In algebraic topology, new spaces are often constructed by characterizing the maps they admit to or from them rather than specifying the set of elements. For example, let X and Y be two space. Classically, the product $X \times Y$ is defined to be the set of pairs

$$X \times Y = \{ (x, y) \mid x \in X,\ y \in Y \}$$

with the product topology. However, we can instead define $X \times Y$ to be the space with the following mapping properties (i) - (ii):

(i) There are two maps

$$X \xleftarrow{pr_1} X \times Y \xrightarrow{pr_2} Y.$$

(ii) For every pair of maps

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

from some space Z, there exists a <u>unique</u> map

$$Z \xrightarrow{h} X \times Y$$

such that the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & Y \end{tikzcd}$$

commutes.

The properties (i) – (iii) define the product up to unique isomorphism. Indeed, suppose that both

$$X \xleftarrow{pr_1} X \times Y \xrightarrow{pr_2} Y$$

and

$$X \xleftarrow{pr_1'} (X \times Y)' \xrightarrow{pr_2'} Y$$

satisfy (i) – (iii). Then there are unique maps h and h' such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, equal] & X \times Y \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[d, shift right, "h"'] & Y \arrow[d, equal] \\ X & (X \times Y)' \arrow[l, "pr_1'"'] \arrow[r, "pr_2'"] \arrow[u, shift right, "h'"'] & Y \end{tikzcd}$$

It follows that also the diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "pr_1"'] \arrow[d, "h' \circ h"] \arrow[dr, "pr_2"] & \\ X & X \times Y \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & Y \end{tikzcd}$$

commutes. But so does the diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "pr_1"'] \arrow[d, "id"] \arrow[dr, "pr_2"] & \\ X & X \times Y \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & Y \end{tikzcd}$$

Hence, by the uniqueness in (ii), we conclude that

$$h' \circ h = id_{X \times Y}.$$

Similarly, we find that

$$h \circ h' = id_{(X \times Y)'}$$

which shows that h and h' are isomorphisms. This situation is typical: If we define a space by characterizing the maps to or from it, we do not get a uniquely defined point-set space — we get a space defined up to unique isomorphism.

Reversing the direction of the maps in (i) and (ii), we get the definition of the coproduct:

(i') There are two maps

$$X \xrightarrow{in_1} X \amalg Y \xleftarrow{in_2} Y .$$

(ii') For every pair of maps

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

to some space Z, there exists a <u>unique</u> map

$$X \amalg Y \xrightarrow{k} Z$$

such that the diagram

$$\begin{tikzcd} X \arrow[r, "in_1"] \arrow[dr, "f"'] & X \amalg Y \arrow[d, "k"] & Y \arrow[l, "in_2"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes.

The coproduct $X \amalg Y$ of two spaces X and Y is, up to unique isomorphism, equal to the disjoint union of X and Y with the topology, where $U \subset X \amalg Y$ is open if and only if $U \cap X \subset X$ and $U \cap Y \subset Y$ are open.

The definition (i)-(ii) of the product and (i')-(ii') of the coproduct makes sense in every category. Here are some examples:

Category	Product	Coproduct
Sets	product set $X \times Y$	disjoint union $X \amalg Y$
Topological spaces	product space $X \times Y$	disjoint union $X \amalg Y$
Abelian groups	product group $A \times B$	direct sum $A \oplus B$
Groups	product group $G \times H$	free product $G * H$
Commutative rings	product ring $R \times S$	tensor product $R \otimes S$

