2008.07.03

Simplicial spaces naturally arise from bi-simplicial set. By definition, a bi-simplicial set is a functor

$$X[-,-] : \Delta^{op} \times \Delta^{op} \longrightarrow \text{Sets}$$

and a map of bi-simplicial sets is a natural transformation. We may view the bi-simplicial set $X[-,-]$ as a simplicial object in the category of simplicial sets in two ways:

$$\Delta^{op} \longrightarrow \text{Sets}^{\Delta^{op}}$$

$$[m] \longmapsto ([n] \longmapsto X[m,n])$$

$$[n] \longmapsto ([m] \longmapsto X[m,n])$$

Composing with the geometric realization functor, we obtain two simplicial spaces

$$\Delta^{op} \longrightarrow \mathcal{T}$$

$$[m] \longmapsto |X[m,-]|$$

$$[n] \longmapsto |X[-,n]|$$

The bi-simplicial set $X[-,-]$ also determines the simplicial set

$$\Delta^{op} \longrightarrow \text{Sets}$$

$$[k] \longmapsto X[k,k]$$

called the diagonal simplicial set. The following theorem is a generalization of a theorem of Eilenberg-Zilber.

<u>Thm</u> Let $X[-,-]$ be a bi-simplicial set. There are canonical and natural homeomorphisms

$$\big|[m] \mapsto |[n] \mapsto X[m,n]|\big|$$

$$\cong |[k] \mapsto X[k,k]|$$

$$\cong \big|[n] \mapsto |[m] \mapsto X[m,n]|\big| .$$

<u>Proof</u> Suppose first that

$$X[-,-] = \Delta[r][-] \times \Delta[s][-] .$$

Then we have canonical and natural homeomorphisms

$$|[m] \mapsto |[n] \mapsto \Delta[r][m] \times \Delta[s][n]||$$

$$\cong |[m] \mapsto \Delta[r][m] \times |[n] \mapsto \Delta[s][n]||$$

$$\cong |[m] \mapsto \Delta[r][m]| \times |[n] \mapsto \Delta[s][n]|$$

$$\cong |[k] \mapsto \Delta[r][k] \times \Delta[s][k]|.$$

Here the third homeomorphism is the combinatorial lemma which we proved earlier on p. 87. The first and second homeomorphism are both cases of the following: let $Y[-]$ be a simplicial k-space and Z a fixed k-space. Then

$$|Y[-]| \times Z \cong |Y[-] \times Z|.$$

Indeed, this follows from the prop. on p. 73 and the canonical homeomorphism between the adjoints:

$$\underline{\mathrm{Sin}}(\underline{\mathrm{Hom}}_{\mathcal{K}}(Z,-))[-]$$

$$\cong \underline{\mathrm{Hom}}_{\mathcal{K}}(Z, \underline{\mathrm{Sin}}(-)[-]).$$

Next, let $X[-,-]$ be a general

bi-simplicial set. Then, as in the prop. on p. 81, we have a canonical isomorphism of bi-simplicial sets

$$\operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} \Delta[r][-]\times\Delta[s][-] \xrightarrow{\cong} X[-,-].$$

It follows that we have canonical and natural isomorphisms

$$|[m]\mapsto|[n]\mapsto X[m,n]||$$

$$\xleftarrow{\cong} |[m]\mapsto|[n]\mapsto \operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} \Delta[r][m]\times\Delta[s][n]||$$

$$\xleftarrow{\cong} \operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} |[m]\mapsto|[n]\mapsto \Delta[r][m]\times\Delta[s][n]||$$

$$\cong \operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} |[k]\mapsto \Delta[r][k]\times\Delta[s][k]|$$

$$\xrightarrow{\cong} |[k]\mapsto \operatorname*{colim}_{\Delta[r][-]\times\Delta[s][-]\to X[-,-]} \Delta[r][k]\times\Delta[s][k]|$$

$$\xrightarrow{\cong} |[k]\mapsto X[k,k]|.$$

Here the first and last homeomorphisms are induced from the isomorphism of bi-simplicial set on top of p. 204. The second and fourth homeomorphisms are induced by the canonical maps and are homeomorphisms because geometric realization has a right adjoint. In the case of the fourth homeomorphism, we also use that the functor

$$\mathrm{Sets}^{\Delta^{op} \times \Delta^{op}} \xrightarrow{\Delta_*} \mathrm{Sets}^{\Delta^{op}}$$

that takes a bi-simplicial set to the associated diagonal simplicial set has a right adjoint given by the right Kan extension

$$\mathrm{Sets}^{\Delta^{op}} \xrightarrow{\Delta^!} \mathrm{Sets}^{\Delta^{op} \times \Delta^{op}}.$$

