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We will study the space

$$|X[-]| = \Big(\coprod_{n \geq 0} X[n] \times \Delta[n]\Big) / \sim$$

in more detail. We say that $x \in X[m]$ is <u>degenerate</u> if

$$x \in \operatorname{im}(\theta^* : X[n] \to X[m])$$

for some $\theta : [m] \to [n]$ with $n < m$. If $x \in X[m]$ is not degenerate, we say that $x \in X[m]$ is <u>non-degenerate</u>. In particular, every $x \in X[0]$ is non-degenerate. Similarly, we say that a point $z \in \Delta[n]$ is <u>non-degenerate</u>, if $z$ is an interior point. The unique point $z \in \Delta[0]$ is non-degenerate. We say that $(x, z) \in X[n] \times \Delta[n]$ is a <u>non-degenerate</u> point if both $x$ and $z$ are non-degenerate. Now, the point

$$\xi \in |X[-]|$$

is an equivalence class of points $(x, z) \in X[n] \times \Delta[n]$. We will show that every such equivalence class

$\xi \in |X[-]|$ contains a unique non-degenerate point $(x, z) \in X[n] \times \Delta[n]$. We first prove two lemmas.

<u>Lemma</u> 1 Let $x \in X[m]$. Then there exists a unique map $\theta : [m] \to [n]$ and a unique non-degenerate element $y \in X[n]$ such that $\theta^*(y) = x$. Moreover, $\theta : [m] \to [n]$ is surjective.

<u>Proof</u> We first prove uniqueness. So suppose that both

$$x = \theta^*(y), \quad \theta : [m] \to [n], \quad y \in X[n]$$

$$x = \theta'^*(y') : \theta' : [m] \to [n'], \quad y' \in X[n']$$

with $y$ and $y'$ non-degenerate. We must show that $n = n'$, $\theta = \theta'$, and $y = y'$. The maps $\theta$ and $\theta'$ are necessarily surjective because $y$ and $y'$ are non-degenerate. Hence, there exist maps $\sigma : [n] \to [m]$ and $\sigma' : [n'] \to [m]$ such that $\theta \circ \sigma = id_{[n]}$ and $\theta' \circ \sigma' = id_{[n']}$. We now calculate:

$$(\theta' \circ \sigma)^*(y') = \sigma^*(\theta'^*(y')) = \sigma^*(x)$$

$$= \sigma^*(\theta^*(y)) = (\theta \circ \sigma)^*(y) = y$$

$$(\theta \circ \sigma')^*(y) = \sigma'^*(\theta^*(y)) = \sigma'^*(x)$$

$$= \sigma'^*(\theta'^*(y')) = (\theta' \circ \sigma')^*(y') = y'.$$

Since $y$ and $y'$ are both non-degenerate, we conclude that $n = n'$, that $y = y'$, and that $\theta' \circ \sigma = \theta \circ \sigma' = \mathrm{id}_{[n]}$. Finally, for every $i \in [m]$, we can find $\sigma : [n] \to [m]$ with $\theta \circ \sigma = \mathrm{id}_{[n]}$ and $(\sigma \circ \theta)(i) = i$. Then also $\theta' \circ \sigma = \mathrm{id}_{[n]}$ as we proved above. But then

$$\theta'(i) = \theta'((\sigma \circ \theta)(i))$$

$$= (\theta' \circ \sigma)(\theta(i)) = \theta(i),$$

It follows that $\theta = \theta'$.

To prove existence, we let $n \leq m$ be minimal with property that there exists $\theta : [m] \to [n]$ and

$y \in X[n]$ such that $x = \theta^*(y)$. Since $n$ is minimal, $y$ is necessarily non-degenerate. //

Lemma 2 Let $z \in \Delta[k]$. Then there exists a unique map $\psi : [m] \to [k]$ and a unique interior point $w \in \Delta[m]$ such that $\psi_*(w) = z$. Moreover, $\psi : [m] \to [k]$ is injective.

