付 録 A　一般相対論と宇宙論の基本的な幾何学量

A.1　一般相対論に関する公式と量

計量

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{A.1}$$

ただし，平坦時空では $(g_{\mu\nu}) = \mathrm{diag.}(-1, 1, 1, 1)$.
計量の逆行列

$$g_{\mu\lambda} g^{\lambda\nu} = \delta^\nu_\mu \tag{A.2}$$

計量の行列式とその微分

$$g = \det(g_{\mu\nu}), \quad dg = g g^{\mu\nu} dg_{\mu\nu} \tag{A.3}$$

接続係数

$$\Gamma^\mu_{\lambda\nu} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\lambda\nu} \right) \tag{A.4}$$

共変微分

$$\begin{aligned} \nabla_\lambda C^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} = {} & \partial_\lambda C^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} + \Gamma^{\mu_1}_{\lambda\rho} C^{\rho\mu_2\cdots}{}_{\nu_1\nu_2\cdots} + \Gamma^{\mu_2}_{\lambda\rho} C^{\mu_1\rho\cdots}{}_{\nu_1\nu_2\cdots} + \cdots \\ & - \Gamma^\rho_{\lambda\nu_1} C^{\mu_1\mu_2\cdots}{}_{\rho\nu_2\cdots} - \Gamma^\rho_{\lambda\nu_2} C^{\mu_1\mu_2\cdots}{}_{\nu_1\rho\cdots} - \cdots \end{aligned} \tag{A.5}$$

測地線方程式

$$\frac{dx^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\lambda} \frac{dx^\lambda}{d\lambda} = 0 \tag{A.6}$$

ここで，λ はアフィンパラメータ.
エネルギー・運動量テンソル

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p\, g^{\mu\nu} + P^\mu{}_i P^\nu{}_j \Sigma^{ij} \tag{A.7}$$

曲率テンソル

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\beta\nu} - \partial_\beta \Gamma^\mu_{\alpha\nu} + \Gamma^\mu_{\alpha\lambda} \Gamma^\lambda_{\beta\nu} - \Gamma^\mu_{\beta\lambda} \Gamma^\lambda_{\alpha\nu} \tag{A.8}$$

リッチ・テンソル

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \tag{A.9}$$

スカラー曲率

$$R = R^\mu{}_\mu \tag{A.10}$$

アインシュタイン・テンソル

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{A.11}$$

アインシュタイン方程式

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{A.12}$$

A.2　一様等方時空の幾何学量

3 次元一様等方計量

$$\gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{A.13}$$

ロバートソン・ウォーカー計量

$$ds^2 = -c^2 dt^2 + a^2(t) \gamma_{ij} dx^i dx^j \tag{A.14}$$

接続係数

$$\begin{aligned} &\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = \Gamma^i_{00} = 0, \quad \Gamma^0_{ij} = \frac{a\dot{a}}{c} \gamma_{ij}, \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{ca} \delta^i{}_j, \\ &\Gamma^i_{jk} = \frac{1}{2} \gamma^{il} (\gamma_{lk,j} + \gamma_{jl,k} - \gamma_{jk,l}) \equiv {}^{(3)}\Gamma^i_{jk} \end{aligned} \tag{A.15}$$

曲率テンソル

$$\begin{aligned} &R^0{}_{00i} = R^0{}_{0ij} = R^0{}_{ijk} = R^i{}_{0jk} = R^i{}_{j0k} = 0, \quad R^0{}_{i0j} = \frac{a\ddot{a}}{c^2} \gamma_{ij}, \quad R^i{}_{00j} = \frac{\ddot{a}}{c^2 a} \delta^i{}_j, \\ &R^i{}_{jkl} = \left(\frac{\dot{a}^2}{c^2} + K \right) (\delta^i{}_k \gamma_{jl} - \delta^i{}_l \gamma_{jk}) \end{aligned} \tag{A.16}$$

リッチ・テンソル

$$R^0{}_0 = \frac{3\ddot{a}}{c^2 a}, \quad R^i{}_0 = R^0{}_i = 0, \quad R^i{}_j = \frac{1}{c^2} \left[\frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a} \right)^2 + \frac{2c^2}{a^2} K \right] \delta^i{}_j \tag{A.17}$$

スカラー曲率

$$R = \frac{6}{c^2} \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{c^2}{a^2} K \right] \tag{A.18}$$

アインシュタイン・テンソル

$$G^0{}_0 = -\frac{3}{c^2} \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{c^2 K}{a^2} \right], \quad G^i{}_0 = G^0{}_i = 0, \quad G^i{}_j = -\frac{1}{c^2} \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{c^2 K}{a^2} \right] \delta^i{}_j \tag{A.19}$$

A.3 線形摂動時空の幾何学量（コンフォーマル・ニュートンゲージ）

ロバートソン・ウォーカー計量とその線形摂動

$$ds^2 = a^2(\tau)\left[-(1+2\Phi)d\tau^2 + \gamma_{ij}(1+2\Psi)\,dx^i dx^j\right] \tag{A.20}$$

接続係数

$$\Gamma^0_{00} = \mathcal{H} + \Phi' \tag{A.21}$$
$$\Gamma^0_{0i} = \Gamma^0_{i0} = \Phi_{|i} \tag{A.22}$$
$$\Gamma^0_{ij} = \gamma_{ij}\left[\mathcal{H}(1-2\Phi) + \Psi' + 2\mathcal{H}\Psi\right] \tag{A.23}$$
$$\Gamma^i_{00} = \Phi^{|i} \tag{A.24}$$
$$\Gamma^i_{0j} = \Gamma^i_{j0} = \delta^i_j(\mathcal{H} + \Psi') \tag{A.25}$$
$$\Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk} + \delta^i_j\Psi_{|k} + \delta^i_k\Psi_{|j} - \gamma_{jk}\Psi^{|i} \tag{A.26}$$

