Properly discontinuous isometric group actions on pseudo-Riemannian manifolds
(擬リーマン多様体への固有不連続かつ等長的な群作用について)

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Contents

1. Introduction

We study the finiteness of the groups acting isometrically and properly discontinuously on pseudo-Riemannian manifolds. We explain the background of our research. Let us start with reminding ourselves what is called the *existence problem of a compact Clifford–Klein form.* Let G be a Lie group, and H a closed subgroup of G. If the left action of a discrete subgroup Γ of G on the homogeneous space G/H is properly discontinuous (namely, for any compact subset $K \subset G/H$, only finitely many elements γ of Γ satisfy $\gamma(K) \cap K \neq \emptyset$), the quotient space $\Gamma \backslash G/H$ is called a *Clifford–Klein form.*

Problem 1 (Existence problem of a compact Clifford–Klein form). *Does a homogeneous space G/H have compact Clifford–Klein forms?*

Whether the answer to this problem is affirmative depends on the choice of the pair (G, H). For instance, let G be a semisimple Lie group, and H a maximal compact subgroup of G. Then the homogeneous space G/H carries the structure of a Riemannian symmetric space. Borel–Harish-Chandra [**6**], Mostow–Tamagawa [**21**], and Borel [**5**] proved that *the Riemannian symmetric space G/H always admits compact Clifford–Klein forms.*

Non-Riemannian symmetric spaces do not necessarily admit a compact Clifford–Klein form. Let q and n be positive integers with $q \leq n$. Throughout this dissertation, we use the notation $O_q(n)$ for the indefinite orthonormal group $O(n-q, q)$, namely, the Lie group of linear transformations of $\mathbb{R}^n$ preserving a non-degenerate symmetric bilinear form of index q. The homogeneous space $O_q(n+1)/O_q(n)$ is called a *pseudo-sphere.* Note that the pseudo-sphere $O_q(n+1)/O_q(n)$ has an $O_q(n+1)$-invariant geodesically complete pseudo-Riemannian metric of positive constant curvature. The following theorem implies non-existence of a compact Clifford–Klein form for the pseudo-sphere with $n \geq 2q$ since the pseudo-sphere is non-compact:

Theorem 1.1 (Calabi–Markus [**7**](the case $q = 1$), Wolf [**26**]). *If $n \geq 2q$, there exists no infinite subgroup of $O_q(n+1)$ whose restricted left action on $O_q(n+1)/O_q(n)$ is properly discontinuous.*

Although the group $O_q(n+1)$ is endowed with co-compact lattices, none of them acts properly discontinuously on the pseudo-sphere $O_q(n+1)/O_q(n)$ via the left action. Kulkarni [**20**] and Kobayashi [**15**], [**16**] generalized Theorem 1.1 to homogeneous spaces. Kobayashi [**15**] particularly gave an extension of Theorem 1.1 in the reductive case.

The pseudo-sphere is a homogeneous space, and at the same time, is a geodesically complete pseudo-Riemannian manifold of positive constant curvature. In the paper [**17**], [**18**], Kobayashi asked if one could understand Theorem 1.1 in a geometric way. Kobayashi [**18**] proposed the following conjecture in concrete terms:

Conjecture (Kobayashi [**18**]). *Let n and q be positive integers with $n \geq 2q$. Assume that M is an n-dimensional geodesically complete pseudo-Riemannian manifold of index q. Suppose that we have a positive lower bound on the sectional curvature of M. Then,*

(i) *M is never compact;*
(ii) *if $n \geq 3$, the fundamental group of M is always finite.*

We should remark that the conjecture holds. In the case $\dim(M) = 2$, the Gauss–Bonnet formula for pseudo-Riemannian manifolds (see Avez [**1**], Chern [**9**]) leads us to non-existence of compact quotients. The following theorem implies that, if a pseudo-Riemannian manifold M of dimension $n \geq 3$ satisfies the assumptions of the conjecture, M has constant curvature:

THEOREM 1.2 (Kulkarni [**19**]). *Let M be a pseudo-Riemannian manifold with the metric indefinite of dimension $n \geq 3$. If the sectional curvature of M is either bounded from above or below, then M is of constant curvature.*

Note that, in the case $\dim(M) \geq 3$, the universal covering space of a geodesically complete pseudo-Riemannian manifold of positive constant curvature is isometric to a pseudo-sphere. By using Theorem 1.1, the proof is complete in the case $\dim(M) \geq 3$.

Unexpectedly we can solve Kobayashi's conjecture. This is because we assume the strong curvature condition in the conjecture. Since the conjecture is analogous to the Myers theorem in Riemannian geometry, the aim of the conjecture must be to understand the topology of pseudo-Riemannian manifolds of variable curvature. We should probably consider the following problem instead of the conjecture:

PROBLEM 2 (cf. Kobayashi [**17**, Problem 2.2.6]). *Let M be a non-compact geodesically complete pseudo-Riemannian manifold. Find a curvature condition of M which is satisfied even if we perturb the metric of M, and which leads us to the finiteness of the fundamental group of M.*

In this dissertation, we provide partial answers to Problem 2.

We investigate the properly discontinuous isometric action on a pseudo-Riemannian manifold that is not necessarily homogeneous. Here we give the definition of the Calabi–Markus phenomenon for pseudo-Riemannian manifolds. We say that the *Calabi–Markus phenomenon occurs* in a pseudo-Riemannian manifold if no groups but finite ones can act isometrically, effectively, and properly discontinuously on it.

On the Lorentzian case, we obtain a general extension of Theorem 1.1. García-Río and Kupeli [**11**, Definition 2.6] defined the following class of Lorentzian manifolds:

DEFINITION 1.1. Let F be a manifold, and $\{g_t\}_{t\in\mathbb{R}}$ a smooth family of Riemannian metrics of F. A *parametrized Lorentzian product* is a product manifold $\mathbb{R} \times F$ with a Lorentzian metric written as $-dt^2 + g_t$, where t is the parameter of $\mathbb{R}$.

Let us introduce our result.

THEOREM 1.3 (Mukuno [**22**]). *Let F be a closed manifold, and $\{g_t\}_{t\in\mathbb{R}}$ a smooth family of Riemannian metrics on F. Assume that there exist positive constants t_0 and c such that a parametrized Lorentzian product $(\mathbb{R} \times F, -dt^2 + g_t)$ satisfies the following condition $(\mathrm{H})_{t_0,c}$:*

$$\frac{t}{|t|} \frac{\partial g_t(X, X)}{\partial t} \geq 2c g_t(X, X)$$

for any vector field X on F and any $t \in \mathbb{R}$ with $|t| \geq t_0$. Then the Calabi–Markus phenomenon occurs in $(\mathbb{R} \times F, -dt^2 + g_t)$.

Note that the condition $(\mathrm{H})_{t_0,c}$ implies that $g_t(X, X)$ grows exponentially with $|t|$. We check that Theorem 1.3 is a generalization of Theorem 1.1. We call the pseudo-sphere $O_1(n+2)/O_1(n+1)$ the *de Sitter space*. Let $\mathbb{S}^n$ be the n-dimensional sphere, and $g_{\mathbb{S}^n}$ the standard metric of the sphere. Realized as the Lorentzian warped product $(\mathbb{R}\times\mathbb{S}^n, -dt^2+\cosh^2 t\, g_{\mathbb{S}^n})$, the de Sitter space $O_1(n+2)/O_1(n+1)$ fulfills the assumption of Theorem 1.3. Therefore the Calabi–Markus phenomenon occurs in the de Sitter space. We should remark that a parametrized Lorentzian product satisfies the condition $(\mathrm{H})_{t_0,c}$ under the assumptions of some curvature condition and the existence of a totally geodesic hypersurface (see the proof of Lemma 4.7 and Remark 4.2). Then the Calabi–Markus phenomenon occurs. We have given a partial solution to Problem 2 on parametrized Lorentzian products.

The proof of Theorem 1.3 is based upon the following observation: The de Sitter space is realized as the hypersurface $\{x = (x_0, x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+2} \mid -x_0^2 + x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$ in the Minkowski space. Note that a geodesic in the de Sitter space is the intersection of a 2-dimensional linear subspace and the de Sitter space. Theorem 1.1 follows from the fact that any spacelike geodesic in the de Sitter space meets the closed spacelike hypersurface $\{0\} \times \mathbb{S}^n$. This fact is the key point of the proof of Theorem 1.3. We should remark that, in the proof of Theorem 1.3, the condition $(\mathrm{H})_{t_0,c}$ enables us to control the behavior of a spacelike geodesic.

As an application of Theorem 1.3, we construct an inhomogeneous Lorentzian manifold with the non-compact isometry group in which the Calabi–Markus phenomenon occurs. This example leads us to that Theorem 1.3 is a new non-trivial extension of Theorem 1.1.

A natural question is whether we extend Theorem 1.3 to pseudo-Riemannian manifolds. We have observed the Calabi–Markus phenomenon for a certain class of pseudo-Riemannian warped products.

Suppose that (B, g_B) and (F, g_F) are Riemannian manifolds. Let ω be a smooth positive function on B. We denote by $(-B) \times_\omega F$ a *pseudo-Riemannian warped product* $(B\times F, g^\omega = -g_B + \omega^2 g_F)$. For example, the pseudo-sphere $O_q(n+1)/O_q(n)$ is realized as the warped product $(-\mathbb{H}^q) \times_{\cosh(r)} \mathbb{S}^{n-q}$, where $\mathbb{S}^{n-q}$ is an $(n-q)$-dimensional sphere with the standard metric, $\mathbb{H}^q$ is a q-dimensional hyperbolic space, and r is the distance function from a point in $\mathbb{H}^q$. We give the following generalization of Theorem 1.1 to pseudo-Riemannian warped products:

THEOREM 1.4. *Let (F, g_F) be a closed Riemannian manifold, and (B, g_B) a complete Riemannian manifold with* $\dim(F) \geq \dim(B)$. *Suppose that there exists a positive smooth strictly convex function ω on B with a minimum point b_0. Assume that $\exp_{b_0} : T_{b_0}B \to B$ is a diffeomorphism. Then $(-B) \times_\omega F$ is geodesically complete, and the Calabi-Markus phenomenon occurs in $(-B) \times_\omega F$.*

Note that the strict convexity of ω relates to some condition of the sectional curvature of $(-B) \times_\omega F$. The details are found in Section 3.1.

A Lorentzian manifold (M, g) is *globally hyperbolic* if M contains a Cauchy hypersurface, i.e. a subset S of M such that every inextendible timelike curve intersects S at exactly one point. We investigate the fundamental group of globally hyperbolic Lorentzian manifolds with some curvature condition. Bernal–Sánchez [**4**] proved that any globally hyperbolic Lorentzian manifold is diffeomorphic to the product manifold of the real line and a Cauchy hypersurface. To study the fundamental group of a globally hyperbolic Lorentzian manifold is to study one of its

Cauchy hypersurface. A Lorentzian manifold $(M,\, g)$ is *lightlike geodesically complete* if any inextendible lightlike geodesic is defined on the real line. For a point $p \in M$ and a timelike vector $T \in T_pM$, we denote by $L_p(T)$ the set of lightlike tangent vectors v of T_pM with $\langle v,\, T\rangle = -1$. We obtain the following result:

Theorem 1.5. *Let $(M,\, g)$ be a lightlike geodesically complete globally hyperbolic Lorentzian manifold of dimension $n > 2$. Assume that there exist a positive constant Q, a point $p \in M$, and a timelike tangent vector $T \in T_pM$ satisfying that, for any $v \in L_p(T)$ and any $s \in \mathbb{R}$, the Ricci tensor $\mathrm{Ric}(d\exp(sv)/ds,\, d\exp(sv)/ds)$ has a positive lower bound $(n-2)Q^2$. Then a Cauchy hypersurface of M is compact, and its fundamental group is finite.*

The Lorentzian product manifold $(\mathbb{R} \times \mathbb{S}^n,\, -dt^2 + g_{\mathbb{S}^n})$ of the real line $(\mathbb{R},\, dt^2)$ and the n-dimensional sphere $(\mathbb{S}^n,\, g_{\mathbb{S}^n})$ satisfies the assumptions of Theorem 1.5. However the de Sitter space does not fulfill the requirements of Theorem 1.5 since the Ricci tensor $\mathrm{Ric}(v,\, v) = 0$ for any lightlike tangent vector v. Therefore Theorem 1.5 is not a generalization of the result of Calabi–Markus [**7**]. We ask if we can obtain a generalization of Theorem 1.5 whose assumption the de Sitter space satisfies.

Outline of the paper. In the second section, we review the preceding results related to Theorem 1.1. In the third section, we investigate some properties of pseudo-Riemannian warped products, and prove Theorem 1.4. In the fourth section, we show Theorem 1.3. Moreover we give an inhomogeneous Lorentzian manifold where the Calabi–Markus phenomenon occurs. In the fifth section, we introduce the fundamental results on globally hyperbolic Lorentzian manifolds, and prove Theorem 1.5. In the sixth section, we pose two questions for our further research.

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2. Preceding Results on the Calabi–Markus phenomenon

In this section, we recall the preceding results, namely, the theorems of Calabi–Markus [**7**], Wolf [**26**], Kulkarni [**20**], and Kobayashi [**15**]. Moreover we write the original proofs of their results.

2.1. Constant Curvature Case. Let q and n be positive integers with $q \leq n$. For any $x = (x_1,\, x_2,\, \cdots,\, x_{n+1}),\ y = (y_1,\, y_2,\, \cdots,\, y_{n+1}) \in \mathbb{R}^{n+1}$, a symmetric bilinear form B_q is defined by

$$B_q(x,\, y) = -x_1y_1 - x_2y_2 - \cdots - x_qy_q + x_{q+1}y_{q+1} + \cdots + x_{n+1}y_{n+1}.$$

The quadric $\{x \in \mathbb{R}^{n+1} | B_q(x,\, x) = 1\}$ is denoted by $\mathbb{S}^n_q$. The quadric $\mathbb{S}^n_q$ carries the structure of a pseudo-Riemannian manifold of index q induced by B_q. The pseudo-Riemannian manifold $\mathbb{S}^n_q$ is the pseudo-sphere. The indefinite orthogonal group $O_q(n+1)$ is the group consisting of the linear transformations of $\mathbb{R}^{n+1}$ which preserve the bilinear form B_q. The group $O_q(n+1)$ acts on $\mathbb{S}^n_q$ as matrices isometrically and transitively. In fact the isometry group of $\mathbb{S}^n_q$ is equal to $O_q(n+1)$. We see that $\mathbb{S}^n_q$ is realized as the homogeneous space $O_q(n+1)/O_q(n)$. Note that the pseudo-sphere $\mathbb{S}^n_q$ is diffeomorphic to $\mathbb{R}^q \times \mathbb{S}^{n-q}$.

Calabi–Markus [**7**] and Wolf [**26**] proved the following theorem:

THEOREM 1.1 (Calabi–Markus [**7**, Theorem 1](the case $q = 1$), Wolf [**26**, Theorem 1]). *Assume that $2q \leq n$. Let Γ be a subgroup of the isometry group $O_q(n+1)$ of $\mathbb{S}^n_q$. If the isometric action of Γ on $\mathbb{S}^n_q$ is properly discontinuous, Γ is finite.*

PROOF. Let Π be a linear subspace of codimension q in $\mathbb{R}^{n+1}$ such that the restricted bilinear form $B_q|_{\Pi\times\Pi}$ is positive definite. We write $\mathbb{S}^{n-q}_0$ for the intersection $\Pi \cap \mathbb{S}^n_q$. Note that $\mathbb{S}^{n-q}_0$ is isometric to the $(n-q)$-dimensional sphere. For any isometry ϕ of $\mathbb{S}^n_q$, let us show that $\phi(\mathbb{S}^{n-q}_0) \cap \mathbb{S}^{n-q}_0 \neq \emptyset$. Here we can regard ϕ as an element of $O_q(n+1)$. Then we have

$$\begin{aligned}\dim(\Pi \cap \phi(\Pi)) &= \dim \Pi + \dim \phi(\Pi) - \dim(\Pi + \phi(\Pi)) \\ &\geq 2(n-q+1) - (n+1) = n - 2q + 1 > 0.\end{aligned}$$

It follows that the subspace $\Pi \cap \phi(\Pi)$ is not empty. Since $\Pi \cap \phi(\Pi)$ intersects $\mathbb{S}^n_q$, we obtain $\Pi \cap \phi(\Pi) \cap \mathbb{S}^n_q = \mathbb{S}^{n-q}_0 \cap \phi(\mathbb{S}^{n-q}_0) \neq \emptyset$.

