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The notion of homotopy equivalence of two spaces is too fine: It is too difficult for two spaces to be homotopy equivalent. The better notion is that of weak equivalence.

Def A map $f: X \to Y$ of topological spaces is a weak equivalence if, for all $x \in X$ and $n \geq 0$, the map

$$f_*: \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a bijection. //

Rem The long-exact sequence

$$\cdots \to \pi_n(F(f,y),(x,\xi)) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,y) \to \cdots$$

shows that f is a weak equivalence if and only if, for all $x \in X$ and $n \geq 0$,

$$\pi_n(F(f,y),(x,\xi)) = 0.$$

We note that, in particular, $F(f,y)$ must be non-empty, for all $y \in Y$. //

The following definition was given by Serre in his thesis. Let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Def A map $p : X \to Y$ of topological spaces is a Serre fibration if, for every $n \geq 0$ and every commutative diagram

$$\begin{tikzcd} D^n \arrow[r, "g"] \arrow[d, "\iota_0"'] & X \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r, "h"] & Y \end{tikzcd}$$

there exists a map

$$\tilde{h} : D^n \times [0,1] \longrightarrow X$$

such that $g = \tilde{h} \circ \iota_0$ and $h = p \circ \tilde{h}$. //

Rem Since S^n, D^n, $S^n \times [0,1]$, and $D^n \times [0,1]$ are all compact Hausdorff spaces, the map $f : X \to Y$ is a weak equivalence (resp. Serre fibration) if and only if the induced map $f : k(X) \to k(Y)$ of the associated k-spaces is a weak equivalence (resp. Serre fibration). //

<u>Lemma</u> Let $p : X \to Y$ be a Serre fibration. Then, for every $y \in Y$, the map

$$f : p^{-1}(y) \longrightarrow F(p, y)$$

defined by $f(x) = (x, \bar{y})$ is a weak equivalence.

<u>Proof</u> We must show that, for every $x \in p^{-1}(y)$ and $n \geq 0$, the map

$$f_* : \pi_n(p^{-1}(y), x) \longrightarrow \pi_n(F(p, y), (x, \bar{y}))$$

is a bijection. We first prove that is surjective. By adjunction, a map

$$\omega : (S^n, \infty) \longrightarrow (F(p, y), (x, \bar{y}))$$

is equivalent to a commutative diagram of maps

$$\begin{tikzcd} S^n \times \{1\} \cup \{\infty\} \times [0,1] \arrow[r, "\omega'"] \arrow[d, hook, "\iota"] & X \arrow[d, "p"] \\ S^n \times [0,1] \arrow[r, "\omega''"] & Y \end{tikzcd}$$

with $\omega'(\infty, s) = x$ and $\omega''(\infty, s) = y$, for all $s \in [0,1]$, and $\omega''(z, 0) = y$, for all $z \in S^n$. We claim there exists a map

$$\tilde{\omega} : S^n \times [0,1] \longrightarrow X$$

such that $\tilde{\omega} \circ \iota = \omega'$ and $p \circ \tilde{\omega} = \omega''$. It follows, by adjunction, that $\tilde{\omega}$ gives a (base-point preserving) homotopy from $f \circ \eta$ to ω with $\eta : (S^n, \infty) \to (p^{-1}(y), x)$ defined by $\eta(z) = \tilde{\omega}(z, 0)$. To prove the claim, we first let

$$\varkappa : D^n \longrightarrow S^n$$

be the map $\varkappa(x) = \tan\left(\frac{\pi}{2}\|x\|\right) \cdot x$. Then the diagram

$$\begin{tikzcd}
D^n \times \{1\} \cup \partial D^n \times [0,1] \arrow[r, "\varkappa \times id"] \arrow[d] & S^n \times \{1\} \cup \{\infty\} \times [0,1] \arrow[d] \\
D^n \times [0,1] \arrow[r, "\varkappa \times id"] & S^n \times [0,1]
\end{tikzcd}$$

