2008. 06. 19

We continue with some more of the abstract homotopy theory connected with model categories. Let $\mathcal{E}$ be a model category. We define the homotopy category $Ho(\mathcal{E})$ as follows: The objects are

$$\mathrm{ob}\, Ho(\mathcal{E}) = \mathrm{ob}(\mathcal{E}).$$

To define $\mathrm{Hom}_{Ho(\mathcal{E})}(X, Y)$, we choose, for every object X, two factorizations

$$\begin{tikzcd}
 & Q(X) \arrow[dr, twoheadrightarrow, "\sim"', "p_X"] & \\
\emptyset \arrow[ur, rightarrowtail] \arrow[rr] & & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & R(X) \arrow[dr, twoheadrightarrow] & \\
X \arrow[ur, rightarrowtail, "\sim", "i_X"'] \arrow[rr] & & *
\end{tikzcd}$$

We refer to $Q(X)$ as a cofibrant replacement of X and $R(X)$ as a fibrant replacement. We note that $RQ(X)$ is both fibrant and cofibrant:

$$\begin{tikzcd}
 & & RQ(X) \arrow[dr, twoheadrightarrow] & \\
 & Q(X) \arrow[ur, rightarrowtail, "\sim"] \arrow[rr] \arrow[dr, twoheadrightarrow, "\sim"] & & * \\
\emptyset \arrow[ur, rightarrowtail] \arrow[rr] & & X &
\end{tikzcd}$$

We then define

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}(X, Y)$$

$$= \mathrm{Hom}_{\mathcal{E}}(RQ(X), RQ(Y)) / \sim$$

to be the set of equivalence classes for the common equivalence relation of left or right homotopy.

We choose, for every map $f : X \to Y$ in $\mathcal{E}$, two liftings

$$\begin{tikzcd}
\emptyset \arrow[rr, tail] \arrow[d, tail] & & Q(Y) \arrow[d, two heads, "p_Y", "\sim"'] \\
Q(X) \arrow[r, two heads, "p_X", "\sim"'] \arrow[urr, dashed, "Q(f)"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "i_X", "\sim"'] & Y \arrow[r, tail, "i_Y", "\sim"'] & R(Y) \arrow[d, two heads] \\
R(X) \arrow[rr, two heads] \arrow[urr, dashed, "R(f)"] & & *
\end{tikzcd}$$

Any two liftings $Q(f)$ and $Q(f)'$ of $f \circ p_X$ are left homotopic, and any

two lifts $R(f)$ and $R(f)'$ of $i_Y \circ f$ are right homotopic; see [Quillen, Lemma 7].
In particular, we have

$$Q(f\circ g) \overset{\ell}{\sim} Q(f)\circ Q(g)$$

$$R(f\circ g) \overset{r}{\sim} R(f)\circ R(g)$$

where $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$ indicates left and right homotopy. Hence, there is a well-defined functor

$$\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$$

with $\gamma(X) = X$ and $\gamma(f) =$ class of $RQ(f)$.

<u>Thm</u> Let $\mathcal{C}$ be a model category. Then the functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ satisfies:

(i) For every weak equivalence f in $\mathcal{C}$, $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

(ii) If $F : \mathcal{C} \to \mathcal{A}$ is a functor such that, for every weak equivalence f in $\mathcal{C}$, $F(f)$ is an isomorphism in $\mathcal{A}$, then there exists a unique functor

$$\Theta : Ho(\mathcal{E}) \longrightarrow \mathcal{A}$$

such that $F = \Theta \circ \gamma$.

<u>Proof</u> Part (i) follows from Whitehead's theorem. To prove (ii), we define

$$\Theta : Ho(\mathcal{E}) \longrightarrow \mathcal{A}$$

as follows: $\Theta(X) = F(X)$, and if

$$\begin{array}{ccc} \alpha & \in & \operatorname{Hom}_{Ho(\mathcal{E})}(X, Y) \\ \| & & \| \\ \text{class of } f & \in & \operatorname{Hom}_{\mathcal{E}}(RQ(X), RQ(Y))/\sim \end{array}$$

then we define

$$\begin{tikzcd}
F(X) \arrow[rr, "\Theta(\alpha)"] & & F(Y) \\
F(Q(X)) \arrow[u, "\sim"', "F(p_X)"] \arrow[d, "\sim"', "F(i_{Q(X)})"] & & F(Q(Y)) \arrow[u, "\sim"', "F(p_Y)"] \arrow[d, "\sim"', "F(i_{Q(Y)})"] \\
F(RQ(X)) \arrow[rr, "F(f)"] & & F(RQ(Y))
\end{tikzcd}$$

