distribution of P and p_i is the row vector of size $\#B_i$ consisting of $p(a)$, $a \in B_i$. We put

$$J_r = \begin{pmatrix} I_0 & 0 & \cdots & 0 \\ 0 & e^{-j2\pi r/N} I_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-j2\pi r(N-1)/N} I_{N-1} \end{pmatrix}, \quad \text{for } 0 \le r \le N-1, \tag{4.2}$$

where I_i is the identity matrix of size $\#B_i \times \#B_i$. We define $rds_{f,\gamma}(\eta)$ for a block η by

$$rds_{f,\gamma}(\eta) = \exp\Big(-\mathrm{i}2\pi\ell\big(i_G(\eta)\big)f/f_S\Big)\mathrm{RDS}_f\big(\gamma(\eta)\big),$$

which has an important role as well as RDS_f in the following sections where RDS_f is defined by (1.4).

Theorem 4.1 [47] Let (G,γ) be an FSTD and let $P \in \Pi_G$. Let p be the stationary distribution of P. Let $\mathcal{S}(G) = \{a_1, a_2, \cdots, a_J\}$ and let $\vec{e}$ be the vector with all components 1 of size J. Let N be the period of G. We define D, F_r, H, C and $G(f)$ as follows:

$$\begin{aligned} D &= \mathrm{diag}\big(p(a_1), p(a_2), \ldots, p(a_J)\big), \\ F_r &= J_r^* p^t p J_r, \qquad \text{for } r = 0, 1, \ldots, N-1, \\ H &= P - \sum_{r=0}^{N-1} \exp(\mathrm{i}2\pi r/N) J_r^* \vec{e} p J_r, \\ C &= D - P^t D P, \\ G(f) &= \big(\exp(\mathrm{i}2\pi f)I - H\big)^{-1}, \end{aligned} \tag{4.3}$$

where the asterisk means the conjugate transpose of a matrix. Then the power spectral density (the continuous part of the spectrum) of (G,P,γ) at a frequency f, denoted $w_c^{(G,P,\gamma)}(f)$, and the spectral line (the discrete part of the spectrum) of (G,P,γ) at f, denoted $w_d^{(G,P,\gamma)}(f)$, are given as follows:

$$w_c^{(G,P,\gamma)}(f) = \vec{\gamma}^* G(f/f_S)^* C G(f/f_S)\vec{\gamma}, \tag{4.4}$$

$$w_d^{(G,P,\gamma)}(f) = \begin{cases} \vec{\gamma}^* F_r \vec{\gamma}, & \text{if } f = r f_S/N, \quad r = 0, 1, \ldots, N-1; \\ 0, & \text{otherwise,} \end{cases} \tag{4.5}$$

where f_S is the symbol frequency. □

Let k be a nonnegative integer and n a positive integer with $\gcd(k,n) = 1$. We note that if $w_d^{(G,P,\gamma)}(kf_S/n) > 0$, then n must be a divisor of N by (4.5).

We can find other formulae for the power spectrum in [48], [49] and [50].

Definition 4.1 [23] Let k be a nonnegative integer and n a positive integer. The FSTD (G,γ) satisfies a *coboundary condition at* kf_S/n if there is a function $\phi : \mathcal{S}(G) \to \boldsymbol{C}$ such that

$$\gamma(a) = \exp(-\mathrm{i}2\pi k/n)\phi(b) - \phi(a)$$

for $(a,b) \in \mathcal{E}(G)$. The function ϕ is called a coboundary function of G at f. □

Theorem 4.2 [23] Let (G,γ) be an irreducible FSTD. Let k be a nonnegative integer and n a positive integer with $\gcd(k,n) = 1$. Let $P \in \Pi_G$. Put $f = kf_S/n$. Then the following conditions are equivalent:

(1) for every cycle η in G of length a multiple of n, $\mathrm{RDS}_f\big(\gamma(\eta)\big) = 0$;

(2) (G,γ) has a finite range of RDS_f values;

(3) (G,γ) satisfies a coboundary condition at f;

(4) (G,P,γ) has a spectral null at f, that is, $w_c^{(G,P,\gamma)}(f) = 0$ and $w_d^{(G,P,\gamma)}(f) = 0$.

□

Remark **4.1** Yasuda and Inose [17], [18] first proved *(1)* ⇔ *(2)* ⇔ *(4)* for $k = 0$. (The same results appeared in [19] and [20].) Yoshida and Yajima [22] first proved *(1)* ⇔ *(2)* ⇔ *(4)* for $k \ge 0$.

Remark **4.2** Although the FSTD (G,γ) itself has no probabilistic structure, if the spectrum of (G,P,γ) vanishes at f for some transition probability matrix P compatible with G then we may say that (G,γ) has a first-order spectral null at f because only the condition (4) depends on P.

In the strict sense, it was proved that the above four statements are equivalent for a Mealy FSSM model, not for a Moore FSSM model in [23], [17], [18], [22]. But it is easy to see that the same result holds for a Moore FSSM model.

Let (G_1,γ_1) and (G_2,γ_2) be state-transition diagrams. Assume that there are two functions f_S and f_E such that f_S is a function of states of G_1 into states of G_2, f_E is a function of edges of G_1 into edges of G_2 and

$$f_S(i_G(e)) = i_G(f_E(e)), \quad f_S(t_G(e)) = t_G(f_E(e)), \quad \gamma_1(e) = \gamma_2(f_E(e)),$$

for every edge e in G_1. Then we say that (G_1,γ_1) is *label-preserving graph homomorphic* to (G_2, γ_2). If f_S and f_E are bijective then we say that (G_1, γ_1) is *label-preserving graph isomorphic* to (G_2, γ_2).

4.3 Spectral lines

For an FSTD (G, γ), (G, P, γ) represents the encoded message which is generated by an encoder when the message from a source is input to the encoder, where P is determined by the source statistics. Our problem is to characterize an encoder such that the encoded message may have information about clock, independent of the source statistics. We interpret this problem as a problem of characterizing an FSTD (G, γ) such that (G, P, γ) has a spectral line of amplitude not less than a given value at a given frequency f for every $P \in \Pi_G$. (Notice that if an FSTD does not satisfy any coboundary condition at f, then the FSTD has a frequency content at f by Theorem 4.2. But it is possible that the amplitude of the frequency content can be very small depending on the transition probability matrix of the underlying Markov chain and this may be undesirable in practical applications in which we need to extract information about clock from the received or reproduced message. So we require not only that the spectral line at f is nonzero but also that the amplitude of the spectral line is not less than the given value.)

The main theorem of this section is Theorem 4.4. We begin this section with lemmas which will be used in proving the theorem and which are also helpful in understanding spectral lines.

Let $\eta = a_0 a_1 \cdots a_{L-1}$ be a cycle in G. We define a transition probability matrix P_η of G corresponding to a cycle as follows:

$$P_\eta(c, c') = \begin{cases} \dfrac{M_\eta(c, c')}{N_\eta(c)}, & \text{if } N_\eta(c) > 0, \\ 0, & \text{if } N_\eta(c) = 0; \end{cases}$$

We put

$$N_\eta(b) = \#\{i : 0 \le i \le L-1 \text{ and } a_i = b\}$$

for $b \in \mathcal{S}(G)$ and

$$M_\eta(c, c') = \#\big\{i : 0 \le i \le L-1 \text{ and } (a_i, a_{i+1}) = (c, c')\big\}$$

for $(c, c') \in \mathcal{S}(G) \times \mathcal{S}(G)$. We define a matrix P_η by for $(c, c') \in \mathcal{S}(G) \times \mathcal{S}(G)$, and

a vector q_η by $q_\eta(b) = N_\eta(b)/\lg(\eta)$ for $b \in \mathcal{S}(G)$. Then P_η is a transition probability matrix of G and q_η is the stationary distribution of P_η.

Proposition 4.1 Let G be an irreducible directed graph and let $P \in \Pi_G$. Then there is a sequence $(\zeta_i)_{i \in \mathbf{N}}$ of cycles such that $P_{\zeta_i} \to P$ as $i \to \infty$.

Proof: Since the underlying Markov chain is ergodic (because G is irreducible), there is an infinite path x such that for every state σ

$$\frac{\#\{j : \sigma = x_j, 0 \le j \le i-1\}}{i} \to p(\sigma)$$

and for every edge (σ, τ)

$$\frac{\#\{j : (\sigma, \tau) = (x_j, x_{j+1}), 0 \le j \le i-1\}}{\#\{j : \sigma = x_j, 0 \le j \le i-1\}} \to P(\sigma, \tau)$$

as $i \to \infty$, where p is the stationary distribution of P. For each i, we get a cycle ζ_i by concatenating $x_0 x_1 \cdots x_{i-1}$ and a simple block. Then $P_{\zeta_i} \to P$ as $i \to \infty$. □

For a subset X, we write $\operatorname{conv} X$ and $\operatorname{cl} X$ for the set of all convex combinations of elements in X and for the closure of X, respectively.

Let η be a cycle of a labeled graph. In the remaining of this chapter quotient $\mathrm{RDS}_f(\gamma(\eta))/\lg(\eta)$ plays an important role.

Related results to the following lemma are found in [51], [52].

Lemma 4.1 Let (G, γ) be an irreducible FSTD. Put

$$X = \left\{ \frac{\mathrm{RDS}_0\big(\gamma(\eta)\big)}{\lg(\eta)} : \eta \text{ is a simple cycle in } G \right\},$$
$$W = \left\{ \frac{\mathrm{RDS}_0\big(\gamma(\eta)\big)}{\lg(\eta)} : \eta \text{ is a cycle in } G \right\}.$$

Then $\operatorname{conv} X = \operatorname{cl} W$.

Proof: See Appendix A.5. □

The following lemma relates RDS_f of a cycle to an equation which is similar to the equation defining the coboundary condition (cf. Definition 4.1).

Lemma 4.2 Let (G, γ) be an irreducible FSTD and let $d : \mathcal{E}(G) \to \mathbf{C}$. The following two conditions are equivalent:

(1) for every cycle η in G,

$$\mathrm{RDS}_0\big(\gamma(\eta)\big) = \sum_\eta d;$$

(2) there is a function $\phi : \mathcal{S}(G) \to C$ such that for every $(a,b) \in \mathcal{E}(G)$,

$$\gamma(a) = \phi(b) - \phi(a) + d(a,b).$$

Proof: See Appendix A.6. □

From the following proposition and Theorem 4.4 proved later, we note that if an FSTD has a spectral line at $f = f_S k'/n'$, then the period of the FSTD must be a multiple of n'.

Proposition 4.2 Let (G,γ) be an FSTD. Let k be a nonnegative integer and n a positive integer with $\gcd(k,n) = 1$. Put $f = kf_S/n$. Assume that $\mathrm{RDS}_f\big(\gamma(\eta)\big) \neq 0$ for every cycle η in G of length a multiple of n. Then the period of G is a multiple of n.

Proof: If $n = 1$, then it is trivial. Assume that $n > 1$. Since the period is the greatest common divisor of cycle lengths, if $\gcd\big(\lg(\eta), n\big) = n$ for every cycle η then the period is a multiple of n. Suppose that the period of N is not any multiple of n and, hence, there is a cycle $\eta = a_0 a_1 \cdots a_{m-1}$ such that $\gcd(m,n) < n$. Put $n' = n/\gcd(m,n)$ and $z = \exp(-\mathrm{i}2\pi k/n)$. Note that $n' > 1$. Put $\zeta = \underbrace{\eta \cdot \eta \cdot \cdots \cdot \eta}_{n' \text{ times}} = b_0 b_1 \cdots b_{n'm-1}$. Then we have

$$\begin{aligned}
\mathrm{RDS}_f\big(\gamma(\zeta)\big) &= \sum_{i=0}^{n'm-1} z^i \gamma(b_i) \\
&= \sum_{i=0}^{n'-1}\sum_{h=0}^{m-1} z^{im+h}\gamma(a_h) \\
&= \left(\sum_{i=0}^{n'-1} z^{im}\right)\left(\sum_{h=0}^{m-1} z^h \gamma(a_h)\right). \qquad (4.6)
\end{aligned}$$

We claim that the powers $1, z^m, \ldots, z^{(n'-1)m}$ of z are all distinct. Suppose that $\exp(-\mathrm{i}2\pi i_1 mk/n) = \exp(-\mathrm{i}2\pi i_2 mk/n)$ for $0 \le i_1 < i_2 \le n'-1$. This means that n' divides $(i_2 - i_1)km/\gcd(m,n)$. But this can not happen because $\gcd\big(n', m/\gcd(n,m)\big) = 1$, $\gcd(n',k) = 1$ and $i_2 - i_1 < n'$. On the other hand, $(z^{im})^{n'} = \exp(-\mathrm{i}2\pi n' mk/n) = \exp\big(-\mathrm{i}2\pi km/\gcd(n,m)\big) = 1$, that is z^{im} is an n'-th root of 1 for $i = 0, 1, \ldots, n'-1$. Thus $\{z^{im} : i = 0, 1, \ldots, n'-1\}$ is the set of all n'-th roots of 1 and this implies $\sum_{i=0}^{n'-1} z^{im} = 0$. Hence, we get $\mathrm{RDS}_f\big(\gamma(\zeta)\big) = 0$ by (4.6). □

Let η be a cycle. Theorem 4.3 relates quotient $\mathrm{RDS}_f(\gamma(\eta))/\lg(\eta)$ with a transition probability matrix.

Lemma 4.3 Let (G,γ) be an FSTD. Let k be a nonnegative integer and n a positive integer. Let ℓ be an indexing of $\mathcal{S}(G)$ with respect to a partition by period. Put $f = kf_S/n$ and $z = \exp(-\mathrm{i}2\pi k/n)$. Let $\eta = a_0 a_1 \ldots a_{L-1}$ be a block in G. Then

(1) if the period of G is a multiple of n, then $z^{\ell(a_i)} = z^{i+\ell(a_0)}$ for $0 \le i \le L-1$;

(2) if η is a cycle and L is a multiple of n, then for every positive integer K

$$\mathrm{RDS}_f\big(\gamma(\underbrace{\eta \cdot \eta \cdot \cdots \cdot \eta}_{K \text{ times}})\big) = K\,\mathrm{RDS}_f\big(\gamma(\eta)\big).$$

(Hence, $rds_{f,\gamma}(\underbrace{\eta \cdot \eta \cdot \cdots \cdot \eta}_{K \text{ times}}) = K\,rds_{f,\gamma}(\eta)$.)

Proof: Note that if K is a multiple of n, then $z^K = 1$. Using this, we can prove this lemma by induction. □

Theorem 4.3 Let (G,γ) be an irreducible FSTD. Let k be a nonnegative integer and n a positive integer. Put $f = kf_S/n$. Assume that the period of G is a multiple of n. Then

(1) for every cycle η with $\mathrm{Eg}(\eta) = \mathcal{E}(G)$,

$$\left|\frac{rds_{f,\gamma}(\eta)}{\lg(\eta)}\right|^2 = w_d^{(G,P_\eta,\gamma)}(f);$$

(2) for every $P \in \Pi_G$ and every positive ϵ, there is a cycle η such that

$$\left| w_d^{(G,P,\gamma)}(f) - \left|\frac{rds_{f,\gamma}(\eta)}{\lg(\eta)}\right|^2 \right| < \epsilon.$$

Proof: We can prove (1) by straightforward computations using (4.2), (4.5) and Lemma 4.3 (1).

Since $w_c^{(G,P,\gamma)}(f)$ is a continuous function of the stationary distribution of P, (2) follows from (1) and Proposition 4.1. □

For $f = 0$, the number $\mathrm{RDS}_0\big(\gamma(\eta)\big)/\lg(\eta)$ in the above theorem is a special case of the "weight-per-symbol" introduced independently by Marcus and Tuncel[52] for other purposes.

Example 4.1 The binary (d,k) code sequence is defined as a binary sequence in which every two consecutive 1's are separated by at least d and at most k 0's. The sequence can be determined by an output sequence of an FSTD. For example, an FSTD which generates the $(1,3)$ code sequence is shown in Fig. 4.1. Let (G_1,γ_1) be an FSTD which generates the (d,k) code sequence for d and k with $0 \le d < k \le \infty$. We note that for

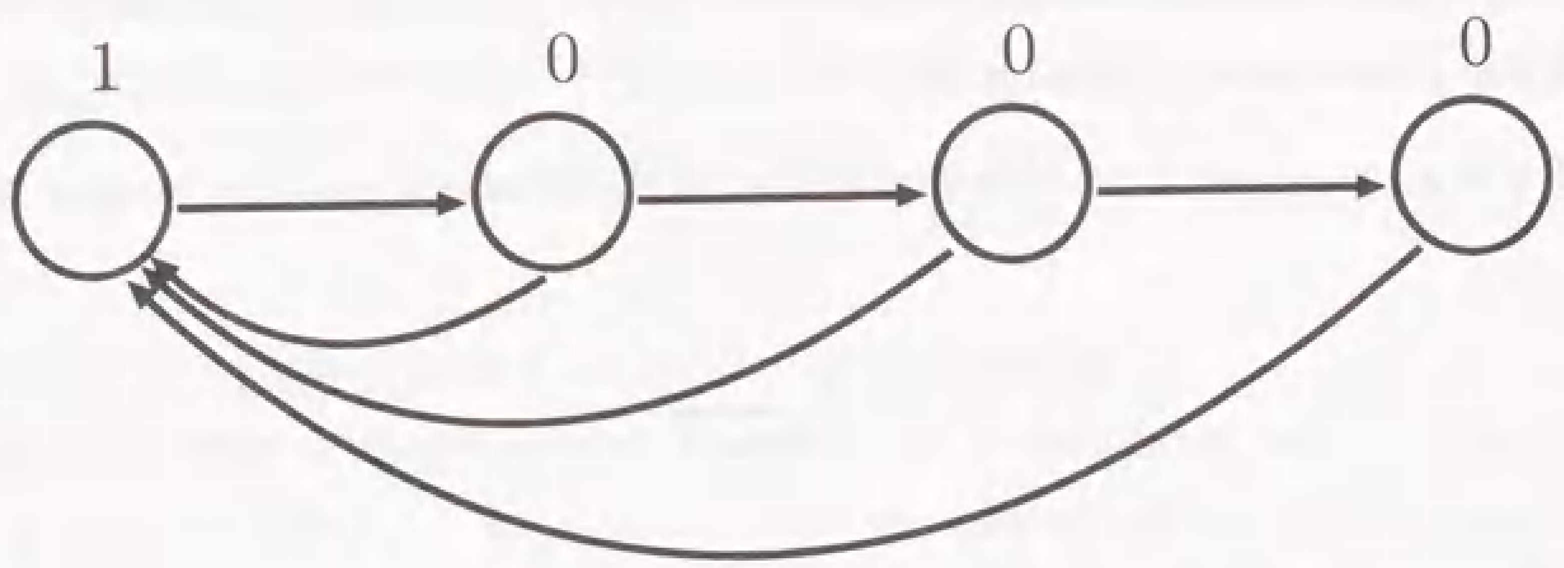


Figure 4.1: FSTD (G_1, γ_1) which generates binary $(1,3)$ code sequences.

every binary sequence $a_0 a_1 \cdots a_{L-1}$ generated by a cycle in G_1,

$$\frac{1}{k+1} \leq \frac{\sum_{i=0}^{L-1} a_i}{L} \leq \frac{1}{d+1}.$$

Thus

$$\left(\frac{1}{k+1}\right)^2 \leq w_d^{(G_1,P,\gamma_1)}(0) \leq \left(\frac{1}{d+1}\right)^2$$

for every $P \in \Pi_{G_1}$ by Theorem 4.3. □

Now we present the main theorem of this section. It is similar to Theorem 4.2 in its statements. But it needs a quite different long proof which contains many different techniques. Moreover the proof shows how Theorem 4.4 differs from Theorem 4.2 and gives how to estimate the amplitude of the spectral line of an FSTD satisfying one of conditions in Theorem 4.4. Therefore we describe the proof completely here.

By $\langle x, y\rangle$ we mean the inner product of $x \in \boldsymbol{C}$ and $y \in \boldsymbol{C}$, that is, $\langle x, y\rangle = \operatorname{Re} x \operatorname{Re} y + \operatorname{Im} x \operatorname{Im} y$. For a complex number e and a real number c, we write

$$H(e,c) = \left\{x \in \boldsymbol{C} : \langle e, x\rangle \geq c\right\}.$$

We also write

$$A \backslash B = \{x | x \in A \text{ and } x \notin B\}.$$

Theorem 4.4 Let (G,γ) be an irreducible FSTD. Let k be a nonnegative integer and n a positive integer with $\gcd(k,n) = 1$. Let N be the period of G and ℓ an indexing of $\mathcal{S}(G)$ with respect to a partition by period. Put $f = kf_S/n$ and $z = \exp(-\mathrm{i}2\pi k/n)$. Let c be a nonnegative real number. Then the following conditions are equivalent:

(1) for every cycle η of length a multiple of n,

$$\left|\frac{rds_{f,\gamma}(\eta)}{\lg(\eta)}\right| \geq c;$$

(2) there are an $e \in \boldsymbol{C}$ with $|e| = 1$ and a function $d : \mathcal{E}(G) \to \boldsymbol{C}$ such that $d\big(\mathcal{E}(G)\big) \subset H(e,c)$ and for every cycle η of length a multiple of n,

$$rds_{f,\gamma}(\eta) = \sum_{\eta} d; \tag{4.7}$$

(3) N is a multiple of n and there are an $e \in \boldsymbol{C}$ with $|e| = 1$ and two functions $d : \mathcal{E}(G) \to \boldsymbol{C}$ and $\phi : \mathcal{S}(G) \to \boldsymbol{C}$ such that for every $(a,b) \in \mathcal{E}(G)$,

$$\gamma(a) = z\phi(b) - \phi(a) + z^{-\ell(a)} d(a,b)$$

and $d(a,b) \in H(e,c)$;

(4) for every $P \in \Pi_G$

$$w_d^{(G,P,\gamma)}(f) \geq c^2.$$

Proof: First we prove this theorem for the case of dc, that is, $k = 0$ and $n = 1$. We need the following two lemmas:

Lemma 4.4 Let (G,γ) be an irreducible FSTD which satisfies *(2)* in Theorem 4.4. Then there is a function $\phi : \mathcal{S}(G) \to \boldsymbol{C}$ such that for every block $\eta = a_0 a_1 \cdots a_{L-1}$

$$\mathrm{RDS}_0\big(\gamma(\eta)\big) - \sum_{i=0}^{L-2} d(a_i, a_{i+1}) = \phi(a_{L-1}) - \phi(a_0) + \gamma(a_{L-1}).$$

Proof of Lemma 4.4: Using Lemma 4.2, this is proved by straightforward computations. □

Let $\zeta = a_0 a_1 \ldots a_{L-1}$ be a cycle in G. If $I = \#\{i : 0 \leq i \leq L-1, (a_i, a_{i+1}) \notin \mathcal{E}(H) \text{ and } a_{i+1} \in \mathcal{S}(H)\}$, then we say that ζ visits H I times. We say that ζ is *simple outside* H if every edge and every state outside H appear at most once in ζ. For a cycle

$\zeta = a_0 a_1 \cdots a_{L-1}$ and a function $d : \mathcal{E}(G) \to \boldsymbol{C}$, we define $\sum_{\zeta}^{[H]} d$ by

$$\sum_{\zeta}^{[H]} d = \sum_{\substack{m=0 \\ (a_m, a_{m+1}) \in \mathcal{E}(H)}}^{L-1} d(a_m, a_{m+1}).$$

We note that for cycles $\eta = c_0 c_1 \cdots c_{L-1}$ and ζ if $\zeta = c_K c_{K+1} \cdots c_{L-1} c_0 c_1 \cdots c_{K-1}$ for some K, then

$$\mathrm{RDS}_0\big(\gamma(\zeta)\big) = \mathrm{RDS}_0\big(\gamma(\eta)\big), \qquad \sum_{\zeta} d = \sum_{\eta} d \quad \text{and} \quad \sum_{\zeta}^{[H]} d = \sum_{\eta}^{[H]} d. \tag{4.8}$$

Lemma 4.5 Let H be a proper subgraph of G consisting of irreducible subgraphs of G. Let η be a cycle with $\mathrm{Eg}(\eta) \backslash \mathcal{E}(H) \neq \emptyset$. Assume that (H, γ) satisfies *(2)* in Theorem 4.4 and that η is simple outside H and does not contain any cycle in H. Then there are simple cycles $\alpha_1, \alpha_2, \ldots, \alpha_K$ which satisfy the following properties:

(q1) α_m visits each irreducible componentof H at most once for each m;

(q2) $\displaystyle\sum_{m=1}^{K} \left(\mathrm{RDS}_0\big(\gamma(\alpha_m)\big) - \sum_{\alpha_m}^{[H]} d \right) = \mathrm{RDS}_0\big(\gamma(\eta)\big) - \sum_{\eta}^{[H]} d;$

(q3) $\displaystyle\sum_{m=1}^{K} \#\big(\mathrm{Eg}(\alpha_m) \backslash \mathcal{E}(H)\big) = \#\big(\mathrm{Eg}(\eta) \backslash \mathcal{E}(H)\big);$

(q4) $\mathrm{Eg}(\alpha_m) \backslash \mathcal{E}(H) \neq \emptyset$ for each m.

Proof of Lemma 4.5: Let F be an irreducible component of H. Assume that η visits F at least twice. We may assume that $\big(t_G(\eta), i_G(\eta)\big) \notin \mathcal{E}(F)$ and $t_G(\eta) \in \mathcal{S}(F)$ because of (4.8). We can write $\eta = \xi_1 \cdot \eta_a \cdot \xi_2 \cdot \eta_b$ such that η_a and η_b are blocks in F and every state in ξ_2 is outside F. Since F is irreducible, there are simple blocks ζ_1 and ζ_2 in F such that $i_G(\zeta_1) = i_G(\eta_a)$, $t_G(\zeta_1) = t_G(\eta_b)$, $i_G(\zeta_2) = i_G(\eta_b)$ and $t_G(\zeta_2) = t_G(\eta_a)$. Then $\xi_1 \cdot \zeta_1$ and $\xi_2 \cdot \zeta_2$ are blocks in G. By Lemma 4.4 there is a function $\phi : \mathcal{S}(F) \to \boldsymbol{C}$ such that for every block $\eta = c_0 c_1 \cdots c_{L-1}$ in F

$$\mathrm{RDS}_0\big(\gamma(\eta)\big) - \sum_{i=0}^{L-2} d(c_i, c_{i+1}) = \phi(c_{L-1}) - \phi(c_0) + \gamma(c_{L-1}).$$

Therefore, putting $t_G(\eta_b) \cdot \xi_1 \cdot i_G(\eta_a) = a_0 a_1 \cdots a_{L-1}$, we have

$$\begin{aligned}
\mathrm{RDS}_0\big(\gamma(\eta)\big) - \sum_{\eta}^{[F]} d &= \mathrm{RDS}_0\big(\gamma(\xi_1)\big) - \sum_{\substack{i=0 \\ (a_i, a_{i+1}) \in \mathcal{E}(F)}}^{L-2} d(a_i, a_{i+1}) \\
&\quad + \phi\big(t_G(\eta_a)\big) - \phi\big(i_G(\eta_a)\big) + \gamma\big(t_G(\eta_a)\big) \\
&\quad + \mathrm{RDS}_0\big(\gamma(\xi_2)\big) + \phi\big(t_G(\eta_b)\big) - \phi\big(i_G(\eta_b)\big) + \gamma\big(t_G(\eta_b)\big) \\
&= \mathrm{RDS}_0\big(\gamma(\xi_1)\big) - \sum_{\substack{i=0 \\ (a_i, a_{i+1}) \in \mathcal{E}(F)}}^{L-2} d(a_i, a_{i+1}) \\
&\quad + \phi\big(t_G(\eta_b)\big) - \phi\big(i_G(\eta_a)\big) + \gamma\big(t_G(\eta_b)\big) \\
&\quad + \mathrm{RDS}_0\big(\gamma(\xi_2)\big) + \phi\big(t_G(\eta_a)\big) - \phi\big(i_G(\eta_b)\big) + \gamma\big(t_G(\eta_a)\big) \\
&= \mathrm{RDS}_0\big(\gamma(\xi_1 \cdot \zeta_1)\big) - \sum_{\xi_1 \cdot \zeta_1}^{[F]} d \\
&\quad + \mathrm{RDS}_0\big(\gamma(\xi_2 \cdot \zeta_2)\big) - \sum_{\xi_2 \cdot \zeta_2}^{[F]} d.
\end{aligned}$$

We also have

$$\#\big(\mathrm{Eg}(\eta) \backslash \mathcal{E}(F)\big) = \#\big(\mathrm{Eg}(\xi_1 \cdot \zeta_1) \backslash \mathcal{E}(F)\big) + \#\big(\mathrm{Eg}(\xi_2 \cdot \zeta_2) \backslash \mathcal{E}(F)\big),$$

and $\xi_2 \cdot \zeta_2$ visits F once, neither $\xi_1 \cdot \zeta_1$ nor $\xi_2 \cdot \zeta_2$ contains cycles in F, both $\xi_1 \cdot \zeta_1$ and $\xi_2 \cdot \zeta_2$ are simple outside F and they contain some edges outside F. Next, we apply the above argument to $\xi_1 \cdot \zeta_1$, and so on. Therefore we get cycles $\beta_1, \beta_2, \ldots, \beta_{K'}$ which are simple outside F, contain no cycle in F and satisfy **(q1)**, **(q2)**, **(q3)** and **(q4)** for F. We can get cycles $\alpha_1, \alpha_2, \ldots, \alpha_K$ which are simple outside H, contain no cycle in H and satisfy **(q1)**, **(q2)**, **(q3)** and **(q4)**, by applying this construction to each β_i for $i = 1, 2, \ldots, K'$ and each irreducible component of H because we do not change any part of η outside F in the above construction. Since α_m is simple outside H, α_m does not intersect itself outside H. Since α_m contains no cycle in H and visits each irreducible component of H at most once, α_m does not intersect itself in any irreducible component of H. Since H consists of irreducible subgraphs of G, every state in H is in some irreducible component of H. Therefore each state in α_m is in some irreducible component of H or not in H. Thus α_m does not intersect itself, that is, α_m is simple. □

(1) ⇒ (2): We define X and W by

$$X = \left\{ \frac{\mathrm{RDS}_0\bigl(\gamma(\eta)\bigr)}{\lg(\eta)} : \eta \text{ is a simple cycle in } G \right\},$$
$$W = \left\{ \frac{\mathrm{RDS}_0\bigl(\gamma(\eta)\bigr)}{\lg(\eta)} : \eta \text{ is a cycle in } G \right\}.$$

Then by Lemma 4.1 we have $\operatorname{conv} X = \operatorname{cl} W$. Hence, $\operatorname{conv} X$ is a closed and convex set whose distance from the origin is at least c. Thus there is an $e \in \boldsymbol{C}$ with $|e| = 1$ such that $\langle e, x\rangle \geq c$ for every $x \in \operatorname{conv} X$.

By executing the following algorithm, we assign a complex number to each edge in $\mathcal{E}(G)$:

1. Let $i = 0$ and let G_0 be the empty graph.

2. $i \leftarrow i + 1$.

3. Let
$$A_i = \bigl\{\eta : \eta \text{ is a simple cycle in } G \text{ with } \mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1}) \neq \emptyset\bigr\}.$$

4. Define $g_i : A_i \to \boldsymbol{C}$ by
$$g_i(\eta) = \frac{\mathrm{RDS}_0\bigl(\gamma(\eta)\bigr) - \sum_{\eta}^{[G_{i-1}]} d}{\#\bigl(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\bigr)}, \qquad \text{for } \eta \in A_i. \tag{4.9}$$

5. Choose a cycle $\tilde{\eta}_i \in A_i$ such that
$$\bigl\langle g_i(\tilde{\eta}_i), e\bigr\rangle = \min_{\eta \in A_i}\bigl\langle g_i(\eta), e\bigr\rangle$$
and the cycle $\tilde{\eta}_i$ visits each irreducible component of G_{i-1} at most once.

6. Let $d(a, b) = g_i(\tilde{\eta}_i)$ for $(a, b) \in \mathrm{Eg}(\tilde{\eta}_i)\backslash\mathcal{E}(G_{i-1})$.

7. Let G_i be the subgraph which consists of all edges in $\mathrm{Eg}(\tilde{\eta}_1), \mathrm{Eg}(\tilde{\eta}_2), \cdots, \mathrm{Eg}(\tilde{\eta}_i)$.

8. If $G \neq G_i$ then go to the step 2.

In this algorithm, we have to prove that we can execute the step 5 at any time because every cycle η_i with $\bigl\langle g_i(\eta_i), e\bigr\rangle = \min_{\eta \in A_i}\bigl\langle g_i(\eta), e\bigr\rangle$ would visit some irreducible components of G_{i-1} more than once. If we can execute the step 5 in this algorithm at any time, the algorithm terminates because the number of edges in G is finite. When the algorithm terminates, the assignment of $d(b, b')$ to each edge $(b, b') \in \mathcal{E}(G)$ is completed.

By induction we prove that for each i, $\tilde{\eta}_i$ can be chosen as described in the step 5 and the statement *(2)* in Theorem 4.4 holds for G_i (though G_i is not necessarily irreducible). We can execute the step 5 for $i = 1$. By (4.9) and the step 6, we have

$$d(a, b) = \frac{\mathrm{RDS}_0\bigl(\gamma(\tilde{\eta}_1)\bigr)}{\#\mathrm{Eg}(\tilde{\eta}_1)} \in W \tag{4.10}$$

for $(a, b) \in \mathrm{Eg}(\tilde{\eta}_1)$. Since e is chosen so that $\operatorname{cl} W \subset H(e, c)$, $d\bigl(\mathrm{Eg}(\tilde{\eta}_1)\bigr) \subset H(e, c)$. On the other hand, by (4.10) we get

$$\mathrm{RDS}_0\bigl(\gamma(\tilde{\eta}_1)\bigr) = \#\mathrm{Eg}(\tilde{\eta}_1) d(a, b) = \sum_{\tilde{\eta}_1} d.$$

Hence the statement *(2)* holds for $i = 1$.

Assume that $i \geq 2$ and let η be a cycle in A_i. We have simple cycles $\alpha_1, \alpha_2, \cdots, \alpha_K$ which satisfy **(q1)**, **(q2)**, **(q3)** and **(q4)** for G_{i-1} and η by Lemma 4.5 because η is simple and G_{i-1} consists of cycles. Since $\alpha_m \in A_i$ for $m = 1, 2, \cdots, K$, we get

$$\begin{aligned}
g_i(\eta) &= \frac{\mathrm{RDS}_0\bigl(\gamma(\eta)\bigr) - \sum_{\eta}^{[G_{i-1}]} d}{\#\bigl(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\bigr)} \\
&= \sum_{m=1}^{K} \frac{1}{\#\bigl(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\bigr)} \left(\mathrm{RDS}_0\bigl(\gamma(\alpha_m)\bigr) - \sum_{\alpha_m}^{[G_{i-1}]} d\right) \\
&= \sum_{m=1}^{K} \frac{\#\bigl(\mathrm{Eg}(\alpha_m)\backslash\mathcal{E}(G_{i-1})\bigr)}{\#\bigl(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\bigr)} \frac{\mathrm{RDS}_0\bigl(\gamma(\alpha_m)\bigr) - \sum_{\alpha_m}^{[G_{i-1}]} d}{\#\bigl(\mathrm{Eg}(\alpha_m)\backslash\mathcal{E}(G_{i-1})\bigr)} \\
&= \sum_{m=1}^{K} \frac{\#\bigl(\mathrm{Eg}(\alpha_m)\backslash\mathcal{E}(G_{i-1})\bigr)}{\#\bigl(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\bigr)} g_i(\alpha_m).
\end{aligned}$$

Since $\#\bigl(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\bigr) = \sum_{m=1}^{K} \#\bigl(\mathrm{Eg}(\alpha_m)\backslash\mathcal{E}(G_{i-1})\bigr)$, $g_i(\eta)$ is a convex combination of $g_i(\alpha_1), g_i(\alpha_2), \ldots, g_i(\alpha_K)$. Hence $\min_m\bigl\langle g_i(\alpha_m), e\bigr\rangle \leq \bigl\langle g_i(\eta), e\bigr\rangle$. Since α_m visits each irreducible component of G_{i-1} at most once for $m = 1, 2, \ldots, K$, it follows that $\tilde{\eta}_i$ can be chosen as described in the step 5. Since G_{i-1} is a subgraph of G_i, $A_i \subset A_{i-1}$. From

(4.9), we can write

$$\mathrm{RDS}_0\big(\gamma(\eta)\big) = \#\big(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-2})\big)g_{i-1}(\eta) + \sum_{\eta}^{[G_{i-2}]} d.$$

Since G_{i-1} consists of G_{i-2} and $\widetilde{\eta}_{i-1}$, we note that

$$\#\big(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-2})\big) = \#\big(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\big) + \#\Big(\big(\mathrm{Eg}(\eta)\cap\mathrm{Eg}(\widetilde{\eta}_{i-1})\big)\backslash\mathcal{E}(G_{i-2})\Big),$$

and since we assign $g_{i-1}(\widetilde{\eta}_{i-1})$ to edges in $\mathrm{Eg}(\widetilde{\eta}_{i-1})\backslash\mathcal{E}(G_{i-2})$,

$$\sum_{\eta}^{[G_{i-2}]} d - \sum_{\eta}^{[G_{i-1}]} d = -\#\Big(\big(\mathrm{Eg}(\eta)\cap\mathrm{Eg}(\widetilde{\eta}_{i-1})\big)\backslash\mathcal{E}(G_{i-2})\Big)\, g_{i-1}(\widetilde{\eta}_{i-1}).$$

From these three equations, we have

$$\begin{aligned}
g_i(\eta) &= \frac{1}{\#\big(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\big)}\Big(\#\big(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-2})\big)g_{i-1}(\eta) + \sum_{\eta}^{[G_{i-2}]} d - \sum_{\eta}^{[G_{i-1}]} d\Big)\\
&= \frac{1}{\#\big(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\big)}\cdot\\
&\quad\bigg(\Big(\#\big(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\big) + \#\Big(\big(\mathrm{Eg}(\eta)\cap\mathrm{Eg}(\widetilde{\eta}_{i-1})\big)\backslash\mathcal{E}(G_{i-2})\Big)\Big)g_{i-1}(\eta)\\
&\quad -\#\Big(\big(\mathrm{Eg}(\eta)\cap\mathrm{Eg}(\widetilde{\eta}_{i-1})\big)\backslash\mathcal{E}(G_{i-2})\Big)g_{i-1}(\widetilde{\eta}_{i-1})\bigg)\\
&= g_{i-1}(\eta) + \frac{\#\Big(\big(\mathrm{Eg}(\eta)\cap\mathrm{Eg}(\widetilde{\eta}_{i-1})\big)\backslash\mathcal{E}(G_{i-2})\Big)}{\#\big(\mathrm{Eg}(\eta)\backslash\mathcal{E}(G_{i-1})\big)}\big(g_{i-1}(\eta) - g_{i-1}(\widetilde{\eta}_{i-1})\big).
\end{aligned}$$

Since $0 \le \big\langle g_{i-1}(\eta) - g_{i-1}(\widetilde{\eta}_{i-1}), e\big\rangle$ by the construction and $c \le \big\langle g_{i-1}(\eta), e\big\rangle$ by the induction hypothesis, we get $c \le \big\langle g_i(\eta), e\big\rangle$, that is, $g_i(\eta) \in H(e, c)$.

Next we prove that d defined in the above algorithm satisfies (4.7) for every cycle in G_i. From (4.9) we have

$$\mathrm{RDS}_0\big(\gamma(\widetilde{\eta}_i)\big) = \#\big(\mathrm{Eg}(\widetilde{\eta}_i)\backslash\mathcal{E}(G_{i-1})\big)g_i(\widetilde{\eta}_i) + \sum_{\widetilde{\eta}_i}^{[G_{i-1}]} d$$

and since $g_i(\widetilde{\eta}_i)$ is assigned to all edges in $\mathrm{Eg}(\widetilde{\eta}_i)\backslash\mathcal{E}(G_{i-1})$,

$$= \sum_{\widetilde{\eta}_i} d. \tag{4.11}$$

Let η be a simple cycle in G_i. If η is in an irreducible component of G_{i-1}, then it is trivial by the induction hypothesis. Therefore we consider the case where η visits G_{i-1} and contains some edges outside G_{i-1}. We may assume that $\big(t_G(\widetilde{\eta}_i), i_G(\widetilde{\eta}_i)\big) \notin \mathcal{E}(G_{i-1})$ and $t_G(\widetilde{\eta}_i) \in \mathcal{S}(G_{i-1})$ because of (4.8). Since G_{i-1} consists of cycles, every block of G_{i-1} is contained in some irreducible component of G_{i-1}. Hence $\widetilde{\eta}_i$ is written as

$$\widetilde{\eta}_i = \widetilde{\eta}_i^{\mathrm{out},1}\cdot\widetilde{\eta}_i^{\mathrm{in},1}\cdot\widetilde{\eta}_i^{\mathrm{out},2}\cdot\widetilde{\eta}_i^{\mathrm{in},2}\cdot\cdots\cdot\widetilde{\eta}_i^{\mathrm{out},K}\cdot\widetilde{\eta}_i^{\mathrm{in},K}$$

where $\widetilde{\eta}_i^{\mathrm{in},m}$ is a simple block in an irreducible component G_{i-1} and $\widetilde{\eta}_i^{\mathrm{out},m}$ is a simple block consisting of states not in G_{i-1}. Since $\widetilde{\eta}_i$ visits each irreducible component of G_{i-1} at most once, η coincides with $\widetilde{\eta}_i$ outside G_{i-1}. Hence η is written as

$$\eta = \widetilde{\eta}_i^{\mathrm{out},1}\cdot\eta^{\mathrm{in},1}\cdot\widetilde{\eta}_i^{\mathrm{out},2}\cdot\eta^{\mathrm{in},2}\cdot\cdots\cdot\widetilde{\eta}_i^{\mathrm{out},K}\cdot\eta^{\mathrm{in},K}$$

where $\eta^{\mathrm{in},m}$ is a simple block contained in an irreducible component of G_{i-1} which contains $\widetilde{\eta}_i^{\mathrm{in},m}$, with $i_G(\widetilde{\eta}_i^{\mathrm{in},m}) = i_G(\eta^{\mathrm{in},m})$ and $t_G(\widetilde{\eta}_i^{\mathrm{in},m}) = t_G(\eta^{\mathrm{in},m})$. By applying Lemma 4.4 to each irreducible component of G_{i-1}, we get a function $\phi : \mathcal{S}(G_{i-1}) \to C$ such that for every block $\zeta = a_0a_1\cdots a_{L-1}$ in G_{i-1}

$$\mathrm{RDS}_0\big(\gamma(\zeta)\big) - \sum_{i=0}^{L-2} d(a_i, a_{i+1}) = \phi(a_{L-1}) - \phi(a_0) + \gamma(a_{L-1}).$$

Hence

$$\begin{aligned}
&\mathrm{RDS}_0\big(\gamma(\widetilde{\eta}_i)\big) - \sum_{\widetilde{\eta}_i} d - \Big(\mathrm{RDS}_0\big(\gamma(\eta)\big) - \sum_{\eta} d\Big)\\
&= \sum_{m=1}^{K}\Bigg(\mathrm{RDS}_0\big(\gamma(\widetilde{\eta}_i^{\mathrm{in},m})\big) - \sum_{(a,b)\in\overline{\mathcal{E}}(\widetilde{\eta}_i^{\mathrm{in},m})} d(a,b)\Bigg)\\
&\qquad - \sum_{m=1}^{K}\Bigg(\mathrm{RDS}_0\big(\gamma(\eta^{\mathrm{in},m})\big) - \sum_{(a,b)\in\overline{\mathcal{E}}(\eta^{\mathrm{in},m})} d(a,b)\Bigg)\\
&= \sum_{m=0}^{K}\Big(\phi\big(t_G(\widetilde{\eta}_i^{\mathrm{in},m})\big) - \phi\big(i_G(\widetilde{\eta}_i^{\mathrm{in},m})\big) + \gamma\big(t_G(\widetilde{\eta}_i^{\mathrm{in},m})\big)\Big)\\
&\qquad - \sum_{m=0}^{K}\Big(\phi\big(t_G(\eta^{\mathrm{in},m})\big) - \phi\big(i_G(\eta^{\mathrm{in},m})\big) + \gamma\big(t_G(\eta^{\mathrm{in},m})\big)\Big)\\
&= 0,
\end{aligned}$$

where for a block $\zeta = a_0a_1\cdots a_{L-1}$, $\overline{\mathcal{E}}(\zeta)$ means the set $\big\{(a_i, a_{i+1}) : 0 \le i \le L-2\big\}$.

Therefore by (4.11) we get $\mathrm{RDS}_0\big(\gamma(\eta)\big) - \sum_\eta d = 0$.

Next, let η be a cycle in G_i which is not simple. We can write $\eta = \eta_1 \cdot \eta_2 \cdot \eta_3$ where η_2 is a simple cycle and $i(\eta_2) = i(\eta_3)$. Then

$$\mathrm{RDS}_0\big(\gamma(\eta)\big) - \sum_\eta d = \mathrm{RDS}_0\big(\gamma(\eta_1\cdot\eta_3)\big) - \sum_{\eta_1\cdot\eta_3} d + \mathrm{RDS}_0\big(\gamma(\eta_2)\big) - \sum_{\eta_2} d.$$

Thus, by applying the above argument repeatedly, we can conclude that d satisfies (4.7) for G_i.

(2) ⇒ (1): Let $\eta = a_0a_1\cdots a_{L-1}$ be a cycle in G. We have

$$\begin{aligned}\left|\frac{\mathrm{RDS}_0\big(\gamma(\eta)\big)}{\lg(\eta)}\right| &= \frac{\left|\sum_{i=0}^{L-1} d(a_i,a_{i+1})\right|}{L}\\ &\ge \frac{\left\langle\sum_{i=0}^{L-1} d(a_i,a_{i+1}), e\right\rangle}{L}\\ &= \frac{\sum_{i=0}^{L-1}\big\langle d(a_i,a_{i+1}), e\big\rangle}{L},\end{aligned}$$

and since $d(a_i,a_{i+1}) \in H(e,c)$,

$$\ge c.$$

(2) ⇔ (3): This follows from Lemma 4.2.

(1) ⇔ (4): This follows from Theorem 4.3.

Next we prove this theorem for the case of $f = kf_S/n$. If one of the statements of Theorem 4.4 holds at f, then n is a divisor of N, by Theorem 4.1 and Proposition 4.2. Define $\tilde\gamma : \mathcal{S}(G) \to C$ by $\tilde\gamma(\sigma) = z^{\ell(\sigma)}\gamma(\sigma)$, $\sigma \in \mathcal{S}(G)$. Then it is easy to see, using Lemma 4.3 and Theorem 4.3, that for every condition of Theorem 4.4, $(G,\tilde\gamma)$ satisfies the condition at $f = 0$ (but n is not changed) if and only if (G,γ) satisfies the condition at f. □

Remark **4.3** We note that e and d depend on the indexing of $\mathcal{S}(G)$ in the strict sense as follows: let ℓ and ℓ' be indexings of $\mathcal{S}(G)$ with respect to partitions by period. For $a \in \mathcal{S}(G)$ with $\ell(a) = 0$, put $I = \ell'(a)$. Then $z^{\ell'(b)} = z^{\ell(b)+I}$ for $b \in \mathcal{S}(G)$. Define $d' : \mathcal{E}(G) \to H(z^Ie, c)$ by $d'(a,b) = z^Id(a,b)$ for $(a,b) \in \mathcal{E}(G)$. Then we have

$$\begin{aligned}\gamma(a) &= z\phi(b) - \phi(a) + z^{-\ell(a)}d(a,b)\\ &= z\phi(b) - \phi(a) + z^{-\ell(a)-I}z^Id(a,b)\\ &= z\phi(b) - \phi(a) + z^{-\ell'(a)}d'(a,b),\end{aligned}$$

and

$$z^{\ell'(a_0)}\sum_{i=0}^{L-1} z^i\gamma(a_i) = z^Iz^{\ell(a_0)}\sum_{i=0}^{L-1} z^i\gamma(a_i) = z^I\sum_{i=0}^{L-1} d(a_i,a_{i+1}) = \sum_{i=0}^{L-1} d'(a_i,a_{i+1}).$$

□

Using this result and the ACH procedure we can construct codes which satisfy our constraints. For example suppose that we want to construct a code having a spectral line of amplitude not less than c for $c > 0$. Since the condition (3) in Theorem 4.4 is described in terms of graphs, we can generate a finite graph G which satisfies the condition. By Proposition 2.1 there is a j such that the capacity of G^j is greater than 1. The constraint represented by G^j is a sofic system and may not be of finite type. However, by approximating G^j from the inside (e.g., applying Marcus' approximation theorem[11] to G^j), we can get a constraint H of finite type which is a sub-constraint of G^j and whose capacity is greater than 1. Then we obtain a finite state encoder E by applying the ACH procedure to H. It is easy to see that if a graph H_1 is obtained from another graph H_2 by the ACH procedure and H_2 satisfies the condition (3), then H_1 also satisfies the condition (3). Therefore the encoder E has a spectral line at f of amplitude not less than c. This code construction scheme can also be applied to Theorem 4.5 and Theorem 4.6 in the next section.