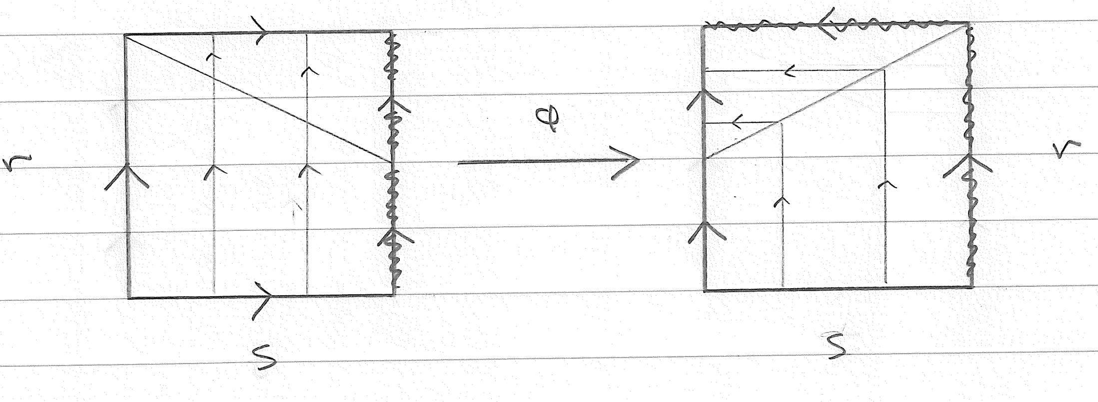
is a push-out. Hence, it will suffice to find a map $\tilde{k} : D^n \times [0,1] \to X$ that makes the following diagram commute:

$$\begin{tikzcd}
D^n \times \{1\} \cup \partial D^n \times [0,1] \arrow[r, "\omega' \circ (\pi \times id)"] \arrow[d] & X \arrow[d, "\Phi"] \\
D^n \times [0,1] \arrow[r, "\omega'' \circ (\pi \times id)"] \arrow[ur, dashed, "\tilde{k}"] & Y
\end{tikzcd}$$

We choose a homeomorphism

$$\varphi : [0,1] \times [0,1] \longrightarrow [0,1] \times [0,1]$$

as indicated by the picture



We also define the quotient map

$$c : \partial D^n \times [0,1] \times [0,1] \longrightarrow D^n \times [0,1]$$

by $c(z, r, s) = (rz, s)$. Then the unique map Φ that makes the diagram

$$\begin{tikzcd}
\partial D^n \times [0,1] \times [0,1] \arrow[r, "\mathrm{id} \times \varphi"] \arrow[d, "c"] & \partial D^n \times [0,1] \times [0,1] \arrow[d, "c"] \\
D^n \times [0,1] \arrow[r, "\Phi"] & D^n \times [0,1]
\end{tikzcd}$$

is a homeomorphism, and the following diagram commutes

$$\begin{tikzcd}
D^n \times \{1\} \arrow[r, "\Phi"] \arrow[d, hook] & D^n \times \{1\} \cup \partial D^n \times [0,1] \arrow[d, hook] \\
D^n \times [0,1] \arrow[r, "\Phi"] & D^n \times [0,1]
\end{tikzcd}$$

Hence, the problem at hand is reduced to finding $\tilde{h} : D^n \times [0,1] \to X$ that makes the following diagram commute:

$$\begin{tikzcd}
D^n \times \{1\} \arrow[r, "w' \circ (r \times \mathrm{id}) \circ \Phi"] \arrow[d, hook] & X \arrow[d, "p"] \\
D^n \times [0,1] \arrow[r, "w'' \circ (r \times \mathrm{id}) \circ \Phi"'] \arrow[ur, dashed, "\tilde{h}"] & Y
\end{tikzcd}$$

But this map $\tilde{h}$ exists by the definition of a Serre fibration. Hence, f_* is onto.

To prove that f_* is injective, let

$$\eta, \eta' : (S^n, \infty) \to (p^{-1}(y), x)$$

be two maps, and let

$$h : (S^n, \infty) \wedge ([0,1]_+, +) \to (F(p, y), (x, \bar{y}))$$

be a homotopy from $f \circ \eta$ to $f \circ \eta'$. We must show that there exists a homotopy

$$k : (S^n, \infty) \wedge ([0,1]_+, +) \to (p^{-1}(y), x)$$

from η to η'. By adjunction, the homotopy h is equivalent to a commutative diagram

$$\begin{tikzcd} S^n \times \{1\} \times [0,1] \cup \{\infty\} \times [0,1] \times [0,1] \arrow[r, "h'"] \arrow[d, "i"] & X \arrow[d, "p"] \\ S^n \times [0,1] \times [0,1] \arrow[r, "h''"] & Y \end{tikzcd}$$

where $h'(\infty, s, t) = x$ and $h''(\infty, s, t) = y$, for all $s, t \in [0,1]$, $h''(z, s, 0) = h''(z, s, 1) = y$, for all $z \in S^n$ and $s \in [0,1]$, and $h''(z,0,t) = y$, for all $t \in [0,1]$. We see as before that, since p is a Serre fibration, there exists

$$\tilde{h} : S^n \times [0,1] \times [0,1] \longrightarrow X$$

such that $h' = \tilde{h} \circ \iota$ and $h'' = p \circ \tilde{h}$. Then $k(z,t) = \tilde{h}(z,0,t)$ is the desired homotopy. //