Some categories do not have products or coproducts. For example, a group G defines a category $\underline{\underline{G}}$ with one object $*$, with

$$\mathrm{Hom}_{\underline{\underline{G}}}(*, *) = G,$$

and with composition defined by

$$g \circ g' = gg'$$

If $* \times *$ exists, then $* \times * = *$ since this is the only object. The projections are given by two elements of G, say,

$$* \xleftarrow{g_1} * \xrightarrow{g_2} *$$

Now, condition (ii) says that, for all $f_1, f_2 \in G$, there must exist a unique $h \in G$ such that

$$\begin{tikzcd} & * \arrow[dl, "f_1"'] \arrow[d, "h"] \arrow[dr, "f_2"] & \\ * & * \arrow[l, "g_1"] \arrow[r, "g_2"'] & * \end{tikzcd}$$

commutes, i.e. such that $f_1 = g_1 h$ and $f_2 = g_2 h$. But then

$$g_1^{-1} f_1 = h = g_2^{-1} f_2,$$

for all $f_1, f_2 \in G$, which holds if and only if G is the trivial group. Hence, for $G \neq 1$, the category $\underline{\underline{G}}$ does not have products (or coproducts).

Products and coproducts are examples of limits and colimits which we now define. A __diagram__ in a category $\mathcal{C}$ is a functor

$$X : I \to \mathcal{C}$$

from a small category I to $\mathcal{C}$. We call I the index category. The __limit__ of X is the object

$$\lim_{I} X$$

of $\mathcal{C}$ characterized up to unique isomorphism by the following properties (i)–(ii):

(i) For every object $i \in \operatorname{ob} I$, there is a morphism in $\mathcal{C}$

$$pr_i : \lim_{I} X \to X_i,$$

such that, for every morphism $\alpha : i \to i'$ in I, the diagram

$$\begin{tikzcd}
 & X_i \arrow[dd, "\alpha_*"] \\
\lim_{I} X \arrow[ur, "pr_i"] \arrow[dr, "pr_{i'}"'] & \\
 & X_{i'}
\end{tikzcd}$$

commutes.

(ii) Suppose that for all $i \in$ ob I, there is a morphism in $\mathcal{E}$

$$f_i : Z \longrightarrow X_i$$

from some object Z in $\mathcal{E}$ such that, for all morphisms $\alpha : i \to i'$ in I, the diagram

$$\begin{tikzcd}
 & X_i \arrow[dd, "\alpha_*"] \\
Z \arrow[ur, "f_i"] \arrow[dr, "f_{i'}"'] & \\
 & X_{i'}
\end{tikzcd}$$

commutes. Then there exists a <u>unique</u> morphism

$$h : Z \longrightarrow \lim_{I} X$$

in $\mathcal{E}$ such that, for all $i \in \mathrm{ob}\, I$,

$$f_i = h \circ pr_i . \quad //$$

The definition of the colimit is obtained by reversing the direction of all arrows above. Hence, the <u>colimit</u> of $X : I \to \mathcal{E}$ is the object

$$\mathrm{colim}_{I}\, X$$

of $\mathcal{E}$ characterized up to unique isomorphism by the following properties (i')–(ii'):

(i') For every object $i \in I$, there is a morphism in $\mathcal{E}$

$$in_i : X_i \longrightarrow \mathrm{colim}_{I}\, X$$

such that, for every morphism $\alpha : i \to i'$ in I, the diagram

$$\begin{tikzcd} X_i \arrow[d, "\alpha_*"'] \arrow[dr, "in_i"] & \\ X_{i'} \arrow[r, "in_{i'}"'] & \operatorname{colim}_I X \end{tikzcd}$$

commutes.

(ii') Suppose that, for all $i \in \mathrm{ob}\, I$, there is a morphism in $\mathcal{E}$

$$g_i : X_i \longrightarrow Z$$

to some object Z in $\mathcal{E}$ such that, for every morphism $\alpha : i \to i'$ in I, the diagram

$$\begin{tikzcd} X_i \arrow[d, "\alpha_*"'] \arrow[dr, "g_i"] & \\ X_{i'} \arrow[r, "g_{i'}"'] & Z \end{tikzcd}$$