We must show that $\theta(\alpha)$ only depends on α and not on the chosen map f. So suppose both f and g represent α. Then f and g are left homotopic:

$$\begin{tikzcd}
RQ(X) \amalg RQ(X) \arrow[rr, "f+g"] \arrow[d, "\nabla"'] \arrow[drr, hook, "d^0+d^1"] & & RQ(Y) \\
RQ(X) & & Cyl(RQ(X)) \arrow[ll, "\sigma", "\sim"'] \arrow[u, "h"']
\end{tikzcd}$$

Now,

$$F(\sigma) \circ F(d^0) = F(\sigma \circ d^0)$$

$$= F(id_{RQ(X)}) = id_{F(RQ(X))}$$

and since $F(\sigma)$ is an isomorphism, we have $F(d^0) = F(\sigma)^{-1}$. Similarly, $F(d^1) = F(\sigma)^{-1}$, so $F(d^0) = F(d^1)$. This shows that

$$\begin{array}{ccccc}
F(f) & = & F(h \circ d^0) & = & F(h) \circ F(d^0) \\
 & & & & \| \\
F(g) & = & F(h \circ d^1) & = & F(h) \circ F(d^1)
\end{array}$$

//

Rem Whitehead's theorem actually shows that the map f in $\mathcal{C}$ is a weak equivalence if and only if $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. //

Lemma Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories, and let (F, G, a) be an adjunction from $\mathcal{C}$ to $\mathcal{D}$. Then the following are equivalent:

(i) F preserves cofibrations and trivial cofibrations.

(ii) G preserves fibrations and trivial fibrations.

Pf We show (i) $\Rightarrow$ (ii). Let $p: X \to Y$ be a fibration in $\mathcal{D}$. We wish to show that $G(p): G(X) \to G(Y)$ is a fibration in $\mathcal{C}$. This is equivalent to showing that, in every diagram of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"', "\sim"] & G(X) \arrow[d, "G(p)"] \\
B \arrow[r, "g"] \arrow[ur, dashed, "h"] & G(Y)
\end{tikzcd}$$

the lifting h exists. By adjunction, this is equivalent to the statement that, in every diagram

$$\begin{tikzcd} F(A) \arrow[r, "f'"] \arrow[d, "F(i)"'] & X \arrow[d, "p"] \\ F(B) \arrow[r, "g'"] \arrow[ur, dashed, "h'"] & Y \end{tikzcd}$$

where $i : A \overset{\sim}{\rightarrowtail} B$ is a trivial cofibration, the lifting h' exists. By assumption $F(i) : F(A) \overset{\sim}{\rightarrowtail} F(B)$ is a trivial cofibration, so h' exists. //

<u>Def</u> Suppose that (F, G, α) satisfies the equivalent statements (i) and (ii) of the lemma. Then we say that that F is a <u>left Quillen functor</u>, that G is a <u>right Quillen functor</u>, and that (F, G, α) is a <u>Quillen adjunction</u>.

<u>Lemma</u> A left Quillen functor preserves weak equivalences between cofibrant objects. A right Quillen functor preserves weak equivalences

between fibrant objects.

Proof Let $f : A \xrightarrow{\sim} B$ be a weak equivalence between cofibrant objects. We factor

$$\begin{tikzcd}
 & C \arrow[dr, two heads, "q", "\sim"'] & \\
A \amalg B \arrow[ur, tail, "j"] \arrow[rr, "\cancel{f+id}"] & & B
\end{tikzcd}$$

and consider the push-out diagram

$$\begin{tikzcd}
\emptyset \arrow[r, tail] \arrow[d, tail] & A \arrow[d, tail, "i_1"] \arrow[ddr, tail, bend left, "\sim"] \arrow[dddrr, bend left=40, "f", "\sim"'] & & \\
B \arrow[r, tail, "i_2"] \arrow[drr, tail, "\sim"] \arrow[ddrrr, bend right, "id_B"', "\sim"] & A \amalg B \arrow[dr, tail, "j"] & & \\
 & & C \arrow[dr, two heads, "q", "\sim"'] & \\
 & & & B
\end{tikzcd}$$