See e.g. MacLane: "Categories for the working mathematician," Chap. X. Finally, the third homeomorphism is induced by the homeomorphisms

$$|[m] \mapsto |[n] \mapsto \Delta[r][m] \times \Delta[s][n]||$$

$$\cong |[k] \mapsto \Delta[r][k] \times \Delta[s][k]|$$

which are natural with respect to maps in $\Delta \times \Delta$. This proves the first homeomorphism of the statement of the theorem. The second homeomorphism is proved analogously. //

Lemma Let $i: A[-] \to X[-]$ be a map of simplicial sets such that, for all $n \geq 0$, the map $i: A[n] \to X[n]$ is injective. Then

$$|i|: |A[-]| \longrightarrow |X[-]|$$

is a Serre cofibration.

Proof We define

$$I' = \{ \partial\Delta[n][-] \longrightarrow \Delta[n][-] \mid n \geq 0 \}$$

$$|I'| = \{ \partial\Delta[n] \longrightarrow \Delta[n] \mid n \geq 0 \}.$$

Then the geometric realization of an I'-cofibration is an $|I'|$-cofibration. Moreover, since the maps in $|I'|$ are homeomorphic to the maps in I, the class of $|I'|$-cofibrations is equal to the class of Serre cofibrations.

Let $i : A[-] \to X[-]$ be a map of simplicial sets. We claim that the following are equivalent:

(i) For all $n \geq 0$, $i : A[n] \to X[n]$ is injective.

(ii) The map $i : A[-] \to X[-]$ is a relative I'-cell complex.

This will prove the lemma. It is clear that (ii) implies (i). So we assume (i) and construct

$$A[-] = X_0[-] \xrightarrow{i_1} X_1[-] \xrightarrow{i_2} \cdots$$

together with push-out diagrams

$$\begin{CD} \coprod_\alpha \partial\Delta[n_\alpha][-] @>>> X_{n-1}[-] \\ @VVV @VV{i_n}V \\ \coprod_\alpha \Delta[n_\alpha][-] @>>> X_n[-] \end{CD}$$

and maps $j_n : X_n[-] \to X[-]$ such that $j_0 = i$, $j_n \circ i_n = j_{n-1}$, and

$$j : \operatorname{colim}_n X_n[-] \to X[-]$$

is an isomorphism. We construct $X_n[-]$ and the maps i_n and j_n by induction on $n \geq 0$. Let $X_0[-] = A[-]$ and $j_0 = i$. We assume that $X_n[-]$ has been constructed together with an injective map $j_n \colon X_n[-] \to X[-]$ such that

$$j_n \colon X_n[k] \longrightarrow X[k]$$

is a bijection, for $0 \leq k < n$. We define $S_n \subset X[n]$ to be the set of non-degenerate simplices in $X[n]$ that are not in the image of $j_n \colon X_n[n] \to X[n]$. We then define $X_{n+1}[-]$, i_{n+1}, and j_{n+1} by the push-out

$$\begin{tikzcd}
\coprod_{x \in S_n} \partial\Delta[n][-] \arrow[r, "\Sigma \sigma_x"] \arrow[d] & X_n[-] \arrow[d, "i_{n+1}"] \arrow[dd, bend left, "j_n"] \\
\coprod_{x \in S_n} \Delta[n][-] \arrow[r] \arrow[dr, bend right, "\Sigma \sigma_x"'] & X_{n+1}[-] \arrow[d, "j_{n+1}"] \\
 & X[-]
\end{tikzcd}$$

Then $j_{n+1} \colon X_{n+1}[k] \to X[k]$ is surjective, for $0 \leq k < n+1$, by construction, and injective, for all $k \geq 0$, since j_n is injective and since we have only added non-degenerate simplices not in the image of j_n. This shows (ii). //

Cor Let $X[-,-]$ be a bi-simplicial set. Then the simplicial k-spaces

$$[m] \longmapsto |[n] \longmapsto X[m,n]|$$

$$[n] \longmapsto |[m] \longmapsto X[m,n]|$$

are Reedy cofibrant.