commutes. Then there exists a unique morphism

$$k : \operatorname*{colim}_I X \longrightarrow Z$$

in $\mathcal{E}$ such that, for all $i \in \mathrm{ob}\, I$,

$$g_i = k \circ \mathrm{in}_i \ .$$ //

If $\mathcal{E}$ is the category of topological spaces and continuous maps, then the limit is the subspace

$$\lim_I X \subset \prod_{i \in \mathrm{ob}\, I} X_i$$

of all tuples $(x_i)_{i \in I}$ such that, for all morphisms $\alpha : i \to i'$ in I,

$$\alpha_*(x_i) = x_{i'} \ .$$

The map

$$\mathrm{pr}_i : \lim_I X \longrightarrow X_i$$

is the restriction of the projection

$$\mathrm{pr}_i : \prod_{i \in I} X_i \longrightarrow X_i$$

to this subspace.

The colimit of a diagram of spaces is the quotient space

$$\operatorname*{colim}_{I} X = \Big(\coprod_{i \in \mathrm{ob}\, I} X_i \Big) \Big/ \approx$$

of the disjoint union by the equivalence relation $\approx$ generated by the relation $\sim$ that, for every morphism $\alpha : i \to i'$ in I, identifies

$$\begin{array}{ccc} x_i & \in & X_i \\ \wr & & \big\downarrow \alpha_* \\ \alpha_*(x_i) & \in & X_{i'} \end{array}.$$

Hence, in general

$$\begin{array}{ccc} x_i & \in & X_i \\ \wr\wr & & \\ x_{i'} & \in & X_{i'} \end{array}$$

if and only if there exist morphisms in I

$$i = i_0 \xrightarrow{\alpha_1} j_1 \xleftarrow{\beta_1} i_1 \xrightarrow{\alpha_2} j_2 \xleftarrow{\beta_2} i_2 \rightarrow \cdots \leftarrow i_{n-1} \xrightarrow{\alpha_n} j_n \xleftarrow{\beta_n} i_n = i'$$

and elements $x_{i_s} \in X_{i_s}$ such that,

$$x_i = x_{i_0} \quad , \quad x_{i'} = x_{i_n}$$

and such that, for all $1 \leq s \leq n$,

$$\alpha_{s*}(x_{i_{s-1}}) = \beta_{s*}(x_{i_s}) . \qquad //$$

We give some examples of limits and colimits:

1) Let I be a discrete category in the sense that every morphism in I is an identity morphism. Then the functor

$$X : I \longrightarrow \mathcal{C}$$

determines and is determined by the set $\{X_i\}_{i \in I}$ of objects of $\mathcal{C}$. Moreover, the limit and colimit

$$\lim_{I} X = \prod_{i \in \mathrm{ob}\, I} X_i$$

$$\operatorname{colim}_{I} X = \coprod_{i \in \mathrm{ob}\, I} X_i$$

are equal to the product and coproduct respectively. We note that the space BI of the discrete category I is homeomorphic to the discrete space $\mathrm{ob}\, I$. Hence, the name.

2) Let G be a group, and $\underline{\underline{G}}$ the category with one object $*$ and $\mathrm{Hom}_{\underline{\underline{G}}}(*,*) = G$. Then the functor

$$\underline{\underline{X}} : \underline{\underline{G}} \longrightarrow \mathcal{E}$$

determines and is determined by the object $X = \underline{\underline{X}}(*)$ of $\mathcal{E}$ with the G-action

$$G \longrightarrow \mathrm{Aut}_{\mathcal{E}}(X)$$

$$g \longmapsto \underline{\underline{X}}(g).$$

If $\mathcal{E}$ is the category of spaces, then the limit and colimit are

$$\lim_{\underline{\underline{G}}} \underline{\underline{X}} = X^G$$

$$\operatorname*{colim}_{\underline{G}} \underline{X} = X/G$$

the spaces of fixed points and orbits of the G-space X, respectively.

