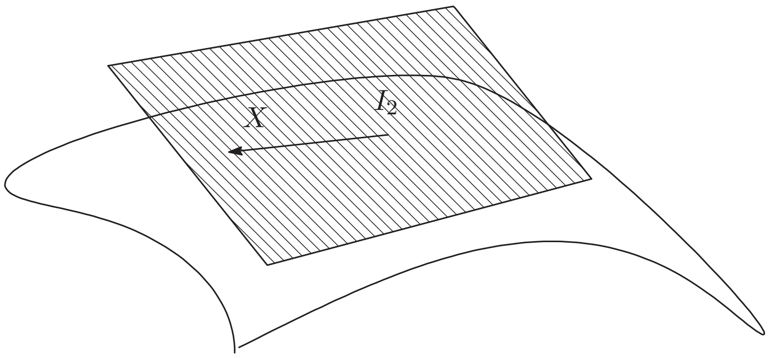
2 Linear fractional transformations

2.1 Lie algebra as the tangent space

Calculus of several variables (partial differentiation)



Tangent space

Example 2.1. *Let*

$$G = SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R}) \mid ad - bc = 1 \right\}.$$

How to compute the tangent space at the identity $I_2 \in G$?

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R})$ and $g = I_2 + \varepsilon X$. We will write down the condition to be $g \in G$ up to $O(\varepsilon^2)$, i.e., just take linear terms of ε and ignore ε^2 or higher. This procedure corresponds to the concept of tangent space. For $g = I_2 + \varepsilon X = \begin{pmatrix} 1+\varepsilon a & \varepsilon b \\ \varepsilon c & 1+\varepsilon d \end{pmatrix}$,

$$\det g = (1+\varepsilon a)(1+\varepsilon b) - \varepsilon b \varepsilon c = 1 + \varepsilon(a+d) + \varepsilon^2(ad - bc),$$

and thus

$$\det g \equiv 1 \mod O(\varepsilon^2) \iff a + d = 0.$$

So, the tangent space of $G = SL(2,\mathbb{R})$ at the identity $I_2 \in G$ is

$$\mathfrak{sl}(2,\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R}) \mid a + d = 0 \right\}.$$

Remark 2.2. *S =special, G =general, L =linear,*
$GL(n,\mathbb{R}) = \{g \in M(n,\mathbb{R}) \mid \det g \neq 0\}$*:group and*
$SL(n,\mathbb{R}) = \{g \in M(n,\mathbb{R}) \mid \det g = 1\}$*:group,*
$SL(n,\mathbb{R}) \lhd GL(n,\mathbb{R})$.

Example 2.3 (From Lie groups to Lie algebras)**.** *(0) The tangent space of* $SL(n,\mathbb{R})$ *at* I_n *is*

$$\{X \in M(n,\mathbb{R}) \mid tr(X) = 0\}.$$

(1) The tangent space of $O(n)$ *at* I_n *is*

$$\{X \in M(n,\mathbb{R}) \mid X + {}^tX = 0_n\},$$

where

$$O(n) = \{g \in M(n,\mathbb{R}) \mid {}^tgg = I_n\}.$$

(2) The tangent space of $U(n)$ *at* I_n *is*

$$\{X \in M(n,\mathbb{C}) \mid X + {}^t\overline{X} = 0_n\},$$

where

$$U(n) = \{g \in M(n,\mathbb{C}) \mid {}^t\overline{g}g = I_n\}.$$

(3) The tangent space of $U(1,1)$ *at* I_2 *is* $\mathfrak{u}(1,1)$*, where*

$$U(1,1) = \left\{ g \in M(2,\mathbb{C}) \mid g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

$$\begin{aligned} \mathfrak{u}(1,1) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{C}) \mid a + \overline{a} = 0,\ d + \overline{d} = 0,\ b = \overline{c} \right\} \\ &= \left\{ \begin{pmatrix} i\alpha & \beta + i\gamma \\ \beta - i\gamma & i\delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R} \right\} \end{aligned}$$

Proof. (0) Exercise.

(1) Take $X \in M(n,\mathbb{R})$, and consider

$$g = I_n + \varepsilon X.$$

Then ${}^tg = I_n + \varepsilon\, {}^tX$ and

$$\begin{aligned} {}^tgg &= (I_n + \varepsilon\, {}^tX)(I_n + \varepsilon\, {}^tX) \\ &= I_n + \varepsilon(X + {}^tX) + \varepsilon^2\, {}^tXX \\ &\equiv I_n + \varepsilon(X + {}^tX) \mod O(\varepsilon^2). \end{aligned}$$

The condition $g \in O(n)$, i.e, ${}^tgg = I_n$ is equivalent to $X + {}^tX = 0$.

(2) Report (in such a case, ε to be real). If $g = I_n + \varepsilon X$, then $\overline{g} = I_n + \varepsilon\overline{X}$ and ${}^t\overline{g} = I_n + \varepsilon\, {}^t\overline{X}$.