Proof Let $Y[-] = |[q] \longmapsto X[-,q]|$. Then

$$\begin{aligned} L_n Y &= \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [p] \\ p<n}} Y[p] \\ &= \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [p] \\ p<n}} |[q] \longmapsto X[p,q]| \\ &\xrightarrow{\simeq} |[q] \longmapsto \operatorname*{colim}_{\substack{[n] \twoheadrightarrow [p] \\ p<n}} X[p,q]| \\ &= |[q] \longmapsto L_n X[-,q]| \end{aligned}$$

Moreover, the map $L_n Y \to Y[n]$ is identified with the realization of the map

$$L_n X[-,-] \longrightarrow X[n,-].$$

But the map

$$L_n X[-,q] \longrightarrow X[n,q]$$

is equal to the inclusion of the degenerate n-simplices in $X[-,q]$. In particular, it is an injective map. Hence, the lemma shows that the map $L_n Y \to Y[n]$ is a Serre cofibration as desired. //

The purpose of our long digression into basic homotopy theory was to prove the following result.

<u>Lemma</u> Let $f\colon X[-,-] \to Y[-,-]$ be a map of bi-simplicial sets and assume that, for all $m \geq 0$, the induced map

$$|[n] \mapsto X[m,n]| \longrightarrow |[n] \mapsto Y[m,n]|$$

is a weak equivalence. Then the induced map of realizations

$$|[m] \mapsto |[n] \mapsto X[m,n]|| \to |[m] \mapsto |[n] \mapsto Y[m,n]||$$

is a weak equivalence.

Proof This follows from the corollaries on pages 199 and 209. //

We will use the lemma above to prove Quillen's Theorem A. Let

$$f : \mathcal{E} \longrightarrow \mathcal{E}'$$

be a functor between small categories. The induced map of spaces

$$Bf : B\mathcal{E} \longrightarrow B\mathcal{E}'$$

is a weak equivalence, if for all $x' \in \operatorname{ob} \mathcal{E}'$, the mapping fiber $F(Bf, x')$ is contractible. We approximate the mapping fiber by the space of the under-category $x' \backslash f$ defined as follows: The objects of $x' \backslash f$ are the pairs $(x, u : x' \to f(x))$ of an object

X in $\mathcal{E}$ and a map $u : X' \to f(X)$ in $\mathcal{E}'$, and a morphism in $X' \backslash f$ from $(X, u : X' \to f(X))$ to $(Y, v : X' \to f(Y))$ is a map $w : X \to Y$ in $\mathcal{E}$ such that $f(w) \circ u = v$. The functor

$$\mathrm{pr}_1 : X' \backslash f \longrightarrow \mathcal{E}$$

defined by $\mathrm{pr}_1(X, u) = X$ induces a map of spaces

$$B\mathrm{pr}_1 : B(X' \backslash f) \longrightarrow B\mathcal{E}.$$

There is a natural transformation η from the constant functor

$$\overline{X'} : X' \backslash f \longrightarrow \mathcal{E}'$$

to the composite functor

$$f \circ \mathrm{pr}_1 : X' \backslash f \longrightarrow \mathcal{E}'$$

given by

$$\eta_{(X,u)} = u : X' \longrightarrow f(X) = (f \circ \mathrm{pr}_1)(X, u).$$

Hence, we get a homotopy

$$h : B(X'\backslash f) \times [0,1] \longrightarrow B\mathcal{E}'$$

from the constant map X' to the map $Bf \circ Bpr_1$. The map Bpr_1 and the homotopy h define a map

$$B(X'\backslash f) \longrightarrow F(Bf, X')$$

which, in certain cases, is a weak equivalence. In particular, we have:

<u>Thm</u> (Quillen's Theorem A) Let $f : \mathcal{E} \to \mathcal{E}'$ be a functor and assume that, for all $X' \in \mathrm{ob}\,\mathcal{E}'$, the space $B(X'\backslash f)$ is contractible. Then

$$Bf : B\mathcal{E} \longrightarrow B\mathcal{E}'$$

is a weak equivalence.

