The morphisms in the simplicial index category Δ are generated by

$$d^i : [n-1] \to [n] \quad , \quad 0 \le i \le n,$$

$$s^i : [n+1] \to [n] \quad , \quad 0 \le i \le n,$$

where d^i (resp. s^i) is the unique injective (resp. surjective) non-decreasing map that omits (resp. repeats) $i \in [n]$, subject to the relations

$$d^j d^i = d^i d^{j-1} \qquad \text{if } i < j$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j \\ \cancel{\mathrm{id}} & \text{if } i = j \text{ or } i = j+1 \\ d^{i-1} s^j & \text{if } i > j+1 \end{cases}$$

$$s^j s^i = s^{i-1} s^j \qquad \text{if } i > j.$$

So a functor

$$X[-] : \Delta^{op} \to \mathcal{C}$$

amounts to objects $X[n]$, $n \geq 0$, together with maps

$$d_i : X[n] \to X[n-1] \qquad 0 \le i \le n$$

$$s_i : X[n] \to X[n+1] \qquad 0 \le i \le n$$

that satisfy the dual relations

$$d_i d_j = d_{j-1} d_i \qquad \text{if } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ id & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

Let A be an associative unital ring. We define the Hochschild complex of A to be the simplicial abelian group

$$[n] \longmapsto hh(A)[n]$$

with

$$hh(A)[n] = \underbrace{A \otimes \cdots \otimes A}_{n+1}$$

and

$$d_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, & 0 \le i < n, \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_n, & i = n, \end{cases}$$

$s_i(a_0\otimes\cdots\otimes a_n) = a_0\otimes\cdots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\cdots\otimes a_n \quad 0\le i\le n.$

There is a canonical isomorphism

$$\pi_q\,|hh(A)[-]| \approx H_q(hh(A)[-];\, d)$$

where

$$d : hh(A)[n] \to hh(A)[n-1]$$

$$d = \sum_{0\le i\le n} (-1)^i d_i .$$

This is true for every simplicial abelian group.

Ex If A is commutative

$$hh_0(A) = A$$

$$hh_1(A) \xleftarrow{\sim} \Omega^1_A .$$

class of $a_0\otimes a_1 \leftarrow\!\shortmid\ a_0 da_1$ //

The Hochschild construction is Morita invariant:

Prop Suppose that the categories

of (left) modules over the rings A and B are equivalent. Then there is a natural weak equivalence

$$|hh(A)[-]| \simeq |hh(B)[-]|.$$

Pf The categories ${}_A\mathcal{M}$ and ${}_B\mathcal{M}$ of left A-modules and left B-modules are equivalent if and only if there exists a pair of bi-modules

P A-B-bi-module

Q B-A-bi-module

and isom. of bi-modules

$P \otimes_B Q \xrightarrow{\sim} A$ A-A-bimod.

$Q \otimes_A P \xrightarrow{\sim} B$ B-B-bimod.

The category equivalences are then

${}_A\mathcal{M} \to {}_B\mathcal{M}$ ${}_B\mathcal{M} \to {}_A\mathcal{M}$

$M \mapsto Q \otimes_A M$ $N \mapsto P \otimes_B N$

Consider the bi-simplicial abelian group

$$E[m,n] = P \otimes A^{\otimes m} \otimes Q \otimes B^{\otimes n}$$

where

$$d_i : E[m,n] \to E[m-1,n] \qquad 0 \leq i \leq m$$

$$d_i' : E[m,n] \to E[m,n-1] \qquad 0 \leq i \leq n$$

are given by

$$d_i(p \otimes a_1 \otimes \cdots \otimes a_m \otimes q \otimes b_1 \otimes \cdots \otimes b_n)$$

$$= \begin{cases} pa_1 \otimes \cdots \otimes a_m \otimes q \otimes b_1 \otimes \cdots \otimes b_n & i=0 \\ p \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_m \otimes q \otimes \cdots \otimes b_n & 0<i<m \\ p \otimes \cdots \otimes a_m q \otimes b_1 \otimes \cdots \otimes b_n & i=m \end{cases}$$

