Earle slices associated with involutions for once punctured torus

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Abstract

In this paper, we will study Earle slices of quasi-Fuchsian space for once punctured torus associated with involutions of its fundamental group induced by orientation reversing diffeomorphism of this surface. First we classify Earle slices into two types: rhombic Earle slices and rectangular Earle slices. The main purpose of this paper is to study the configuration of Earle slices. Especially, we obtain a necessary and sufficient condition for two Earle slices to intersect each other. We also show that the union of all Earle slices is connected. In the end, we describe Earle slices using trace coordinates of quasi-Fuchsian space.

0 Introduction

The main part of this paper is contained in [9], which will be appeared in Tokyo Journal of Mathematics.

To obtain global coordinates of the Teichmüller space $\mathcal{T}(S)$ of a topological surface S of finite type, there are several ways to embed the Teichmüller space into the representation space $\mathcal{R}(S) = Hom(\pi_1(S), PSL(2,\mathbb{C}))/PSL(2,\mathbb{C})$. Among them, Earle slices, as well as Bers slices and Maskit slices, are realizations of the Teichmüller space as slices of the set of faithful discrete representations in $\mathcal{R}(S)$. To be more precise, recall that the interior of the set of discrete faithful representations is the quasi-Fuchsian space $\mathcal{QF}(S)$, which can be naturally identified with $\mathcal{T}(S)\times\mathcal{T}(\bar{S})$ (where $\bar{S}$ is S with its orientation reversed). Under this identification, a Bers slice is defined to be the slice $\mathcal{T}(S) \times \{Y\}$ of $\mathcal{QF}(S)$ for a fixed $Y \in \mathcal{T}(\bar{S})$. On the other hand, an Earle slice is defined to be the slice $\{(X, \varphi \cdot X) : X \in \mathcal{T}(S)\}$ of $\mathcal{QF}(S)$, where φ is an orientation reversing homeomorphism from S onto S. Especially, the case where φ is an involution (i.e. $\varphi^2 = id$) is important. In fact, in this case, for every hyperbolic manifold corresponding to an element of the Earle slice, there is an isometric automorphism of order 2 which interchanges the two boundary components X and $\varphi \cdot X$. Therefore, the study of the distributions of Earle slices associated with involutions is nothing but the study of the distributions of such symmetric groups in the quasi-Fuchsian space.

In this paper, we let S be a once punctured tours and study the Earle slices associated with involutions for S. We will classify these Earle slices into two types: rhombic Earle slices and rectangular Earle slices. Here rhombic Earle slices are associated with involutions exchanging some pair (α, β) of generators of $\pi_1(S)$, while rectangular Earle slices are associated with involutions which take α to α and β to β^{-1} for some pair (α, β) of generators of $\pi_1(S)$. Komori and Series [8] studied one rhombic Earle slice, and Komori [7] studied one rectangular Earle slice. On the other hand, in this paper we study the configuration and the distribution of all Earle slices associated with involutions for once punctured torus.

The main results of this paper are as follows:

Theorem 0.1 (Theorems 5.1, 5.5 and 5.6). *1. When two Earle slices intersect, they intersect at a single point.*

2. For any rhombic Earle slice, there exist exactly four rhombic Earle slices and one rectangular Earle slice that intersect it.

3. For any rectangular Earle slice, there exists a unique rhombic Earle slice and no different rectangular Earle slice that intersect it.

Theorem 0.2 (Theorem 5.7 and Corollary 5.8). *The union of all rhombic Earle slices is connected. Moreover, the union of all Earle slices is connected.*

The paper is organized as follows. In Sections 1 and 2, we set up notations and recall some definitions we will use later. In Section 3, we give the classification of Earle slices. In Section 4, we consider the action of mapping class group on the quasi-Fuchsian space and determine the stabilizer subgroup of each Earle slice.

