Amplification of Gaussian beams through a gaseous medium

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An amplification of the Gaussian beams in an active gaseous medium, possesing a quadratic radial variation of the population inversion, is considered. The gain curve is shown to have an asymmetrical shape. For a given population inversion, the maximum gain is calculated to be attainable at higher frequency than that of the atomic resonance, both in unsaturated and saturated operations.

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I. INTRODUCTION

The amplification of Gaussian beams in homogeneous gas media can be treated in terms of ABCD matrices,[1,2] in which the propagation constant is substituted with a complex number $k=\beta+i\alpha$. This k is generally dependent not only upon the distance from the axis, but also upon frequency and saturation of the lasing field. In the high-intensity development, nonlinear effect of the saturation plays an important role.

In our approach to describe the electric field, we start with the scalar wave equation,[3] $-\nabla^2 E+(1/c_0^2)(\partial^2 E/\partial t^2)=-\mu_0\,\partial^2 P/\partial t^2$, and solve this by using the following assumptions: (i) the electric field is a single monochromatic Gaussian mode and (ii) the medium suspending the beam has the population inversion N given by[4] $N=N_0 J_0(a_0 r/b)$, where $a_0=2.405$, r is the radial distance from the cylindrical axis, and b is the inner radius of the laser tube. The polarization of the medium is calculated from Lamb's theory with the help of a rotating-wave approximation.

II. THEORY

The Gaussian-beam formalism can be applied if higher-order profiles and diffraction losses are small.[3] The field distribution of a fundamental complex Gaussian mode is given by

$$E(r,z)=\frac{1}{w}\exp\left[-i(kz+\Phi)-r^2\left(\frac{1}{w^2}+\frac{ik}{2R}\right)\right],$$

where an amplitude factor is dropped. k is the wave number, z is the distance along the optic axis, w is the $1/e$ radius of the beam, Φ is the additional phase shift, and R is the radius of curvature of the phase front of the spherical wave. A real Gaussian beam traveling in the positive z direction may be written

$$E(r,z,t)=E\exp\left(-\frac{r^2}{w^2}\right)\sin\left[\omega t-k\left(z+\frac{r^2}{2R}\right)-\Phi\right], \qquad (1)$$

in which E, w, R, and Φ are functions of z.

The scalar wave equation in cylindrical coordinates is given by

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right)-\frac{\partial^2 E}{\partial z^2}+\frac{1}{c_0^2}\frac{\partial^2 E}{\partial t^2}=-\mu_0\frac{\partial^2 P}{\partial t^2}. \qquad (2)$$

By letting

$$S=E\exp\left(-\frac{r^2}{w^2}\right)\sin\left[\omega t-k\left(z+\frac{r^2}{2R}\right)-\Phi\right]$$

and

$$C=E\exp\left(-\frac{r^2}{w^2}\right)\cos\left[\omega t-k\left(z+\frac{r^2}{2R}\right)-\Phi\right],$$

we have the left-hand side of Eq. (2) in a power series of r as

$$\left[\left(\frac{4E}{w^2}+E(k+\Phi')^2-\frac{E\omega^2}{c_0^2}\right)S+\left(\frac{2Ek}{R}+2E'(k+\Phi')\right)C\right]$$
$$+r^2\left[\left(-\frac{8E}{w^4}+\frac{Ek^2}{R^2}-\frac{4E'w'}{w^3}-\frac{E(k+\Phi')^2}{w^2}-\frac{EkR'(k+\Phi')}{R^2}+\frac{E\omega^2}{c_0^2w^2}\right)S+\left(-\frac{6Ek}{Rw^2}-\frac{2E'(k+\Phi')}{w^2}-\frac{E'kR'}{R^2}+\frac{4Ew'(k+\Phi')}{w^3}\right)C\right]+r^4[\cdots],$$

where the second derivatives $\partial^2/\partial z^2$ are dropped.

The macroscopic polarization of the medium consisting of two-level atoms is calculated from the following equation[5]:

$$P=\frac{N_0\mu_{ab}^2 J_0(a_0 r/b)}{\hbar u\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\Gamma_0 C+(\omega-\omega_0+kV)S}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\times\left(1+\frac{\mu_{ab}^2E^2}{\hbar^2\gamma_a\gamma_b}\frac{\Gamma_0^2}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\right)^{-1}\exp\left(-\frac{V^2}{u^2}\right)dV, \qquad (3)$$

in which the velocity of atoms is assumed to have the Maxwell-Boltzmann distribution with the mean squared velocity u^2. μ_{ab} is the matrix element of the dipole moment, γ_a and γ_b are the decay constants of the upper and lower levels, respectively, ω_0 is the resonant angular frequency of the atomic transition, and $\Gamma_0=\frac{1}{2}(\gamma_a+\gamma_b)$.