曲率テンソル

$$R^0{}_{00i} = R^0{}_{0ij} = 0 \tag{A.27}$$
$$R^0{}_{i0j} = \left[\mathcal{H}'(1-2\Phi+2\Psi) + \mathcal{H}(\Psi'-\Phi') + \Psi''\right]\gamma_{ij} - \Phi_{|ij} \tag{A.28}$$
$$R^i{}_{00j} = \delta^i_j\mathcal{H}' - \Phi^{|i}{}_{|j} - \mathcal{H}\delta^i_j\Phi' + \delta^i_j(\Psi'' + \mathcal{H}\Psi') \tag{A.29}$$
$$R^0{}_{ijk} = \mathcal{H}\left(\gamma_{ij}\Phi_{|k} - \gamma_{ik}\Phi_{|j}\right) + \left(\gamma_{ik}\Psi_{|j} - \gamma_{ij}\Psi_{|k}\right)' \tag{A.30}$$
$$R^i{}_{0jk} = \mathcal{H}\left(\delta^i_j\Phi_{|k} - \delta^i_k\Phi_{|j}\right) + \left(\delta^i_k\Psi_{|j} - \delta^i_j\Psi_{|k}\right)' \tag{A.31}$$
$$R^i{}_{j0k} = \mathcal{H}\left(\gamma_{jk}\Phi^{|i} - \delta^i_k\Phi_{|j}\right) + \left(\delta^i_k\Psi_{|j} - \gamma_{jk}\Psi^{|i}\right)' \tag{A.32}$$
$$\begin{aligned} R^i{}_{jkl} = {}& \left[K + \mathcal{H}^2(1-2\Phi+2\Psi) + 2\mathcal{H}\Psi'\right]\left(\delta^i_k\gamma_{jl} - \delta^i_l\gamma_{jk}\right) \\ & + \delta^i_l\Psi_{|jk} - \delta^i_k\Psi_{|jl} + \gamma_{jk}\Psi^{|i}{}_{|l} - \gamma_{jl}\Psi^{|i}{}_{|k} \end{aligned} \tag{A.33}$$

リッチ・テンソル

$$R_{00} = -3\mathcal{H}' + \triangle\Phi + 3\mathcal{H}(\Phi' - \Psi') - 3\Psi'' \tag{A.34}$$
$$R_{0i} = 2\mathcal{H}\Phi_{|i} - 2\Psi'_{|i} \tag{A.35}$$
$$\begin{aligned} R_{ij} = {}& \gamma_{ij}\left[2K + \left(\mathcal{H}' + 2\mathcal{H}^2\right)(1-2\Phi+2\Psi) - \mathcal{H}\Phi' + \Psi'' + 5\mathcal{H}\Psi' - \triangle\Psi\right] \\ & - \Phi_{|ij} - \Psi_{|ij} \end{aligned} \tag{A.36}$$

リッチ・テンソル (添字を上げた形)

$$R^0{}_0 = \frac{1}{a^2}\left[3\mathcal{H}'(1-2\Phi) - \triangle\Phi - 3\mathcal{H}\Phi' + 3\Psi'' + 3\mathcal{H}\Psi'\right] \tag{A.37}$$

$$R^0{}_i = -\frac{1}{a^2}\left[2\mathcal{H}\Phi_{|i} - 2\Psi'_{|i}\right] \tag{A.38}$$

$$R^i{}_0 = \frac{1}{a^2}\left[2\mathcal{H}\Phi^{|i} - 2\Psi^{|i\prime}\right] \tag{A.39}$$

$$\begin{aligned} R^i{}_j = \frac{1}{a^2}\Big\{&\delta^i_j\left[2K + \left(\mathcal{H}' + 2\mathcal{H}^2\right)(1-2\Phi) - \mathcal{H}\Phi' + \Psi'' + 5\mathcal{H}\Psi' - (\triangle + 4K)\Psi\right] \\ &- \Phi^{|i}{}_{|j} - \Psi^{|i}{}_{|j}\Big\} \end{aligned} \tag{A.40}$$

スカラー曲率

$$R = \frac{2}{a^2}\left[3K + 3\left(\mathcal{H}' + \mathcal{H}^2\right)(1-2\Phi) - \triangle\Phi - 3\mathcal{H}\Phi' + 3\Psi'' + 9\mathcal{H}\Psi' - 2(\triangle + 3K)\Psi\right] \tag{A.41}$$

アインシュタイン・テンソル

$$G^0{}_0 = -\frac{1}{a^2}\left[3K + 3\mathcal{H}^2(1-2\Phi) - 6\mathcal{H}\Psi' + 2(\triangle + 3K)\Psi\right] \tag{A.42}$$

$$G^0{}_i = -\frac{1}{a^2}\left(2\mathcal{H}\Phi_{|i} - 2\Psi'_{|i}\right) \tag{A.43}$$

$$G^i{}_0 = \frac{1}{a^2}\left(2\mathcal{H}\Phi^{|i} - 2\Psi^{|i\prime}\right) \tag{A.44}$$

$$\begin{aligned} G^i{}_j = -\frac{1}{a^2}\Big\{&\left[K + \left(2\mathcal{H}' + \mathcal{H}^2\right)(1-2\Phi) - 2\mathcal{H}\Phi' - \triangle\Phi + 2\Psi'' + 4\mathcal{H}\Psi' - (\triangle + 2K)\Psi\right]\delta^i_j \\ &+ \Phi^{|i}{}_{|j} + \Psi^{|i}{}_{|j}\Big\} \end{aligned} \tag{A.45}$$