Let Γ be a discrete subgroup of the isometry group of $\mathbb{S}^n_q$ acting on $\mathbb{S}^n_q$ properly discontinuously. It follows that $\{\gamma \in \Gamma | \mathbb{S}^{n-q}_0 \cap \gamma(\mathbb{S}^{n-q}_0) \neq \emptyset\} = \Gamma$. The proper-discontinuity of the action of Γ on $\mathbb{S}^n_q$ implies the finiteness of the subgroup Γ. □

Kulkarni [**20**] proved the converse of Theorem 1.1.

THEOREM 2.1 (Kulkarni [**20**, Theorem 5.7]). *Assume that $2q > n$. Then there exists an infinite subgroup Γ of the isometry group $O_q(n+1)$ of $\mathbb{S}^n_q$ such that Γ acts on $\mathbb{S}^n_q$ properly discontinuously.*

Before beginning with the proof of Theorem 2.1, we prove the following lemma:

LEMMA 2.1. *Let Γ be a subgroup of $O_q(n+1)$. Assume that $\mathbb{R}^{n+1}$ decomposes into a direct sum of Γ-invariant proper linear subspaces V_1 and V_2. Suppose that each vector of V_1 is perpendicular to each vector of V_2 with respect to B_q, namely, that $B_q(v_1, v_2) = 0$ for any $v_1 \in V_1$ and any $v_2 \in V_2$. Set $S_i = V_i \cap \mathbb{S}^n_q$ for $i = 1, 2$. Then S_i is Γ-invariant. Moreover, Γ acts on $\mathbb{S}^n_q$ properly discontinuously if and only if Γ acts on S_i properly discontinuously for $i = 1, 2$.*

PROOF OF LEMMA 2.1. Since V_i is Γ-invariant, so is S_i.

Assume that Γ acts on $\mathbb{S}^n_q$ properly discontinuously. Take any compact subset C_i of S_i for $i = 1, 2$. It is clear that each C_i is also a compact subset in $\mathbb{S}^n_q$. The proper-discontinuity of the action of Γ on $\mathbb{S}^n_q$ implies that $\{\gamma \in \Gamma | C_i \cap \gamma(C_i) \neq \emptyset\}$ is finite. Therefore Γ acts on S_i properly discontinuously for $i = 1, 2$.

Suppose that Γ acts on S_i properly discontinuously for $i = 1, 2$. Take any compact subset K of $\mathbb{S}^n_q$. Let π_i be the orthonormal projection of $\mathbb{R}^{n+1}$ onto V_i. The subset K_i is given by $K_i = \{v \in K | B_q(\pi_i(v), \pi_i(v)) \geq 1/2\}$ for $i = 1, 2$. We should remark that

$$B_q(v, v) = B_q(\pi_1(v), \pi_1(v)) + B_q(\pi_2(v), \pi_2(v)) = 1,$$

for $v \in \mathbb{S}^n_q$. Then $K = K_1 \cup K_2$. Put $\zeta_\Gamma(K) = \{\gamma \in \Gamma | \gamma(K) \cap K \neq \emptyset\}$. By the decomposition of $\mathbb{R}^{n+1}$, we have

$$v = \pi_1(v) + \pi_2(v) \tag{1}$$

for any $v \in \mathbb{R}^{n+1}$. Replacing v by γv in the equation (1), we have $\gamma v = \pi_1(\gamma v) + \pi_2(\gamma v)$ for any $\gamma \in \Gamma$. When an element γ of Γ acts on the equation (1), we see that $\gamma v = \gamma\pi_1(v) + \gamma\pi_2(v)$. It follows that $\pi_1(\gamma v) + \pi_2(\gamma v) = \gamma\pi_1(v) + \gamma\pi_2(v)$. For $i = 1, 2$, as V_i is a Γ-invariant, the relation $\pi_i \circ \gamma = \gamma \circ \pi_i$ holds. Here we take any $\gamma \in \zeta_\Gamma(K)$. Since $K = K_1 \cup K_2$, for some i_0 there exists a point $x \in K_{i_0}$ with $\gamma x \in K$. The relation $\pi_{i_0} \circ \gamma = \gamma \circ \pi_{i_0}$ implies that $\gamma x \in K_{i_0}$. It follows that $\zeta_\Gamma(K) = \zeta_\Gamma(K_1) \cup \zeta_\Gamma(K_2)$. Let L_i be the image of K_i under the projection π_i for $i = 1, 2$.

It is sufficient to prove that $\zeta_\Gamma(K_i)$ is a subset of $\zeta_\Gamma(L_i)$. This is because $\zeta_\Gamma(L_i)$ is finite by the proper-discontinuity of the action of Γ on S_i. We show that $\zeta_\Gamma(K_i)$ is a subset of $\zeta_\Gamma(L_i)$. For any $\gamma \in \zeta_\Gamma(K_i)$, we have $\gamma(K_i) \cap K_i \neq \emptyset$. As $\pi_i(\gamma(K_i) \cap K_i) \subset \gamma(L_i) \cap L_i$, we have $\gamma(L_i) \cap L_i \neq \emptyset$. $\square$

Let us prove Theorem 2.1.

Proof of Theorem 2.1. We decompose $\mathbb{R}^{n+1}$ into the orthogonal sum $W \oplus (\bigoplus_{i=1}^{(n-q)} V_i)$ with $\dim V_i = 2$ and $\dim W = 2q-n+1$ so that there exist coordinates (x_i, y_i) of V_i with $B_q(x_i, y_i) = x_i \cdot y_i$, and that $B_q|_{W\times W}$ is negative definite. We define the linear transformation $\phi \in O_q(n+1)$ of $\mathbb{R}^{n+1}$ by $V_i \ni (x_i, y_i) \mapsto (ex_i, e^{-1}y_i) \in V_i$ for any i, and by the identity action on W. By Lemma 2.1, the action of $\{\phi^m\}_{m\in\mathbb{Z}}$ on $\mathbb{S}^n_q$ is properly discontinuous. $\square$

2.2. Homogeneous Case. We recall Kobayashi's theorem for the Calabi–Markus phenomenon in the reductive case. The following observation indicates that we can study the proper-discontinuity of the action of a cocompact discrete subgroup of a Lie group by using the theory of Lie groups:

Observation 2.1. *Let L be a Lie group and Γ a discrete subgroup of L such that L/Γ is compact. Assume that L acts on a manifold M. Then the following conditions are equivalent:*

- *L acts on M properly (that is, for any compact subset $K \subset M$, the set of the elements g of L satisfying $g(K) \cap K \neq \emptyset$ is compact);*
- *Γ acts on M properly discontinuously.*

We explain some fundamental properties of linear reductive groups. Let n, p, and q be positive integers. The automorphism θ of the real general linear group $GL(n,\mathbb{R})$ is given by $\theta(g) = {}^t g^{-1}$ for $g \in GL(n,\mathbb{R})$.

Definition 2.1. A Lie group G is *linear reductive* if G is realized as a closed subgroup of $GL(n,\mathbb{R})$ such that $\theta(G) = G$, and that the number of the connected components of G is finite.

Here are some examples of linear reductive Lie groups: $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $SO(n, \mathbb{R})$, $U(n)$, $O_q(p+q)$ et cetera. The automorphism of G induced from θ is called the *Cartan involution*, denoted by the same letter θ. We also use θ for the automorphism on the Lie algebra $\mathfrak{g}$ of G associated with θ.

Since $\theta^2 = 1$, let $\mathfrak{k}$ (resp. $\mathfrak{p}$) be the eigenspace of θ with the eigenvalue 1 (resp. -1). Then $\mathfrak{g}$ decomposes into a direct sum of $\mathfrak{k}$ and $\mathfrak{p}$. This decomposition is called the *Cartan decomposition*. Here let K be a connected subgroup of G whose Lie algebra is $\mathfrak{k}$. Note that K is a maximal compact subgroup of G. We denote by $d(G)$ the dimension of $\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. The dimension of the subspace $\mathfrak{a}$ is called the *real rank* of G, denoted by $\mathbb{R}\text{-rank}(G)$. It is known that the real rank of G is independent of the choice of $\mathfrak{a}$. A subspace $\mathfrak{a}$ in $\mathfrak{g}$ is a *maximally split abelian subspace* if $\mathfrak{a}$ is conjugate to a maximal abelian subspace in $\mathfrak{p}$ by G. Let M and M' be the centralizer and the normalizer of $\mathfrak{a}$ in K, respectively. The quotient group M'/M is called the *Weyl group*, denoted by W. Then the Weyl group is finite. The Weyl group acts on a maximal abelian subspace effectively. We denote by A the connected subgroup of G whose Lie algebra is $\mathfrak{a}$. Then it is known that $G = KAK$.

Definition 2.2. Let G be a linear reductive Lie group, and H a closed subgroup of G with finite connected components. We call H *reductive* in G if H is stable under the Cartan involution θ of G.

Note that a reductive subgroup H is also a linear reductive Lie group. It is known that, if G is a linear reductive Lie group and H is a reductive subgroup, the homogeneous space G/H carries a structure of a pseudo-Riemannian manifold of index $(d(G) - d(H))$ such that G acts isometrically on G/H via the left action. Moreover G/H is compact if and only if $d(G) = d(H)$ (see [**15**, Theorem 4.7]). Take a maximally split abelian subspace $\mathfrak{a}'$ of the Lie algebra $\mathfrak{h}$ of H. Then there exists $g \in G$ such that $\mathrm{Ad}(g)\mathfrak{a}' \subset \mathfrak{a}$. For this g, we denote by $\mathfrak{a}(H)$ the set $\{w(\mathrm{Ad}(g)\mathfrak{a}')|w \in W\}$.

Kobayashi [**15**] gave the following theorem for proper actions:

Theorem 2.2 (Kobayashi [**15**, Corollary 3.1]). *Let G be a linear reductive Lie group. Suppose that H and L are reductive subgroups of G. Then the following conditions are equivalent:*

- *L acts on G/H properly;*
- $\mathfrak{a}(L) \cap \mathfrak{a}(H) = \{0\}$.

By using Theorem 2.2, Kobayashi [**15**] proved the following theorem on the Calabi–Markus phenomenon:

Theorem 2.3 (Kobayashi [**15**, Corollary 4.4]). *Let G be a linear reductive group, and H a reductive subgroup of G. The following conditions are equivalent:*

(i) $\mathbb{R}\text{-rank}(G) = \mathbb{R}\text{-rank}(H)$*;*
(ii) *no infinite subgroup of G acts properly discontinuously on G/H.*

Proof of Theorem 2.3. Let $\mathfrak{a}$ and $\mathfrak{a}'$ be maximally split abelian subspaces of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We denote by K a maximal compact subgroup of G.

We show that (i) implies (ii). As $\mathbb{R}\text{-rank}\, H = \mathbb{R}\text{-rank}\, G$, we see that $\mathfrak{a}'$ becomes a maximally split abelian subgroup in $\mathfrak{g}$. Therefore we have

$$G = K \exp \mathfrak{a}' K = KHK.$$

For any $g \in G$, there exist k_1, $k_2 \in K$ such that $gk_2 \in k_1 H$. Namely, for any $g \in G$, we have $g(KH/H) \cap KH/H \neq \emptyset$. Let Γ be a discrete subgroup of G acting on the homogeneous space G/H properly discontinuously via the left action. It follows that $\{\gamma \in \Gamma | \gamma(KH/H) \cap KH/H \neq \emptyset\} = \Gamma$. Here we see that KH/H is a compact subset in G/H as K is compact. Acting on G/H properly discontinuously, Γ is finite.

We show that (ii) implies (i). Assume that $\mathbb{R}\text{-rank}\, H < \mathbb{R}\text{-rank}\, G$. It follows that $\mathfrak{a}' \neq \mathfrak{a}$. Therefore there exists an element $X \in \mathfrak{a}$ which does not belong to $\mathfrak{a}(H)$. By Theorem 2.2, the subgroup $\exp(\mathbb{R}X)$ acts on G/H properly. Here Γ stands for the infinite discrete subgroup $\exp(\mathbb{Z}X)$ of G. It follows that Γ acts on G/H properly discontinuously. $\square$

Let n and q be positive integers with $n \geq 2q$. We consider the pseudo-sphere $\mathbb{S}^n_q = O_q(n+1)/O_q(n)$. We have

$$\mathbb{R}\text{-rank}\, O_q(n+1) - \mathbb{R}\text{-rank}\, O_q(n) = q - q = 0.$$

Theorem 1.1 follows from Theorem 2.3.

3. Pseudo-Riemannian warped products

In this section, we give a generalization of Theorem 1.1 to some class of pseudo-Riemannian warped products.

In order to state our result, we set up notation and terminology. Suppose that $(B,\, g_B)$ and $(F,\, g_F)$ are Riemannian manifolds. Let ω be a smooth positive function on B. Recall that the pseudo-Riemannian manifold $(B \times F,\, g^\omega = -g_B + \omega^2 g_F)$, denoted by $(-B) \times_\omega F$, is called a pseudo-Riemannian warped product. B (resp. F, ω) is called the *base* (resp. *fiber*, *warping function*) of the warped product $(-B) \times_\omega F$. We say that ω is *strictly convex* if, for any geodesic $\gamma(s)$ in $(B,\, g_B)$, the function $\omega(\gamma(s))$ is strictly convex with respect to s. We prove the following theorem in this section:

Theorem 1.4. *Let $(F,\, g_F)$ be a closed Riemannian manifold, and $(B,\, g_B)$ a complete Riemannian manifold with $\dim(F) \geq \dim(B)$. Suppose that there exists a positive smooth strictly convex function ω on B with a minimum point b_0. Assume that $\exp_{b_0} : T_{b_0}B \to B$ is a diffeomorphism. Then $(-B) \times_\omega F$ is geodesically complete, and the Calabi-Markus phenomenon occurs in $(-B) \times_\omega F$.*

3.1. Preliminaries. In this subsection, we present some preliminaries to prove Theorem 1.4. The results and their proofs of this subsection are found in O'Neil [**23**]. We write their proofs in more detail than O'Neil [**23**].

We introduce fundamental notions of pseudo-Riemannian geometry. Let $(M,\, g)$ be a pseudo-Riemannian manifold. A pseudo-Riemannian manifold $(M,\, g)$ is often denoted by M for short. A tangent vector $v \in T_pM$ is said to be *timelike, lightlike, causal*, and *spacelike* if $g(v,\, v)$ is negative, zero, non-positive, and positive, respectively. There exists a unique Levi-Civita connection ∇ for $(M,\, g)$. Then the Levi-Civita connection satisfies the following equation, called *Koszul formula* (see for instance O'Neil [**23**, p. 61]):

$$\begin{aligned} 2g(\nabla_X Y,\, Z) =& Xg(Y,\, Z) + Yg(X,\, Z) - Zg(X,\, Y) \\ &- g(X,\, [Y,\, Z]) + g(Y,\, [Z,\, X]) + g(Z,\, [X,\, Y]) \end{aligned}$$

for any vector field X, Y, and Z on M. Let $\gamma(s)$ be a smooth curve in M. Throughout this dissertation, we denote by $\dot{\gamma}(s)$ the tangent vector of the curve $\gamma(s)$ at time s. We can define the induced linear connection on the bundle $\pi : \gamma^*TM \to I$. For simplicity of notation, we use the same letter ∇ for the induced linear connection. A curve $\gamma(s)$ is a *geodesic* if $\nabla_{\partial/\partial s}\dot{\gamma}(s) = 0$. A pseudo-Riemannian manifold $(M,\, g)$ is *geodesically complete* if any inextendible geodesic is defined on the real line. For any point p of M, a subspace Π in T_pM is said to be *degenerate, non-degenerate, indefinite*, and *definite* if the metric restricted to Π is degenerate, non-degenerate, indefinite, and definite, respectively. The *curvature tensor* R is defined by $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z$ for any vector field X, Y, and Z on M. Note that the curvature tensor R is a $(1, 3)$ tensor field on M. Let Π be a non-degenerate 2-dimensional subspace in a tangent space of M. Then $K(u, v)$ is given by

$$K(u, v) = \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2},$$

where (u, v) is a basis for Π. The number $K(u, v)$ is independent of the choice of (u, v). We call $K(u, v) = K(\Pi)$ the *sectional curvature*. Let f be a smooth function of a pseudo-Riemannian manifold (M, g). The *Hessian* $\operatorname{Hess} f$ of f at a point p on M is defined by $(\operatorname{Hess} f)_p(u, v) = g(p)(\nabla_u\nabla f, v)$ for any tangent vector $u, v \in T_pM$. For a geodesic $\gamma(s)$ from p with an initial velocity $v \in T_pM$, it follows that $(\operatorname{Hess} f)_p(v, v) = d^2 f(\gamma(s))/ds^2|_{s=0}$.