Example 4.2 Let (G_2,γ_2) be the FSTD in Fig. 4.2. Then for every cycle η in G_2, $\mathrm{RDS}_{f_S/2}\big(\gamma_2(\eta)\big)/\lg(\eta) \ge 1/2$. Thus $w_d^{(G_2,P,\gamma_2)}(f_S/2) \ge 1/4$ for every $P \in \Pi_{G_2}$. Let P_2 be the transition probability matrix which gives the maximum probabilistic entropy of G_2. The spectral density function of (G_2,P_2,γ_2) is shown in Fig. 4.3 and (G_2,P_2,γ_2) has the spectral lines of amplitude 0.076393 at dc and 0.523607 at $f_S/2$. □

Next we consider combinations of spectral lines at several given frequencies. Let k_1 and k_2 be nonnegative integers, and n_1 and n_2 positive integers with $\gcd(k_1,n_1) = \gcd(k_2,n_2) = 1$, $k_1/n_1 \ne k_2/n_2$, $k_1 < n_1$ and $k_2 < n_2$. Put $f_1 = k_1f_S/n_1$ and $f_2 = k_2f_S/n_2$. Suppose that $k_1 = 0$ (hence, $k_2 > 0$). Let a and b be channel symbols with $a \ne b$. Put

$$\eta = \underbrace{baa\cdots a}_{n_2L}\underbrace{baa\cdots a}_{n_2L}\underbrace{aa\cdots a}_{2n_2L},\qquad \xi = \underbrace{aa\cdots a}_{2n_2L}b\underbrace{a\cdots a}_{2n_2L-1}, \tag{4.12}$$

where L is a positive integer such that neither $\mathrm{RDS}_0(\eta)$ nor $\mathrm{RDS}_0(\xi)$ is equal to 0, that is, $(4Ln_2-1)a \ne b$ and $(4Ln_2-1)a \ne b$. We note that $\mathrm{RDS}_{f_2}(\eta) = 2(b-a)$

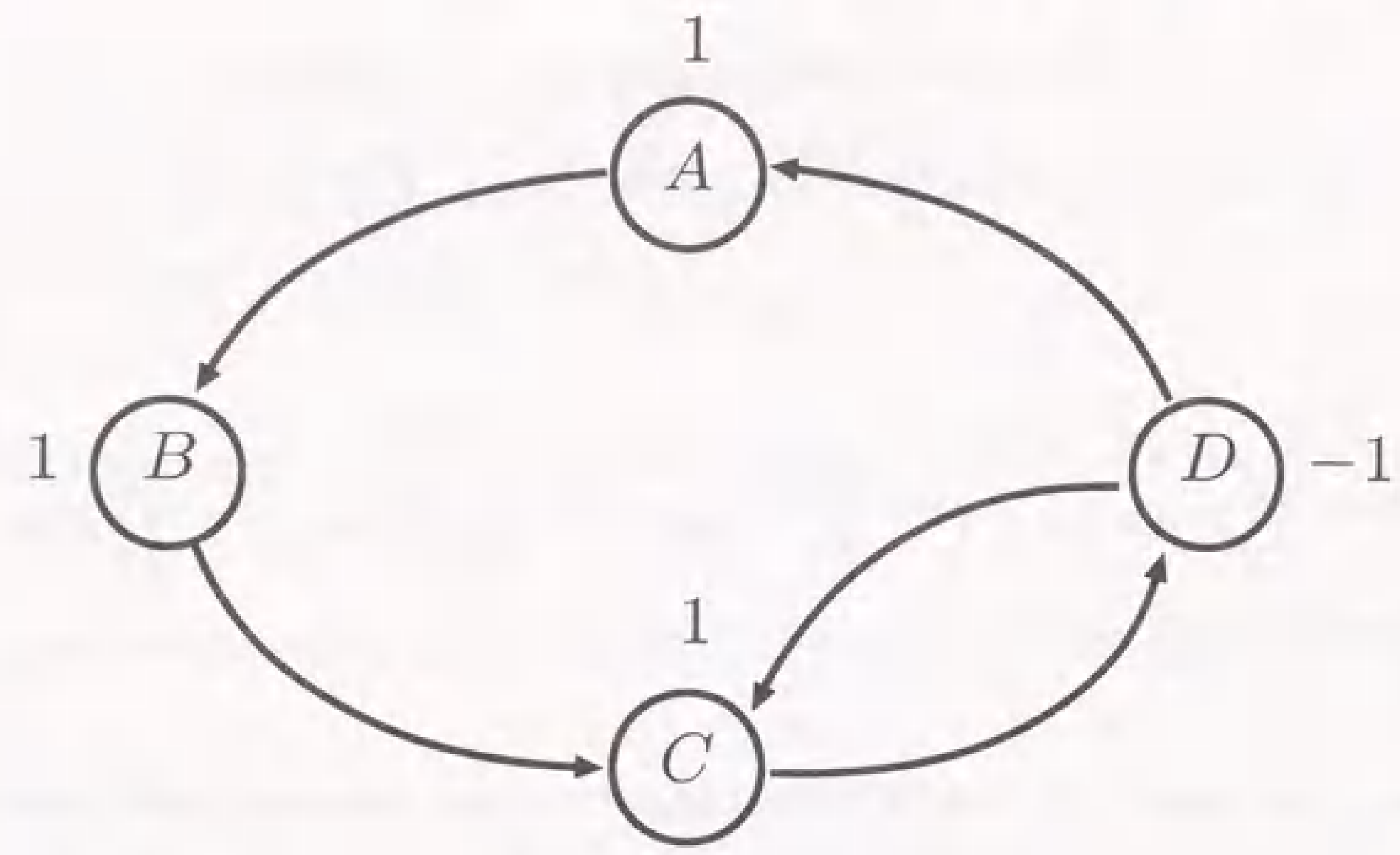


Figure 4.2: FSTD (G_2, γ_2) which has spectral line at dc.

and $\mathrm{RDS}_{f_2}(\xi) = b - a$. We can construct an FSTD (G, γ) which generates sequences consisting of only η and ξ. Hence, by Theorem 4.4, there are positive numbers c_1 and

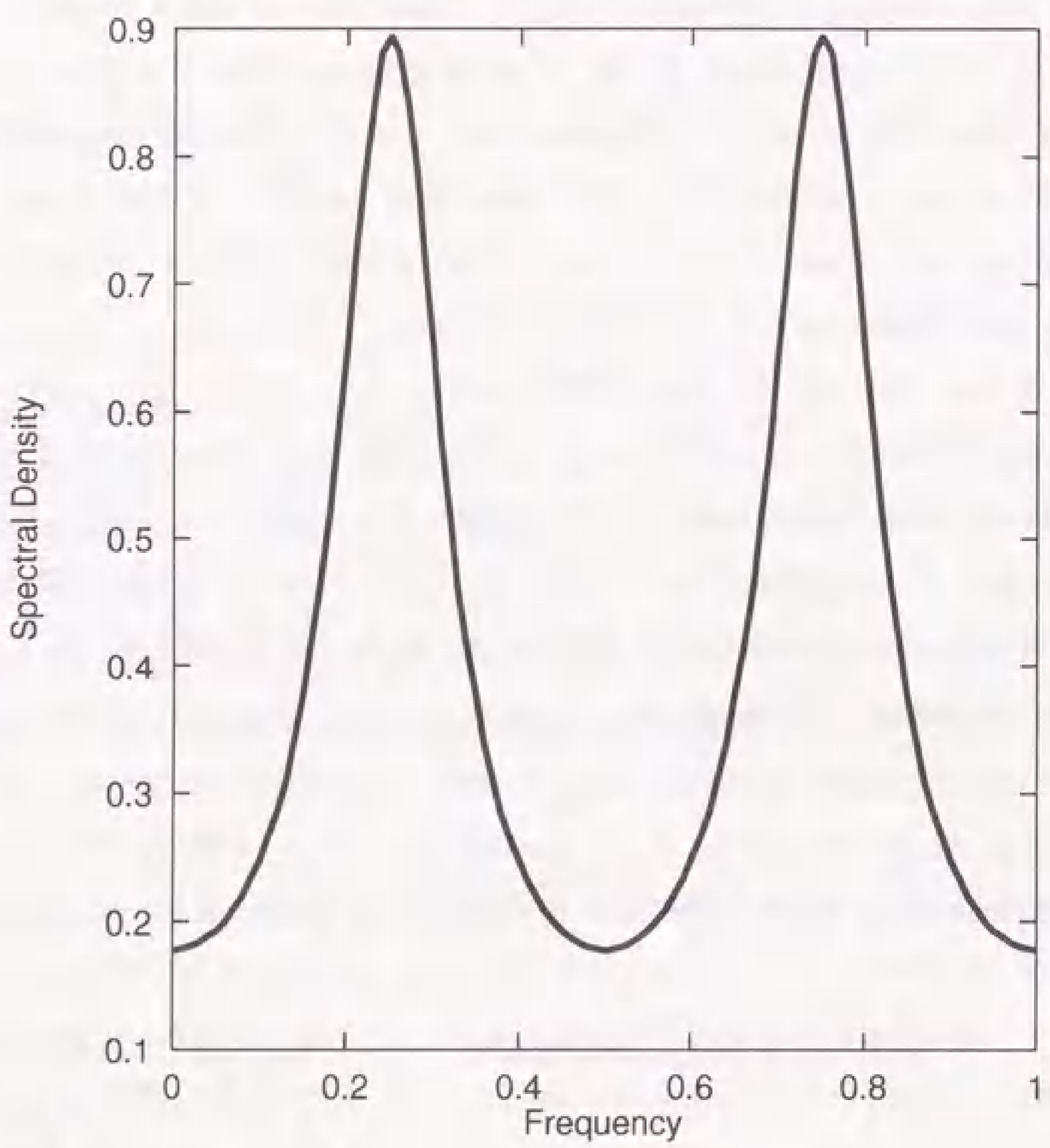


Figure 4.3: Spectral Density of (G_2, γ_2).

c_2 such that for every $P \in \Pi_G$, (G, P, γ) has spectral lines of amplitude not less than c_1 at f_1 and of amplitude not less than c_2 at f_2, simultaneously. When $n_1 > 0$ and $n_2 > 0$, take η and ξ similarly in (4.12) but put $L = n_1$. Then we get a similar result.

If $\gcd(n_1, n_2) < \min(n_1, n_2)$ then by an argument similar to the above we can see that there are an FSTD (G, γ) and a positive number c such that (G, γ) has a spectral null at f_1 and $w_d^{(G,P,\gamma)}(f_2) \geq c$ for every $P \in \Pi_G$.

In a recording system with buried servos, a servo signal is added to encoded sequences and the servo signal is retrieved from reproduced signals by using an electric filter [12]. Theorem 4.4 show that we can generate signals such that reproduced signals from them have some amount of frequency content at a frequency $f = kf_S/n$, without adding any signal of frequency f to them. In fact, if (G, γ) is an encoder (an FSTD) which satisfies the conditions of Theorem 4.4 at f and if we use a coding scheme in which a message from a source is encoded by (G, γ) and the encoded message is recorded on a medium, then we can retrieve a signal of frequency f from a reproduced signal by an electric filter. We can apply this scheme to recording systems whose medium is digital, for example, an optical-magnetic recording system.

4.4 Biased coboundary conditions and spectral lines

In this section we characterize (G, P, γ)'s whose spectral lines do not depend on P in terms of graphs (Theorem 4.5 and Theorem 4.6). These characterization are related to spectral density null (not necessarily spectral null). We also give some examples of FSTD's having spectral lines.

Definition 4.2 Let (G, γ) be an FSTD and ℓ an indexing of $\mathcal{S}(G)$ with respect to a partition by period. Let $d \in \boldsymbol{C}$. Let k be a nonnegative integer and n a positive integer. We say that (G, γ) satisfies a *biased coboundary condition with respect to* d *at* kf_S/n if there is a function $\phi : \mathcal{S}(G) \to \boldsymbol{C}$ such that

$$\gamma(a) = z\phi(b) - \phi(a) + z^{-\ell(a)}d \qquad \text{for each } (a, b) \in \mathcal{E}(G) \tag{4.13}$$

where $z = \exp(-\mathrm{i}2\pi k/n)$. We call ϕ *the biased coboundary function.* □

Remark **4.4** For the case of dc, this condition first appeared in [53] in a different context [53, Theorem 3] and an FSTD satisfying a biased coboundary condition at dc is said to have bounded running digit sum. □

The following example shows that there is an FSTD satisfies both a coboundary condition at a frequency f and a biased coboundary condition at f with respect to

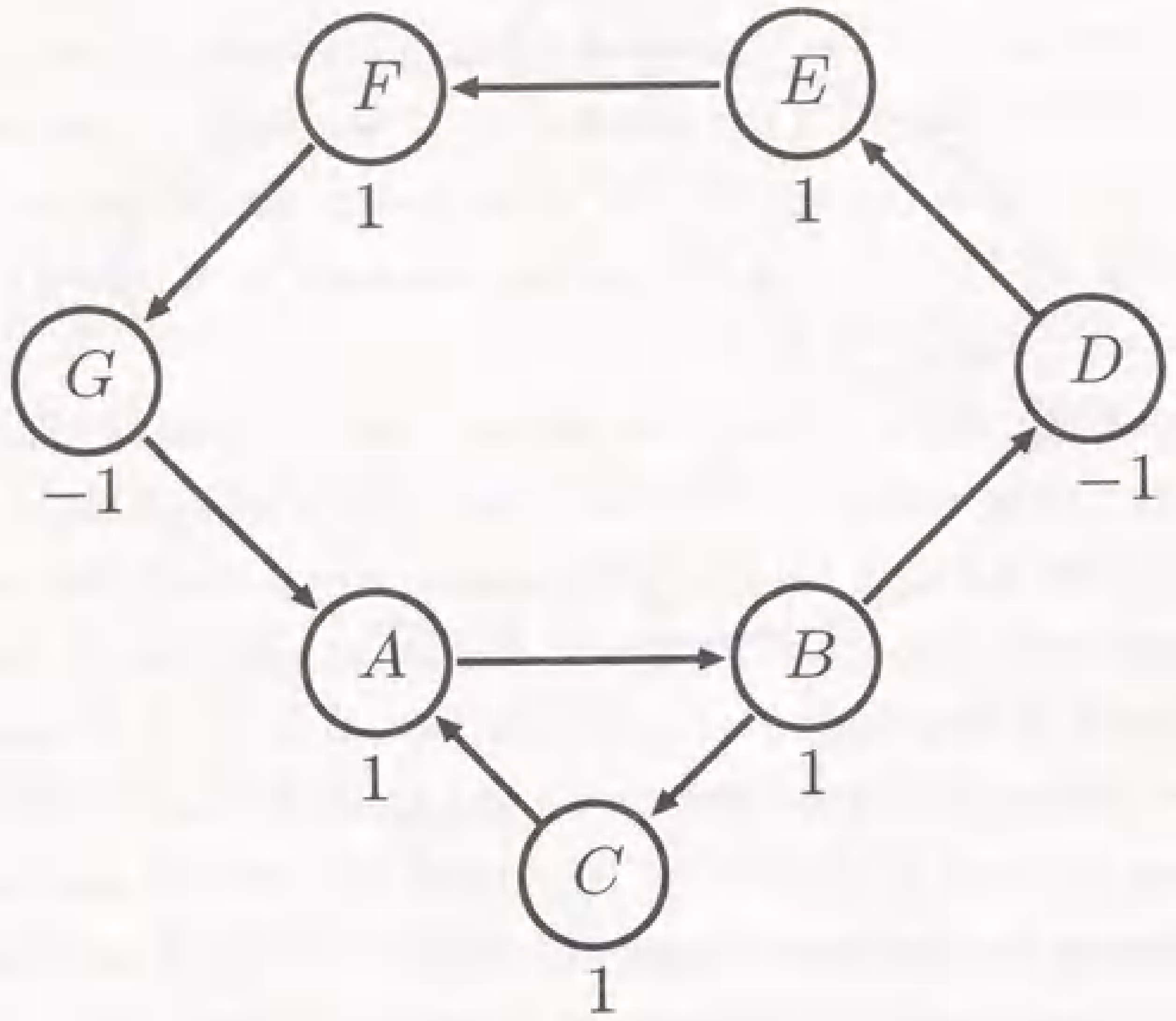


Figure 4.4: FSTD (G_3, γ_3) which has spectral null at $f_S/6$.

some nonzero constant. That is, the FSTD has a spectral null at f and satisfies the biased coboundary condition at f with respect to the constant. (Thus, we need to the condition about period in Theorem 4.4 (3) and also in Theorem 4.6 (3) below.)

Example 4.3 Let (G_3, γ_3) be the FSTD in Fig. 4.4. Put $z = \exp(-i2\pi/6)$. We can see that $\mathrm{RDS}_{f_S/6}\big(\gamma_3(\eta)\big) = 0$ for every cycle η in G_3 of length a multiple of 6. (Actually, (G_3, γ_3) is a finite subgraph of the canonical graph $G_0^{f_S/6}$, which is defined in [23].) Thus, (G_3, γ_3) has a spectral null at $f_S/6$ by Theorem 4.2. The period of G_3 is 3. Let ℓ_3 be an indexing of $\mathcal{S}(G_3)$ with respect to a partition by period. Define $\tilde{\gamma}_3 : \mathcal{S}(G_3) \to \boldsymbol{C}$ by $\tilde{\gamma}_3(a) = \gamma_3(a) + z^{-\ell_3(a)}$ for $a \in \mathcal{S}(G_3)$. Fig. 4.5 shows the FSTD $(G_3, \tilde{\gamma}_3)$, where I in the figure is an integer determined by ℓ_3. We can check that $\mathrm{RDS}_{f_S/6}\big(\tilde{\gamma}_3(\eta)\big) = 0$ for every cycle η of length a multiple of 6. Therefore, there is a coboundary function $\psi : \mathcal{S}(G_3) \to \boldsymbol{C}$ such that $\tilde{\gamma}_3(a) = z\psi(b) - \psi(a)$ for $(a, b) \in \mathcal{E}(G_3)$. Thus, by the construction, we have $\gamma_3(a) = z\psi(b) - \psi(a) - z^{-\ell_3(a)}$ for $(a, b) \in \mathcal{E}(G_3)$, that is, (G_3, γ_3) satisfies a biased coboundary condition with respect to -1 at $f_S/6$. □

Main results of this section are Theorem 4.5 and Theorem 4.6. We need the following two lemmas in proving the theorems.

Lemma 4.6 Let (G, γ) be an irreducible FSTD. Let k be a nonnegative integer and n a positive integer with $\gcd(k, n) = 1$. Put $f = kf_S/n$. Then the following conditions are equivalent for $c \geq 0$:

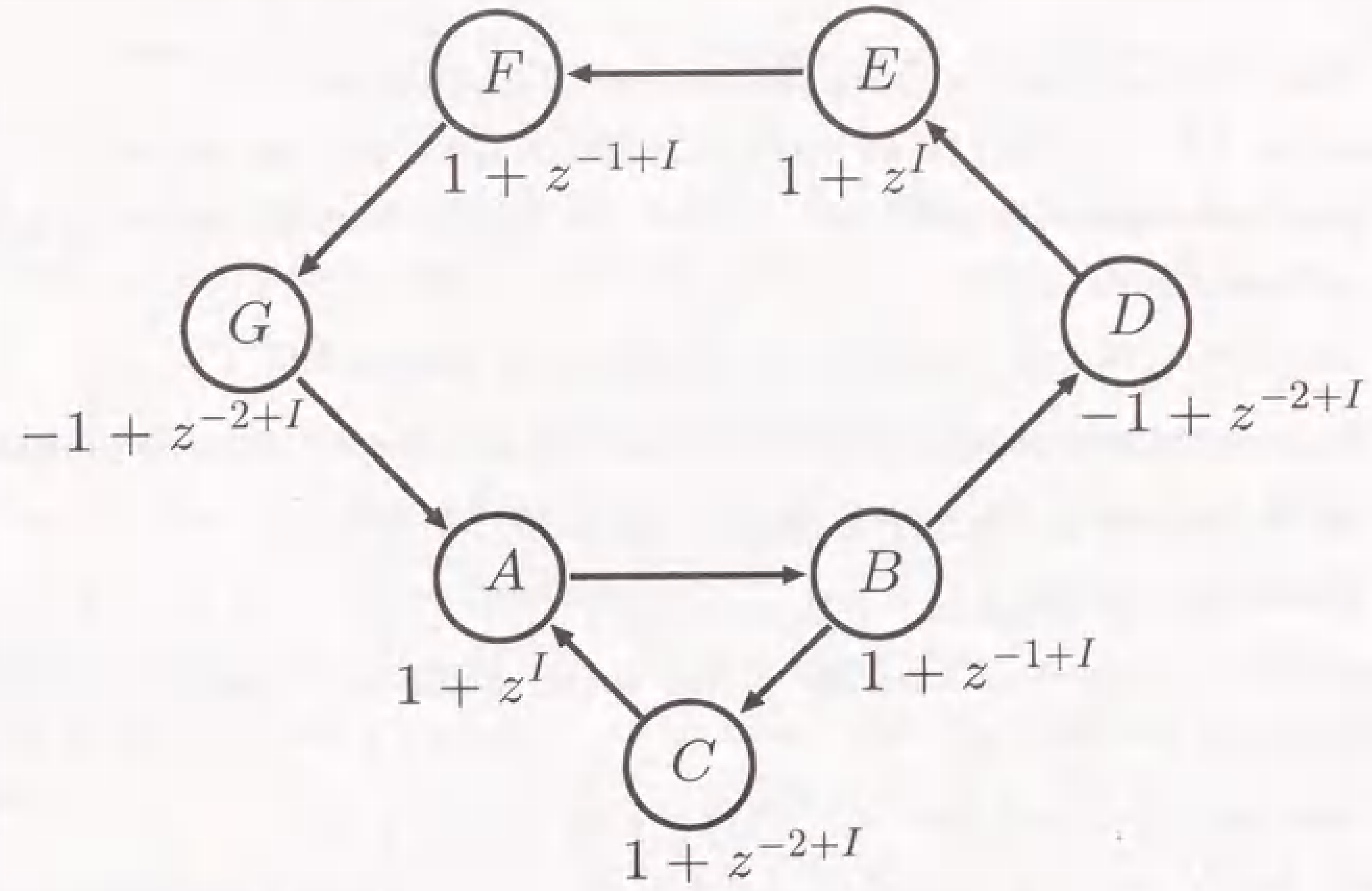


Figure 4.5: FSTD (G_3, γ_3') which has spectral null at $f_S/6$.

(1) for every cycle η of length a multiple of n,

$$\left|\frac{rds_{f,\gamma}(\eta)}{\lg(\eta)}\right| = c;$$

(2) there is a $d \in \boldsymbol{C}$ with $|d| = c$ such that for every cycle η of length a multiple of n,

$$rds_{f,\gamma}(\eta) = d\lg(\eta);$$

(3) there are a nonnegative number K and a $d \in \boldsymbol{C}$ with $|d| = c$ such that for every block η in G,

$$\left|rds_{f,\gamma}(\eta) - d\lg(\eta)\right| \leq K;$$

Proof: See Appendix A.7. □

For a set $A \subset \mathcal{S}(G)$ and a vector $\vec{v}$ with the index set $\mathcal{S}(G)$, we say that $\vec{v}$ is constant on A if $\vec{v}(a) = \vec{v}(b)$ for $a, b \in A$.

Lemma 4.7 Let G be an irreducible directed graph and N the period of G. Let $0 \leq f \leq f_S$. Let γ and $\gamma' : \mathcal{S}(G) \to \boldsymbol{C}$ such that $\gamma - \gamma'$ is constant on each B_i, where $\{B_0, B_1, \ldots, B_{N-1}\}$ is the partition of $\mathcal{S}(G)$ by period. Then $w_c^{(G,P,\gamma)} = w_c^{(G,P,\gamma')}$ for every $P \in \Pi_G$.

Proof: See Appendix A.8. □

The following theorem characterizes spectral density null.

Theorem 4.5 Let (G,γ) be an irreducible FSTD. Let k be a nonnegative integer and n a positive integer with $\gcd(k,n)=1$. Put $f=kf_S/n$. Then the following conditions are equivalent:

(1) for some $c\geq 0$, (G,γ) satisfies the conditions in Lemma 4.6;

(2) (G,γ) satisfies a biased coboundary condition with respect to some constant at f;

(3) $w_c^{(G,P,\gamma)}(f)=0$ for every $P\in\Pi_G$;

(4) $w_c^{(G,P,\gamma)}(f)=0$ for some $P\in\Pi_G$.

Furthermore, if (G,γ) satisfies one of the above conditions, then (G,γ) satisfies the following:

(5) there is a $c\geq 0$ such that $w_d^{(G,P,\gamma)}(f)=c$ for every $P\in\Pi_G$.

Proof: Let N be the period of G. Let $\{B_0,B_1,\ldots,B_{N-1}\}$ be a partition of $\mathcal{S}(G)$ by period and ℓ the indexing of $\mathcal{S}(G)$ with respect to the partition. For $d\in\boldsymbol{C}$, define $\gamma_d:\mathcal{S}(G)\to\boldsymbol{C}$ by $\gamma_d(\sigma)=\gamma(\sigma)-z^{-\ell(\sigma)}d$, $\sigma\in\mathcal{S}(G)$.

(2) ⇒ (3): We note that (G,γ_d) satisfies a coboundary condition at f. Since $\gamma-\gamma_d$ is constant on each B_i, (G,P,γ) and (G,P,γ_d) have the same spectral density function for every $P\in\Pi_G$ by Lemma 4.7. Thus (3) follows from Theorem 2.2.

(3) ⇒ (4): Trivial.

(4) ⇒ (2): If N is not a multiple of n, then there is no spectral line at f and, hence, (2) follows from Theorem 2.2.

Assume that N is a multiple of n. Let $P\in\Pi_G$ with $w_c^{(G,P,\gamma)}(f)=0$ and put $d=pJ_K\gamma$, where $K=Nk/n$ and p is the stationary distribution of P. Since (G,P,γ) and (G,P,γ_d) have the same spectral density function and (G,P,γ_d) has no spectral line at f by (4.5), (G,γ_d) satisfies a coboundary condition at f and, hence, (2) follows.

(1) ⇒ (2): If $c=0$, then (2) follows from Theorem 4.2 and if $c>0$, then (2) follows in the same way as Theorem 4.4 is proved.

(2) ⇒ (1): Assume that N is not a multiple of n. Then there is no spectral line at f. Since (2) implies (3), (G,γ) has a spectral null at f. Therefore (G,γ) satisfies (1), by Theorem 4.2. If N is a multiple of n, then (G,γ_d) satisfies a coboundary condition and, thus, (1) follows from Theorem 4.2 and Lemma 4.3.

Assume that (G,γ) satisfies the conditions (1) in Lemma 4.6. Then (5) follows from Theorem 4.3. □

Remark **4.5** Assume that $c=0$. Let (G_2,γ_2) be the FSTD given in Figure 2 and P_2 the transition matrix given in Example 4.2. As shown in Fig. 4.3, the spectral density function of (G_2,P_2,γ_2) takes a nonzero value at all frequencies and there are spectral lines only at frequencies $0, f_S/2$. Hence, (5) does not imply any of the other conditions in Theorem 4.5.

Remark **4.6** For an aperiodic FSTD (that is, the period of the underlying directed graph is one) with $f=0$, *(2)* ⇔ *(3)* in the above theorem was proved implicitly in [20]. □

The next theorem is a special case of Theorem 4.4 and characterizes (G,P,γ)'s which have spectral lines of amplitude a given positive value at f independent of P.

Theorem 4.6 Let (G,γ) be an irreducible FSTD. Let k be a nonnegative integer and n a positive integer with $\gcd(k,n)=1$. Put $z=\exp(-i2\pi k/n)$ and $f=kf_S/n$. Let N be the period of G and c a positive real number. Then the following conditions are equivalent:

(1) (G,γ) satisfies the conditions in Lemma 4.6;

(2) (G,γ) satisfies a biased coboundary condition and there is a cycle η of length a multiple of n such that
$$\left|\frac{rds_{f,\gamma}(\eta)}{\lg(\eta)}\right|=c;$$

(3) N is a multiple of n and (G,γ) satisfies a biased coboundary condition with respect to some $d\in\boldsymbol{C}$ with $|d|=c$;

(4) for every $P\in\Pi_G$
$$w_d^{(G,P,\gamma)}(f)=c^2;$$

(5) for some $P\in\Pi_G$
$$w_c^{(G,P,\gamma)}(f)=0 \quad\text{and}\quad w_d^{(G,P,\gamma)}(f)=c^2.$$

Proof: We can prove *(1)* ⇔ *(3)* ⇔ *(4)* in the same way as Theorem 4.4 is proved.

(3) ⇒ (2): Since *(1)* ⇔ *(3)*, *(2)* holds.

(2) ⇒ (3): Let ℓ be an indexing of $\mathcal{S}(G)$ with respect to a partition by period. Let η be a cycle of length a multiple of n such that $|rds_{f,\gamma}(\eta)/\lg(\eta)|=c$. Let $d\in\boldsymbol{C}$ and assume that (G,γ) satisfies a biased coboundary condition with respect to d. Let $P\in\Pi_G$. By Theorem 4.5, $w_c^{(G,P,\gamma)}(f)=0$. But, since $\left|rds_{f,\gamma}(\eta)\right|>0$, (G,P,γ) does not have any spectral null at f by Theorem 4.2. Hence, there is a spectral line at f. Since the spectral line can appear only at $0, f_S/N,\cdots,(N-1)f_S/N$, N is a multiple of n. By

Lemma 4.3 (1), we have

$$\begin{aligned} rds_{f,\gamma}(\eta) &= \sum_{i=0}^{L-1} z^{\ell(a_i)}\left(z\phi(a_{i+1}) - \phi(a_i) + z^{-\ell(a_i)}d\right) \\ &= \sum_{i=0}^{L-1} z^{\ell(a_{i+1})}\phi(a_{i+1}) - \sum_{i=0}^{L-1} z^{\ell(a_i)}\phi(a_i) + Ld \\ &= Ld. \end{aligned}$$

Hence $|d| = \left|rds_{f,\gamma}(\eta)/\lg(\eta)\right| = c$.

(3) ⇒ (5): Let $P \in \Pi_G$. By Theorem 4.5, $w_c^{(G,P,\gamma)}(f) = 0$. Since *(3)⇔ (4), (5)* follows.

(5) ⇒ (3): By Theorem 4.5, (G,γ) satisfies a biased coboundary condition with respect to a constant $d \in \boldsymbol{C}$. Since $w_d^{(G,P,\gamma)}(f) = c^2 > 0$, N is a multiple of n by (4.5). We note that $|d| = c$ since *(3) ⇔ (4)*. □

Assume that the period of G is a multiple of n and let c be a positive number. Formally, it remains to show necessary and sufficient conditions for an FSTD to satisfy the following:

(A) for all cycle η,

$$\left|\frac{rds_{f,\gamma}(\eta)}{\lg(\eta)}\right| \leq c.$$

The next condition is necessary and sufficient to satisfy (A) by the statement (2) of Theorem 4.3:

(B) for every $P \in \Pi_G$

$$w_d^{(G,P,\gamma)}(kf_S/n) \leq c.$$

The next two conditions are necessary but not sufficient:

(C) there are an $e \in \boldsymbol{C}$ with $|e| = 1$ and $d : \mathcal{E}(G) \to H(e, -c)$ such that for every cycle $\eta = a_0 a_1 \cdots a_{L-1}$

$$rds_{f,\gamma}(\eta) = \sum_{i=0}^{L-1} d(a_i, a_{i+1});$$

(D) there are an $e \in \boldsymbol{C}$ with $|e| = 1$, $d : \mathcal{E}(G) \to H(e, -c)$ and $\phi : \mathcal{S}(G) \to \boldsymbol{C}$ such that

$$\gamma(a) = z\phi(b) - \phi(a) + z^{-\ell(a)}d(a,b) \quad \text{for all } (a,b) \in \mathcal{E}(G).$$

We can prove these conditions by the same argument in the proof of Theorem 4.4 with slight modifications. For example we can derive (C) from (A) by replacing min with max and "≤" with "≥" in the proof. Let (G_4, γ_4) be the FSTD shown in Fig. 4.6. Define

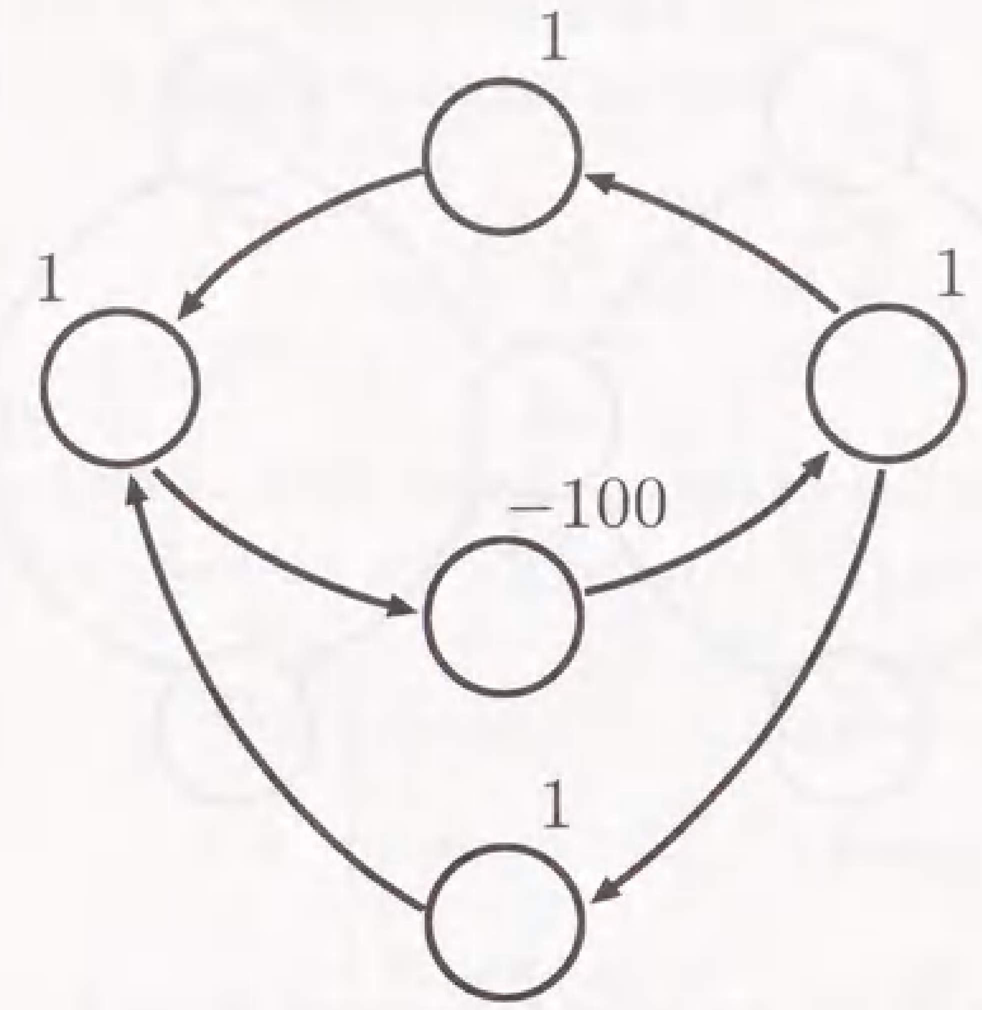


Figure 4.6: FSTD (G_4, γ_4)

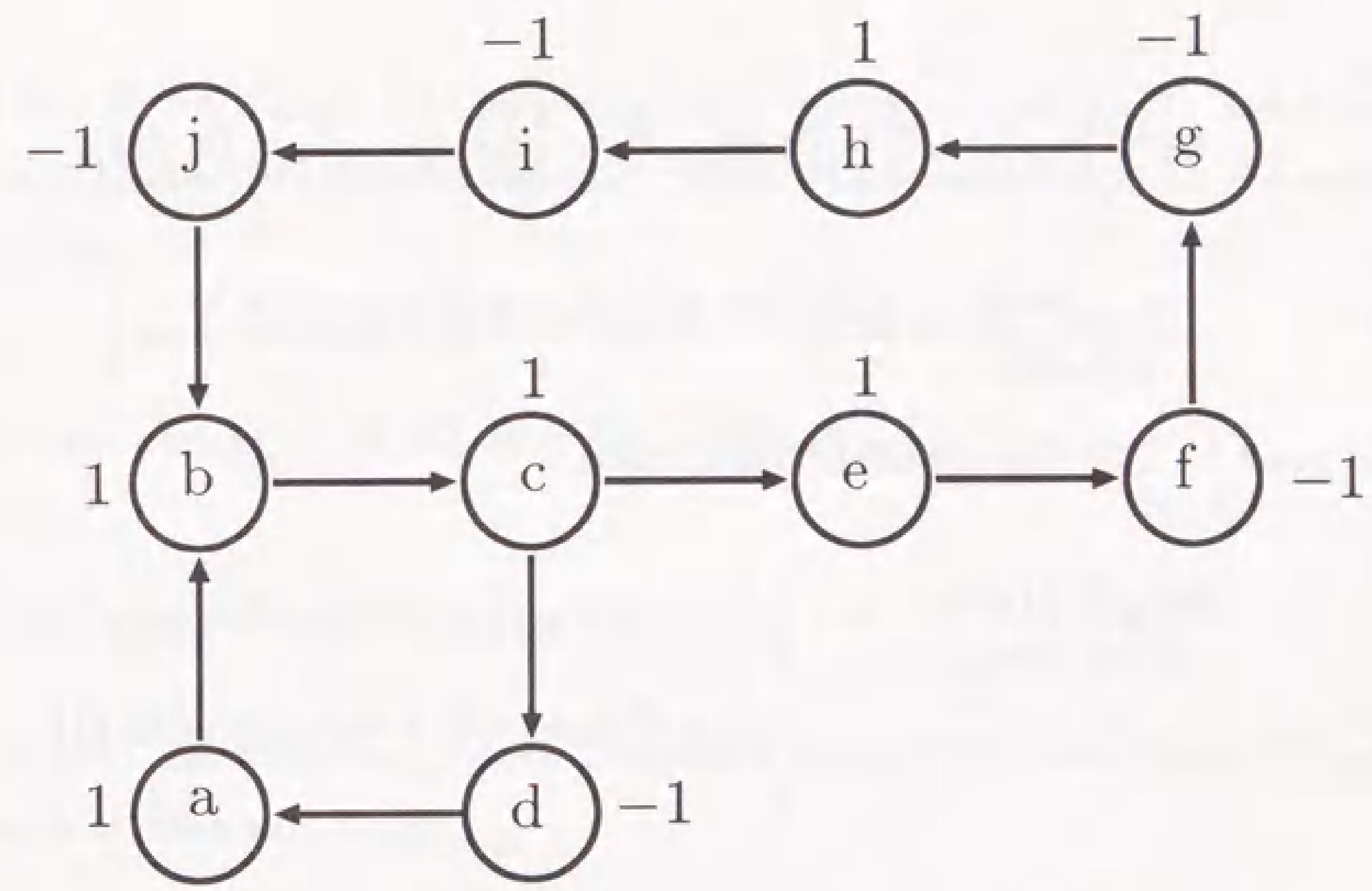


Figure 4.7: FSTD (G_5, γ_5)

$d_4 : \mathcal{E}(G_4) \to \boldsymbol{C}$ by $d_4(a,b) = \gamma(a)$ for $(a,b) \in \mathcal{E}(G_4)$. It is obvious that $d_4\big(\mathcal{E}(G_4)\big) \subset H(-1,-1)$ and that for all η, $\mathrm{RDS}_0\big(\gamma_4(\eta)\big) = \sum_\eta d_4$. But $\left|\mathrm{RDS}_0\big(\gamma_4(ABDE)\big)/4\right| > 1$. Thus neither (C) nor (D) is sufficient to satisfy (A).

In the condition (C), d could take its value in the circle of radius c at the origin. But we have the following example. Let (G_5, γ_5) be an FSTD given in Fig. 4.7. Let d_5 be a function on $\mathcal{E}(G_5)$ satisfying (C). Suppose that $d_5\big(\mathcal{E}(G_5)\big)$ is included in the circle of radius 1/2. The period of G_5 is 4 and a partition of $\mathcal{S}(G_5)$ by period is given as

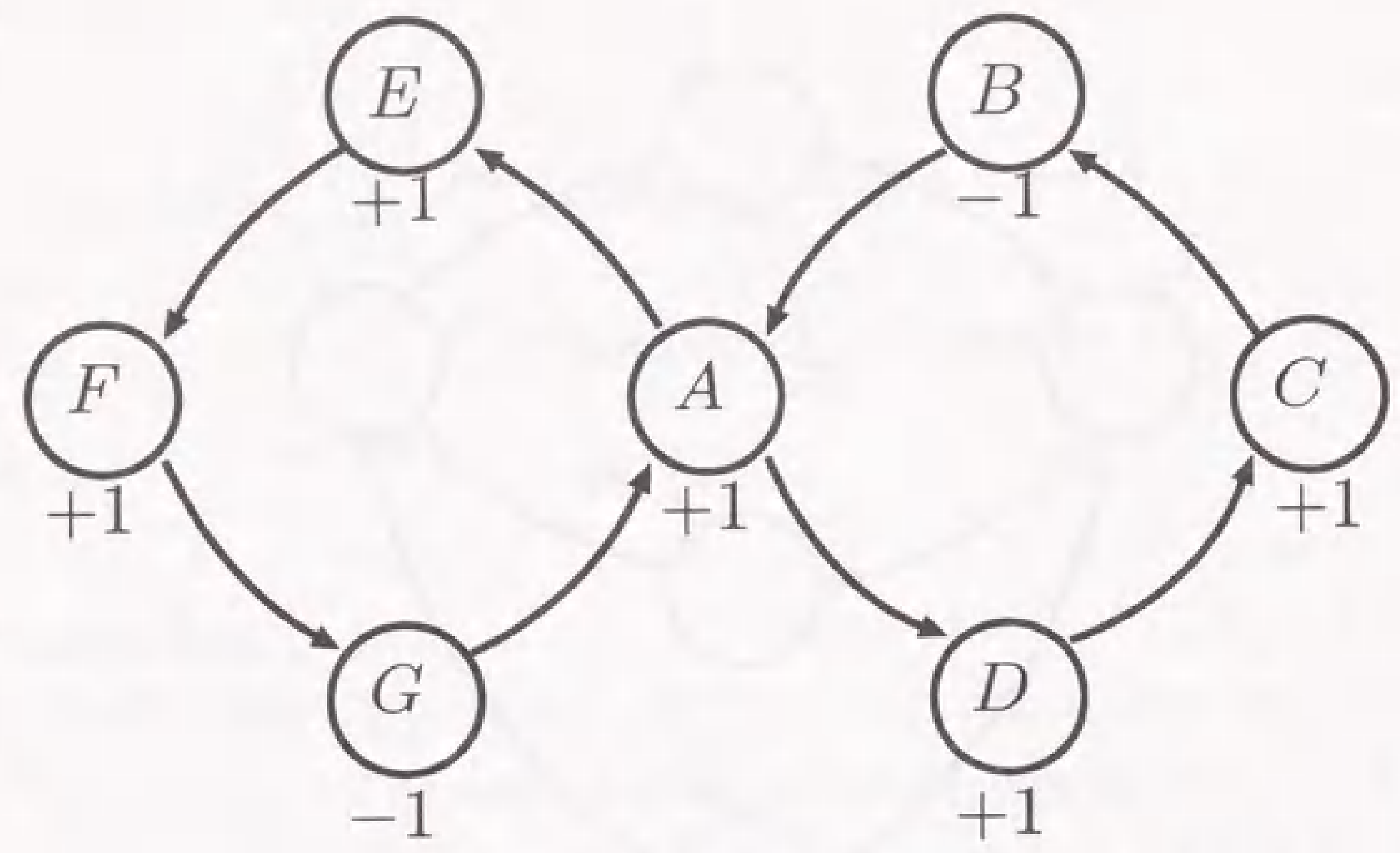


Figure 4.8: FSTD (G_6, γ_6) which has spectral line at $f_S/2$.

follows:

$$B_0 = \{b, g\}, \quad B_1 = \{c, b\}, \quad B_2 = \{d, e, i\}, \quad B_3 = \{a, f, j\}.$$

Since

$$\frac{rds_{f,\gamma_5}(bcda)}{\lg(bcda)} = \frac{d_5(b,c) + d_5(c,d) + d_5(d,a) + d_5(a,b)}{4} = \frac{1}{2},$$

$d_5(b,c)$ must be 1/2. On the other hand, since

$$\begin{aligned}\frac{rds_{f,\gamma_5}(bcefghij)}{\lg(bcefghij)} &= \frac{1}{8}\Big(d_5(b,c) + d_5(c,e) + d_5(e,f) + d_5(f,g) \\ &\quad + d_5(g,h) + d_5(h,i) + d_5(i,j) + d_5(j,b)\Big) \\ &= \frac{j}{2},\end{aligned}$$

$d_5(b,c)$ must be $j/2$. But this is impossible.

Example 4.4 Let (G_6, γ_6) be the FSTD given in Fig. 4.8. It is easy to see that $|\mathrm{RDS}_{f_S/2}(\gamma_6(\eta))| = 1/2$ for every cycle η in G_6. Thus $w_d^{(G_6,P,\gamma_6)}(f_S/2) = 1/4$ for every $P \in \Pi_{G_6}$. Let P_6 be the transition probability matrix which gives the maximum probabilistic entropy of G_6. The spectral density function of (G_6, P_6, γ_6) is shown in Fig. 4.9 and (G_6, P_6, γ_6) has the spectral lines of amplitude 1/4 at dc, 0 at $f_S/4$, 1/4 at $f_S/2$ and 0 at $3f_S/4$. □

Our results can be applied to the case of continuous signals as follows.

Example 4.5 Let G_7 be the directed graph given in Fig. 4.10. Let $T = 1/f_S$. The period of G_7 is 5. Consider a signal $x(t)$ given by

$$x(t) = \sum_{n=-\infty}^{+\infty} y(t - nT; s_n),$$

where $\{s_i\}_{i\in\mathbf{Z}}$ is a Markov chain for some probability transition matrix of G_7 and $y(t;\sigma)$ is defined as follows: for $t < 0$ or $t > T$, $y(t;\sigma) = 0$; for $0 \le t \le T$,

$$y(t;\sigma) = \begin{cases} \sin \omega_0 t & \text{if } \sigma = \sigma_1, \sigma_2; \\ \sin(\omega_0 t + \pi/2) & \text{if } \sigma = \sigma_3, \sigma_6; \\ \sin(\omega_0 t + 3/2\pi) & \text{if } \sigma = \sigma_5, \sigma_4; \\ \sin(\omega_0 t + \pi) & \text{if } \sigma = \sigma_7. \end{cases}$$

Then $x(t)$ is a QPSK signal. Put $\boldsymbol{y}(t) = \big(y(t;\sigma_1), y(t;\sigma_2), \cdots, y(t;\sigma_7)\big)^t$. Let $P \in \Pi_{G_7}$, and let p be the stationary distribution of P. The continuous part $w_x^c(t)$ of the spectrum of this signal is

$$w_x^c(f) = \frac{1}{T} Y^*(f) G(fT)^* C G(fT) Y(f), \tag{4.14}$$

and the discrete part $w_x^d(f)$ of the spectrum of this signal is

$$w_x^{(qh+r)}(f) = \left|\frac{1}{T} p J_r Y\big((qh+r)f_0\big)\right|^2, \quad q = \cdots, -1, 0, 1, \cdots, \quad r = 0, 1, \cdots, N-1, \tag{4.15}$$

where J_r, G and C are matrices given in Theorem 4.1, and $Y(f)$ is the Fourier Transform of $\boldsymbol{y}(t)$, which is given as follows:

$$\begin{aligned} Y(f) &= \big(Y_{\sigma_1}(f), Y_{\sigma_2}(f), Y_{\sigma_3}(f), Y_{\sigma_4}(f), Y_{\sigma_5}(f), Y_{\sigma_6}(f), Y_{\sigma_7}(f)\big)^t \\ Y_{\sigma_1}(f) = Y_{\sigma_2}(f) &= \frac{\pi}{j2} e^{-j\pi fT}\big(F(f - f_0) - F(f + f_0)\big) \\ Y_{\sigma_3}(f) = Y_{\sigma_6}(f) &= \frac{\pi}{j2} e^{-j\pi fT}\big(F(f + f_0) - F(f - f_0)\big) \\ Y_{\sigma_4}(f) = Y_{\sigma_5}(f) &= \frac{\pi}{2} e^{-j\pi fT}\big(F(f + f_0) - F(f - f_0)\big) \\ Y_{\sigma_7}(f) &= \frac{\pi}{2} e^{-j\pi fT}\big(F(f - f_0) - F(f + f_0)\big) \\ F(f) &= \frac{\sin 2\pi fT}{2\pi f}. \end{aligned}$$

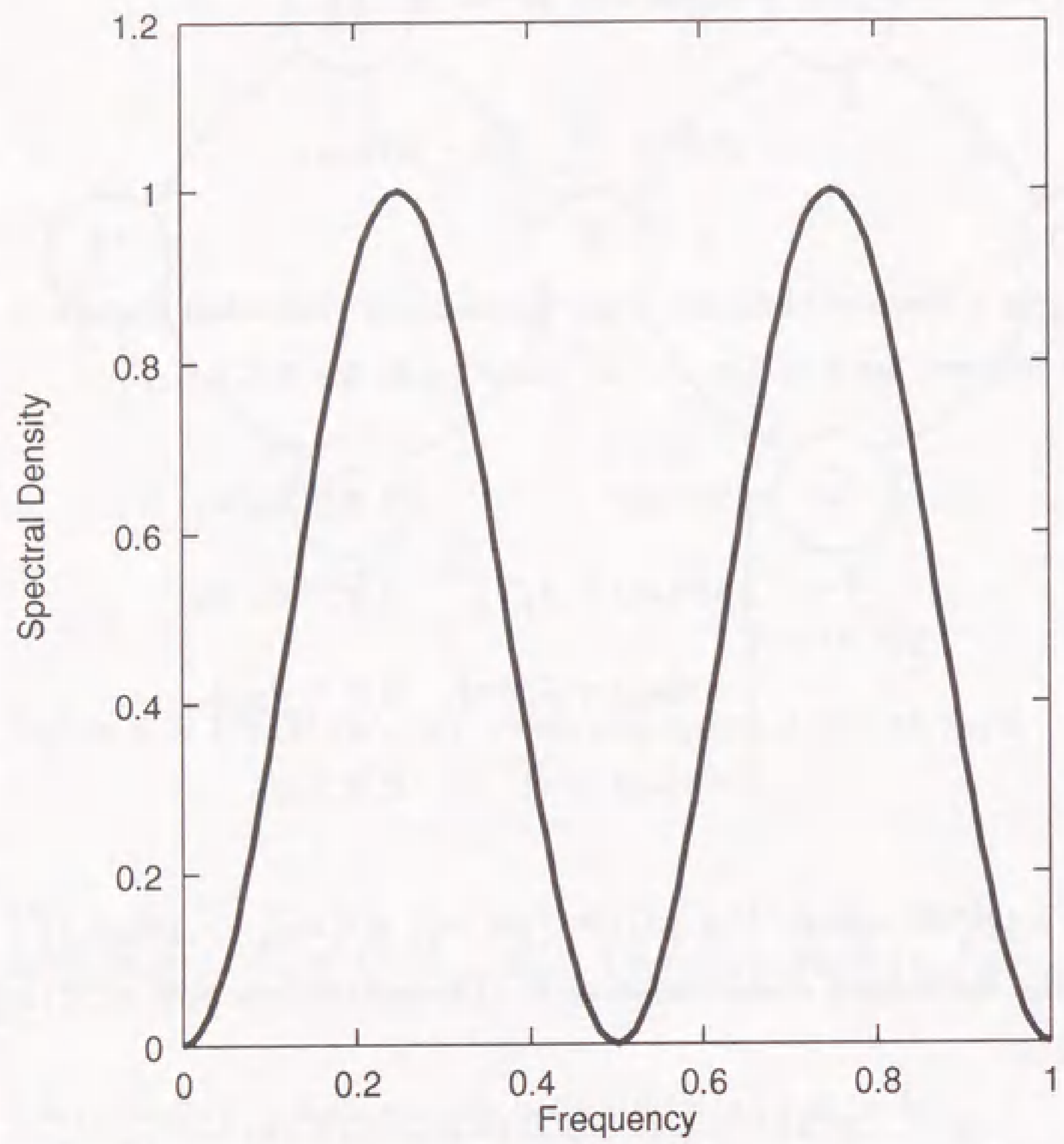


Figure 4.9: Spectral Density of (G_6, γ_6).

Let f_0 be the carrier frequency. Assume that $T = 5/f_0$ and put $g(f) = \pi e^{-j\pi fT}\big(F(f - f_0) - F(f + f_0)\big)/2$. Then the spectral line of this signal at f_0 is

$$\left|\frac{1}{T} pJ_0Y(f_0)\right|^2,$$

which is $1/T^2$ times the spectral line of the FSTD (G_7, P, γ_7) in Fig. 4.11 at dc. We can check that for every cycle η in G_7,

$$\left|\frac{rds_{0,\gamma_7}(\eta)}{\lg(\eta)}\right| = \frac{|g(f_0)|}{5}.$$

Thus, it follows that for every $P \in \Pi_{G_7}$ the signal $x(t)$ has the spectral line of amplitude $\pi g(f_0)/10$. On the other hand, the continuous part $w_x^c(f)$ of the spectrum of the signal

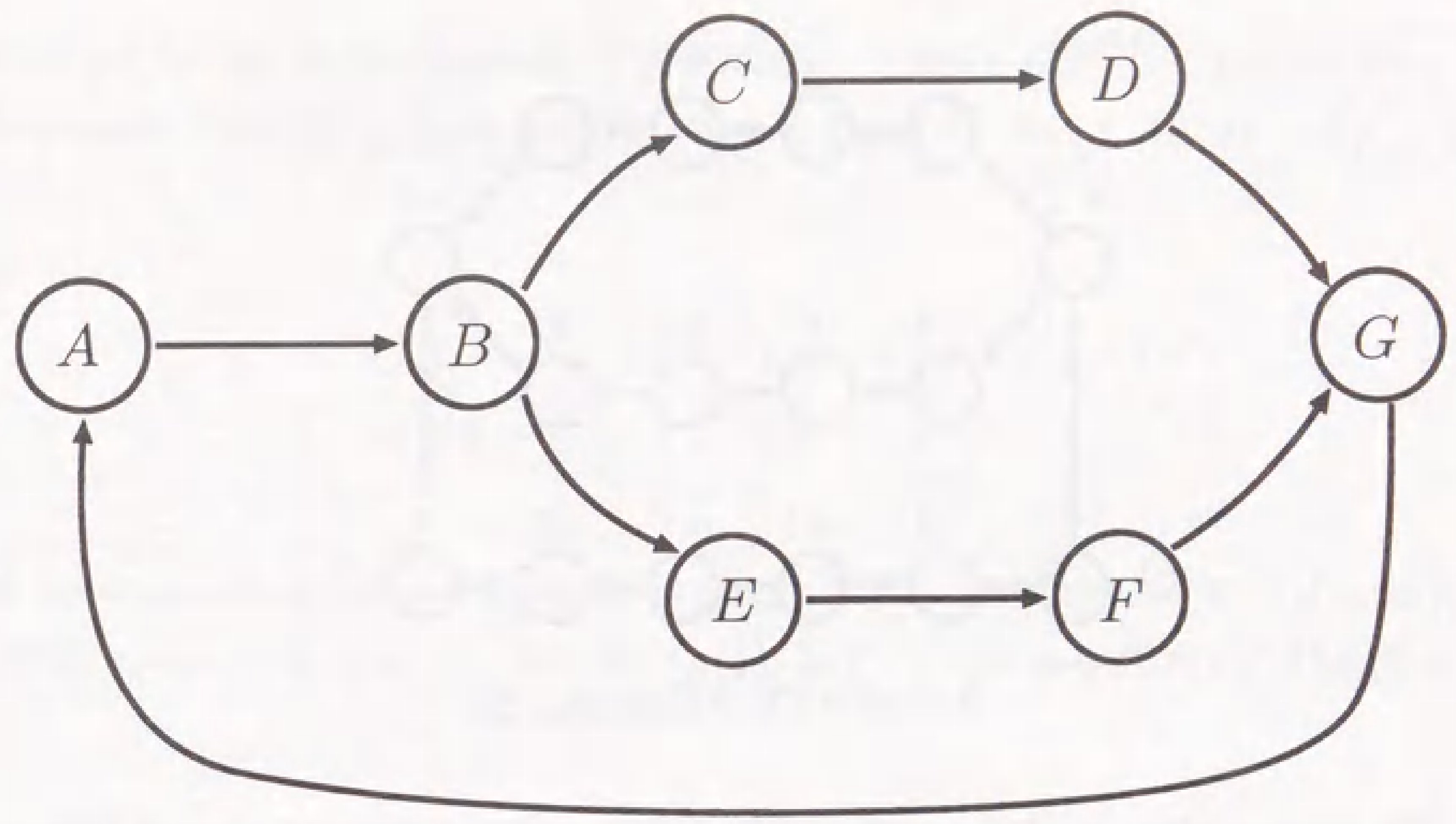


Figure 4.10: Directed Graph G_7.