Expanding E and P in a power series to the quadratic of r to compare them term by term on both sides of the wave equation (2), we find the following four differential equations:

$$\frac{4E}{w^2}+E(k+\Phi')^2-\frac{E\omega^2}{c_0^2}=\omega^2AE\int_{-\infty}^{\infty}\frac{\omega-\omega_0+kV}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\,\frac{\exp(-V^2/u^2)}{1+(P/P_{sat})L(\omega,V)}\,dV, \qquad (4)$$

$$\frac{2Ek}{R}+2E'(k+\Phi')=\omega^2AE\int_{-\infty}^{\infty}\frac{\Gamma_0}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\,\frac{\exp(-V^2/u^2)}{1+(P/P_{sat})L(\omega,V)}\,dV, \qquad (5)$$

$$\frac{8E}{w^4}+\frac{E}{w^2}\left((k+\Phi')^2-\frac{\omega^2}{c_0^2}\right)-\frac{Ek^2}{R^2}+\frac{4E'w'}{w^3}+\frac{EkR'(k+\Phi')}{R^2}$$
$$=\omega^2AE\left(\frac{a_0^2}{4b^2}+\frac{1}{w^2}\right)\int_{-\infty}^{\infty}\frac{\omega-\omega_0+kV}{\Gamma_0^2+(\omega-\omega_0+kV)^2}$$
$$\times\frac{\exp(-V^2/u^2)}{1+(P/P_{sat})L(\omega,V)}\,dV$$
$$-\frac{2}{w^2}\omega^2AE\int_{-\infty}^{\infty}\frac{\omega-\omega_0+kV}{\Gamma_0^2+(\omega-\omega_0+kV)^2}$$
$$\times\frac{(P/P_{sat})L(\omega,V)\exp(-V^2/u^2)}{[1+(P/P_{sat})L(\omega,V)]^2}\,dV, \qquad (6)$$

and

$$\frac{6Ek}{Rw^2}+\frac{2E'(k+\Phi')}{w^2}+\frac{E'kR'}{R^2}-\frac{4Ew'(k+\Phi')}{w^3}$$
$$=\omega^2AE\left(\frac{a_0^2}{4b^2}+\frac{1}{w^2}\right)\int_{-\infty}^{\infty}\frac{\Gamma_0}{\Gamma_0^2+(\omega-\omega_0+kV)^2}$$
$$\times\frac{\exp(-V^2/u^2)}{1+(P/P_{sat})L(\omega,V)}\,dV$$
$$-\frac{2}{w^2}\omega^2AE\int_{-\infty}^{\infty}\frac{\Gamma_0}{\Gamma_0^2+(\omega-\omega_0+kV)^2}$$
$$\times\frac{(P/P_{sat})L(\omega,V)\exp(-V^2/u^2)}{[1+(P/P_{sat})L(\omega,V)]^2}\,dV, \qquad (7)$$