From now on, we restrict our attention to pseudo-Riemannian warped products. The natural projection from $B \times F$ to B (resp. F) is denoted by π_B (resp. π_F). Let (b, f) be a point of $B \times F$. A tangent vector in $T_{(b, f)}(B \times \{f\})$ (resp. $T_{(b, f)}(\{b\} \times F)$) is said to be *horizontal* (resp. *vertical*). We identify $T_{(b, f)}(B \times \{f\})$ (resp. $T_{(b, f)}(\{b\} \times F)$) with T_bB (resp. T_fF) by the differential map of π_B (resp. π_F). The projection from $T_{(b, f)}(B \times F)$ to $T_{(b, f)}(B \times \{f\})$ (resp. $T_{(b, f)}(\{b\} \times F)$) is denoted by H (resp. I). For a vector field X on the base B (resp. fiber F), a vector field $\overline{X}$ on the product manifold $B \times F$ is a *lift* of a vector field X on B (resp. F) if $I(\overline{X}) = 0$ and $H(\overline{X}) = X$ (resp. $H(\overline{X}) = 0$ and $I(\overline{X}) = X$). We always identify a vector field on B or F with its lift. The set of lifts of vector fields on F (resp. B) is denoted by $\mathcal{L}(F)$ (resp. $\mathcal{L}(B)$).

Proposition 3.1 (O'Neil [**23**, Chapter 7, Proposition 35]). *Let ∇, ∇^B, and ∇^F be the Levi-Civita connections of $(-B)\times_\omega F$, (B, g_B), and (F, g_F), respectively. For $X, Y \in \mathcal{L}(B)$ and $V, W \in \mathcal{L}(F)$, the following equations hold:*

(i) $\nabla_X Y = \nabla^B_X Y$;
(ii) $\nabla_X V = \nabla_V X = (X\omega/\omega)V$;
(iii) $H(\nabla_V W) = \omega g_F(V, W)\nabla^B\omega$;
(iv) $I(\nabla_V W) = \nabla^F_V W$.

Proof. First, we show (i). By $[V, X] = [V, Y] = 0$ and the Koszul formula,

$$2g^\omega(\nabla_X Y, V) = -Vg^\omega(X, Y) + g^\omega(V, [X, Y]).$$

Since X and Y are lifts of vector fields on the base B, we have $Vg^\omega(X, Y) = 0$. Now that $[X, Y]$ is horizontal, $g^\omega(V, [X, Y]) = 0$. Hence $g^\omega(\nabla_X Y, V) = 0$ for any $V \in \mathcal{L}(F)$. Therefore $\nabla_X Y$ is horizontal. It follows that $\nabla_X Y = \nabla^B_X Y$ as the Levi-Civita connection is uniquely defined by the metric g_B.

Second, we show (ii). By $[X, V] = 0$, we have $\nabla_X V = \nabla_V X$. Since (i) implies $g^\omega(\nabla_X V, Y) = 0$ for any $Y \in \mathcal{L}(B)$, we see that $\nabla_X V$ is vertical. The Koszul formula implies that for any $W \in \mathcal{L}(F)$,

$$\begin{aligned} 2g^\omega(\nabla_X V, W) &= Xg^\omega(V, W) = X(\omega^2 g_F(V, W)) \\ &= 2\omega X(\omega)g_F(V, W) = 2\frac{X(\omega)}{\omega}g^\omega(V, W). \end{aligned}$$

Hence,

$$\nabla_X V = \frac{X(\omega)}{\omega} V.$$

Next, we show (iii). By (ii), for any $X \in \mathcal{L}(B)$,

$$\begin{aligned} g^\omega(\nabla_V W,\, X) &= -g^\omega(W,\, \nabla_V X) = -g^\omega\left(W,\, \frac{X(\omega)}{\omega} V\right) \\ &= -g^\omega(W,\, V)\frac{g^\omega(X,\, \nabla\omega)}{\omega} = -g^\omega\left(\frac{g^\omega(V,\, W)}{\omega}\nabla\omega,\, X\right). \end{aligned}$$

Therefore,

$$H(\nabla_V W) = -\frac{g^\omega(V,\, W)}{\omega}\nabla\omega = \frac{g^\omega(V,\, W)}{\omega}\nabla^B\omega = \omega g_F(V,\, W)\nabla^B\omega.$$

Finally, we show (iv). We see that $I \circ \nabla$ is the Levi-Civita connection on $(\{b\} \times F,\, \omega(b)^2 g_F)$. Hence, $I(\nabla_V W)$ equals $\nabla^F_V W$. □

Here we should remark that the strict convexity of ω follows if some curvature condition is satisfied. Let R and K be the curvature tensor and the sectional curvature of $(-B) \times_\omega F$, respectively. For $X \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$, we have

$$\begin{aligned} R(V,\, X)X &= \nabla_V\nabla_X X - \nabla_X\nabla_V X - \nabla_{[V,\, X]}X \\ &= \frac{(\nabla^B_X X)\omega}{\omega}V - \frac{X^2\omega}{\omega}V \\ &= -\frac{\operatorname{Hess}^B \omega(X,\, X)}{\omega}V, \end{aligned}$$

where $\operatorname{Hess}^B \omega$ is the Hessian of ω on B. For any 2-dimensional subspace Π spanned by $X \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$, it follows that

$$K(\Pi) = \frac{\operatorname{Hess}^B \omega(X,\, X)}{\omega g_B(X,\, X)}.$$

Therefore $K(\Pi) > 0$ implies that ω is a strictly convex function on B.

Now, we consider a geodesic $\gamma(s)$ in $(-B) \times_\omega F$.

Proposition 3.2 (O'Neil [**23**, Chapter 7, Proposition 38]). *A geodesic $\gamma(s) = (\gamma_B(s),\, \gamma_F(s))$ in $(-B) \times_\omega F$ satisfies the following two conditions:*

$$\nabla^B_{\partial/\partial s}\dot{\gamma_B}(s) = -\frac{C}{\omega(\gamma_B(s))^3}\nabla^B\omega(\gamma_B(s)); \tag{2}$$

$$\nabla^F_{\partial/\partial s}\dot{\gamma_F}(s) = -\frac{2}{\omega(\gamma_B(s))}\frac{d\omega(\gamma_B(s))}{ds}\dot{\gamma_F}(s), \tag{3}$$

where $C = \omega(\gamma_B(s))^4 g_F(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s))$. Moreover C is constant.

Proof. First, we consider the case where $\dot{\gamma_B}(0)$ and $\dot{\gamma_F}(0)$ are nonzero. By Proposition 3.1, we have

$$\begin{aligned} \nabla_{\partial/\partial s}\dot{\gamma}(s) =&\nabla^B_{\partial/\partial s}\dot{\gamma_B}(s) + \frac{2}{\omega(\gamma_B(s))}\frac{d}{ds}\omega(\gamma_B(s))\dot{\gamma_F}(s) \\ &- \frac{g^\omega(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s))}{\omega(\gamma_B(s))}\nabla^B\omega + \nabla^F_{\partial/\partial s}\dot{\gamma_F}(s). \end{aligned} \tag{4}$$

The left side is 0 since $\gamma(s)$ is a geodesic. Seeing a horizontal part and a vertical part of the equation (4), we have the equations (2) and (3) in this case.

Next, we consider the case where $\dot{\gamma}(0)$ is horizontal. By Proposition 3.1 (i), $B \times \{\gamma_F(0)\}$ is totally geodesic. Hence, we obtain $\gamma_F(s) = \gamma_F(0)$ and $\nabla^B_{\partial/\partial s}\dot{\gamma_B}(s) = 0$. This completes the proof of this case.

Finally, we consider the case where $\dot{\gamma}(0)$ is vertical and nonzero. If $\{\gamma_B(0)\} \times F$ includes the geodesic $\gamma(s)$ on a sufficiently small open interval around 0, the equations (2) and (3) hold by the preceding argument. Thus we consider the other case where there exists a sequence $\{s_i\}$ which converges to zero such that $\dot{\gamma}(s_i)$ is neither horizontal nor vertical. Since γ_B and γ_F are smooth, the equations (2) and (3) hold in this case.

Now, we show that C is constant.

$$
\begin{aligned}
\frac{dC}{ds} =& \frac{d\omega(\gamma_B(s))^4 g_F(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s))}{ds} \\
=& 4\omega(\gamma_B(s))^3 \frac{d\omega(\gamma_B(s))}{ds} g_F(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s)) + 2\omega(\gamma_B(s))^4\; g_F(\nabla^F_{\partial/\partial s}\dot{\gamma_F}(s),\, \dot{\gamma_F}(s)) \\
=& \left(4\omega(\gamma_B(s))^3 \frac{d\omega(\gamma_B(s))}{ds} - 4\omega(\gamma_B(s))^3 \frac{d\omega(\gamma_B(s))}{ds}\right) g_F(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s)) = 0.
\end{aligned}
$$

The proof is complete. $\square$

3.2. Proof of Theorem 1.4. In this subsection, we prove Theorem 1.4.

We show that the Calabi–Markus phenomenon occurs in $(-B) \times_\omega F$. The following theorem gives a criterion for when the Calabi–Markus phenomenon occurs in a general setting including that of Theorem 1.4:

THEOREM 3.1. *Let $(B,\, g_B)$ be a complete Riemannian manifold, and F a closed manifold such that $\dim F \geq \dim B$. Take a family $\{g_F(b)\}_{b\in B}$ of Riemannian metrics on F smoothly parametrized by elements of B, namely satisfying that, for any tangent vector v on F, $g_F(b)(v,\, v)$ is a smooth function with respect to $b \in B$. Consider a pseudo-Riemannian manifold $(B \times F,\, g = -g_B + g_F)$, denoted by $(-B) \times_{g_F} F$. Suppose that there exists a point $b_1 \in B$ such that $\exp_{b_1} : T_{b_1}B \to B$ is a diffeomorphism, and that $F_1 = \{b_1\} \times F$ is a totally geodesic submanifold in $(-B) \times_{g_F} F$. Take any non-constant geodesic $\tau : [0,\, \infty) \to B$ starting from b_1. Assume that*

$$\frac{d}{ds} g_F(\tau(s))(X,\, X) > 0,$$

for any $s > 0$ and any non-zero tangent vector field X on F. Then the Calabi–Markus phenomenon occurs in $(-B) \times_{g_F} F$.

PROOF OF THEOREM 3.1. Suppose that $F_1 \cap \phi(F_1) = \emptyset$ for some isometry ϕ of $(-B) \times_{g_F} F$. Then we see that b_1 does not belong to $\pi_B(\phi(F_1))$. Since F_1 is compact, there exists a point x_1 of F_1 satisfying that $d_B(b_1,\, \pi_B(\phi(F_1))) = d_B(b_1,\, \pi_B(\phi(x_1)))$, where d_B is the distance of $(B,\, g_B)$. The function r_{b_1} on B is given by $r_{b_1}(q) = d_B(b_1,\, q)$ for $q \in B$. Then we have $d\pi_B(T_{\phi(x_1)}\phi(F_1)) \subset (\nabla r_{b_1})^\perp$. Since $\dim F_1 > \dim B - 1$, there exists a tangent vector $v \in T_{\phi(x_1)}\phi(F_1)$ with $d\pi_B(v) = 0$.

We calculate the Hessian of the function $r_{b_1} \circ \pi_B$ on $(-B) \times_{g_F} F$ at $\phi(x_1)$. Take a geodesic $\gamma : (-\epsilon,\, \epsilon) \to (-B) \times_{g_F} F$ satisfying that $\gamma(0) = \phi(x_1)$, and that $\dot{\gamma}(0) = v$ for some $\epsilon > 0$. Since F_1 is totally geodesic, $\gamma((-\epsilon,\, \epsilon))$ is included in

$\phi(F_1)$. Since the point $\phi(x_1)$ is a minimum point of $(r_{b_1} \circ \pi_B)|_{\phi(F_1)}$, we have

$$\mathrm{Hess}(r_{b_1} \circ \pi_B)(v, v) = \frac{d^2}{dt^2}\Big|_{t=0} r_{b_1}(\pi_B(\gamma(t))) \geq 0.$$

Let $\tau_B : [0, L] \to B$ be the minimal geodesic from b_1 to $\pi_B(\phi(x_1))$ with unit speed in (B, g_B), and $\tau_F : (-\epsilon, \epsilon) \to F$ a curve such that $\tau_F(0) = \pi_F(\phi(x_1))$, and that $\dot{\tau_F}(0) = d\pi_F(v)$. We should remark that $(\tau_B(s), \pi_F(\phi(x_1)))$ is a geodesic in $(-B) \times_{g_F} F$. We define a variation $\alpha : (-\epsilon, \epsilon) \times [0, L] \to (-B) \times_{g_F} F$ of the geodesic $(\tau(s), \pi_F(\phi(x_1)))$ by $\alpha(t, s) = (\tau_B(s), \tau_F(t))$. Then we obtain

$$\begin{aligned}
\mathrm{Hess}(r_{b_1} \circ \pi_B)(v, v) &= g(\nabla_v \nabla(r_{b_1} \circ \pi_B), v) \\
&= -g\left(\nabla_{\partial/\partial t} \frac{\partial \alpha(t, s)}{\partial s}, \frac{\partial \alpha(t, s)}{\partial t}\right)\Big|_{(t, s)=(0, L)} \\
&= -g\left(\nabla_{\partial/\partial s} \frac{\partial \alpha(t, s)}{\partial t}, \frac{\partial \alpha(t, s)}{\partial t}\right)\Big|_{(t, s)=(0, L)} \\
&= -\frac{1}{2}\frac{\partial}{\partial s} g\left(\frac{\partial \alpha(t, s)}{\partial t}, \frac{\partial \alpha(t, s)}{\partial t}\right)\Big|_{(t, s)=(0, L)} \\
&= -\frac{1}{2}\frac{d}{ds}\Big|_{s=L} g_F(\tau_B(s))(d\pi_F(v), d\pi_F(v)) < 0.
\end{aligned}$$

This is impossible. □

We check that Theorem 3.1 implies that the Calabi–Markus phenomenon occurs in the setting of Theorem 1.4. We write F_0 instead of $\pi_B^{-1}(b_0)$. Since b_0 is a critical point of ω, we see that the fiber F_0 is totally geodesic by using Proposition 3.1 (iii). Take any geodesic $\tau : [0, \infty) \to B$ starting from b_0 with unit speed. Then

$$\frac{d^2}{ds^2}\omega(\tau(s)) > 0$$

by the strict convexity of ω. Since b_0 is a critical point of ω, we have

$$\frac{d}{ds}\omega(\tau(s)) > 0,$$

for $s > 0$. Therefore we see that the Calabi–Markus phenomenon occurs in $(-B) \times_\omega F$.

We show that $(-B) \times_\omega F$ is geodesically complete under the assumptions of Theorem 1.4. For a point $b \in B$ and a non-negative number r, the function $\omega^b_{\inf}(r)$ is given by $\omega^b_{\inf}(r) = \inf\{\omega(p) | d_B(p, b) = r, p \in B\}$. We should remark that $\omega^b_{\inf}(r)$ could take an infinite value. Romero–Sánchez [**24**] gave a sufficient condition of the geodesic completeness of $(-B) \times_\omega F$.

THEOREM 3.2 (Romero–Sánchez [**24**, Theorem 3.9]). *Assume that (B, g_B) is a non-compact complete Riemannian manifold. Let ω be a smooth positive function on B. Suppose that there exists a point $b_2 \in B$ such that*

$$\int_0^\infty (E + (\omega^{b_2}_{\inf}(r))^{-2})^{-1/2} dr = \infty \tag{5}$$

for any positive number E. Then $(-B) \times_\omega F$ is geodesically complete.