The main results appear in Section 5. We give a necessary and sufficient condition for two Earle slices to intersect each other. More precisely, we first obtain a necessary and sufficient condition for two rhombic Earle slices to intersect each other. We next show that two rectangular Earle slices do not intersect. Finally we obtain a necessary and sufficient condition for one rhombic Earle slice and one rectangular Earle slice to intersect each other.

Besides, we show that for any Earle slice, there exists a unique Earle slice of different kind that intersects it. We also show that for any rhombic Earle slice, there exist exactly four distinct rhombic Earle slices that intersect it. In the end of Section 5, we show that the union of all rhombic Earle slices is connected. As a consequence, the union of all Earle slices is also connected.

Finally in Section 6, we describe Earle slices using trace coordinates of the quasi-Fuchsian space.

1 Preliminary

In this section, we give some basic concepts and theories related to Kleinian groups and quasi-Fuchsian spaces, which mainly come from Matsuzaki and Taniguchi [12].

1.1 Möbius transformations

Let $\hat{\mathbb{R}}^3$ denote $\mathbb{R}^3 \cup \{\infty\}$. A *similarity* S of $\mathbb{R}^3$ is given by

$$S(p) = \lambda \cdot Ap + b \quad (\lambda > 0, A \in O(3), b \in \mathbb{R}^3); \quad S(\infty) = \infty,$$

where $O(3)$ is the orthogonal transformation group of $\mathbb{R}^3$. The *fundamental reflection* J is defined by

$$J(p) = \frac{p}{|p|^2}; \quad J(0) = \infty, J(\infty) = 0.$$

Both S and J are automorphisms of $\hat{\mathbb{R}}^3$.

Definition 1.1. A *Möbius transformation* of $\hat{\mathbb{R}}^3$ is an orientation-preserving automorphism of $\hat{\mathbb{R}}^3$ composed of a finite number of similarities and fundamental reflections. We denote by $\text{Möb}(\hat{\mathbb{R}}^3)$ the group of all Möbius transformations of $\hat{\mathbb{R}}^3$.

Let $\mathbb{H}^3$ be $\{p = (p_1, p_2, p_3) \in \mathbb{R}^3 | p_3 > 0\}$. The set of all Möbius transformations of $\mathbb{H}^3$ is denoted by $\text{Möb}(\mathbb{H}^3)$ and defined by

$$\text{Möb}(\mathbb{H}^3) = \{T \in \text{Möb}(\hat{\mathbb{R}}^3) | T(\mathbb{H}^3) = \mathbb{H}^3\}.$$

We call the upper half space $\mathbb{H}^3$ equipped with the metric

$$ds_H^2 = \frac{|dp|^2}{p_3^2}$$

the *upper half-space model* of the hyperbolic space $\mathbb{H}^3$, and the metric ds_H^2 is called the hyperbolic metric of $\mathbb{H}^3$.

Let $\mathrm{Isom}^+(\mathbb{H}^3)$ be the group of all orientation-preserving hyperbolic isometries of $\mathbb{H}^3$. Then from the definition of upper half-space model, we can naturally identified $\mathrm{Möb}(\mathbb{H}^3)$ with $\mathrm{Isom}^+(\mathbb{H}^3)$.

After identifying the relative boundary of $\mathbb{H}^3$ in $\hat{\mathbb{R}^3}$ with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and through *Poincaré extension* we have the following theorem.

Theorem 1.2 ([12]). *We can identify* $\mathrm{Möb}(\mathbb{H}^3)$ *with the linear fractional transformation group*

$$\mathrm{Möb}(\hat{\mathbb{C}}) = \left\{ \frac{az+b}{cz+d} |\ a,b,c,d \in \mathbb{C}, ad-bc = 1 \right\}$$

and therefore with

$$PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\pm I.$$

Remark 1. We embed $\hat{\mathbb{C}}$ in $\hat{\mathbb{R}^3}$ in the natural way by $(p_1,p_2) \in \mathbb{C} \mapsto (p_1,p_2,0) \in \mathbb{R}^3$ and $\infty \in \hat{\mathbb{C}} \mapsto \infty \in \hat{\mathbb{R}^3}$. The *Poincaré extension* of

$$\gamma(z) = \frac{az+b}{cz+d} \quad (ad-bc=1)$$

is given by

$$T(\zeta) = \frac{a\zeta+b}{c\zeta+d} = \frac{(az+b)(\overline{cz+d}) + a\bar{c}t^2 + tj)}{|cz+d|^2 + |c|^2t^2} \quad (\zeta = z+tj, z \in \mathbb{C}, t>0).$$