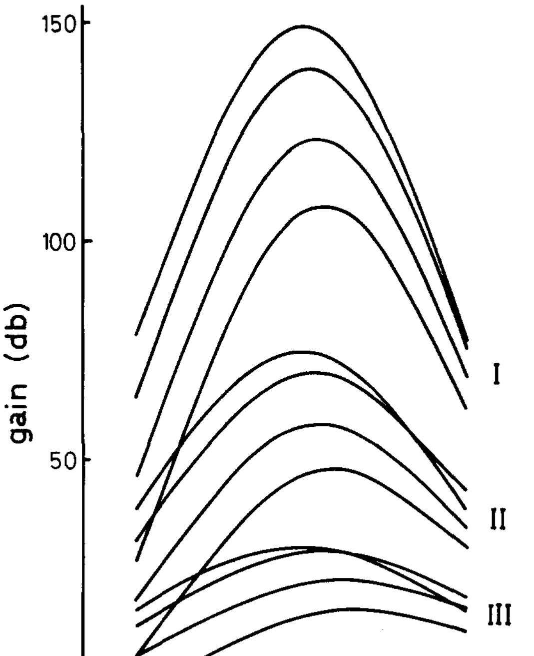

FIG. 1. Unsaturated gain curves. These are computed gains for the Xe 3.51-μ transition with the parameters $\gamma_a/\Gamma_0=6.4\times10^{-2}$, $\gamma_b/\Gamma_0=1.94$, $ku/\Gamma_0=37.0$, and $P_{sat}=3$ W/m^2. Horizontal line is in units of $\Delta\nu_n$(3.73 MHz). Group I is for the population inversion $N=10^{11}$ (atoms/cm^3), group II is for 5×10^{10} (atoms/cm^3), and group III is for 2×10^{10} (atoms/cm^3). Each of the curves in one group, from the upper one to the lower one, are for the tube with inner radius 10 cm, 2 mm, 1 mm, and 0.67 mm, respectively.

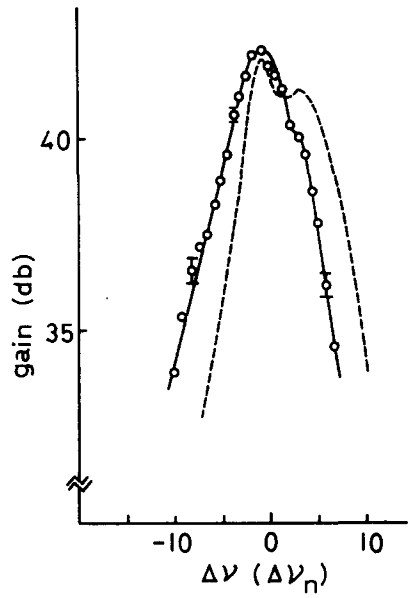

FIG. 2. Observed and computed values of the gain of the Xe amplifier. Circles indicate the observed gains of the Xe 3.51-μ transition for the tube with length 160 cm and inner diameter 4 mm from Ref. 7. Calculated values of this amplifier with the input-beam parameters given in Ref. 5. are shown by a broken line.

where $A=\mu_0N_0\mu_{ab}^2/\hbar u\sqrt{\pi}$, $P=\frac{1}{2}\epsilon_0c_0E^2$, $P_{sat}=\frac{1}{2}\epsilon_0c_0\hbar^2\gamma_a\gamma_b/\mu_{ab}^2$, and $L(\omega,V)=\Gamma_0^2/[\Gamma_0^2+(\omega-\omega_0+kV)^2]$.

These four expressions from (4) to (7) are the basic equations to describe propagations of a Gaussian beam.

A. Unsaturated region

In the unsaturated region we may put $P/P_{sat}=0$. Neglecting the products of first derivatives Φ'^2, $E'w'$, $E'R'$, etc., we have the following relations from Eqs. (4)–(7):

$$\frac{4E}{w^2}+E\left(k^2-\frac{\omega^2}{c_0^2}\right)+2Ek\Phi'$$
$$=\omega^2AE\int_{-\infty}^{\infty}\frac{\omega-\omega_0+kV}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\times\exp\left(-\frac{V^2}{u^2}\right)dV, \qquad (8)$$

$$\frac{2Ek}{R}+2E'k$$
$$=\omega^2AE\int_{-\infty}^{\infty}\frac{\Gamma_0}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\exp\left(-\frac{V^2}{u^2}\right)dV, \qquad (9)$$

$$\frac{8E}{w^4}+\frac{E}{w^2}\left(k^2-\frac{\omega^2}{c_0^2}+2k\Phi'\right)-\frac{Ek^2}{R^2}+\frac{Ek^2R'}{R^2}$$
$$=\omega^2AE\left(\frac{a_0^2}{4b^2}+\frac{1}{w^2}\right)\int_{-\infty}^{\infty}\frac{\omega-\omega_0+kV}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\exp\left(-\frac{V^2}{u^2}\right)dV, \qquad (10)$$

and

$$\frac{6Ek}{Rw^2}+\frac{2E'k}{w^2}-\frac{4Ekw'}{w^3}$$
$$=\omega^2AE\left(\frac{a_0^2}{4b^2}+\frac{1}{w^2}\right)\int_{-\infty}^{\infty}\frac{\Gamma_0\exp(-V^2/u^2)}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\,dV. \qquad (11)$$