3) Let I be the category

$$\{1\} \xleftarrow{\alpha_1} \emptyset \xrightarrow{\alpha_2} \{2\}.$$

Then the diagram

$$X : I \longrightarrow \mathcal{E}$$

determines and is determined by the two maps

$$X_1 \xleftarrow{\alpha_{1*}} X_\emptyset \xrightarrow{\alpha_{2*}} X_2 .$$

In this case, the colimit is called the <u>push</u>-<u>out</u>:

$$\begin{array}{ccc} X_\emptyset & \xrightarrow{\alpha_{1*}} & X_1 \\ \downarrow{\scriptstyle \alpha_{2*}} & & \downarrow{\scriptstyle in_1} \\ X_2 & \xrightarrow{in_2} & \operatorname*{colim}_{I} X \end{array}$$

The morphism

$$in_\emptyset = in_1 \circ \alpha_{1*} = in_2 \circ \alpha_{2*}$$

is not displayed. The limit of X is the object $X_\emptyset$ together with the maps α_{1*}, α_{2*}, and $id\, X_\emptyset$.

4) Let I' be the category

$$\{1\} \xrightarrow{\beta_1} \{1,2\} \xleftarrow{\beta_2} \{2\} .$$

Then the diagram

$$X : I' \longrightarrow \mathcal{E}$$

determines and is determined by the two maps

$$X_1 \xrightarrow{\beta_{1*}} X_{12} \xleftarrow{\beta_{2*}} X_2 .$$

The limit, in this case, is called the pull-back :

$$\begin{array}{ccc} \lim\limits_{I} X & \xrightarrow{pr_1} & X_1 \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle \beta_{1*}} \\ X_2 & \xrightarrow{\beta_{2*}} & X_{12} \end{array}$$

The morphism

$$pr_{12} = \beta_{1*} \circ pr_1 = \beta_{2*} \circ pr_2$$

is not displayed. The colimit is the object X_{12} together with the maps β_{1*}, β_{2*}, and $\mathrm{Id}_{X_{12}}$.

5) Let $\emptyset$ be the empty category, and let $\mathcal{E}$ be any category. Then there is a unique functor

$$X : \emptyset \longrightarrow \mathcal{E},$$

the limit

$$\lim_{\emptyset} X \in \mathrm{ob}\,\mathcal{E},$$

if it exists, is the terminal object of $\mathcal{E}$, and the colimit

$$\operatorname{colim}_{\emptyset} X \in \mathrm{ob}\,\mathcal{E},$$

if it exists, is the initial object of $\mathcal{E}$.

6) The limit (resp. colimit) of

the diagram

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

that consists of two parallel morphisms in the category $\mathcal{E}$ is called the <u>equalizer</u> (resp. <u>coequalizer</u>) of the two morphisms. It is equal to the pull-back

$$\begin{array}{ccc} \lim (X \overset{f}{\underset{g}{\rightrightarrows}} Y) & \xrightarrow{pr} & X \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle (id, f)} \\ X & \xrightarrow{(id, g)} & X \times Y \end{array}$$

(resp. to the push-out

$$\begin{array}{ccc} X \amalg Y & \xrightarrow{f + id} & Y \\ \downarrow{\scriptstyle g+id} & & \downarrow{\scriptstyle in} \\ Y & \xrightarrow{in} & \operatorname{colim} (X \overset{f}{\underset{g}{\rightrightarrows}} Y) \end{array}$$).

7) Let $\mathcal{E}$ be a category and suppose that all products and equalizers exist in $\mathcal{E}$. Then

every (small) diagram

$$X : I \to \mathcal{E}$$

has a limit

$$\lim_{I} X \in \mathrm{ob}\, \mathcal{E}.$$

It is equal to the equalizer of the two maps

$$\prod_{i \in \mathrm{ob}\, I} X_i \overset{f}{\underset{g}{\rightrightarrows}} \prod_{\alpha : i \to i'} X_{i'}$$