Cor Let $p : X \to Y$ be a Serre fibration, let $x \in X$, and let $y = p(x)$. Then there is a long-exact sequence

$$\cdots \to \pi_{n+1}(Y,y) \xrightarrow{\delta'} \pi_n(p^{-1}(y),x) \xrightarrow{j} \pi_n(X,x) \xrightarrow{p_*} \pi_n(Y,y) \to \cdots$$

where $j : p^{-1}(y) \to X$ is the inclusion.

Proof We have

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(Y,y) \arrow[r, "\delta"] & \pi_n(F(p,y),(x,\bar{y})) \arrow[r, "i"] & \pi_n(X,x) \arrow[r, "p_*"] & \pi_n(Y,y) \arrow[r] & \cdots \\
& & \pi_n(p^{-1}(y),x) \arrow[u, "\sim"', "f_*"] & & &
\end{tikzcd}$$

and $j = i \circ f$. We define $\delta' = f_*^{-1} \circ \delta$. //

We next define a class of maps in $\mathcal{K}$ called the Serre cofibrations. Let

$$I = \{ \partial D^n \rightarrowtail D^n \mid n \geq 0 \}$$

be the set of the standard inclusions of the boundary ∂D^n of D^n in D^n. We note that $\partial D^0 = \emptyset$. We refer to I as the set of generating cofibrations.

Def (i) The map $f : A \to B$ is a <u>relative</u> <u>I-cell complex</u> if there exists a sequence of maps

$$A = B_0 \xrightarrow{i_0} B_1 \xrightarrow{i_1} B_2 \xrightarrow{i_2} \cdots$$

together with push-out diagrams

$$\begin{tikzcd}
\coprod_\alpha U_\alpha \arrow[r, "\Sigma \varphi_\alpha"] \arrow[d, "\coprod g_\alpha"] & B_{m-1} \arrow[d, "i_{m-1}"] \\
\coprod_\alpha V_\alpha \arrow[r, "\Sigma \psi_\alpha"] & B_m
\end{tikzcd}$$

with $g_\alpha : U_\alpha \to V_\alpha$ in I, and maps

$$j_m : B_m \longrightarrow B$$

such that $j_0 = f$, $j_m \circ i_{m-1} = j_{m-1}$, and such that the induced map

$$j : \operatorname*{colim}_m B_m \longrightarrow B$$

is a homeomorphism.

(ii) The map $f : A \to B$ is a <u>Serre cofibration</u> (or I-cofibration) if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}_A"] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & B' \arrow[r] & B \end{tikzcd}$$

where $f' : A' \to B'$ is a relative I-cell complex. //

We refer to the commutative diagram in (ii) by saying that the map $f : A \to B$ is a <u>retract</u> of the map $f' : A' \to B'$.

A space X is an I-cell complex if the map $\emptyset \to X$ is a relative I-cell complex. In particular, the geometric realization $|X[-]|$ of a simplicial set $X[-]$ is an I-cell complex.

The classes of weak equivalences, Serre fibrations, and Serre cofibrations satisfy the axioms of a Quillen model structure which we now define.

Def (Quillen) A model category is a category $\mathcal{E}$ together with three classes of maps called the weak equivalences ($\xrightarrow{\sim}$), fibrations ($\twoheadrightarrow$), and cofibrations ($\rightarrowtail$) that satisfy the following axioms:

M1: All small limits and colimits exist in $\mathcal{E}$.

M2: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps in $\mathcal{E}$. If two of the maps f, g, $g \circ f$ are weak equivalences, then so is the third.

M3: If f and g are maps in $\mathcal{E}$ such that f is a retract of g, and if g is a weak equivalence, a fibration, or a cofibration, then so is f.