$x(t)$ is given by

$$w_x^c(f_0) = \frac{1}{T} Y^*(f_0)G(5)^*CG(5)Y(f_0).$$

In this equation $G(5)^*CG(5)$ is just the value of the spectral density of (G_7, γ_7) at 5 (that is, at dc). By Theorem 4.6 and the above construction, we can conclude that $w_x^c(f_0) = 0$. □

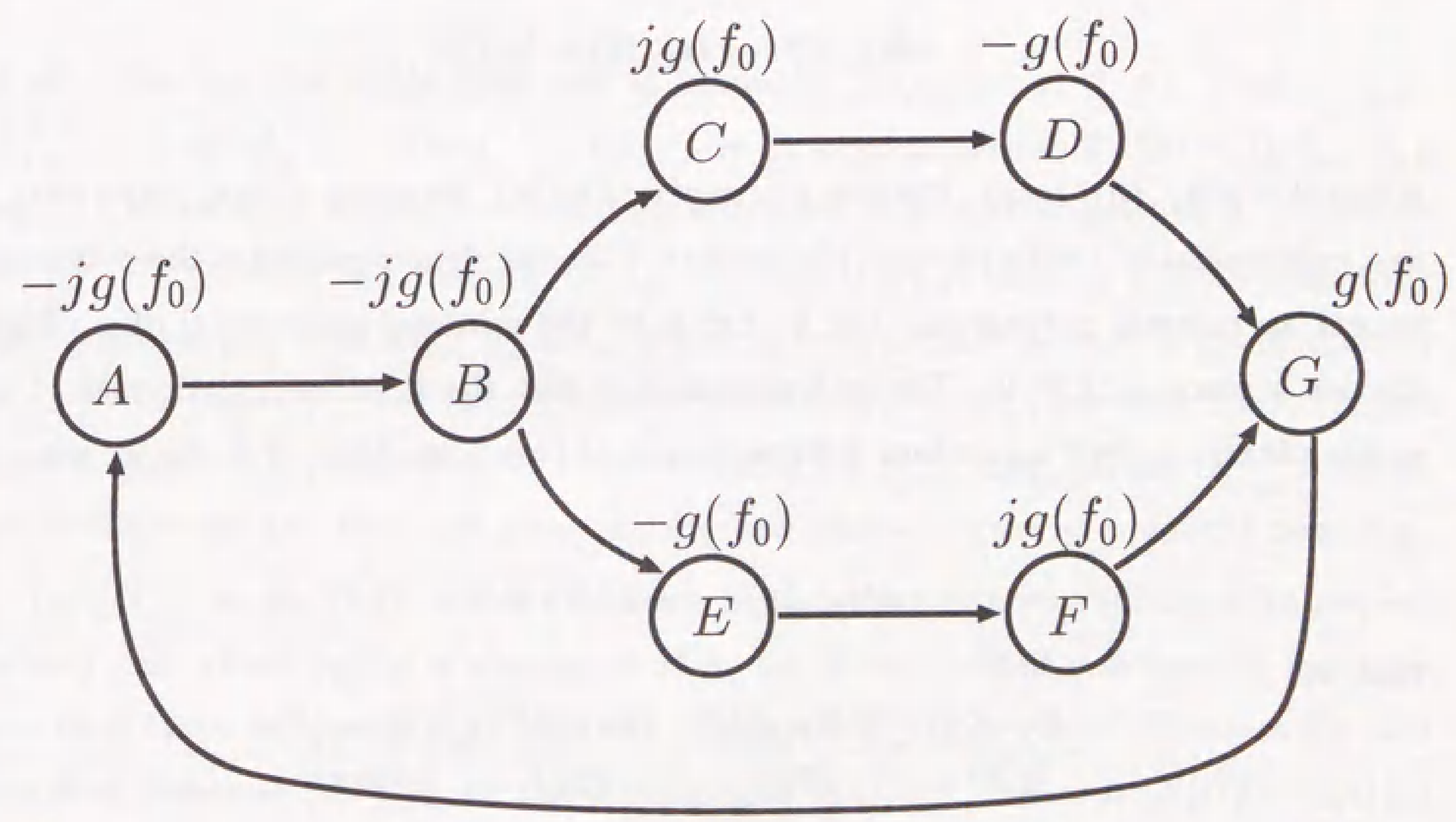


Figure 4.11: FSTD (G_7, γ_7).

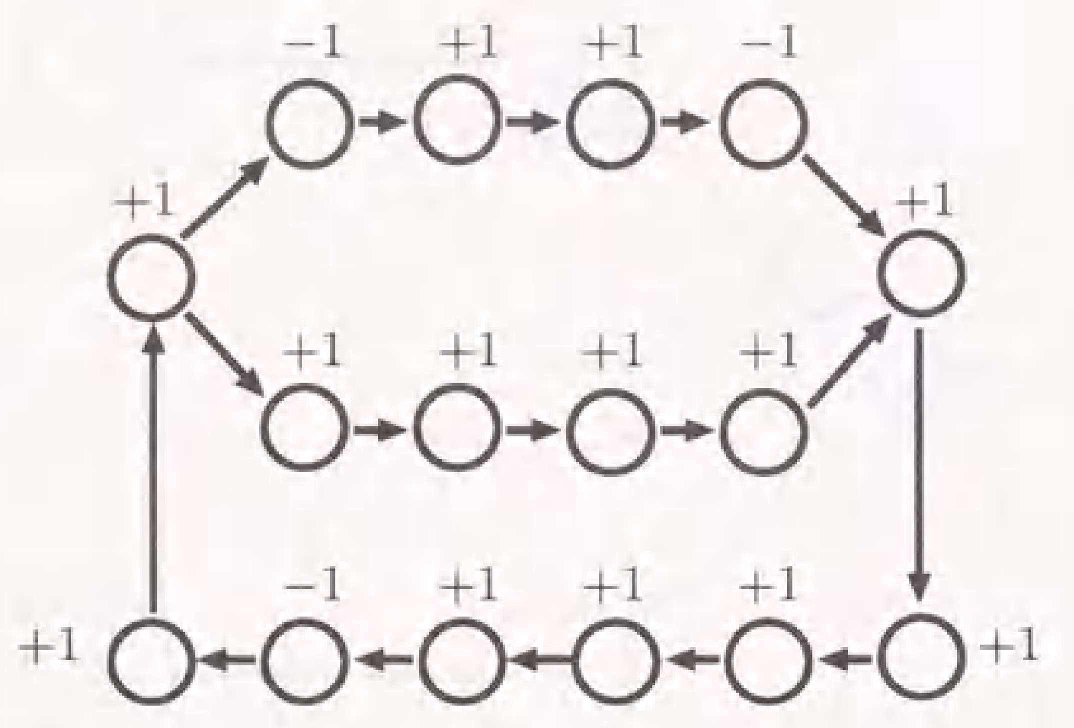


Figure 4.12: FSTD (G_8, γ_8).

We discuss the encoded message which has spectral lines at several frequencies, independent of the source statistics, simultaneously.

Corollary 4.1 Let (G, γ) be an irreducible FSTD. Let k and k' be nonnegative integers and n a positive integer with $\gcd(k, n) = \gcd(k', n) = 1$. Put $f = kf_S/n$ and $f' = k'f_S/n$. Assume that for some $c > 0$ and every $P \in \Pi_G$ $w_d^{(G,P,\gamma)}(f) = c$ and that $\gamma(a)$ is a rational number for each $a \in \mathcal{S}(G)$. Then for some c' and every $P \in \Pi_G$, $w_d^{(G,P,\gamma)}(f') = c'$.

Proof: Let η and $\tilde{\eta}$ be cycles in G. Put $\xi = \underbrace{\eta \cdot \eta \cdot \cdots \cdot \eta}_{\lg(\tilde{\eta}) \text{ times}}$ and $\tilde{\xi} = \underbrace{\tilde{\eta} \cdot \tilde{\eta} \cdot \cdots \cdot \tilde{\eta}}_{\lg(\eta) \text{ times}}$. By Theorem 4.6 we then have

$$rds_{f,\gamma}(\xi) = rds_{f,\gamma}(\tilde{\xi}) = \lg(\xi)d,$$

where $d = rds_{f,\gamma}(\eta)/\lg(\eta)$. Putting $z = \exp(-\mathrm{i}2\pi k/n)$, we think of $rds_{f,\gamma}(\eta) - rds_{f,\gamma}(\tilde{\eta})$ as a polynomial of z with rational coefficients. Clearing denominators of the polynomial, we get an integral polynomial, say g. Let h be the minimal polynomial of z. Then h divides g since $g(z) = 0$. The polynomial h is also the minimal polynomial of $z' = \exp(-\mathrm{i}2\pi k'/n)$. By Proposition 4.2 the period of G is a multiple of n. So we get

$$rds_{f',\gamma}(\xi) - rds_{f',\gamma}(\tilde{\xi}) = 0,$$

that is,

$$\frac{rds_{f',\gamma}(\eta)}{\lg(\eta)} = \frac{rds_{f',\gamma}(\xi)}{\lg(\xi)} = \frac{rds_{f',\gamma}(\tilde{\xi})}{\lg(\tilde{\xi})} = \frac{rds_{f',\gamma}(\tilde{\eta})}{\lg(\tilde{\eta})}$$

where the first and last equality follow from Lemma 4.3 (2). □

Remark **4.7** In the above corollary, if $\gcd(k', n) > 1$, then $w_d^{(G,P,\gamma)}(f')$ depends on $P \in \Pi_G$ in general. Let (G_8, γ_8) be the FSTD given in Fig. 4.12. Put $f = f_S/6$ and $f' = f_S/3$. Let

$$\begin{aligned} \zeta &= +1+1+1+1+1-1+1+1+1+1+1+1 \\ \xi &= +1+1+1+1+1-1+1-1+1+1-1+1 \end{aligned}$$

which are sequences generated by cycles in (G_8, γ_8). We can show that $\mathrm{RDS}_f\big(\gamma_8(\eta)\big)/\lg(\eta) = 1/3$ for every cycle η in G_8. But $\mathrm{RDS}_{f'}(\zeta)/\lg(\zeta) = 1/6$ and $\mathrm{RDS}_{f'}(\xi)/\lg(\xi) = 1/3$. □

4.5 Canonical graphs for spectral density nulls

Next, we describe canonical graphs for a spectral lines. Let $d \in \boldsymbol{C}$ and let A be a finite subset of $\boldsymbol{C}$. Let k be a nonnegative integer and n a positive integer with $\gcd(k, n) = 1$. Put $f = kf_S/n$ and $z = \exp(-\mathrm{i}2\pi k/n)$. Let $G_{f,d}^A$ be a graph such that its state set is a subset of $\boldsymbol{C} \times A \times \{0, 1, \cdots, n-1\}$ and there is an edge going from (ξ_1, a_1, i_1) to (ξ_2, a_2, i_2) only if

$$a_1 = z\xi_2 - \xi_1 + z^{-i_1}d \qquad \text{and} \qquad (i_1 + 1) \bmod n = i_2,$$

and all states are reachable from one of states in $\{(0, a, 0) : a \in A\}$. Define $\gamma_{f,d}^A : \mathcal{S}(G_{f,d}^A) \to A$ by $\gamma_{f,d}^A(\xi, a, i) = a$. Let (G, γ) be an irreducible sub-FSTD of $(G_{f,d}^A, \gamma_{f,d}^A)$. Then, (G, γ) satisfies a biased coboundary condition at f with respect to d. Hence we have $w_d^{(G,P,\gamma)}(f) = |d|^2$ for every $P \in \Pi_G$. By Remark 4.3, there is a label-preserving graph isomorphism from G_{f,d_1}^A to G_{f,d_2}^A where $d_1 = z^{i_1}d$ and $d_2 = z^{i_2}d$ for integers i_1 and i_2. (Note that $G_{f,d}^A$ does not necessarily contain cycles: e.g., for every finite symbol sequence $a_0a_1 \cdots a_{L-1}$ consisting of 1 and -1, $1/\sqrt{2} \neq \sum_{i=0}^{L-1} a_i/L$. Hence $G_{0,1/\sqrt{2}}^{\{-1,1\}}$ has no cycle by Theorem 4.6. But such a canonical graph seems to have no practical meaning.)

Let (G, γ) be an FSTD which satisfies a biased coboundary condition with respect to d at f and whose period is a multiple of n. Let ϕ be a biased coboundary function. Let a be a state in G with $\ell(a) = 0$. By the proof of *(2) ⇒ (3)* of Theorem 4.4 and Remark of Theorem 3 in [23], we can assume that $\phi(a) = 0$. Define $h : \mathcal{S}(G) \to \mathcal{S}(G_{f,d}^A)$ by

$$h(b) = (\phi(b), \gamma(b), \ell(b) \bmod n) \qquad \text{for } b \in \mathcal{S}(G).$$

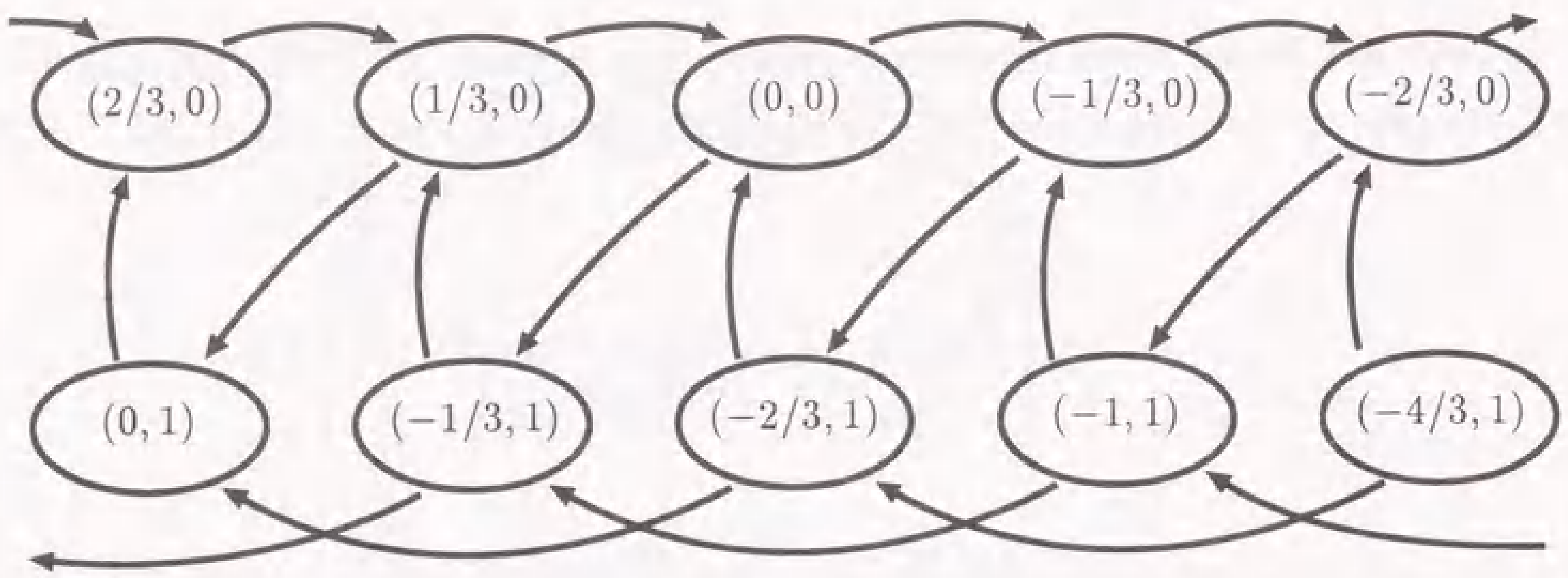


Figure 4.13: Canonical Graph $G^{\{0,1\}}_{0,1/2}$.

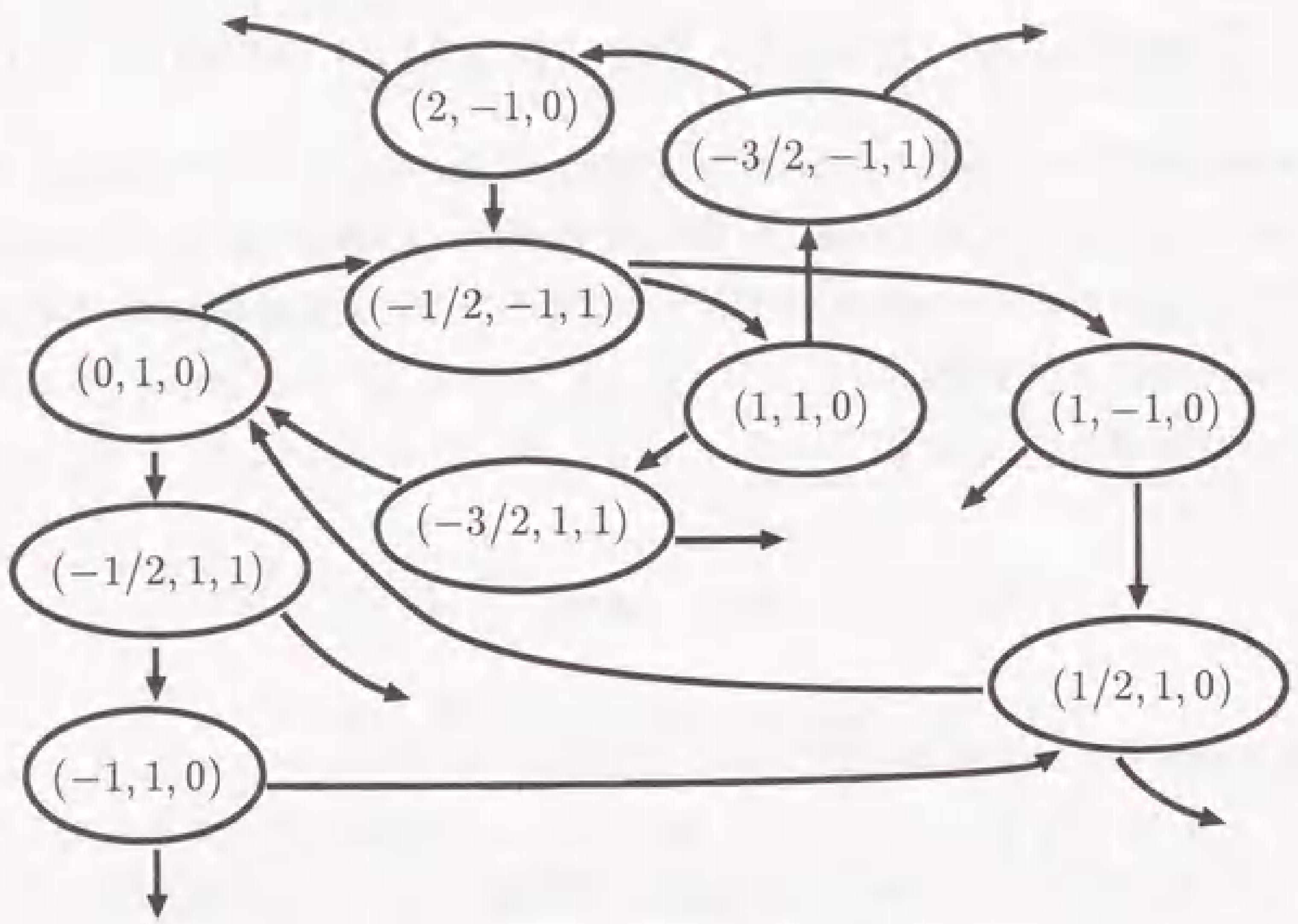


Figure 4.14: Canonical Graph $G^{\{-1,1\}}_{1/2,1/2}$.

Then, we note that h is a label-preserving homomorphism from G to $G^A_{f,d}$. Hence, we get the following proposition:

Proposition 4.3 Let A be a finite subset of $\boldsymbol{C}$. Let k be a nonnegative integer and n a positive integer with $\gcd(k, n) = 1$. Let $d \in \boldsymbol{C}$. Put $f = kf_S/n$ and $z = \exp(-\mathrm{i}2\pi k/n)$. Then

(1) there is a label-preserving graph isomorphism between G^A_{f,d_1} and G^A_{f,d_2} for $d_1 = z^{i_1}d$ and $d_2 = z^{i_2}d$ with $0 \le i_1, i_2 \le n-1$;

(2) for every irreducible FSTD (G, γ) which satisfies a biased coboundary condition with respect to d at f and whose period is a multiple of n, there is a label-preserving graph homomorphism from G to $G^A_{f,d}$.

Example 4.6 Fig. 4.13 shows a part of $G^{\{0,1\}}_{0,1/2}$ and Fig. 4.14 shows a part of $G^{\{-1,1\}}_{1/2,1/2}$. □

Let (G, γ) be an encoder (an FSTD) which satisfies the condition of Theorem 4.6 at kf_S/n and assume that we encode the source message by (G, γ) and then record or send the encoded message. Since each reproduced signal has a spectral line at kf_S/n and no frequency content near kf_S/n (Theorem 4.5), it may be possible to extract, from the reproduced signal, a servo signal of frequency kf_S/n which is better (e.g., has smaller phase jitter) than that extracted in the coding schema based on Theorem 4.4 stated at the end of the previous section.

4.6 Summary

We have considered a problem of characterizing encoders such that the encoded message emitted from those encoders has some amount of information about clock, independent of the source statistics. We assume that f is a submultiple of the symbol frequency. We have presented the relations between the spectral line and the RDS_f per symbol of a cycle. Next we have given necessary and sufficient conditions for the encoder to have a spectral line of amplitude not less than a given positive value at f, independent of the source statistics. We have defined a spectral density null at f and a biased coboundary condition with respect to a complex number at f, and proved that they are equivalent. We also have presented some necessary and sufficient conditions for the encoder in order that the amplitude of the spectral line is equal to a given positive value at f. We have investigated FSTD's which have some spectral lines at some frequencies simultaneously. Those results can be applied to the design of transmission or recording systems in which we need to retrieve information about clock only from the received or reproduced message.

Chapter 5

Higher Order Spectral Density Nulls and Spectral Lines

5.1 Introduction

A spectral null has been extended to a higher order spectral null by several authors [25,26,28,27,24,29]. We shall show the spectral density null also can be extended to a higher order spectral density null.

First we shall prepare terminologies and notation, and present basic lemmas. In Section 5.3 we shall give characterizations of a higher order spectral density null at f (Theorem 5.1). We shall derive a lower bound on the Euclidean distance of codes with an integer alphabet having a higher order spectral density null (Theorem 5.1). In Section 5.4, we shall define the canonical graph for a higher order biased coboundary condition at f (Definition 5.4), and explain how to construct the code having both a higher order spectral density null and a nonzero spectral line from the canonical graph.

The main result of this chapter, Theorem 5.1, also appears in [24], exactly speaking, the final version of [24], which the author of this thesis could see after he had done his work. His work was done (in fact, he first presented that result in [54]) with reference to only the earlier version of [24] which does not include Theorem 5.1.

5.2 Preliminaries

Let G be a finite directed graph and let γ be a complex valued function on $\mathcal{S}(G)$. Let N be the period of G. As stated in Section 4.2, $\mathcal{S}(G)$ is partitioned into N subsets, $B_0, B_1, \ldots, B_{N-1}$ such that Eq. (4.1) holds. Let ℓ be the indexing of $\mathcal{S}(G)$. Assume that n is a divisor of N. Then, by Lemma 4.3, we have

$$\exp(-i2k\pi(\ell(\sigma)+1)/n) = \exp(-i2k\pi(\ell(\tau))/n), \qquad \text{for every edge } (\sigma,\tau) \text{ in } G. \quad (5.1)$$

Let $P \in \Pi_G$. If the power spectral density of (G, P, γ) and its first $2K-1$ derivatives vanish at f, then (G,γ) is said to have an *order-K spectral density null at f*. If (G,γ) has an order-K spectral density null at f and (G, P, γ) has no spectral line at f, then (G,γ) is said to have an *order-K spectral null at f*.

We use the following terminologies from [24].

Definition 5.1 Let $\boldsymbol{x} = x_0 \cdots x_{L-1}$ be a sequence of complex numbers. The *order-K running-digital-sum* of $\boldsymbol{x}$ at f, denoted $\mathrm{RDS}_f^{(K)}(\boldsymbol{x})$, is defined recursively by

$$\begin{aligned} \mathrm{RDS}_f^{(0)}(\boldsymbol{x}) &= \omega^{L-1} x_{L-1} \\ \mathrm{RDS}_f^{(j)}(\boldsymbol{x}) &= \sum_{m=0}^{L-1} \mathrm{RDS}_f^{(j-1)}(x_0 \cdots x_m), \qquad j \geq 1. \end{aligned}$$

The *order-K moment of $\boldsymbol{x}$ at f*, denoted $M_f^{(K)}(\boldsymbol{x})$, is defined by

$$M_f^{(K)}(\boldsymbol{x}) = \sum_{i=0}^{L-1} i^K \omega^i x_i$$

where we put $0^0 = 1$.

Definition 5.2 We say that an FSTG (G,γ) satisfies an *order-K biased coboundary condition* at the frequency f if there are functions $\phi_j : \mathcal{S}(G) \to \boldsymbol{C}$, $j = 1, 2, \ldots, K$, and complex numbers $d_1, d_2, \ldots, d_K \in \boldsymbol{C}$ such that for every edge (σ,τ) in G and for $j = 0, 1, \ldots, K-1$

$$\phi_j(\sigma) = \omega\phi_{j+1}(\tau) - \phi_{j+1}(\sigma) + \omega^{-\ell(\sigma)} d_{j+1}, \qquad (5.2)$$

where $\phi_0 = \gamma$. The function ϕ is called an order-K biased coboundary function at f.

When $d_1 = d_2 = \cdots = d_K = 0$, we say that (G,γ) satisfies an order-K coboundary

condition at f (Definition 3 of [24]). The order-K biased coboundary condition is considered a generalization of the order-K coboundary condition, or the biased coboundary condition Definition'4.2(Definition 4 of [30]), which is the order-1 biased coboundary condition.

Remark **5.1** In the above definition of the order-K biased coboundary condition, if n is a divisor of the period of G, then we can take $\phi_1, \phi_2, \ldots, \phi_K$ so that $d_2 = d_3 = \cdots = d_K = 0$ as follows: Define $\phi'_1, \phi'_2, \ldots, \phi'_{K-1}$ by $\phi'_j(\sigma) = \phi_j(\sigma) - \omega^{-\ell(\sigma)} d_{j+1}$ for $\sigma \in \mathcal{S}(G)$. Then, from (5.1), we have that for every edge (σ,τ) in G and for $j = 1, 2, \ldots, K$

$$\begin{aligned} \omega\phi'_j(\tau) - \phi'_j(\sigma) &= \omega\left(\phi_j(\tau) - \omega^{-\ell(\tau)} d_{j+1}\right) - \left(\phi_j(\sigma) - \omega^{-\ell(\sigma)} d_{j+1}\right) \\ &= \omega\phi_j(\tau) - \phi_j(\sigma) \\ &= \phi_{j-1}(\sigma) - \omega^{-\ell(\sigma)} d_j = \phi'_{j-1}(\sigma). \end{aligned}$$

□

5.3 Higher order spectral density nulls

In this section, we characterize FSTG's that have order-K spectral density nulls at rational submultiples of the symbol frequency. Throughout of the remainder of this chapter, let K be a positive integer.

Theorem 5.1 Let (G,γ) be an irreducible FSTG with an alphabet consisting of complex numbers. Then the following statements are equivalent:

(i) For every $P \in \Pi_G$

$$\Phi^{(j)}(f) = 0, \qquad j = 0, 1, \ldots, 2K-1,$$

where $\Phi(f)$ is the power spectral density of (G, P, γ) and $\Phi^{(j)}(f)$ is the j-th derivative of $\Phi(f)$;

(ii) (G,γ) satisfies an order-K biased coboundary condition at f;

(iii) For every difference cycle $\boldsymbol{e}$ of length a multiple of n,

$$\mathrm{RDS}_f^{(j)}(\boldsymbol{e}) = 0, \qquad j = 1, 2, \ldots, K;$$

(iv) For every difference cycle $\boldsymbol{e}$ of length a multiple of n,

$$M_f^{(j)}(\boldsymbol{e}) = 0, \qquad j = 0, 1, \ldots, K-1.$$

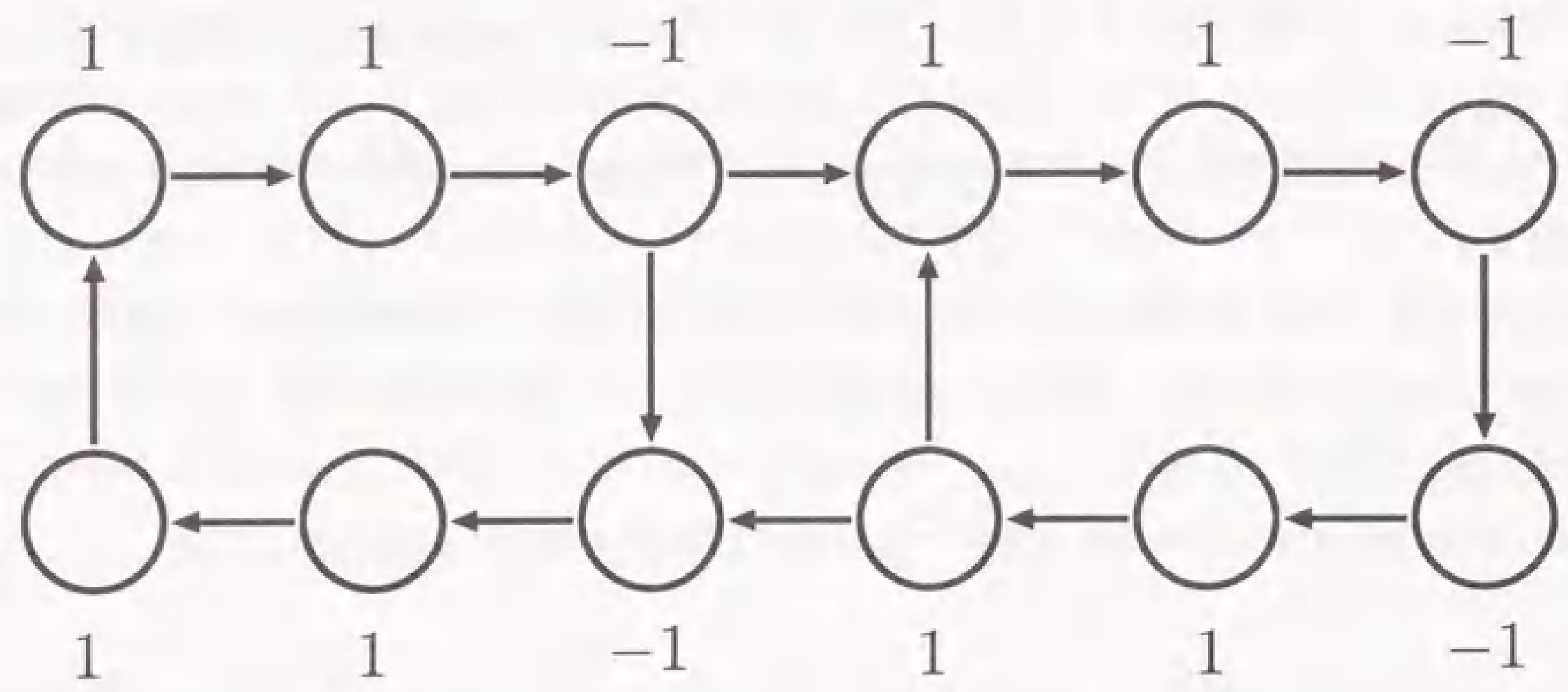


Figure 5.1: FSTG with second order biased coboundary condition.

Proof: See Appendix A.9. □

By this theorem we are justified in saying that an FSTG (or a code) has an order-K spectral density null at f.

Example 5.1 Let (G,γ) be the FSTG given in Figure 5.1. It is easy to see that for any cycle $\boldsymbol{s}$ of G

$$\frac{\mathrm{RDS}_0(\gamma(\boldsymbol{s}))}{\lg(\boldsymbol{s})} = \frac{1}{3}$$

Define a function $\psi : \mathcal{S}(G) \to R$ by $\psi(A) = 0$ and

$$\psi(\tau) = \gamma(\sigma) + \psi(\sigma) - \frac{1}{3}$$

for every edge (σ,τ). Although G contains cycles, the above definition of ψ does not give rise to contradiction. Hence (G,γ) satisfies a biased coboundary condition at dc with a biased term $1/3$. It is shown that for any cycle $\boldsymbol{s}$ of G, $\mathrm{RDS}_0(\psi(\boldsymbol{s})) = 0$. Therefore, (G,ψ) has a coboundary function at dc. Thus (G,γ) satisfies a order-2 biased coboundary condition at dc.

Remark **5.2** Let (G,γ) be an FSTG satisfying an order-K biased coboundary condition at f. Let $\phi_1,\phi_2,\ldots,\phi_K$ be biased coboundary functions and $d_1,d_2,\ldots,d_K$ biased terms. Assume that n is not a divisor of the period of G. For every $P \in \Pi_G$, (G,P,γ) has no spectral line at f and, hence, (G,P,γ) has an order-K spectral null at f by Theorem 5.1. Hence (G,γ) has order-K coboundary functions by Theorem 4 of [29] or Theorem 2 of [24]. Thus, by this and Remark 5.1, if (G,γ) satisfies an order-K biased coboundary condition then we can take coboundary functions and biased terms such that $d_2 = d_3 = \cdots = d_K = 0$.

For block codes, Theorem 5.1 is described as follows:

Corollary 5.1 Let $\mathcal{C}$ be a fixed length block code of length a multiple of n. Then the following conditions are equivalent:

(i) For every code words $\boldsymbol{x},\boldsymbol{y} \in \mathcal{C}$,

$$M_f^{(j)}(\boldsymbol{x}) = M_f^{(j)}(\boldsymbol{y}), \qquad j = 0,1,\ldots,K-1;$$

(ii) For every code words $\boldsymbol{x},\boldsymbol{y} \in \mathcal{C}$

$$\mathrm{RDS}_f^{(j)}(\boldsymbol{x}) = \mathrm{RDS}_f^{(j)}(\boldsymbol{y}), \qquad j = 1,2,\ldots,K;$$

(iii) For every independent identically distributed process,

$$\Phi^{(j)}(f) = 0, \qquad j = 0,1,\ldots,2K-1,$$

where $\Phi(f)$ is the power spectral density of the code driven by the process.

Proof: Apply Theorem 5.1 to the FSTG determined by $\mathcal{C}$ as shown by Fig. 6 of [23], noting that there is only one state that has more than one outgoing edges in the FSTG. □

Let C be a block code of length L. Under the following two assumptions:

(**R**1) the codewords are equiprobable;

(**R**2) for all i the number of codewords $\boldsymbol{x} = (x_1,x_2,\ldots,x_n)$ with $x_i = 1$ is equal to the number of codewords with $x_i = -1$,

Immink and Beenker showed that if

$$M_0^{(j)}(\boldsymbol{x}) = 0, \qquad \text{for every } \boldsymbol{x} \in C \text{ and } j = 0,1,\ldots,K-1, \tag{5.3}$$

then the code has an order-K spectral null at dc [26]. As they pointed out, the restriction (**R**1) of the above statement can be relaxed. It is noticed that the restriction (**R**2) can also be omitted. In fact, referring [20], it is shown that if a code is driven by a sequence of independent identically distributed symbols, (5.3) is a sufficient condition for the block code to have an order-K spectral null at dc. On the other hand, putting

$$\begin{array}{rcl} \boldsymbol{x} &=& (\ \ 1\ -1\ \ 1\ -1\ -1\ -1\ \ \ 1\ \ 1\ ) \\ \boldsymbol{y} &=& (\ -1\ \ \ 1\ \ 1\ -1\ -1\ \ \ 1\ -1\ \ 1\ ), \end{array}$$

we have $\mathrm{RDS}_0(\boldsymbol{x}) = \mathrm{RDS}_0(\boldsymbol{y}) = 0$ and $\mathrm{RDS}_0^{(2)}(\boldsymbol{x}) = \mathrm{RDS}_0^{(2)}(\boldsymbol{y}) = -2$. Hence the spectrum of the code consisting of $\boldsymbol{x}$ and $\boldsymbol{y}$ has an order-2 spectral null at dc by Corollary 5.1 and Theorem 3 of [23]. This implies that (5.3) is a proper sufficient condition for a code to have an order-K spectral null at dc.

Immink and Beenker derived also a lower bound on the minimum Hamming distance of higher order spectral null codes with symbols ± 1 [26]. Karabed and Siegel have extended it to integer alphabets at frequencies of rational submultiple kf_S/n of the symbol frequency as follows.

Theorem 5.2 *(Karabed and Siegel[24])* Let $\boldsymbol{x} = x_0 x_1 \dots x_{L-1}$ and $\boldsymbol{y} = y_0 y_1 \cdots y_{L-1}$ be distinct sequences of integers. If $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfy

$$\sum_{i=0}^{n} i^j \omega^i (x_i - y_i) = 0, \qquad j = 0, 1, \dots, K-1,$$

then

$$d^2(\boldsymbol{x}, \boldsymbol{y}) \geq 2K$$

where $d(\boldsymbol{x}, \boldsymbol{y})$ is the Euclidean distance of $\boldsymbol{x}$ and $\boldsymbol{y}$.

From this and Theorem 5.1, it immediately follows that:

Proposition 5.1 Let (G, γ) be an irreducible FSTG with an integer alphabet. Assume that (G, γ) has an order-K spectral density null at f. Let $\boldsymbol{s} = s_0 s_1 \cdots s_{L-1}$ and $\boldsymbol{t} = t_0 t_1 \cdots t_{L-1}$ be distinct cycles of (G, γ) with $s_0 = t_0$. Then

$$d^2(\gamma(\boldsymbol{s}), \gamma(\boldsymbol{t})) \geq 2K.$$

5.4 Canonical graphs for higher order spectral density nulls

Spectral null constraints and (d, k)-constraints are frequently used in channel coding. Sets of (d, k)-sequences are represented by FSTG's, and each of the FSTG's plays an important role in constructing a code for the constraint represented by the FSTG [5]. Spectral null constraints, however, can not be represented by any FSTG. In [23] Siegel and Marcus introduced transition graphs, which are called canonical graphs, with a countable number of states for spectral null constraints. They are useful in constructing codes with spectral nulls. Canonical graphs were extended in [29], [24] [28] to higher order spectral null constraints, and in the previous chapter [30] to spectral density null and nonzero spectral lines. In this section, we give canonical graphs for higher order spectral density nulls and nonzero spectral lines. The canonical graphs explain the structure of FSTG's having those constraints in a different way from the way in which the results in the previous section do. Then we discuss the code construction problem, using the canonical graphs and the results in Section 3.

Definition 5.3 Let $d_1, d_2, \dots, d_K$ be complex numbers with $d_1 \neq 0$. A countable-state transition graph $\mathcal{G}$ is *canonical for a spectral density null at f with constants $d_1, d_2, \dots, d_K$* if:

(I) Every irreducible finite subgraph of $\mathcal{G}$ has an order-K spectral density null and a spectral line of amplitude $|d_1|$ at f;

(II) Let (G, γ) be an FSTG (G, γ) having an order-K spectral density null and a nonzero spectral line at f. Let ℓ be the indexing of $\mathcal{S}(G)$ by period. Let $\phi'_1, \phi'_2, \dots, \phi'_K$ be biased coboundary functions and $d'_1, d'_2, \dots, d'_K$ biased terms. Assume that $d'_1 = d_1$ and there is a state σ_0 with $\ell(\sigma_0) = 0$ such that $\phi'_j(\sigma_0) + d'_{j+1} = d_{j+1}$ for $j = 1, 2, \dots, K-1$. Then there is a label preserving graph homomorphism from (G, γ) to $\mathcal{G}$.

Definition 5.4 Let $\mathcal{A}$ be an alphabet, i.e., a finite subset of complex numbers. Let $d_1, d_2, \dots, d_K$ be complex numbers with $d_1 \neq 0$ and put $\boldsymbol{d} = (d_1, d_2, \dots, d_K)$. We define $h^f_{\boldsymbol{d}} : C^K \times \mathcal{A} \times \{0, 1, \dots, n-1\} \times \mathcal{A} \to C^K \times \mathcal{A} \times \{0, 1, \dots, n-1\}$ by

$$\begin{array}{rcl} (s_1, s_2, \dots, s_K, a, j, b) & \mapsto & \big(\omega^{-1}(s_1 + a - \omega^{-j} d_1), \omega^{-1}(s_2 + s_1 - \omega^{-j} d_2), \dots, \\ & & \omega^{-1}(s_K + s_{K-1} - \omega^{-j} d_K), b, (j+1) \bmod n\big). \end{array}$$

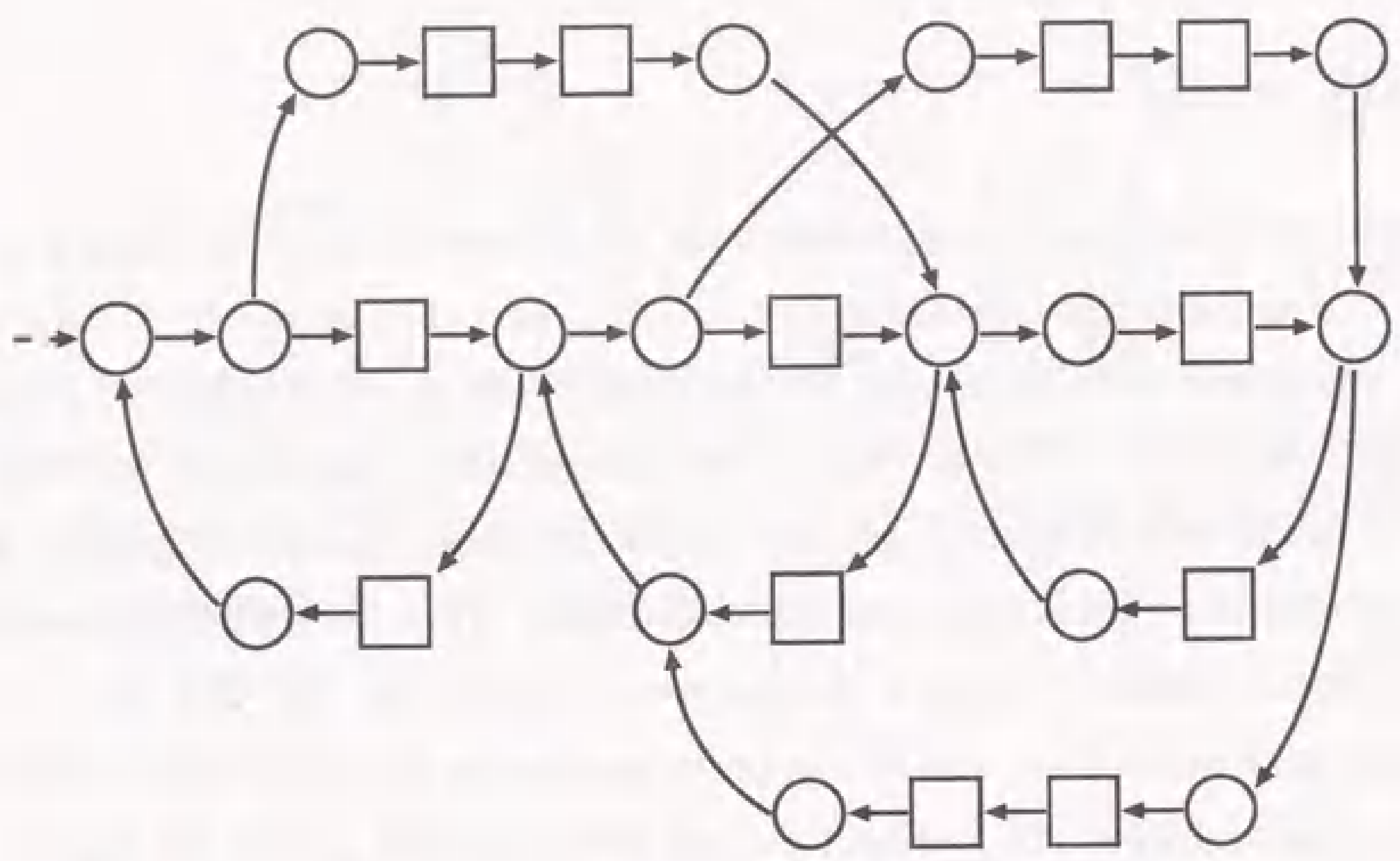


Figure 5.2: $G^0_{(1/3,1/3)}$

We define a function T by

$$T(B) = \{h^f_{\boldsymbol{d}}(c,a) | c \in B, a \in \mathcal{A}\}, \qquad \text{for } B \subset C^K \times \mathcal{A} \times \{0,1,\ldots,n-1\}.$$

We let $G^f_{\boldsymbol{d}}$ denote the countable-state transition graph defined by:

- The state set is $\cup_{i=0}^{\infty} T^i(\{(0,0,\ldots,0,a) | a \in \mathcal{A}\})$.
- The label of a state $(s_1, s_2, \ldots, s_K, a, j)$ is a.
- There is an edge from σ to τ if and only if $h^f_{\boldsymbol{d}}(\sigma, a) = \tau$ where a is the label of τ.

Then we have:

Proposition 5.2 Let $d_1, d_2, \ldots, d_K$ be complex numbers with $d_1 \neq 0$. Then $G^f_{d_1,d_2,\ldots,d_K}$ is canonical for an order-K spectral density null at f with constants $d_1, d_2, \ldots, d_K$.

Proof: See Appendix A.4. □

Example 5.2 Canonical graphs for spectral density nulls. A part of $G^0_{(1/3,1/3)}$ and a part of $G^{f_S/2}_{(1/2,0)}$ are shown in Figure 5.2 and Figure 5.3, respectively. In both graphs, circles and squares represent states labeled $+1$ and states labeled -1, respectively.

Assume that we need a code such that encoded sequences have a higher order spectral density null and a nonzero spectral line at a specified frequency. Given a word length L a multiple of n, we can get a block code by collecting words of length L

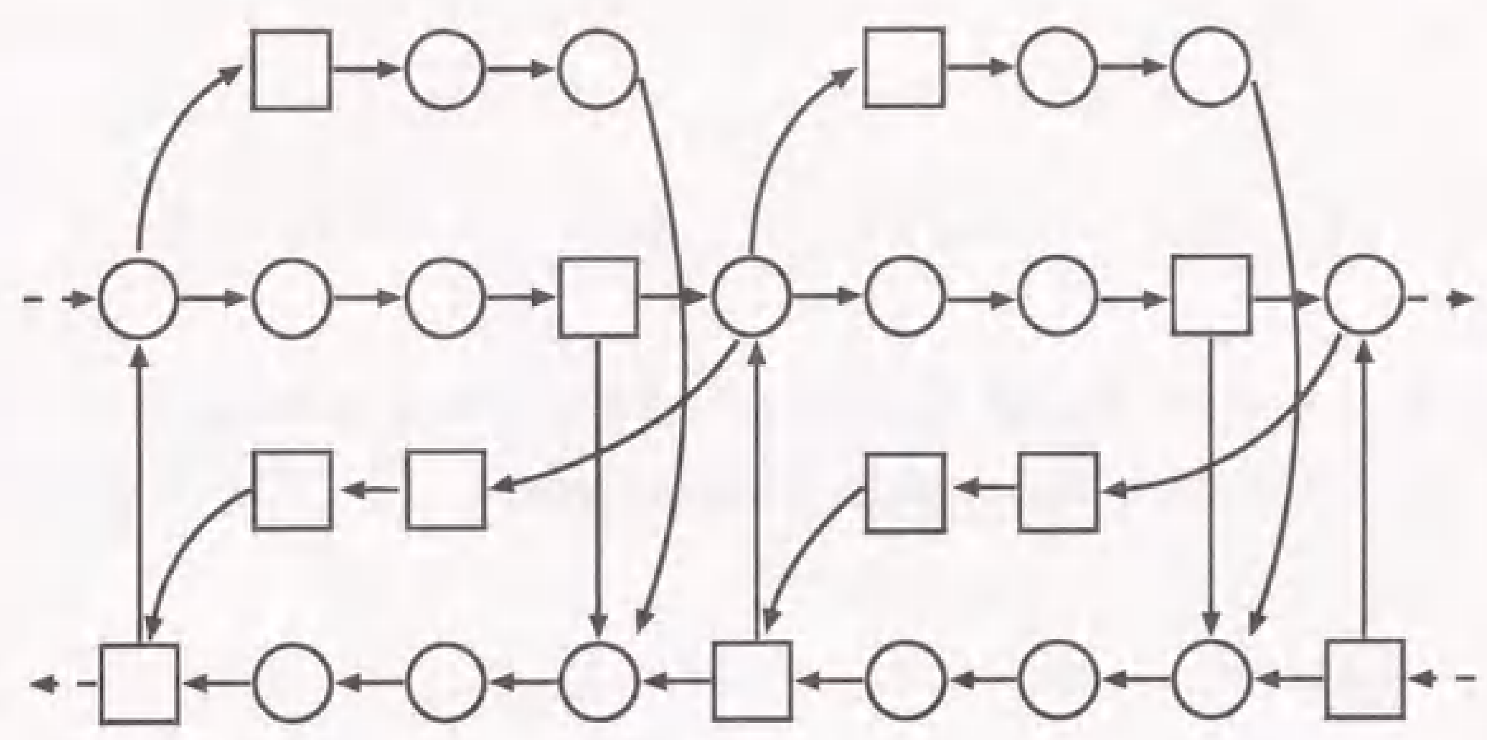


Figure 5.3: $G^{f_S/2}_{(1/2,0)}$

satisfying the condition (i) (or (ii)) in Corollary 5.1. We can get also a finite-state code or a variable length code, based on the canonical graphs for the spectral density null constraint, as follows:

(I) Choose $d_1(\neq 0), d_2, \ldots, d_K$ such that $G^f_{d_1,d_2,\ldots,d_K}$ contains irreducible finite subgraphs;

(II) Fix an irreducible FSTG (G,γ) contained in $G^f_{d_1,d_2,\ldots,d_K}$;

(III) Construct a code by applying any one of methods in [7], [5] (and [11]), [37],[45] to (G,γ).

In each step of (I), (II) and (III) there remain many options which we have to choose according to other requirements to the code. Suppose that a code is constructed by (I), (II) and (III). Let (G,γ) be an FSTG fixed in (II). Let T be the adjacency matrix of G, i.e., the matrix of size $\#\mathcal{S}(G) \times \#\mathcal{S}(G)$ such that for $\sigma,\tau \in \mathcal{S}(G)$, $T(\sigma,\tau) = 1$ if there is an edge from σ to τ and $T(\sigma,\tau) = 0$ otherwise. Then the rate of the code can not exceed $\log_2 \lambda$, where λ is the maximal (real) eigenvalue of T. But for each $\epsilon > 0$ the code construction algorithm, which was given in [5] and was modified in [11], can give a code whose rate is greater than $\log_2 \lambda - \epsilon$, at the expense of complexity of the encoder.

In some cases the code may be required to contain a given code word $\boldsymbol{s} = s_0 s_1 \cdots s_L$. Padding $\boldsymbol{s}$ with some word, if necessary, we assume that the length of $\boldsymbol{s}$ is a multiple of n. We can construct a desired code by using the above methods because we can find a canonical graph which contains a cycle generating $\boldsymbol{s}$ as follows: Put $L = \lg(\boldsymbol{s})$ and $\phi_0(0) \cdots \phi_0(L-1) = \boldsymbol{s}$. Determine d_{j+1} by

$$d_{j+1} = \frac{\mathrm{RDS}_f(\phi_j(0)\phi_j(1)\cdots\phi_j(L-1))}{L}$$

and ϕ_{j+1} by

$$\phi_{j+1}(m+1) = \overline{\omega}(\phi_j(m) + \phi_{j+1}(m) - \omega^{-m} d_{j+1}), \quad m = 0, 1, \ldots, L-1$$

for $j = 0, 1, \ldots, K-1$ where we put $\phi_{j+1}(0) = 0$. Since L is a multiple of n, we have $\phi_{j+1}(L) = \phi_{j+1}(0)$. Let $\boldsymbol{d} = (d_1, d_2, \ldots, d_K)$. Then we note

$$\begin{aligned} & h^f_{\boldsymbol{d}}(\phi_1(m), \ldots, \phi_K(m), s_m, m \bmod n, s_{m+1}) \\ & \quad = (\phi_1(m+1), \ldots, \phi_K(m+1), s_{m+1}, m+1 \bmod n). \end{aligned}$$

Thus $G^f_{\boldsymbol{d}}$ is a desired canonical graph.

5.5 Summary

We have defined an order-K spectral density null at f and an order-K biased coboundary condition at f, and proved that they are equivalent. This characterization is an extension of the characterization of given by Theorem 4.5 about a first order spectral density null. Other characterizations of the order-K spectral density null also have been obtained. We have given a lower bound on the minimum Euclidean distance of a code having an order-K spectral density null. Canonical graphs for higher order spectral density nulls with nonzero spectral lines have been defined. These graphs can be used in constructing codes with higher order spectral density nulls and nonzero spectral lines.

Chapter 6

Irreducible Components of Canonical Graphs for Spectral Nulls

6.1 Introduction

In this chapter we shall consider a problem of identifying all irreducible components of canonical graphs.

Marcus and Siegel[23] introduced canonical graphs for a spectral null constraint at a frequency f and showed a few properties about the structure of them. The graphs are infinite graphs with labeled edges having the following properties: 1) every finite-state code is label-preserving graph homomorphic to the finite subgraph of a canonical graph for a spectral null at f; 2) every finite subgraph of the canonical graph for the spectral null at f has a spectral null at f. We may consider that each irreducible component of canonical graphs represents a class of spectral null codes because of the property 1). We can use canonical graphs in the design of spectral null codes because of the property 2). For example, we can construct a finite-state spectral null code from an irreducible finite subgraph of the canonical graph by using code construction methods presented in [11], [37], [5]. Therefore it is a fundamental problem for the coding theory for constrained channels to investigate the structure of canonical graphs.

Since canonical graphs are infinite directed graphs, and generally consist of disjoint irreducible components which are also infinite directed graphs, the problem is very different from the identification problem for finite directed graphs. We shall show that if n is a prime number or a double of a prime number then we can identify all irreducible components of canonical graphs for first-order spectral null at $f_S k/n$ (Theorem 6.1 and Theorem 6.2). We have defined a canonical graph for a spectral density null at a specified frequency f with a nonzero spectral line in Chapter 4 and 5. However, since the period of every irreducible component of the canonical graph is a multiple of n, there is essentially only one irreducible component, that is, all irreducible components of the canonical graph are label-preserving graph isomorphic to one another (Remark 6.4). Therefore we shall consider the problem of identifying the irreducible components of the canonical graphs only for spectral nulls here.