defined by

$$f \circ pr_{\alpha : i \to i'} = pr_{i'}$$

$$g \circ pr_{\alpha : i \to i'} = \alpha_* \circ pr_i .$$

Dually, if $\mathcal{E}$ has all coproducts and coequalizers, then every (small) diagram

$$X : I \to \mathcal{E}$$

has a colimit

$$\operatorname*{colim}_{I} X \in \mathrm{ob}\, \mathcal{E}$$

given by the coequalizer of the two maps

$$\coprod_{\alpha : i \to i'} X_i \overset{h}{\underset{k}{\rightrightarrows}} \coprod_{i \in \mathrm{ob}\, I} X_i$$

defined by

$$h \circ \mathrm{in}_{\alpha : i \to i'} = \mathrm{in}_i$$

$$k \circ \mathrm{in}_{\alpha : i \to i'} = \mathrm{in}_{i'} \circ \alpha_*$$

The proof is an exercise in the definitions. See MacLane: Categories for the working mathematician, GTM 5, Springer, p. 109 for the solutions.

We return to colimits of diagrams of sets, which are given by the same formula as for topological spaces except that one forgets the topology. In general, the relation $\sim$ is not an equivalence relation and the equivalence relation $\approx$ it generates can be difficult to understand. There is one case, however, where $\approx$ is easy to understand. We say that I is <u>filtered</u> if

(a) $I \neq \emptyset$

(b) For every pair of objects i and j in I, there exists two morphisms

$$\begin{tikzcd} i \arrow[dr] & \\ & k \\ j \arrow[ur] & \end{tikzcd}$$

to a common object in I.

(c) For every pair of parallel morphisms $i \overset{\alpha}{\underset{\beta}{\rightrightarrows}} j$ in I, there exists a morphism $\gamma : j \to k$ such that $\gamma \circ \alpha = \gamma \circ \beta$.

We prove the following result:

Lemma Let I be a filtered small category and

$$X : I \to \text{Sets}$$

a diagram of sets. Then:

(i) The colimit is the quotient

$$\operatorname*{colim}_I X = \Big(\coprod_{i \in \mathrm{ob}\, I} X_i\Big) \Big/ \equiv$$

by the equivalence relation that identifies $x_i \in X_i$ and $x_j \in X_j$ if there exists $\alpha : i \to k$ and $\beta : j \to k$ such that $\alpha_*(x_i) = \beta_*(x_j)$.

(ii) Any two elements $x_i \in X_i$ and $x_j \in X_j$ are equivalent under $\equiv$ to two elements of a common set X_k.

(iii) Two elements $x_i, x_i' \in X_i$ are equivalent if and only if there exists $\alpha : i \to j$ such that $\alpha_*(x_i) = \alpha_*(x_i')$.

Pf (i) To see that $\equiv$ is an equivalence relation we use

$$\begin{tikzcd}
 & & p & & \\
 & & n \arrow[u] & & \\
 & l \arrow[ur] & & m \arrow[ul] & \\
i \arrow[ur] & & j \arrow[ul] \arrow[ur] & & k \arrow[ul]
\end{tikzcd}$$

.

(iii) By definition, $x_i, x_i' \in X_i$ are equivalent if there exists $\alpha : i \to j$ and $\beta : i \to j$ such that $\alpha_*(x_i) = \beta_*(x_i')$. Now choose $\gamma : j \to k$ such that $\gamma \circ \alpha = \gamma \circ \beta$. Then $(\gamma \circ \alpha)_*(x_i) = (\gamma \circ \beta)(x_i')$. //

We say that J is finite if it has only finitely many objects and morphism. The following result is very useful:

Prop Let I be a filtered small category and J a finite category, and let

$$X : I \times J \longrightarrow \mathrm{Sets}$$

be a diagram of sets. Then the canonical map

$$\operatorname*{colim}_I \lim_J X_{i,j} \longrightarrow \lim_J \operatorname*{colim}_I X_{i,j}$$

is a bijection.

