

The geometric realization $|X[-]|$ of the simplicial set may also be characterized by the set of maps it admits from it. We first introduce the following abstract notion which is due to Dan Kan.

Def An adjunction from the category $\mathcal{C}$ to the category $\mathcal{D}$ is a triple (F, G, a) that consists of two functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ and a natural bijection

$$a = a_{(X,Y)} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

The functor F is said to be left adjoint of the functor G, and the functor G is said to be right adjoint of the functor F. //

We emphasize that the bijection a must be a natural transformation between the two functors

$$(X, Y) \longmapsto \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$$

$$(X, Y) \longmapsto \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

from $\mathcal{C}^{op} \times \mathcal{D}$ to Sets. This means that, for every morphism $\varphi : X \to X'$ in $\mathcal{C}$ and $\psi : Y \to Y'$ in $\mathcal{D}$, the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F(X'), Y) & \xrightarrow{a(X',Y)} & \mathrm{Hom}_{\mathcal{C}}(X', G(Y)) \\
\downarrow F(\varphi)^* & & \downarrow \varphi^* \\
\mathrm{Hom}_{\mathcal{D}}(F(X), Y) & \xrightarrow{a(X,Y)} & \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \\
\downarrow \psi_* & & \downarrow G(\psi)_* \\
\mathrm{Hom}_{\mathcal{D}}(F(X), Y') & \xrightarrow{a(X,Y')} & \mathrm{Hom}_{\mathcal{C}}(X, G(Y'))
\end{array}$$

commutes. We define the <u>unit</u> of the adjunction (F, G, a) to be the natural transformation

$$\eta_X : X \to G(F(X))$$

defined by

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) & \xrightarrow{a(X,F(X))} & \mathrm{Hom}_{\mathcal{C}}(X, G(F(X))) \\
\cup & & \cup \\
\mathrm{id}_{F(X)} & \longmapsto & \eta_X
\end{array}$$

Now, if $f : F(X) \to Y$ is any morphism in $\mathcal{D}$, the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) & \xrightarrow{a_{(X,F(X))}} & \mathrm{Hom}_{\mathcal{C}}(X, G(F(X))) \\ \downarrow f_* & & \downarrow G(f)_* \\ \mathrm{Hom}_{\mathcal{D}}(F(X), Y) & \xrightarrow{a_{(X,Y)}} & \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \end{array}$$

shows that

$$a_{(X,Y)}(f) = G(f) \circ \eta_X .$$

We conclude that the natural transformation $a = a_{(X,Y)}$ determines and is determined by the natural transformation $\eta = \eta_X$. Dually, the <u>counit</u> of (F, G, a) is the natural transformation

$$\varepsilon_Y : F(G(Y)) \longrightarrow Y$$

defined by

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F(G(Y)), Y) & \xleftarrow{a^{-1}_{(G(Y),Y)}} & \mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y)) \\ \in & & \in \\ \varepsilon_Y & \longleftarrow\!\shortmid & \mathrm{id}_{G(Y)} \end{array}$$

and, if $g : X \to G(Y)$ is any morphism in $\mathcal{C}$, the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F(G(Y)), Y) & \xleftarrow{a^{-1}_{(G(Y), Y)}} & \mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y)) \\
\downarrow F(g)^* & & \downarrow g^* \\
\mathrm{Hom}_{\mathcal{D}}(F(X), Y) & \xleftarrow{a^{-1}_{(X,Y)}} & \mathrm{Hom}_{\mathcal{C}}(X, G(Y))
\end{array}$$

shows that

$$a^{-1}_{(X,Y)}(g) = \varepsilon_Y \circ F(g).$$

Hence, the natural transformation $a^{-1} = a^{-1}_{(X,Y)}$ determines and is determined by the natural transformation ε_Y. Moreover, the statement that $a_{(X,Y)}$ and $a^{-1}_{(X,Y)}$ are each others inverse maps is equivalent to the statement that the two compositions

$$G(Y) \xrightarrow{\eta_{G(Y)}} G(F(G(Y))) \xrightarrow{G(\varepsilon_Y)} G(Y)$$

$$F(X) \xrightarrow{F(\eta_X)} F(G(F(X))) \xrightarrow{\varepsilon_{F(X)}} F(X)$$

are equal to the identity morphisms of $G(Y)$ and $F(X)$, respectively.

Examples 1) Let $\mathcal{C}$ be the category of sets and maps, and let $\mathcal{D}$ be the category of abelian groups and homomorphisms. Then there is an adjunction (F, G, a), where $F(X)$ is the free abelian group generated by the set X, and $G(Y)$ is the underlying set of the abelian group Y. The map

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{a(X,Y)} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

takes the group homomorphism $f: F(X) \to Y$ to the set map $a(f): X \to G(Y)$ defined by

$$a(f)(x) = f(x).$$

It is a bijection because the group homomorphism f is uniquely determined by its value on the basis X of $F(X)$.