$$d_i'(p \otimes a_1 \otimes \cdots \otimes a_m \otimes q \otimes b_1 \otimes \cdots \otimes b_n)$$

$$= \begin{cases} p \otimes \cdots \otimes a_m \otimes q b_1 \otimes b_2 \otimes \cdots \otimes b_n & i=0 \\ p \otimes \cdots \otimes a_m \otimes q \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n & 0<i<n \\ b_n p \otimes a_1 \otimes \cdots \otimes a_m \otimes q \otimes \cdots \otimes b_{n-1} & i=n \end{cases}$$

The homotopy groups of the realization

$$\| [m] \mapsto \| [n] \mapsto E[m,n] \| \|$$

$$\approx \| [n] \mapsto \| [m] \mapsto E[m,n] \| \|$$

are isomorphic to the homology groups of the total complex of the associated bi-complex

$$E[m,n] \xrightarrow{d} E[m-1,n]$$

$$E[m,n] \xrightarrow{d'} E[m,n-1] .$$

There are two spectral sequences for calculating this homology:

$$E^2_{s,t} = H_s(H_t(E[-,-];d);d')$$

$$\Rightarrow \pi_{s+t} | E[-,-] |$$

$$'E^2_{s,t} = H_s(H_t(E[-,-];d');d)$$

$$\Rightarrow \pi_{s+t} | E[-,-] |$$

and

$$E^2_{s,t} = \begin{cases} hh_s(B), & t=0 \\ 0, & t>0 \end{cases}$$

$$ {}'E^2_{s,t} = \begin{cases} hh_s(A) , & t=0 \\ 0 , & t>0 . \end{cases} $$

So

$$ hh_q(A) \approx \pi_q |E[-,-].| \approx hh_q(B) . \quad // $$

This suggests that we can define $hh(A)[-]$ as a functor of the category ${}_A\mathcal{M}$ of modules rather than as a functor of the ring A.

Let $\mathcal{E}$ be a category where :

$Hom_{\mathcal{E}}(c, c')$: abelian group

$Hom_{\mathcal{E}}(c', c) \times Hom_{\mathcal{E}}(c'', c')$: bilinear.

$\xrightarrow{\circ} Hom_{\mathcal{E}}(c'', c)$

Call $\mathcal{E}$ a <u>ringoid</u>. Indeed, a ringoid with one object is a ring. We define

$$ hh(\mathcal{E})[-] : \Delta^{op} \to \mathcal{A}b $$

$hh(\mathcal{C})[n]$

$$= \bigoplus_{(c_0,\dots,c_n)} \mathrm{Hom}_{\mathcal{C}}(c_0,c_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(c_n,c_{n-1})$$

$$d_i : hh(\mathcal{C})[n] \to hh(\mathcal{C})[n-1]$$

maps the summand indexed by $(c_0,\dots,c_n)$ to the summand indexed by $(c_0,\dots,\hat{c}_i,\dots,c_n)$ by the map

$$d_i(f_0 \otimes f_1 \otimes \cdots \otimes f_n)$$

$$= \begin{cases} f_0 \otimes \cdots \otimes f_i f_{i+1} \otimes \cdots \otimes f_n & 0 \leq i < n \\ f_n f_0 \otimes f_1 \otimes \cdots \otimes f_{n-1} & i = n \end{cases} ;$$

$$s_i : hh(\mathcal{C})[n] \to hh(\mathcal{C})[n+1]$$

maps the summand $(c_0,\dots,c_n)$ to the summand $(c_0,\dots,c_i,c_i,\dots,c_n)$ by

$$s_i(f_0 \otimes f_1 \otimes \cdots \otimes f_n)$$

$$= f_0 \otimes \cdots \otimes f_i \otimes id_{c_i} \otimes f_{i+1} \otimes \cdots \otimes f_n \quad 0 \leq i \leq n.$$