In this chapter we shall consider directed graphs with labeled edges as models of constrained channels. In section 6.2, we shall give notation, background, and an example that explains our motivation. In section 6.3, irreducible components of canonical graphs for first-order spectral nulls at f are specified in terms of sequences generated by cycles. Canonical graphs for higher order spectral nulls were introduced by Monti and Pierobon[28], Karabed and Siegel[24], and Eleftheriou and Cideciyan[29]. We shall also identify irreducible components of canonical graphs for higher order spectral nulls at dc (Theorem 6.4) in Section 6.4.

6.2 Preliminaries

6.2.1 Notation and background

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be blocks. We mean the block consisting of only one digit by the boldface character of the digit, e.g., $\mathbf{1}$. If there are blocks $\boldsymbol{c}_1, \boldsymbol{c}_2$ such that $\boldsymbol{a} = \boldsymbol{c}_1 \cdot \boldsymbol{b} \cdot \boldsymbol{c}_2$, we say that $\boldsymbol{b}$ appears in $\boldsymbol{a}$ where $\boldsymbol{c}_1$ or $\boldsymbol{c}_2$ may be empty blocks. Put $\boldsymbol{a} = a_0 a_1 \cdots a_{L-1}$. We define $\boldsymbol{a}^{[i]}$ and $\boldsymbol{a}^i$ by

$$\begin{aligned} \boldsymbol{a}^{[i]} &= a_i a_{i+1} \cdots a_{L-1} a_0 \cdots a_{i-1}, \\ \boldsymbol{a}^i &= \underbrace{\boldsymbol{a} \cdot \boldsymbol{a} \cdots \boldsymbol{a}}_{i \text{ times}}. \end{aligned}$$

Definition 6.1 Let $\mathcal{A}$ be an alphabet. By $S_{\mathcal{A}}$ we mean the set of all non-empty blocks consisting of symbols in $\mathcal{A}$,
Assume that $\boldsymbol{a} \in Z_{\mathcal{A}}$. We define $-\boldsymbol{a} = a'_0 a'_1 \cdots a'_{L-1}$ by $a'_i = -a_i$ for each i. Let (G_1, γ_1) and (G_2, γ_2) be state-transition diagrams. If (G_1, γ_1) is label-preserving graph isomorphic to (G_2, γ_2) then we write $(G_1, \gamma_1) \cong (G_2, \gamma_2)$.

If j is divisible by i, we write $i \mid j$. If $n \mid (i-j)$, we write $i \equiv j \bmod n$. A block $\boldsymbol{a}$ is said to be *generated* by $\boldsymbol{x}$ if $\gamma(\boldsymbol{x}) = \boldsymbol{a}$.

The next corollary follows from the proof of Theorem 3[23] and Proposition 1[23]:

Corollary 6.1 Let (G, γ) be an irreducible state-transition-diagram. Let N be the period of G. Suppose that (G, γ) satisfies a coboundary condition at f.

1) If $n \not| N$, then the coboundary function ψ of G is determined uniquely: for every cycle $\boldsymbol{x}$ of G with $n \not| \lg(\boldsymbol{x})$ we have

$$\psi(i(\boldsymbol{x})) = \frac{\mathrm{RDS}_f(\gamma(\boldsymbol{x}))}{\omega^{\lg(\boldsymbol{x})} - 1};$$

2) Suppose that $n | N$. Let $B_0, B_1, \ldots, B_{N-1}$ be the partition of the set of states in G by period. For state σ let $\ell(\sigma)$ be the number such that $\sigma \in B_{\ell(\sigma)}$. Then the coboundary function ψ of G is determined up to constant, that is, for every constant c the map ψ' defined by $\psi'(\sigma) = \psi(\sigma) + \omega^{-\ell(\sigma)} c$ for state σ in G is also a coboundary function of G; conversely, if $\tilde{\psi}$ is a coboundary function of G then there is a constant c such that $\psi(\sigma) - \tilde{\psi}(\sigma) = \omega^{-\ell(\sigma)} c$ for every state σ in G.

Throughout this chapter we employ the following assumption.

Assumption We assume that the channel symbol alphabet is $\{-1, 1\}$.
That is, we only consider state-transition diagrams whose edges are labeled 1 or -1. Although results in this chapter strongly depend on this assumption, the assumption is satisfied in applications of practical interest.

If the coefficient of the leading term of a polynomial is 1 then the polynomial is said to be monic. Let $Z[\omega]$ be the set of polynomials of ω with integer coefficients. The minimal polynomial $\Phi_n(x)$ of $\exp(-2\pi\mathrm{i}/n)$ is the monic polynomial of least order in $Z[\exp(-2\pi\mathrm{i}/n)]$ with $\Phi_n(\exp(-2\pi\mathrm{i}/n)) = 0$.

Definition 6.2 We denote the minimal polynomial of $\exp(2\pi\mathrm{i}/n)$ by $\Phi_n(x)$.
For integer l with $\gcd(l, n) = 1$, $\Phi_n(x)$ is also the minimal polynomial of $\exp(-2\pi\mathrm{i}l/n)$. Let $\phi(n)$ be the Euler number of n (the number of integers less than and prime to n). The order of the minimal polynomial of ω is $\phi(n)$. Hence $Z[\omega]$ can be regarded as the set of polynomials with integer coefficients of order less than $\phi(n)$. Then $Z[\omega]$ is isomorphic to $Z(x)/(\Phi_n(x))$ where $Z(x)$ is the ring consisting of polynomials with integer coefficients. We can calculate $\Phi_n(x)$ inductively. If n is a prime number then $\Phi_n(x) = 1 + x + \cdots x^{n-1}$. We also have $\Phi_4(x) = x^2 + 1$ and $\Phi_6(x) = x^2 - x + 1$. For every $m > 0$ the polynomial $\Phi_m(x)$ is an irreducible polynomial, that is, $\Phi_m(x)$ can not be represented as any product of the polynomials of order greater than 0. For these subjects see [55, Chapter 3, Section 6.6, Section 8.4].

The following graph was introduced by Ashley in [23] as an extension of G_p^f in [23].

Definition 6.3 [23, p. 565] By $g_f(\omega)$ we mean the least common multiple of polynomials $\omega^l - 1$, $l = 1, 2, \ldots, n-1$. We define a function $h_f : Z[\omega] \times \{-1, 1\} \to Z[\omega]$ as follows

$$h_f(\sigma, b) = \omega^{-1}(\sigma + g_f(\omega) b) \qquad \text{for } \sigma \in Z[\omega] \text{ and } b \in \{-1, 1\}. \tag{6.1}$$

Let G_f be a labeled graph such that the state set of G_f is $Z[\omega]$ and there is an edge from σ to τ with label b if and only if $\tau = h_f(\sigma, b)$.

Remark **6.1** Assume that $n > 1$. Let σ and τ be states of G_f and assume that there is a path from σ to τ. Let $\boldsymbol{u} \in S_{\{-1,1\}}$ be a block generated by the path. Assume that $\lg(\boldsymbol{u})$ is divisible by n. Applying (6.1) to each transition on the path, we note that τ is given by

$$\tau = \begin{cases} \sigma + \mathrm{RDS}_f(\boldsymbol{u}) & \text{if } p = 0; \\ \sigma + g_f(\omega)\mathrm{RDS}_f(\boldsymbol{u}) & \text{if } 1 \le p \le n-1. \end{cases}$$

Hence, if $\mathrm{RDS}_f(\boldsymbol{u}) = 0$ then $\tau = \sigma$. Since $\mathrm{RDS}_f(\mathbf{1}^n) = 0$, every irreducible component of G_f has a cycle of length n. Therefore for every irreducible component the period of the component is not greater than n. Assume that $\sigma \ne \tau$. Let $\boldsymbol{v} \in S_{\{-1,1\}}$ be a block such that $n \mid \lg(\boldsymbol{u} \cdot \boldsymbol{v})$. Then $\mathrm{RDS}_f(\boldsymbol{u} \cdot \boldsymbol{v} \cdot -\boldsymbol{u} \cdot -\boldsymbol{v}) = 0$. By definition, there is a path that starts from τ and generates $\boldsymbol{u} \cdot \boldsymbol{v} \cdot -\boldsymbol{u} \cdot -\boldsymbol{v}$. Thus σ and τ are in the same irreducible component of G_f.

Remark **6.2** Let $\boldsymbol{s}$ be a block with $n \not| \lg(\boldsymbol{s})$. Let $\sigma = g_f(\omega)\mathrm{RDS}_f(\boldsymbol{s})/(\omega^{\lg(\boldsymbol{s})} - 1)$. From the definition of g_f we note that $\sigma \in Z[\omega]$. By calculation we can see that there is a cycle in G_f which starts from σ and generates $\boldsymbol{s}$. We also note that there is a cycle in G_f which starts from $-\sigma$ and generates $-\boldsymbol{s}$.

The following is a modification of Proposition 2[23].

Proposition 6.1 Let l_1 and l_2 be integers which are prime to n. Then $G_{f_S l_1/n} \cong G_{f_S l_2/n}$.

Proof: Let $\omega_1 = \exp(-2\pi i l_1/n)$ and $\omega_2 = \exp(-2\pi i l_2/n)$. Let $f_1 = f_S l_1/n$ and $f_2 = f_S l_2/n$. It follows from assumption that there is an isomorphism F_1 between $Z[\omega_1]$ and $Z(x)/(\Phi_1(x))$ such that for every polynomial $g(x) \in Z(x)/(\Phi_n(x))$ we have $F_1(g(x)) = g(\omega_1)$. There is also an isomorphism between $Z[\omega_2]$ and $Z(x)/(\Phi_n(x))$ with the same property. Therefore we have $g_{f_1}(x) = g_{f_2}(x)$. Since isomorphisms preserve operations addition and multiplication, we note that $G_{f_1} \cong G_{f_2}$. □

From Remark 6.1, we have the following

Proposition 6.2 G_f is a disjoint union of irreducible components.

The following definition is a modification of Definition 8 in [23].

Definition 6.4 A labeled graph G is *canonical for a spectral null constraint at* f if

1) every finite subgraph of G has a spectral null at f;
2) every finite graph which has a spectral null at f is label-preserving graph homomorphic to a subgraph of G.

Proposition 6.3 [23, Proposition 1] G_f is period-p canonical for a first-order spectral null constraint at f.

Remark **6.3** A function $\phi^{(p)} : Z[\omega] \to C$ defined in the following equation is a coboundary function of G_f [23]: for every state σ in G_f,

$$\phi^{(p)}(\sigma) = \begin{cases} \sigma, & \text{if } p = 0; \\ \sigma / g_f(\omega), & \text{if } 1 < p. \end{cases}$$

Our problem is interesting in itself and, moreover, may be useful in practice because we can construct spectral null codes at $f = f_S k/n$ from canonical graphs as follows:

Step A: For some p $(0 \le p \le n-1)$, choose an irreducible finite subgraph (G, γ) of G_f.

Step B: Construct a code C by code construction schemes due to Adler, Coppersmith and Hassner [5], Marcus [11], Beal [40], Franaszek [6], Lempel and Cohn [9] or Karabed and Marcus [37].

Finite subgraphs chosen in Step A are sofic systems. Since we have Proposition 2.5, we can apply code construction procedures listed in Step B.

6.2.2 An example

The following example tell us that canonical graphs may be reducible (cf. Proposition 6.2).

Example 6.1 The set of states in $G_{f_S/2}$ is Z, and it is partitioned into three sets given as follows:

$$\begin{aligned} S_1 &= \{2l : l \in Z\}, \\ S_2 &= \{(-1)^{l+1}(2l+1) : l \in Z\}, \\ S_3 &= \{(-1)^l(2l+1) : l \in Z\}. \end{aligned}$$

Each of these sets is the set of states of an irreducible component of $G_{f_S/2}$. In Fig. 6.1 a part of $G_{f_S/2}$ is given, where solid lines mean edges with symbol 1 and dotted lines means edges with symbol -1. Note that states are located in unusual order.

Proof: Clearly, we have $S_1 \cap (S_2 \cup S_3) = \emptyset$. Therefore we show first that the set of odd integers is partitioned into S_2 and S_3. Suppose that there are integers l and k such that

$$(-1)^{l+1}(2l+1) = (-1)^k(2k+1). \tag{6.2}$$

We note that $|l+k| = 1$. We also note that $l \ne 0$ and $k \ne 0$. Therefore, we have that $lk < 0$. So, we should have $(2l+1)(2k+1) < 0$. From this and (6.2) it follows that $(-1)^{l+1} = (-1)^{k+1}$. This equation requires that $l-k$ should be an even integer. But this contradicts with $|l+k| = 1$. Thus $S_2 \cap S_3 = \emptyset$. Consider an odd integer $2m+1$.

If m is an even integer then $2m+1 \in S_3$. If m is an odd integer then $2m+1 \in S_2$. Therefore our claim holds.

Since $\overline{\omega} = -1$, we can prove the following equations by calculation:

$$h_{f_S/2}((-1)^{l+1}(2l+1), 1) \;=\; \begin{cases} (-1)^{(l+1)+1}(2(l+1)+1) & \text{if } l \text{ is even;} \\ (-1)^{(l-1)+1}(2(l-1)+1) & \text{if } l \text{ is odd;} \end{cases}$$

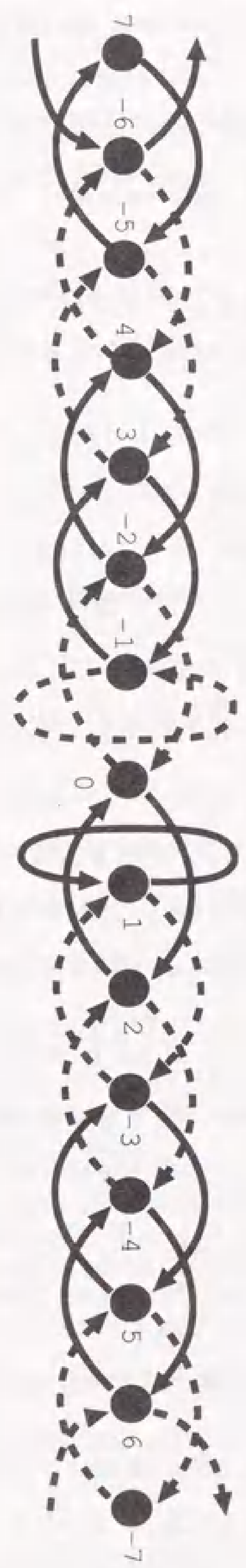


Figure 6.1: Example 6.1

$$h_{f_S/2}((-1)^{l+1}(2l+1), -1) \;=\; \begin{cases} (-1)^{(l-1)+1}(2(l-1)+1) & \text{if } l \text{ is even;} \\ (-1)^{(l+1)+1}(2(l+1)+1) & \text{if } l \text{ is odd.} \end{cases}$$

Therefore, we note that a subgraph of $G_{f_S/2}$ with state set S_2 is an irreducible components. It is proved similarly that there are two other irreducible components in G_f with state sets S_1 and S_3 respectively. □

From Proposition 6.2 G_f is a disjoint union of irreducible components. The structure of $G_{f_S/2}$ is understood intuitively as shown above. But for other frequency f, it is difficult to understand the structure of G_f intuitively or graphically. Hence we need a systematic way to grasp all irreducible components of canonical graphs.

In order to identify an irreducible component of G_f, it is enough to know a state in the irreducible component because the state and the transition rule (6.1) determine the irreducible component completely.

From the next proposition, we note that graphs which we have to consider are actually infinite graphs. Hence we can not analyze our problem in terms of transition matrix.

Proposition 6.4 For every irreducible component I of G_f the set of states in I is a countably infinite set.

Proof: For G_0, it is obvious. Assume that $1 < n$. Let $\boldsymbol{s} = \mathbf{1} \cdot -\mathbf{1}^{n-1}$. Let σ be a state of G_f and let l be a positive integer. By definition there is a path that starts from σ and generates $\boldsymbol{s}^l$. Let τ be the terminal state of the path. Since $\overline{\omega}^n = 1$ and $1 + \overline{\omega} + \cdots + \overline{\omega}^{n-1} = 0$, we have $\mathrm{RDS}_f(\boldsymbol{s}^l) = 2l$. Hence by Remark 6.1

$$|\sigma - \tau| = \begin{cases} |g_f(\omega)|2l & \text{if } 0 < p \leq n-1; \\ 2l & \text{if } p = 0. \end{cases}$$

□

6.2.3 Irreducible components with period n

First we consider irreducible components with period n. Assume that $1 < n$. Let I and J be irreducible components. Assume that the period of I is n and that of J is less than n. Let σ_I and σ_J be states of I and J respectively. Let ϕ_I and ϕ_J be coboundary functions of I and J respectively. Then, by Corollary 6.1 we may assume that $\phi_J(\sigma_J) = \phi_I(\sigma_I)$. Since ϕ_J is injective by Remark 6.3, it can be shown that $\phi_J^{-1} \circ \phi_I$ is well-defined and gives a label-preserving graph homomorphism of I to J. The homomorphism can not be an isomorphism because the period of I is not equal to that of J. But if the period of J is also n, then we can show that $I \cong J$:

Proposition 6.5 Assume that $1 < n$. Let I and J be two irreducible components. If both the period of I and that of J are equal to n, then $I \cong J$.

Proof: See Appendix A.9. □

Remark **6.4** We can also prove a similar result about canonical graphs for spectral density nulls with nonzero spectral lines.

Let I be an irreducible component of G_f which contains state 0. If n is a prime number then the period of I is n [56, Proposition 9]. But, if $n = 6$ then the period of I is 3. In fact we do not know if for any n there is an irreducible component with period n in $G_{f_{sk/n}}$. But we have the following.

Proposition 6.6 Assume that $n > 1$. There is a canonical graph which contains an irreducible component with period n.

Proof: Let $\tilde{G}_f$ be a labeled graph with state set $Z[\omega]$ such that there is an edge from state σ to τ with label $b \in \{-1, 1\}$ if and only if

$$\tau = \omega^{-1}(\sigma + 3g_f(\omega)b). \tag{6.3}$$

We note that $\tilde{G}_f$ is canonical for a spectral null constraint at f.

Let $\boldsymbol{c} = c_0c_1\cdots c_{l-1}$ be a cycle with $n \not| l$. Let $\boldsymbol{s} = s_0s_1\cdots s_{l-1}$ be the block generated by $\boldsymbol{c}$. Since Eq. (6.3) holds for every state-transition on $\boldsymbol{c}$, we note that $(\omega^l - 1)i_{\tilde{G}_f}(c_0) = 3g_f(\omega)\mathrm{RDS}_f(\boldsymbol{s})$. Hence we have $i_{\tilde{G}_f}(c_0) = 3g'_f(\omega)\mathrm{RDS}_f(\boldsymbol{s})$, where $g'_f(\omega)$ is a unique element in $Z[\omega]$ such that $(\omega^l - 1)g'_f(\omega) = g_f(\omega)$. We regard $\mathrm{RDS}_f(\boldsymbol{s})$ and $i_{\tilde{G}_f}(c_0)$ as polynomials of ω. Since every coefficient of polynomial $i_{\tilde{G}_f}(c_0)$ is 3, we should have $i_{\tilde{G}_f}(c_0) \neq 2\omega$. Therefore state 2ω can not be in any irreducible component with period less than n. On the other hand for every n with $n > 1$ there is an irreducible component which contains state 2ω. □

Let $\tilde{I}_f = (\tilde{S}, \tilde{E})$ be an irreducible component with period n of $\tilde{G}_f$ in the above proof. Let σ be a state of $\tilde{I}_f$ and let e_1 and e'_1 be distinct edges which emanate from σ. Let I_1 be an irreducible component of graph $(\tilde{S}, \tilde{E}\backslash\{e_1\})$ which contains state σ. Let I_2 be an irreducible component of graph $(\tilde{S}, \tilde{E}\backslash\{e_2\})$ which contains state σ. We can prove that I_1 or I_2 is an irreducible component in which for every integer m there is a state τ with $|\tau| \geq m$, where $|c|$ means the absolute value of a complex number c. Without loss of generality we may assume that I_1 is such an irreducible component. Let G' be a finite graph with period n which has a spectral null at f. Let $\boldsymbol{x}$ be the longest simple cycle of G' (a simple cycle is a cycle in which every state in G' appears at most once). We have $|\overline{\omega}(\sigma + (\overline{\omega} - 1)b)| \geq \big||\sigma| - |(\overline{\omega} - 1)b|\big| \geq \big||\sigma| - 2\big|$. Therefore, we can prove that G' is label-preserving graph homomorphic to a subgraph of I_1 which contains a state τ with $|\tau| \geq 2\,\mathrm{lg}(\boldsymbol{x})$. Hence, I_1 is period-n canonical for a spectral null at f (see [23] for the definition of "period-p canonical"). We let θ denote the procedure by which we obtained I_1 from I and write $I_1 = \theta(I, \sigma)$. Let σ_1 be a state of I_1 with $\sigma_1 \neq \sigma$. We can construct $\theta(I_1, \sigma_1)$ as well as $\theta(I, \sigma)$ and note that $\theta(I_1, \sigma_1)$ is also period-n canonical for a spectral null at f. Finally, we get a sequence of period-n canonical graphs $I_0 = \tilde{I}_f, I_1, \ldots$ and a sequence of states $\sigma_0 = \sigma, \sigma_1, \ldots$ such that $I_i = \theta(I_{i-1}, \sigma_{i-1})$ for each i with $i \geq 1$. Then for every i, a graph which consists of $\theta(I_i, \sigma_i)$ and all irreducible components with period less than n is canonical for a spectral null at f. It, however, is enough to consider only $\tilde{I}_f$ as a period-n canonical graph when we use canonical graphs in order to design spectral null codes, because we only need finite parts of a canonical graph but the entire configuration of the graph. We note that $\tilde{G}_f$ in the proof of Proposition 6.6 has an injective coboundary function. Therefore if there is an irreducible component with period n in G_f, then we can prove it by the same way of the proof of Proposition 2 or Proposition 3 that every irreducible component with period n is label-preserving graph isomorphic to $\tilde{I}_f$. This means that we may assume that there is only one irreducible component with period n. Thus we identify in the following irreducible components with period less than n.

6.2.4 Irreducible components with period less than n

Next we consider irreducible components with period less than n. We give fundamental properties of them here.

Definition 6.5 By $\mathcal{N}$ we mean the set of all blocks $\boldsymbol{a}$ with $n \not| \mathrm{lg}(\boldsymbol{a})$.

Definition 6.6 Let G be a graph. We define $C(G)$ by

$$C(G) = \{\boldsymbol{s} : \boldsymbol{s} \text{ is a block generated by a cycle in } G\}.$$

Proposition 6.7 Assume that $1 < n$. Let $\boldsymbol{s} \in \mathcal{N}$. Then there is an irreducible component I such that $\boldsymbol{s} \in C(I)$.

Proof: Let $\boldsymbol{t} = \boldsymbol{1}^n$. We consider a graph G consisting of two cycles $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\boldsymbol{x}$ generates $\boldsymbol{s}$, $\boldsymbol{y}$ generates $\boldsymbol{t}$ and these cycles have only one common state and no common edge. Let $\boldsymbol{z}$ be a cycle in G with $n|\,\mathrm{lg}(\boldsymbol{z})$ and let $\boldsymbol{u}$ be a block generated by $\boldsymbol{z}$. We show that $\mathrm{RDS}_f(\boldsymbol{u}) = 0$. If there are integers j and l such that $\boldsymbol{z}^{[j]} = \boldsymbol{y}^l$, it is clear. We may assume that $i(\boldsymbol{z})$ is a state in $\boldsymbol{x}$. Since $\mathrm{RDS}_f(\boldsymbol{a} \cdot \boldsymbol{1}^n \cdot \boldsymbol{b}) = \mathrm{RDS}_f(\boldsymbol{a} \cdot \boldsymbol{b})$ for any pair of blocks $\boldsymbol{a}$ and $\boldsymbol{b}$, we note there are integers j and l such that $\mathrm{RDS}_f(\boldsymbol{u}) = \mathrm{RDS}_f((\boldsymbol{s}^{[j]})^l)$. Since $\mathrm{lg}(\boldsymbol{u}) \equiv \mathrm{lg}((\boldsymbol{s}^{[j]})^l) \bmod n$, by the proof of Proposition 4.2 we have $\mathrm{RDS}_f(\boldsymbol{u}) = 0$. Therefore by Proposition 6.3 G is label-preserving graph homomorphic to a finite subgraph of G_f. This means that G_f has an irreducible component I such that $\boldsymbol{s} \in C(I)$. □

Proposition 6.8 Assume that $1 < n$. Let I and I' be irreducible components with period less than n. Let ϕ and ϕ' be coboundary functions of I and I' respectively. Assume that there are states σ in I and σ' in I' such that $\phi(\sigma) = \phi'(\sigma')$. Then $I \cong I'$.

Proof: Let e be an edge emanating from σ. Let $\tau = t_G(e)$ and let b be the label of e. There is a state τ' in I' such that there is an edge which goes from σ' to τ' with label b. We have $\phi'(\tau') = \omega^{-1}(\phi'(\sigma') + b) = \omega^{-1}(\phi(\sigma) + b) = \phi(\tau)$ by coboundary condition and assumption. Since ϕ' and ϕ are injective by Remark 6.3, they give a label-preserving graph isomorphism of I into I'. □

Corollary 6.2 Assume that $1 < n$. Let $\boldsymbol{s}, \boldsymbol{t} \in \mathcal{N}$. Let I be an irreducible component such that $\boldsymbol{s} \in C(I)$. If

$$\frac{\mathrm{RDS}_f(\boldsymbol{s})}{\omega^{\lg(\boldsymbol{s})} - 1} = \frac{\mathrm{RDS}_f(\boldsymbol{t})}{\omega^{\lg(\boldsymbol{t})} - 1},$$

then $\boldsymbol{t} \in C(I)$.

Corollary 6.3 Assume that $1 < n$. Let $\boldsymbol{s} \in \mathcal{N}$. Let I and J be irreducible components of G_f. If $\boldsymbol{s} \in C(I) \bigcap C(J)$, then $I \cong J$.

Corollary 6.4 Assume that $n > 1$. Let I be an irreducible component of G_f. Let $\boldsymbol{a} \in S_{\{-1,1\}}$ and $\boldsymbol{s} \in S_{\{-1,1\}}$. Assume that $\mathrm{RDS}_f(\boldsymbol{s}) = 0$ and $n \mid \lg(\boldsymbol{s})$. Then $\boldsymbol{a} \in C(I)$ if and only if $\boldsymbol{a} \cdot \boldsymbol{s} \in C(I)$.

Definition 6.7 Let $\boldsymbol{a} \in \mathcal{N}$. We define $I_f(\boldsymbol{a})$ to be the irreducible components in which a cycle generates $\boldsymbol{a}$.

Definition 6.8 Let $\boldsymbol{a} \in \mathcal{N}$. By $C_f(\boldsymbol{a})$ we mean the set of all blocks generated by cycle in $I_f(\boldsymbol{a})$, that is, $C(I_f(\boldsymbol{a}))$.

By Proposition 6.7 and Corollary 6.3 $I_f(\boldsymbol{a})$ is well-defined.

Proposition 6.9 If $\boldsymbol{b} \in \mathcal{N}, \boldsymbol{a}^i \in C_f(\boldsymbol{b})$ and $n \not| i \lg(\boldsymbol{a})$, then $\boldsymbol{a} \in C_f(\boldsymbol{b})$.

Proof: We have

$$\begin{aligned}
\mathrm{RDS}_f(\boldsymbol{a}^i) &= \mathrm{RDS}_f(\boldsymbol{a}^{i-1}) + \omega^{(i-1)\lg(\boldsymbol{a})}\mathrm{RDS}_f(\boldsymbol{a}) \\
&= \mathrm{RDS}_f(\boldsymbol{a}^{i-2}) + (\omega^{(i-2)\lg(\boldsymbol{a})} + \omega^{(i-1)\lg(\boldsymbol{a})})\mathrm{RDS}_f(\boldsymbol{a}) \\
&= \cdots \\
&= (1 + \omega^{\lg(\boldsymbol{a})} + \cdots + \omega^{(i-1)\lg(\boldsymbol{a})})\mathrm{RDS}_f(\boldsymbol{a}).
\end{aligned}$$

Since $\omega^{i\lg(\boldsymbol{a})} - 1 = (1 + \omega^{\lg(\boldsymbol{a})} + \cdots + \omega^{(i-1)\lg(\boldsymbol{a})})(\omega^{\lg(\boldsymbol{a})} - 1)$, we have $\mathrm{RDS}_f(\boldsymbol{a}^i)/(\omega^{i\lg(\boldsymbol{a})} - 1) = \mathrm{RDS}_f(\boldsymbol{a})/(\omega^{\lg(\boldsymbol{a})} - 1)$. Thus we have $\boldsymbol{a} \in C(I)$ by Corollary 6.2.

□

Let I be an irreducible component of a canonical graph and assume that the period of I is less than n. We can choose an FSTD H which contains a cycle of length not a multiple of n. Let $\boldsymbol{s}$ be a block generated by the cycle. Since G_f is canonical, there is an irreducible component of G_f such that $\boldsymbol{s} \in C(J)$. By Corollary 6.3, we have $I \cong J$. Therefore we only consider irreducible components of G_f.

Let G_p^f be a labeled graph defined in [23].

Proposition 6.10 *[23, p. 565]* For a positive integer p with $0 \le p \le n-1$, G_p^f is label-preserving graph isomorphic to a subgraph of G_f.

Definition 6.9 Assume that $1 < n$. Define a function $h : \boldsymbol{C} \times \{-1, 1\} \to \boldsymbol{C}$ by $h^f(\sigma, b) = \omega^{-1}(\sigma + b)$. For $\boldsymbol{s} \in \mathcal{N}$ we define a set S as follows:

$$\begin{aligned}
S_0 &= \left\{\frac{\mathrm{RDS}_f(\boldsymbol{s})}{\omega^{\lg(\boldsymbol{s})} - 1}\right\}, \\
S_i &= \bigcup_{\sigma \in S_{i-1}} \{h^f(\sigma, -1), h^f(\sigma, 1)\}, \quad i = 1, 2, \ldots, \\
S &= \bigcup_{i=0}^{\infty} S_i.
\end{aligned}$$

Then $\mathcal{H}_f(\boldsymbol{s})$ is the CSTD such that its state set is S and there is an edge from σ to τ with label b if and only if $h(\sigma, b) = \tau$.

By Remark 6.1, $\mathcal{H}_f(\boldsymbol{s})$ is an irreducible graph. By the proof of Theorem 3[23] (or by the proof of PropositionLemma 4.2) we note that if $1 < n$ then $\boldsymbol{s} \in C(\mathcal{H}_f(\boldsymbol{s}))$ for every $\boldsymbol{s} \in \mathcal{N}$.

Proposition 6.11 Assume that $1 < n$. Let I be an irreducible component with period less than n. Then :

1) for every block $\boldsymbol{s} \in C(I) \bigcap \mathcal{N}$, we have $\mathcal{H}_f(\boldsymbol{s}) \cong I$;

2) for every pair of blocks $\boldsymbol{s}$ and $\boldsymbol{t} \in C(I) \bigcap \mathcal{N}$, we have $\mathcal{H}_f(\boldsymbol{s}) = \mathcal{H}_f(\boldsymbol{t})$.

Proof: The proof of the state 1) is similar to the proof of Proposition 6.8.

Suppose that $\boldsymbol{s}$ and $\boldsymbol{t}$ are blocks in $C(I) \bigcap \mathcal{N}$. Then from 1) we have $\mathcal{H}_f(\boldsymbol{s}) \cong \mathcal{H}_f(\boldsymbol{t})$ and, so, $\boldsymbol{s} \in \mathcal{H}_f(\boldsymbol{t})$. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ by cycles in $\mathcal{H}_f(\boldsymbol{s})$ and $\mathcal{H}_f(\boldsymbol{t})$ respectively such that $\boldsymbol{x}$ and $\boldsymbol{x}'$ generate block $\boldsymbol{s}$. Let ϕ_s and ϕ_t be coboundary functions of $\mathcal{H}_f(\boldsymbol{s})$ and $\mathcal{H}_f(\boldsymbol{t})$ respectively. By Corollary 6.1, they are uniquely determined and are identity functions on the set of states. Therefore $i(\boldsymbol{x}) = \phi(i(\boldsymbol{x})) = \phi(i(\boldsymbol{x}')) = i(\boldsymbol{x}')$. Thus by definition $\mathcal{H}_f(\boldsymbol{s}) = \mathcal{H}_f(\boldsymbol{t})$. □

By Proposition 6.11 we note that identifying all irreducible components with period less than n is equivalent to identifying all elements in $\{\mathcal{H}_f(\boldsymbol{s}) : \boldsymbol{s} \in \mathcal{N}\}$.

6.2.5 homomorphism between irreducible components

Here we consider label-preserving graph homomorphism of irreducible components. Let I and I' be irreducible components with period less than n. Suppose that I is label-preserving graph homomorphic to I'. Let $\boldsymbol{s} \in C(I) \bigcap \mathcal{N}$. Then $\boldsymbol{s} \in C(I')$. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be cycles in I and I' respectively which generate $\boldsymbol{s}$. Let ϕ and ϕ' be coboundary functions of I and I' respectively. By Corollary 6.1 we have $\phi(i(\boldsymbol{x})) = \phi'(i(\boldsymbol{x}'))$. Therefore we should have $I \cong I'$ by Proposition 6.8. That is, if $I \not\cong I'$ then I is not label-preserving graph homomorphic to I'. As shown in the following, every irreducible component with period n is, however, label-preserving graph homomorphic to every irreducible component with period less than n.

Assume that $1 < n$. Let I and J be irreducible components. Assume that the period of I is n and that of J is less than n. Let σ_I and σ_J be states of I and J respectively. Let ϕ_I and ϕ_J be coboundary functions of I and J respectively. By Corollary 6.1, we may assume that $\phi_J(\sigma_J) = \phi_I(\sigma_I)$. Since ϕ_J is injective by Remark 6.3, it can be shown that $\phi_J^{-1} \circ \phi_I$ is well-defined and gives a label-preserving graph homomorphism of I to J. The homomorphism can not be an isomorphism because the period of I is not equal to that of J.

6.3 Irreducible components of canonical graphs for first-order spectral nulls

Let $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{N}$. We show a necessary and sufficient condition that there exists an irreducible component I of G_f such that $\boldsymbol{a}, \boldsymbol{b} \in C(I)$. For every irreducible component I with period less than n, we give a block $\boldsymbol{a}$ such that $\boldsymbol{a} \in C(I)$. By giving a set of such blocks or a super set of the set, we identify all irreducible components of G_f.

For the case $n = 1$, our problem is obvious. Hence in this section we assume that $n \geq 2$, that is, $f > 0$.

6.3.1 Blocks generated by cycles

We first characterize two blocks which are generated by cycle in the same irreducible component.

Proposition 6.12 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be blocks in $\mathcal{N}$. Then, $\boldsymbol{b} \in C_f(\boldsymbol{a})$ if and only if a block $\boldsymbol{s}$ and an integer l exist such that $n \mid \lg(\boldsymbol{s})$ and

$$\mathrm{RDS}_f(\boldsymbol{s}) = \frac{\mathrm{RDS}_f(\boldsymbol{a})}{1-\omega^{\lg(\boldsymbol{a})}} - \frac{\mathrm{RDS}_f(\boldsymbol{b}^{[l]})}{1-\omega^{\lg(\boldsymbol{b})}}. \tag{6.4}$$

Moreover, there is an algorithm by which for every integer i we can determine whether there exists a block $\boldsymbol{s}$ satisfying (6.4).

Since the proof of this proposition gives the algorithm, we describe the proof here.

Proof: Since $\boldsymbol{a} \in C(I)$, the period of $I_f(\boldsymbol{a})$ is less than n. By Corollary 6.1 there is a unique coboundary function ψ of I at f. Let $\boldsymbol{x}$ be a cycle which generates $\boldsymbol{a}$.

First assume that $\boldsymbol{b} \in C_f(\boldsymbol{a})$. Let $\boldsymbol{y}$ be a cycle which generates $\boldsymbol{b}$. There is an l with $0 \leq l < \gcd(n, r)$ such that there is a path $\boldsymbol{z}$ with $n \mid \lg(\boldsymbol{z})$ which goes from $i_G(\boldsymbol{x})$ to $i_G(\boldsymbol{y}^{[l]})$ because the period of $I_f(\boldsymbol{a})$ is not greater than $\gcd(n, r)$. Let $\boldsymbol{s}' = s'_0 s'_1 \cdots s'_{L-1}$ be the block generated by $\boldsymbol{z}$ (see Fig. 6.2). Since the coboundary equation holds on every transition in $I_f(\boldsymbol{a})$, we have $\mathrm{RDS}_f(\boldsymbol{s}') = \sum_{j=0}^{L-1} \omega^j s'_j = \sum_{j=0}^{L-1} \omega^j(\omega\psi(t_G(z_j)) - \psi(i_G(z_j))) = \psi(i_G(\boldsymbol{y}^{[l]})) - \psi(i_G(\boldsymbol{x}))$. By Corollary 6.1, we have $\psi(i_G(\boldsymbol{x})) = \mathrm{RDS}_f(\boldsymbol{a})/(\omega^{\lg(\boldsymbol{a})} - 1)$ and $\psi(i_G(\boldsymbol{y}^{[l]})) = \mathrm{RDS}_f(\boldsymbol{b}^{[l]})/(\omega^{\lg(\boldsymbol{b})} - 1)$. Therefore we get Eq. (6.4) by putting $\boldsymbol{s} = -\boldsymbol{s}'$.

Conversely, we assume that there exist a block $\boldsymbol{s}$ and an integer l such that $n \mid \lg(\boldsymbol{s})$ and Eq. (6.4) holds. Put $\boldsymbol{s}' = -\boldsymbol{s}$. By the definition of G_f there is a path $\boldsymbol{z}$ which starts from $i_G(\boldsymbol{x})$ and which generates $\boldsymbol{s}'$. We also have $\psi(t_G(\boldsymbol{z})) - \psi(i_G(\boldsymbol{x})) = \mathrm{RDS}_f(\boldsymbol{s}')$ from the coboundary equation. By assumption we get $\psi(t_G(\boldsymbol{z})) = \mathrm{RDS}_f(\boldsymbol{b}^{[l]})/(\omega^{\lg(\boldsymbol{b})} - 1)$. By Remark 6.1, there is a path which goes from $t_G(\boldsymbol{z})$ to $i(\boldsymbol{x})$ and which generates $-\boldsymbol{s}'$. Therefore we have $\psi(t_G(\boldsymbol{z})) = \mathrm{RDS}_f(-\boldsymbol{s}' \cdot \boldsymbol{a} \cdot \boldsymbol{s}')/(\omega^{\lg(-\boldsymbol{s}' \cdot \boldsymbol{a} \cdot \boldsymbol{s}')} - 1)$ from Corollary 6.1. Thus we have $\boldsymbol{b} \in C_f(\boldsymbol{a})$ from Corollary 6.2 because $-\boldsymbol{s}' \cdot \boldsymbol{a} \cdot \boldsymbol{s}' \in C(I)$.

Next we prove that there is an algorithm by which for every integer i we can determine whether there exists a block $\boldsymbol{s}$ satisfying (6.4). Without loss of generality, we may assume that $i = 0$ in (6.4) by replacing $\boldsymbol{b}$ with $\boldsymbol{b}^{[i]}$ because $\boldsymbol{b} \in C_f(\boldsymbol{a}) \Leftrightarrow \boldsymbol{b}^{[i]} \in C_f(\boldsymbol{a})$. For a block $\boldsymbol{a} \in S_{\{-1,1\}}$, we regard $\mathrm{RDS}_f(\boldsymbol{a})$ as a polynomial of ω, that is, an element in $Z[\omega]$. Let $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2^n-1}\}$ be the set of all n-blocks in $S_{\{-1,1\}}$. If $\boldsymbol{c}$ and $\boldsymbol{c}'$ are blocks in $S_{\{-1,1\}}$ with $n \mid \lg(\boldsymbol{c})$, then $\mathrm{RDS}_f(\boldsymbol{c} \cdot \boldsymbol{c}') = \mathrm{RDS}_f(\boldsymbol{c}) + \mathrm{RDS}_f(\boldsymbol{c}')$. Hence, for every block $\boldsymbol{s} \in S_{\{-1,1\}}$ with $n \mid \lg(\boldsymbol{s})$, $\mathrm{RDS}_f(\boldsymbol{s})$ can be written as a linear combination of $\mathrm{RDS}_f(\boldsymbol{a}_0), \ldots, \mathrm{RDS}_f(\boldsymbol{a}_{2^n-1})$ with integer coefficients. Conversely, for every linear combination of $\mathrm{RDS}_f(\boldsymbol{a}_0)$, $\ldots$, $\mathrm{RDS}_f(\boldsymbol{a}_{2^n-1})$ with integer coefficients there is a block $\boldsymbol{s}$ of length a multiple of n such that $\mathrm{RDS}_f(\boldsymbol{s})$ is equal to the linear combination because for every j with $0 \leq j \leq 2^n - 1$, there is an l with $\mathrm{RDS}_f(\boldsymbol{a}_j) = -\mathrm{RDS}_f(\boldsymbol{a}_l)$. By the definition of $g_f(\omega)$, there are polynomials $f_{\boldsymbol{a}}(\omega)$ and $f_{\boldsymbol{b}}(\omega)$ such that $(\omega^{\lg(\boldsymbol{a})} - 1) f_{\boldsymbol{a}}(\omega) = (\omega^{\lg(\boldsymbol{b})} - 1) f_{\boldsymbol{b}}(\omega) = g_f(\omega)$. Hence there is a block $\boldsymbol{s} \in S_{\{-1,1\}}$ of length a multiple of n such that (6.4) holds, if and only if there is a block $\boldsymbol{s}$ of length a multiple of n such that

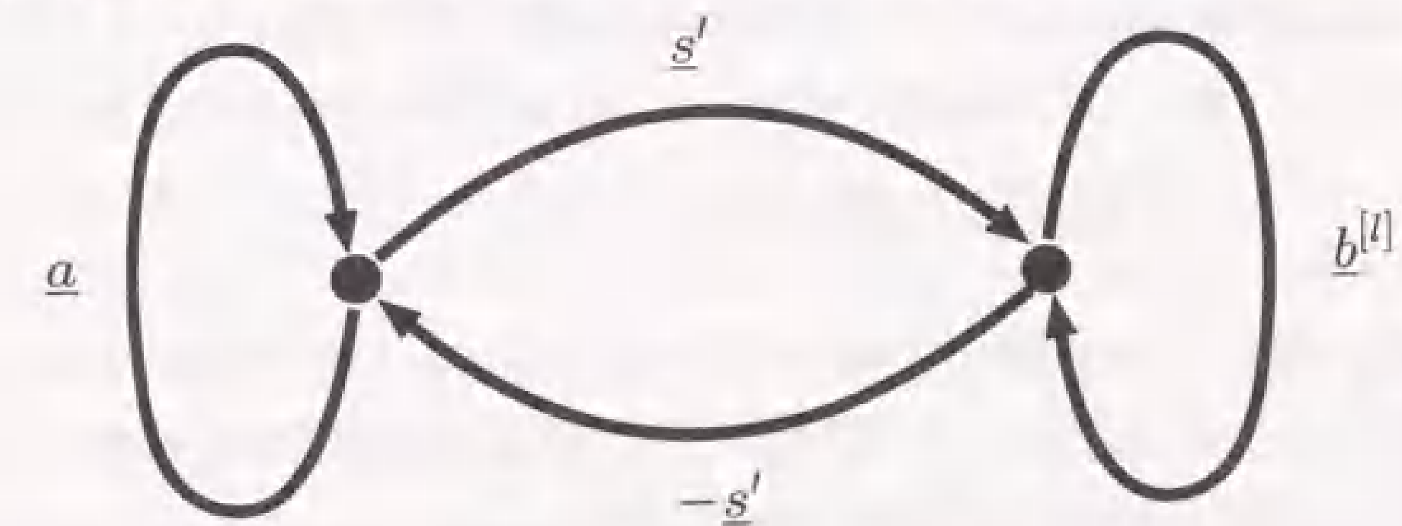


Figure 6.2: (G, γ) in proof of Proposition 6.12

$f_{\boldsymbol{a}}(\omega)\mathrm{RDS}_f(\boldsymbol{a}) + f_{\boldsymbol{b}}(\omega)\mathrm{RDS}_f(\boldsymbol{b}) = g_f(\omega)\mathrm{RDS}_f(\boldsymbol{s})$. This is also equivalent to that there are integers $i_0, i_1, \ldots, i_{2^n-1}$ such that

$$f_{\boldsymbol{a}}(\omega)\mathrm{RDS}_f(\boldsymbol{a}) + f_{\boldsymbol{b}}(\omega)\mathrm{RDS}_f(\boldsymbol{b}) = g_f(\omega)\left(\sum_{j=0}^{2^n-1} i_j \mathrm{RDS}_f(\boldsymbol{a}_j)\right). \tag{6.5}$$

Regarding elements in $Z[\omega]$ as integral vectors of size $\phi(n)$, (6.5) is an integral linear equation with respect to unknown variables $i_0, i_1, \ldots, i_{2^n-1}$, where $\phi(n)$ is the Euler number of n. Since it is possible to determine whether (6.5) has a solution in integral vectors, we have completed the proof. □

Remark **6.5** Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be blocks in $\mathcal{N}$ with $\lg(\boldsymbol{a}) = \lg(\boldsymbol{b})$. Put $r = \lg(\boldsymbol{a})$. Let $\boldsymbol{t}$ be a block in $S_{\{-1,1\}}$ with $n|\lg(\boldsymbol{t})$. Suppose that $(\mathrm{RDS}_f(\boldsymbol{b}) - \mathrm{RDS}_f(\boldsymbol{a}))/(\omega^r - 1) = \mathrm{RDS}_f(\boldsymbol{t})$. Then $I_f(\boldsymbol{a})$ contains cycles $\boldsymbol{x} = x_0 \cdots x_{r-1}$ and $\boldsymbol{y} = y_0 \cdots y_{r-1}$ which generate $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. By examining the above proof, we note that there is a path of length a multiple of n which goes from $i(\boldsymbol{x})$ to $i(\boldsymbol{y})$ and generates $\boldsymbol{t}$. Since the period of $I_f(\boldsymbol{a})$ is a divisor of n, we note that $i(\boldsymbol{x})$ and $i(\boldsymbol{y})$ belong to the same periodic component. Therefore for every l with $0 \le l \le r-1$, $i(x_l)$ and $i(y_l)$ belong to the same periodic component. So, for every l with $0 \le l \le r-1$ there is a path z_l of length a multiple of n which goes from $i(x_l)$ to $i(y_l)$. Let $\boldsymbol{t}_l$ be the block generated by $\boldsymbol{z}_l$. Then by the above proof, we also note that $(\mathrm{RDS}_f(\boldsymbol{b}^{[l]}) - \mathrm{RDS}_f(\boldsymbol{a}^{[l]}))/(\omega^r - 1) = \mathrm{RDS}_f(\boldsymbol{t}_l)$.

The dimension of (6.4) is very large even if n is not so large, e.g., $n = 10$. Moreover, when we execute a program which implements the algorithm, we have to treat very large numbers. Therefore this algorithm is not practical.

Corollary 6.5 Let $\boldsymbol{a} \in \mathcal{N}$ and put $r = \lg(\boldsymbol{a})$. Assume that there are integers j and l such that $0 \le j < l \le r-1$, $\gcd(n, r) \mid (l-j)$, and $a_j = -a_l$. Let I be an irreducible component. Define a block $\boldsymbol{b} = b_0 b_1 \cdots b_{r-1}$ as follows:

$$b_i = \begin{cases} -a_i & \text{if } i = j \text{ or } i = l; \\ a_i & \text{otherwise}, \end{cases}$$

where $\boldsymbol{a} = a_0 a_1 \cdots a_{r-1}$. Then $\boldsymbol{a} \in C(I)$ if and only if $\boldsymbol{b} \in C(I)$.

Proof: Without loss of generality, we may assume that $a_j = -1$. Then we have

$$\mathrm{RDS}_f(\boldsymbol{a}) - \mathrm{RDS}_f(\boldsymbol{b}) = -2\omega^j\left(1 - \omega^{l-j}\right).$$

By assumption there is an $m \in Z$ such that $rm \equiv (l-j) \bmod n$. Let g be a polynomial given by $g(D) = -2 - 2D - \cdots - 2D^{m-1}$ where D is indeterminate. Then we have $-2(1-\omega^{l-j}) = g(\omega^r)(1-\omega^r)$. Therefore we have

$$\frac{-2\omega^j\left(1-\omega^{l-j}\right)}{1-\omega^r} = \omega^j g(\omega^d).$$

Since $\mathrm{RDS}_f(\mathbf{1}^i \cdot -\mathbf{1} \cdot \mathbf{1}^{n-i-1}) = -2\omega^i$, we can construct a block $\boldsymbol{s}$ such that $n \mid \lg(\boldsymbol{s})$ and $\mathrm{RDS}_f(\boldsymbol{s}) = \omega^j g(\omega^r)$. Thus this corollary follows from Proposition 6.12. □

In Corollary 6.5, we say that $\boldsymbol{b}$ is obtained from $\boldsymbol{a}$ by an *exchange of symbols.* If there are blocks $\boldsymbol{a} = \boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_L = \boldsymbol{b}$ such that $\boldsymbol{a}_i$ is obtained from $\boldsymbol{a}_{i-1}$ by an exchange of symbols for every $i = 1, 2, \ldots, L$, then we say that $\boldsymbol{b}$ is obtained from $\boldsymbol{a}$ by exchanges of symbols.

Definition 6.10 For integers i and n, we define $R(i, n)$ to be the remainder when we divide i by n.

Definition 6.11 Let $\boldsymbol{a} = a_0 a_1 \cdots a_{L-1}$ be a block. Define $p(\boldsymbol{a})$ and $m(\boldsymbol{a})$ by $p(\boldsymbol{a}) = \#\{i : a_i = 1\}$ and $m(\boldsymbol{a}) = \#\{i : a_i = -1\}$.

Corollary 6.6 Let $\boldsymbol{s} \in \mathcal{N}$ and assume that $\gcd(\lg(\boldsymbol{s}), n) = 1$. Let I be an irreducible component. Then $\boldsymbol{s} \in C(I)$ if and only if $\mathbf{1}^{R(p(\boldsymbol{s}),n)} \cdot -\mathbf{1}^{R(m(\boldsymbol{s}),n)} \in C(I)$.

Proof: By exchanges of symbols we have $\boldsymbol{s} \in C(I)$ if and only if $\mathbf{1}^{p(\boldsymbol{s})} \cdot -\mathbf{1}^{m(\boldsymbol{s})} \in C(I)$. Since $\mathrm{RDS}_f(\mathbf{1}^n) = \mathrm{RDS}_f(-\mathbf{1}^n) = 0$, we have that $\boldsymbol{s} \in C(I)$ if and only if $\mathbf{1}^{R(p(\boldsymbol{s}),n)} \cdot -\mathbf{1}^{R(m(\boldsymbol{s}),n)} \in C(I)$ from Corollary 6.4. □

To prove Lemma 6.2 we need the following lemma.

Lemma 6.1 Let $\boldsymbol{s}$ be a block with $\gcd(n, \lg(\boldsymbol{s})) = 1$. Then there is a block $\boldsymbol{t}$ such that $\boldsymbol{t} \in C_f(\boldsymbol{s})$ and $\lg(\boldsymbol{t}) = n-1$.

Proof: By assumption there is an integer i such that $i\lg(\boldsymbol{s}) \equiv n-1 \bmod n$. Put $\boldsymbol{s}^i = u_0 u_1 \cdots u_{i\lg(\boldsymbol{s})-1}$. By Corollary 6.5 $\mathbf{1}^{p(\boldsymbol{s})} \cdot -\mathbf{1}^{m(\boldsymbol{s})}$ is obtained from $\boldsymbol{s}^i$ by exchanges of symbols. Since $\mathrm{RDS}_f(\mathbf{1}^n) = \mathrm{RDS}_f(-\mathbf{1}^n) = 0$, by Corollary 6.4 we may assume that $0 \le i_1, i_2 \le n-1$. Since $i_1 + i_2 \le 2n-2$, we conclude that $i_1 + i_2 = n-1$. □

Lemma 6.2 Assume that $n > 1$. Let $\boldsymbol{s} \in \mathcal{N}$ and $d = \gcd(n, \lg(\boldsymbol{s}))$. Then there is a block $\boldsymbol{t} \in C_f(\boldsymbol{t})$ with $\lg(\boldsymbol{t}) = n-d$.

Proof: See Appendix A.11 □

Definition 6.12 Let $\mathcal{R}$ be a relation in $\mathcal{N}$ defined by "there is an irreducible component I such that $\boldsymbol{a} \in C(I)$ and $\boldsymbol{b} \in C(I)$" where $\boldsymbol{a}$ and $\boldsymbol{b}$ are blocks in $\mathcal{N}$.

By Lemma 6.2 and Corollary 6.3 we note that the number of irreducible components is finite for every n. Let $T = \{\boldsymbol{s} \in \mathcal{N} : \lg(\boldsymbol{s}) = n-d \text{ for some divisor } d \text{ of } n\}$.

Then identifying all irreducible components with period less than n is equivalent to identifying all $\mathcal{R}$-equivalence classes of T. From Proposition 6.12 we can identify these equivalence classes by checking $\mathcal{R}$-equivalence of all pairs of blocks in T with a computer. In the case where n is a prime integer or equal to $2d$ for a prime integer d, however, our identification problem has been solved theoretically.

6.3.2 Case 1: n is a prime number

Throughout this subsection, we assume that n is a prime number. Under this assumption our problem is solved completely(Theorem 6.1). Corollary 6.6 guarantees that the number of irreducible components is finite. We identify an irreducible component with period n and irreducible components with period less than n separately (Proposition 6.13 and Theorem 6.1). For irreducible components with period less than n, we reduce our problem to a problem of linear independence of vectors (Lemma 6.4, Lemma 6.5, Lemma 6.6).