<u>Proof</u> If $X[-]$ is a simplicial set, we define bi-simplicial sets $XL[-,-]$ and $XR[-,-]$ by

$$\begin{tikzcd} \Delta^{op} \times \Delta^{op} \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] \arrow[rr, bend left, "{XL[-,-]}"] \arrow[rr, bend right, "{XR[-,-]}"'] & \Delta^{op} \arrow[r, "{X[-]}"] & \mathrm{Sets} \end{tikzcd}$$

We define $T(f)[-,-]$ to be the bi-simplicial set whose (m,n)-simplices are pairs of diagrams

$$(Y_m \to \cdots \to Y_0 \to f(X_0),\ X_0 \to \cdots \to X_n)$$

in $\mathcal{E}'$ and $\mathcal{E}$, respectively. The bi-simplicial structure maps (θ^*, η^*) are defined in the unique way that makes the maps

$$N(\mathcal{E}'^{op})L[-,-] \xleftarrow{p_1} T(f)[-,-] \xrightarrow{p_2} N(\mathcal{E})R[-,-]$$

given by

$$p_1(Y_m \to \cdots \to Y_0 \to f(X_0),\ X_0 \to \cdots \to X_n)$$

$$= Y_m \to \cdots \to Y_0$$

$$p_2(Y_m \to \cdots \to Y_0 \to f(X_0),\ X_0 \to \cdots \to X_n)$$

$$= X_0 \to \cdots \to X_n$$

maps of bi-simplicial sets. We now show that both p_1 and p_2 induce weak equivalences of geometric realizations. (For a bi-simplicial set

$X[-,-]$, we write $|X[-,-]|$ for the space $|[k] \mapsto X[k,k]|$.) For p_1, we realize first in the n-direction:

$$\begin{array}{ccc} |[n] \mapsto T(f)[m,n]| & \xrightarrow{p_1} & N(\mathcal{E}'^{op})[m] \\ \| & & \| \\ \coprod\limits_{Y_m \to \cdots \to Y_0} B(Y_0 \backslash f) & \longrightarrow & \coprod\limits_{Y_m \to \cdots \to Y_0} * \end{array}$$

The resulting map is a weak equivalence by our assumption that $B(Y_0 \backslash f)$ be contractible, for all $Y_0 \in \mathrm{ob}\,\mathcal{E}'$. Hence, the lemma on pages 210-211 shows that

$$|T(f)[-,-]| \underset{\sim}{\xrightarrow{|p_1|}} B(\mathcal{E}'^{op})$$

We have also used the thm. on p. 202. For p_2, we realize in the m-direction:

$$\begin{array}{ccc} |[m] \mapsto T(f)[m,n]| & \xrightarrow{p_2} & N(\mathcal{E})[n] \\ \| & & \| \\ \coprod\limits_{X_0 \to \cdots \to X_n} B(f(X_0) \backslash id_{\mathcal{E}'^{op}}) & \longrightarrow & \coprod\limits_{X_0 \to \cdots \to X_n} * \end{array}$$

The resulting map is a weak equivalence since $f(X_0)\backslash id_{\mathcal{E}'^{op}}$ has the initial object $(f(X_0), id : f(X_0) \to f(X_0))$ so $B(f(X_0)\backslash id_{\mathcal{E}'^{op}})$ is contractible. By the lemma on pages 210-211, we conclude

$$|T(f)[-,-]| \xrightarrow[\sim]{|p_2|} B(\mathcal{E}).$$

Finally, the diagram

$$\begin{tikzcd}
B(\mathcal{E}'^{op}) \arrow[d, equal] & {|T(f)[-,-]|} \arrow[l, "p_1"', "\sim"] \arrow[r, "p_2", "\sim"'] \arrow[d, "f'"] & B(\mathcal{E}) \arrow[d, "Bf"] \\
B(\mathcal{E}'^{op}) & {|T(id_{\mathcal{E}'})[-,-]|} \arrow[l, "p_1"', "\sim"] \arrow[r, "p_2", "\sim"'] & B(\mathcal{E}')
\end{tikzcd}$$

shows that Bf is a weak equivalence as stated. //

We will state but not prove the more general theorem B. A map $v : X' \to Y'$ in $\mathcal{E}'$ defines a functor

$$v^* : Y'\backslash f \longrightarrow X'\backslash f$$

that takes $(X, u : Y' \to f(X))$ to $(X, u\circ v : X' \to f(X))$. We call v^* the base-change functor associated with v.

<u>Thm</u> (Quillen's thm. B) Let $f: \mathcal{C} \to \mathcal{C}'$ be a functor, and assume that, for every map $v: X' \to Y'$ in $\mathcal{C}'$, the base-change functor $v^*: Y' \backslash f \to X' \backslash f$ induces a weak equivalence

$$Bv^* : B(Y' \backslash f) \xrightarrow{\sim} B(X' \backslash f).$$

Then, for every $X' \in \operatorname{ob} \mathcal{C}'$, the canonical map

$$B(X' \backslash f) \to F(Bf, X')$$

is a weak equivalence.

<u>Proof</u> See Quillen: Higher algebraic K-theory I, LNM 341. //