Under the identification in Theorem 1.2 and the above fact, we shall now classify elements of $\mathrm{Möb}(\mathbb{H}^3) \cong \mathrm{Möb}(\hat{\mathbb{C}})$ in terms of conjugation form.

Definition 1.3. Any element $\gamma \neq id$ of $\mathrm{Möb}(\hat{\mathbb{C}})$ is conjugate to either

1. $\gamma_0(z) = z+1$ or
2. $\gamma_0(z) = \lambda z \quad (\lambda \in \mathbb{C} - \{0,1\})$.

We call γ *parabolic* if it is conjugate to (1), and *elliptic* if it is conjugate to (2) with $|\lambda| = 1$, and *loxodromic* if it is conjugate to (2) with $|\lambda| \neq 1$. Particularly, we also call γ *hyperbolic* if it is conjugate to (2) with $\lambda > 1$ or $0 < \lambda < 1$.

We also can give the classification in terms of fixed points in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$.

Theorem 1.4 ([12])**.** *Any element $\gamma \neq id$ of* $\mathrm{Isom}^+(\mathbb{H}^3)$ *is parabolic, loxodromic or elliptic if and only if the number of its fixed points in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ is one, two or infinity respectively. Moreover, it has a fixed point in $\mathbb{H}^3$ if and only if γ has finite order.*

Definition 1.5. A group Γ is called *torsion free* if Γ has no elements of finite order other than the identify.

Remark 2. Any element $\gamma \neq id$ of $\mathrm{M\ddot{o}b}(\hat{\mathbb{C}})$ is of finite order if and only if it is elliptic.

1.2 Kleinian groups and Kleinian manifolds

A subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}^3)$ is said to act on $\mathbb{H}^3$ *discontinuously* if for any compact $K \subset \mathbb{H}^3$, the set $\{\gamma \in \Gamma : K \cap \gamma(K) \neq \emptyset\}$ is finite.

Definition 1.6. A subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}^3)$ is called a *Kleinian group* if Γ act on $\mathbb{H}^3$ discontinuously. Under the identification of $\mathrm{Isom}^+(\mathbb{H}^3)$ with $\mathrm{M\ddot{o}b}(\mathbb{H}^3)$, $\mathrm{M\ddot{o}b}(\hat{\mathbb{C}})$ and $PSL(2, \mathbb{C})$, the corresponding subgroups in these groups are also called Kleinian groups.

A subgroup Γ of $PSL(2, \mathbb{C})$ is *discrete* if $\{\gamma_n\}_{n=1}^{\infty}$ are in Γ with $\gamma_n \to \gamma \in PSL(2, \mathbb{C})$ as $n \to \infty$ implies that $\gamma_n = \gamma$ for all sufficient large n.

Theorem 1.7 ([12])**.** *A Kleinian group is discrete subgroup of $PSL(2, \mathbb{C})$, and vice versa.*

The Γ-*orbit* of some $x \in \mathbb{H}^3$ is defined by $\Gamma x = \{\gamma(x) | \gamma \in \Gamma\}$. The we can define the *limit set* $\Lambda(\Gamma)$ of Γ by

$$\Lambda(\Gamma) = \overline{\Gamma x} \cap \hat{\mathbb{C}}.$$

Definition 1.8. For a Kleinian group Γ, the complement of the limit set $\hat{\mathbb{C}} - \Lambda(\Gamma)$ is denoted by $\Omega(\Gamma)$ and called the *region of discontinuity* of Γ. In the case $\Omega(\Gamma) = \emptyset$, Γ is called a Kleinian group of the *first kind*, and otherwise of the *second kind*.