First we consider irreducible components with period n. By Proposition 6.6 we know that for every n there exists an irreducible component with period n of an canonical graph for a spectral null at $f = f_S k/n$. We, however, can prove that G_f contains an irreducible component with period n under the assumption that n is a prime number.

Lemma 6.3 Let $\boldsymbol{s} \in S_{\{-1,1\}}$. Assume that $\mathrm{RDS}_f(\boldsymbol{s}) = 0$. Then $n | \lg(\boldsymbol{s})$.

Proof: See Appendix A.12. □

Proposition 6.13 Let I be the irreducible component which contains state 0. Then the period of I is n.

Proof: Suppose that the period of I is less than n. There is a cycle $\boldsymbol{x}$ such that $i_G(\boldsymbol{x}) = 0$ and $n \not| \lg(\boldsymbol{x})$. Let $\boldsymbol{a}$ be a block generated by $\boldsymbol{x}$. By Corollary 6.1 we have $\mathrm{RDS}_f(\boldsymbol{a}) = 0$. But this contradicts with Lemma 6.3. □

By $\widetilde{\mathcal{H}}_f$ we mean the irreducible component that contains state 0. From Proposition 6.5 and Proposition 6.13 we may assume that an irreducible component with period n.

Next we consider the identification of each element in $\{\mathcal{H}_f(\boldsymbol{s}) : \boldsymbol{s} \in \mathcal{N}\}$. Let $Z_n = Z/nZ$.

Definition 6.13 For block $\boldsymbol{a} \in S_{\{-1,1\}}$ we define a two dimensional vector $\varphi(\boldsymbol{a})$ by $\varphi(\boldsymbol{a}) = (R(p(\boldsymbol{a}), n) \;\; R(m(\boldsymbol{a}), n))$.

We mean multiplication in Z_n by $*$. For $i \in Z_n$ and for $v = (v_1 \;\; v_2) \in Z_n^2$ we write $i * v = (i * v_1 \;\; i * v_2)$.

Lemma 6.4 Let $\boldsymbol{s}, \boldsymbol{t}$ and $\boldsymbol{u} \in \mathcal{N}$. Let I be an irreducible component. Assume that $\boldsymbol{s}$ and $\boldsymbol{t}$ are blocks in $C(I)$. Assume that there are integers i and $j \in Z_n$ such that $i * \varphi(\boldsymbol{s}) + j * \varphi(\boldsymbol{t}) = \varphi(\boldsymbol{u})$. Then $\boldsymbol{u} \in C(I)$.

Proof: Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be cycles in I which generate $\boldsymbol{s}$ and $\boldsymbol{t}$ respectively. Since $\gcd(n, \lg(\boldsymbol{s})) = 1$ and $\boldsymbol{1}^n \in C(I)$, the period of I is 1. Therefor there is a path $\boldsymbol{z}$ such that $i_G(\boldsymbol{z}) = i_G(\boldsymbol{x})$, $t_G(\boldsymbol{z}) = i_G(\boldsymbol{y})$ and $n \mid \lg(\boldsymbol{z})$. Let $\boldsymbol{v} \in S_{\{-1,1\}}$ be the block generated by $\boldsymbol{z}$. Then there is a path $\boldsymbol{w}$ from $i_G(\boldsymbol{y})$ to $i_G(\boldsymbol{x})$ generating $-\boldsymbol{v}$ by Remark 6.1. A concatenation $\boldsymbol{x}^i \cdot \boldsymbol{z} \cdot \boldsymbol{y}^j \cdot \boldsymbol{w}$ is a cycle of I and generates $\boldsymbol{v}' = \boldsymbol{s}^i \cdot \boldsymbol{v} \cdot \boldsymbol{t}^j \cdot -\boldsymbol{v}$. By Corollary 6.6 we have $\boldsymbol{v}' \in C(I) \Leftrightarrow \boldsymbol{1}^{R(p(\boldsymbol{v}'),n)} \cdot -\boldsymbol{1}^{R(m(\boldsymbol{v}'),n)} \in C(I)$ and $\boldsymbol{u} \in C(I) \Leftrightarrow$

$\boldsymbol{1}^{R(p(\boldsymbol{u},n))} \cdot -\boldsymbol{1}^{R(m(\boldsymbol{u},n))} \in C(I)$. By the definition of φ, we have

$$\begin{aligned} \varphi(\boldsymbol{s}^i \cdot \boldsymbol{v} \cdot \boldsymbol{t}^j \cdot -\boldsymbol{v}) &= \varphi(\boldsymbol{s}^i \cdot \boldsymbol{t}^j) \\ &= i * \varphi(\boldsymbol{s}) + j * \varphi(\boldsymbol{t}) \\ &= \varphi(\boldsymbol{u}). \end{aligned}$$

Thus we have $\boldsymbol{u} \in C(I)$. □

Lemma 6.5 There is no irreducible component I such that both $\boldsymbol{1}$ and $-\boldsymbol{1}$ are blocks in $C(I)$.

Proof: See Appendix A.13. □

Lemma 6.6 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be blocks in $\mathcal{N}$. Then, $\varphi(\boldsymbol{a})$ and $\varphi(\boldsymbol{b})$ are linearly dependent if and only if there is an irreducible component I such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are blocks in $C(I)$.

Proof: See Appendix A.14. □

The following theorem is one of main results of this section.

Theorem 6.1 Every irreducible component of G_f is label-preserving graph isomorphic to one of the following diagrams:

$$\widetilde{\mathcal{H}}_f, \mathcal{H}_f(\boldsymbol{1}), \mathcal{H}_f(-\boldsymbol{1}), \mathcal{H}_f(\boldsymbol{1} \cdot -\boldsymbol{1}), \mathcal{H}_f(\boldsymbol{1} \cdot -\boldsymbol{1}^2), \ldots, \mathcal{H}_f(\boldsymbol{1} \cdot -\boldsymbol{1}^{n-2}).$$

Proof: Let $E = \{\boldsymbol{1}, -\boldsymbol{1}, \boldsymbol{1} \cdot -\boldsymbol{1}, \boldsymbol{1} \cdot -\boldsymbol{1}^2, \ldots, \boldsymbol{1} \cdot -\boldsymbol{1}^{n-2}\}$. From Proposition 6.5 and Proposition 6.13, we have only to prove that E is a set of representatives of $\mathcal{R}$-equivalence classes of $\mathcal{N}$.

It is easy to see that for every pair of distinct blocks $\boldsymbol{a}$ and $\boldsymbol{b}$ in E, $\varphi(\boldsymbol{a})$ and $\varphi(\boldsymbol{b})$ are linearly independent. Let $\boldsymbol{b} \in \mathcal{N}$. We prove that an $\boldsymbol{a} \in E$ and an $i \in Z_n$ exist such that $\varphi(\boldsymbol{b}) = i * \varphi(\boldsymbol{a})$. Put $\varphi(\boldsymbol{b}) = (c \;\; d)$. Note that $c + d \equiv \lg(\boldsymbol{b}) \not\equiv 0 \bmod n$. If $c = 0$, then $\varphi(\boldsymbol{b}) = d * \varphi(-\boldsymbol{1})$. We assume that $c \neq 0$. Let e be the multiplicative inverse of c in Z_n. Then we have $\varphi(\boldsymbol{b}) = c * (1 \;\; d * e)$. We note that $d * e \not\equiv n - 1$ because $c * (1 + d * e) = c + d \neq 0$. Thus this theorem follows from Lemma 6.6. □

Let $\boldsymbol{s} \in E$ and let $I = I_f(\boldsymbol{a})$. Since $I \cong \mathcal{H}_f(\boldsymbol{s})$ by Proposition 6.11 and every state of I can be represented by an integer vector, we prefer using I to using $\mathcal{H}_f(\boldsymbol{s})$ in designing spectral null codes. that I contains $-\omega^{(n-1)\lg(\boldsymbol{s})}\mathrm{RDS}_f(\boldsymbol{s})$ as a state. Hence we can draw I easily from definition.

Example 6.2 Let $f = f_S/3$. From Theorem 6.1 we note that there are three irreducible components in the canonical graph, which is given in Fig. 1.15, for a spectral null at f. We give the components, $\mathcal{H}_f(-\boldsymbol{1})$, $\mathcal{H}_f(\boldsymbol{1})$, $\mathcal{H}_f(\boldsymbol{1} \cdot -\boldsymbol{1})$ and $\widetilde{\mathcal{H}}_f$, in Fig. 6.3, where solid arrows mean edges with label 1 and dotted arrows mean edges with label -1.

Suppose that we are given blocks $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{N}$. By Proposition 6.12, we can determine whether $\boldsymbol{a}\mathcal{R}\boldsymbol{b}$, i.e., $\boldsymbol{a}$ and $\boldsymbol{b}$ are generated by cycles in the same irreducible component

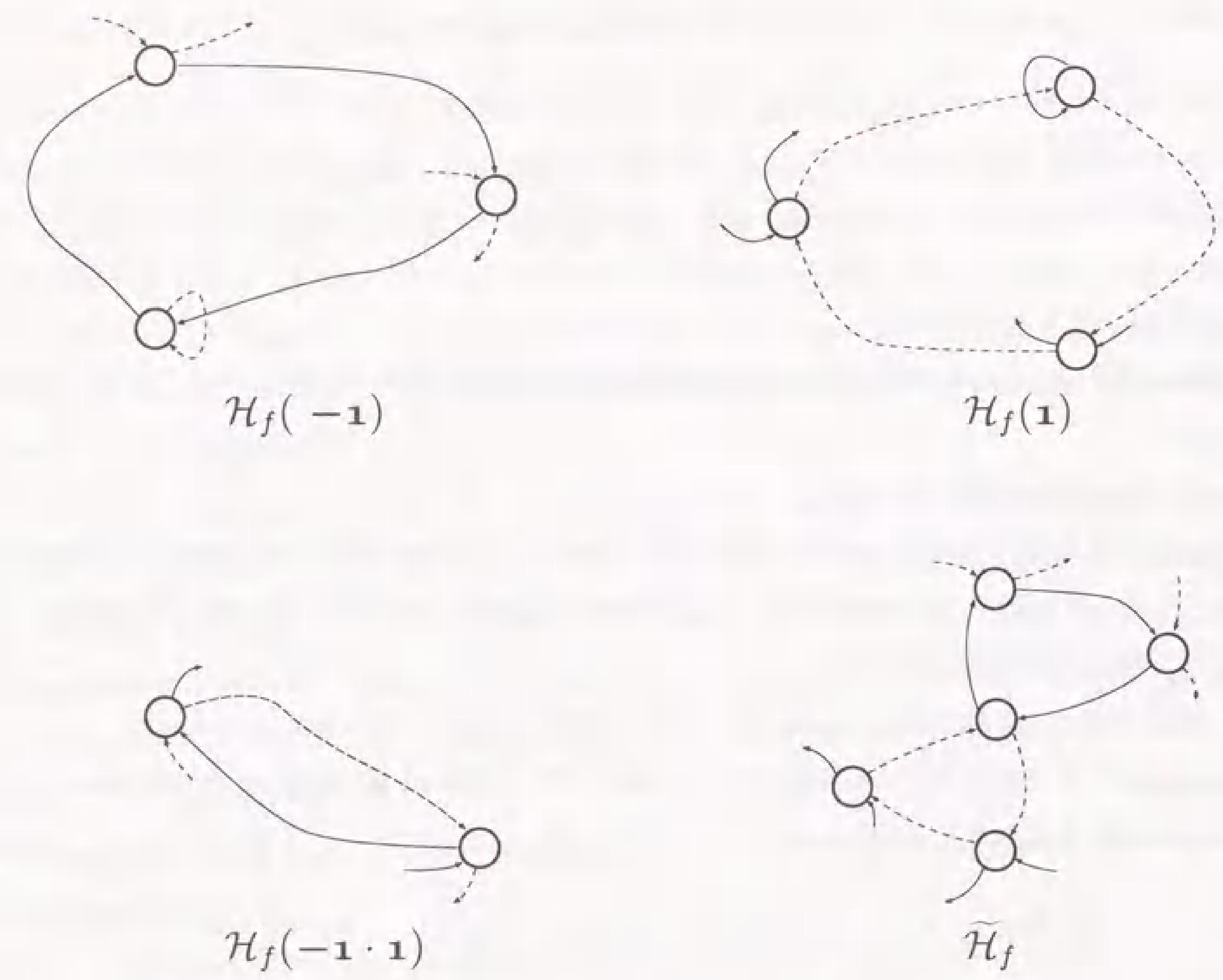


Figure 6.3: Irreducible components of canonical graph for spectral null at $f_S/3$

of the canonical graph. However by using the result in this subsection we can determine this much easier as follows. If $\varphi(\boldsymbol{a}) = \varphi(\boldsymbol{b})$ then $\boldsymbol{a}\mathcal{R}\boldsymbol{b}$. Let E be the set defined in the graph of Theorem 6.1. It is very easy to find integers i_a and i_b such that $i_a\varphi(\boldsymbol{a}) \in E$ and $i_b\varphi(\boldsymbol{b}) \in E$. Then by Lemma 6.6 we note that $i_a\varphi(\boldsymbol{a}) = i_b\varphi(\boldsymbol{b})$ if and only if $\boldsymbol{a}\mathcal{R}\boldsymbol{b}$.

6.3.3 Case 2: $n = 2d$ where d is a prime number

Let d be a positive integer. In this subsection we assume that $n = 2d$. Under this assumption we can also identify theoretically all irreducible components of canonical graphs for a spectral null at f. First we give three examples which tell us differences between the case $n = 2d$ and the previous case.

Example 6.3 Let $f = f_S/4$. Then each irreducible component of G_f is label-preserving graph isomorphic to one of the following diagrams:

$$\widetilde{\mathcal{H}}_f,\ \mathcal{H}_f(\mathbf{1}),\ \mathcal{H}_f(-\mathbf{1}),\ \mathcal{H}_f(\mathbf{1}\cdot-\mathbf{1}).$$

In these diagrams there is no pair of diagrams I and J such that $I \cong J$. We note that $\mathbf{1}\cdot-\mathbf{1}^2 \in C_f(\mathbf{1})$ (cf. Theorem 6.1).

Proof: See Appendix A.15. □

Example 6.4 Assume that $n = 6$. The FSTD shown in Fig. 6.4 is a finite subgraph of an irreducible component of $G_{f_S/6}$. Polynomials of ω in circles mean states. From the figure, we note that $G_{f_S/6}$ has an irreducible component I such that $\mathbf{1} \in C(I)$ and $-\mathbf{1} \in C(I)$ (cf. Lemma 6.5 and Theorem 6.1).

Example 6.5 Assume that $n = 6$. The FSTD in Fig. 6.5 is a finite subgraph of an irreducible component of $G_{f_S/6}$ which contains state 0. From the figure we note that the period of $\widetilde{\mathcal{H}}_{f_S/6}$ is not 6 (cf. Proposition 6.13).

Example 6.6 A part of the canonical graph for a spectral null at $f = f_S/6$ in Fig. 6.6 where solid arrows mean edges with label 1 and dotted arrows mean edges with label -1. Irreducible components of the graph will be identified in Example 6.7 after we prove Theorem 6.2.

In the rest of this subsection we identify theoretically irreducible components. The main result of this part is Theorem 6.2. Proof methods which we use in the following depend strongly on the form of the minimum polynomial of ω. Furthermore, they are different from the methods which we employed in the previous subsection. We need the following lemmas to prove the theorem.

Lemma 6.7

$$\Phi_n(D) = \sum_{i=0}^{d-1}(-1)^i D^i. \tag{6.6}$$

Proof: Since $\Phi_2(D) = D+1$, $\Phi_d(D) = D^{d-1}+D^{d-2}+\cdots+1$ and $D^{2d-1}+D^{2d-2}+\cdots+1 = \Phi_{2d}(D)\Phi_2(D)\Phi_d(D)$ [55, Chapter 6], we have (6.6) by calculation. □

Lemma 6.8 Let $\boldsymbol{s} = (\mathbf{1}\cdot-\mathbf{1})^{(d-1)/2}\cdot\mathbf{1}$. Let σ be a state of an irreducible component of G_f. Then $-\sigma$ is also a state of the component and there are two paths $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $i(\boldsymbol{x}) = i(\boldsymbol{y}) = \sigma$, $t(\boldsymbol{x}) = t(\boldsymbol{y}) = -\sigma$ and $\boldsymbol{x}$ and $\boldsymbol{y}$ generate $\boldsymbol{s}$ and $-\boldsymbol{s}$ respectively.

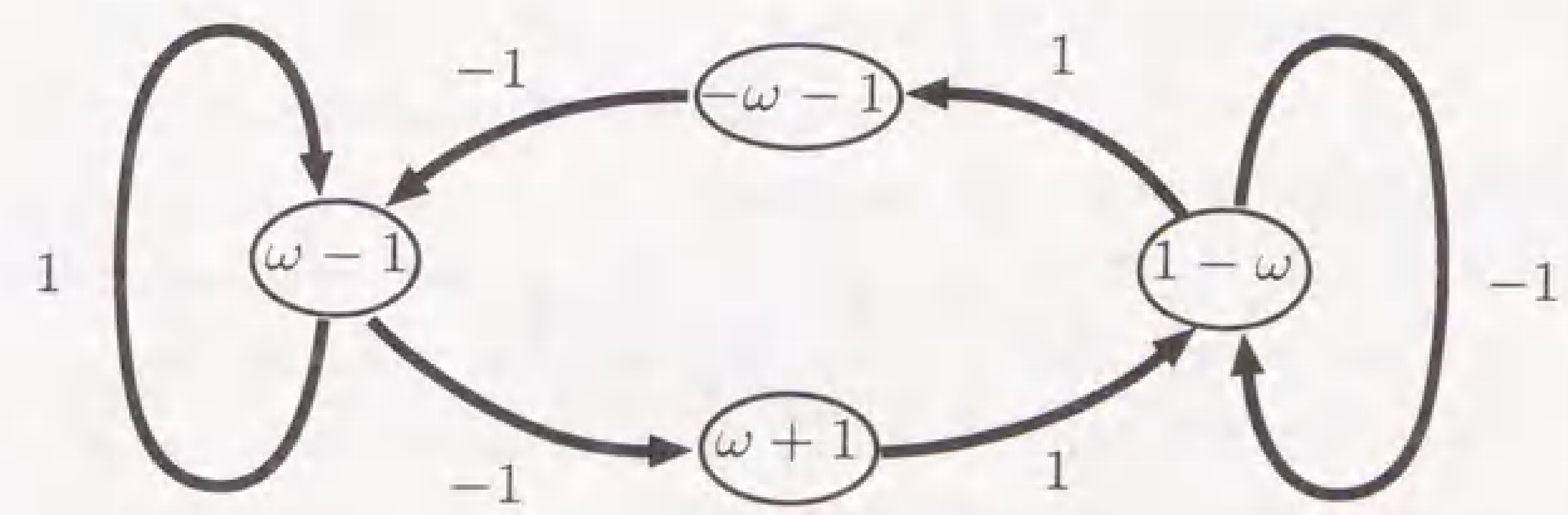


Figure 6.4: Subgraph of $G_1^{f_S/6}$

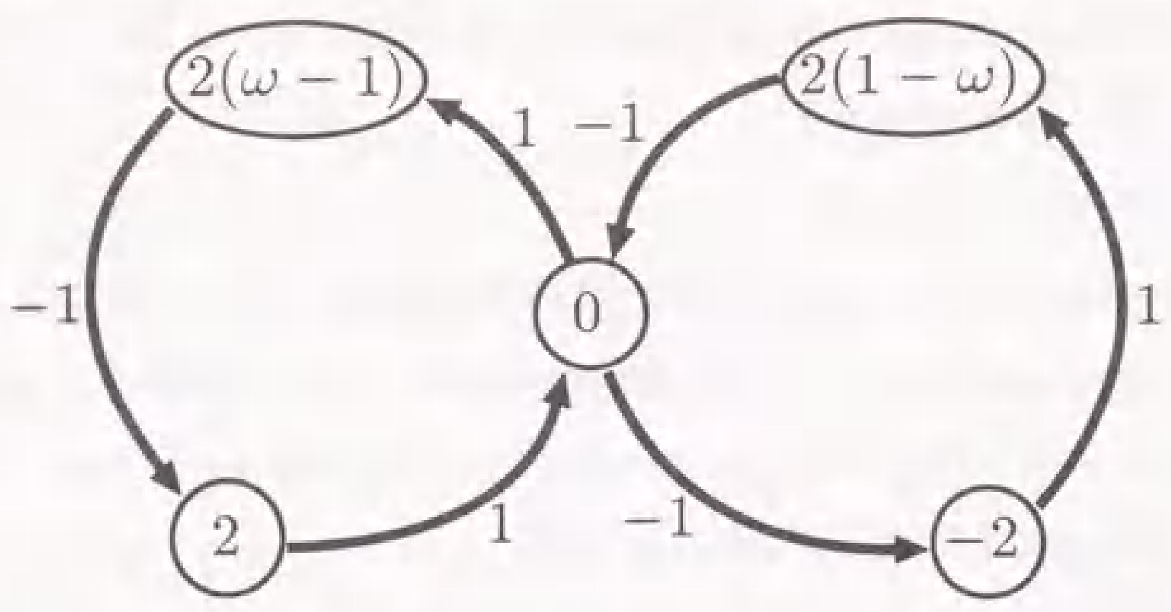


Figure 6.5: Subdgraph of $G_3^{f_S/6}$ containing state 0

Proof: By the definition of G_f, there is a path $\boldsymbol{x}$ which starts from σ and generates $\boldsymbol{s}$. Then by applying the transition rule of G_f we get $t(\boldsymbol{x}) = -\sigma$ because $\mathrm{RDS}_f(\boldsymbol{s}) = 0$ from Lemma 6.7 and $\omega^d = -1$ from assumption. Similarly, we get a path $\boldsymbol{y}$ which satisfies conditions described in the statements of this lemma. □

Lemma 6.9 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be blocks in $S_{\{-1,1\}}$ with $\lg(\boldsymbol{a}) = \lg(\boldsymbol{b}) = d$. Assume that we have $\boldsymbol{a} \neq \boldsymbol{b}^{[i]}$ and $\boldsymbol{a} \neq -\boldsymbol{b}^{[i]}$ for every i. Then $\boldsymbol{a} \notin C_f(\boldsymbol{b})$.

Proof: See Appendix A.16. □

Lemma 6.10 Let L be an odd integer with $1 \leq L \leq d-2$. Let $\boldsymbol{a} = -\mathbf{1} \cdot \mathbf{1}^L$. Then $\mathbf{1} \notin C_f(\boldsymbol{a})$.

Proof: See Appendix A.17. □

Lemma 6.11 Let L be an integer with $L + 1 \not\equiv 0 \mod d$. Then there is an integer M with $0 \leq M \leq d-2$ such that $-\mathbf{1} \cdot \mathbf{1}^M \in C_f(-\mathbf{1} \cdot \mathbf{1}^L)$.

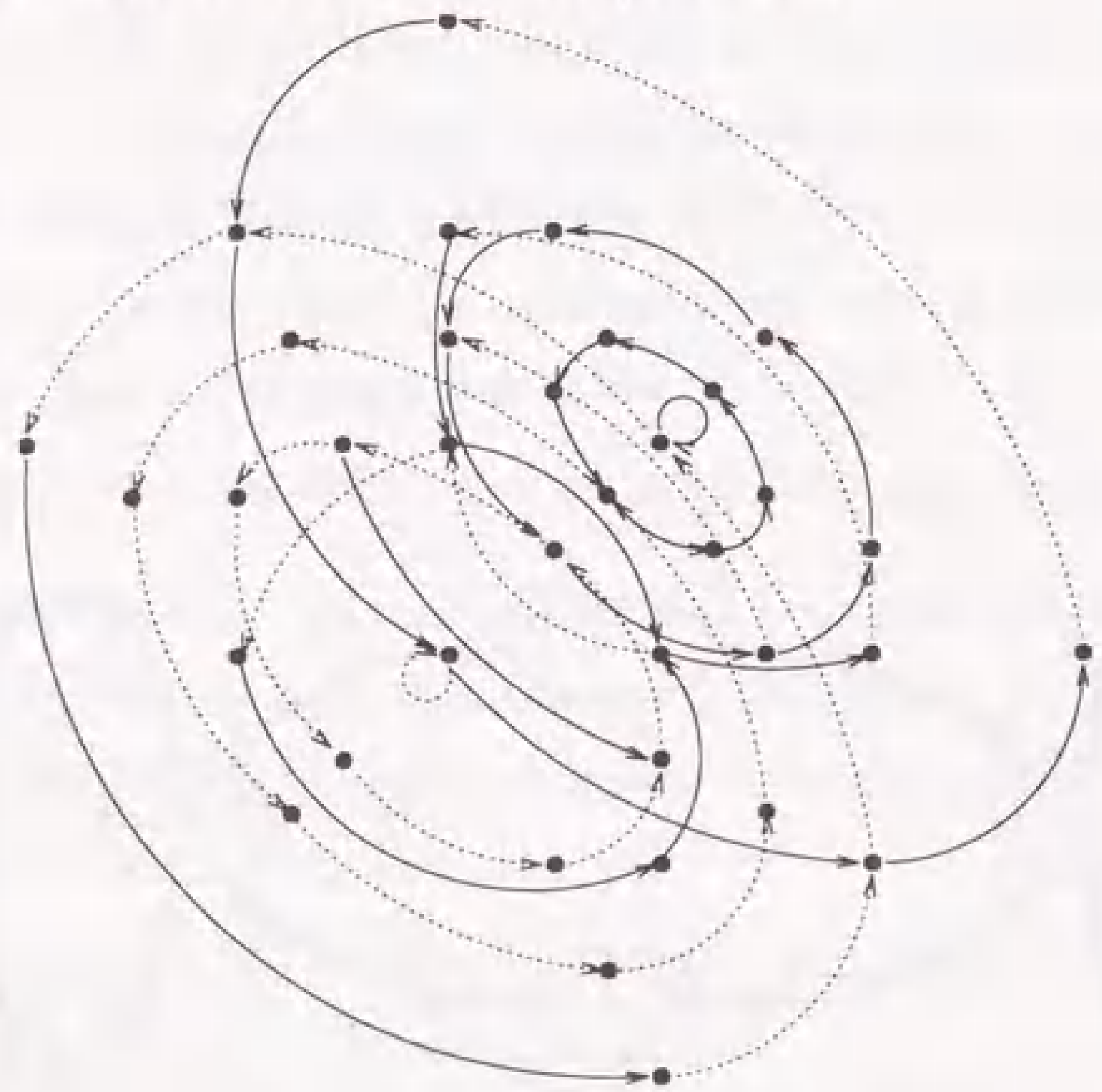


Figure 6.6: Canonical graph for spectral null at $f_S/6$

Proof: See Appendix A.18. □

Definition 6.14 Let $\boldsymbol{a} = a_0 a_1 \cdots a_{L-1} \in S_{\{-1,1\}}$. We define $N_{e,1}(\boldsymbol{a})$ and $N_{o,1}(\boldsymbol{a})$ as follows:

$$N_{e,1}(\boldsymbol{a}) = \#\{i : a_{2i} = 1\}, \quad N_{o,1}(\boldsymbol{a}) = \#\{i : a_{2i+1} = 1\}.$$

We define $N_{e,-1}(\boldsymbol{a})$ and $N_{o,-1}(\boldsymbol{a})$ similarly.

Lemma 6.12 Let $\boldsymbol{a}$ be a block such that $d \not| \lg(\boldsymbol{a})$ and $2 | \lg(\boldsymbol{a})$. Assume $N_{o,-1}(\boldsymbol{a}) = N_{e,-1}(\boldsymbol{a})$. Then $\mathbf{1} \in C_f(\boldsymbol{a})$.

Proof: See Appendix A.19. □

Lemma 6.13 Let L_1 and L_2 be distinct odd integers with $1 \leq L_1, L_2 \leq d-2$. Then we have $-\mathbf{1} \cdot \mathbf{1}^{L_1} \notin C_f(-\mathbf{1} \cdot \mathbf{1}^{L_2})$.

Proof: See Appendix A.20. □

Lemma 6.14 Let $\boldsymbol{a} \in S_{\{-1,1\}}$. Assume that $d \not| \lg(\boldsymbol{a})$ and $2 \not| \lg(\boldsymbol{a})$. Then $\mathbf{1} \in C_f(\boldsymbol{a})$.

Proof: See Appendix A.21. □

We define a relation R between two blocks $\boldsymbol{a}$ and $\boldsymbol{b}$ by "there is an integer i such that $\boldsymbol{a} = \boldsymbol{b}^{[i]}$ or $\boldsymbol{a} = -\boldsymbol{b}^{[i]}$." Then R is an equivalence relation. Let F'_d be a set of blocks in $S_{\{-1,1\}}$ of length d. We partition the set $F'_d - \{\mathbf{1}^d, -\mathbf{1}^d\}$ into R-equivalence classes. Let F_d be a set of representatives of those equivalence classes. Let $E_d = \{\mathbf{1}, -\mathbf{1} \cdot \mathbf{1}, -\mathbf{1} \cdot \mathbf{1}^3, \cdots, -\mathbf{1} \cdot \mathbf{1}^{d-2}\}$.

Lemma 6.15 Let $\boldsymbol{a} \in \mathcal{N}$. Then there is a block $\boldsymbol{b} \in E_d \bigcup F_d$ such that $\boldsymbol{b} \in C_f(\boldsymbol{a})$.

Proof: See Appendix A.22. □

The following is one of our main results.

Theorem 6.2 Let $E_d \bigcup F_d = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_K\}$. Then every irreducible component is label-preserving graph isomorphic to exactly one of the following graphs:

$$\tilde{I}_f, I_f(\boldsymbol{a}_1), I_f(\boldsymbol{a}_2), \ldots, I_f(\boldsymbol{a}_K). \tag{6.7}$$

Proof: By Proposition 6.5 and Lemma 6.15, the graphs in (6.7) are all of irreducible components of canonical graphs for a spectral null at f. So, it is enough to prove that for every pair of distinct blocks $\boldsymbol{a}$ and $\boldsymbol{b}$ in $E_d \bigcup F_d$ we have $\boldsymbol{a} \notin C_f(\boldsymbol{b})$. If $\boldsymbol{a} \in F_d$ and $\boldsymbol{b} \in F_d$ with $\boldsymbol{a} \neq \boldsymbol{b}$, we have $\boldsymbol{b} \notin C_f(\boldsymbol{a})$ by Lemma 6.9. If $\boldsymbol{a} \in E_d$ and $\boldsymbol{b} \in E_d$ with $\boldsymbol{a} \neq \boldsymbol{b}$, we have $\boldsymbol{b} \notin C_f(\boldsymbol{a})$ by Lemma 6.10 and Lemma 6.13.

Suppose $\boldsymbol{a} \in E_d$, $\boldsymbol{b} \in F_d$ and $\boldsymbol{b} \in C_f(\boldsymbol{a})$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be cycles which generate $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Since $\gcd(\lg(\boldsymbol{a}), \lg(\boldsymbol{b})) = 1$, the period of $I_f(\boldsymbol{a})$ is 1. Therefore, there are paths $\boldsymbol{w}$ and $\boldsymbol{w}'$ such that $i(\boldsymbol{x}) = i(\boldsymbol{w}) = t(\boldsymbol{w}')$, $i(\boldsymbol{y}) = t(\boldsymbol{w}) = i(\boldsymbol{w}')$, $n \mid \lg(\boldsymbol{w})$ and $n \mid \lg(\boldsymbol{w}')$. Put $\boldsymbol{z} = \boldsymbol{x} \cdot \boldsymbol{w} \cdot \boldsymbol{y} \cdot \boldsymbol{w}'$. Then $d \not| \lg(\boldsymbol{z})$, $2 \not| \lg(\boldsymbol{z})$ and $\boldsymbol{z}$ is a cycle of $I_f(\boldsymbol{a})$. Therefore we would have $\mathbf{1} \in C_f(\boldsymbol{a})$ by Lemma 6.14. But this contradicts with Lemma 6.10. Thus $\boldsymbol{a} \notin C_f(\boldsymbol{b})$. □

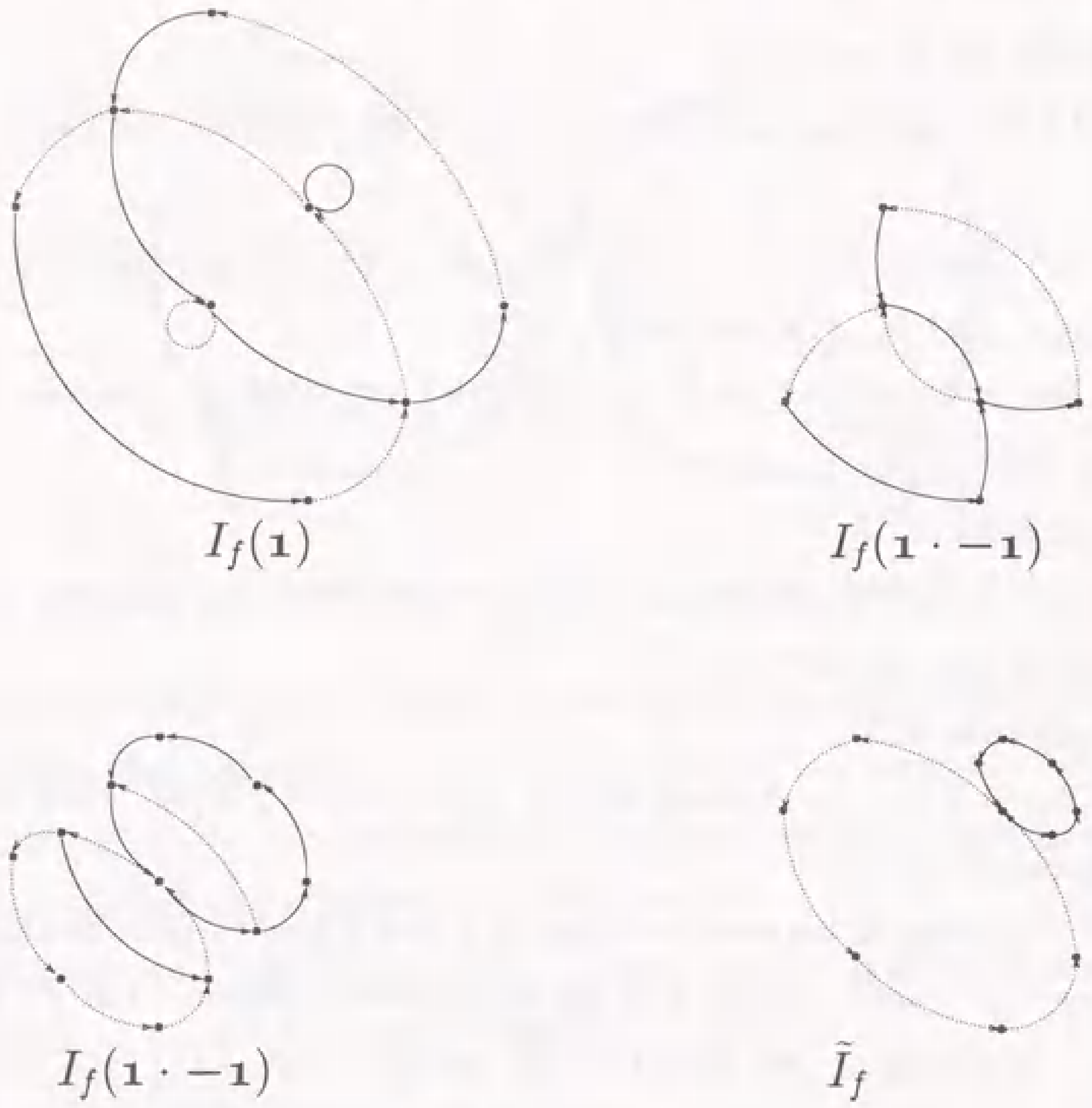


Figure 6.7: Irreducible components of canonical graph for spectral null at $f_S/6$

Example 6.7 The canonical graph for a spectral null at $f = f_S/6$ is given in Fig. 6.6. From Theorem 6.2, we note that there are four irreducible components, $I_f(\mathbf{1})$, $I_f(\mathbf{1}\cdot -\mathbf{1})$, $I_f(\mathbf{1}\cdot -\mathbf{1}\cdot \mathbf{1})$ and $\tilde{I}_f$, in the graph. They are shown in Fig. 6.7.

For any given two blocks we can determine whether they are generated by cycles in the same irreducible component as follows, by using the results in this subsection. We assume that blocks $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{N}$ are given. First we find $\boldsymbol{c}_a \in E_d \bigcup F_d$ with $\boldsymbol{c}_a R \boldsymbol{a}$ as follows. We may assume that $\lg(\boldsymbol{a}) < n$ by Lemma 6.2. If $\boldsymbol{a} = \mathbf{1}^K$ for some K then $\boldsymbol{c}_a = \mathbf{1}$. If $d \mid \lg(\boldsymbol{a})$ and $\boldsymbol{a} \neq \mathbf{1}^K$ for any K then $\boldsymbol{c}_a = \boldsymbol{a}$ or $\boldsymbol{c}_a = -\boldsymbol{a}$ by Lemma 6.9. If $d \not| \lg(\boldsymbol{a})$ and $2 \not| \lg(\boldsymbol{a})$ then $\boldsymbol{c}_a = \mathbf{1}$ by Lemma 6.14. If $2 \mid \lg(\boldsymbol{a})$, $d \not| \lg(\boldsymbol{a})$ and $\boldsymbol{a} \neq \mathbf{1}^K$ for any K then we can find a block $\boldsymbol{c}_a$ with $\boldsymbol{c}_a R \boldsymbol{a}$ by tracing the proof of Lemma 6.11. We can also find a block $\boldsymbol{c}_b \in E_d \bigcup F_d$ with $\boldsymbol{c}_b R \boldsymbol{b}$ as well. Then by Theorem 6.2, $\boldsymbol{a} R \boldsymbol{b}$ if and only if $\boldsymbol{c}_a = \boldsymbol{c}_b$.

6.3.4 Number of irreducible components

Let $N(n)$ be the number of irreducible components of $G_{f_S/n}$ with period less than n. It follows from Proposition 6.1 that for every integer l with $\gcd(l, n) = 1$ $N(n)$ also

n	$N(n)$	n	$N(n)$
2	2	12*	5
3	3	13	13
4	3	14	12
5	5	15*	25
6	3	16*	20
7	7	17	17
8*	6	18*	13
9*	11	19	19
10	5	20*	154
11	11	–	—

Table 6.1: Table of $N(n)$

represents the number of irreducible components of $G_{f_S l/n}$ with period less than n. If n is a prime number then we have

$$N(n) = n \tag{6.8}$$

by Theorem 6.1.

Let d be an odd prime number and assume that $n = 2d$. Let E_d and F_d be the sets defined in the previous subsection. By Theorem 6.2, $E_d \bigcup F_d$ is a set of representative blocks of Q-equivalence classes of blocks of length less than n. We have $N(n) = \#(E_d \bigcup F_d)$ and so

$$N(n) = \#E_d + \#F_d = \frac{d+1}{2} + \frac{2^{d-1}-1}{d}. \tag{6.9}$$

We have also proved that $N(4) = 3$ in Example 6.3.

We implemented the algorithm given in the proof of Proposition 6.12 as a computer program. Using the program we calculated $N(n)$ for $n \leq 20$. These values are given in Table 6.1. In the table n^* means that $N(n)$ is given by (6.8) or (6.9). For $n > 20$ it is very difficult to execute the program because the size of (6.4) is very large ($2^n \times \phi(n)$, where $\phi(n)$ is the euler number of n) and very large coefficients appear in solving the equation.

6.4 Irreducible components of canonical graphs for second-order spectral nulls at dc

First, we review definitions and results only on second-order spectral nulls at dc. For complete descriptions of Definition 6.15, 6.16, 6.18, 6.19, and Theorem 6.3, see [24] or

[29]. We show results on (weekly) connected subgraphs of canonical graphs for second-order spectral nulls at dc. Then we prove all of them are really irreducible.

We assume that the channel symbol alphabet is $\{-1, 1\}$ in this section as well as in the previous section. But in the following we have to consider blocks consisting of symbols in another alphabet.

For a block $\boldsymbol{a}$, we mean $\mathrm{RDS}_0(\boldsymbol{a})$ by $\mathrm{RDS}(\boldsymbol{a})$.

Definition 6.15 Let $\boldsymbol{a} = a_0 a_1 \cdots a_{L-1} \in S_Z$. We define $rds(\boldsymbol{a})$ and $\mathrm{RDS}^{(2)}(\boldsymbol{a})$ as follows:

$$\begin{aligned} rds(\boldsymbol{a}) &= \mathrm{RDS}(a_0)\mathrm{RDS}(a_0 a_1) \cdots \mathrm{RDS}(\boldsymbol{a}), \\ \mathrm{RDS}^{(2)}(\boldsymbol{a}) &= \mathrm{RDS}(rds(\boldsymbol{a})) = \sum_{i=0}^{L-1} \sum_{j=0}^{i} a_j. \end{aligned}$$

Proposition 6.14 Let $\boldsymbol{s} \in S_Z$ and assume that $\mathrm{RDS}(\boldsymbol{s}) = 0$. Let m be a positive integer. Then $\mathrm{RDS}^{(2)}(\boldsymbol{s}^m) = m\mathrm{RDS}^{(2)}(\boldsymbol{s})$.

Definition 6.16 A state-transition-diagram (G, γ) satisfies a *second-order coboundary condition at dc* if there are functions ϕ_1 and ϕ_2 such that for every edge e, the following equations hold:

$$\begin{aligned} \gamma(e) &= \phi_1(t_G(e)) - \phi_1(i_G(e)), && (6.10) \\ \phi_1(i_G(e)) &= \phi_2(t_G(e)) - \phi_2(i_G(e)). && (6.11) \end{aligned}$$

We call ϕ_1 and ϕ_2 a pair of coboundary functions of (G, γ) for a second-order spectral null at dc.

Remark **6.6** Let $\boldsymbol{x} = x_0 x_1 \cdots x_{L-1}$ be a cycle of G. Then, by summing up (6.11) along $\boldsymbol{x}$, we have

$$\sum_{j=0}^{L-1} \phi_1(i_G(x_j)) = \phi_2(t_G(x_{L-1})) - \phi_2(i_G(x_0)) = 0.$$

From (6.10), we have $\phi_1(t_G(x_j)) = \sum_{l=0}^{j} \gamma(x_l) + \phi_1(i_G(x_0))$, $j = 0, 1, \ldots, L-1$. By summing up these equations along $\boldsymbol{x}$ we have $\sum_{j=0}^{L-1} \phi_1(i_G(x_j)) = \mathrm{RDS}^{(2)}(\gamma(\boldsymbol{x})) + \lg(\boldsymbol{x})\phi_1(i_G(x_0))$. Hence we get

$$\phi_1(i_G(x_0)) = -\frac{\mathrm{RDS}^{(2)}(\gamma(\boldsymbol{x}))}{\lg(\boldsymbol{x})}. \qquad (6.12)$$

Therefore we note that ϕ_1 is determined uniquely if G has a cycle. Define a function η on the set of edges in G by $\eta(e) = \phi_1(i_G(e))$ for edge e in G. Then ϕ_2 can be regarded as a coboundary function of (G, η) for a (first-order) spectral null at dc. Therefore by Corollary 6.1 we note that ϕ_2 is determined up to constant, that is, for every constant c if we define a function ϕ_2' by $\phi_2'(\sigma) = \phi_2(\sigma) + c$ for state σ in G, then the pair of ϕ_1 and ϕ_2' is also a pair of coboundary functions of G; conversely, if there is another pair of coboundary functions ϕ_1 and $\tilde{\phi}_2$ of G then there is a constant c such that $\phi_2(\sigma) - \tilde{\phi}_2(\sigma) = c$ for every state σ in G.

Definition 6.17 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be blocks of numbers with $\lg(\boldsymbol{a}) = \lg(\boldsymbol{b})$. We define a block $\boldsymbol{a} - \boldsymbol{b} = d_0 d_1 \cdots d_{\lg(\boldsymbol{a})-1}$ by $d_i = a_i - b_i$, $0 \le i \le \lg(\boldsymbol{a}) - 1$.

Notice that $\mathrm{RDS}(\boldsymbol{a} \pm \boldsymbol{b}) = \mathrm{RDS}(\boldsymbol{a}) \pm (\boldsymbol{b})$ and $\mathrm{RDS}^{(2)}(\boldsymbol{a} \pm \boldsymbol{b}) = \mathrm{RDS}^{(2)}(\boldsymbol{a}) \pm^{(2)} (\boldsymbol{b})$.

Theorem 6.3 [24, Theorem 2] Let (G, γ) be an FSTD and let P be a transition probability matrix compatible with G. Then the following are equivalent:

1) (G, P, γ) has a second-order spectral null at dc.

2) (G, γ) satisfies a second-order coboundary condition at dc.

3) (G, γ) satisfies the following: a) there is some block $\boldsymbol{c} \in C(G)$ such that $\mathrm{RDS}(\boldsymbol{c}) = 0$; b) for every pair of cycles $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $i_G(\boldsymbol{x}) = i_G(\boldsymbol{y})$ and $\lg(\boldsymbol{x}) = \lg(\boldsymbol{y})$, we have $\mathrm{RDS}^{(2)}(\gamma(\boldsymbol{x}) - \gamma(\boldsymbol{y})) = 0$.

This theorem is a part of Theorem 2[24].

Next we define canonical graphs for second-order spectral nulls at dc as well as first-order spectral nulls.

Definition 6.18 Let p be a positive integer. We define a CSTD $G_p^{(2)}$ as follows:

- the state set of $G_p^{(2)}$ is Z^2;
- there is an edge from state (σ_1, σ_2) to (τ_1, τ_2) with label b if and only if $\tau_1 = \sigma_1 + pb$ and $\tau_2 = \sigma_2 + \sigma_1$.

Definition 6.19 A CSTD $\mathcal{G}$ is a period-p canonical graph for a second-order spectral null constraint at dc if :

- every FSTD contained in $\mathcal{G}$ has a second-order spectral null at dc;
- every FSTD with period p which produces a second-order spectral null at dc is label-preserving graph homomorphic to a subgraph of $\mathcal{G}$.

Then $G_p^{(2)}$ is a period-p canonical graph for a second-order spectral null constraint at dc [24, Definition 8].

We identify all irreducible components of $G_p^{(2)}$ here. Our results in this section may be of practical interest in designing second-order spectral null codes. First we collect basic observations on canonical graphs for second-order spectral nulls at dc.

We note the following.

Proposition 6.15 [24, Definition 8] A pair of coboundary functions ψ_1 and ψ_2 of $G_p^{(2)}$ is given by

$$\psi_1(\sigma) = \frac{\sigma_1}{p}, \quad \psi_2(\sigma) = \frac{\sigma_2}{p}, \qquad \text{for state } \sigma = (\sigma_1, \sigma_2). \qquad (6.13)$$

Proposition 6.16 Let σ be a state in $G_p^{(2)}$ and let $\boldsymbol{s}$ and $\boldsymbol{t}$ be blocks in $S_{\{-1,1\}}$ with $\lg(\boldsymbol{s}) = \lg(\boldsymbol{t})$. Let ϕ_1 and ϕ_2 be the pair of coboundary functions of $G_p^{(2)}$ given by (6.13). Assume that $\mathrm{RDS}(\boldsymbol{s}) = 0$ and $\phi_1(\sigma) = -\mathrm{RDS}^{(2)}(\boldsymbol{s})/\lg(\boldsymbol{s})$. Then the following hold.

1) there is a cycle in $G_p^{(2)}$ which starts from σ and which generates $\boldsymbol{s}$;

2) if $\mathrm{RDS}(\boldsymbol{t}) = 0$ and $\mathrm{RDS}^{(2)}(\boldsymbol{s} - \boldsymbol{t}) = 0$ then there is a cycle which starts from σ and generates $\boldsymbol{t}$;

3) $\boldsymbol{s} \in C(G^{(2)}_{\mathrm{lg}(\boldsymbol{s})})$.

Proof: 1) can be proved by calculation and 2) immediately follows from 1) since $\mathrm{RDS}^{(2)}(\boldsymbol{s} - \boldsymbol{t}) = \mathrm{RDS}^{(2)}(\boldsymbol{s}) - \mathrm{RDS}^{(2)}(\boldsymbol{t})$.

Consider the state τ in $G^{(2)}_{\mathrm{lg}(\boldsymbol{s})}$ given by $\tau = (-\mathrm{RDS}^{(2)}(\boldsymbol{s}), 0)$. By Proposition 6.15 we have $\phi_1(\sigma) = -\mathrm{RDS}^{(2)}(\boldsymbol{s})/\mathrm{lg}(\boldsymbol{s})$. Hence 3) follows from 1). □

Let p be a positive integer. For a state σ in $G_p^{(2)}$, let $L_p(\sigma)$ be a (weekly) connected subgraph containing σ, that is, a subgraph which consists of all states τ (and all edges connected to those states) such that there are paths from τ to σ or paths from σ to τ. Notice that if there is a path from σ to state τ or from τ to σ in $L_p(\sigma)$ then $L_p(\sigma) = L_p(\tau)$.

Furthermore we have the following.

Proposition 6.17 Let p and p' be positive integers. Let σ and σ' be states in $G_p^{(2)}$ and $G_{p'}^{(2)}$ respectively. Let ϕ_1 and ϕ_2 be the pair of coboundary functions of $L_p(\sigma)$ given by (6.13) and let ϕ_1' and ϕ_2' be that of $L_{p'}(\sigma')$. If $\phi_1(\sigma) = \phi_1'(\sigma')$ then $L_p(\sigma) \cong L_{p'}(\sigma')$.

Proof: Define a function $\tilde{\phi}_2'$ by $\tilde{\phi}_2'(\tau') = \phi_2'(\tau') + (\phi_2(\sigma) - \phi_2'(\sigma'))$ for state τ' in $L_{p'}(\sigma')$. By Remark 6.6, we note that the pair of ϕ_1' and $\tilde{\phi}_2'$ is also a pair of coboundary functions of $L_{p'}(\sigma')$. Define ψ and ψ' by $\psi = (\phi_1, \phi_2)$ and $\psi' = (\phi_1', \tilde{\phi}_2')$. We note that ψ and ψ' are injective and $\psi(\sigma) = \psi'(\sigma')$. Let $b \in \{-1, 1\}$. Let τ and τ' be states in $L_p(\sigma)$ and $L_{p'}(\sigma')$ respectively. Then there are edges e and e' with label b such that $i_{G_p^{(2)}}(e) = \tau$ and $i_{G_{p'}^{(2)}}(e') = \tau'$. Put $\eta = t_{G_p^{(2)}}(e)$ and $\eta' = t_{G_{p'}^{(2)}}(e')$. We note that if $\psi(\tau) = \psi'(\tau')$ then $\psi(\eta) = \psi'(\eta')$ and vice versa. Therefore it is easy to see that $\psi^{-1} \circ \psi'$ is well-defined and gives a label-preserving graph isomorphism of $L_{p'}(\sigma')$ to $L_p(\sigma)$. □

Corollary 6.7 Let p and k be positive integers. Then $G_p^{(2)}$ is label-preserving graph isomorphic to a subgraph of $G_{pk}^{(2)}$.

Proof: Let ϕ_1 and ϕ_2 be the pair of coboundary functions of $G_p^{(2)}$ given by (6.13) and let ϕ_1' and ϕ_2' be that of $G_{pk}^{(2)}$. Let (σ_1, σ_2) be a state in $G_p^{(2)}$. There is a state (σ_1', σ_2') in $G_{pk}^{(2)}$ such that $\sigma_1' = \sigma_1 k$. Then we have $\phi_1((\sigma_1, \sigma_2)) = \phi_1'((\sigma_1', \sigma_2'))$. Therefore this corollary follows from Proposition 6.17. □

Corollary 6.8 Let $\boldsymbol{s}$ be a block in $S_{\{-1,1\}}$ such that $\mathrm{RDS}(\boldsymbol{s}) = 0$. Let p and p' be positive integers and let σ and σ' be states in $G_p^{(2)}$ and $G_{p'}^{(2)}$ respectively. If $\boldsymbol{s} \in C(L_p(\sigma)) \cap C(L_{p'}(\sigma'))$, then $L_p(\sigma) \cong L_{p'}(\sigma')$.

Proof: Let ϕ_1 and ϕ_2 be a pair of coboundary functions of $G_p^{(2)}$ and let ϕ_1' and ϕ_2' be that of $G_{p'}^{(2)}$. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be cycles in $L_p(\sigma)$ and $L_{p'}(\sigma')$ respectively which generate $\boldsymbol{s}$. By Remark 6.6 we may assume that ϕ_1, ϕ_2, ϕ_1' and ϕ_2' are functions given by (6.13) and we have $\phi_1(i_{G_p^{(2)}}(\boldsymbol{x})) = \phi_1'(i_{G_{p'}^{(2)}}(\boldsymbol{x}'))$. By Proposition 6.17 we have $L_p(\sigma) = L_p(i_{G_p^{(2)}}(\boldsymbol{x})) \cong L_{p'}(i_{G_{p'}^{(2)}}(\boldsymbol{x}')) = L_{p'}(\sigma')$. □

From this corollary and 3) of Proposition 6.16, we can get all irreducible components of $G_p^{(2)}$, $p = 1, 2, \ldots$ in principle by generating irreducible components which contain state $(-\mathrm{RDS}^{(2)}(\boldsymbol{s}), 0)$ for all blocks $\boldsymbol{s} \in S_{\{-1,1\}}$ with $\mathrm{RDS}(\boldsymbol{s}) = 0$. However, we will prove it in the following (Proposition 6.17 and Corollary 6.8) that some of them are label-preserving graph isomorphic to one another. We prove that for every p, the number of irreducible components of $G_p^{(2)}$ (more strictly, the number of label-preserving graph isomorphic classes of irreducible components of $G_p^{(2)}$) is finite.