2) Let $f: A \to B$ be a ring homomorphism, let $\mathcal{C}$ be the category of left A-modules and A-linear maps, and let $\mathcal{D}$ be the cate-

gory of left B-modules and B-linear maps. Then there is an adjunction (F, G, a) from $\mathcal{C}$ to $\mathcal{D}$, where $F(X)$ is the left B-module $B \otimes_A X$, where $G(Y)$ is Y cosidered as a left A-module by $a \cdot y = f(a)y$, and where

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{a(X,Y)} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

takes the B-linear map $f \colon B \otimes_A X \to Y$ to the A-linear map $a(f) \colon X \to G(Y)$ defined by

$$a(f)(x) = f(1 \otimes x).$$

It is a bijection because f is uniquely determined by $a(f)$ by the formula

$$f(b \otimes x) = b f(1 \otimes x) = b\, a(f)(x).$$

We also define $H \colon \mathcal{C} \to \mathcal{D}$ by

$$H(Z) = \mathrm{Hom}_A(B, Z)$$

where the left B-module structure

is defined by

$$(bf)(b') = f(b'b).$$

Then there is an adjunction (G, H, a') from $\mathcal{D}$ to $\mathcal{C}$, where

$$\mathrm{Hom}_{\mathcal{C}}(G(Y), Z) \xrightarrow{a'_{(Y,Z)}} \mathrm{Hom}_{\mathcal{D}}(Y, H(Z))$$

takes the A-linear map $g: G(Y) \to Z$ to the B-linear map (check this!) $a'(g): Y \to \mathrm{Hom}_A(B, Z)$ defined by

$$a'(g)(y)(b) = g(by).$$

We leave it as an exercise to check that $a'_{(X,Y)}$ is well-defined and a bijection.

3) Let $\mathcal{C}$ be a category, let I be a small (index) category, and let $\mathcal{C}^I$ be the category of all functors $X: I \to \mathcal{C}$ with the natural transformations as morphisms. We define the diagonal functor

$$\Delta : \mathcal{C} \to \mathcal{C}^I$$

by $\Delta(X)(i) = X$ and $\Delta(f)(\alpha) = id_X$.
Suppose that the colimit of every I-diagram $\mathcal{X}: I \to \mathcal{E}$ exists. Then we obtain an adjunction

$$(\operatorname{colim}_I, \Delta, a)$$

from $\mathcal{E}^I$ to $\mathcal{E}$, where

$$\operatorname{Hom}_{\mathcal{E}}(\operatorname{colim}_I \mathcal{X}, Y) \xrightarrow{a(\mathcal{X}, Y)} \operatorname{Hom}_{\mathcal{E}^I}(\mathcal{X}, \Delta(Y))$$

takes $k: \operatorname{colim}_I \mathcal{X} \to Y$ to the natural transformation $a(k): \mathcal{X} \to \Delta(Y)$ that at $i \in \operatorname{ob} I$ is defined by the composite morphism

$$a(k)_i : \mathcal{X}_i \xrightarrow{in_i} \operatorname{colim}_I \mathcal{X} \xrightarrow{k} Y.$$

It is a bijection by the definition of the colimit. In fact, we can reformulate the definition of the colimit by saying that colim_I is the left adjoint functor of the diagonal functor Δ. The morphisms

$$in_i : \mathcal{X}_i \to \operatorname{colim}_I \mathcal{X}$$

form the unit of the adjunction

$$\eta_{\mathcal{X}} : \mathcal{X} \to \Delta(\operatorname*{colim}_I \mathcal{X}).$$

Similarly, the limit $\lim_I$ is the right adjoint of the diagonal Δ. //

The following result is very useful.

Prop Suppose that in the diagram of categories and functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] \arrow[d, shift left, "H"] & \mathcal{B} \arrow[l, shift left, "F'"] \arrow[d, shift left, "G"] \\ \mathcal{C} \arrow[r, shift left, "K"] \arrow[u, shift left, "H'"] & \mathcal{D} \arrow[l, shift left, "K'"] \arrow[u, shift left, "G'"] \end{tikzcd}$$

the functor F (resp. G, resp. H, resp. K) is left adjoint to the functor F' (resp. G', resp. H', resp. K').