Pf The limit over J is equal to an iterated pull-back. So it suffices to show

$$\operatorname*{colim}_I (X_i \times_{Z_i} Y_i) \xrightarrow{\sim} \operatorname*{colim}_I X_i \times_{\operatorname*{colim}_I Z_i} \operatorname*{colim}_I Y_i$$

We define the inverse map. Let $(\bar{x}, \bar{y})$ be an element of the right-hand set. Then $\bar{x}$ is represented by $x_i \in X_i$, $\bar{y}$ is represented by $y_j \in Y_j$ and $\alpha_{i*}(x_i) \in Z_i$ is equivalent to $\beta_{j*}(y_j) \in Z_j$. We first choose

$$\begin{tikzcd} i \arrow[dr, "u"] & \\ & k \\ j \arrow[ur, "v"'] & \end{tikzcd}$$

Then $x_k = u_*(x_i) \in X_k$ also represents $\bar{x}$ and $y_k = v_*(y_j) \in Y_k$ also represents $\bar{y}$. Moreover,

$$\alpha_{k*}(x_k) \equiv \beta_{k*}(y_k)$$

in Z_k. Therefore, we can choose $w: k \to m$ such that

$$(w \circ \alpha_k)_*(x_k) = (w \circ \beta_k)(y_k).$$

Let $x_m = w_*(x_k)$ and $y_m = w_*(y_k)$. Then

$$\alpha_{m*}(x_m) = \beta_{m*}(y_m)$$

so (x_m, y_m) represents an element

$$\xi \in \operatorname{colim}_I (X_i \times_{Z_i} Y_i).$$

This defines a map

$$\operatorname{colim}_I X_i \times_{\operatorname{colim}_I Z_i} \operatorname{colim}_I Y_i \to \operatorname{colim}_I (X_i \times_{Z_i} Y_i)$$

which is the inverse of the canonical map. //

Cor Let U : Groups $\to$ Sets be the forgetful functor, and let $X : I \to$ Groups be a diagram of groups indexed by a filtered category I. Then the canonical map

$$\operatorname*{colim}_I UX \to U \operatorname*{colim}_I X$$

is a bijection.

Pf We define a group structure on the set $\operatorname{colim}_I UX$ as follows: For every $i \in I$, the multiplication in the group X_i defines a map of sets

$$\mu_i : UX_i \times UX_i \to UX_i$$

and these maps constitute a natural transformation of functors from I to Sets. Hence, we get an induced map of colimits

$$\operatorname*{colim}_I (UX_i \times UX_i) \xrightarrow{\mu_*} \operatorname*{colim}_I UX_i .$$

We also have the canonical map

$$\operatorname*{colim}_I (UX_i \times UX_i) \longrightarrow \operatorname*{colim}_I UX_i \times \operatorname*{colim}_I UX_i$$

which is a bijection by the proposition. Hence, the composition

$$\operatorname*{colim}_I UX_i \times \operatorname*{colim}_I UX_i \xleftarrow{\cong} \operatorname*{colim}_I (UX_i \times UX_i)$$

$$\xrightarrow{\mu_*} \operatorname*{colim}_I UX_i$$

defines a multiplication on the set $\operatorname{colim}_I UX_i$. It is easy to see that this gives $\operatorname{colim}_I UX_i$ a group structure and that, with this group structure, $\operatorname{colim}_I UX_i$ is the colimit of the diagram of groups X. //

Rem The requirement that I be filtered is necessary. Let, for example, I be the discrete category with two objects 1 and 2. Then

$$\begin{array}{ccc} \operatorname*{colim}_I UX & \longrightarrow & U \operatorname*{colim}_I X \\ \| & & \| \\ UX_1 \amalg UX_2 & \neq & U(X_1 * X_2) \end{array}$$

Rem The analogous statements hold with the category of groups replaced by the category of abelian groups, rings, commutative rings, monoids, modules, etc. The proofs are similar.