Let p be a positive integer and let σ be a state in $G_p^{(2)}$. By the definition of $G_p^{(2)}$ if (σ_1, σ_2) and (τ_1, τ_2) are states in $L_p(\sigma)$ then $\sigma_1 \equiv \sigma_2 \bmod p$. Moreover, we have the following.

Proposition 6.18 Let p be a positive integer. Let $\sigma = (\sigma_1, \sigma_2)$ and $\sigma' = (\sigma_1', \sigma_2')$ be states in $G_p^{(2)}$. If $\sigma_1 \equiv \sigma_1' \bmod p$ then $L_p(\sigma) \cong L_p(\sigma')$. Moreover, if $L_p(\sigma) \cong L_p(\sigma')$ and $C(L_p(\sigma)) \neq \emptyset$ then $\sigma_1 \equiv \sigma_1' \bmod p$.

Proof: Suppose that $\sigma_1 \equiv \sigma_1' \bmod p$. Then, from the definition of $G_p^{(2)}$, $L_p(\sigma')$ contains a state (τ_1, τ_2) with $\tau_1 = \sigma_1$. By Proposition 6.15 and Proposition 6.17 we have $L_p(\sigma) \cong L_p(\sigma')$.

Suppose that $L_p(\sigma) \cong L_p(\sigma')$ and $C(L_p(\sigma)) \neq \emptyset$. Then there is a block $\boldsymbol{s}$ in $S_{\{-1,1\}}$ with $\boldsymbol{s} \in C(L_p(\sigma)) \cap C(L_p(\sigma'))$. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be cycles in $L_p(\sigma)$ and $L_p(\sigma')$ respectively which generate $\boldsymbol{s}$. Put $(\tau_1, \tau_2) = i_{L_p(\sigma)}(\boldsymbol{x})$ and $(\tau_1', \tau_2') = i_{L_p(\sigma')}(\boldsymbol{x}')$. By Remark 6.6 and Proposition 6.15 we must have $-\mathrm{RDS}^{(2)}(\boldsymbol{s})/\mathrm{lg}(\boldsymbol{s}) = \tau_1/p = \tau_1'/p$ and hence, $\tau_1 = \tau_1'$. Therefore by the definition of $G_p^{(2)}$ we get $\sigma_1 \equiv \sigma_1' \bmod p$. □

We note that there is no path between $L_p((i, 0))$ and $L_p((j, 0))$ for any pair of distinct integers i and j with $0 \leq i, j \leq p - 1$. Therefore we have only to consider the irreducibility of $L_p((i, 0))$, $i = 0, 1, \ldots, p - 1$. In fact, they are all irreducible (Theorem 6.4). To prove this we need Proposition 6.19 and Lemma 6.16.

Proposition 6.19 Let p be a positive integer and let η be a state in $G_p^{(2)}$. If $L_p(\eta)$ has a cycle, then $L_p(\eta)$ is irreducible.

Proof: See Appendix A.23. □

Therefore in order to prove that $L_p((i, 0))$, $i = 0, 1, \ldots, p - 1$ are irreducible we show that each of them contains a cycle.

Lemma 6.16 Let $\boldsymbol{s}$ and $\boldsymbol{t}$ be blocks in $S_{\{-1,1\}}$ such that $\mathrm{RDS}(\boldsymbol{s}) = \mathrm{RDS}(\boldsymbol{t}) = 0$. Let I be an irreducible component such that $\boldsymbol{s} \in C(I)$. Then $\boldsymbol{t} \in C(I)$ if and only if

$$\frac{\mathrm{RDS}^{(2)}(\boldsymbol{s})}{\mathrm{lg}(\boldsymbol{s})} - \frac{\mathrm{RDS}^{(2)}(\boldsymbol{t})}{\mathrm{lg}(\boldsymbol{t})} \in Z. \tag{6.14}$$

Proof: Let ϕ_1 and ϕ_2 be a pair of coboundary functions of I. Let $\boldsymbol{x}$ be a cycle which generates $\boldsymbol{s}$. By Remark 6.6, we have $\phi_1(i_I(\boldsymbol{x})) = -\mathrm{RDS}^{(2)}(\boldsymbol{s})/\lg(\boldsymbol{s})$. Put $m = \mathrm{RDS}^{(2)}(\boldsymbol{s})/\lg(\boldsymbol{s}) - \mathrm{RDS}^{(2)}(\boldsymbol{t})/\lg(\boldsymbol{t})$.

Suppose that $\boldsymbol{t} \in C(I)$. Let $\boldsymbol{y}$ be a cycle which generates $\boldsymbol{t}$. We have $\phi_1(i_I(\boldsymbol{y})) = -\mathrm{RDS}^{(2)}(\boldsymbol{t})/\lg(\boldsymbol{t})$. There is a path $z_0 z_1 \cdots z_{L-1}$ which goes from $i_I(\boldsymbol{x})$ to $i_I(\boldsymbol{y})$ and let $\boldsymbol{v}$ be the block generated by the path. From (6.10), we have $m = \phi_1(i_I(\boldsymbol{y})) - \phi_1(i_I(\boldsymbol{x})) = \sum_{l=0}^{L-1}(\phi_1(t_I(z_l)) - \phi_1(i_I(z_l))) = \mathrm{RDS}(\boldsymbol{v}) \in Z$.

Conversely suppose that m is an integer. Let p be an integer such that I is a subgraph of $G_p^{(2)}$. From (6.10) there is a path which goes from $i_I(\boldsymbol{x})$ to a state σ and which generates a block $\boldsymbol{d} \in S_{\{-1,1\}}$, where σ is the state in $G_p^{(2)}$ with $\phi_1(\sigma) = \phi_1(i_I(\boldsymbol{x})) + m = -\mathrm{RDS}^{(2)}(\boldsymbol{t})/\lg(\boldsymbol{t})$ and $\boldsymbol{d}$ is given as follows:

$$\boldsymbol{d} = \begin{cases} \boldsymbol{1}^m & \text{if } m \geq 0; \\ -\boldsymbol{1}^{-m} & \text{if } m < 0. \end{cases}$$

By Proposition 6.16 there is a cycle in $L_p(i_I(\boldsymbol{x}))$ which starts from σ and which generates $\boldsymbol{t}$. Since $L_p(i_I(\boldsymbol{x}))$ is irreducible by Proposition 6.19, we have $\boldsymbol{t} \in C(I)$. □

Let p be a positive even integer. For $i = 0, 1, \ldots, p/2 - 1$ we define $\zeta_p(i)$ as follows:

$$\zeta_p(i) = -\boldsymbol{1}^{i+1} \cdot \boldsymbol{1} \cdot -\boldsymbol{1}^{p/2-i-1} \cdot \boldsymbol{1}^{p/2-1}.$$

We have

$$\mathrm{RDS}^{(2)}(\zeta_p(i)) - \mathrm{RDS}^{(2)}(\zeta_p(i+1)) = \mathrm{RDS}^{(2)}(\zeta_p(i) - \zeta_p(i+1)) = 2, \quad \text{for every } i \text{ with } 0 \leq i \leq p/2 - 2. \tag{6.15}$$

Then we have the following result.

Theorem 6.4 Let p be a positive integer. Then $G_p^{(2)}$ has p irreducible components (more strictly, there are p label-preserving graph isomorphic classes of irreducible components of $G_p^{(2)}$), which are $L_p((i,0))$, $i = 0, 1, \ldots, p-1$.

Proof: Let ϕ_1 and ϕ_2 be the pair of coboundary functions of $G_p^{(2)}$ given by (6.13). We have

$$\begin{aligned} \mathrm{RDS}^{(2)}(\boldsymbol{1}^{2p} \cdot -\boldsymbol{1}^{2p}) &= \mathrm{RDS}(rds(\boldsymbol{1}^{2p} \cdot -\boldsymbol{1}^{2p})) \\ &= \mathrm{RDS}(1 \cdot 2 \cdots (2p-1) \cdot 2p \cdot (2p-1) \cdots 2 \cdot 1 \cdot 0) \\ &= \sum_{i=1}^{2p} i + \sum_{i=1}^{2p-1} i \\ &= 4p^2. \end{aligned}$$

By Proposition 6.15 and Proposition 6.16 a cycle starting from $(-p^2, 0)$ in $G_p^{(2)}$ generates

$\boldsymbol{1}^{2p} \cdot -\boldsymbol{1}^{2p}$. Since $(\zeta_{4p}(2p-1))^{[2p]} = \boldsymbol{1}^{2p} \cdot -\boldsymbol{1}^{2p}$, we have

$$\mathrm{RDS}^{(2)}(\zeta_{4p}(2p-1))/4p \in Z \tag{6.16}$$

by Proposition 6.15 and Lemma 6.16. Let $i \in Z$ with $0 \leq i \leq p-1$. From (6.15) we have $\mathrm{RDS}^{(2)}(\zeta_{4p}(l))/4p - \mathrm{RDS}^{(2)}(\zeta_{4p}(l+2))/4p = 1/p$ for l with $0 \leq l \leq 2p-3$. So, by (6.16) there is an integer j_i with $0 \leq j_i \leq 2p-1$ such that $-\mathrm{RDS}^{(2)}(\zeta_{4p}(j_i))/4p - i/p \in Z$. This means that $-\mathrm{RDS}^{(2)}(\zeta_{4p}(j_i))/4 \in Z$. Therefore by Proposition 6.15 there is a state σ in $G_p^{(2)}$ with $\phi_1(\sigma) = (-\mathrm{RDS}^{(2)}(\zeta_{4p}(j_i))/4)/p$. Hence by Proposition 6.16 there is a cycle which starts from σ and generates $\zeta_{4p}(j_i)$. Since $p|(-\mathrm{RDS}^{(2)}(\zeta_{4p}(j_i))/4 - i)$, by Proposition 6.18 we note that $\zeta_{4p}(j_i) \in C(L_p((i,0)))$. Thus by Proposition 6.19, we conclude that $L_p((j,0))$, $j = 0, 1, \ldots, p-1$ are all irreducible.

Let i and j be integers such $i \not\equiv j \bmod p$. By Proposition 6.18 we note that $L_p((i,0)) \not\cong L_p((j,0))$ because $C(L_p((i,0)))$ is not empty. By definition there is no path from $L_p((i,0))$ to $L_p((j,0))$. □

The following lemma gives a representative block for each irreducible component.

Lemma 6.17 Let $\boldsymbol{s} \in S_{\{-1,1\}}$ be a block such that $\mathrm{RDS}(\boldsymbol{s}) = 0$, and put $L = \lg(\boldsymbol{s})$. Let I be an irreducible component such that $\boldsymbol{s} \in C(I)$. Then there is a unique integer i such that $0 \leq i < L/2$ and $\zeta_L(i) \in C(I)$.

Proof: Since $\mathrm{RDS}(\boldsymbol{s}) = 0$, we have $\#\{j : s_j = 1\} = \#\{j : s_j = -1\}$ where we put $\boldsymbol{s} = s_0 s_1 \cdots s_{L-1}$. Suppose that for any i and j $(0 \leq j \leq L/2-1)$ $\boldsymbol{s}^{[i]} \neq \zeta_L(j)$. Therefore, we may assume that there are blocks $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$ (some of them can be empty blocks) such that $\boldsymbol{s} = \boldsymbol{w}_1 \cdot -\boldsymbol{1} \cdot \boldsymbol{1} \cdot \boldsymbol{w}_2 \cdot \boldsymbol{1} \cdot -\boldsymbol{1} \cdot \boldsymbol{w}_3$. Put $\boldsymbol{t} = \boldsymbol{w}_1 \cdot \boldsymbol{1} \cdot -\boldsymbol{1} \cdot \boldsymbol{w}_2 \cdot -\boldsymbol{1} \cdot \boldsymbol{1} \cdot \boldsymbol{w}_3$. Then we have

$$\begin{aligned} \mathrm{RDS}^{(2)}(\boldsymbol{s} - \boldsymbol{t}) &= \mathrm{RDS}^{(2)}(\boldsymbol{0}^{\lg(\boldsymbol{w}_1)} \cdot -\boldsymbol{2} \cdot \boldsymbol{2} \cdot \boldsymbol{0}^{\lg(\boldsymbol{w}_2)} \cdot \boldsymbol{2} \cdot -\boldsymbol{2} \cdot \boldsymbol{0}^{\lg(\boldsymbol{w}_3)}) \\ &= \mathrm{RDS}(\boldsymbol{0}^{\lg(\boldsymbol{w}_1)} \cdot -\boldsymbol{2} \cdot \boldsymbol{0}^{\lg(\boldsymbol{w}_2)+1} \cdot \boldsymbol{2} \cdot \boldsymbol{0}^{\lg(\boldsymbol{w}_3)+1}) \\ &= 0. \end{aligned}$$

Hence by Proposition 6.16, $\boldsymbol{t} \in C(I)$. Therefore by repeating this transformation, we get integers j and M $(0 \leq j, M < L/2)$ such that $\boldsymbol{1}^M \cdot -\boldsymbol{1}^j \cdot \boldsymbol{1} \cdot -\boldsymbol{1}^{L/2-j} \cdot \boldsymbol{1}^{L/2-M-1} \in C(I)$. Let $\boldsymbol{u}$ be the block that we have gotten. Then, $\boldsymbol{u}^{[M]} = \zeta_L(j-1)$ if $j > 0$, and $\boldsymbol{u}^{[M+1]} = \zeta_L(L/2-1)$ if $j = 0$.

Assume that $\zeta_L(j) \in C(I)$ and $\zeta_L(j') \in C(I)$. By Lemma 6.16, $\mathrm{RDS}^{(2)}(\zeta_L(j))/L - \mathrm{RDS}^{(2)}(\zeta_L(j'))/L$ is an integer, that is, $\mathrm{RDS}^{(2)}(\zeta_L(j)) \equiv \mathrm{RDS}^{(2)}(\zeta_L(j')) \bmod L$. By (6.15), we have $\left|\mathrm{RDS}^{(2)}(\zeta_L(j)) - \mathrm{RDS}^{(2)}(\zeta_L(j'))\right| \leq L-2$. Thus $j = j'$. □

By Lemma 6.17 and the proof of Theorem 6.4 we note that for every i with $0 \leq i \leq p-1$ there is a unique j such that $\zeta_{4p}(j) \in L_p((i,0))$ and $j \in \{1, 3, \ldots, 2p-1\}$. By Corollary 6.8 this correspondence is injective.

Definition 6.20 For block $s \in S_{\{-1,1\}}$ with $\mathrm{RDS}(s) = 0$, we define $\theta(s) = \zeta_{\lg(s)}(i)$ where i is the integer determined in Lemma 6.17.

The following describe cycle lengths in $G_p^{(2)}$.

Corollary 6.9 Let p be a positive even integer. In $G_p^{(2)}$ there are exactly $p/2$ irreducible components (more strictly, $p/2$ label-preserving graph isomorphic classes of irreducible components) which contain cycles of length p.

Proof: From Proposition 6.16, every block s of length p with $\mathrm{RDS}(s) = 0$ is generated by a cycle in $G_p^{(2)}$. From Theorem 6.4 and Lemma 6.17 it is enough to consider $L_p((i,0))$, $i = 0, 1, \ldots, p-1$ and $\zeta_p(i)$, $i = 0, 1, \ldots, p/2-1$. If $\zeta_p(i) \in C(L_p((j,0))) \bigcap C(L_p((j',0)))$ then $L_p((j,0)) \cong L_p((j',0))$ by Corollary 6.8. From Lemma 6.17 we note that if $L_p((i,0))$ contains a cycle of length p then there is a unique integer j such that $\zeta_p(j) \in L_p((i,0))$. Thus this corollary holds. □

For every irreducible component I and for every block $t \in C(I)$ we have $\mathrm{RDS}(t) = 0$ by Theorem 4.2. So, if p is an odd integer, there is no irreducible component which contains a cycle of length p because for every block $s \in S_{\{-1,1\}}$ of length p we have $\mathrm{RDS}(s) \neq 0$. Moreover, we have the following:

Theorem 6.5 Let q be a positive odd integer. Then $G_q^{(2)}$ does not contain any cycle of length $2q$.

Proof: We have

$$\begin{aligned}\mathrm{RDS}^{(2)}(\zeta_{2q}(q-1)) &= \mathrm{RDS}^{(2)}(-\mathbf{1}^q \cdot \mathbf{1}^q) \\ &= \mathrm{RDS}(-1 \cdot -2 \cdots -q \cdot -(q-1) \cdots -1 \cdot 0) \\ &= \sum_{i=1}^{q}(-i) + \sum_{i=1}^{q-1}(-i) \\ &= -q^2.\end{aligned}$$

Therefore by (6.15) we note that for every i with $0 \leq i \leq q-1$, $\mathrm{RDS}^{(2)}(\zeta_{2q}(i))$ is an odd integer. Let $t \in S_{\{-1,1\}}$ be a block such that $\lg(t) = 2q$ and $\mathrm{RDS}(t) = 0$. By Lemma 6.16 and Lemma 6.17 $\mathrm{RDS}^{(2)}(\theta(t))/2q - \mathrm{RDS}^{(2)}(t)/2q \in Z$. This equation requires that $\mathrm{RDS}^{(2)}(t)$ should be an odd integer.

Suppose that there is an integer j such that $L_q((j,0))$ contains a cycle of length $2q$. Let s be a block generated by the cycle. Let u be a block generated by a cycle starting from state $(j,0)$. By Remark 6.6 and Proposition 6.15 we have $j/q = -\mathrm{RDS}^{(2)}(u)/\lg(u)$, and by Lemma 6.16

$$\frac{\mathrm{RDS}^{(2)}(s)}{2q} - \frac{\mathrm{RDS}^{(2)}(u)}{\lg(u)} = \frac{\mathrm{RDS}^{(2)}(s)}{2q} + \frac{j}{q} \in Z.$$

Thus $\mathrm{RDS}^{(2)}(s)$ should be an even integer. But this contradicts with the above discussion. □

Let s and t be blocks with $\lg(s) = \lg(t)$. In this section we have shown explicitly or implicitly necessary and sufficient conditions that there is an irreducible component I such that $s \in C(I)$ and $t \in C(I)$. We summarize these in the following proposition.

Proposition 6.20 Let s and t be blocks such that $\lg(s) = \lg(t)$ and $\mathrm{RDS}(s) = \mathrm{RDS}(t) = 0$. Then the following conditions are equivalent:

1) $\theta(s) = \theta(t)$;

2) $\lg(s) \mid \mathrm{RDS}^{(2)}(s-t)$;

3) there is an irreducible component I such that $s \in C(I)$ and $t \in C(I)$.

Proof: By Lemma 6.16 we have 2) $\Leftrightarrow$ 3).

Let I and I' be irreducible components such that $s \in C(I)$ and $t \in C(I')$ respectively. If $\theta(s) = \theta(t)$ then $\theta(s) \in C(I) \bigcap C(I')$ by Lemma 6.17. From Corollary 6.8, we have $I \cong I'$. If $I \cong I'$ then we have $\theta(s) = \theta(t)$ by Lemma 6.17. Therefore we have 1) $\Leftrightarrow$ 3). □

We now consider another canonical graph. Let $G^{(2)}$ be a CSTD such that the state set of $G^{(2)}$ is Q^2 and there is an edge going from state (σ_1, σ_2) to (τ_1, τ_2) with label b if and only if $(\tau_1, \tau_2) = (\sigma_1 + b, \sigma_2 + \sigma_1)$, where Q is the set of rational numbers. We note that for every positive integer p $G^{(2)}$ is period-p canonical for a second order spectral null at dc. A pair of functions ϕ_1 and ϕ_2 defined in the following is a pair of coboundary functions of $G^{(2)}$:

$$\phi_1(\sigma) = \sigma_1, \quad \phi_2(\sigma) = \sigma_2 \qquad \text{for state } \sigma = (\sigma_1, \sigma_2).$$

By modifying the proof of Proposition 6.17, we can prove that for every state $\sigma = (i/p, j/q)$ the irreducible component containing state σ is label-preserving graph isomorphic to $L_{pq}((iq, 0))$. So, by Theorem 6.4 we have the following:

Proposition 6.21 $G^{(2)}$ has infinitely many irreducible components.

Example 6.8 $G_2^{(2)}$ has two irreducible components, $L_2((0,0))$ and $L_2((1,0))$: they are shown in Fig. 6.8 and Fig. 6.9, where solid arrows mean edges with label 1 and dotted arrows mean edges with label -1. $G_4^{(2)}$ has four irreducible components. By definition $L_4((0,0)) \cong L_2((0,0))$. By Proposition 6.17, we have $L_4((2,0)) \cong L_2((1,0))$. In Fig. 6.10 $L_4((1,0))$ is given.

Example 6.9 $G_6^{(2)}$ has six irreducible components. One of them is $L_6((0,0))$ and we have $L_6((0,0)) \cong L_4((0,0)) \cong L_2((0,0))$. By Proposition 6.17 we have $L_6((3,0)) \cong L_2((1,0))$. $L_6((1,0))$ and $L_6((2,0))$ are given in Fig. 6.11 and Fig. 6.12.

6.5 Summary

We have investigated the structure of canonical graphs. First we have considered the problem of identifying all irreducible components (all label-preserving graph isomorphic classes of irreducible components) of canonical graphs for first order spectral nulls at $f = f_S k/n$. We have proved that there is only one irreducible component with period n. If n is a prime number or the double of a prime number then we have identified all irreducible components of canonical graphs. For $n = 4$, we also have identified irreducible components. For $n = 6, 8, 9, 12, 15, 16, 19, 20$ we have identified all irreducible components of canonical graphs for spectral nulls at $f = f_S k/n$ by using computer.

We also have identified all irreducible components of canonical graphs for second order spectral nulls at dc.

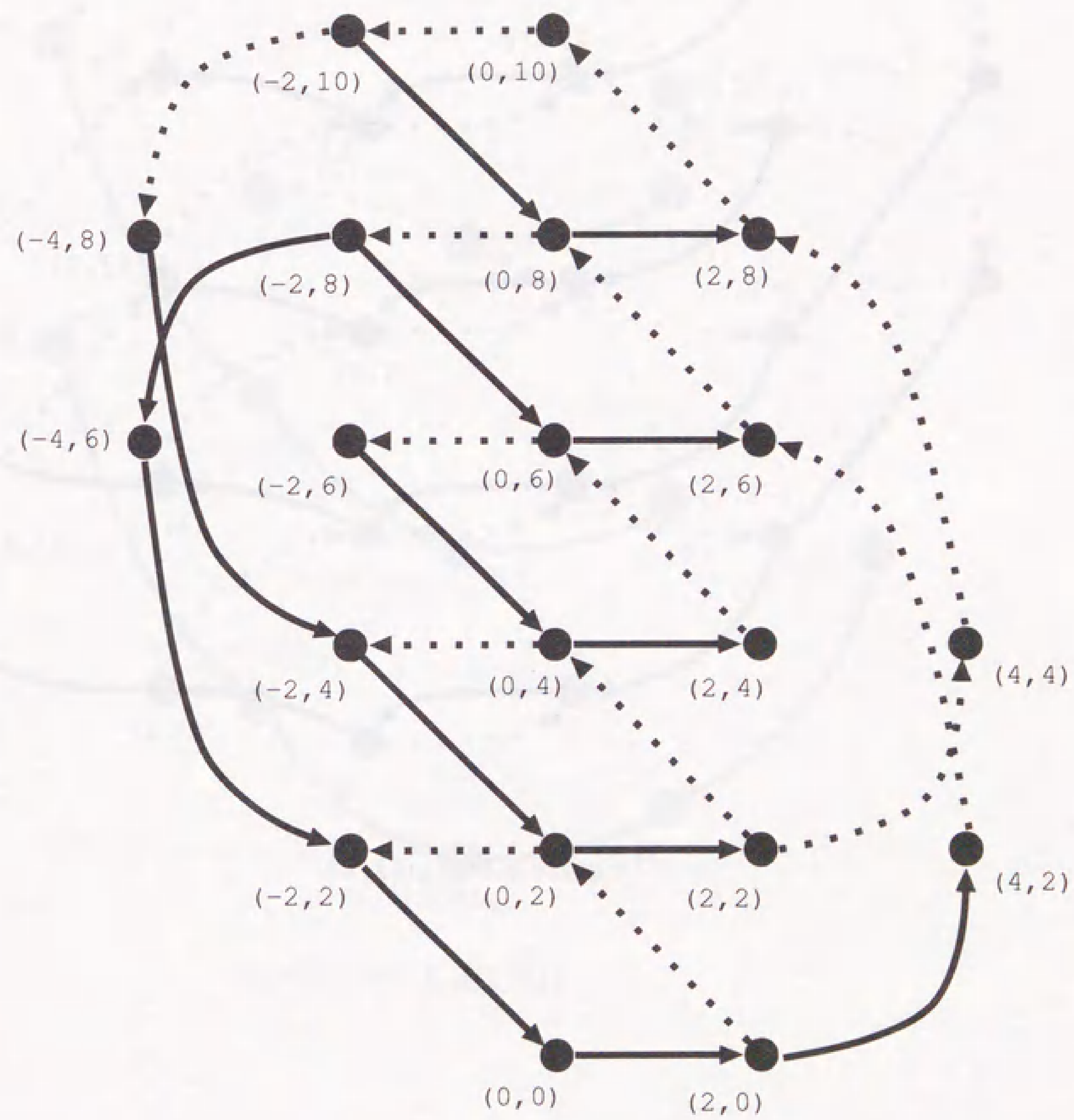


Figure 6.8: $L_2((0,0))$

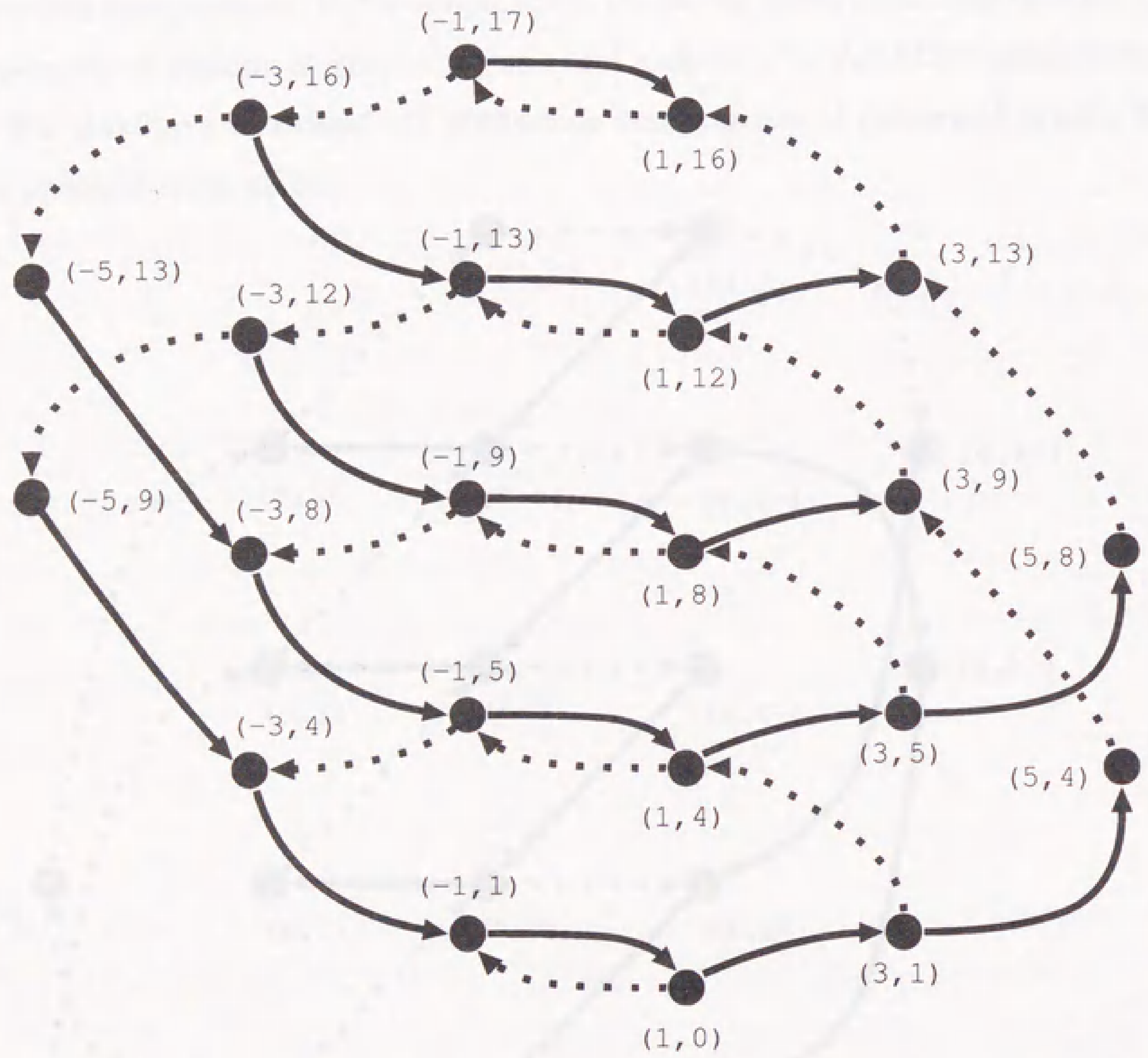


Figure 6.9: $L_2((1,0))$

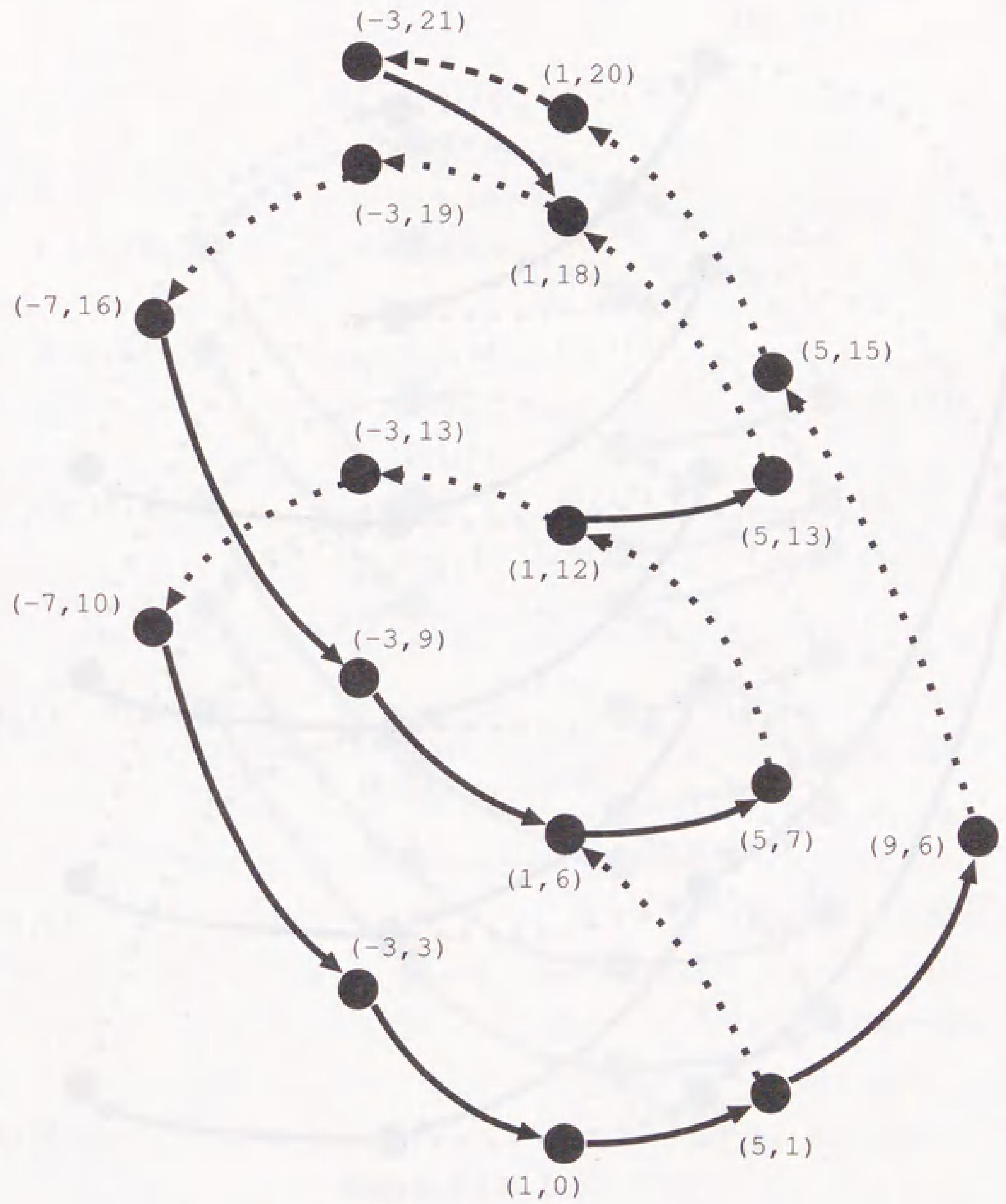


Figure 6.10: $L_4((1,0))$

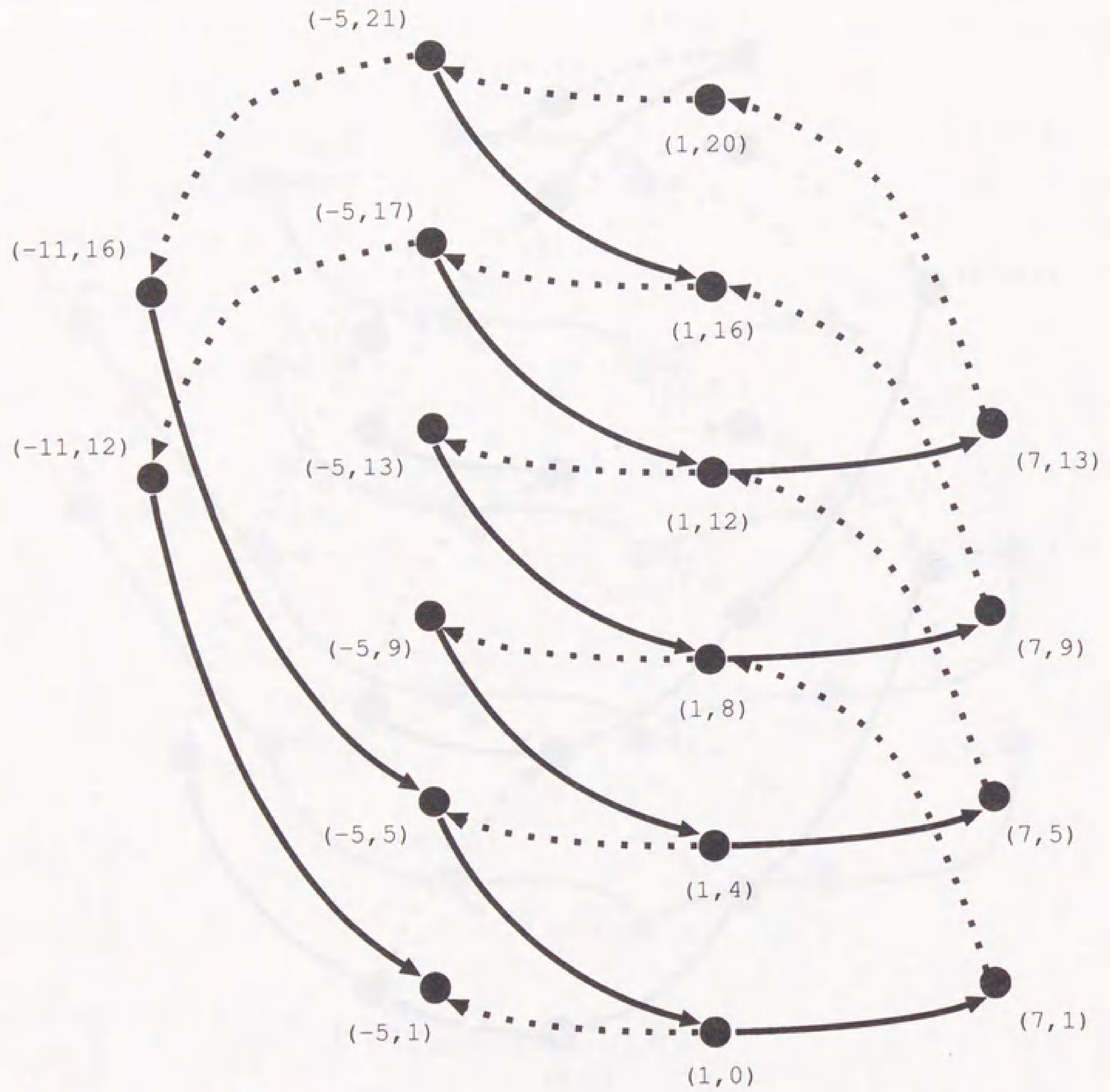


Figure 6.11: $L_6((1,0))$

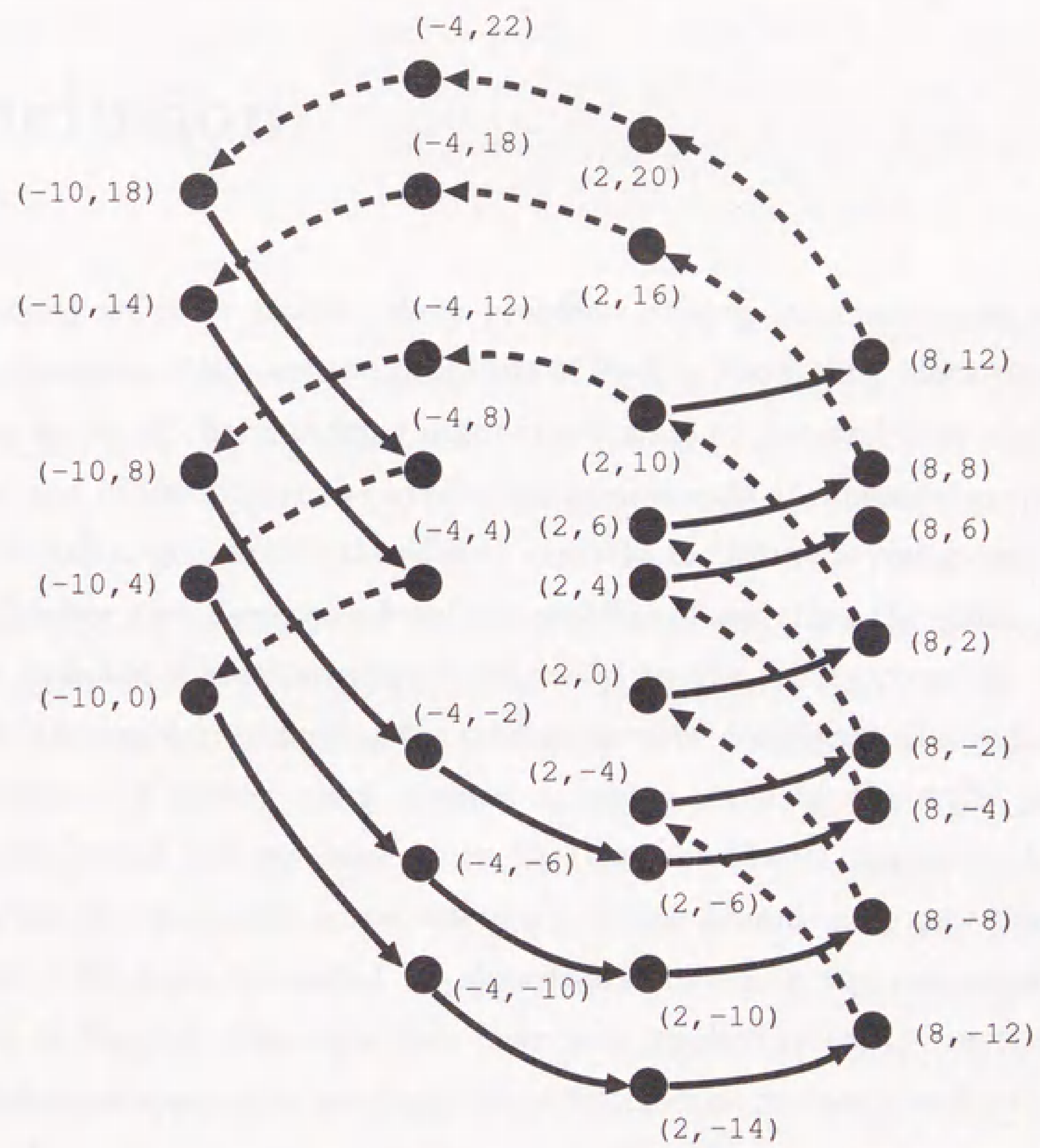


Figure 6.12: $L_6((5,0))$

Chapter 7

Conclusion

In this thesis we have studied three problems arising in constructing codes for constrained channels. They are the problems of finding the sliding block decoder with the minimum scope, of characterizing input-constraints on spectral lines and spectral density nulls, and of identifying the irreducible components of canonical graphs for spectral nulls. We shall conclude this thesis with remarks for future investigation.

In Chapter 3 we have considered the problem of searching the sliding block decoder mapping with the minimum scope constructed by the ACH procedure. This problem reduces to the problem of finding the minimum scope conjugacy that induces the sliding block decoder. A sliding block decoder mapping given by the ACH procedure have been characterized and we have given the two bounds for approximate eigenvectors which induce the minimum scope conjugacy. These bounds guaranty that our problem is solvable. We have presented an algorithm for finding the minimum scope of the conjugacy in Fig. 3.7. The algorithm have been applied to (d, k)-constraints. We have shown that some apparently good splitting strategies do not work well for all constrained systems. However, no systematic efficient way of finding the minimum scope conjugacy has been found yet. Since there are some greedy algorithms for the problem which work well in many cases, it is important to characterize the classes of constraints for which these greedy algorithms can obtain the minimum scope conjugacies. It has been known by experience the ACH procedure usually gives efficient codes for constraints of finite type. For sofic constraint systems we can construct codes by applying the ACH procedure in conjunction with Marcus' approximation theorem[11]. Although this method can give a code with a sliding block decoder for any sofic constrained system, the code may be inefficient. Therefore we should further investigate code construction schemes for sofic constrained systems.

We have considered the constraint which requires that the message sequence should have a spectral line of amplitude not less than a given positive value c at a specified frequency f in Chapter 4, where f is given by $f = f_S k/n$ for integers k and n with $\gcd(k,n) = 1$. We have proved that this constraint is equivalent to the condition that for every cycle the absolute value of RDS_f per symbol is not less than c. We also have presented other several equivalent conditions. We have introduced the spectral density null at f, which means that the derivative of the spectral distribution function vanishes at f. We also have introduced the biased coboundary condition with respect to a complex number at f, which is considered as an extension of a coboundary condition. We have proved that the spectral density null constraint and the biased coboundary condition are equivalent. We also have proved that for a code and for a positive number c the following conditions are equivalent: a) the code has a spectral line of amplitude c at f, independent of the source statistics; b) the code satisfies a biased coboundary condition with respect to a complex number d with $|d| = c$; c) for every cycle the absolute value of the RDS_f per symbol is equal to c. Our results imply that we can encode data sequences so that encoded sequences may contain the servo signal for head positioning control even if the channel is intrinsically digital as in, for example, an optical storage system.

It was shown experimentally that some spectral null codes at dc exhibits spectral lines at low frequencies for equiprobable information sources[14], and Immink proved this theoretically[15, Chapter 11], [16]. Such a code was used in the design of a storage system which employs saturation recording with the thin medium[14]. A code which has a spectral line of large amplitude might have a larger signal-to-noise ratio (SNR) than the above code, where SNR is defined to be the amplitude of the spectral line divided by the power spectral density. Moreover, if in addition the code has a spectral density null then SNR must be infinite. Therefore, spectral density null codes are ideal in the sense of SNR and so should be examined in practical storage devices.

In Chapter 6 we have considered the problem of identifying all irreducible components of the canonical graphs for spectral null constraints. We have shown that this identification problem is solvable for the first order spectral null. Moreover we have given all the irreducible components on the two assumptions. If n is a prime number there are n irreducible components of the canonical graph for the spectral null at $f = f_S k/n$. If n is the double of a prime number then the number of irreducible components is given by (6.9). For n with $n \leq 20$ we also have identified all irreducible components by using computer. We have identified all irreducible components of the canonical graphs for second order spectral nulls at dc. However, we have not yet understand completely the structure of the canonical graphs. Even the general solution of the identification

problem for the canonical graphs for first-order spectral nulls has not been obtained yet. Although we have identified the irreducible components of the canonical graphs for first order spectral nulls in some cases, we have few results for second order spectral nulls at frequencies other than zero. The canonical graphs for higher order spectral null constraints at frequencies other than zero should be further studied. If we can identify the irreducible components, next we should investigate the properties of each of them. That is, we should study the problem of characterizing the properties of each irreducible component of the canonical graphs for spectral null constraints, for example, in terms of spectrum or asymptotic channel capacity.

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Appendix A

Appendix

A.1 Proof of Theorem 3.1

The "if" part holds since the composition of m block maps of $(0,1)$ type is of $(0,m)$ type and the composition of m block maps of $(0,0)$ type is of $(0,0)$ type. Hence we prove the "only if " part. If $m=0$ then it is trivial since π is only a renaming of edges. Therefore we assume that $m \geq 1$. Let $\alpha \in S(G)$. For each $e_1...e_m \in L_G$ with $t_G(e_m)=\alpha$, we put

$$\begin{aligned} \{x_1,...,x_k\} &= \{\pi(e_1...e_m e) : e \in \mathcal{F}_G(\alpha)\}, \\ B_j(e_1...e_m) &= \{e \in \mathcal{F}_G(\alpha) : \pi(e_1...e_m e) = x_j\}, \end{aligned}$$

for $1 \leq j \leq k$. Let $\mathcal{B}(e_1...e_m) = \{B_j(e_1...e_m)\}_{1\leq j\leq k}$. Then $\mathcal{B}(e_1...e_m)$ is a partition of $\mathcal{F}_G(\alpha)$. Putting

$$\mathcal{B}_\alpha = \bigwedge_{\substack{e_1\cdots e_m \\ t_G(e_m)=\alpha}} \mathcal{B}(e_1\cdots e_m),$$

$\mathcal{B}_\alpha$ is a partition of $\mathcal{F}_G$. Let G' be the graph obtained from G by the state-splitting defined by the family of partitions $\{\mathcal{B}_\alpha\}_{\alpha\in S(G)}$. Let $\eta : \Lambda_G \to \Lambda_{G'}$ be the conjugacy corresponding to the state-splitting. Let $\pi' = \pi \circ \eta^{-1}$. By construction $\pi : \Lambda_{G'} \to \Lambda_H$ is of $(0, m-1)$ type and the scope of $(\pi')^{-1}$ is at most 2.

Next we show that $(\pi')^{-1}$ is of $(0,0)$ type. Assume that this is not true, that is, there exist $a_1 \in E(G)$ and $a_2', a_2'' \in \mathcal{F}_H(t_H(a_1))$ such that $\eta\circ\pi^{-1}(a_1a_2') \neq \eta\circ\pi^{-1}(a_1a_2'')$. Let d_1 be an edge with $d_1 = \pi^{-1}(a_1)$ and let $d_2'...d_{m+2}'$ and $d_2''...d_{m+2}''$ be blocks in Λ_G such that $\pi(d_2'...d_{m+2}') = a_2'$ and $\pi(d_2''...d_{m+2}'') = a_2''$. Since π is of $(0,m)$ type and π^{-1} is of (0,0) type it follows that $\pi^{-1}(a_2') = d_2'$ and $\pi^{-1}(a_2'') = d_2''$ so that we have $\eta(d_1d_2') \neq \eta(d_1d_2'')$. This means that for some $e_1...e_m \in L_G$ with $t_G(e_m) = t_G(d_1)$, $\pi(e_1...e_m d_2') \neq \pi(e_1...e_m d_2'')$. It is clear that $\pi(e_1...e_m d_2'...d_{m+1}')\pi(d_2''...d_{m+2}'') = y_1...y_{m+1} \in L_H$. We have

$\pi(\pi^{-1}(y_1...y_{m+1})) = y_1 = \pi(e_1...e_m d_2')$. On the other hand we have $\pi(\pi^{-1}(y_1...y_{m+1})) = \pi(\pi^{-1}(y_1...y_m\pi(d_2''...d_{m+2}''))) = \pi(e_1...e_m\pi^{-1}(\pi(d_2''...d_{m+2}''))) = \pi(e_1...e_m d_2'')$. This is a contradiction.

Repeating this argument m times we can obtain H from G by a state-splitting of m rounds. □

A.2 Proof of Lemma 3.1

Proposition 3.1 is proved by applying the following lemma.

First we give necessary definitions. Let N be a positive integer and let $G = (V, E)$ be a directed graph such that $\log N \leq h(\Lambda_G)$. Let v be an approximate eigenvector for G with respect to N. Let $\bar{G} = (\bar{V}, \bar{E})$ be the subgraph of G consists of $\bar{V} = \{\alpha \in V : v(\alpha) > 0\}$ and $\bar{E} = \{e \in E : v(i_G(e)) > 0, v(t_G(e)) > 0\}$. A graph $H = (\bar{V}, F)$ is called a *nonzero graph* of G for v if $\bar{E} \subset F$ and $i_G|_{\bar{E}} = i_H|_{\bar{E}}$ and $t_G|_{\bar{E}} = t_H|_{\bar{E}}$. Of course $\bar{G}$ is a trivial nonzero graph of G for v. Let $\bar{v}$ be the vector defined by $\bar{v} = v|_{\bar{V}}$. Then $\bar{v}$ is an approximate eigenvector for any nonzero graph of G for v.

Lemma A.2.1 Let N be a positive integer and let $G = (V, E)$ be a directed graph with $\log N \leq h(\Lambda_G)$. Let v be an approximate eigenvector for G with respect to N. Let $G' = (V', E')$ be a directed graph obtained from G by a state-splitting of one round compatible with v and let v' be an approximate eigenvector for G' induced from v by the state-splitting. Let $H = (\bar{V}, F)$ be a nonzero graph of G for v. Then by a state-splitting of one round compatible with $\bar{v}$, we can transform H into a nonzero graph H' of G' for v', and $\bar{v}'$ is an approximate eigenvector for H' induced from $\bar{v}$ by the state-splitting.

Proof: We shall construct a state-splitting of one round for H according to the state-splitting of one round for G. For each $\alpha \in V$ let $w(\alpha)$ and $w'(\alpha)$ be the numbers such that $w(\alpha) \leq w'(\alpha)$, $\mathcal{F}_G(\alpha)$ is partitioned into $w'(\alpha)$ sets as $\mathcal{F}_G(\alpha) = B_1(\alpha) \cup ... \cup B_{w'(\alpha)}(\alpha)$ with $v'((\alpha, k)) > 0, 1 \leq k \leq w(\alpha)$ and $v'((\alpha, k)) = 0, w(\alpha) < k \leq w'(\alpha)$. For $\alpha \in \bar{V}$ let $\{D_1(\alpha), ..., D_{w(\alpha)}(\alpha)\}$ be the partition of $\mathcal{F}_H(\alpha)$ defined as follows:

$$\begin{aligned} D_k(\alpha) &= B_k(\alpha) \cap \bar{E}, \qquad 1 \leq k < w(\alpha); \\ D_{w(\alpha)}(\alpha) &= \mathcal{F}_H(\alpha) \setminus (D_1(\alpha) \cup ... \cup D_{w(\alpha)-1}(\alpha)). \end{aligned}$$

Let $H' = (U', F')$ be the graph obtained by the state-splitting corresponding to the family of partitions $\{\{D_1(\alpha), ..., D_{w(\alpha)}(\alpha)\} : \alpha \in \bar{V}\}$. Then it is easy to see that this state-splitting is compatible with $\bar{v}$ and $\bar{v}'$ is an approximate eigenvector induced from $\bar{v}$ by the state-splitting. We can put $U' = \bar{V}'$. If $(e, j) \in \bar{E}'$ with $e \in B_k(i_G(e))$, then (e, j) goes from $(i_G(e), k)$ to $(t_G(e), j)$ in G' and $v'((i_G(e), k)) > 0$ and $v'((t_G(e), j)) > 0$.

This implies that $D_k(i_G(e))$ and $D_j(t_G(e))$ exist and that $e \in \bar{E}$ because $v(i_G(e)) > 0$ and $v(t_G(e)) > 0$. Hence $e \in D_k(i_G(e))$ so that (e, j) goes from $(i_G(e), k)$ to $(t_G(e), j)$ in H'. Thus H' is a nonzero graph of G' for v'. □

A.3 Proof of Theorem 3.2

By Proposition 3.1 there exists a nice graph $H = (U, F)$ obtained from $\bar{G} = (\bar{V}, \bar{E})$ by a state-splitting of m rounds which is strongly compatible with $\bar{v}$ and induces a vector with all components 1. Each state in H has at least N outgoing edges. Let $\pi : \Lambda_{\bar{G}} \to \Lambda_H$ be the conjugacy induced by the state-splitting.