PROOF OF THEOREM 3.2. We give the original proof of Romero–Sánchez [**24**]. Assume that $(-B) \times_\omega F$ is not geodesically complete. Let $\gamma : [0, L) \to (-B) \times_\omega F$ be an inextendible geodesic for some positive number $L > 0$. We denote by D a constant $g(\dot{\gamma}(s),\, \dot{\gamma}(s))$. Then we have

$$g_B(\dot{\gamma_B}(s),\, \dot{\gamma_B}(s)) = -D + \omega(\gamma_B(s))^2 g_F(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s)) = -D + \frac{C}{\omega(\gamma_B(s))^2},$$

where C is the constant appearing in Proposition 3.2. We obtain

$$\begin{aligned}
&\frac{1}{\sqrt{C}} \int_{d_B(b_2,\, \gamma_B(0))}^{d_B(b_2,\, \gamma_B(s))} \left(\frac{|D|}{C} + \frac{1}{\omega_{\inf}^{b_2}(r)^2} \right)^{-1/2} dr \\
&\leq \int_{d_B(b_2,\, \gamma_B(0))}^{d_B(b_2,\, \gamma_B(s))} \left(-D + \frac{C}{\omega_{\inf}^{b_2}(r)^2} \right)^{-1/2} dr \\
&\leq \int_0^s \left(-D + \frac{C}{\omega(\gamma_B(u))^2} \right)^{-1/2} \sqrt{g_B(\dot{\gamma_B}(u),\, \dot{\gamma_B}(u))} du \\
&= s,
\end{aligned}$$

for any $s \in [0,\, L)$. We note that there is a sequence $\{s_n\}_{n=1}^{\infty}$ such that s_n approaches to L, and that $\gamma_B(s_n)$ is not contained in any compact set of B as n increases. Since $(B,\, g_B)$ is complete, $\lim_{s \to L} d_B(b_2,\, \gamma_B(s)) = \infty$. Taking $|D|/C$ as E in the equation (5), we have reached a contradiction.

□

Under the assumptions of Theorem 1.4, let us prove that the equation (5) holds. Since ω is strictly convex with the minimum point b_0, it is clear that $\omega_{\inf}^{b_0}(r) \geq \omega(b_0)$. For any positive number E, we have

$$\int_0^{\infty} (E + (\omega_{\inf}^{b_0}(r))^{-2})^{-1/2} dr \geq \int_0^{\infty} (E + (\omega(b_0))^{-2})^{-1/2} dr = \infty.$$

Since we can take b_0 as b_2 in Theorem 3.2, the geodesic completeness of $(-B) \times_\omega F$ follows. The proof of Theorem 1.4 is complete.

4. Parametrized Lorentzian products

In this section, we present our result on the Calabi–Markus phenomenon of parametrized Lorentzian products.

The partial derivative in the direction t is denoted by ∂_t. For $t \neq 0$, $\partial_{|t|}$ is defined to be the "signed" partial derivative $(t/|t|)\partial_t$. For positive constants t_0 and c, we say that a parametrized Lorentzian product satisfies the condition $(\mathrm{H})_{t_0,c}$ if the inequality $\partial_{|t|} g_t(X,\, X) \geq 2c g_t(X,\, X)$ holds for any vector field X on F and $|t| \geq t_0$. In this section, we prove the following theorem:

THEOREM 1.3 (Mukuno [**22**]). *Let F be a connected closed manifold, and $\{g_t\}_{t\in\mathbb{R}}$ a smooth family of Riemannian metrics on F. Assume that there exist positive constants t_0 and c such that a parametrized Lorentzian product $(\mathbb{R}\times F,\, -dt^2+g_t)$ satisfies $(\mathrm{H})_{t_0,c}$. Then the Calabi–Markus phenomenon occurs in $(\mathbb{R}\times F,\, -dt^2+g_t)$.*

In Subsection 4.1, we give a proof of Theorem 1.3. Moreover we present a new example where the Calabi–Markus phenomenon occurs. A Lorentzian manifold $(M,\, g)$ is *time-oriented* if there exists a non-vanishing timelike vector field on M. We recall that a Lorentzian manifold M is globally hyperbolic if M contains a Cauchy hypersurface. A Lorentzian manifold $(M,\, g)$ is *non-spacelike geodesically complete* if any inextendible causal geodesic is defined on the real line. We prove the following proposition:

Proposition 4.1 (Mukuno [**22**]). *Let M be a time-oriented, non-spacelike geodesically complete, globally hyperbolic Lorentzian manifold of dimension greater than 1. Suppose that for any $p \in M$ we have a positive lower bound d^2 on the sectional curvature for any indefinite 2-dimensional subspace in T_pM. Assume that M admits a connected closed totally geodesic Cauchy hypersurface F in M. Let $I_F(p)$ be the set consisting of the points in F which a timelike geodesic joins to p. We impose the following technical conditions on $I_F(p)$ for any $p \in M - F$:*

(i) *$I_F(p)$ is geodesically connected;*
(ii) *no lightlike geodesic ray from p meets $I_F(p)$.*

Then the Calabi–Markus phenomenon occurs in M.

In Subsection 4.3, we explain that the assumptions of Proposition 4.1 imply those of Theorem 1.3; therefore the Calabi–Markus phenomenon occurs. Furthermore, we give another short, self-contained proof of Proposition 4.1. In Subsection 4.4, we construct an inhomogeneous Lorentzian manifold where the Calabi–Markus phenomenon occurs.

4.1. Preparation of the proof of Theorem 1.3. We denote by $M^{1,\,n}$ a parametrized Lorentzian product $(\mathbb{R}\times F,\, h_{M^{1,\,n}} = -dt^2+g_t)$ satisfying $(\mathrm{H})_{t_0,c}$. We examine spacelike geodesics in $M^{1,\,n}$ to prove Theorem 1.3.

Let $\gamma(s)$ be a curve in $M^{1,\,n}$, and $\gamma_F(s)$ the curve in F obtained by projecting the curve $\gamma(s)$ onto F. We write $\pi_{\mathbb{R}}$ for the natural projection of $M^{1,\,n} = \mathbb{R}\times F$ onto $\mathbb{R}$. Take a local coordinate system $(x^0,\, x^1,\, \dots, x^n)$ of $M^{1,\,n}$ such that $x^0 = \pi_{\mathbb{R}}$, and that $(x^1,\, x^2,\, \dots, x^n)$ comes from a local coordinate system of F. Set coordinate vector fields $\partial_i = \partial/\partial x^i$, the i-th component of the curve $\gamma^i(s) = x^i(\gamma(s))$, the i-th component of the velocity $\dot{\gamma^i}(s) = d\gamma^i(s)/ds$, and the second derivative $\ddot{\gamma^i}(s) = d^2\gamma^i(s)/ds^2$ for $0 \le i \le n$. The symbols h_{ij}, h^{ij}, and Γ^k_{ij} stand for $h_{M^{1,\,n}}(\partial_i,\, \partial_j)$, the $(i,\, j)$ entry of the inverse matrix of $(h_{ij})_{0\le i,\, j\le n}$, and the Christoffel symbols for $0 \le i,\, j,\, k \le n$, respectively, where $h_{M^{1,\,n}}$ is the Lorentzian metric of $M^{1,\,n}$.

In what follows, we make estimates on the lengths of spacelike geodesics of $M^{1,\,n}$. Recall that t_0 and c are the constants in the condition $(\mathrm{H})_{t_0,c}$.

Lemma 4.1. *The length of a spacelike geodesic γ in $\pi_{\mathbb{R}}^{-1}((-\infty,\, -t_0] \cup [t_0,\, \infty))$ is less than π/c.*

Proof. We require γ to be parametrized by arc length. For simplicity, let the domain of γ be $[0,\, L]$. As γ maintains unit speed,

$$h_{M^{1,\,n}}(\dot\gamma(s),\, \dot\gamma(s)) = -|\dot{\gamma^0}(s)|^2 + g_{\gamma^0(s)}(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s)) = 1, \tag{6}$$

for all $s \in (0,\, L)$. We assume that γ stays in $\pi_{\mathbb{R}}^{-1}([t_0,\, \infty))$. Here we should notice that

$$\Gamma^0_{00} = \Gamma^0_{i0} = \Gamma^0_{0j} = 0,\ \Gamma^0_{ij} = \frac{1}{2}\partial_0 h_{ij}\ (1 \le i,\, j \le n).$$

Therefore we obtain

$$\ddot{\gamma^0}(s) + \sum_{0\le i,j\le n} \Gamma^0_{ij}\dot{\gamma^i}(s)\dot{\gamma^j}(s) = \ddot{\gamma^0}(s) + \frac{1}{2}(\partial_t g)_{\gamma^0(s)}(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s)). \tag{7}$$

Since γ is a geodesic, the left-hand side is zero. By $(\mathrm{H})_{t_0,c}$, (6), and (7),

(8)
$$\ddot{\gamma}^0(s) = -\frac{1}{2}(\partial_t g)_{\gamma^0(s)}(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s)) \leq -c g_{\gamma^0(s)}(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s)) = -c(1 + |\dot{\gamma}^0(s)|^2).$$

Note that the inequality (8) makes it obvious that γ^0 is strictly concave for any spacelike geodesic γ whose image is in $\pi_{\mathbb{R}}^{-1}((-\infty,\, -t_0] \cup [t_0,\, \infty))$. Putting $G(s) = \cot(cs)$, we check at once that $dG(s)/ds = -c(1 + G(s)^2)$. There exists a positive number $s_0 < \pi/c$ such that $\dot{\gamma}^0(0) = G(s_0)$. Then by the comparison argument we have $\dot{\gamma}^0(s) \leq G(s + s_0)$ as long as $\gamma^0(s) \geq t_0$ (see [**13**, Cororallies 1. 6. 2]). It immediately follows that

$$\gamma^0(s) \leq \gamma^0(0) + \int_0^s \cot c(x + s_0)\; dx$$

whenever s satisfies $\gamma^0(s) \geq t_0$. If s goes to $\pi/c - s_0$, the right-hand side approaches $-\infty$. As $\gamma^0(s) \geq t_0$, the length of γ is less than $\pi/c - s_0 < \pi/c$. The same argument applies to the case $\gamma \subset \pi_{\mathbb{R}}^{-1}((-\infty,\, -t_0])$ as well. $\square$

Write F_t for the spacelike submanifold $\pi_{\mathbb{R}}^{-1}(\{t\})$ in $M^{1,\,n}$, which is isometric to the Riemannian manifold $(F,\, g_t)$. Let π_t be the natural projection of $M^{1,\,n} = \mathbb{R} \times F$ onto F_t.

Lemma 4.2. *Let $\gamma : [0,\, L] \to M^{1,\,n}$ be a spacelike geodesic in $\pi_{\mathbb{R}}^{-1}((-\infty,\, -t_0] \cup [t_0,\, \infty))$ such that $\dot{\gamma}^0(0) = 0$. Then there is the constant $(1 + \pi)/c$ that dominates the length of the spacelike curve $\pi_{\gamma^0(L)}(\gamma)$.*

Proof. We give the proof in the case where $\gamma \subset \pi_{\mathbb{R}}^{-1}([t_0,\, \infty))$, for the same proof works in the other case as well. We assume that γ is parametrized by arc length. Then we see that

$$1 + |\dot{\gamma}^0(s)|^2 = g_{\gamma^0(s)}(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s)).$$

By $(\mathrm{H})_{t_0,c}$, for any vector field X on F and positive numbers t_1, t_2 with $t_2 \geq t_1 \geq t_0$, we have $g_{t_2}(X,\, X) \geq e^{2c(t_2 - t_1)} g_{t_1}(X,\, X)$. We see that γ^0 is strictly concave from what has already been proved in Lemma 4.1. Then γ^0 is strictly decreasing, as $\dot{\gamma}^0(0) = 0$. It follows that

$$\begin{aligned}
\sqrt{g_{\gamma^0(L)}(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s))} &= \sqrt{(1 + |\dot{\gamma}^0(s)|^2)\left(\frac{g_{\gamma^0(s)}(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s))}{g_{\gamma^0(L)}(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s))}\right)^{-1}} \\
&\leq \left(\sqrt{1 + |\dot{\gamma}^0(s)|^2}\right) e^{-c(\gamma^0(s) - \gamma^0(L))} \\
&\leq (1 + |\dot{\gamma}^0(s)|)\, e^{-c(\gamma^0(s) - \gamma^0(L))}.
\end{aligned}$$

Integrating the above inequality, we find that

$$\int_0^L \sqrt{g_{\gamma^0(L)}(\dot{\gamma_F}(s),\, \dot{\gamma_F}(s))}\; ds \leq \int_0^L e^{-c(\gamma^0(s) - \gamma^0(L))} ds \tag{9}$$

$$+ \int_0^L |\dot{\gamma}^0(s)|\; e^{-c(\gamma^0(s) - \gamma^0(L))}\; ds. \tag{10}$$

As $e^{-c(\gamma^0(s)-\gamma^0(L))} \leq 1$, we see that

$$\int_0^L e^{-c(\gamma^0(s)-\gamma^0(L))} ds \leq L.$$

Due to Lemma 4.1, L is bounded above by π/c. We estimate the second term of the right-hand side of the inequality (9).

$$\begin{aligned}\int_0^L |\dot{\gamma}^0(s)|\ e^{-c(\gamma^0(s)-\gamma^0(L))}\ ds &= -\int_0^L \dot{\gamma}^0(s)\ e^{-c(\gamma^0(s)-\gamma^0(L))}\ ds \\ &= -\int_{\gamma^0(0)}^{\gamma^0(L)} e^{-c(s-\gamma^0(L))}\ ds \leq \frac{1}{c}.\end{aligned}$$

The proof is complete. □

REMARK 4.1. Fix $T \geq t_0$. Let $\gamma : [0, 1] \to \pi_{\mathbb{R}}^{-1}([T, \infty))$ be a spacelike geodesic. We extend γ as long as $\gamma \subset \pi_{\mathbb{R}}^{-1}([T, \infty))$. Let $\overline{\gamma}$ be the maximal extension of γ, and I the domain of $\overline{\gamma}$. We prove that each of the endpoints of $\overline{\gamma}$ reaches F_T. By Lemma 4.1, I is a bounded interval. Suppose that an endpoint of I does not belong to I. For simplicity, we put $I = [0, L)$, where $L > 0$. We show that, if u approaches L, $\overline{\gamma}(u)$ converges. As $\overline{\gamma}^0$ is concave and bounded, the limit $\lim_{u\to L} \overline{\gamma}^0(u)$ exists. We regard $\overline{\gamma}_F$ as a curve in (F, g_{t_0}). By Lemma 4.2, the length of $\overline{\gamma}_F$ is finite. Since F is compact, the limit $\lim_{u\to L} \overline{\gamma}_F(u)$ lies in F. Therefore we can extend $\overline{\gamma}$ until the endpoint continuously. We say that an open set U of a Lorentzian manifold M is *convex* if for any point p of U there exists an open set V of T_pM such that the restriction of $\exp_p$ to V is a diffeomorphism onto U, and that $v \in V$ implies $tv \in V$ for any $t \in [0, 1]$. The existence of a convex neighborhood of any point of $M^{1,n}$ leads us to the extension of the geodesic γ to the limit $\lim_{u\to L} \overline{\gamma}(u)$. This contradicts our assumption. It follows that I is closed. The maximality of $\overline{\gamma}$ completes the proof.