Proof We can write $z$ uniquely as

$$z = \sum_{j \in [k]} a_j \cdot j$$

where

$$a_j \in [0, 1]$$

$$\sum_{j \in [k]} a_j = 1.$$

The point $z \in \Delta[k]$ is interior if and only if $a_j \neq 0$, for all $j \in [k]$. Define $0 \leq m \leq k$ by

$$m + 1 = \#\{ j \in [k] \mid a_j \neq 0 \}$$

and let $\psi : [m] \hookrightarrow [k]$ be the unique non-decreasing map such that $j \in \psi([m])$ if and only if $a_j \neq 0$. Finally, define $w \in \Delta[m]$ by

$$w = \sum_{i \in [m]} a_{\psi(i)} \cdot i \,.$$

Then $w \in \Delta[m]$ is interior and $\psi_*(w) = z$. It is clear that $\psi$ and $w$ are unique with this property and that $\psi$ is injective. //

We now prove the promised result:

Prop Every $\xi \in |X[-1]|$ is represented by a unique non-degenerate point $(x, z) \in X[n] \times \Delta[n]$.

Proof We will define a map

$$\bar{h} : |X[-1]| \to \coprod_{n \geq 0} X[n] \times \Delta[n]$$

that is injective and whose image is the subset that consists of all non-degenerate points. The map $\bar{h}$, of course, will <u>not</u> be continuous. We first define the (discontinuous) map

$$\coprod_{n\geq 0} X[n]\times\Delta[n] \xrightarrow{g} \coprod_{n\geq 0} X[n]\times\Delta[n]$$

as follows: let $(x,z)\in X[k]\times\Delta[k]$. By Lemma 2, there exists unique $\psi:[m]\hookrightarrow[k]$ and $w\in\Delta[m]$ interior such that $\psi_*(w)=z$. We then define

$$g(x,z) = (\psi^*(x), w)$$

and note that

$$g(x,z)=(\psi^*(x),w)\sim(x,\psi_*(w)) = (x,z).$$

We next define the map

$$\coprod_{n\geq 0} X[n]\times\Delta[n] \xrightarrow{f} \coprod_{n\geq 0} X[n]\times\Delta[n]$$

as follows. Let $(a,w)\in X[m]\times\Delta[m]$. By Lemma 1, there exist unique $\theta:[m]\twoheadrightarrow[n]$ and $y\in\Delta[n]$ non-degenerate such that $\theta^*(y)=a$. We then define

$$f(a,w) = (y, \theta_*(w))$$

and note that

$$f(a,w) = (y, \theta_*(w)) \sim (\theta^*(y), w) = (a,w).$$

We also note that, since $\theta$ is surjective, if $w \in \Delta[m]$ is interior, then also $\theta_*(w) \in \Delta[n]$ is interior. Hence, the composition

$$\coprod_{n \geq 0} X[n] \times \Delta[n] \xrightarrow{h = f \circ g} \coprod_{n \geq 0} X[n] \times \Delta[n]$$

takes the point $(x,z)$ to the non-degenerate point $h(x,z)$. If $(x,z)$ was already non-degenerate, then $h(x,z) = (x,z)$. Therefore, the image of $h$ is exactly the subset of non-degenerate points. We finally prove that $h$ maps equivalent points to the same point. So let

$$\alpha : [k] \longrightarrow [k']$$

$$(x,z) \in X[k] \times \Delta[k]$$

$$(x', z') \in X[k'] \times \Delta[k']$$

with $x = \alpha^*(x')$ and $z' = \alpha_*(z)$. Then

$$(x, z) = (\alpha^*(x'), z) \sim (x', \alpha_*(z)) = (x', z')$$

and we wish to show that

$$h(x, z) = h(x', z').$$

The definition of $f$ and $g$ uses the following maps:

$$\begin{tikzcd}
{[k]} \arrow[d, "\alpha"] \arrow[r, hook, "\psi"] & {[m]} \arrow[r, two heads, "\theta"] & {[n]} \\
{[k']} \arrow[r, hook, "\psi'"] & {[m']} \arrow[r, two heads, "\theta'"] & {[n']}
\end{tikzcd}$$