Let $\psi : U \to \bar{V}$ be defined by $\psi(\gamma) = i_{\bar{G}}(\pi^{-1}(a)), \gamma \in U$, where $a \in F$ with $i_H(a) = \gamma$. Clearly ψ is well defined and for each $\alpha \in \bar{V}$, α is split into the $v(\alpha)$ states in $\psi^{-1}(\alpha)$. If $e_1...e_m e_{m+1}$ and $e_1...e_m e'_{m+1}$ are in $L_{\bar{G}}$, then $i_H(\pi(e_1...e_m e_{m+1})) = i_H(\pi(e_1...e_m e'_{m+1}))$. This is proved by induction from the definition of the conjugacy corresponding to a state-splitting. Hence we can define $\phi : L_{\bar{G}}(m) \to U$ by

$$\phi(e_1...e_m) = i_H(\pi(e_1...e_m e_{m+1})), \qquad e_1...e_m e_{m+1} \in L_{\bar{G}}.$$

Since π is of $(0, m)$ type and π^{-1} is of $(0, 0)$ type, it follows that for $e_1...e_m \in L_{\bar{G}}$,

$$\psi(\phi(e_1...e_m)) = i_{\bar{G}}(e_1)$$

so that for $\alpha \in \bar{V}$,

$$\{\phi(e_1...e_m) : i_{\bar{G}}(e_1) = \alpha, e_1...e_m \in L_{\bar{G}}\} = \psi^{-1}(\alpha).$$

It also follows that for $e_1...e_{2m}, e'_1...e'_{2m} \in L_{\bar{G}}$, if $e_1...e_m \neq e'_1...e'_m$ then $\pi(e_1...e_{2m}) \neq \pi(e'_1...e'_{2m})$. It is easily seen that for $e_1...e_{2m}, e_1...e_{m+1}e'_{m+2}...e'_{2m} \in L_{\bar{G}}$, $\phi(e_{m+1}...e_{2m}) = \phi(e_{m+1}e'_{m+2}...e'_{2m})$ if and only if $\pi(e_1...e_{2m}) = \pi(e_1...e_{m+1}e'_{m+2}...e'_{2m})$. Using these facts, for $\gamma \in U$ we get

$$\begin{aligned} N^m &\leq |\{a_1...e_m \in L_H : i_H(a_1) = \gamma\}| \\ &= |\{\pi(e_1...e_{2m}) : \phi(e_1...e_m) = \gamma, e_1...e_{2m} \in L_{\bar{G}}\}| \\ &= \sum_{\substack{\phi(e_1...e_m)=\gamma \\ e_1...e_m \in L_{\bar{G}}}} |\{\pi(e_1...e_{2m}) : e_1...e_{2m} \in L_{\bar{G}}\}| \\ &= \sum_{\substack{\phi(e_1...e_m)=\gamma \\ e_1...e_m \in L_{\bar{G}}}} |\{\phi(e_{m+1}...e_{2m}) : e_{m+1}...e_{2m} \in L_{\bar{G}}, i_{\bar{G}}(e_{m+1}) = t_{\bar{G}}(e_m)\}| \\ &= \sum_{\substack{\phi(e_1...e_m)=\gamma \\ e_1...e_m \in L_{\bar{G}}}} |\psi^{-1}(t_{\bar{G}}(e_m))| \end{aligned}$$

$$= \sum_{\substack{\phi(e_1 \dots e_m)=\gamma \\ e_1 \dots e_m \in L_{\bar{G}}}} v(t_{\bar{G}}(e_m)). \tag{A.3.1}$$

For $\alpha \in \bar{V}$, we can put $\{\phi(e_1 \dots e_m) : i_{\bar{G}}(e_1) = \alpha, e_1 \dots e_m \in L_{\bar{G}}\} = \{\gamma_{\alpha,1}, \cdots, \gamma_{\alpha,v(\alpha)}\}$ because the left-hand side is equal to $\psi^{-1}(\alpha)$. For $\alpha \in \bar{V}$ and $1 \le i \le v(\alpha)$, we define $C(\alpha, i) = \{e_1 \dots e_m \in L_{\bar{G}} : \phi(e_1 \dots e_m) = \gamma_{\alpha,i}\}$. Then $\{C(\alpha,i)\}_{1 \le i \le v(\alpha)}$ can be thought as a partition of $\mathcal{F}_{(\bar{G})^m}(\alpha)$ and by (A.3.1) it follows that $\bar{v}$ is a 1-round vector for $(\bar{G})^m$ with respect to N^m. Therefore v is a 1-round vector for G^m with respect to N^m. □

A.4 Proof of Proposition 5.2

Let $\boldsymbol{d} = (d_1, d_2, \dots, d_K)$. Let (G, γ) be an irreducible finite subgraph of $G^f_{\boldsymbol{d}}$. For a state $\sigma = (s_1, s_2, \dots, s_K, a, j)$ in $G^f_{\boldsymbol{d}}$, let $\tilde{\ell}(\sigma) = j$. By the definition of $h^f_{\boldsymbol{d}}$, the period of G is a multiple of n and there is a state σ_0 in G with $\tilde{\ell}(\sigma_0) = 0$. Let ℓ be the indexing of $S(G)$ by period and we can assume that $\ell(\sigma_0) = 0$ without loss of generality. We note that $\omega^{-\tilde{\ell}(\sigma)} = \omega^{-\ell(\sigma)}$ for $\sigma \in S(G)$ by Lemma 4.4(Lemma 3 of [30]). We define $\phi_1, \phi_2, \dots, \phi_K$ by $\phi_m(s_1, s_2, \dots, s_K, a, j) = s_m$ for $(s_1, s_2, \dots, s_K, a, j) \in S(G)$. Then by the definition of $h^f_{\boldsymbol{d}}$ we have

$$\phi_j(\sigma) = \omega\phi_{j+1}(\tau) - \phi_{j+1}(\sigma) + \omega^{-\ell(\sigma)} d_{j+1}$$

for each edge (σ, τ) in G where $\phi_0 = \gamma$. Hence (G, γ) has an order-K spectral density null by Theorem 5.1. Let $P \in \Pi_G$. Let p be the stationary distribution of P. Let N be the period of G, and $\{B_0, B_1, \dots, B_{N-1}\}$ a partition by period. Put $S(G) = \{\sigma_1, \sigma_2, \dots, \sigma_J\}$ and $p = (p_0, p_1, \dots, p_{N-1})$, where p_i is a row vector of size $\# B_i$. Since N is a multiple of n, we note that $\omega^N = 1$. Since (G, γ) satisfies a biased coboundary condition, we can write $\vec{\gamma} = (\omega P - I)\vec{\phi}_1 + \vec{d}_1$ where $\vec{d}_1 = (\omega^{-\ell(\sigma_1)} d_1, \omega^{-\ell(\sigma_2)} d_1, \dots, \omega^{-\ell(\sigma_J)} d_1)^t$, $\vec{\gamma} = (\gamma(\sigma_1), \gamma(\sigma_2), \dots, \gamma(\sigma_J))^t$ and $\vec{\phi}_1 = (\phi_1(\sigma_1), \phi_1(\sigma_2), \dots, \phi_1(\sigma_J))^t$. Putting $K = kN/n$, we have

$$\begin{aligned} pJ_K(\omega P - I) &= \omega(p_0, \omega p_1, \dots, \omega^{N-1} p_{N-1})P - pJ_K) \\ &= (\omega(\omega^{N-1} p_0, p_1, \dots, \omega^{N-2} p_{N-1}) - pJ_K \\ &= (p_0, \omega p_1, \dots, \omega^{N-1} p_{N-1}) - pJ_K \\ &= 0, \end{aligned}$$

where J_K is the matrix defined in (A.9.6). Therefore $\vec{\gamma}^* J_K^* p^t p J_K \vec{\gamma} = \vec{d}_1^* J_K^* p^t p J_K \vec{d}_1$, and this is the amplitude of a spectral line at f [47]. Thus $G^f_{\boldsymbol{d}}$ satisfies the condition (I).

Let $\phi'_1, \phi'_2, \dots, \phi'_K$ be the order-K biased coboundary functions and $d'_1, d'_2, \dots, d'_{K-1}$ the biased terms. Since for every $P \in \Pi_G$, (G, P, γ) has a nonzero spectral line at f, the period of G is a multiple of n. Define ϕ_j by $\phi_j(\sigma) = \phi'_j(\sigma) - \omega^{-\ell(\sigma)}\phi'_j(\sigma_0)$, $\sigma \in S(G)$, $j = 1, 2, \dots, K$. Then by (5.1) we have that for every edge (σ, τ)

$$\begin{aligned} \gamma(\sigma) &= \omega\phi'_1(\tau) - \phi'_1(\sigma) + \omega^{-\ell(\sigma)} d_1 \\ &= \omega\phi_1(\tau) + \omega^{-(\ell(\tau)-1)}\phi'_1(\sigma_0) - \phi_1(\sigma) - \omega^{-\ell(\sigma)}\phi'_1(\sigma_0) + \omega^{-\ell(\sigma)} d_1 \\ &= \omega\phi_1(\tau) - \phi_1(\sigma) + \omega^{-\ell(\sigma)} d_1 \end{aligned}$$

and for $j = 1, 2, \dots, K-1$

$$\begin{aligned} \phi_j(\sigma) &= \omega\phi'_{j+1}(\tau) - \phi'_{j+1}(\sigma) + \omega^{-\ell(\sigma)}\left(d'_{j+1} + \phi'_j(\sigma_0)\right) \\ &= \omega\phi_{j+1}(\tau) - \phi_{j+1}(\sigma) + \omega^{-\ell(\sigma)} d_{j+1}. \end{aligned}$$

It is easy to check that a mapping

$$\sigma \mapsto (\phi_1(\sigma), \dots, \phi_K(\sigma), \gamma(\sigma), \ell(\sigma) \bmod n)$$

is a desired homomorphism. Thus $G^f_{\boldsymbol{d}}$ satisfies the condition (II). □

A.5 Proof of Lemma 4.1

Let η be a cycle in G. If η is simple, then $\mathrm{RDS}_0(\gamma(\eta))/\lg(\eta) \in X$. Assume that η is not simple. We can write $\eta = \eta_1 * \eta_2 * \eta_3$ where η_2 is a simple cycle and $i(\eta_2) = i(\eta_3)$. Then, we have $\mathrm{RDS}_0(\gamma(\eta)) = \mathrm{RDS}_0(\gamma(\eta_1 * \eta_3)) + \mathrm{RDS}_0(\gamma(\eta_2))$. Thus there are simple cycles $\zeta'_1, \zeta'_2, \dots, \zeta'_L$ such that $\mathrm{RDS}_0(\gamma(\eta)) = \sum_{i=1}^{L} \mathrm{RDS}_0(\gamma(\zeta'_i))$ and $\lg(\eta) = \sum_{i=1}^{L} \lg(\zeta'_i)$. Thus we have

$$\frac{\mathrm{RDS}_0(\gamma(\eta))}{\lg(\eta)} = \sum_{i=1}^{L} \frac{\lg(\zeta'_i)}{\lg(\eta)} \left(\frac{\mathrm{RDS}_0(\gamma(\zeta'_i))}{\lg(\zeta'_i)} \right) \in \operatorname{conv} X,$$

that is, $W \subset \operatorname{conv} X$. Since X is a finite set, $\operatorname{conv} X$ is closed. Hence, $\operatorname{cl} W \subset \operatorname{conv} X$.

Next, we show the converse inclusion. Let $\{\zeta_1, \zeta_2, \cdots, \zeta_I\}$ be the set of all simple cycles in G. Since G is irreducible, there is a cycle in which the states $i(\zeta_1), i(\zeta_2), \dots, i(\zeta_I)$ appear, say θ. We put $L = \lg(\zeta_1) \cdot \lg(\zeta_2) \cdot \cdots \cdot \lg(\zeta_I)$. Assume that we are given I nonnegative integers, say $m_1, m_2, \dots, m_I$ with $\sum_{h=1}^{I} m_h > 0$. Let K be an arbitrary positive integer and ξ_K a cycle of length $\sum_{i=1}^{I} K m_i L + \lg(\theta)$ obtained by connecting the cycles $\underbrace{\zeta_i * \zeta_i * \cdots * \zeta_i}_{K m_i L / \lg(\zeta_i) \text{ times}}$ by θ. Then

$$\frac{\mathrm{RDS}_0(\gamma(\xi_K))}{\lg(\xi_K)} = \frac{\sum_{h=1}^{I} \frac{K m_h L}{\lg(\zeta_h)} \mathrm{RDS}_0(\gamma(\zeta_h)) + \mathrm{RDS}_0(\gamma(\theta))}{\sum_{h=1}^{I} K m_h L + \lg(\theta)}$$

$$= \sum_{h=1}^{I} \frac{m_h}{\sum_{i=1}^{I} m_i + \frac{\lg(\theta)}{KL}} \frac{\mathrm{RDS}_0\big(\gamma(\zeta_h)\big)}{\lg(\zeta_h)} + \frac{\mathrm{RDS}_0\big(\gamma(\theta)\big)}{K\sum_{i=1}^{I} m_i L + \lg(\theta)}.$$

Hence

$$\frac{\mathrm{RDS}_0\big(\gamma(\xi_K)\big)}{\lg(\xi_K)} \longrightarrow \sum_{h=1}^{I} \frac{m_h}{\sum_{i=1}^{I} m_i} \frac{\mathrm{RDS}_0\big(\gamma(\zeta_h)\big)}{\lg(\zeta_h)} \quad \text{as} \quad K \to \infty.$$

This means that the set of all convex combinations of elements in X with rational coefficients is contained in $\mathrm{cl}\, W$. Therefore, since its closure is equal to $\mathrm{conv}\, X$, $\mathrm{conv}\, X$ is also contained in $\mathrm{cl}\, W$. ∎

A.6 Proof of Lemma 4.2

(1) ⇒ (2): We define a set $Y(G)$ by

$$Y(G) = \big\{\big((a,b),(b,c)\big) : (a,b),(b,c) \in E(G)\big\}$$

and a function $\tilde{\gamma} : E(G) \to \boldsymbol{C}$ by

$$\tilde{\gamma}\big((a,b)\big) = \gamma(a) - d(a,b), \qquad (a,b) \in E(G).$$

We define a graph $\tilde{G}$ with $S(\tilde{G}) = E(G)$ and $E(\tilde{G}) = Y(G)$ such that each $\big((a,b),(b,c)\big) \in Y(\tilde{G})$ is an edge going from (a,b) to (b,c). It is easy to see that $\mathrm{RDS}_0\big(\tilde{\gamma}(\eta)\big) = 0$ for every cycle η in $\tilde{G}$. Therefore the FSTD $(\tilde{G}, \tilde{\gamma})$ has a coboundary function $\tilde{\phi} : E(G) \to \boldsymbol{C}$ by Theorem 4.2. Since for $(a,b) \in E(G)$ and $c, c' \in \mathcal{F}_G(b)$, $\tilde{\phi}\big((b,c)\big) = \tilde{\phi}\big((b,c')\big)$, we can define a function $\phi : S(G) \to \boldsymbol{C}$ by $\phi(a) = \tilde{\phi}\big((a,b)\big)$, where $(a,b) \in E(G)$. We have $\tilde{\gamma}\big((a,b)\big) = \phi(b) - \phi(a)$ for $(a,b) \in E(G)$, so that $\gamma(a) - d(a,b) = \phi(b) - \phi(a)$ for every $(a,b) \in E(G)$.

(2) ⇒ (1): This is proved by straightforward computations. ∎

A.7 Proof of Lemma 4.6

Let N be the period of G. We define $\tilde{\gamma} : S(G) \to \boldsymbol{C}$ by $\tilde{\gamma}(a) = \exp\big(-\mathrm{i}2\pi\ell(a)k/n\big)\gamma(a)$, $a \in S(G)$, where ℓ is an indexing of a partition of $S(G)$ by period. We also define $\gamma_d : S(G) \to \boldsymbol{C}$ by $\gamma_d(\sigma) = \gamma(\sigma) - \exp(\mathrm{i}2\pi\ell(\sigma)k/n)d$, $\sigma \in S(G)$. It is easy to see that if N is a multiple of n then for every block η in G, $rds_{f,\gamma}(\eta) - d\lg(\eta) = rds_{f,\gamma_d}(\eta)$ by Lemma 4.3 *(1)*.

If $c = 0$, the equivalence follows from Theorem 4.2.

Assume that $c > 0$. We can prove that *(1) ⇔ (2)* in the same way as Theorem 4.4 is proved. But we here show a simpler proof of *(1) ⇒ (2)*.

(1) ⇒ (2): Since N is a multiple of n by Proposition 4.2, $rds_{f,\gamma}(\eta) = \mathrm{RDS}_0\big(\tilde{\gamma}(\eta)\big)$ for every block η in G, by Lemma 4.3 *(1)*. Let η be a cycle in G and put

$$d = \frac{\mathrm{RDS}_0\big(\tilde{\gamma}(\eta)\big)}{\lg(\eta)}.$$

Suppose that we have a cycle ζ in G such that

$$\frac{\mathrm{RDS}_0\big(\tilde{\gamma}(\zeta)\big)}{\lg(\zeta)} \neq d.$$

Since $|\mathrm{RDS}_0\big(\tilde{\gamma}(\zeta)\big)/\lg(\zeta)| = |d| = c$, by Lemma 4.1 there must be a cycle ζ' with

$$\left|\frac{\mathrm{RDS}_0\big(\tilde{\gamma}(\zeta')\big)}{\lg(\zeta')}\right| < c.$$

Thus we have *(1) ⇒ (2)*.

(2) ⇒ (3): Let η be a block. Since it follows that N is a multiple of n from Proposition 4.2, for every cycle η we have $|rds_{f,\gamma_d}(\eta)| = 0$. Therefore, it follows that there is a K such that for every block η, $|rds_{f,\gamma_d}(\eta)| \leq K$ from Theorem 4.2. This means *(3)*.

(3) ⇒ (2): For a cycle η and an integer m, we write $C(\eta, m)$ for the cycle obtained by concatenating m η's, i.e., $\underbrace{\eta \cdot \eta \cdot \cdots \cdot \eta}_{m \text{ times}}$. Suppose that there is a cycle η of length a multiple of n such that $rds_{f,\gamma}(\eta) = 0$. We note that $rds_{f,\gamma}\big(C(\eta,m)\big) = m\, rds_{f,\gamma}(\eta) = 0$ by Lemma 4.3 *(2)*. Therefore we have $\left|rds_{f,\gamma}\big(C(\eta,m)\big) - d\lg\big(C(\eta,m)\big)\right| = m|d|\lg(\eta)$, but this contradicts *(3)*. Hence $\left|rds_{f,\gamma}(\eta)\right| > 0$ for every cycle η of length a multiple of n, so that N is a multiple of n by Proposition 4.2. Therefore for every block η, $|rds_{f,\gamma}(\eta) - d\lg(\eta)| = |rds_{f,\gamma_d}(\eta)| \leq K$. Thus it follows that for every cycle η, $rds_{f,\gamma_d}(\eta) = 0$ from Theorem 4.2. This means *(2)*. ∎

A.8 Proof of Lemma 4.7

Let ℓ be an indexing of $S(G)$ with respect to a partition by period and we write $S(G) = \{a_1, a_2, \ldots, a_J\}$. Let $d \in \boldsymbol{C}$ and $P \in \Pi_G$. Put $\tilde{z} = \exp(-\mathrm{i}2\pi k/n)$ and $\vec{d} = \vec{\gamma} - \vec{\gamma}'$. Let p be the stationary distribution of P and $\vec{e}$ the column vector with all components 1 of size J. We put

$$U = \sum_{r=0}^{N-1} \exp(\mathrm{i}2\pi r/N) J_r^* \vec{e} p J_r = \begin{pmatrix} \boldsymbol{u}(a_1) \\ \boldsymbol{u}(a_2) \\ \vdots \\ \boldsymbol{u}(a_J) \end{pmatrix},$$
$$\vec{v}_r = -\left(\bar{z}(P-U)\right)^r \vec{d} \qquad \text{for } r = 1, 2, \ldots$$

where $\boldsymbol{u}(a_1), \boldsymbol{u}(a_2), \ldots, \boldsymbol{u}(a_J)$ are row vectors of U. It is easy to see that $\boldsymbol{u}(a) = \boldsymbol{u}(b)$ for $a, b \in B_i$. Hence for every vector $\vec{u}$, $U\vec{u}$ is constant on B_i. If $\vec{u}$ is a vector which is constant on B_i for each i, then we note that for i and $a, a' \in B_i$, $(P\vec{u})(a) = (P\vec{u})(a')$. Hence it is easy to see that for every r and i, $\vec{v}_r$ is constant on B_i. Since $\bar{z}(P-U)$ has all its eigenvalues with modulus less than 1 [47, Appendix A], we have

$$\left(\bar{z}(P-U) - I\right)^{-1} = -\sum_{i=0}^{\infty} \left(\bar{z}(P-U)\right)^i. \tag{A.8.2}$$

Put $\vec{w} = \left(\bar{z}(P-U) - I\right)^{-1} \vec{d}$. Since $\vec{w}$ is given by the summation of $\vec{v}_r$'s, $\vec{w}$ is constant on B_i for each i.

We define a binary relation $\mathcal{R}$ on $S(G)$ as follows: $a\mathcal{R}a'$ if and only if there is a state b with $a, a' \in \mathcal{F}_G(b)$. Let $\{R_0, R_1, \ldots, R_{h-1}\}$ be the equivalence classes defined by the transitive closure of the relation $\mathcal{R}$. Then $\{R_0, R_1, \ldots, R_{h-1}\}$ is finer than or equal to $\{B_0, B_1, \ldots, B_{N-1}\}$. Under the same notations of Theorem 4.1, we can write

$$D^{-1}P^tDP = \begin{pmatrix} H_0 & 0 & \ldots & 0 \\ 0 & H_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & H_{h-1} \end{pmatrix}, \tag{A.8.3}$$

where H_i is a transition probability matrix of size $\#R_i \times \#R_i$ for $i = 0, 1, \cdots, h-1$ [20] and we assume that R_i, $i = 0, 1, \cdots, h-1$ are indexed such that if $R_i \subset B_l$, $R_{i'} \subset B_{l'}$ and $i < i'$, then $l \leq l'$. Therefore

$$\begin{aligned} CG(k/n)\vec{\gamma} &= D\left(I - D^{-1}P^tDP\right)\left(\bar{z}^{-1}I - (P-U)\right)^{-1}\left(\vec{\gamma}' + \vec{d}\right) \\ &= D\left(I - D^{-1}P^tDP\right)\left(\bar{z}^{-1}I - (P-U)\right)^{-1}\vec{\gamma}' + D\left(I - D^{-1}P^tDP\right)\vec{w}\bar{z} \\ &= CG(k/n)\vec{\gamma}'. \end{aligned}$$

Thus it follows that $w_c^{(G,P,\gamma)}(f) = w_c^{(G,P,\gamma')}(f)$ for all f from (4.4). □

A.9 Proof of Theorem 5.1

First we give Lemma A.9.2 and prove Lemma A.9.3.

Lemma A.9.2 *([29, Lemma 1], [24, Proposition 1])* For every sequence $\boldsymbol{x}$ of complex numbers the following statements are equivalent:

(i) $\mathrm{RDS}_f^{(j)}(\boldsymbol{x}) = 0, \quad \text{for } j = 1, 2, \ldots, K;$

(ii) $M_f^{(j)}(\boldsymbol{x}) = 0, \quad \text{for } j = 0, 1, \ldots, K-1.$

Let $\boldsymbol{s} = s_0 \cdots s_{L-1}$ and $\boldsymbol{t} = t_0 \cdots t_{L-1}$ be cycles in G with $s_0 = t_0$. The *difference cycle* $\boldsymbol{e} = e_0 e_1 \cdots e_{L-1}$ of $\boldsymbol{s}$ and $\boldsymbol{t}$ is the sequence defined by

$$e_i = \gamma(s_i) - \gamma(t_i), \qquad i = 0, 1, \ldots, L-1.$$

We have the following lemma about difference cycles and RDS_f:

Lemma A.9.3 Let (G, γ) be an irreducible FSTG with an alphabet consisting of complex numbers. Let k be a nonnegative integer and n a positive integer. Let ℓ be the indexing of $S(G)$ by period. Put $f = kf_S/n$ and $\omega = \exp(-\mathrm{i}2\pi k/n)$. Assume that n is a divisor of the period of G. Then the following statements are equivalent:

(i) For every difference cycle $\boldsymbol{e}$ of length a multiple of n

$$\mathrm{RDS}_f(\boldsymbol{e}) = 0;$$

(ii) There is a constant $d \in \boldsymbol{C}$ such that for every cycle $\boldsymbol{s} = s_0 s_1 \cdots s_{L-1}$ of length a multiple of n,

$$\omega^{\ell(s_0)} \frac{\mathrm{RDS}_f\left(\gamma(\boldsymbol{s})\right)}{\lg(\boldsymbol{s})} = d.$$

Proof: Since the period of G is a multiple of n, we have $\omega^{\ell(s_0)}\mathrm{RDS}_f(\gamma(\boldsymbol{s})) = \sum_{i=0}^{L-1} \omega^{\ell(s_i)}\gamma(s_i)$ for any block $\boldsymbol{s} = s_0 s_1 \cdots s_{L-1}$ by (5.1). Hence it is enough to prove this lemma for the case of dc.

(1) ⇒ (2) : Let $\boldsymbol{s} = s_0 s_1 \cdots s_{L-1}$ and $\boldsymbol{t} = t_0 t_1 \cdots t_{M-1}$ be cycles. Put

$$\boldsymbol{s}' = \underbrace{\boldsymbol{s} \cdot \boldsymbol{s} \cdots \boldsymbol{s}}_{M \text{ times}} \quad \text{and} \quad \boldsymbol{t}' = \underbrace{\boldsymbol{t} \cdot \boldsymbol{t} \cdots \boldsymbol{t}}_{L \text{ times}}.$$

Then $\lg(\boldsymbol{s}') = \lg(\boldsymbol{t}')$ and we have

$$\frac{\mathrm{RDS}_0\left(\gamma(\boldsymbol{s}')\right)}{\lg(\boldsymbol{s}')} = \frac{\mathrm{RDS}_0\left(\gamma(\boldsymbol{s})\right)}{\lg(\boldsymbol{s})}, \qquad \frac{\mathrm{RDS}_0\left(\gamma(\boldsymbol{t}')\right)}{\lg(\boldsymbol{t}')} = \frac{\mathrm{RDS}_0\left(\gamma(\boldsymbol{t})\right)}{\lg(\boldsymbol{t})}. \tag{A.9.4}$$

First suppose that $s_0 = t_0$. Let $\boldsymbol{e}$ be the difference cycle of $\gamma(\boldsymbol{s}')$ and $\gamma(\boldsymbol{t}')$. Since $0 = \mathrm{RDS}_0(\boldsymbol{e}) = \mathrm{RDS}_0\big(\gamma(\boldsymbol{s}')\big) - \mathrm{RDS}_0\big(\gamma(\boldsymbol{t}')\big)$ by assumption, we have

$$\frac{\mathrm{RDS}_0\left(\gamma(\boldsymbol{s}')\right)}{\lg(\boldsymbol{s}')} = \frac{\mathrm{RDS}_0\left(\gamma(\boldsymbol{t}')\right)}{\lg(\boldsymbol{t}')}.$$

Thus by (A.9.4), we get

$$\frac{\mathrm{RDS}_0\left(\gamma(\boldsymbol{s})\right)}{\lg(\boldsymbol{s})} = \frac{\mathrm{RDS}_0\left(\gamma(\boldsymbol{t})\right)}{\lg(\boldsymbol{t})}.$$

Suppose that $s_0 \neq t_0$. Since G is irreducible, there are blocks $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ such that $\boldsymbol{u}_1 \cdot \boldsymbol{t} \cdot \boldsymbol{u}_2$ is a cycle, $\boldsymbol{s}$ and $\boldsymbol{u}_1$ have the same initial state and so do $\boldsymbol{t}$ and $\boldsymbol{u}_2$. Then $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2$ is a cycle. Hence, from the fact proved above, we have

$$\begin{aligned}\frac{\mathrm{RDS}_0\big(\gamma(\boldsymbol{s})\big)}{\lg(\boldsymbol{s})} &= \frac{\mathrm{RDS}_0\big(\gamma(\boldsymbol{u}_1 \cdot \boldsymbol{u}_2)\big)}{\lg(\boldsymbol{u}_1 \cdot \boldsymbol{u}_2)} \\ &= \frac{\mathrm{RDS}_0\big(\gamma(\boldsymbol{u}_2 \cdot \boldsymbol{u}_1)\big)}{\lg(\boldsymbol{u}_2 \cdot \boldsymbol{u}_1)} \\ &= \frac{\mathrm{RDS}_0\big(\gamma(\boldsymbol{t})\big)}{\lg(\boldsymbol{t})}.\end{aligned}$$

(2)⇒(1) Let $\boldsymbol{e} = e_0 \cdots e_{L-1}$ be a difference cycle. Let $\boldsymbol{s} = s_0 \cdots s_{L-1}$ and $\boldsymbol{t} = t_0 \cdots t_{L-1}$ be cycles such that $s_0 = t_0$ and $e_i = \gamma(s_i) - \gamma(t_i)$ for $i = 0, \ldots, L-1$. Then $\mathrm{RDS}_0(\boldsymbol{e}) = \mathrm{RDS}_0(\gamma(\boldsymbol{s})) - \mathrm{RDS}_0(\gamma(\boldsymbol{t}))$. Since $\mathrm{RDS}_0(\gamma(\boldsymbol{s})) = \mathrm{RDS}_0(\gamma(\boldsymbol{t})) = dL$, we have $\mathrm{RDS}_0(\boldsymbol{e}) = 0$. ∎

Proof of Theorem 5.1

Let N be the period of G. For states σ and τ, we define a relation between σ and τ if and only if there is a state η such that $\sigma, \tau \in \mathcal{F}_G(\eta)$. Let $\mathcal{R} = \{R_0, R_1, \ldots, R_{h-1}\}$ be the partition of $S(G)$ determined by the transitive closure of the relation. Let $\mathcal{B} = \{B_0, B_1, \ldots, B_{N-1}\}$ be the partition of $S(G)$ by period. The partition $\mathcal{R}$ is finer than or equal to $\mathcal{B}$. For a vector $\vec{v}$ we say that $\vec{v}$ is constant on a set $B \subset S(G)$ if $\vec{v}(\sigma) = \vec{v}(\tau)$ for every $\sigma, \tau \in B$. We write $S(G) = \{\sigma_1, \sigma_2, \ldots, \sigma_J\}$.

(ii)⇒(i): Let $\phi_1, \phi_2, \ldots, \phi_K$ be biased coboundary functions and $d_1, d_2, \ldots, d_K$ biased terms. Fix $P \in \Pi_G$ and let $p = (p_1, p_2, \ldots, p_J)$ be the distribution of $S(G)$ such that $pP = p$. Noting that $\sum_{\tau \in \mathcal{F}_G(\sigma)} P(\sigma, \tau) = 1$ for every $\sigma \in S(G)$, by (5.2) we have

$$\begin{aligned}& \sum_{\tau \in \mathcal{F}_G(\sigma)} P(\sigma, \tau) \left(\omega \phi_{j+1}(\tau) - \phi_{j+1}(\sigma) + \omega^{-\ell(\sigma)} d_{j+1}\right) \\ = & \sum_{\tau \in \mathcal{F}_G(\sigma)} \omega P(\sigma, \tau) \phi_{j+1}(\tau) - \phi_{j+1}(\sigma) + \omega^{-\ell(\sigma)} d_{j+1} \\ = & \ \phi_j(\sigma),\end{aligned}$$

that is,

$$\vec{\phi}_j = (\omega P - I)\vec{\phi}_{j+1} + \vec{d}_{j+1}, \qquad j = 0, 1, \ldots, K-1, \tag{A.9.5}$$

where $\vec{\phi}_0 = (\gamma(\sigma_1), \ldots, \gamma(\sigma_J))^t$, $\vec{d}_j = \left(\omega^{-\ell(\sigma_1)} d_j, \omega^{-\ell(\sigma_2)} d_j, \ldots, \omega^{-\ell(\sigma_J)} d_j\right)^t$ and $\vec{\phi}_j = \left(\phi_j(\sigma_1), \phi_j(\sigma_2), \ldots, \phi_j(\sigma_J)\right)^t$. Rewriting (5.2), we have

$$\phi_{j+1}(\tau) = \omega^{-1}\left(\phi_{j+1}(\sigma) + \phi_j(\sigma) - \omega^{-\ell(\sigma)} d_{j+1}\right), \quad j = 0, 1, \ldots, K-1.$$

This implies that $\vec{\phi}_1, \vec{\phi}_2, \ldots, \vec{\phi}_K$ are constant on $\mathcal{F}_G(\sigma)$ for every $\sigma \in \mathcal{F}(G)$, and so constant on each R_i by the definition of $\mathcal{R}$. Without loss of generality, we assume that states are ordered such that $\ell(\sigma_i) \leq \ell(\sigma_m)$ for $1 \leq i < m \leq J$. Let

$$\begin{aligned} H &= \sum_{r=0}^{N-1} e^{\mathrm{i}2\pi r/N} J_r^* w p J_r \\ J_r &= \begin{pmatrix} I_0 & 0 & \cdots & 0 \\ 0 & e^{-\mathrm{i}2\pi r/N} I_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\mathrm{i}\pi r(N-1)/N} I_{N-1} \end{pmatrix} \quad r = 0, 1, \ldots, N-1, \end{aligned} \tag{A.9.6}$$

where w is the column vector with all components 1 and I_L is the unit matrix of size $\#B_L \times \#B_L$. Putting $\vec{d}'_j = \vec{d}_j + \omega H \vec{\phi}_j$ for $j = 1, 2, \ldots, K$, we get

$$\vec{\phi}_j = (\omega(P - H) - I)\,\vec{\phi}_{j+1} + \vec{d}'_{j+1}, \qquad j = 0, 1, \ldots, K-1 \tag{A.9.7}$$

from (A.9.5). We note that for $\sigma_j, \sigma_l \in S(G)$ if $\sigma_j, \sigma_l \in B_i$ for some i, then $\boldsymbol{a}(\sigma_j) = \boldsymbol{a}(\sigma_l)$, where $\boldsymbol{a}(\sigma_m)$ is the m-th row vector of H. Therefore for any vector $\vec{v}$, $H\vec{v}$ is constant on each B_i. Hence $\vec{d}'_1, \ldots, \vec{d}'_K$ are constant on B_i. By the same argument of the proof of Lemma 4.7(Lemma 7 of [30]), if $\vec{v}$ is constant on each B_i, then the vectors

$$\omega(P - H)\vec{v}, \quad \omega\big(I - \omega(P - H)\big)^{-1}\vec{v} \tag{A.9.8}$$

also are constant on B_i. Substituting (A.9.7) for itself, we get

$$\vec{\gamma} = (-1)^j\big(I - \omega(P - H)\big)^j \vec{\phi}_j + \vec{d}''_j, \quad j = 1, 2, \ldots, K$$

where $\vec{d}''_j = \sum_{m=0}^{j-1}\big(\omega(P - H) - I\big)^m \vec{d}'_{m+1}$. Put $G(z) = z\,(I - z(P - H))^{-1}$. Then

$$\{G(\omega)\}^j \vec{\gamma} = \omega^j (-1)^j \vec{\phi}_j + \omega^j \big(I - \omega(P - H)\big)^{-j} \vec{d}''_j, \quad j = 1, 2, \ldots, K. \tag{A.9.9}$$

By (A.9.8), the second term in the right-hand side of (A.9.9) is constant on B_i. Since $\mathcal{R}$ is finer than or equal to $\mathcal{B}$ and $\vec{\phi}_j$ is constant on R_i, the vector $\{G(\omega)\}^j \vec{\gamma}$ is constant on R_i. We write $D_p = \mathrm{diag}(p_1, p_2, \ldots, p_J)$. Then, by Appendix of [20] we have

$$D_p^{-1} P^t D_p P = \begin{pmatrix} H_0 & 0 & \cdots & 0 \\ 0 & H_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{h-1} \end{pmatrix},$$

where H_i is an irreducible probability matrix of size $\#R_i \times \#R_i$ for each i and we assume that $R_0, R_1, \ldots, R_{h-1}$ are indexed such that if $R_i \subset B_l, R_{i'} \subset B_{l'}$ and $i < i'$, then $l \le l'$. Thus we have

$$D_p \left(I - D_p^{-1} P^t D_p P\right) \{G(\omega)\}^j \vec{\gamma} = 0$$

for $j = 1, 2, \ldots, K$ since $\{G(\omega)\}^j \vec{\gamma}$ is constant on R_i. Similarly, we can prove that

$$\vec{\gamma}^* \{G^*(\omega)\}^j (D_p - P^t D_p P) = 0, \qquad j = 1, 2, \ldots, K.$$

Thus by the equation (I-2) in [28, Appendix], we have $\Phi^{(j)}(f) = 0$ for $j = 1, 2, \ldots, 2K-1$.
(i)⇒(ii): If n is not a divisor of the period of G, then there is no spectral line and, hence, (G, γ) has an order-K spectral null at f. Thus (ii) follows from Theorem 2 of [24] or Theorem 4 of [29].

Assume that n is a divisor of the period. Let $P \in \Pi_G$ with $\Phi^{(j)}(f) = 0$ for $j = 0, 1, \ldots, 2K-1$ and put $d = pJ_M\gamma$, where J_r is the matrix defined in (A.9.6), p is the stationary distribution of P and we put $M = Nk/n$. Define $\gamma_d : S(G) \to \boldsymbol{C}$ by $\gamma_d(\sigma) = \gamma(\sigma) - \omega^{-\ell(\sigma)} d$. Since (G, P, γ) and (G, P, γ_d) have the same power spectral density function by Lemma 4.7(Lemma 7 of [30]) and (G, P, γ_d) has no spectral line at f by the equation (17) of [47], (G, γ_d) satisfies an order-K coboundary condition at f. Hence, (G, γ_d) has coboundary functions $\phi_1, \phi_2, \ldots, \phi_K$. We note that $\phi_1, \phi_2, \ldots, \phi_K$ are also biased coboundary functions with biased terms $d, 0, \ldots, 0$.
(ii)⇒(iii) Assume that n is not a divisor of the period of G. Then there is no spectral line at f. As already proved, (G, γ) satisfies the statement (i) of this theorem. Hence (G, γ) has an order-K spectral null at f. By Theorem 4 of [29], there are functions $\psi_1, \psi_2, \ldots, \psi_{K-1}$ such that for every cycle $\boldsymbol{s} = s_0 s_1 \cdots s_{L-1}$ of length a multiple of n,

$$\begin{aligned} \mathrm{RDS}_f(\gamma(\boldsymbol{s})) &= 0, \\ \mathrm{RDS}_f^{(j)}(\gamma(\boldsymbol{s})) &= -\sum_{i=1}^{j-1} \psi_{j-1}(s_0) \binom{n+i}{i}, \qquad j = 2, 3, \ldots, K. \end{aligned}$$

Hence for cycles $\boldsymbol{s} = s_0 s_1 \cdots s_{L-1}$ and $\boldsymbol{t} = t_0 t_1 \cdots t_{L-1}$ with $s_0 = t_0$, we have

$$\mathrm{RDS}_f^{(j)}(\gamma(\boldsymbol{s})) - \mathrm{RDS}_f^{(j)}(\gamma(\boldsymbol{t})) = 0, \qquad j = 1, 2, \ldots, K.$$

Assume that n is a divisor of the period of G. Let $\phi_1 : S(G) \to \boldsymbol{C}$ be a biased coboundary function and d a number such that for every (σ, τ)

$$\gamma(\sigma) = \omega\phi_1(\tau) - \phi_1(\sigma) + \omega^{-\ell(\sigma)} d. \tag{A.9.10}$$

Define $\gamma'(\sigma)$ by $\gamma'(\sigma) = \gamma(\sigma) - \omega^{-\ell(\sigma)} d$ for $\sigma \in S(G)$. Then we note that (G, γ') satisfies an order-K coboundary condition from (A.9.10) and Remark 5.1. By using Theorem 4 of [29] again, for every difference cycle $\boldsymbol{e}$ of (G, γ') we have also

$$\mathrm{RDS}_f^{(j)}(\boldsymbol{e}) = 0, \qquad j = 1, 2, \ldots, K. \tag{A.9.11}$$

For every block $\boldsymbol{s} = s_0 \cdots s_{L-1}$, $\mathrm{RDS}_f(\gamma'(\boldsymbol{s})) = \mathrm{RDS}_f(\gamma(\boldsymbol{s})) - L\omega^{-\ell(s_0)} d$ for every block $\boldsymbol{s}$ by (5.1). Thus, for every blocks $\boldsymbol{s}$ and $\boldsymbol{t}$ with the same initial state and $\mathrm{lg}(\boldsymbol{s}) = \mathrm{lg}(\boldsymbol{t})$, we have

$$\mathrm{RDS}_f\big(\gamma(\boldsymbol{s})\big) - \mathrm{RDS}_f\big(\gamma(\boldsymbol{t})\big) = \mathrm{RDS}_f\big(\gamma'(\boldsymbol{s})\big) - \mathrm{RDS}_f\big(\gamma'(\boldsymbol{t})\big). \tag{A.9.12}$$

Then (iii) follows from (A.9.11) and (A.9.12).
(iii)⇒(ii): First we prove the next claim:
Claim: if (G, γ) satisfies (iii), then there is a biased coboundary function ϕ and for every difference cycle $\boldsymbol{e}$ of length a multiple of n in (G, ϕ),

$$\mathrm{RDS}_f^{(j)}(\boldsymbol{e}) = 0, \qquad j = 1, 2, \ldots, K-1.$$

Proof of Claim: From Lemma A.9.3 and Theorem 4.6(Theorem 6 of [30]), we note that (G, γ) satisfies a biased coboundary condition. Let ϕ be the biased coboundary function and d the biased term. If n is not a divisor of the period of G, then we may assume that $d = 0$ by Theorem 3 of [23] and Theorem 4.5(Theorem 5 of [30]). Define a function $\gamma' : S(G) \to \boldsymbol{C}$ by

$$\gamma'(\sigma) = \begin{cases} \gamma(\sigma) - \omega^{-\ell(\sigma)} d, & \text{if } n \text{ is a divisor of the period;} \\ \gamma(\sigma), & \text{otherwise,} \end{cases}$$

for $\sigma \in S(G)$. Then we note that for every blocks $\boldsymbol{s}$ and $\boldsymbol{t}$ having the same initial state

$$\mathrm{RDS}_f\big(\gamma(\boldsymbol{s})\big) - \mathrm{RDS}_f\big(\gamma(\boldsymbol{t})\big) = \mathrm{RDS}_f\big(\gamma'(\boldsymbol{s})\big) - \mathrm{RDS}_f\big(\gamma'(\boldsymbol{t})\big) \tag{A.9.13}$$

and (G, γ') satisfies a coboundary condition at f.

Let $\boldsymbol{s} = s_0 s_1 \cdots s_{L-1}$ and $\boldsymbol{t} = t_0 t_1 \cdots t_{L-1}$ be cycles in G. Assume that L is a multiple of n and $s_0 = t_0$. Using Lemma A.24.4, by induction we can show that for $m = 1, 2, \ldots, K-1$,

$$\mathrm{RDS}_f^{(m)}\big(\phi(\boldsymbol{s})\big) - \mathrm{RDS}_f^{(m)}\big(\phi(\boldsymbol{t})\big)$$

$$= \mathrm{RDS}_f^{(m+1)}\big(\gamma'(\boldsymbol{s})\big) + g_m(L)\phi(s_0) - \mathrm{RDS}_f^{(m-1)}\big(\phi(s_1 s_2 \cdots s_L)\big)$$
$$-\Big(\mathrm{RDS}_f^{(m+1)}\big(\gamma'(\boldsymbol{t})\big) + g_m(L)\phi(t_0) - \mathrm{RDS}_f^{(m-1)}\big(\phi(t_1 t_2 \cdots t_L)\big)\Big)$$
$$= \mathrm{RDS}_f^{(m+1)}(\gamma'(\boldsymbol{s})) - \mathrm{RDS}_f^{(m+1)}(\gamma'(\boldsymbol{t}))$$

by (A.9.13)

$$= \mathrm{RDS}_f^{(m+1)}\big(\gamma(\boldsymbol{s})\big) - \mathrm{RDS}_f^{(m+1)}\big(\gamma(\boldsymbol{t})\big)$$

by the assumption

$$= 0$$

where $g_m(j)$ is a function defined in (A.24.31). This completes the proof of the claim.

Let ϕ_1 be a biased coboundary function obtained by applying the above claim to (G, γ). Next apply the claim to $(G, \phi_1), (G, \phi_2), \ldots,$ and (G, ϕ_{K-1}). Finally, we note that (G, γ) satisfies an order-K biased coboundary condition.

(iv) $\Leftrightarrow$ (iii) This follows from Lemma A.9.2. □

A.10 Proof of Proposition 6.5

Let $B_0, B_1, \ldots, B_{n-1}$ and $B'_0, B'_1, \ldots, B'_{n-1}$ be partitions of $S(I)$ and $S(J)$ by period, respectively. Fix states $\sigma_0 \in B_0$ and $\sigma'_0 \in B'_0$. Let ϕ and ϕ' be coboundary functions of I and J respectively. By Remark 6.3 we may assume that both ϕ and ϕ' are injective. Let $d = \phi'(\sigma'_0) - \phi(\sigma_0)$. For state σ, we denote the integer l such that $\sigma \in B_l$ by $\ell(\sigma)$.

We define a function ψ of $S(I)$ to $S(J)$ as follows: for every state σ $\psi(\sigma) = (\phi')^{-1}(\omega^{-\ell(\sigma)}d + \phi(\sigma))$. We show that ψ is well-defined. Clearly, $\psi(\sigma_0)$ is well-defined. Let σ and σ' be states in I and J respectively. Consider an edge emanating from σ. Let b be the label of the edge and let τ be the terminal end state of the edge. Let τ' be a state in J such that there is an edge going from σ' to τ' with label b. Assume that $\psi(\sigma) = \sigma'$. Since $\omega\phi(\tau) - \phi(\sigma) = b$, $\omega\phi'(\tau') - \phi'(\sigma') = b$ and $\phi'(\sigma') - \phi(\sigma) = \omega^{\ell(\sigma)}d$, we have $\phi'(\tau') - \phi(\tau) = \omega^{-(\ell(\sigma)+1)}d$. Since $\omega^{-n} = 1$ and the period of I is n, we conclude by induction that ψ is well-defined. We also note that ψ defines a label-preserving graph homomorphism of I to J and is surjective.

Suppose that there are states σ and η in I such that $\psi(\sigma) = \psi(\eta)$. Let τ' and τ'' be states in I and assume that there is an edge from τ' to τ''. Since ψ is a graph homomorphism of I to J and the period of I and that of J are equal to n, if $\tau' \in B_l$ and $\psi(\tau') \in B'_l$ then $\tau'' \in B_j$ and $\psi(\tau'') \in B'_j$ where j is the integer such that $0 \le j \le n-1$ and $j \equiv l + 1 \bmod n$. Since $\sigma_0 \in B_0$, $\psi(\sigma_0) = \sigma'_0 \in B'_0$ and there are paths $\boldsymbol{x}$ and $\boldsymbol{x}'$ such that $i_I(\boldsymbol{x}) = i_I(\boldsymbol{x}') = \sigma_0$, $t_I(\boldsymbol{x}) = \sigma$ and $t_I(\boldsymbol{x}') = \eta$, we have $\psi(\sigma) \in B'_{\ell(\sigma)}$ and $\psi(\eta) \in B'_{\ell(\eta)}$. Since $\psi(\sigma) = \psi(\eta)$, we get $\ell(\sigma) = \ell(\eta)$. Since both $(\phi')^{-1}$ and ϕ are injective, we have $\sigma = \eta$. Thus ψ gives a label-preserving graph isomorphism of I to J. □

A.11 Proof of Lemma 6.2

Let $l = \lg(\boldsymbol{s})$ and $\boldsymbol{s} = s_0 s_1 \cdots s_{l-1}$. Let $n' = n/d$ and $l' = l/d$. Let $\boldsymbol{a}_i = s_i s_{i+d} \cdots s_{(l'-1)d+i}$ for $i = 0, 1, \ldots, d-1$. By Lemma 6.1 we note that for each $i = 0, 1, \ldots, d-1$ there is a block $\boldsymbol{b}_i \in C_{f'}(\boldsymbol{a}_i)$ with $\lg(\boldsymbol{b}_i) = n' - 1$, where $f' = f_S k/n'$. It follows from Proposition 6.12 that for every $i = 0, 1, \ldots, d-1$ there is a block $\boldsymbol{c}_i$ which satisfies the following:

$$\frac{\mathrm{RDS}_{f'}(\boldsymbol{a}_i)}{\omega'^{l'} - 1} - \frac{\mathrm{RDS}_{f'}(\boldsymbol{b}_i)}{\omega'^{n'-1} - 1} = \mathrm{RDS}_{f'}(\boldsymbol{c}_i), \quad n' | \lg(\boldsymbol{c}_i), \tag{A.11.14}$$

where $\omega' = \exp(-2\pi\sqrt{-1}k/n')$. Since we have $\mathrm{RDS}_{f'}(\boldsymbol{c}) = \mathrm{RDS}_{f'}(\boldsymbol{c} \cdot \mathbf{1}^{n'}) = \mathrm{RDS}_{f'}(\boldsymbol{c} \cdot -\mathbf{1}^{n'})$ for every block $\boldsymbol{c}$, we may assume that $\lg(\boldsymbol{c}_0) = \lg(\boldsymbol{c}_1) = \cdots = \lg(\boldsymbol{c}_{d-1})$. Put $L = \lg(\boldsymbol{c}_0)$. Let $\boldsymbol{u} = u_0 u_1 \cdots u_{dL-1}$ be a block obtained from $\boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_{d-1}$ by interleaving them, that is, $u_{id+j} = c_{j,i}$ where $\boldsymbol{c}_j = c_{j,0} c_{j,1} \cdots c_{j,L-1}$ for $i = 0, 1, \ldots, L-1$ and for $j = 0, 1, \ldots, d-1$. By multiplying ω^i to (A.11.14) for each i and by summing them up, we have

$$\sum_{i=0}^{d-1} \omega^i \left(\frac{\mathrm{RDS}_{f'}(\boldsymbol{a}_i)}{\omega'^{l'} - 1} - \frac{\mathrm{RDS}_{f'}(\boldsymbol{b}_i)}{\omega'^{n'-1} - 1} \right) = \sum_{i=0}^{d-1} \omega^i \mathrm{RDS}_{f'}(\boldsymbol{c}_i). \tag{A.11.15}$$

The right hand side of (A.11.15) is

$$\begin{aligned} \sum_{i=0}^{d-1} \omega^i \mathrm{RDS}_{f'}(\boldsymbol{c}_i) &= \sum_{i=0}^{d-1} \omega^i \sum_{j=0}^{L-1} u_{i+jd} \exp(-2\pi\sqrt{-1}jk/n') \\ &= \sum_{i=0}^{d-1} \omega^i \sum_{j=0}^{L-1} u_{i+jd} \exp(-2\pi\sqrt{-1}jdk/n) \\ &= \sum_{i=0}^{d-1} \sum_{j=0}^{L-1} u_{i+jd} \exp(-2\pi\sqrt{-1}(i+jd)k/n) \\ &= \sum_{i=0}^{dL-1} u_i \omega^i \\ &= \mathrm{RDS}_f(\boldsymbol{u}). \end{aligned}$$

Similarly, the terms in the left hand side of (A.11.15) are given as follows:

$$\sum_{i=0}^{d-1} \omega^i \frac{\mathrm{RDS}_{f'}(\boldsymbol{a}_i)}{\omega'^{l'} - 1} = \frac{\mathrm{RDS}_f(\boldsymbol{s})}{\omega^l - 1} \quad \text{and} \quad \sum_{i=0}^{d-1} \omega^i \frac{\mathrm{RDS}_{f'}(\boldsymbol{b}_i)}{\omega'^{n'-1} - 1} = \frac{\mathrm{RDS}_f(\boldsymbol{t})}{\omega^{n-d} - 1},$$

where $\boldsymbol{t}$ is a block obtained from $\boldsymbol{b}_0, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{d-1}$ by interleaving them. Therefore we have $\mathrm{RDS}_f(\boldsymbol{s})/(\omega^l - 1) - \mathrm{RDS}_f(\boldsymbol{t})/(\omega^{n-d} - 1) = \mathrm{RDS}_f(\boldsymbol{u})$. Thus by Proposition 6.12 we have $\boldsymbol{t} \in C_f(\boldsymbol{s})$. □

A.12 Proof of Lemma 6.3

Put $L = \lg(s)$ and $s = s_0 s_1 \cdots s_{L-1}$. From assumption it follows that $\sum_{i=0}^{\lg(s)-1} s_i D^i$ is divisible by the minimal polynomial of ω, $r(D) = 1 + D + \cdots + D^{n-1}$. Using $D^n = 1$, we note that $G(D) = \sum_{i=0}^{n-1} a_i D^i$ is divisible by $r(D)$ where

$$a_j = \sum_{\substack{0 \le i \le L-1 \\ n|(i-j)}} s_i, \qquad j = 0, 1, \ldots, n-1.$$

Hence there should be an integer c such that

$$G(D) = cp(D). \tag{A.12.16}$$

Suppose that $n \not| L$. Hence we may write $L = ln + q$ with $0 < q < n$ and $0 \le l$. Let

$$\begin{aligned} m_1 &= \#\{j | s_{jn} = 1,\ 0 \le j \le l\}, \\ m_{-1} &= \#\{j | s_{jn} = -1,\ 0 \le j \le l\}, \\ m'_1 &= \#\{j | s_{jn+q} = 1,\ 0 \le j \le l-1\}, \\ m'_{-1} &= \#\{j | s_{jn+q} = -1,\ 0 \le j \le l-1\}. \end{aligned}$$

Then we have $m_1 + m_{-1} = l + 1$, $a_0 = m_1 - m_{-1}$, $m'_1 + m'_{-1} = l$ and $a_q = m'_1 - m'_{-1}$.

Suppose that l is an odd integer. If m_1 is an odd integer, then so is m_{-1}. If m_1 is an even integer, then so is m_{-1}. Hence a_0 is an even integer. We can also prove that a_q is an odd integer. Thus $a_0 \ne a_q$. But this contradicts with the existence of the integer c satisfying (A.12.16).

In the case where l is an even integer we can also prove similarly that $n|L$. □

A.13 Proof of Lemma 6.5

Assume that $n = 2$. The state set of $G_{fs/2}$ is Z. We have $h_{fs/2}(1,1) = 1$ and $h_{fs/2}(-1,-1) = -1$. That is, states 1 and -1 have self cycles which generate $\mathbf{1}$ and $-\mathbf{1}$ respectively. Let S_2 and S_3 be sets defined in Example 6.1. Then we note that $-1 \in S_2$ and $1 \in S_3$. Thus this lemma follows from Example 6.1.