Let d_T be the "intrinsic" distance on F_T defined by the Riemannian metric of F_T. Combining Lemma 4.2 and Remark 4.1, we obtain

COROLLARY 4.1. *Fix a positive constant $T > t_0$, and let γ be a spacelike geodesic with endpoints y, z. If γ is included in $\pi_{\mathbb{R}}^{-1}([T, \infty))$ (resp. in $\pi_{\mathbb{R}}^{-1}((-\infty, -T])$, then the distance $d_T(\pi_T(y), \pi_T(z))$ (resp. $d_{-T}(\pi_{-T}(y), \pi_{-T}(z))$ is dominated by $2(1+\pi)/c$.*

PROOF. It is sufficient to show the case where $\gamma \subset \pi_{\mathbb{R}}^{-1}([T, \infty))$. By Remark 4.1, we can extend the geodesic γ until each of the endpoints of the geodesic reaches F_T. We write this extension of γ as $\overline{\gamma}$. The symbol $L(-)$ stands for the length of a spacelike curve. We have

$$d_T(\pi_T(y), \pi_T(z)) \leq L(\pi_T(\gamma)) \leq L(\pi_T(\overline{\gamma})).$$

Let $[-L_0, L_1]$ be the domain of $\overline{\gamma}$ such that $\dot{\overline{\gamma}}^0(0) = 0$. According to Lemma 4.2, each of $L(\pi_T(\overline{\gamma}|_{[-L_0, 0]}))$ and $L(\pi_T(\overline{\gamma}|_{[0, L_1]}))$ is bounded above by $(1+\pi)/c$. □

4.2. Proof of Theorem 1.3. This subsection is devoted to the proof of Theorem 1.3.

We denote by diam$(-)$ the diameter of a metric space. First we note

Lemma 4.3. *Let $(X,\, d)$ be a connected compact metric space. For any family $\{A_i\}_{i=1}^{k}$ of connected closed subsets of X such that $X = \bigcup_{i=1}^{k} A_i$, we have*

$$\operatorname{diam}(X) \leq \sum_{i=1}^{k} \operatorname{diam}(A_i).$$

Proof. Consider two points $y,\, z \in X$ such that $d(y,\, z) = \operatorname{diam}(X)$. Replacing the indices properly, we may require A_1 to contain y. We have only to consider the case where z belongs to some A_i with $i \neq 1$. Rearranging $A_2,\, \dots,\, A_k$ in an appropriate manner allows us to assume that $z \in A_l$ for some l, and that $A_i \cap A_{i+1} \neq \emptyset$ for any positive integer $i \leq l-1$. Put $w_1 = y$ and $w_{l+1} = z$, and take $w_{i+1} \in A_i \cap A_{i+1}$ for $i = 1,\, 2,\, \dots,\, l-1$. Then we obtain

$$\operatorname{diam}(X) = d(y,\, z) \leq \sum_{i=1}^{l} d(w_i,\, w_{i+1}) \leq \sum_{i=1}^{l} \operatorname{diam}(A_i) \leq \sum_{i=1}^{k} \operatorname{diam}(A_i).$$

□

We return to our manifold $M^{1,\, n}$. Recall that F_T is the spacelike submanifold $\pi_{\mathbb{R}}^{-1}(\{t\})$ in $M^{1,\, n}$. Next we observe

Lemma 4.4. *For any positive numbers T and ϵ, we can cover F_T by finitely many compact subsets $V_1,\, V_2,\, \dots,\, V_{n(T,\, \epsilon)}$ such that any $p,\, q \in V_i$ can be joined by a spacelike geodesic contained in $\pi_{\mathbb{R}}^{-1}((T-\epsilon,\, T+\epsilon))$.*

Proof. Any $x \in F_T$ admits a compact convex neighborhood W_x in $\pi_{\mathbb{R}}^{-1}((T-\epsilon,\, T+\epsilon))$ [**23**, Chapter 5, Proposition 7]). For any $p,\, q \in V_x := F_T \cap W_x$, the geodesic from p to q in W_x is spacelike: Indeed if γ is a non-spacelike geodesic, γ^0 is a strictly monotonic function, and this contradicts the fact that $\pi_{\mathbb{R}}(p) = \pi_{\mathbb{R}}(q)$. Compactness of F_T implies that there are $x_1,\, x_2,\, \dots,\, x_{n(T,\, \epsilon)}$ such that $F_T = \bigcup_{i=1}^{n(T,\, \epsilon)} V_{x_i}$. This completes the proof. □

Let us prove Theorem 1.3. We denote by $\operatorname{Isom}(M^{1,\, n})$ the isometry group of $M^{1,\, n}$. We write $\operatorname{Isom}^+(M^{1,\, n})$ for the subgroup of $\operatorname{Isom}(M^{1,\, n})$ consisting of those elements that preserve time-orientation. The Calabi–Markus phenomenon emerges if a certain compact set K meets the image of K under any $\phi \in \operatorname{Isom}(M^{1,\, n})$. Since the index of $\operatorname{Isom}^+(M^{1,\, n})$ in $\operatorname{Isom}(M^{1,\, n})$ is at most two, the last claim remains valid even when $\operatorname{Isom}(M^{1,\, n})$ is replaced by $\operatorname{Isom}^+(M^{1,\, n})$. Therefore it suffices to prove the following lemma:

Lemma 4.5. *For some sufficiently large $T_2 > 0$, the preimage $K = \pi_{\mathbb{R}}^{-1}([-T_2,\, T_2])$ satisfies $\phi(K) \cap K \neq \emptyset$ for any $\phi \in \operatorname{Isom}^+(M^{1,\, n})$.*

Proof. Take $T_1 > t_0$ arbitrarily, and $\epsilon > 0$ so that $T_1 - \epsilon > t_0$. As $(\mathrm{H})_{t_0, c}$ implies that $\operatorname{diam}(F_T)$ has an at least exponential growth with respect to $T \geq t_0$, we can find $T_2 > T_1$ such that $\operatorname{diam}(F_{T_2}) > 2n(T_1,\, \epsilon)\,(1+\pi)/c$, where $n(T_1,\, \epsilon)$ is the constant appearing in Lemma 4.4.

Suppose by contradiction that

$$\phi(\pi_{\mathbb{R}}^{-1}([-T_2,\, T_2])) \cap \pi_{\mathbb{R}}^{-1}([-T_2,\, T_2]) = \emptyset$$

for some isometry $\phi \in \mathrm{Isom}^+(M^{1,n})$. We have $\phi(F_{T_1}) \subset \pi_{\mathbb{R}}^{-1}((-\infty, -T_2] \cup [T_2, \infty))$ as $T_1 \in (0, T_2)$. We consider only the case where $\phi(F_{T_1}) \subset \pi_{\mathbb{R}}^{-1}([T_2, \infty))$, for the same argument applies to the remaining case as well. We should remark that, for any timelike geodesic $\gamma : \mathbb{R} \to M^{1,n}$, γ^0 is surjective. It follows that $\pi_{\mathbb{R}}(\phi^{-1}({\pi_{T_2}}^{-1}(\{p\}))) = \mathbb{R}$ for all $p \in F_{T_2}$, since ${\pi_{T_2}}^{-1}(\{p\})$ is totally geodesic. Hence we obtain $\pi_{T_2}(\phi(F_{T_1})) = F_{T_2}$ as ϕ is an isometry.

On the other hand, we should notice that F_{T_1} can be covered by finitely many compact subsets $V_1, V_2, \ldots, V_{n(T_1, \epsilon)}$ such that any $p, q \in V_i$ can be joined by a spacelike geodesic contained in $\pi_{\mathbb{R}}^{-1}((T_1 - \epsilon, T_1 + \epsilon))$. For each i, compactness of $\pi_{T_2}(\phi(V_i))$ guarantees the existence of $y_i, z_i \in V_i$ with

$$\mathrm{diam}(\pi_{T_2}(\phi(V_i))) = d_{T_2}(\pi_{T_2}(\phi(y_i)), \pi_{T_2}(\phi(z_i))).$$

By the choice of ϵ, we can find a spacelike geodesic γ_i from y_i to z_i in $\pi_{\mathbb{R}}^{-1}((t_0, \infty))$. We note that ${\gamma_i}^0$ is concave from the proof of Lemma 4.1. Then we see that the image of γ_i is included in $\pi_{\mathbb{R}}^{-1}([T_1, \infty))$. Recall that $\phi(F_{T_1})$ is included in $\pi_{\mathbb{R}}^{-1}([T_2, \infty))$. As ϕ preserves time-orientation, $\phi(\gamma_i)$ is included in $\pi_{\mathbb{R}}^{-1}((T_2, \infty))$. Corollary 4.1 leads us to

$$\mathrm{diam}(\pi_{T_2}(\phi(V_i))) = d_{T_2}(\pi_{T_2}(\phi(y_i)), \pi_{T_2}(\phi(z_i))) \le 2(1+\pi)/c.$$

Due to Lemma 4.3, we obtain

$$\mathrm{diam}\left(\bigcup_{i=1}^{n(T_1, \epsilon)} \pi_{T_2}(\phi(V_i))\right) \le 2n(T_1, \epsilon)\,(1+\pi)/c < \mathrm{diam}(F_{T_2}).$$

This contradicts the fact that $F_{T_2} = \pi_{T_2}(\phi(F_{T_1})) = \bigcup_{i=1}^{n(T_1, \epsilon)} \pi_{T_2}(\phi(V_i))$. □

4.3. Proof of Proposition 4.1. Let M be a Lorentzian manifold and $\gamma : [0, L] \to M$ a smooth curve, where L is a positive number. A vector field $X(s)$ along $\gamma(s)$ is *parallel* if $\nabla_{\partial/\partial s} X(s) = 0$. Given a vector $v \in T_{\gamma(0)}M$, there exists a unique parallel vector field $X(s)$ along $\gamma(s)$ with $X(0) = v$. A variation $\alpha : (-\epsilon, \epsilon) \times [0, L] \to M$ of $\gamma(s)$ is a smooth map such that $\alpha(0, s) = \gamma(s)$ for any $s \in [0, L]$, where ϵ is a positive number. Then a *variation vector field* $X : [0, L] \to TM$ along $\gamma(s)$ is given by $X(s) = \partial\alpha(t, s)/\partial t|_{t=0}$. Assume that $\gamma(s)$ is a geodesic in M. A vector field $Y(s)$ along the geodesic $\gamma(s)$ is called a *Jacobi field* if $Y(s)$ satisfies the following differential equation:

$$\nabla_{\partial/\partial s}\nabla_{\partial/\partial s} Y(s) + R(Y(s), \dot\gamma(s))\dot\gamma(s) = 0.$$

Let $\alpha(t, s)$ be a variation of the geodesic $\gamma(s)$ such that for any fixed $t \in (-\epsilon, \epsilon)$ the curve $\alpha(t, s)$ is a geodesic with respect to s. Then the variation vector field $Y(s) = \partial\alpha(t, s)/\partial t|_{t=0}$ along $\gamma(s)$ is a Jacobi field.

We prove Proposition 4.1. Let (M, h) be a Lorentzian manifold satisfying the assumptions of Proposition 4.1. First we prove that (M, h) is realized as a parametrized Lorentzian product $(\mathbb{R} \times F, -dt^2 + g_t)$, where $\{g_t\}_{t \in \mathbb{R}}$ is some smooth family of Riemannian metrics of F. Let $\pi : NF \to F$ be the normal bundle over F. By the timelike completeness, we can define the normal exponential map $\exp^\perp : NF \to M$. The following lemma is probably known, but we do not have a reference:

LEMMA 4.6. *The map* $\exp^\perp : NF \to M$ *is a diffeomorphism.*

PROOF. First, we show that $\exp^{\perp}$ is a local diffeomorphism. This reduces to proving that there exists no focal point of F. Let $\gamma : [0, \infty) \to M$ be a timelike geodesic normal to F with $\gamma(0) \in F$, and $Y(s)$ a Jacobi field along γ with $Y(0) \in T_{\gamma(0)}N$. As F is totally geodesic, we may assume that $\nabla_{\partial/\partial s}Y(0) = 0$. Then both $Y(s)$ and $\nabla_{\partial/\partial s}Y(s)$ are spacelike. Set a smooth function $f(s) = h(Y(s), Y(s))$. We have

$$\frac{d^2 f(s)}{ds^2} = 2\{h(\nabla_{\partial/\partial s}Y(s), \nabla_{\partial/\partial s}Y(s)) - h(R(Y(s), \dot\gamma(s))\dot\gamma(s), Y(s))\} \geq 0.$$

Since $df(0)/ds = 0$, f is a monotonically increasing function. If there is a positive number s_0 such that $f(s_0) = 0$, we see that $f(s) = 0$ for any $s \geq 0$ by using the property of Jacobi fields.

Next we show that $\exp^{\perp}$ is a bijection. For any point of $M - F$, it is suffices to find a unique geodesic from the point perpendicular to F. Since such a geodesic exists due to Kim–Kim [**14**, Proposition 3.4], all that we have to do is to prove the uniqueness. We may assume that $\exp^{\perp}(v_0) = \exp^{\perp}(v_1)$, denoted by p, for some future-directed vectors $v_0, v_1 \in NF$. Let $T_p^{\mathrm{past}}M$ be the set of past-directed timelike tangent vectors of T_pM. Put $D(p) = \{v \in T_p^{\mathrm{past}}M \mid \exp_p(v) \in F\}$. As F is a Cauchy hypersurface, for any $v \in T_p^{\mathrm{past}}M$, there is a unique positive number a such that $av \in D(p)$. Since the exponential maps $\exp_p$ restricted to both $T_p^{\mathrm{past}}M$ and its boundary are transverse regular on F, the set $D(p)$ is a hypersurface in $T_p^{\mathrm{past}}M$, whose boundary is the set of the lightlike vectors $v \in T_pM$ satisfying that $\exp_p(v) \in F$. The curvature condition of M implies that the map $\exp_p|_{T_p^{\mathrm{past}}M}$ is a local diffeomorphism (see [**23**, Chapter 10, Corollary 20]). Recall that $I_F(p)$ is the set of the points in F which a timelike geodesic joins to p. We see that the restricted map $\exp_p|_{D(p)} : D(p) \to I_F(p)$ is proper by the condition (ii) of Proposition 4.1. It follows that the map $\exp_p|_{D(p)}$ is a covering map. We can take a geodesic τ connecting $\pi(v_0)$ and $\pi(v_1)$ in $I_F(p)$ by using the condition (i). Let $\overline{\tau}$ be a lifting of τ. We set $\gamma_u(s) = \exp_p(s\overline{\tau}(u))$ for $s \in [0, 1]$. We define by $V_u(s) = \partial_u\gamma_u(s)$ the variation vector field V_s along the geodesic γ_u. Let $V_u^{\perp}(s)$ be the component of $V_u(s)$ perpendicular to $\dot\gamma_u(s)$. We should notice that $\nabla_{\partial/\partial s}V_u^{\perp}(s)$ is spacelike. Here R stands for the curvature tensor of M. We have

$$\begin{aligned} L^-(\gamma_u)\frac{d^2L^-(\gamma_u)}{du^2} = \int_0^1 & h(R(V_u^{\perp}(s), \dot\gamma_u(s))\dot\gamma_u(s), V_u^{\perp}(s)) \\ & - h(\nabla_{\partial/\partial s}V_u^{\perp}(s), \nabla_{\partial/\partial s}V_u^{\perp}(s))\, ds. \end{aligned}$$

Hence $L^-(\gamma_u)$ is concave. Since the ends of the domain are critical points of $L^-(\gamma_u)$, the function $L^-(\gamma_u)$ is a constant. We have $v_0 = v_1$ as there exists no focal point of F. □

There exists the normal vector field $n : F \to NF$ with $h(n(p), n(p)) = -1$ as M is time-orientable. Let ψ be the map given by $\psi : \mathbb{R} \times F \ni (t, p) \mapsto t\,n(p) \in NF$. According to Gauss's lemma, we have $(\exp^{\perp} \circ \psi)^*h = -dt^2 + g_t$ where t is the parameter of $\mathbb{R}$, and $\{g_t\}_{t\in\mathbb{R}}$ is a smooth family of Riemannian metrics of F. By abuse of notation, we write h and F instead of $(\exp^{\perp} \circ\psi)^*h$ and $\{0\} \times F$, respectively.

Recall that $\pi_{\mathbb{R}}$ is the natural projection of $\mathbb{R} \times F$ onto $\mathbb{R}$. Next we prove

LEMMA 4.7. *For any spacelike geodesic γ in $\pi_{\mathbb{R}}^{-1}((0, \infty))$ (resp. $\pi_{\mathbb{R}}^{-1}((-\infty, 0))$), the second derivative $\ddot\gamma^0(s)$ is negative (resp. positive).*

Proof. Since $\ddot{\gamma^0}(s) = -(\partial_t g)_{\gamma^0(s)}(\dot{\gamma_F}(s), \dot{\gamma_F}(s))/2$ as in the proof of Lemma 4.1, we investigate the partial derivative of g_t with respect to t. Take a point $x \in F$ and a non-zero tangent vector $w \in T_x F$ arbitrarily. Let γ^x be the curve defined by $\gamma^x(s) = (s, x) \in \mathbb{R} \times F$ for $s \in \mathbb{R}$. Then γ^x is a geodesic in $(\mathbb{R} \times F, h)$. Put $Y_w(s) = \partial_u \gamma^{\alpha(u)}(s)|_{u=0}$, where α is a curve $\alpha : (-\epsilon, \epsilon) \to F$ such that $\alpha(0) = x$, and that $\dot{\alpha}(0) = w$. We see that Y_w is a Jacobi field along the geodesic γ^x such that $h(Y_w(s), Y_w(s)) = g_s(w, w)$ and that $\nabla_{\partial/\partial s} Y_w(s)$ is space-like. We have $\partial_s^2 h(Y_w(s), Y_w(s)) \geq d^2 h(Y_w(s), Y_w(s))$ by the curvature condition, where d is the constant appearing in Proposition 4.1. Since F is totally geodesic, $h(Y_w(0), \nabla_{\partial/\partial s} Y_w(0)) = 0$. We obtain

$$\frac{\partial_{|s|} g_s(w, w)}{g_s(w, w)} = \frac{\partial_{|s|} h(Y_w(s), Y_w(s))}{h(Y_w(s), Y_w(s))} \geq |d \tanh(d\, s)|. \tag{11}$$

The lemma is proved. □

Remark 4.2. The inequality (11) indicates that $\{g_t\}_{t\in\mathbb{R}}$ satisfies $(\mathrm{H})_{t_0,c}$. Therefore we can apply Theorem 1.3 and Proposition 4.1 follows.