(i) A natural transformation

$$\varphi_A : (G \circ F)(A) \to (K \circ H)(A)$$

determines and is determined by

a natural transformation

$$\varphi'_D : (F' \circ G')(D) \to (H' \circ K')(D).$$

(ii) The natural transformation

$$\varphi_A : (G \circ F)(A) \to (K \circ H)(A)$$

is an isomorphism if and only if the corresponding natural transformation

$$\varphi'_D : (F' \circ G')(D) \to (H' \circ K')(D)$$

is an isomorphism.

<u>Proof</u> (i) Let $\varphi_A : G(F(A)) \to K(H(A))$ be a natural transformation of functors from $\mathcal{A}$ to $\mathcal{D}$. The diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(G(F(A)), D) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{A}}(A, F'(G'(D))) \\
\uparrow \varphi_A^* & & \uparrow \Phi'_{(A,D)} \\
\mathrm{Hom}_{\mathcal{D}}(K(H(A)), D) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{A}}(A, H'(K'(D)))
\end{array}$$

where the horizontal maps are the

natural bijections which are part of the given adjunctions, defines a natural transformation $\Phi'_{(A,D)}$ of functors from $\mathcal{A}^{op} \times \mathcal{D}$ to Sets. The natural transformation $\Phi'_{(A,D)}$ determines the natural transformation

$$\varphi'_D : H'(K'(D)) \to F'(G'(D))$$

defined by

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(H'(K'(D)), F'(G'(D))) & \ni & \varphi'_D \\ \uparrow \Phi'_{(H'(K'(D)),D)} & & \uparrow \\ \mathrm{Hom}_{\mathcal{A}}(H'(K'(D)), H'(K'(D))) & \ni & \mathrm{id}_{H'(K'(D))} \end{array}$$

Conversely, φ'_D determines $\Phi'_{(A,D)}$ by

$$\Phi'_{(A,D)}(f) = \varphi'_D \circ f .$$

The natural transformation $\Phi'_{(A,D)}$, in turn, determines the natural transformation φ^*_A which determines the natural transformation φ_A.

(ii) Exercise. //

Let I be a small (index) category. A functor $F: \mathcal{E} \to \mathcal{D}$ induces a functor $F^I: \mathcal{E}^I \to \mathcal{D}^I$ defined by

$$F^I(\mathcal{X})_i = F(\mathcal{X}_i).$$

If (F, G, a) is an adjunction from $\mathcal{E}$ to $\mathcal{D}$, then (F^I, G^I, a^I) is an adjunction from $\mathcal{E}^I$ to $\mathcal{D}^I$, where

$$\mathrm{Hom}_{\mathcal{D}^I}(F^I(\mathcal{X}), \mathcal{X}') \xrightarrow{a^I_{(\mathcal{X},\mathcal{X}')}} \mathrm{Hom}_{\mathcal{E}^I}(\mathcal{X}, G(\mathcal{X}'))$$

takes the natural transformation $f: F^I(\mathcal{X}) \to \mathcal{X}'$ to the natural transformation $a(f): \mathcal{X} \to G^I(\mathcal{X}')$ defined by

$$a(f)_i = a(f_i): \mathcal{X}_i \to G(\mathcal{X}'_i).$$

As an application of the proposition above we have:

<u>Cor</u> Let (F, G, a) be an adjunction from the category $\mathcal{E}$ to the category $\mathcal{D}$, let I be a small category.

(i) Suppose that all I-diagrams in $\mathcal{E}$ and $\mathcal{D}$ have a colimit. Then, for every I-diagram $\mathcal{X} : I \to \mathcal{C}$ in $\mathcal{C}$, the canonical map

$$\operatorname*{colim}_I F^I(\mathcal{X}) \longrightarrow F(\operatorname*{colim}_I \mathcal{X})$$

is an isomorphism.

(ii) Suppose all I-diagrams in $\mathcal{C}$ and $\mathcal{D}$ have a limit. Then, for every I-diagram $\mathcal{X}' : I \to \mathcal{D}$ in $\mathcal{D}$, the canonical map

$$G(\lim_I \mathcal{X}') \longrightarrow \lim_I G^I(\mathcal{X}')$$

is an isomorphism.

<u>Proof</u> (i) We apply the proposition to:

$$\begin{tikzcd}
\mathcal{C}^I \arrow[rr, shift left, "\operatorname{colim}_I"] \arrow[d, shift left, "F^I"] & & \mathcal{C} \arrow[ll, shift left, "\Delta"] \arrow[d, shift left, "F"] \\
\mathcal{D}^I \arrow[rr, shift left, "\operatorname{colim}_I"] \arrow[u, shift left, "G^I"] & & \mathcal{D} \arrow[ll, shift left, "\Delta"] \arrow[u, shift left, "G"]
\end{tikzcd}$$

where $\Delta \circ G = G^I \circ \Delta$.

(ii) Similar. $\quad /\!/$