(3) We start with a coordinate-free computation

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{C}), \ g = I_2 + \varepsilon X \ (\varepsilon \in \mathbb{R}, \varepsilon = \overline{\varepsilon}), \ {}^t\overline{g} = I_2 + \varepsilon {}^t\overline{X}.$$

The condition $g \in U(1,1)$, i.e., ${}^t g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is equivalent to

$$(I_2 + \varepsilon\, {}^t\overline{X}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (I_2 + \varepsilon X) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The LHS is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \varepsilon\, {}^t\overline{X} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X + \varepsilon^2\, {}^t\overline{X} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X.$$

Thus

$$\mathfrak{u}(1,1) = \left\{ X \in M(2,\mathbb{C}) \mid {}^t\overline{X} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X = 0_2 \right\}.$$

Now, we use the expression in coordinates. Put the expression $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the equation above, we obtain

$${}^t\overline{X} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X = \begin{pmatrix} \overline{a} & -\overline{c} \\ \overline{b} & -\overline{d} \end{pmatrix} + \begin{pmatrix} a & b \\ -c & -d \end{pmatrix},$$

which is 0_2 if and only if

$$\overline{a} + a = 0, \quad -\overline{c} + b = 0, \ \overline{b} - c = 0, \ \text{and} \ -\overline{d} - d = 0.$$

□

Definition 2.4. The tangent space at the origin of a (Lie) group G is called the *Lie algebra* of G. It is often written as $\mathfrak{g}$ (the corresponding lower German), e.g., the Lie algebra of

$$G = SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R}) \mid ad - bc = 1 \right\}$$

is

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R}) \mid a + d = 0 \right\}.$$

Exercise 2.5. *Define* $[\ ,\ ]$ *by* $[A,B] = AB - BA$. *Prove*

(1) *If* $A, B \in \mathfrak{o}(n)$, *then* $[A,B] \in \mathfrak{o}(n)$.

(2) *If* $A, B \in \mathfrak{u}(n)$, *then* $[A,B] \in \mathfrak{u}(n)$.

Exercise 2.6. *Let* $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. *Prove* $H, X, Y \in \mathfrak{sl}(2,\mathbb{R})$ *and compute*

$$[H,X] =?$$

$$[H,Y] =?$$

$$[X,Y] =?$$

2.2 Linear fractional transformation

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ and x be a variable.

$$\rho(g)x = \frac{ax+b}{cx+d}$$

is called *linear fractional transformation*.

Exercise 2.7. *Prove*

$$\rho(g_1 g_2)x = \rho(g_1)\rho(g_2)x \text{ for } g_1, g_2 \in SL(2,\mathbb{C}). \tag{2.1}$$

Proof(elegant). Use the homogeneous/inhomogeneous coordinates of projective line $\mathbf{P}^1$, and use the associativity of the multiplications of matrices

$$(AB)\mathbf{v} = A(B(\mathbf{v})),\ A, B \in SL(2,\mathbb{C}),\ \mathbf{v} \in M(2,1).$$

□

Proof(elephant). Let $g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in SL(2,\mathbb{C})$ and

$$g_1 g_2 = \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

The left-hand side of (2.1) is

$$\frac{ix+j}{kx+l} = \frac{(ae+bg)x+af+bh}{(ce+dg)x+cf+dh},$$

and the right-hand side of (2.1) is

$$\rho(g_1)\frac{ex+f}{gx+h} = \frac{a\cdot\frac{ex+f}{gx+h}+b}{c\cdot\frac{ex+f}{gx+h}+d} = \frac{(ae+bg)x+af+bh}{(ce+dg)x+cf+dh}.$$

Thus

$$\rho(g_1g_2)x = \rho(g_1)\rho(g_2)x \text{ for } g_1, g_2 \in SL(2).$$

□

Let $f = f(x)$ be a function on x. We define $\pi(g)$ by

$$(\pi(g)f)(x) := f(\rho(g^{-1})x).$$

Exercise 2.8. *Prove*

$$\pi(g_1g_2) = \pi(g_1)\pi(g_2) \textit{ for } g_1, g_2 \in SL(2,\mathbb{C}).$$

Meaning of Exercise 2.8 is

$$(\pi(g_1g_2)f)(x) = (\pi(g_1)(\pi(g_2)f))(x).$$

Proof.

$$\begin{aligned} RHS &= (\pi(g_1)f)(\rho(g_2^{-1})x) \qquad \text{by def. of } \pi(g_2) \\ &= f(\rho(g_1^{-1})\rho(g_2^{-1})x) \qquad \text{by def. of } \pi(g_1) \\ &= f(\rho(g_1^{-1}g_2^{-1})x) \qquad \text{by ex. 2.7} \\ &= LHS \qquad \text{by def. of } \pi(g_1g_2). \end{aligned}$$