From now on, we give another simple proof of Proposition 4.1 without Theorem 1.3. Take any isometry ϕ of M and $T > 0$. Suppose that $\phi(F) \cap K_T = \emptyset$, where $K_T = \pi_{\mathbb{R}}^{-1}([-T, T])$. It suffices to consider the case where $\phi(F) \subset \pi_{\mathbb{R}}^{-1}((T, \infty))$. For any $p \in \phi(F)$, let $\{e_i(p)\}_{i=1}^n$ be an orthonormal basis of $T_p\phi(F)$. As $\phi(F)$ is totally geodesic, we have

$$\int_{\phi(F)} \Delta_{\phi(F)}(\pi_{\mathbb{R}}|_{\phi(F)}(p))\, dp = \int_{\phi(F)} \sum_{i=1}^n (\operatorname{Hess} \pi_{\mathbb{R}})(e_i(p), e_i(p))\, dp, \tag{12}$$

where $\operatorname{Hess} \pi_{\mathbb{R}}$ is the Hessian of $\pi_{\mathbb{R}}$ on $(\mathbb{R} \times F, h)$, and $\Delta_{\phi(F)}$ is the Laplacian on $\phi(F)$. By Lemma 4.7 the right-hand side of the equality (12) is negative. The divergence theorem implies that the left side of the equality (12) is zero. This is a contradiction. Hence for any isometry ϕ of M we have $\phi(F) \cap K_T \neq \emptyset$. The proof of Proposition 4.1 is complete.

4.4. Construction of an inhomogeneous example. We present an inhomogeneous example by modifying the Lorentzian metric of the de Sitter space. Let n be a positive integer. We denote by Q the hypersurface $\{(x_0, x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+2} | -x_0^2 + x_1^2 + \cdots + x_{n+1}^2 = 1\}$. The de Sitter space $\mathbb{S}_1^{n+1}$ is the hypersurface Q with the metric induced from the Minkowski space $\mathbb{R}_1^{n+2} = (\mathbb{R}^{n+2}, B_1)$, where B_1 is the bilinear form defined in Section 2. Take a basis $n_1, n_2, e_1, e_2, \cdots, e_n$ of the Minkowski space such that $B_1(n_1, n_1) = 0$, $B_1(n_2, n_2) = 0$, $B_1(n_1, n_2) = -2$, and that $\{e_i\}_{i=1}^n$ is an orthonormal basis of the vector subspace $\{x \in \mathbb{R}^{n+2} | B_1(x, n_1) = B_1(x, n_2) = 0\}$. We denote by Q^+ the open set $\{x \in \mathbb{S}_1^{n+1} | B_1(x, n_1) > 0\}$ of Q. Then the map $\chi : \mathbb{R}^{n+1} \to Q^+$ is given by

$$\chi(t, v_1, \cdots, v_n) = \frac{e^t - e^{-t}(v_1^2 + \cdots + v_n^2)}{2} n_1 - \frac{e^{-t}}{2} n_2 + e^{-t} \sum_{i=1}^n v_i e_i.$$

We see that χ is a diffeomorphism, and that $\chi^*(g_{\mathbb{S}_1^{n+1}}|_{Q^+})(t, v_1, \cdots, v_n) = -dt^2 + e^{-2t}(dv_1^2 + \cdots + dv_n^2)$, where $g_{\mathbb{S}_1^{n+1}}$ is the Lorentzian metric of the de Sitter space.

Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth function such that $\eta(t) = 0$ if $t > -2$, and that $\eta(t) = 1$ if $t < -3$. For any positive number ϵ, we write X_ϵ^+ for the Lorentzian manifold $(Q^+, (\chi^{-1})^*(e^{-\epsilon\eta(t)}(-dt^2 + e^{-2t}(dv_1^2 + \cdots + dv_n^2))))$. The metric of X_ϵ^+ is denoted by $g_{X_\epsilon^+}$. The Lorentzian metric g on Q is given by $g(p) = g_{X_\epsilon^+}(p)$ if $p \in Q^+$, and by $g(p) = g_{\mathbb{S}_1^{n+1}}(p)$ if $p \in Q - Q^+$. Then g is a smooth Lorentzian metric of Q. Hence X_ϵ stands for the Lorentzian manifold (Q, g).

If $\epsilon > 0$ is sufficiently small, the sectional curvature of X_ϵ is bounded from below by a positive constant since the de Sitter space has positive constant curvature. $X_\epsilon - X_\epsilon^+$ has constant curvature, but X_ϵ^+ not. It follows that X_ϵ is inhomogeneous.

From what follows, we check that X_ϵ satisfies the assumptions of Proposition 4.1. We should remark that the metric of X_ϵ is conformally equivalent to the one of the de Sitter space. If a curve in X_ϵ is timelike, so is it in the de Sitter space. Therefore the Cauchy hypersurfaces in the de Sitter space are the ones in X_ϵ. It follows that X_ϵ is globally hyperbolic. For any $\delta > 0$, we define the submanifold $\mathbb{S}_\delta = \{(x_0, x_1, \cdots, x_{n+1}) \in Q||x_0| < \delta\}$ in X_ϵ. If $\delta < (e^2 - e^{-2})/2$, $\mathbb{S}_\delta$ is represented as a warped product $(\sinh^{-1}((-\delta, \delta)) \times \mathbb{S}^n, -dt^2 + \cosh^2(t) g_{\mathbb{S}^n})$ in X_ϵ, where $g_{\mathbb{S}^n}$ is the standard metric of the n-dimensional sphere $\mathbb{S}^n$. Then it is easy to check that $\{0\} \times \mathbb{S}^n$ is totally geodesic. We see that $I_{\{0\}\times\mathbb{S}^n}(p)$ satisfies the conditions (i) and (ii) of Proposition 4.1 for any $p \in X_\epsilon - (\{0\} \times \mathbb{S}^n)$.

We denote by $\mathrm{Lor}(M)$ the set of all Lorentzian metrics on M. We can define the C^1 *topology* on $\mathrm{Lor}(M)$ (see for instance [**3**, Section 3.2]).

THEOREM 4.1 (Beem–Ehrich [**2**, Corollary 3.8]). *Let (M, g) be a globally hyperbolic time-oriented Lorentzian manifold. If (M, g) is non-spacelike geodesically complete, there exists a C^1 neighborhood $U(g)$ of g in* $\mathrm{Lor}(M)$ *such that each $g_1 \in U(g)$ is non-spacelike complete.*

From Theorem 4.1, X_ϵ is non-spacelike geodesically complete if ϵ is sufficiently small. By using Proposition 4.1, the Calabi–Markus phenomenon occurs in X_ϵ.

Finally we note that the isometry group of X_ϵ is not compact. For any $u \in \mathbb{R}$ and any $i = 1, 2, \cdots, n$, the linear transformation ϕ_u^i of $\mathbb{R}_1^{n+2}$ is defined by

$$\begin{aligned}
&\phi_u^i(n_1) = n_1;\\
&\phi_u^i(n_2) = n_2 + 2ue_i + u^2 n_1;\\
&\phi_u^i(e_i) = e_i + un_1;\\
&\phi_u^i(e_j) = e_j \quad if\ j \neq i.
\end{aligned}$$

We have $\phi_u^i(Q) = Q$ for any $u \in \mathbb{R}$. Since

$$\chi^{-1} \circ \phi_u^i \circ \chi(t, v_0, v_1, \cdots, v_n) = (t, v_1, \cdots, v_{i-1}, v_i - u, v_{i+1}, \cdots, v_n),$$

the action restricted to X_ϵ of ϕ_u^i is isometric. Therefore $\{\phi_u^i\}_{u\in\mathbb{R}}$ is a one-parameter non-compact subgroup in the isometry group of X_ϵ for any i.

5. Globally hyperbolic manifolds under some Ricci curvature condition

In this section, we will prove the finiteness of the fundamental group on a certain class of globally hyperbolic Lorentzian manifolds. Let (M, g) be a Lorentzian manifold. For any point $p \in M$, we take a tangent vector u of T_pM. The curvature operator $R_u : T_pM \to T_pM$ is defined by $R_u(X) = R(X, u)u$ for $X \in T_pM$. The *Ricci tensor* $\mathrm{Ric}(u, u)$ is the trace of the operator R_u. For any $p \in M$, we denote by $T_p^- M$ the set of timelike tangent vectors in T_pM. Put $T^-M = \bigsqcup_{p\in M} T_p^- M$. If a Lorentzian manifold (M, g) is time-oriented, T^-M has two connected components.

We call one connected component of $T^{-}M$ the *future*, and the other connected component the *past*. A smooth curve in M is said to be *future-directed timelike* (resp. *past-directed timelike*) if the velocity of the curve is timelike, and belongs to the future (resp. the past). A Lorentzian manifold (M, g) is lightlike geodesically complete if any inextendible lightlike geodesic is defined on the real line. We prove the following theorem in this section:

THEOREM 1.5. *Let (M, g) be a lightlike geodesically complete globally hyperbolic Lorentzian manifold of dimension $n > 2$. Assume that there exist a positive constant Q, a point $p \in M$, and a timelike tangent vector $T \in T_pM$ satisfying that, for any $v \in L_p(T)$ and any $s \in \mathbb{R}$, the Ricci tensor $\mathrm{Ric}(d\exp(sv)/ds,\, d\exp(sv)/ds)$ has a positive lower bound $(n-2)Q^2$. Then a Cauchy hypersurface of M is compact, and its fundamental group is finite.*

Let us introduce the outline of our proof of Theorem 1.5. First we apply the argument of the proof of Penrose's singularity theorem. Namely we show that a Cauchy hypersurface is compact by using the causal theory and the theory of Jacobi fields. Next we consider the universal covering space $\widetilde{M}$. Since $\widetilde{M}$ also satisfies the assumptions of Theorem 1.5, its Cauchy hypersurfaces are compact. Note that the action of $\pi_1(M)$ on $\widetilde{M}$ never destroys the causal relation of $\widetilde{M}$. This leads us to the finiteness of the fundamental group $\pi_1(M)$.

5.1. Preliminaries. In this subsection, we recall some results to prove Theorem 1.5. The results and their proofs of this subsection are found in O'Neil [**23**], Galloway [**10**], or Harris [**12**]. We write their proofs in more detail than the above references. In what follows, (M, g) stands for a time-oriented Lorentzian manifold of dimension $n > 2$. For a smooth function $f(s)$, we denote by $f'(s)$ and by $f''(s)$ the first derivative $df(s)/ds$ and the second derivative $d^2f(s)/ds^2$, respectively.

Here we recall the fundamental notions on Lorentzian geometry. For $p, q \in M$, we write $p \ll q$ (resp. $p < q$) if there exists a piecewise smooth future-directed timelike (resp. causal) curve from p to q. The notation $p \leq q$ means that $p = q$ or $p < q$. For $p \in M$, the *chronological future* $I^{+}(p)$, the *chronological past* $I^{-}(p)$, the *causal future* $J^{+}(p)$, and the *causal past* $J^{-}(p)$ are defined by $I^{+}(p) = \{q \in M \,|\, p \ll q\}$, $I^{-}(p) = \{q \in M \,|\, q \ll p\}$, $J^{+}(p) = \{q \in M \,|\, p \leq q\}$, and $J^{-}(p) = \{q \in M \,|\, q \leq p\}$. The chronological future $I^{+}(S)$ and the causal future $J^{+}(S)$ of a subset $S \subset M$ are defined by $I^{+}(S) = \bigcup_{p\in S} I^{+}(p)$ and by $J^{+}(S) = \bigcup_{p\in S} J^{+}(p)$. The chronological past $I^{-}(S)$ and the causal past $J^{-}(S)$ of S are defined as well. Note that $I^{+}(S)$ is an open subset. A subset S is a *future set* (resp. *past set*) if $I^{+}(S) = S$ (resp. $I^{-}(S) = S$).

As an example, let $\mathbb{R}^n_1$ be an n-dimensional Minkowski space $(\mathbb{R}^n, -dx_1^2 + dx_2^2 + dx_3^2 + \cdots + dx_n^2)$. Then the chronological future $I^{+}(o)$ of the origin o is the future-directed timelike cone $\{v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n | -v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2 < 0,\, v_1 > 0\}$, and the causal future $J^{+}(o)$ of the origin o is equal to the closure of $I^{+}(o)$. In general, the causal set of a subset S is included in the closure of the chronological set of S. Moreover, under the assumption of globally hyperbolicity, the following proposition holds:

PROPOSITION 5.1 (O'Neil [**23**, p.412], Galloway [**10**, Proposition 2.6]). *Let M be a globally hyperbolic Lorentzian manifold. Then $J^{+}(p)$ (resp. $J^{-}(p)$) is the closure of $I^{+}(p)$ (resp. $I^{-}(p)$) for any point $p \in M$.*

Therefore, if M is globally hyperbolic, we have $\partial I^+(p) = J^+(p) - I^+(p)$ and $\partial I^-(p) = J^-(p) - I^-(p)$ for any point $p \in M$.

A subset $A \subset M$ is *achronal* if $p \not\ll q$ for any $p,\, q \in A$. For example a Cauchy hypersurface is achronal. Note that an open subset U in a time-oriented Lorentzian manifold M has time-orientation induced by M. For any subset A of U, we denote by $I^+(A,\, U)$ (resp. $I^-(A,\, U)$) the chronological future (resp. past) of A in U. A point p of the closure of A is an *edge point* of A if for any open neighborhood U of p, there exists a timelike curve $\gamma \subset U$ from one point of $I^-(p,\, U)$ to another point of $I^+(p,\, U)$ such that $\gamma \cap A = \emptyset$. The following proposition gives an answer to the question of when an achronal set is a topological hypersurface:

Proposition 5.2 (O'Neil [**23**, Chapter 14, Proposition 25]). *Let a subset A in M be achronal. Then A is a topological hypersurface if and only if A and the set of the edge points of A do not intersect.*

Proof. Assume that A is a topological hypersurface. Take any point p of A. Then there exists a connected open neighborhood U of $p \in A$ such that the coordinate map $\phi : U \to \mathbb{R}^n$ satisfies that $\phi(A \cap U)$ is included in a hyperplane, and that $U - A$ has two connected components. Since A is achronal, we see that $I^+(p,\, U) \cap A = \emptyset$, and that $I^-(p,\, U) \cap A = \emptyset$. Take a causal curve γ from an element of $I^+(p,\, U)$ to another element of $I^-(p,\, U)$. Then γ meets A. Hence p is not an edge of A.

Assume that A and the set of the edge points of A do not intersect. Take any point $p \in A$. Then there exists an open neighborhood U of p such that the coordinate map $\phi : U \to \mathbb{R}^n$, and that $\partial/\partial x_1$ is timelike, where we set $\phi = (x_1,\, x_2,\, \cdots,\, x_n)$. There exist positive numbers $a,\, b,\, \delta$, a ball N in $\mathbb{R}^{n-1}$, and a small neighborhood $V \subset U$ satisfying the following conditions:

(1) $\phi(V) = (a - \delta,\, b + \delta) \times N \subset \mathbb{R} \times \mathbb{R}^{n-1}$;
(2) the slice $x_0 = a$ in V is in $I^-(p,\, U)$;
(3) the slice $x_0 = b$ in V is in $I^+(p,\, U)$.

Take any point $y \in N$. The curve $c_y : [a,\, b] \to M$ is given by $c_y(s) = \phi^{-1}(s,\, y)$. Since p is not an edge point of A, the curve c_y and the subset A meet. The achronality of A implies that the intersection of c_y and A is a single point. We denote the point by $(h(y),\, y)$, where h is a function on N into $(a,\, b)$.