Suppose that there is an irreducible component I such that $\mathbf{1}$ and $-\mathbf{1}$ are blocks in $C(I)$. By Proposition 6.12 there is a block $s = s_0 s_1 \cdots s_{L-1}$ such that $n|L$ and $(1-\omega)\mathrm{RDS}_f(s) = 2$. This means that the polynomial $q(D) = (1-D)\sum_{i=0}^{L-1} s_i D^i - 2$ should be divisible by the minimal polynomial of ω, $r(D) = 1 + D + \cdots + D^{n-1}$. By using $D^n = 1$, we obtain a polynomial $q'(D)$ of order less than n from $q(D)$. The polynomial $q'(D)$ should also be divisible by $r(D)$. Therefore there should be an integer c such that

$q'(D) = c\sum_{i=0}^{n-1} D^i$. Since $D^n = 1$, we may write $\sum_{i=0}^{L-1} s_i D^i = \sum_{i=0}^{n-1} x_i D^i$. So, we have $q'(D) = (1-D)\sum_{i=0}^{n-1} x_i D^i - 2$ and, hence, $x_0 - x_{n-1} - 2 + \sum_{i=1}^{n-1}(x_i - x_{i-1})D^i = c\sum_{i=0}^{n-1} D^i$. By comparing coefficients in the left and the right hand sides of the previous equation, we have $c = x_0 - x_{n-1} - 2$ and $c = x_i - x_{i-1}$ for $i = 1, 2, \ldots, n-1$. By summing up these equations, we get $nc = -2$. But if $n \ge 3$, there is no integer c for which this equation holds. □

A.14 Proof of Lemma 6.6

Assume that $\varphi(\boldsymbol{a})$ and $\varphi(\boldsymbol{b})$ are linearly dependent, that is, there are nonzero integers i and j in Z_n such that $i * \varphi(\boldsymbol{a}) + j * \varphi(\boldsymbol{b}) = 0$. Let I be an irreducible component such that $\boldsymbol{b} \in C(I)$. Let l be an additive inverse of i in Z_n. Then we have $j * \varphi(\boldsymbol{b}) = \varphi(\boldsymbol{a}^l)$. Therefore from Lemma 6.4 we have $\boldsymbol{a}^l \in C(I)$. Thus we have $\boldsymbol{a} \in C(I)$ by Proposition 6.9.

Next assume that $\varphi(\boldsymbol{a})$ and $\varphi(\boldsymbol{b})$ are linearly independent, that is, there are integers i, i', j and j' in Z_n such that $i * \varphi(\boldsymbol{a}) + j * \varphi(\boldsymbol{b}) = (1\ 0)$ and $i' * \varphi(\boldsymbol{a}) + j' * \varphi(\boldsymbol{b}) = (0\ 1)$. Suppose that there is an irreducible component I such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are blocks in $C(I)$. Then from Lemma 6.4 it follows that $\mathbf{1}$ and $-\mathbf{1}$ are blocks in $C(I)$. But this contradicts with Lemma 6.5. □

A.15 Proof of Example 6.3

By Lemma 6.2 we have only to consider classification of blocks of length less than 4 by the relation $\mathcal{R}$. Since $n = 4$, we have $\omega^2 = -1$ and $\overline{\omega} = \omega^3 = -\omega$. We use these in rearranging expressions in the following.

Let $F \subset S_{\{-1,1\}}$ be the set of blocks of length less than 4. Let $\boldsymbol{t} = t_0 t_1 \cdots t_{L-1} \in \mathcal{N}$ and let $In(\boldsymbol{t})$ be the initial state of the cycle generating $\boldsymbol{t}$. We determine $In(\boldsymbol{t})$ as a unique polynomial of ω which is of order less than 2 and which satisfies

$$(\omega^L - 1)In(\boldsymbol{t}) = (\overline{\omega}^2 - 1)\sum_{i=0}^{L-1} \omega^i t_i.$$

For every l with $n \not| l$ there is an m such that $ml \equiv 2 \bmod 4$ and, so, $(\overline{\omega}^2 - 1) = \overline{\omega}^2(1 - \omega^L)(1 + \omega^L + \cdots + \omega^{(m-1)L})$. Therefore $In(\boldsymbol{t})$ is well-defined. We have

$$\omega + 1 = S(\mathbf{1}) = S(-\mathbf{1} \cdot \mathbf{1} \cdot -\mathbf{1}), \qquad -\omega - 1 = S(-\mathbf{1}) = S(\mathbf{1} \cdot -\mathbf{1} \cdot \mathbf{1}). \tag{A.15.17}$$

Define sets A, B and C as follows:

$$\begin{aligned} A &= \{\mathbf{1},\ \mathbf{1}\cdot-\mathbf{1}\cdot-\mathbf{1},\ -\mathbf{1}\cdot\mathbf{1}\cdot-\mathbf{1},\ -\mathbf{1}\cdot-\mathbf{1}\cdot\mathbf{1}\}, \\ B &= \{-\mathbf{1},\ -\mathbf{1}\cdot\mathbf{1}\cdot\mathbf{1},\ \mathbf{1}\cdot-\mathbf{1}\cdot\mathbf{1},\ \mathbf{1}\cdot\mathbf{1}\cdot-\mathbf{1}\}, \\ C &= \{-\mathbf{1}\cdot\mathbf{1},\ \mathbf{1}\cdot-\mathbf{1}\}. \end{aligned}$$

Each set of these is a subset of an $\mathcal{R}$-equivalence class. We show that all of A, B and C are actually $\mathcal{R}$-equivalence classes.

Let $\boldsymbol{s} = \boldsymbol{s}_0 \cdot \boldsymbol{s}_1 \cdots \boldsymbol{s}_{L-1} \in S_{\{-1,1\}}$ be a block with $4|\,\mathrm{lg}(\boldsymbol{s})$ where $\boldsymbol{s}_i$, $i = 0, 1, \ldots, L-1$ are blocks of length 4. Then we have $\mathrm{RDS}_f(\boldsymbol{s}) = \sum_{i=0}^{L-1} \mathrm{RDS}_f(\boldsymbol{s}_i)$. Let σ and τ be states in G_f. From the definition of G_f we note that there is a path which goes from σ to τ and generates $\boldsymbol{s}$ if and only if we have

$$\tau = \sigma + \left(\overline{\omega}^2 - 1\right) \mathrm{RDS}_f(\boldsymbol{s}). \tag{A.15.18}$$

Suppose, for example, that A and B are subsets of the same $\mathcal{R}$-equivalence class. Let I be an irreducible component such that $\mathbf{1} \in C(I)$. The period of I is 1 because $\mathbf{1} \in C(I)$. Since $In(\mathbf{1}) = \omega + 1$ and $In(-\mathbf{1}) = -\omega - 1$, there is a path of length a multiple of 4 which goes from state $-\omega - 1$ to state $\omega + 1$. By (A.15.18) this requires that $2\omega + 2$ should be a linear combination, say $g(\omega)$, of elements in a set T with nonnegative integer coefficients where T is a set defined by

$$\begin{aligned} T &= \left\{\left(\overline{\omega}^2 - 1\right) \mathrm{RDS}_f(\boldsymbol{s}) : \boldsymbol{s} \in S_{\{-1,1\}}, \mathrm{lg}(\boldsymbol{s}) = 4\right\} \\ &= \{0, 4, -4, 4\omega, -4\omega, 4\omega + 4, -4\omega + 4, 4\omega - 4, -4\omega - 4\}. \end{aligned}$$

Since elements in T are polynomials of ω of order less than 2 with coefficients in $\{-4, 0, 4\}$, $g(\omega) - 2\omega - 2$ should be a polynomial of ω with coefficients in $\{-2, 2\}$ which is of order less than 2 and which is equal to 0. But this contradicts with that the minimal polynomial of ω is $\omega^2 + 1$. Applying similar arguments, we note that A, B and C are $\mathcal{R}$-equivalence classes. We can show similarly that none of $\mathcal{H}_f(\mathbf{1})$, $\mathcal{H}_f(-\mathbf{1})$ and $\mathcal{H}_f(\mathbf{1}\cdot-\mathbf{1})$ contains state 0. □

A.16 Proof of Lemma 6.9

Suppose that $\boldsymbol{a} \in C_f(\boldsymbol{b})$. By Proposition 6.12 a block $\boldsymbol{t} = t_0 t_1 \cdots t_{L-1} \in S_{\{-1,1\}}$ and an integer i exist such that

$$\mathrm{RDS}_f(\boldsymbol{a}) - \mathrm{RDS}_f(\boldsymbol{b}^{[i]}) = (\omega^d - 1)\mathrm{RDS}_f(\boldsymbol{t}) \tag{A.16.19}$$

and $n|L$ where $L = \mathrm{lg}(\boldsymbol{t})$. By replacing $\boldsymbol{b}$ with $\boldsymbol{b}^{[i]}$ we assume that $i = 0$. By assumption there is a j with $a_j = b_j$. By replacing $\boldsymbol{a}$ with $\boldsymbol{a}^{[j+1]}$ and $\boldsymbol{b}$ with $\boldsymbol{b}^{[j+1]}$ we may assume that $a_{d-1} = b_{d-1}$. Note that (A.16.19) also holds for $i = 0$ from Remark 6.5 after these replacings. Let $g(D) = \sum_{l=0}^{d-2}(a_l - b_l)D^l$. Then $g(D) \neq 0$ by assumption and absolute values of coefficients of $g(D)$ are at most 2. By Lemma 6.7 $\Phi_n(D) = D^{d-1} - D^{d-2} + \cdots + 1 = 0$ and so $D^d - 1 = 0$. Therefore we obtain a polynomial $g'(D) = \sum_{l=0}^{d-2} f_l D^l$ from $\sum_{l=0}^{L-1} t_l D^l$, where $f_l = \sum_{j=0}^{L/d-1}(-1)^j t_{jd+l} + (-1)^{(l+1)} \sum_{j=0}^{L/d-1}(-1)^j t_{jd+d-1}$ for $l = 0, 1, \ldots, d-2$. Since L/d is an even integer and $|t_0| = |t_1| = \cdots = |t_{L-1}|$, we note that every f_i is an even integer. Since $D^d - 1 = -2$, every coefficient of $(D^d - 1)g'(D)$ is a multiple of 4. But this contradicts with that $g(D)$ is a polynomial with coefficients in $\{-2, 0, 2\}$ because $g(D) = (D^d - 1)g'(D)$ and both polynomials in this equation are of degree less than $d - 1$. □

A.17 Proof of Lemma 6.10

Suppose that $\mathbf{1} \in C_f(\boldsymbol{a})$. By Proposition 6.12, a block $\boldsymbol{b} = b_0 b_1 \cdots b_{\mathrm{lg}(\boldsymbol{b})-1} \in S_{\{-1,1\}}$ and an integer i exist such that $n|\,\mathrm{lg}(\boldsymbol{b})$ and $\mathrm{RDS}_f(\mathbf{1}^{L+1}) - \mathrm{RDS}_f(\boldsymbol{a}^{[i]}) = \mathrm{RDS}_f(\boldsymbol{b})(\omega^{L+1} - 1)$. By Remark 6.5 we may assume that $i = 0$. The left hand side of this equation is equal to 2. Regarding $\mathrm{RDS}_f(\boldsymbol{b})$ and $(\omega^{L+1} - 1)\mathrm{RDS}_f(\boldsymbol{b})$ as elements in $Z[\omega]$, we write

$$\begin{aligned} \sum_{i=0}^{d-2} x_i D^i &= \sum_{i=0}^{\mathrm{lg}(\boldsymbol{b})-1} b_i D^i, \\ \sum_{i=0}^{d-2} y_i D^i &= (D^{L+1} - 1) \sum_{i=0}^{\mathrm{lg}(\boldsymbol{b})-1} b_i D^i, \end{aligned}$$

and we have

$$y_0 = 2, \quad y_i = 0, \quad i = 1, 2, \ldots, d-2. \tag{A.17.20}$$

We put $j = d - 1 - (L + 1)$. We can regard $\sum_{i=0}^{d-2} y_i D^i$ as a polynomial obtained from $(D^{L+1} - 1)\sum_{i=0}^{d-2} x_i D^i$ by using $D^{d-1} = D^{d-2} - D^{d-3} + \cdots - 1$. Hence

$$y_i = \begin{cases} x_{i-L-1} - x_i + (-1)^{i+1} x_j & \text{if } L + 1 \le i; \\ -x_i + (-1)^{i+1} x_j & \text{if } L = i; \\ -x_{d-L-1+i} - x_i + (-1)^{i+1} x_j & \text{if } L - 1 \ge i. \end{cases} \tag{A.17.21}$$

We also have

$$\sum_{i=L+1}^{d-2} (-1)^{i+1} x_{i-L-1} = \sum_{i=0}^{d-L-3} (-1)^{i+1} x_i \tag{A.17.22}$$

$$\sum_{i=0}^{L-1} (-1)^{i+1}(-x_{d-L-1+i}) = \sum_{i=d-L-1}^{d-2} x_i. \tag{A.17.23}$$

From (A.17.20), (A.17.21), (A.17.22) and (A.17.23) we have

$$\begin{aligned} 2 &= \sum_{i=0}^{d-2}(-1)^{i+1}y_i \\ &= \sum_{i=L+1}^{d-2}(-1)^{i+1}(x_{i-L-1} - x_i + (-1)^{i+1}x_j) \\ &\quad + \sum_{i=0}^{L-1}(-1)^{i+1}(-x_{d-L-1+i} - x_i + (-1)^{i+1}x_j) - x_L + x_j \\ &= dx_j + \sum_{i=0}^{d-L-3}(-1)^{i+1}x_i - x_j + \sum_{i=d-L-1}^{d-2}(-1)^{i+1}x_i \\ &\quad - \sum_{i=0}^{L-1}(-1)^{i+1}x_i - x_L - \sum_{i=L+1}^{d-2}(-1)^{i+1}x_i, \end{aligned}$$

since j is an even number and L is an odd number,

$$\begin{aligned} &= dx_j + \sum_{i=0}^{d-2}(-1)^{i+1}x_i - \sum_{i=0}^{d-2}(-1)^{i+1}x_i \\ &= dx_j. \end{aligned}$$

But there is no integer x_j which satisfies the above equation because d is an odd prime integer. □

A.18 Proof of Lemma 6.11

It is enough to consider the case where $L \geq d$. By Corollary 6.2, we may assume that $L < n-1$. Let $M = n - L - 2$. Let $\boldsymbol{a} = -\mathbf{1}\cdot\mathbf{1}^L$ and $\boldsymbol{b} = -\mathbf{1}\cdot\mathbf{1}^M$. Then

$$\begin{aligned} \frac{\mathrm{RDS}_f(\boldsymbol{b})}{\omega^{M+1}-1} - \frac{\mathrm{RDS}_f(\boldsymbol{a}^{[L]})}{\omega^{L+1}-1} &= \frac{\mathrm{RDS}_f(\boldsymbol{b})}{\omega^{M+1}-1} - \frac{\mathrm{RDS}_f(\boldsymbol{a}^{[L]})}{\omega^{-M-1}-1} \\ &= \frac{\mathrm{RDS}_f(\boldsymbol{b})}{\omega^{M+1}-1} + \omega^{M+1}\frac{\mathrm{RDS}_f(\boldsymbol{a}^{[L]})}{\omega^{M+1}-1} \\ &= \frac{\mathrm{RDS}_f(\boldsymbol{b}\cdot\boldsymbol{a}^{[L]})}{\omega^{M+1}-1} \\ &= \frac{-2-2\omega^{M+2}}{\omega^{M+1}-1} \\ &= \frac{2(\omega^{d+M+2}-1)}{\omega^{M+1}-1}. \end{aligned}$$

If $M+1$ is an even integer, then there is an integer l such that $l(M+1)/2 \equiv (d+M+2)/2 \bmod d$ because $\gcd((M+1)/2, d) = 1$. If $M+1$ is an odd integer, then there is an integer l such that $l(M+1) \equiv d+M+2 \bmod n$ because $\gcd(M+1, n) = 1$. Therefore, in both cases there is an integer l such that

$$\frac{2(\omega^{d+M+2}-1)}{\omega^{M+1}-1} = 2\left(\omega^{(l-1)(M+1)} + \omega^{(l-2)(M+1)} + \cdots + 1\right).$$

Since $2\omega^i = \mathrm{RDS}_f(-\mathbf{1}^{i-1}\cdot\mathbf{1}\cdot-\mathbf{1}^{n-i-1})$, we can construct a block $\boldsymbol{s}$ such that $n|\lg(\boldsymbol{s})$ and $\mathrm{RDS}_f(\boldsymbol{s}) = 2(\omega^{d+M+2}-1)/(\omega^{M+1}-1)$. Thus we have $\boldsymbol{b} \in C_f(\boldsymbol{a})$ by Proposition 6.12. □

A.19 Proof of Lemma 6.12

Since $N_{e,1}(\boldsymbol{a}) + N_{e,-1}(\boldsymbol{a}) = N_{o,1}(\boldsymbol{a}) + N_{o,-1}(\boldsymbol{a}) = \lg(\boldsymbol{a})/2$, we have $N_{o,1}(\boldsymbol{a}) = N_{e,1}(\boldsymbol{a})$. Since $-\boldsymbol{a} \in C_f(\boldsymbol{a})$ by Lemma 6.8 and Remark 6.2, we may assume that $N_{o,1}(\boldsymbol{a}) \geq N_{o,-1}(\boldsymbol{a})$. If $N_{o,-1}(\boldsymbol{a}) = 0$ then this lemma follows from Corollary 6.4.

Assume that $N_{o,-1}(\boldsymbol{a}) > 0$. Let $d' = (d+1)/2$. Let $\boldsymbol{s} = (-\mathbf{1}\cdot\mathbf{1})^{d'-1}\cdot-\mathbf{1}$. Let $\boldsymbol{v} = \boldsymbol{a}^{d'}$ and $l = N_{o,-1}(\boldsymbol{a})$. By Corollary 6.5, there is a block $\tilde{\boldsymbol{v}}$ such that $\tilde{\boldsymbol{v}}\cdot\mathbf{1}\cdot-\mathbf{1}$ is obtained from $\boldsymbol{v}$ by exchanges of symbols. Repeating such exchanges of symbols we get a block $\boldsymbol{v}'$ such that $\boldsymbol{v}'\cdot\boldsymbol{s}^{2l}$ is obtained from $\boldsymbol{v}$. We have $N_{\alpha,1}(\boldsymbol{v}') = N_{\alpha,1}(\boldsymbol{v}) - l(d'-1)$ and $N_{\alpha,-1}(\boldsymbol{v}') = N_{\alpha,-1}(\boldsymbol{v}) - ld' = 0$ for $\alpha = o, e$. Since $\mathrm{RDS}_f(\boldsymbol{s}^{2l}) = 0$ and $\lg(\boldsymbol{v}') = \lg(\boldsymbol{v}) - nl$, it follows from Corollary 6.4 that $\boldsymbol{v}' = \mathbf{1}^{\lg(\boldsymbol{v})-nl} \in C_f(\boldsymbol{a})$. Since $d'\lg(\boldsymbol{a}) \not\equiv 0 \bmod n$, we have $\mathbf{1} \in C_f(\boldsymbol{a})$ by Corollary 6.4. □

A.20 Proof of Lemma 6.13

We put $\boldsymbol{a} = -\mathbf{1}\cdot\mathbf{1}^{L_1}$ and $\boldsymbol{b} = -\mathbf{1}\cdot\mathbf{1}^{L_2}$. We assume that $\boldsymbol{a} \in C_f(\boldsymbol{b})$ and will show this assumption leads to a contradiction. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be cycles which generate $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Since $\gcd(1+L_1, n) = \gcd(1+L_2, n) = 2$, the period of $I_f(\boldsymbol{b})$ is at most 2.

Assume that $i(\boldsymbol{x})$ and $i(\boldsymbol{y})$ belong to the same periodic component of $I_f(\boldsymbol{b})$. There is a path $\boldsymbol{w}$ such that $n|\lg(\boldsymbol{w})$, $i(\boldsymbol{w}) = i(\boldsymbol{x})$ and $t(\boldsymbol{w}) = i(\boldsymbol{y})$. Let $\boldsymbol{r}$ be a block generated by $\boldsymbol{w}$. Let $i = d+1$ and $j = d-1$. We note that $(1+L_1)i + (1+L_2)j \not\equiv 0 \bmod d$. We consider a block $\boldsymbol{e} = \boldsymbol{a}^i\cdot\boldsymbol{r}\cdot\boldsymbol{b}^j\cdot-\boldsymbol{r}$. This block is generated by a cycle in $I_f(\boldsymbol{b})$. Note that $d \not| \lg(\boldsymbol{e})$ by construction. We have

$$\begin{aligned} N_{e,-1}(\boldsymbol{e}) &= i + j + N_{e,-1}(\boldsymbol{r}) + N_{e,1}(\boldsymbol{r}), \\ N_{o,-1}(\boldsymbol{e}) &= N_{o,-1}(\boldsymbol{r}) + N_{o,1}(\boldsymbol{r}), \\ N_{e,1}(\boldsymbol{e}) &= iN_{e,1}(\boldsymbol{a}) + jN_{e,1}(\boldsymbol{b}) + N_{e,1}(\boldsymbol{r}) + N_{e,-1}(\boldsymbol{r}), \\ N_{o,1}(\boldsymbol{e}) &= iN_{o,1}(\boldsymbol{a}) + jN_{o,1}(\boldsymbol{b}) + N_{o,1}(\boldsymbol{r}) + N_{o,-1}(\boldsymbol{r}). \end{aligned}$$

Since $d|(i+j)$, by Corollary 6.5 we have a block $\boldsymbol{e}'$ such that $\boldsymbol{e}'\cdot(-\mathbf{1}\cdot\mathbf{1})^{i+j}$ is obtained from $\boldsymbol{e}$ by exchanges of symbols. Then

$$N_{e,-1}(\boldsymbol{e}') = N_{e,-1}(\boldsymbol{r}) + N_{e,1}(\boldsymbol{r}),$$

$$\begin{aligned} N_{o,-1}(\boldsymbol{e}') &= N_{o,-1}(\boldsymbol{e}), \\ N_{e,1}(\boldsymbol{e}') &= N_{e,1}(\boldsymbol{e}), \\ N_{o,1}(\boldsymbol{e}') &= N_{o,1}(\boldsymbol{e}) - (i+j). \end{aligned}$$

It follows from Corollary 6.4 that $\boldsymbol{e}' \in C_f(\boldsymbol{b})$ because $\mathrm{RDS}_f((-\mathbf{1}\cdot\mathbf{1})^{i+j}) = 0$. We have $N_{e,1}(\boldsymbol{r}) + N_{e,-1}(\boldsymbol{r}) = N_{o,1}(\boldsymbol{r}) + N_{o,-1}(\boldsymbol{r}) = \lg(\boldsymbol{r})/2$ and so $N_{o,-1}(\boldsymbol{e}') = N_{e,-1}(\boldsymbol{e}')$. Since $d \not| \lg(\boldsymbol{e}')$, we have $\mathbf{1} \in C_f(\boldsymbol{b})$ by Lemma 6.12. But this contradicts with Lemma 6.10.

Next assume that $i(\boldsymbol{x})$ and $i(\boldsymbol{y})$ belong to distinct periodic components. By Lemma 6.8, $-i(\boldsymbol{y})$ is a state of $I_f(\boldsymbol{b})$ and there is a path of length d which goes from $i(\boldsymbol{y})$ to $-i(\boldsymbol{y})$ and generates $(\mathbf{1}\cdot-\mathbf{1})^{(d-1)/2}\cdot\mathbf{1}$. By Remark 6.2 we note that $-i(\boldsymbol{y})$ is the initial state of a cycle generating $\mathbf{1}\cdot-\mathbf{1}^{L_2}$ and, therefore, $\mathbf{1}\cdot-\mathbf{1}^{L_2} \in C_f(\boldsymbol{b})$. Since d is an odd integer and the period of $I_f(\boldsymbol{b})$ is at most 2, $i(\boldsymbol{x})$ and $-i(\boldsymbol{y})$ are in the same periodic component. Therefore, there is a path $\boldsymbol{w}'$ such that $i(\boldsymbol{w}') = i(\boldsymbol{x})$, $t(\boldsymbol{w}') = -i(\boldsymbol{y})$ and $n|\lg(\boldsymbol{w}')$. Let $\boldsymbol{r}'$ be a block generated by $\boldsymbol{w}'$. See Fig. A.22.1. There is a path which generates $-\boldsymbol{r}'$ and goes from $-i(\boldsymbol{y})$ to $i(\boldsymbol{x})$ by Remark 6.1. Hence $-\mathbf{1}\cdot\mathbf{1}^{L_1}\cdot\boldsymbol{r}'\cdot\mathbf{1}\cdot-\mathbf{1}^{L_2}\cdot-\boldsymbol{r}' \in C_f(\boldsymbol{b})$. Since L_1 and L_2 are odd integers, we note that $2 + L_1 + L_2 \neq d$. Since $1 + L_2$ and $\lg(\boldsymbol{r}')$ are even integers, it follows from Corollary 6.5 that the block $\boldsymbol{u} = -\mathbf{1}\cdot\mathbf{1}^{L_1}\cdot\mathbf{1}\cdot-\mathbf{1}^{L_2}\cdot\boldsymbol{r}'\cdot-\boldsymbol{r}'$ is obtained from $-\mathbf{1}\cdot\mathbf{1}^{L_1}\cdot\boldsymbol{r}'\cdot\mathbf{1}\cdot-\mathbf{1}^{L_2}\cdot-\boldsymbol{r}'$ by exchanges of symbols and, therefore, $\boldsymbol{u}$ is also in $C_f(\boldsymbol{b})$. Since $\mathrm{RDS}_f(\boldsymbol{r}'\cdot-\boldsymbol{r}') = 0$ and $n \mid \lg(\boldsymbol{r}'\cdot-\boldsymbol{r}')$, by Corollary 6.4 we have $-\mathbf{1}\cdot\mathbf{1}^{L_1}\cdot\mathbf{1}\cdot-\mathbf{1}^{L_2} \in C_f(\boldsymbol{b})$, that is, $-\mathbf{1}^{1+L_2}\cdot\mathbf{1}^{1+L_1} \in C_f(\boldsymbol{b})$. We may assume that $1 + L_1 \geq 1 + L_2$ because we can exchange $\boldsymbol{a}$ and $\boldsymbol{b}$ if necessary. Let $\boldsymbol{v} = -\mathbf{1}^{1+L_2}\cdot\mathbf{1}^{1+L_1}$. Then $N_{o,-1}(\boldsymbol{v}) = N_{e,-1}(\boldsymbol{v})$. From Lemma 6.12 we have $\mathbf{1} \in C_f(\boldsymbol{b})$ because $d \not| (2 + L_1 + L_2)$ and $2 \mid (2 + L_1 + L_2)$. But this contradicts with Lemma 6.10. □

A.21 Proof of Lemma 6.14

Let $\boldsymbol{s} = (\mathbf{1}\cdot-\mathbf{1})^{(d-1)/2}\cdot\mathbf{1}$. Let σ be the initial state of the cycle generating $\boldsymbol{a}$. By Lemma 6.8 and Remark 6.2, there is a cycle which starts from σ and which generates block $\boldsymbol{v} = \boldsymbol{a}\cdot\boldsymbol{s}\cdot-\boldsymbol{a}\cdot-\boldsymbol{s}$. Put $d' = (d+1)/2$. Since $\lg(\boldsymbol{a}\cdot\boldsymbol{s})$ is an even integer, we have

$$N_{o,1}(\boldsymbol{v}) = N_{o,-1}(\boldsymbol{v}) = N_{o,1}(\boldsymbol{a}) + N_{o,-1}(\boldsymbol{a}) + d', \tag{A.21.24}$$
$$N_{e,1}(\boldsymbol{v}) = N_{e,-1}(\boldsymbol{v}) = N_{e,1}(\boldsymbol{a}) + N_{e,-1}(\boldsymbol{a}) + d' - 1. \tag{A.21.25}$$

By definition we note that $N_{e,1}(\boldsymbol{v}) + N_{e,-1}(\boldsymbol{v}) = N_{o,1}(\boldsymbol{v}) + N_{o,-1}(\boldsymbol{v})$. Therefore, from (A.21.24) and (A.21.25), we have $N_{o,1}(\boldsymbol{v}') = N_{o,-1}(\boldsymbol{v}') = N_{e,1}(\boldsymbol{v}') = N_{e,-1}(\boldsymbol{v}')$. Note that $d \not| \lg(\boldsymbol{v}')$ and $2 \mid \lg(\boldsymbol{v}')$. Thus, by Lemma 6.12, we have $\mathbf{1} \in C_f(\boldsymbol{a})$. □

A.22 Proof of Lemma 6.15

By Lemma 6.2, there is a block $\boldsymbol{c} \in C_f(\boldsymbol{a})$ with $\lg(\boldsymbol{c}) < n$. If $\lg(\boldsymbol{c}) = d$, it is obvious from the definition of F_d. Therefore we assume that $\lg(\boldsymbol{a}) \neq d$ and $\lg(\boldsymbol{a}) < n$.

If $\lg(\boldsymbol{a})$ is an odd integer, then $\mathbf{1} \in C_f(\boldsymbol{a})$ by Lemma 6.14.

We assume that $\lg(\boldsymbol{a})$ is an even number. Let $\boldsymbol{s} = (\mathbf{1}\cdot-\mathbf{1})^{(d-1)/2}\cdot\mathbf{1}$. By Lemma 6.8 and Remark 6.2 we note that $\boldsymbol{v} = \boldsymbol{a}\cdot\boldsymbol{s}\cdot-\boldsymbol{a}\cdot-\boldsymbol{s} \in C_f(\boldsymbol{a})$. We have

$$N_{o,1}(\boldsymbol{v}) = N_{e,-1}(\boldsymbol{v}) = N_{o,1}(\boldsymbol{a}) + N_{e,-1}(\boldsymbol{a}),$$
$$N_{e,1}(\boldsymbol{v}) = N_{o,-1}(\boldsymbol{v}) = N_{e,1}(\boldsymbol{a}) + N_{o,-1}(\boldsymbol{a}) + d.$$

If $N_{e,-1}(\boldsymbol{v}) = N_{o,-1}(\boldsymbol{v})$, then we have $\mathbf{1} \in C_f(\boldsymbol{a})$ by Lemma 6.12.

We consider the case $N_{e,-1}(\boldsymbol{v}) \neq N_{o,-1}(\boldsymbol{v})$. Since $\boldsymbol{v}^{[i]} \in C_f(\boldsymbol{b})$ for every positive integer i, we may assume that $N_{e,-1}(\boldsymbol{v}) < N_{o,-1}(\boldsymbol{v})$. Put $d' = (d+1)/2$ and $\boldsymbol{u} = \boldsymbol{v}^{d'}$. By Corollary 6.5, there is a block $\boldsymbol{u}' \in S_{\{-1,1\}}$ such that $\boldsymbol{u}'\cdot-\boldsymbol{s}^{2N_{e,-1}(\boldsymbol{v})}$ is obtained from $\boldsymbol{u}$ by exchanges of symbols. Since $\mathrm{RDS}_f(-\boldsymbol{s}^{2N_{e,-1}(\boldsymbol{v})}) = 0$, we have $\boldsymbol{u}' \in C_f(\boldsymbol{a})$ by Corollary 6.4. We have

$$\begin{aligned} N_{\alpha,-1}(\boldsymbol{u}') &= N_{\alpha,-1}(\boldsymbol{u}) - d'N_{e,-1}(\boldsymbol{v}), \quad \text{for } \alpha = e, o, \\ N_{\alpha,1}(\boldsymbol{u}') &= N_{\alpha,1}(\boldsymbol{u}) - (d'-1)N_{e,-1}(\boldsymbol{v}), \quad \text{for } \alpha = e, o. \end{aligned}$$

Hence

$$N_{o,1}(\boldsymbol{u}'), N_{e,1}(\boldsymbol{u}'), N_{o,-1}(\boldsymbol{u}') > 0,$$
$$N_{e,1}(\boldsymbol{u}') > N_{o,-1}(\boldsymbol{u}'), \quad N_{e,-1}(\boldsymbol{u}') = 0.$$

Suppose that $d|N_{o,-1}(\boldsymbol{u}')$. Put $j = N_{o,-1}(\boldsymbol{u}')/d$. Then by Corollary 6.5 there is a block $\boldsymbol{t}$ such that $\boldsymbol{t}\cdot(\mathbf{1}\cdot-\mathbf{1})^{jd}$ is obtained from $\boldsymbol{u}'$ by exchanges of symbols. Since $\lg((\mathbf{1}\cdot-\mathbf{1})^d) = n$ and $\mathrm{RDS}_f((\mathbf{1}\cdot-\mathbf{1})^d) = 0$, we have $\boldsymbol{t} \in C_f(\boldsymbol{a})$ by Corollary 6.4. We also have

$$\begin{aligned} N_{o,-1}(\boldsymbol{t}) &= N_{o,-1}(\boldsymbol{u}') - jd = 0, \\ N_{e,1}(\boldsymbol{t}) &= N_{e,1}(\boldsymbol{u}') - jd > 0, \\ N_{o,1}(\boldsymbol{t}) &= N_{o,1}(\boldsymbol{u}'), \\ N_{e,-1}(\boldsymbol{t}) &= 0. \end{aligned}$$

Thus $\mathbf{1} \in C_f(\boldsymbol{a})$ by Proposition 6.9.

Next consider the case $d \not| N_{o,-1}(\boldsymbol{u}')$. If $N_{o,-1}(\boldsymbol{u}') = 1$ then the proof has finished. Therefore we assume that $N_{o,-1}(\boldsymbol{u}') > 1$. Since d is a prime integer, there is a positive integer l such that $N_{o,-1}((\boldsymbol{u}')^l) \equiv 1 \bmod d$. Put $j = (N_{o,-1}((\boldsymbol{u}')^l) - 1)/d$. Since $N_{e,1}((\boldsymbol{u}')^l) > N_{o,-1}((\boldsymbol{u}')^l)$, by Corollary 6.5 there is a block $\boldsymbol{t} \in S_{\{-1,1\}}$ such that

$\boldsymbol{t} \cdot (\mathbf{1} \cdot -\mathbf{1})^{jd}$ is obtained from $(\boldsymbol{u}')^l$ by exchanges of symbols. We have $\boldsymbol{t} \in C_f(\boldsymbol{a})$ by Corollary 6.4. By construction we have

$$N_{o,-1}(\boldsymbol{t}) = N_{o,-1}((\boldsymbol{u}')^l) - jd = 1,$$
$$N_{e,1}(\boldsymbol{t}) = N_{e,1}((\boldsymbol{u}')^l) - jd > 0,$$
$$N_{o,1}(\boldsymbol{t}) = N_{o,1}((\boldsymbol{u}')^l) \quad \text{and} \quad N_{e,-1}(\boldsymbol{t}) = 0.$$

Therefore we may assume that $\boldsymbol{t} = -\mathbf{1} \cdot \mathbf{1}^L$ for some positive integer L. Since $2 | \lg(\boldsymbol{a})$, L is an odd integer. Since $d \not| \lg(\boldsymbol{t})$, it follows from Lemma 6.11 that there is an L' with $-\mathbf{1} \cdot \mathbf{1}^{L'} \in C_f(\boldsymbol{a})$ and $1 \leq L' \leq d-2$. □

A.23 Proof of Lemma 6.19

Let σ and τ be states in $L_p(\eta)$. Assume that there is a cycle starting from σ and let $\boldsymbol{s} \in S_{\{-1,1\}}$ be the block generated by the cycle. Assume that there is a path from σ to τ and let $\boldsymbol{v} \in S_{\{-1,1\}}$ be the block generated by the path. Since $\mathrm{RDS}(\boldsymbol{s}) = 0$, both $\mathbf{1}$ and $-\mathbf{1}$ appear in $\boldsymbol{s}$. Therefore a block $\boldsymbol{v}' \in S_{\{-1,1\}}$ and an integer ℓ exist such that

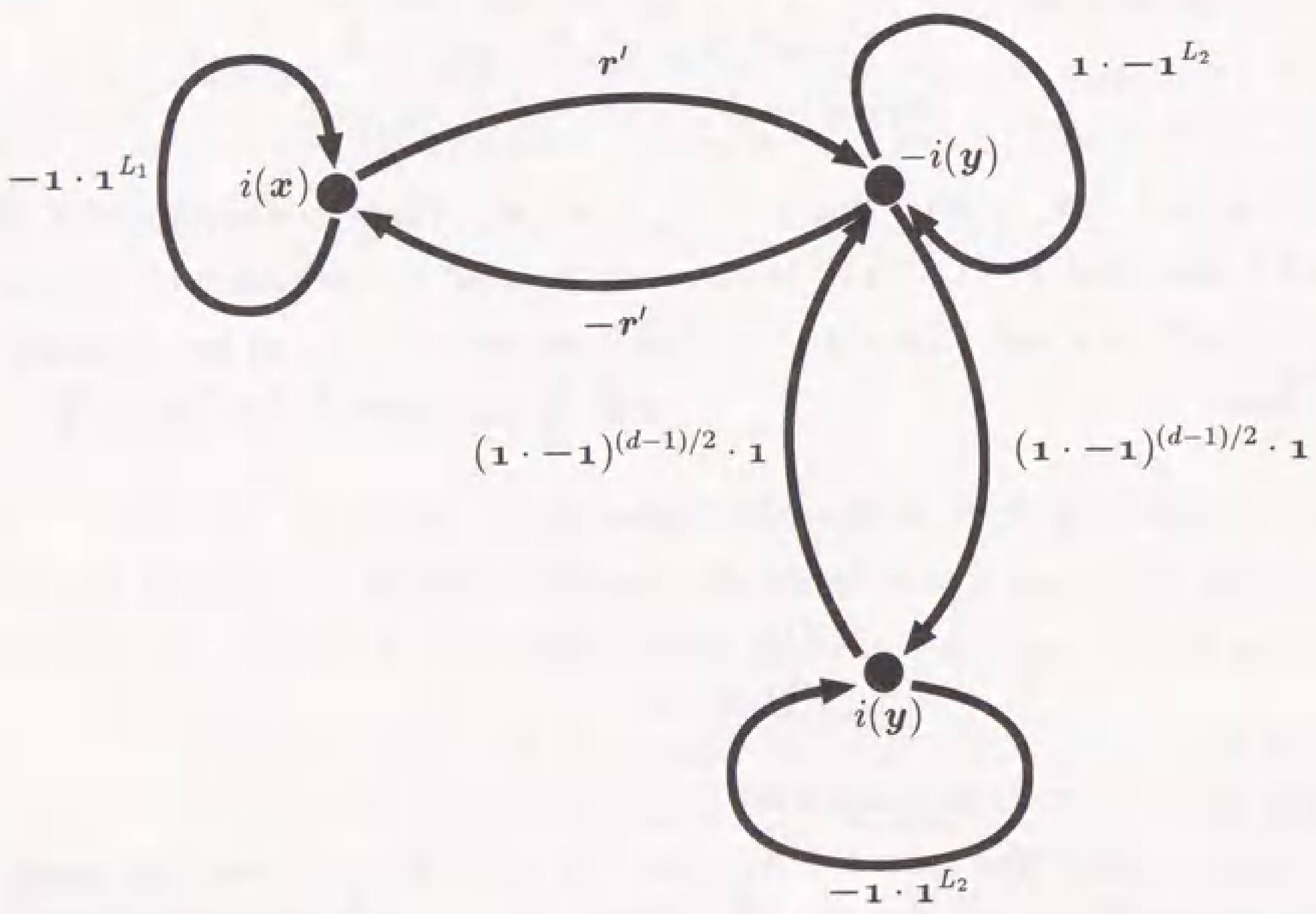

Figure A.22.1: Positions of $i(\boldsymbol{x})$, $i(\boldsymbol{y})$ and $-i(\boldsymbol{y})$

$\ell \lg(\boldsymbol{s}) = \lg(\boldsymbol{v} \cdot \boldsymbol{v}')$, $\mathrm{RDS}(\boldsymbol{s}^\ell - \boldsymbol{v} \cdot \boldsymbol{v}') = 0$, and both $\mathbf{2}$ and $-\mathbf{2}$ appear in $rds(\boldsymbol{s}^\ell - \boldsymbol{v} \cdot \boldsymbol{v}')$. Let $\boldsymbol{s}^\ell = a_0 a_1 \cdots a_{\ell \lg(\boldsymbol{s})-1}$, $\boldsymbol{v} \cdot \boldsymbol{v}' = b_0 b_1 \cdots b_{\ell \lg(\boldsymbol{s})-1}$ and $rds(\boldsymbol{s}^\ell - \boldsymbol{v} \cdot \boldsymbol{v}') = e_0 e_1 \cdots e_{\ell \lg(\boldsymbol{s})-1} \in S_Z$. Note that $e_i = 0$ means $\mathrm{RDS}(a_0 \cdots a_i - b_0 \cdots b_i) = 0$. Since $\boldsymbol{s}^\ell - \boldsymbol{v} \cdot \boldsymbol{v}' \in S_{\{-2,0,2\}}$, we have $rds(\boldsymbol{s}^\ell - \boldsymbol{v} \cdot \boldsymbol{v}') \in S_{\{2l : l \in Z\}}$. We define $\boldsymbol{u}_+ = u'_0 u'_1 \cdots u'_{\ell \lg(\boldsymbol{s})-1}$ and $\boldsymbol{u}_- = u''_0 u''_1 \cdots u''_{\ell \lg(\boldsymbol{s})-1}$ as follows:

$$u'_i = \begin{cases} b_i & \text{if } e_{i-1} > 0 \text{ or } e_i > 0; \\ a_i & \text{otherwise;} \end{cases}$$

$$u''_i = \begin{cases} b_i & \text{if } e_{i-1} < 0 \text{ or } e_i < 0; \\ a_i & \text{otherwise,} \end{cases}$$

for $i = 0, 1, \ldots, \ell \lg(\boldsymbol{s}) - 1$ where we put $e_{-1} = 0$.

We claim that $\mathrm{RDS}(\boldsymbol{s}^\ell - \boldsymbol{u}_+) = \mathrm{RDS}(\boldsymbol{s}^\ell - \boldsymbol{u}_-) = 0$. Let $\{j : e_j = 0\} = \{j_1, j_2, \ldots, j_K\}$ and assume that $j_i < j_{i+1}$ for $i = 1, 2, \ldots, K-1$. Then we have

$$\mathrm{RDS}(\boldsymbol{s}^\ell - \boldsymbol{u}_+) = \sum_{i=1}^{K} \mathrm{RDS}(a_{j_{i-1}+1} \cdots a_{j_i} - u'_{j_{i-1}+1} \cdots u'_{j_i}), \qquad (A.23.26)$$

$$\mathrm{RDS}(\boldsymbol{s}^\ell - \boldsymbol{u}_-) = \sum_{i=1}^{K} \mathrm{RDS}(a_{j_{i-1}+1} \cdots a_{j_i} - u''_{j_{i-1}+1} \cdots u''_{j_i}) \qquad (A.23.27)$$

where we put $j_0 = -1$. Therefore in order to prove our claim we prove the following.

$$\mathrm{RDS}(a_{j_{i-1}+1} \cdots a_{j_i} - u'_{j_{i-1}+1} \cdots u'_{j_i}) = \mathrm{RDS}(a_{j_{i-1}+1} \cdots a_{j_i} - u''_{j_{i-1}+1} \cdots u''_{j_i}) = 0$$
$$\text{for each } i \text{ with } 1 \leq i \leq K-1. \qquad (A.23.28)$$

Since $\boldsymbol{s}^\ell - \boldsymbol{v} \cdot \boldsymbol{v}' \in S_{\{-2,0,2\}}$, we note that if the product of e_{i_1} and e_{i_2} is negative and $i_1 < i_2$ then there is an i' such that $e_{i'} = 0$ and $i_1 < i' < i_2$. Hence if $e_i > 0$ then $u'_i = b_i$, $u'_{i+1} = b_{i+1}$, $u''_i = a_i$ and $u''_{i+1} = a_{i+1}$; if $e_i < 0$ then $u'_i = a_i$, $u'_{i+1} = a_{i+1}$, $u''_i = b_i$ and $u''_{i+1} = b_{i+1}$. If $e_i = e_{i+1} = 0$ then $0 = e_{i+1} - e_i = a_{i+1} - b_{i+1}$ by the definition of rds. Therefore, we have

$$a_i - b_i = a_i - u'_i + a_i - u''_i \quad \text{for } i = 0, 1, \ldots, \ell \lg(\boldsymbol{s}) - 1. \qquad (A.23.29)$$

Since $e_{j_i} = 0$, we have

$$\begin{aligned} 0 &= \mathrm{RDS}(a_0 \cdots a_{j_i} - b_0 \cdots b_{j_i}) \\ &= \mathrm{RDS}(a_0 \cdots a_{j_{i-1}} - b_0 \cdots b_{j_{i-1}}) + \mathrm{RDS}(a_{j_{i-1}+1} \cdots a_{j_i} - b_{j_{i-1}+1} \cdots b_{j_i}), \\ &\text{since } e_{j_{i-1}} = 0, \\ &= \mathrm{RDS}(a_{j_{i-1}+1} \cdots a_{j_i} - b_{j_{i-1}+1} \cdots b_{j_i}), \end{aligned}$$

and by (A.23.29), we have

$$\begin{aligned} &= \mathrm{RDS}(a_{j_{i-1}+1}\cdots a_{j_i} - u'_{j_{i-1}+1}\cdots u'_{j_i}) \\ &\quad +\mathrm{RDS}(a_{j_{i-1}+1}\cdots a_{j_i} - u''_{j_{i-1}+1}\cdots u''_{j_i}). \end{aligned} \tag{A.23.30}$$

If $j_i = j_{i-1}+1$ then it is clear that our claim holds because $a_{j_i} = u'_{j_i} = u''_{j_i}$. Suppose that $e_{j_{i-1}+1} > 0$. Since $\boldsymbol{s}^\ell - \boldsymbol{v}\cdot\boldsymbol{v}' \in S_{\{-2,0,2\}}$, if m is an integer with $j_{i-1} < m < j_i$ then we have $e_m > 0$. By the definition of $\boldsymbol{u}_-$, we have

$$\mathrm{RDS}(a_{j_{i-1}+1}\cdots a_{j_i} - u''_{j_{i-1}+1}\cdots u''_{j_i}) = \mathrm{RDS}(a_{j_{i-1}+1}\cdots a_{j_i} - a_{j_{i-1}+1}\cdots a_{j_i}) = 0.$$

Therefore from (A.23.30), we get (A.23.28). For the case $e_{j_{i-1}+1} < 0$ we also have (A.23.28) similarly.

Put $q = \mathrm{RDS}^{(2)}(\boldsymbol{s}^\ell - \boldsymbol{u}_+)$ and $r = -\mathrm{RDS}^{(2)}(\boldsymbol{s}^\ell - \boldsymbol{u}_-)$. Since both $\mathbf{2}$ and $-\mathbf{2}$ appear in $\boldsymbol{s}^\ell - \boldsymbol{v}\cdot\boldsymbol{v}'$, we have $\boldsymbol{s}^\ell \neq \boldsymbol{u}_+$ and $\boldsymbol{s}^\ell \neq \boldsymbol{u}_-$. Hence $q > 0$ and $r > 0$. By (A.23.29) we have

$$\begin{aligned} \mathrm{RDS}^{(2)}(\boldsymbol{s}^\ell - \boldsymbol{v}\cdot\boldsymbol{v}') &= \mathrm{RDS}^{(2)}(\boldsymbol{s}^\ell - \boldsymbol{u}_+ + \boldsymbol{s}^\ell - \boldsymbol{u}_-) \\ &= \mathrm{RDS}^{(2)}(\boldsymbol{s}^\ell - \boldsymbol{u}_+) + \mathrm{RDS}^{(2)}(\boldsymbol{s}^\ell - \boldsymbol{u}_-) \\ &= q - r. \end{aligned}$$

For every pair of blocks $\boldsymbol{c}$ and $\boldsymbol{d}$ with $\mathrm{RDS}(\boldsymbol{c}) = 0$ we have $\mathrm{RDS}^{(2)}(\boldsymbol{c}\cdot\boldsymbol{d}) = \mathrm{RDS}^{(2)}(\boldsymbol{c}) + \mathrm{RDS}^{(2)}(\boldsymbol{d})$. From Proposition 6.14, we have

$$\begin{aligned} &\mathrm{RDS}^{(2)}\left(\boldsymbol{s}^{\ell+(r-1)\ell+(q-1)l} - \boldsymbol{v}\cdot\boldsymbol{v}'\cdot\boldsymbol{u}_+^{r-1}\cdot\boldsymbol{u}_-^{q-1}\right) \\ &= \mathrm{RDS}^{(2)}\left(\left(\boldsymbol{s}^\ell - \boldsymbol{v}\cdot\boldsymbol{v}'\right)\cdot\left(\boldsymbol{s}^{(r-1)\ell} - \boldsymbol{u}_+^{r-1}\right)\cdot\left(\boldsymbol{s}^{(q-1)\ell} - \boldsymbol{u}_-^{q-1}\right)\right) \\ &= \mathrm{RDS}^{(2)}\left(\boldsymbol{s}^\ell - \boldsymbol{v}\cdot\boldsymbol{v}'\right) + \mathrm{RDS}^{(2)}\left(\boldsymbol{s}^{(r-1)\ell} - \boldsymbol{u}_+^{r-1}\right) \\ &\qquad +\mathrm{RDS}^{(2)}\left(\boldsymbol{s}^{(q-1)\ell} - \boldsymbol{u}_-^{q-1}\right) \\ &= \mathrm{RDS}^{(2)}(\boldsymbol{s}^\ell - \boldsymbol{v}\cdot\boldsymbol{v}') + (r-1)\mathrm{RDS}^{(2)}(\boldsymbol{s}^\ell - \boldsymbol{u}_+) + (q-1)\mathrm{RDS}^{(2)}(\boldsymbol{s}^\ell - \boldsymbol{u}_-) \\ &= q - r + (r-1)q - (q-1)r = 0. \end{aligned}$$

Therefore, by Proposition 6.16 there is a path from τ to σ.

For the case where there is a path from τ to σ, we can also prove that there is a path from σ to τ. □

A.24 Proof of Lemma A.24.4

We define $q_l(i)$ as follows:

$$q_l(i) = \begin{cases} i+2, & \text{if } l = 1; \\ \sum_{j=0}^{i} q_{l-1}(j), & \text{if } l > 1. \end{cases} \tag{A.24.31}$$

Lemma A.24.4 Let (G,ϕ_1) be an FSTD. Assume that (G,γ) satisfies a coboundary condition at f and let ϕ_2 be its coboundary function. Let $s_0\cdots s_L$ be a cycle in G. Then

$$\begin{aligned} \mathrm{RDS}_f^{(l)}\big(\phi_2(s_0\cdots s_i)\big) &= \mathrm{RDS}_f^{(l+1)}\big(\phi_1(s_0\cdots s_i)\big) + q_l(i)\phi_2(s_0) \\ &\quad -\mathrm{RDS}_f^{(l-1)}\big(\phi_2(s_1\cdots s_{i+1})\big), \quad i \geq 0,\ l \geq 1 \end{aligned} \tag{A.24.32}$$

Proof: We prove this lemma by induction. By the assumption

$$\phi_1(s_j) = \omega\phi_2(s_{j+1}) - \phi_2(s_j), \quad j = 0,1,\ldots,L.$$

By multiplying these equations by ω^j and summing up these equations, we get

$$\omega^{i+1}\phi_2(s_{i+1}) = \mathrm{RDS}_f^{(1)}\big(\phi_1(s_0\cdots s_i)\big) + \phi_2(s_0).$$

Hence

$$\begin{aligned} \mathrm{RDS}_f^{(1)}\big(\phi_2(s_0\cdots s_i)\big) &= \sum_{j=0}^{i}\omega^j\phi_2(s_j) \\ &= \sum_{j=0}^{i}\omega^{j+1}\phi_2(s_{j+1}) + \phi_2(s_0) - \omega^{i+1}\phi_2(s_{i+1}) \\ &= \sum_{j=0}^{i}\Big(\mathrm{RDS}_f^{(1)}\big(\phi_1(s_0\cdots s_j)\big) + \phi_2(s_0)\Big) \\ &\quad +\phi_2(s_0) - \omega^{i+1}\phi_2(s_{i+1}) \\ &= \mathrm{RDS}_f^{(2)}\big(\phi_1(s_0\cdots s_i)\big) + q_1(i)\phi_2(s_0) \\ &\quad -\mathrm{RDS}_f^{(0)}\big(\phi_2(s_1\cdots s_{i+1})\big). \end{aligned}$$

Next assume that (A.24.32) is true for $0,1,\ldots,l$. Then

$$\begin{aligned} \mathrm{RDS}_f^{(l+1)}\big(\phi_2(s_0\cdots s_i)\big) &= \sum_{j=0}^{i}\mathrm{RDS}_f^{(l)}\big(\phi_2(s_0\cdots s_i)\big) \\ &= \sum_{j=0}^{i}\Big(\mathrm{RDS}_f^{(l+1)}\big(\phi_1(s_0\cdots s_j)\big) + g_l(j)\phi_2(s_0) \\ &\quad -\mathrm{RDS}_f^{(l-1)}\big(\phi_2(s_1\cdots s_{j+1})\big)\Big) \\ &= \mathrm{RDS}_f^{(l+2)}\big(\phi_1(s_0\cdots s_i)\big) + q_{l+1}(i)\phi_2(s_0) \\ &\quad -\mathrm{RDS}_f^{(l)}\big(\phi_2(s_1\cdots s_{i+1})\big). \end{aligned}$$

□

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Higher order spectral density nulls and spectral lines	IEICE Trans. Fundamentals. (Sept, 1991)	Chapter 5
Irreducible components of canonical diagrams for spectral nulls	IEEE Trans. Inform. Theory. (Accepted for publication)	Chapter 6
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