Definition 1.9. For a torsion free Kleinian group Γ, a manifold $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ possibly with boundary is denoted by M_Γ and called a *Kleinian manifold*. The interior $\mathbb{H}^3/\Gamma$ of M_Γ which admits the hyperbolic structure is denoted by N_Γ.

Definition 1.10. We say a Riemann surface R is *analytically finite* if R is closed except for a finite number of punctures. Moreover, if R is a closed surface of genus g from which n punctures are removed, then we say that R is of (g, n)-type.

Definition 1.11. A Kleinian group is called *geometrically finite* if it has a convex fundamental polyhedron in $\mathbb{H}^3$ with a finite number of sides.

1.3 Quasi-conformal deformation space

A Kleinian group Γ' that is conjugate to Γ by a quasi-conformal automorphism of $\hat{\mathbb{C}}$ is called a *quasi-conformal deformation* of Γ.

For a finite generated Kleinian group Γ, we fix a generator system $\langle \gamma_1, \cdots, \gamma_N \rangle$ once for all. We call a homomorphism $\rho : \Gamma \to PSL(2, \mathbb{C})$ a $PSL(2, \mathbb{C})$-*representation* of Γ. Since a homomorphism is determined by the images of the generators $\{\rho(\gamma_1), \cdots, \rho(\gamma_N)\}$, we may regard ρ as a product manifold $PSL(2, \mathbb{C})^N$. We denote the subset of $PSL(2, \mathbb{C})^N$ consisting of all $PSL(2, \mathbb{C})$-representations of Γ by $\mathrm{Hom}(\Gamma)$.

Definition 1.12. For a non-elementary finitely generated Kleinian group Γ, we consider the following subsets of $\mathrm{Hom}(\Gamma)$:

1. $\mathrm{PHom}(\Gamma) = \{\rho \in \mathrm{Hom}(\Gamma) | \operatorname{Tr}^2 \rho(\gamma) = 4$ for any parabolic $\gamma \in \Gamma\}$
2. $\mathrm{AHom}(\Gamma) = \{\rho \in \mathrm{PHom}(\Gamma) | \rho$ is faithful and discrete$\}$
3. $\mathrm{QHom}(\Gamma) = \{\rho \in \mathrm{AHom}(\Gamma) | \rho$ is a quasi-conformal deformation$\}$.

Here we say that ρ is discrete if $\rho(\Gamma)$ is discrete and that ρ is quasi-conformal deformation of Γ if there exists a quasi-conformal automorphism f of $\hat{\mathbb{C}}$ such that $\rho(\gamma) = f \circ \gamma \circ f^{-1}$ for every $\gamma \in \Gamma$.

We call the conjugacy class

$$\mathcal{QH}(\Gamma) = \mathrm{QHom}(\Gamma)/\sim_{conj}$$

the *quasi-conformal deformation space* of Γ. Here the conjugation is given by some element in $PSL(2, \mathbb{C})$.

1.4 Quasi-Fuchsian space

A Kleinian group Γ is called a *Fuchsian group* if Γ is a discrete subgroup of $PSL(2, \mathbb{R})$. We say Γ is of (g, n)-type if $\mathbb{H}/\Gamma$ is of (g, n)-type.

Definition 1.13. Let Γ be a Fuchsian group of (g, n)-type. A *quasi-Fuchsian space* of Γ is the quasi-conformal deformation space $\mathcal{QH}(\Gamma)$ of Γ. We especially denote it by $\mathcal{QF}(\Gamma)$. A Kleinian group $\rho(\Gamma)$ for every $\rho \in \mathrm{QHom}(\Gamma)$ is called a *quasi-Fuchsian group* of (g, n)-type.