It is sufficient to prove that $h : N \to (a,\, b)$ is continuous. To obtain a contradiction, we suppose that there exists a sequence $\{y_k\}_{k=1}^{\infty}$ in N converging to $y \in N$ such that $h(y_k)$ does not tend to $h(y)$. As $[a,\, b]$ is compact, there exists the subsequence $\{y_{k_i}\}_{i=1}^{\infty}$ such that $h(y_{k_i})$ approaches to a point $r \neq h(y)$. Put $q = c_y(h(y)) \in A$. Then $c_y(r) \in I^+(q,\, U) \cup I^-(q,\, U)$. From the definition of h, it follows that $c_{y_{k_i}}(h(y_{k_i}))$ is included in A. Since $c_{y_{k_i}}(h(y_{k_i}))$ converges to $c_y(r)$, we see that $c_{y_{k_i}}(h(y_{k_i}))$ is included in $I^+(q,\, U) \cup I^-(q,\, U)$ for sufficiently large i. This contradicts the achronality of A. $\square$

Then the following corollary holds:

Corollary 5.1 (O'Neil [**23**, Chapter 14, Corollary 27], Galloway [**10**, Proposition 2. 20]). *Each boundary of the future set and the past set is a closed achronal topological hypersurface.*

Proof. We give the proof for the case of a future set since the same proof works in the case of a past set as well. We denote by F a future set. First we prove that ∂F is achronal. Take any point p of the boundary ∂F. Let q be any point of the chronological future $I^+(p)$. As $I^-(q)$ is a neighborhood of p, there exists a point $r \in F \cap I^-(q)$. Since F is a future set and $r \ll q$, we see that $q \in F$. Hence $I^+(p) \subset F$. Moreover the same argument implies that $I^-(p) \subset M - \overline{F}$, where $\overline{F}$ is the closure of F. Then it is clear that $I^+(\partial F) \subset F$, and that $I^-(\partial F) \subset M - \overline{F}$. It follows that F is achronal. Next we show that ∂F is edgeless. Assume that there exists an edge point p of ∂F. As $p \in \partial F$, by the above argument, we know that $I^+(p) \subset F$, and that $I^-(p) \subset M - \overline{F}$. Then there exists no timelike curve from a point of F to a point of $M - \overline{F}$ without meeting the boundary ∂F. This is a contradiction. Hence the corollary follows from Proposition 5.2. □

Let S be a Cauchy hypersurface in a Lorentzian manifold M. We will check that a Cauchy hypersurface S is a topological hypersurface. Since S is a Cauchy hypersurface, M is the disjoint union of the open sets $I^+(S)$, $I^-(S)$, and the Cauchy hypersurface S. Hence each boundary of $I^+(S)$ and $I^-(S)$ are S. Corollary 5.1 implies that S is a topological hypersurface.

We investigate $\partial I^+(p) = J^+(p) - I^+(p)$ for a point $p \in M$. We call a curve a *pre-geodesic* if the curve admits a reparametrization as a geodesic. For simplicity of notation, $\langle -, - \rangle$ stands for a metric $g(-, -)$. The following lemma is useful to obtain a timelike curve:

LEMMA 5.1 (O'Neil [**23**, Chapter 10, Lemma 45]). *Take a causal curve $\gamma(s)$ in M and a vector field $V(s)$ along the curve $\gamma(s)$. Let $\alpha(t, s)$ be the variation of $\gamma(s)$ with the variation vector field $V(s)$. Assume that $\langle \nabla_{\partial/\partial s} V(s), \dot\gamma(s) \rangle$ is negative. Then for any s and any small positive number t, the velocity $\partial\alpha(t, s)/\partial s$ is timelike.*

PROOF. Let $O(-)$ be the Landau symbol. Then for any small $t > 0$, by using Taylor's theorem, we have

$$\begin{aligned}\left\langle \frac{\partial}{\partial s}\alpha(t, s), \frac{\partial}{\partial s}\alpha(t, s) \right\rangle &= \left\langle \frac{\partial}{\partial s}\alpha(0, s), \frac{\partial}{\partial s}\alpha(0, s) \right\rangle \\ &\quad + t\frac{\partial}{\partial t}\Big|_{t=0} \left\langle \frac{\partial}{\partial s}\alpha(t, s), \frac{\partial}{\partial s}\alpha(t, s) \right\rangle + O(t^2) \\ &= \langle \dot\gamma(s), \dot\gamma(s) \rangle + 2t \left\langle \nabla_{\partial/\partial s} V(s), \dot\gamma(s) \right\rangle + O(t^2).\end{aligned}$$

We see that $\langle \partial\alpha(t, s)/\partial s, \partial\alpha(t, s)/\partial s \rangle < 0$ for sufficiently small $t > 0$ since $\langle \dot\gamma(s), \dot\gamma(s) \rangle \leq 0$ and $\left\langle \nabla_{\partial/\partial s} V(s), \dot\gamma(s) \right\rangle < 0$. □

We prove the following proposition:

PROPOSITION 5.3 (O'Neil [**23**, Chapter 10, Proposition 46]). *Let (M, g) be a Lorentzian manifold. Take a causal curve γ from a point p to another point q which is not a lightlike pre-geodesic. Then there exists a timelike curve from p to q.*

PROOF. Take a causal curve $\gamma : [0, 1] \to M$. We will deform the curve $\gamma(s)$ to a timelike curve with the endpoints fixed. First, we consider two cases.

Case 1: Assume that $\dot\gamma(0)$ or $\dot\gamma(1)$ is timelike. Here we treat only the case $\dot\gamma(1)$ is timelike. This is because the other case can be proved similarly. The vector field $W(s)$ along the curve $\gamma(s)$ is defined by the parallel translation of $\dot\gamma(1)$ along the curve $\gamma(s)$. Then we should remark that $W(s)$ and $\dot\gamma(s)$ are in the same causal cone in the tangent space at $\gamma(s)$. Since $\dot\gamma(1)$ is timelike, there exists $\delta > 0$ such that

$\langle \dot\gamma(s), \dot\gamma(s)\rangle < -\delta$ on the closed interval $[1-\delta, 1]$. Take a smooth function $f(s)$ on $[0, 1]$ satisfying that $f(0) = f(1) = 0$, and that the derivative $f'(s)$ is positive on $[0, 1-\delta]$. We define a vector field $V(s)$ by $V(s) = f(s)W(s)$. Let $\alpha(t, s)$ be a variation of $\gamma(s)$ with the variation vector field $V(s)$. We note that for fixed t, the curve $\alpha(t, s)$ connects $\gamma(0)$ to $\gamma(1)$.

Take any number s in the interval $[0, 1-\delta]$. As $W(s)$ and $\dot\gamma(s)$ have the same causal direction, $\langle W(s), \dot\gamma(s)\rangle$ is negative. Since $f'(s)$ is positive, we have $\left\langle \nabla_{\partial/\partial s}V(s), \dot\gamma(s)\right\rangle = f'(s)\langle W(s), \dot\gamma(s)\rangle < 0$. Therefore for sufficiently small $t > 0$, by Taylor's theorem, there exists a positive number C such that

$$\left\langle \frac{\partial}{\partial s}\alpha(t, s), \frac{\partial}{\partial s}\alpha(t, s)\right\rangle \le \langle \dot\gamma(s), \dot\gamma(s)\rangle - Ct.$$

Therefore for smaller $t > 0$, we see that $\left\langle \frac{\partial}{\partial s}\alpha(t, s), \frac{\partial}{\partial s}\alpha(t, s)\right\rangle$ is negative.

Take any number s in the interval $[1-\delta, 1]$. Then we have

$$\langle \dot\gamma(s), \dot\gamma(s)\rangle < -\delta.$$

Here we see that $\left\langle \nabla_{\partial/\partial s}V(s), \dot\gamma(s)\right\rangle = f'(s)\langle W(s), \dot\gamma(s)\rangle$ is bounded above by a constant. Therefore, by Taylor's theorem, for sufficiently small $t > 0$, there exists a constant D such that

$$\left\langle \frac{\partial}{\partial s}\alpha(t, s), \frac{\partial}{\partial s}\alpha(t, s)\right\rangle \le -\delta + Dt.$$

For smaller $t > 0$, it follows that $\left\langle \frac{\partial}{\partial s}\alpha(t, s), \frac{\partial}{\partial s}\alpha(t, s)\right\rangle$ is negative. The proof of Case 1 is complete.

Case 2: Assume that $\gamma(s)$ is a smooth lightlike curve. Since $\dot\gamma(s)$ is lightlike, $\langle \nabla_{\partial/\partial s}\dot\gamma(s), \dot\gamma(s)\rangle = 0$. Moreover it follows that $\langle \nabla_{\partial/\partial s}\dot\gamma(s), \nabla_{\partial/\partial s}\dot\gamma(s)\rangle \ge 0$. We note that $\nabla_{\partial/\partial s}\dot\gamma(s)$ is not always lightlike unless $\gamma(s)$ is a pre-geodesic.

It is sufficient to prove that there exists a vector field $V(s)$ along the curve $\gamma(s)$ such that $\langle \nabla_{\partial/\partial s}V(s), \dot\gamma(s)\rangle < 0$. Let $W(s)$ be a parallel timelike vector field along the curve $\gamma(s)$. Put $V(s) = f(s)W(s) + g(s)\nabla_{\partial/\partial s}\dot\gamma(s)$, where $f(s)$ and $g(s)$ are the functions defined later. Then we have

$$\begin{aligned}\langle \nabla_{\partial/\partial s}V(s), \dot\gamma(s)\rangle =&\langle f'(s)W(s) + g'(s)\nabla_{\partial/\partial s}\dot\gamma(s) + g(s)\nabla_{\partial/\partial s}\nabla_{\partial/\partial s}\dot\gamma(s), \dot\gamma(s)\rangle \\ =&f'(s)\langle W(s), \dot\gamma(s)\rangle + g'(s)\langle \nabla_{\partial/\partial s}\dot\gamma(s), \dot\gamma(s)\rangle \\ &+ g(s)\langle \nabla_{\partial/\partial s}\nabla_{\partial/\partial s}\dot\gamma(s), \dot\gamma(s)\rangle \\ =&f'(s)\langle W(s), \dot\gamma(s)\rangle - g(s)\langle \nabla_{\partial/\partial s}\dot\gamma(s), \nabla_{\partial/\partial s}\dot\gamma(s)\rangle.\end{aligned}$$

Here the function $h(s)$ is given by

$$h(s) = \frac{\langle \nabla_{\partial/\partial s}\dot\gamma(s), \nabla_{\partial/\partial s}\dot\gamma(s)\rangle}{\langle W(s), \dot\gamma(s)\rangle}.$$

Note that $h(s)$ is not identically zero. Then we can define a smooth function $g(s)$ satisfying that

$$\int_0^1 g(s)h(s)ds = -1.$$

Moreover the function $f(s)$ is given by

$$f(s) = \int_0^s (g(u)h(u) + 1)du.$$

Then it follows that

$$f'(s) > g(s)h(s) = \frac{g(s)\langle \nabla_{\partial/\partial s}\dot{\gamma}(s),\, \nabla_{\partial/\partial s}\dot{\gamma}(s)\rangle}{\langle W(s),\, \dot{\gamma}(s)\rangle}.$$

Therefore we obtain $\langle \nabla_{\partial/\partial s}V(s),\, \dot{\gamma}(s)\rangle < 0$. Applying Lemma 5.1, we complete the proof of Case 2.

We consider the remaining cases. Assume that there exists $s_0 \in (0,\, 1)$ such that $\dot{\gamma}(s_0)$ is timelike. We apply the argument of Case 1 on $[0,\, s_0]$ and $[s_0,\, 1]$ to this case. Then we obtain a desired piecewise timelike curve.

Assume that $\gamma(s)$ is a piecewise lightlike curve. Using the proof of Case 2, we are reduced to the case where $\gamma(s)$ is a piecewise lightlike geodesic. Suppose that there exists only one break point $s_0 \in (0,\, 1)$ of the geodesic $\gamma(s)$. The tangent vector $\triangle\dot{\gamma}(s_0)$ at $\gamma(s_0)$ is given by

$$\triangle\dot{\gamma}(s_0) = \lim_{s \searrow s_0} \dot{\gamma}(s) - \lim_{s \nearrow s_0} \dot{\gamma}(s).$$

We define the vector field $W(s)$ along the curve $\gamma(s)$ by the parallel translation of $\triangle\dot{\gamma}(s_0)$ along the curve $\gamma(s)$. Take any point $s \in [0,\, s_0)$. We have

$$\begin{aligned}\langle W(s),\, \dot{\gamma}(s)\rangle &= \langle \triangle\dot{\gamma}(s_0),\, \lim_{s \nearrow s_0} \dot{\gamma}(s)\rangle \\ &= \langle \lim_{s \searrow s_0} \dot{\gamma}(s),\, \lim_{s \nearrow s_0} \dot{\gamma}(s)\rangle \\ &< 0.\end{aligned}$$

Let s be in $(s_0,\, 1]$. Then we similarly see that

$$\langle W(s),\, \dot{\gamma}(s)\rangle = -\langle \lim_{s \searrow s_0} \dot{\gamma}(s),\, \lim_{s \nearrow s_0} \dot{\gamma}(s)\rangle > 0.$$

We define a piecewise smooth function $f(s)$ with the break point s_0 by imposing that $f(0) = f(1) = 0$, $f'(s) > 0$ on $s \in [0,\, s_0)$, and $f'(s) < 0$ on $s \in (s_0,\, 1]$. Put $V(s) = f(s)W(s)$. Then we have $\langle \nabla_{\partial/\partial s}V(s),\, \dot{\gamma}(s)\rangle < 0$ for any $s \in [0,\, 1] - \{s_0\}$. From Lemma 5.1, we can join $\gamma(0)$ to $\gamma(1)$ by a piecewise timelike curve. Hence the proof of this case follows from Case 1. Moreover, by induction, we can give a proof for the case with many break points. We have proved Proposition 5.3. $\square$

Next we investigate the case where the curve $\gamma(s)$ is a geodesic with a conjugate point.

PROPOSITION 5.4 (O'Neil [**23**, Chapter 10, Proposition 48]). *Let $\gamma : [0,\, L] \to M$ be a lightlike geodesic from p to q. Assume that there exists a conjugate point of p along the geodesic $\gamma(s)$ strictly before q. Then, for any neighborhood U of $\gamma([0,\, L])$, there exists a timelike curve from p to q included in U.*

PROOF. There exists $r \in (0,\, L)$ such that $\gamma(r)$ is the first conjugate point of $\gamma(0)$. We will show that $\gamma|_{[0,\, r+\delta]}$ is deformed to a timelike curve with the endpoints fixed for some $\delta > 0$.

Let $J(s)$ be a non-zero Jacobi field along the geodesic $\gamma(s)$ with $J(0) = J(r) = 0$. Then there exist a vector field $Y(s)$ on $\gamma(s)$ and a positive number $\delta > 0$ such that $J(s) = s(r - s)Y(s)$, and that $Y(s) \neq 0$ for any $s \in [0,\, r + \delta]$. The spacelike vector field $U(s)$ along the geodesic $\gamma(s)$ is defined by $U(s) = Y(s)/\sqrt{\langle Y(s),\, Y(s)\rangle}$. Put $f(s) = s(r - s)\sqrt{\langle Y(s),\, Y(s)\rangle}$. Then it follows that $J(s) = f(s)U(s)$ on $[0,\, r + \delta]$.

For a smooth function $g(s)$ to be defined later, we set

$$V(s) = (f(s) + g(s))U(s) = J(s) + g(s)U(s),$$

for $s \in [0, r + \delta]$. Then we have

$$\begin{aligned}
&\nabla_{\partial/\partial s}\nabla_{\partial/\partial s}V(s) - R(V(s), \dot{\gamma}(s))\dot{\gamma}(s) \\
&= \nabla_{\partial/\partial s}\nabla_{\partial/\partial s}(g(s)U(s)) - g(s)R(U(s), \dot{\gamma}(s))\dot{\gamma}(s) \\
&= g''(s)U(s) + 2g'(s)\nabla_{\partial/\partial s}U(s) + g(s)(\nabla_{\partial/\partial s}\nabla_{\partial/\partial s}U(s) - R(U(s), \dot{\gamma}(s))\dot{\gamma}(s)).
\end{aligned}$$

Since $U(s)$ is a unit spacelike vector field, we have

$$\langle \nabla_{\partial/\partial s}\nabla_{\partial/\partial s}V(s) - R(V(s), \dot{\gamma}(s))\dot{\gamma}(s), V(s)\rangle = (g(s) + f(s))(g''(s) + g(s)h(s)),$$

where $h(s) = \langle \nabla_{\partial/\partial s}\nabla_{\partial/\partial s}U(s) - R(U(s), \dot{\gamma}(s))\dot{\gamma}(s), U(s)\rangle$. As $[0, r+\delta]$ is compact, there exists $a > 0$ such that the continuous function $h(s)$ on $[0, r + \delta]$ is bounded from below by $-a^2$. Here we set $g(s) = b(e^{as} - 1)$, where $b > 0$ satisfies that $b(e^{a(r+\delta)} - 1) = -f(r+\delta)$. Then we have $g''(s) + g(s)h(s) = g(s)(a^2 + h(s)) + a^2 b > 0$. Furthermore $(f(s) + g(s))$ is positive on $(0, r]$, and $f(r + \delta) + g(r + \delta) = 0$. Taking $r + \delta$ as the first solution after r of the equation $f(s) + g(s) = 0$, we can obtain $f(s) + g(s) > 0$ for $s \in (0, r + \delta)$. We have proved that the vector field $V(s)$ satisfies the following conditions:

- $V(s)$ vanishes at 0 and $r + \delta$;
- $\langle V(s), \dot{\gamma}(s)\rangle = 0$ for any s;
- $\langle \nabla_{\partial/\partial s}\nabla_{\partial/\partial s}V(s) - R(V(s), \dot{\gamma}(s))\dot{\gamma}(s), V(s)\rangle > 0$ for any $s \in (0, r + \delta)$.