□

Exercise 2.9. *If you would define*

$$(\pi(g)f)(x) := f(\rho(g)x),$$

then which do you obtain either

$$\pi(g_1g_2) = \pi(g_1)\pi(g_2)$$

or

$$\pi(g_1g_2) = \pi(g_2)\pi(g_1).$$

2.3 Taylor expansion of $\pi(g)$

Let $g = I_2 + \varepsilon H$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we obtain

$$g^{-1} = \frac{1}{1-\varepsilon^2} \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix} \equiv \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix} \mod O(\varepsilon^2)$$

and

$$\begin{aligned} \rho(g^{-1})x &= \frac{(1-\varepsilon)x+0}{0x+(1+\varepsilon)} = \frac{1-\varepsilon}{1+\varepsilon}x \\ &= \frac{(1-\varepsilon)^2}{1-\varepsilon^2}x \equiv (1-2\varepsilon)x \mod O(\varepsilon^2). \end{aligned}$$

$$\begin{aligned} (\pi(g)f)(x) &= f(\rho(g^{-1})x) = f(x - 2\varepsilon x) \\ &\equiv f(x) - 2\varepsilon x f'(x) = f(x) - 2\varepsilon x \partial f(x) \mod O(\varepsilon^2), \end{aligned}$$

where we have used the fact that

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \cdots \equiv f(x) + f'(x)h + O(h^2).$$

Thus $\pi(g)f \equiv f - 2\varepsilon x \partial f \mod O(\varepsilon^2)$.
Summarize: We define $\pi'(H) = -2x\partial$, then

$$\pi(I_2 + \varepsilon H)f \equiv f + \varepsilon \pi'(H)f \mod O(\varepsilon^2).$$

Exercise 2.10. *Compute $\pi'(X)$ and $\pi'(Y)$, so that*

$$\pi(I_2 + \varepsilon X)f \equiv f + \varepsilon \pi'(X)f \mod O(\varepsilon^2),$$

$$\pi(I_2 + \varepsilon Y)f \equiv f + \varepsilon \pi'(Y)f \mod O(\varepsilon^2).$$

The answer will be

$$\pi'(X) = -\partial$$

$$\pi'(Y) = x^2\partial.$$

2.4 Where is λ

Definition 2.11. Fix $\lambda \in \mathbb{C}$. For a function $f = f(x)$ of one variable x, we define another function $F = F\begin{pmatrix} x \\ y \end{pmatrix}$ with two variables x and y by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = y^\lambda f(\frac{x}{y}).$$

Also, we define $\widetilde{\pi}(g)$ by

$$(\widetilde{\pi}(g)F)\begin{pmatrix} x \\ y \end{pmatrix} = F(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}),\ g \in SL(2).$$

$$(\pi_\lambda(g)f)(x) := (\widetilde{\pi}(g)F)\begin{pmatrix} x \\ 1 \end{pmatrix}.$$

What is the effect of λ?

We have

$$\begin{aligned}
(\pi_\lambda(I+\varepsilon H)f)(x) &= (\widetilde{\pi}(I+\varepsilon H)F)\begin{pmatrix} x \\ 1 \end{pmatrix} && \text{by def. of } \pi_\lambda \\
&= F((I+\varepsilon H)^{-1}\begin{pmatrix} x \\ 1 \end{pmatrix}) && \text{by def. of } \widetilde{\pi} \\
&\equiv F\begin{pmatrix} (1-\varepsilon)x \\ 1+\varepsilon \end{pmatrix} \mod O(\varepsilon^2) \\
&\equiv (1+\varepsilon)^\lambda f(\frac{1-\varepsilon}{1+\varepsilon}x) \mod O(\varepsilon^2) && \text{by def. of } F \\
&\equiv (1+\varepsilon\lambda)(f+\varepsilon\pi'(H)f) \mod O(\varepsilon^2) && \text{by §2.3} \\
&\equiv f + \varepsilon(\pi'(H)f + \lambda f) \mod O(\varepsilon^2),
\end{aligned}$$

where we have used the fact that Taylor expansion

$$(1+\varepsilon)^\lambda = 1 + \varepsilon\lambda + O(\varepsilon^2).$$

So, if we define $\pi'_\lambda(H) = \pi_\lambda(H) + \lambda$, then

$$\pi_\lambda(I+\varepsilon H)f \equiv f + \varepsilon\pi'_\lambda(H)f \mod O(\varepsilon^2).$$

Exercise 2.12. *Compute $\pi'_\lambda(X)$ and $\pi'_\lambda(Y)$.*

Now, we give the following table:

	π'	π'_λ
X	$-\partial$	$-\partial$
H	-2∂	$-2x\partial + \lambda$
Y	$x^2\partial$	$x^2\partial - \lambda x$