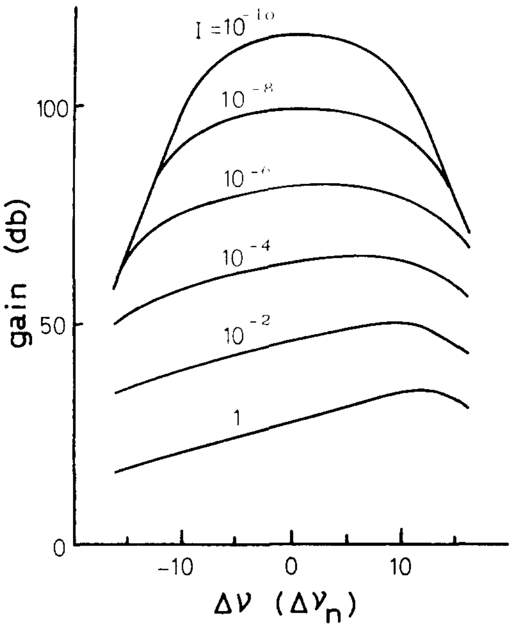

FIG. 3. Saturated gain curves. These are the computed amplifier gains for $N=10^{11}$ with various input powers I (W/m^2).

By introducing a complex-valued parameter q defined by

$$\frac{1}{q}=\frac{1}{R}-i\frac{\lambda}{\pi w^2}, \qquad (12)$$

we find from the above four equations that q satisfies

$$-\frac{1}{q^2}+\frac{1}{q^2}q'=i\frac{\alpha_0^2\omega^2A}{4b^2k^2}\int_{-\infty}^{\infty}\frac{\exp(-V^2/u^2)}{\Gamma_0+i(\omega-\omega_0+kV)}\,dV. \qquad (13)$$

The solution of this first-order differential equation may be given by

$$\frac{1}{q}=\zeta\cot\zeta(z+c), \qquad (14)$$

in which

$$\zeta^2=i\frac{\alpha_0^2\omega^2A}{4b^2k^2}\int_{-\infty}^{\infty}\frac{\exp(-V^2/u^2)}{\Gamma_0+i(\omega-\omega_0+kV)^2}\,dV$$

and c is a constant determined from boundary conditions at $z=z_0$. Integration of Eq. (9) leads to

$$E=E_0\,\mathrm{mod}\left(\frac{\sin\zeta c}{\sin\zeta(z+c)}\right)\exp\left(z\frac{\omega^2A}{2k}\int_{-\infty}^{\infty}\frac{\Gamma_0\exp(-V^2/u^2)\,dV}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\right). \qquad (15)$$

For a large z, the exponential gain is given by

$$\frac{\omega^2A}{2k}\int_{-\infty}^{\infty}\frac{\Gamma_0\exp(-V^2/u^2)}{\Gamma_0^2+(\omega-\omega_0+kV)^2}\,dV-\mathrm{Im}(\zeta). \qquad (16)$$

Computed values of gain in the unsaturated region are plotted in Fig. 1 for a Xe 3.51-μ transition,[6] for various population inversion densities (atoms/cm^3) and inner radii of tubes. The parameters of input beams are taken $R=10^4$ cm and $w=10^4$ cm.

The observed gains[7] of a Xe-136-isotope amplifier is shown in Fig. 2. The experiment is conducted with a tube of 160-cm length and 4-mm inner diameter. One of the flat mirrors of the Xe-136-isotope laser is scanned and the output from this laser is split into two beams. One beam is used as input of the amplifier and the other as a reference beam. The signals before and after the amplifications are recorded by two pen recorders and the amplification is calculated afterwards. The broken line is computed from the input-beam parameters given in Ref. 5.

B. Saturated region

When saturation sets in, we cannot put $P/P_{sat}=0$. Under this condition, it is somewhat difficult to find analytical solutions of Eqs. (4)—(7). Thus we use a computer to evaluate amplifications. Figure 3 depicts saturated gain curves for different input powers (W/m^2) with the input-beam parameters $R=10^4$ cm and $w=10^4$ cm. For these almost plane waves, the maximum gain can be obtained at a higher frequency.

III. CONCLUSION

Gain is sensitive to the input-beam parameters R and w. The maximum gain is shown to be attainable at a higher frequency than the line center for large R and w. When a laser tube is used as an amplifier, this nonlinear amplification gives rise to distorted output profiles and thus should be taken into account.

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