We claim that

$$\mathrm{im}(\psi') = \mathrm{im}(\alpha \circ \psi).$$

Indeed, if we write

$$z = \sum_{j \in [k]} a_j \cdot j$$

then $j \in \mathrm{im}(\psi)$ if and only if

$a_j \neq 0$. Then

$$z' = \alpha_*(z) = \sum_{j' \in [k']} \Big( \sum_{j \in \alpha^{-1}(j')} a_j \Big) \cdot j'$$

and so $j' \in \operatorname{im}(\psi')$ if and only if, for some $j \in \alpha^{-1}(j')$, $a_j \neq 0$, or equivalently, if and only if, $j \in \operatorname{im}(\alpha \circ \psi)$. Hence, there exists a unique surjective map

$$\beta : [m] \twoheadrightarrow [m']$$

such that the diagram

$$\begin{tikzcd} {[k]} \arrow[d, "\alpha"] & {[m]} \arrow[l, hook', "\psi"'] \arrow[r, two heads, "\theta"] \arrow[d, "\beta"] & {[n]} \\ {[k']} & {[m']} \arrow[l, hook', "\psi'"'] \arrow[r, two heads, "\theta'"] & {[n']} \end{tikzcd}$$

commutes. Now,

$$\psi^*(x) = \theta^*(y)$$

$$\psi'^*(x') = \theta'^*(y')$$

with $x \in X[n]$ and $x' \in X[n']$ non-degenerate. But $x = \alpha^*(x')$ so

$$\psi^*(x) = \psi^*(\alpha^*(x')) = (\alpha \circ \psi)^*(x')$$

$$= (\psi' \circ \beta)^*(x') = \beta^*(\psi'^*(x'))$$

$$= \beta^*(\theta'^*(y')) = (\theta' \circ \beta)^*(y').$$

By the uniqueness statement in Lemma 1, we conclude

$$n = n', \quad y = y'.$$

$$\theta = \theta' \circ \beta : [m] \to [n].$$

Hence

$$h(x', z') = (y', \theta'_*(w'))$$

$$= (y, \theta'_*(\alpha_*(w)))$$

$$= (y, \theta_*(w))$$

$$= h(x, z).$$

It follows that $h$ factors through the canonical projection

$$\coprod_{n\geq 0} X[n]\times\Delta[n] \xrightarrow{pr} |X[-]|$$

and induces the desired map

$$\bar{h} : |X[-]| \to \coprod_{n\geq 0} X[n]\times\Delta[n].$$

Finally, since $h(x,\tau)\sim(x,\tau)$, it follows that $pr\circ\bar{h} = id_{|X[-]|}$. //

We define the $m$-<u>skeleton</u>

$$sk_m|X[-]| \subset |X[-]|$$

to be the subspace given by the image of the composition

$$\coprod_{0\leq n\leq m} X[n]\times\Delta[n]$$

$$\hookrightarrow \coprod_{n\geq 0} X[n]\times\Delta[n]$$

$$\twoheadrightarrow |X[-]|$$

of the canonical inclusion and the canonical projection. We let

$$NX[n] \subset X[n]$$

be the subset of non-degenerate elements, and let

$$NX[m] \times \Delta[m] \xrightarrow{\Phi_m} sk_m |X[-]|$$

be the composition

$$NX[m] \times \Delta[m] \hookrightarrow X[m] \times \Delta[m]$$

$$\hookrightarrow \coprod_{0 \le n \le m} X[n] \times \Delta[n]$$

$$\twoheadrightarrow sk_m |X[-]|$$

of the canonical inclusion. Finally, we say that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

is a <u>push-out</u> diagram of spaces, if $\varphi$ and $\psi$ induce a homeomorphism

$$(X \amalg B)/\sim \xrightarrow{\overline{\varphi \amalg \psi}} Y$$

from the quotient space of $X \amalg B$ by the equivalence relation generated by the relation that identifies $f(a) \in B$ and $g(a) \in X$, for all $a \in A$, onto $Y$. The prop. now shows:

<u>Thm</u> Let $X[-]$ be a simplicial set. Then

$$|X[-]| = \bigcup_{m \geq 0} sk_m |X[-]|$$

and the subset $U \subset |X[-]|$ is open if and only if

$$U \cap sk_m |X[-]| \subset sk_m |X[-]|$$

is open, for all $m \geq 0$. Moreover, the diagram

$$\begin{tikzcd}
\coprod_{x \in NX[n]} \partial\Delta[n] \arrow[r, "\varphi_m"] \arrow[d, hook, "\iota_m"] & sk_{m-1}|X[-]| \arrow[d, hook, "\iota_m"] \\
\coprod_{x \in NX[n]} \Delta[n] \arrow[r, "\varphi_m"] & sk_m |X[-]|
\end{tikzcd}$$

is a push-out, for all $m \geq 0$.

<u>Proof</u> The proposition shows that $e_m$ and $\iota_m$ induce a continuous bijection

$$\Big( \big( \coprod_{x \in NX[m]} \Delta[m] \big) \amalg \mathrm{sk}_{m-1} |X[-]| \Big) / \sim$$

$$\longrightarrow \mathrm{sk}_m |X[-]| .$$

One can show that it is a homeomorphism. See e.g. Gabriel-Zisman, Calculus of Fractions and Homotopy Theory. //

<u>Ex</u> Let $X[-] = N[1][-]$ be the nerve of the category $0 \to 1$. Then

$$N[1][0] = \{0, 1\}$$

$$N[1][1] = \{0 \to 0, 0 \to 1, 1 \to 1\}$$

$$N[1][2] = \{0 \to 0 \to 0, 0 \to 0 \to 1, 0 \to 1 \to 1, 1 \to 1 \to 1\}$$

$$\vdots$$

so there only 3 non-degenerate simplices: The two 0-simplices and the 1-simplex $0 \to 1$. So the theorem shows that

$$B[1] := |N[1][-1]|$$

$$= sk_1 |N[1][-1]|$$

since there are no non-degenerate simplices for $m \geq 2$. Now $sk_0 |N[1][-1]|$ is the discrete space with the two points

$$(0, 1) \in N[1][0] \times \Delta[0]$$

$$(1, 1) \in N[1][0] \times \Delta[0] .$$

To find $sk_1 |N[1][-1]|$, we calculate the upper horizontal map $\varphi_1$ in the push-out diagram

$$\begin{tikzcd} \partial\Delta[1] \arrow[r, "\varphi_1"] \arrow[d, hook] & sk_0 |N[1][0]| \arrow[d, hook] \\ \Delta[1] \arrow[r, "\varphi_1"] & sk_1 |N[1][0]| \end{tikzcd}$$

.

Here $\partial\Delta[1] \subset \Delta[1]$ consists of the two points $1 \cdot 0 \in \Delta[1]$ and $1 \cdot 1 \in \Delta[1]$. Let $d^0 : [0] \to [1]$ be the map $d^0(0) = 1$, and let $d^1 : [0] \to [1]$ be the map $d^1(0) = 0$. Then we have

$$d^0_*(1 \cdot 0) = 1 \cdot 1$$

$$d^1_*(1 \cdot 0) = 1 \cdot 0 .$$

On the other hand,

$$d^{0*}(0 \to 1) = 1$$

$$d^{1*}(0 \to 1) = 0 .$$

Hence,

$$\begin{aligned} \varphi_1(1 \cdot 0) &:= \text{class of } (0 \to 1, 1 \cdot 0) \\ &= \text{class of } (0 \to 1, d^1_*(1 \cdot 0)) \\ &= \text{class of } (d^{1*}(0 \to 1), 1 \cdot 0) \\ &= \text{class of } (0, 1 \cdot 0) \end{aligned}$$

and

$$\varphi_1(1 \cdot 1) := \text{class of } (0 \to 1, 1 \cdot 1)$$

$= \text{class of } (0\to 1,\ d^0_*(1\to 0))$

$= \text{class of } (d^{0*}(0\to 1),\ 1\to 0)$

$= \text{class of } (1,\ 1\to 0).$

It follows that $\varphi_1$ is a homeomorphism

$$\varphi_1 : \Delta[1] \xrightarrow{\cong} sk_1 |N[1][-1]|.$$