A linear functor $\phi : \mathcal{C} \to \mathcal{D}$ induces a map

$$\phi_* : hh(\mathcal{C})[-] \to hh(\mathcal{D})[-]$$

that takes the summand of $hh(\mathcal{C})[n]$ indexed by $(c_0, \ldots, c_n)$ to the summand of $hh(\mathcal{D})[n]$ indexed by $(\phi(c_0), \ldots, \phi(c_n))$ by the map

$$\begin{array}{c} \mathrm{Hom}_{\mathcal{C}}(c_0, c_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(c_n, c_{n-1}) \\ \downarrow \phi \otimes \cdots \otimes \phi \\ \mathrm{Hom}_{\mathcal{D}}(\phi(c_0), \phi(c_n)) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{D}}(\phi(c_n), \phi(c_{n-1})) \end{array}.$$

<u>Lemma</u> Let $\eta : \phi_0 \to \phi_1$ be a natural isomorphism. Then η induces a simplicial homotopy

$$H_\eta : hh(\mathcal{C})[-] \times N[1][-] \to hh(\mathcal{D})[-]$$

from ϕ_{0*} to ϕ_{1*}.

<u>Pf</u> We make the maps in $[n]$ go in the opposite direction today. Let

$$\alpha = (i_0 \xleftarrow{\alpha_1} i_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_n} i_n)$$

be an element of $N[i][n]$. Then $H_\eta(-,\alpha)$ takes the summand $(c_0,\dots,c_n)$ to the summand indexed by $(\phi_{i_0}(c_0),\dots,\phi_{i_n}(c_n))$ by the map

$$\mathrm{Hom}_{\mathcal{D}}(\eta^{-i_0},\eta^{i_0}) \otimes \dots \otimes \mathrm{Hom}_{\mathcal{D}}(\eta^{-i_n},\eta^{i_n}).$$

This map is compatible with the simplicial structure maps. //

Cor Let $\phi: \mathcal{C} \to \mathcal{D}$ be a linear equivalence of categories. Then

$$\phi_*: hh(\mathcal{C})[-] \to hh(\mathcal{D})[-]$$

is a homotopy equivalence. //

We say that a full subcategory $\mathcal{C} \subset \mathcal{D}$ is cofinal if, for every $d \in \mathrm{ob}\,\mathcal{D}$, there exists $c \in \mathrm{ob}\,\mathcal{C}$ together with morphisms in $\mathcal{D}$

$$d \xrightarrow{s} c \xrightarrow{r} d$$

such that $rs = \mathrm{id}_d$.

Ex The category $\mathcal{F}_A$ of f.g. free

A-modules is a cofinal subcategory of the category $\mathcal{P}_A$ of f.g. projective A-modules.

Prop Suppose $\mathcal{C} \subset \mathcal{D}$ is cofinal. Then

$$hh(\mathcal{C})[-] \xrightarrow{\sim} hh(\mathcal{D})[-].$$

Pf Similar: exercise. //

We view A as a ringoid with one object. Then there is a map of ringoids

$$A \xrightarrow{c} \mathcal{P}_A$$

unique object $\mapsto$ free module of rk. 1

Prop The induced map

$$hh(A)[-] \xrightarrow{c_*} hh(\mathcal{P}_A)[-]$$

is a weak equivalence.

Pf Let $\mathcal{F}_A^k \subset \mathcal{F}_A$ be the full subcategory of f.g. free A-modules of rank $\leq k$. Then

$$
\begin{tikzcd}
hh(A)[-] \arrow[r, "\sim", "\text{Morita}"'] \arrow[d] & \operatorname{colim}_n hh(M_n(A))[-] \arrow[d, "\sim"', "\text{cofinality}"] \\
hh(F_A)[-] \arrow[d, "\text{cofinality}"] & \operatorname{colim}_n hh(F_A^n)[-] \arrow[l] \\
hh(P_A)[-] &
\end{tikzcd}
$$

//