Applying the left Quillen functor F, we get the following (push-out) diagram

$$\begin{tikzcd}
\emptyset \arrow[r, tail] \arrow[d, tail] & F(A) \arrow[d, tail, "F(i_1)"] \arrow[ddr, tail, bend left, "\sim"] \arrow[dddrr, bend left, "F(f)"] & & \\
F(B) \arrow[r, tail, "F(i_2)"] \arrow[drr, tail, "\sim"] \arrow[ddrrr, tail, bend right, "\sim", "id"'] & F(A \sqcup B) \arrow[dr, tail] & & \\
 & & F(C) \arrow[dr, "F(g)"] & \\
 & & & F(B)
\end{tikzcd}$$

We see that $F(g)$, and hence, $F(f)$ are weak equivalences as desired. //

The left Quillen functor $F : \mathcal{C} \to \mathcal{C}'$ gives rise to a functor

$$LF : Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{C}')$$

called the <u>total left derived functor</u> defined as follows. We may define a functor

$$F' : \mathcal{C} \longrightarrow Ho(\mathcal{C}')$$

by

$$\begin{array}{ccc} F'(X) & = & \gamma'(F(Q(X))) \\ \Big\downarrow F'(f) & & \Big\downarrow \gamma'(F(Q(f))) \\ F'(Y) & = & \gamma'(F(Q(Y))) \end{array}$$

It takes weak equivalences in $\mathcal{E}$ to isomorphisms in $Ho(\mathcal{E}')$, so by the universal property of γ, we get the functor LF such that

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\gamma"] \arrow[dr, "F'"] & \\ Ho(\mathcal{E}) \arrow[r, "LF"'] & Ho(\mathcal{D}) \end{tikzcd}$$

We note that the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathcal{E}' \arrow[d, "\gamma'"] \\ Ho(\mathcal{E}) \arrow[r, "LF"] & Ho(\mathcal{E}') \end{tikzcd}$$

does <u>not</u> commute. However, there

is a natural transformation

$$\varepsilon : LF \circ \gamma \longrightarrow \gamma' \circ F$$

defined by the map

$$\gamma'(F(Q(X))) \xrightarrow{\gamma'(F(p_X))} \gamma'(F(X)).$$

Moreover, $LF \circ \gamma$ is best approximation to $\gamma' \circ F$ from the left in the following sense: If $g : Ho(\mathcal{E}) \to Ho(\mathcal{E}')$ is a functor and $\xi : g \circ \gamma \to \gamma' \circ F$ a natural transformation, there exists a <u>unique</u> natural transformation $\theta : g \longrightarrow LF$ such that the diagram

$$\begin{tikzcd} g \circ \gamma \arrow[d, "\theta \circ \gamma"] \arrow[dr, "\xi"] & \\ LF \circ \gamma \arrow[r, "\varepsilon"'] & \gamma' \circ F \end{tikzcd}$$

commutes. The <u>total right derived functor</u>

$$RG : Ho(\mathcal{E}') \longrightarrow Ho(\mathcal{E})$$

of the right Quillen functor $G : \mathcal{C}' \to \mathcal{C}$ is defined similarly.

We consider the category $\mathcal{K}^{\Delta^{op}}$ of simplicial k-spaces. The geometric realization $|X[-]|$ of the simplicial k-space $X[-]$ is defined by the same formula as for simplicial sets:

$$|X[-]| := \Big(\coprod_{n \geq 0} X[n] \times \Delta[n] \Big) / \sim$$

where $\sim$ is the equivalence relation generated by the relation that identifies $(\theta^* x, z) \in X[m] \times \Delta[m]$ and $(x, \theta_* z) \in X[n] \times \Delta[n]$, for all maps $\theta : [m] \to [n]$ in Δ, $x \in X[n]$, and $z \in \Delta[m]$. However, by contrast to simplicial sets, $X[n]$ is now a k-space and not (necessarily) a discrete set. Geometric realization is a functor

$$|-| : \mathcal{K}^{\Delta^{op}} \longrightarrow \mathcal{K}.$$

It has a right adjoint functor

$$\underline{\mathrm{Sin}}(-)[-] : \mathcal{K} \longrightarrow \mathcal{K}^{\Delta^{op}}$$

defined by

$$\underline{\mathrm{Sin}}(Y)[-] = \underline{\mathrm{Hom}}_{\mathcal{X}}(\Delta[-], Y).$$

We note that $\underline{\mathrm{Sin}}(Y)[n]$ is equal to the set $\mathrm{Sin}(Y)[n]$ but equipped with the function space topology as opposed to the discrete topology. We proceed to define a model structure called the Reedy model structure on $\mathcal{X}^{\Delta^{op}}$. With this model structure, the adjoint pair of functors $(|-|, \underline{\mathrm{Sin}}, a)$ becomes a Quillen adjunction.