M4: Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

where i is a cofibration, p is a fibration, and one of i and p is a weak equivalence, there exists a map

$$h : B \longrightarrow X$$

such that $f = h \circ i$ and $g = p \circ h$.

M5: Every map f can be factored

$$f = p \circ i = q \circ j$$

where p is a fibration and i a cofibration and a weak equivalence, and where q is a fibration and a weak equivalence and j is a cofibration. //

<u>Thm</u> (Quillen) The category $\mathcal{K}$ with the classes of weak equivalences, Serre fibrations, and Serre cofibrations is a model category.

<u>Proof</u> This is not so easy to prove. We refer to Hovey: "Model categories" for the proof. //

Let $i: A \to B$ and $p: X \to Y$ be maps in a category $\mathcal{E}$. We say that i has the <u>left lifting property</u> (LLP) with respect to p and that p has the <u>right lifting property</u> (RLP) with respect to i if, for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y \end{tikzcd}.$$

there exists a map

$$h : B \longrightarrow X$$

such that $f = h \circ i$ and $g = p \circ h$. If $\mathcal{E}$ is a model category, we say that the map $i : A \to B$ is a <u>trivial cofibration</u> ($\overset{\sim}{\rightarrowtail}$) if i is both a cofibration and a weak equivalence. We say that $p : X \to Y$ is a <u>trivial fibration</u> ($\overset{\sim}{\twoheadrightarrow}$) if p is both a fibration and a weak equivalence.

<u>Prop</u> Let $\mathcal{E}$ be a model category.

(i) The map $i : A \to B$ is a cofibration (resp. trivial cofibration) if and only if it has LLP with respect to every map p that is a trivial fibration (resp. fibration).

(ii) The map $p : X \to Y$ is a fibration (resp. trivial fibration) if and only if it has RLP with respect to every map i that is a trivial cofibration (resp. cofibration).

Proof We prove the first part of (i) and leave the second part of (ii) along with (ii) as an exercise. If $i: A \to B$ is a cofibration, then, by M4, it has LLP with respect to all maps p that are trivial fibrations. To prove the converse, we assume $i: A \to B$ has LLP with respect to trivial fibrations. By M5, we can factor i as

$$i = q \circ j$$

where $j: A \rightarrowtail X$ is a cofibration and $q: X \overset{\sim}{\twoheadrightarrow} B$ a trivial fibration. By our assumption on i, there exists a map $h: B \to X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, tail, "j"] \arrow[d, "i"'] & X \arrow[d, two heads, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

We may rewrite this diagram in the following way:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i'"] & A \arrow[r, equal] \arrow[d, tail, "j"] & A \arrow[d, "i'"] \\ B \arrow[r, "h"] \arrow[rr, bend right, "\mathrm{id}"'] & X \arrow[r, "q"] & B \end{tikzcd}$$

This shows that i' is a retract of the cofibration j, and hence, by M3, i' is a cofibration. //

It follows that, in a model category, the two of the three classes of weak equivalences, fibrations, and cofibrations determine the third. Given a push-out (resp. pull-back) diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "h"] & Y' \end{tikzcd},$$

we say that f' is the <u>cobase-change</u> of f along g (resp. f is the <u>base-change</u> of f' along h).

Exercise Use the proposition to show that the cobase-change of a cofibration (resp. trivial cofibration) is a cofibration (resp. trivial cofibration) and that the base-change of a fibration (resp. trivial fibration) is a fibration (resp. trivial fibration). //

Def Let $\mathcal{E}$ be a model category. A cylinder object for $A \in \mathrm{ob}\,\mathcal{E}$ is a diagram of the form

$$\begin{tikzcd} & \mathrm{Cyl}(A) \arrow[dr, "\sigma", "\sim"'] & \\ A \amalg A \arrow[ur, "d^0+d^1", tail] \arrow[rr, "\nabla"] & & A \end{tikzcd}.$$

A path object for $B \in \mathrm{ob}\,\mathcal{C}$ is a diagram

$$\begin{tikzcd} & \mathrm{Path}(B) \arrow[dl, "{(d_0,d_1)}"', two heads] & \\ B \times B & & B \arrow[ul, "s"', "\sim"] \arrow[ll, "\Delta"'] \end{tikzcd}.$$