Let $N(s)$ be a parallel lightlike vector field on the geodesic $\gamma(s)$ such that $\langle N(s), \dot{\gamma}(s)\rangle = -1$. We can construct a variation $\alpha(t, s)$ such that

$$\partial\alpha(t, s)/\partial t|_{t=0} = V(s),\ \nabla_{\partial/\partial t}\partial\alpha(t, s)/\partial t|_{t=0} = \langle \nabla_{\partial/\partial s}V(s), V(s)\rangle N(s).$$

The function $e(t, s)$ is defined by

$$e(t, s) = \left\langle \frac{\partial\alpha(t, s)}{\partial s}, \frac{\partial\alpha(t, s)}{\partial s} \right\rangle.$$

Then we have

$$\begin{aligned}
\frac{1}{2}\frac{\partial e(t, s)}{\partial t}\Big|_{t=0} &= -\langle V(s), \nabla_{\partial/\partial s}\dot{\gamma}(s)\rangle + \frac{d}{ds}\langle V(s), \dot{\gamma}(s)\rangle, \\
\frac{1}{2}\frac{\partial^2 e(t, s)}{\partial t^2}\Big|_{t=0} &= -\langle \nabla_{\partial/\partial s}\nabla_{\partial/\partial s}V(s) - R(V(s), \dot{\gamma}(s))\dot{\gamma}(s), V(s)\rangle \\
&\quad + \langle \nabla_{\partial/\partial s}\nabla_{\partial/\partial t}\partial\alpha(t, s)/\partial t|_{t=0}, \dot{\gamma}(s)\rangle + \frac{d}{ds}\langle \nabla_{\partial/\partial s}V(s), V(s)\rangle.
\end{aligned}$$

As $\langle V(s), \dot{\gamma}(s)\rangle = 0$, we see that $\partial e(t, s)/\partial t|_{t=0} = 0$. We have

$$\begin{aligned}
\langle \nabla_{\partial/\partial s}\nabla_{\partial/\partial t}\partial\alpha(t, s)/\partial t|_{t=0}, \dot{\gamma}(s)\rangle &= \langle N(s), \dot{\gamma}(s)\rangle \frac{d}{ds}\langle \nabla_{\partial/\partial s}V(s), V(s)\rangle \\
&= -\frac{d}{ds}\langle \nabla_{\partial/\partial s}V(s), V(s)\rangle.
\end{aligned}$$

Therefore we obtain $\partial^2 e(t, s)/\partial t^2|_{t=0} < 0$. For small fixed t, the curve $\alpha(t, s)$ is timelike, and connects $\gamma(0)$ to $\gamma(r + \delta)$. Since the the resulting curve $\alpha(t, s)$ is a causal curve with a break point, we apply Proposition 5.3 to the curve. The proof of Proposition 5.4 is complete. □

Now Proposition 5.3 and Proposition 5.4 give the following result:

Theorem 5.1 (O'Neil [**23**, Chapter 10, Theorem 51]). *If $p < q$ and $p \not\ll q$, then there exists a lightlike geodesic from p to q with no conjugate point of p strictly before q.*

We consider conjugate points along a lightlike geodesic. The *index form I* is given by

$$I(X, Y) = \int_0^L \langle \nabla_{\partial/\partial s} X(s), \nabla_{\partial/\partial s} Y(s)\rangle - \langle R(X(s), \dot{\gamma}(s))\dot{\gamma}(s), Y(s)\rangle ds,$$

for any vector field $X(s)$, $Y(s)$ along a geodesic $\gamma : [0, L] \to M$. The following lemma holds:

Lemma 5.2 (Harris [**12**, Null Index Lemma 2.2]). *Assume that a lightlike geodesic $\gamma : [0, L] \to M$ has no conjugate point to $\gamma(0)$. Take a smooth vector field $V(s)$ along the geodesic $\gamma(s)$ with $\langle V(s), \dot{\gamma}(s)\rangle = 0$. Let $J(s)$ be a Jacobi field along the geodesic $\gamma(s)$ satisfying that $J(0) = 0$, that $J(L) = V(L)$, and that $\langle J(s), \dot{\gamma}(s)\rangle = 0$ for $s > 0$. Then $I(V, V) \geq I(J, J)$. Moreover if the equality holds, $V(s) - J(s)$ is parallel to $\dot{\gamma}(s)$ or 0.*

Proof. Denote by E the space of Jacobi fields $J(s)$ such that $J(0) = 0$, and that $\langle J(s), \dot{\gamma}(s)\rangle = 0$ for $s > 0$. Note that the Jacobi field $s\dot{\gamma}(s)$ belongs to E since $\dot{\gamma}(s)$ is lightlike. Let $\{J_i(s)\}_{i=1}^{n-1}$ be the basis of E such that $J_1(s) = s\dot{\gamma}(s)$, where n is the dimension of M. As there exists no conjugate point of $\gamma(0)$, we see that $\{J_i(s)\}_{i=1}^{n-1}$ is linearly independent except at $s = 0$. Therefore there exist some smooth functions $f_1(s), f_2(s), \cdots, f_{n-1}(s)$ such that $V(s) = \sum_{i=1}^{n-1} f_i(s)J_i(s)$. Note that $J(s) = \sum_{i=1}^{n-1} f_i(L)J_i(s)$. Then the following formula holds:

$$I(V, V) = I(J, J) + \int_0^L \langle A(s), A(s)\rangle ds,$$

where $A(s) = \sum_{i=1}^{n-1} f_i'(s)J_i(s)$. For the proof of this formula, we refer the reader to Cheeger–Ebin [**8**, p. 25]. We should remark that $A(s)$ is lightlike or spacelike since $\langle A(s), \dot{\gamma}(s)\rangle = 0$ If the equality holds, $A(s)$ is lightlike or 0. If $A(s)$ is lightlike, $f_i'(s) = 0$ for $i \geq 2$. Hence $V(s) - J(s)$ is parallel to $\dot{\gamma}(s)$. If $A(s) = 0$, the equation $V(s) = J(s)$ holds. □

The following theorem holds under the curvature condition:

Theorem 5.2 (Harris [**12**, Proposition 2.6]). *Let $\gamma : [0, L] \to M$ be a lightlike geodesic. Assume that $Ric(\dot{\gamma}(s), \dot{\gamma}(s))$ is bounded from below by a positive constant $(n - 2)Q^2$. Then γ has a conjugate point if $L \geq \pi/Q$.*

Proof. Suppose that $L \geq \pi/Q$, and that there exists no conjugate point along $\gamma(s)$. Let W be the subspace of the tangent space at $\gamma(0)$ such that $\langle w, \dot{\gamma}(0)\rangle = 0$ for any $w \in W$. We should remark that $\dot{\gamma}(0) \in W$ since $\dot{\gamma}(0)$ is lightlike. Take an orthogonal basis $\{e_i\}_{i=1}^{n-1}$ of W such that $e_1 = \dot{\gamma}(0)$. We construct the orthogonal frame $\{E_i(s)\}_{i=1}^{n-1}$ by the parallel transporting the tangent vector e_i along the geodesic $\gamma(s)$. We define a vector field $V_i(s)$ along the geodesic $\gamma(s)$ by

$V_i(s) = \sin(\pi s/L)E_i(s)$. We have

$$I(V_i,\, V_i) = -\int_0^L \langle \nabla_{\partial/\partial s}\nabla_{\partial/\partial s}V_i(s) + R(V_i(s),\, \dot\gamma(s))\dot\gamma(s),\, V_i(s)\rangle ds$$
$$= \int_0^L \sin^2(\pi s/L)((\pi/L)^2 - \langle R(E_i(s),\, \dot\gamma(s))\dot\gamma(s),\, E_i(s)\rangle)ds.$$

Therefore we obtain

$$\begin{aligned}\sum_{i=2}^{n-1} I(V_i,\, V_i) &= \int_0^L \sin^2(\pi s/L)((n-2)(\pi/L)^2 - \mathrm{Ric}(\dot\gamma(s),\, \dot\gamma(s)))ds \\ &\leq (n-2)\{(\pi/L)^2 - Q^2\} \int_0^L \sin^2(\pi s/L)ds \\ &\leq 0.\end{aligned}$$

Then there exists an integer i such that $I(V_i,\, V_i) \leq 0$.

Suppose that there exists no conjugate point of $\gamma(0)$ along the geodesic $\gamma(s)$. By using Lemma 5.2, we see that $I(V_i,\, V_i) \geq 0$. Hence $I(V_i,\, V_i) = 0$. It follows that $V_i(s)$ is 0 or parallel to $\dot\gamma(s)$. However this is a contradiction by the construction of $V_i(s)$. Therefore there exists a conjugate point of $\gamma(0)$. □

5.2. Proof of Theorem 1.5. In this subsection, we prove Theorem 1.5. Recall that $L_p(T)$ is the set of lightlike tangent vectors v of T_pM with $\langle v,\, T\rangle = -1$. Since M is lightlike geodesically complete, for any point $q \in \partial I^+(p)$ there exists a geodesic ray γ_v with the initial velocity $v \in L_p(T)$ from p through q by Theorem 5.1. We have a positive constant Q such that $\mathrm{Ric}(\dot\gamma_v(s),\, \dot\gamma_v(s)) \geq (n-2)Q^2$ for any $v \in L_p(T)$ and any $s \geq 0$. By Theorem 5.2, the length of the parameter of a geodesic segment γ_v from p to q is less than or equal to π/Q. It follows that $\partial I^+(p)$ is bounded. Hence $\partial I^+(p)$ is compact. By the same argument, $\partial I^-(p)$ is also compact.

Let us prove that Cauchy hypersurfaces are compact. The following theorem is known:

Theorem 5.3 (Brouwer invariance of domain theorem, cf. Vick [**25**, Theorem 1.31]). *Let M and N be topological manifolds with the same dimension. Any injective and continuous map $f : M \to N$ is an open map.*

We have a natural projection onto a Cauchy hypersurface as follows:

Theorem 5.4 (O'Neil [**23**, Chapter 14, Proposition 31]). *Let S be a Cauchy hypersurface in M, and X a timelike vector field on M. For any $p \in M$, a maximal integral curve of X through p meets S at a unique point $\rho(p)$. Then $\rho : M \to S$ is a continuous open map onto S.*

Proof of Theorem 5.4. We present a proof given in O'Neil [**23**]. Let $\psi(t,\, p)$ be the flow of the vector field X such that the initial point $\psi(0,\, p) = p$ belongs to S. We denote by $D(S)$ the maximal domain of the flow $\psi(t,\, p)$ in $\mathbb{R} \times S$. Then we note that ψ is a continuous map on $D(S)$.

We will prove that ψ is injective. Suppose that $\psi(t_0,\, p_0) = \psi(t_1,\, p_1)$ for some $t_0,\, t_1 \in \mathbb{R}$, and $p_0,\, p_1 \in S$. Then we have $p_0 = \psi(t_1 - t_0,\, p_1)$. By the achronality, it follows that $t_0 = t_1$, and that $p_0 = p_1$. As S is a Cauchy hypersurface, we see that ψ is surjective. Since ψ is a continuous bijective map between the same dimensional topological manifolds, the Brouwer invariance of domain theorem implies that ψ is a homeomorphism. Define the natural projection $\pi_S : \mathbb{R} \times S \to S$ by $\pi_S(t, p) = p$. Then we set the map $\rho = \pi_S \circ \psi^{-1} : M \to S$. We check that ρ is a continuous open map onto S. $\square$

Take any Cauchy hypersurface S. We consider the map $\rho|_{\partial I^+(p)}$ restricted to $\partial I^+(p)$. Then $\rho|_{\partial I^+(p)}$ is injective by the achronality of $\partial I^+(p)$. Moreover we note that $\partial I^+(p)$ and S are topological hypersurfaces. By the Brouwer invariance of domain theorem, we see that the image $\rho(\partial I^+(p))$ is an open subset in S. The compactness of $\partial I^+(p)$ implies that the image $\rho(\partial I^+(p))$ is closed. Since S are connected, $\rho(\partial I^+(p)) = S$. Therefore the Cauchy hypersurface S is compact.

We will show that the fundamental group is finite. Let $\widetilde{M}$ be the universal covering space of M with the pull-back metric, and $\pi : \widetilde{M} \to M$ the the universal covering map. Then $\widetilde{M}$ satisfies the assumptions of Theorem 1.5. Take a point $\widetilde{p} \in \widetilde{M}$ with $\pi(\widetilde{p}) = p$. It follows that $\partial I^+(\widetilde{p})$, $\partial I^-(\widetilde{p})$, and Cauchy hypersurfaces of $\widetilde{M}$ are compact. Put $K = \widetilde{M} - (I^+(\widetilde{p}) \cup I^-(\widetilde{p}))$. We should remark that the action of the fundamental group $\pi_1(M)$ on $\widetilde{M}$ as deck transformations is to satisfy that $\gamma x \not\ll x$ for any point $x \in \widetilde{M}$ and any $\gamma \in \pi_1(M)$. This is because there is no closed timelike curve in M. Therefore $\gamma\widetilde{p} \in K$ for any $\gamma \in \pi_1(M)$. It is enough to prove that K is compact since the action of the fundamental group $\pi_1(M)$ is properly discontinuous. A timelike vector field gives the flow ψ on the maximal domain $D(S)$ including S as in the proof of Theorem 5.4. The functions $h_+, h_- : S \to \mathbb{R}$ are defined by $h_+ = \pi_{\mathbb{R}} \circ \psi^{-1} \circ (\rho|_{\partial I^+(\widetilde{p})})^{-1}$ and by $h_- = \pi_{\mathbb{R}} \circ \psi^{-1} \circ (\rho|_{\partial I^-(\widetilde{p})})^{-1}$, where $\pi_{\mathbb{R}}$ is the natural projection of $\mathbb{R} \times S$ onto $\mathbb{R}$. Then h_+ and h_- are continuous and satisfy that $\psi^{-1}(\partial I^+(\widetilde{p})) = \{(h_+(x), x) \in \mathbb{R} \times S | x \in S\}$, and that $\psi^{-1}(\partial I^-(\widetilde{p})) = \{(h_-(x), x) \in \mathbb{R} \times S | x \in S\}$. It follows that $\psi^{-1}(K) = \{(t, x) \in \mathbb{R} \times S | x \in S, h_-(x) \leq t \leq h_+(x)\}$. Therefore K is compact. Hence the fundamental group $\pi_1(M)$ is finite. The proof of Theorem 1.5 is complete.

6. Conclusions

Our results are partial solutions to Problem 2 on three cases: pseudo-Riemannian warped products, parametrized Lorentzian products, and globally hyperbolic Lorentzian manifolds. Moreover we have obtained generalizations of the result of Calabi–Markus [**7**] by the technique of differential geometry. We pose questions toward further researches.

We show that the Calabi–Markus phenomenon occurs in an inhomogeneous Lorentzian manifold with the non-compact isometry group. The first question is as follows:

Question 6.1. *Can we get a non-trivial inhomogeneous pseudo-Riemannian manifold of index $q > 1$ where the Calabi–Markus occurs?*

To solve Question 6.1, we need to extend the result of pseudo-Riemannian warped products.

In the fifth section, we treat the case of globally hyperbolic Lorentzian manifolds. Our result however is not applied to the de Sitter space. Therefore the following question naturally arises:

Question 6.2. *Can we generalize Theorem 1.5 to a certain class of globally hyperbolic Lorentzian manifolds including the de Sitter space?*

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