A left homotopy from $f: A \to B$ to $g: A \to B$ is a diagram

$$\begin{tikzcd} & \mathrm{Cyl}(A) \arrow[dr, "h"] & \\ A \amalg A \arrow[ur, tail, "d^0+d^1"] \arrow[rr, "f+g"] & & B \end{tikzcd}$$

and a <u>right homotopy</u> from f to g is a diagram

$$\begin{tikzcd} & \mathrm{Path}(B) \arrow[dl, two heads, "(d_0,d_1)"'] & \\ B \times B & & A \arrow[ul, "k"'] \arrow[ll, "(f,g)"'] \end{tikzcd}$$

//

<u>Example</u> If $\mathcal{E} = \mathcal{K}$, then

$$\begin{tikzcd} & \mathrm{Path}(B) = \underline{\mathrm{Hom}}_{\mathcal{K}}([0,1], B) \arrow[dl, "(ev_0, ev_1)"'] & \\ B \times B & & B \arrow[ul, "\sim", "s"'] \arrow[ll, "\Delta"'] \end{tikzcd}$$

with $s(b) = \bar{b}$ is a path object for B. Hence right homotopy is the usual notion of homotopy. However, in general,

$$\begin{tikzcd} & A \times [0,1] \arrow[dr, "\sigma = pr_1"] & \\ A \amalg A \arrow[ur, "i_0+i_1"] \arrow[rr, "\nabla"] & & A \end{tikzcd}$$

is not a cylinder object, because the map $i_0 + i_1$ may fail to be a Serre cofibration. //

<u>Def</u> Let $\mathcal{C}$ be a model category, and let $\emptyset$ and $*$ denote the initial and terminal objects. The object A is called <u>cofibrant</u> if $\emptyset \rightarrowtail A$ is a cofibration. The object B is called <u>fibrant</u> if $B \twoheadrightarrow *$ is a fibration.

<u>Example</u> In $\mathcal{C} = \mathcal{K}$, every object is fibrant and the cofibrant objects are the retracts of I-cell complexes. //

<u>Prop</u> Let $\mathcal{C}$ be a model category, let A be a cofibrant object, and let B be a fibrant object. Then both left homotopy and right homotopy are equivalence relations on the set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ and these two equivalence relations are equal.

<u>Proof</u> See Hovey, Prop. 1.2.5. //

<u>Thm</u> (Whitehead) Let $\mathcal{E}$ be a model category, and let $f : X \to Y$ be a map between objects that are both fibrant and cofibrant. Then f is a weak equivalence if and only if f is a homotopy equivalence.

<u>Proof</u> Suppose first that f is a trivial fibration. Since Y is cofibrant,

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "f", "\sim"'] \\
Y \arrow[r, equal] \arrow[ur, dashed, "g"] & Y
\end{tikzcd}$$

show that there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$. To show that $g \circ f$ is left homotopic to id_X, let

$$\begin{tikzcd}
 & \mathrm{Cyl}(X) \arrow[dr, "\sigma", "\sim"'] & \\
X \amalg X \arrow[ur, tail, "d^0 + d^1"] \arrow[rr, "\nabla"] & & X
\end{tikzcd}$$

be a cylinder object for X. Then the diagram

$$\begin{tikzcd}
X \amalg X \arrow[r, "g \circ f + id"] \arrow[d, rightarrowtail, "d^0 + d^1"'] & X \arrow[d, twoheadrightarrow, "f", "\sim"'] \\
Cyl(X) \arrow[r, "f \circ s"] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

gives the desired left homotopy. The dual argument shows that trivial cofibrations, too, are homotopy equivalences. Finally, let $f \colon X \xrightarrow{\sim} Y$ be a weak equivalence. We use M5 to factor f as

$$\begin{tikzcd}
& Z \arrow[dr, twoheadrightarrow, "p", "\sim"'] & \\
X \arrow[ur, rightarrowtail, "i"] \arrow[rr, "f", "\sim"'] & & Y
\end{tikzcd}$$

By M2, also i is a weak equivalence. Moreover, Z is cofibrant, since X is cofibrant and i a cofibration, and Z is fibrant, since Y is fibrant and p a fibration. Hence, i and p are both homotopy equivalences, and therefore, f is a homotopy equivalence.

We refer to Hovey, Prop. 1.2.8 for the proof that every homotopy equivalence is a weak equivalence. //

Ex If $\mathcal{C} = \mathcal{X}$, the theorem states that a map $f : X \to Y$ between two retracts of I-cell complexes is a homotopy equivalence if and only if it is a weak equivalence. //