Remark 3. A (g, n)-type quasi-Fuchsian group is geometrically finite. The limit set $\Lambda(\Gamma)$ of a quasi-Fuchsian group Γ is a Γ-invariant Jordan curve. The region of discontinuity $\Omega(\Gamma)$ of Γ has exactly two simply connected invariant components. The Kleinian manifold $M_\Gamma = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is homeomorphic to $S \times [0, 1]$ for some topological surface S of (g, n)-type. For this reason, we also denote $\mathcal{QF}(\Gamma)$ by $\mathcal{QF}(S)$.

Next we will give a deformation space of a analytically finite Riemann surface.

Definition 1.14. Let S be a topological surface of (g, n)-type. The *Teichmüller space* $\mathcal{T}(S)$ of S is defined to be the set of equivalent classes of (X, f), where X is a hyperbolic Riemann surface and $f : S \to X$ is a homeomorphism. (X, f) is equivalent to (Y, g) if and only if $g \circ f^{-1} : X \to Y$ is homotopic to a conformal map. We denote the equivalent class of (X, f) by $[X, f]$.

Let $\rho \in \mathcal{QF}(S)$ and $\pi_1(S)$ be its fundamental group of S. And we let $\Gamma = \rho(\pi_1(S))$. Then the Kleinian manifold $M_\Gamma = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is bounded by a pair of Riemann surfaces denoted by X and Y. The orientations of X and Y are induced from that of $\Omega(\Gamma)$. There is a homeomorphism $\psi : S \times [0, 1] \to M_\Gamma$ such that ψ restricted to the boundary components are orientation preserving homeomorphism: $\psi|_{S \times \{0\}} : S \to X$ and $\psi_{S \times \{1\}} : \bar{S} \to Y$. After equipping X and Y with markings induced by ψ, we can regard X and Y as elements of $\mathcal{T}(S)$ and $\mathcal{T}(\bar{S})$, respectively.

Theorem 1.15 (Bers [1])**.** *The map $\mathcal{QF}(S) \to \mathcal{T}(S) \times \mathcal{T}(\bar{S})$, $\rho \mapsto (X, Y)$ defined as above is a homeomorphism.*

So we denote by $Q(X, Y) := \rho \in \mathcal{QF}(S)$ the pre-image of (X, Y). For a given $X \in \mathcal{T}(S)$, we define the *Bers slice* $\mathcal{B}_X$ as

$$\mathcal{B}_X = \{Q(X, Y) \in \mathcal{QF}(S) | Y \in \mathcal{T}(\bar{S})\}.$$

Bers slices are complex (3g+n-3)-dimensional slices of $\mathcal{QF}(S)$ and can be identified with $\mathcal{T}(S)$.

Besides Bers slices, there are another typical slices of quasi-Fuchsian space $\mathcal{QF}(S)$, which are so called *Earle slices*. Earle slices can be also identified with $\mathcal{T}(S)$. Given an automorphism θ of $\pi_1(S)$ which induced by an orientation-reversing diffeomorphism of S. Then the Earle slice $\mathcal{E}_\theta$ is defined as

$$\mathcal{E}_\theta = \{Q(X,Y) \in \mathcal{QF}(S) | \exists \text{ a conformal map } h : X \to Y \text{ associated with } \theta\}.$$

Here we say that a conformal map $h : X \to Y$ is associated with θ if $g \circ h \circ f : S \to \bar{S}$ induces the automorphism θ of $\pi_1(S)$, where $f : S \to X$ and $g : \bar{S} \to Y$ are markings.

2 Concepts and theories restricted to once punctured torus

From now on, we concentrate our attention to the case of once punctured torus.

2.1 Teichmüller space of once punctured torus

Let S be an oriented once punctured torus and let $\pi_1(S)$ be its fundamental group. An ordered pair (α, β) of generators of $\pi_1(S)$ is called *canonical* if the algebraic intersection number $i(\alpha, \beta)$ is equal to $+1$ with respect to the given orientation of S. The *Teichmüller space* $\mathcal{T}(S)$ is defined to be the set of equivalence classes of (X, f), where X is a hyperbolic Riemann surface of finite hyperbolic area and $f : \pi_1(S) \to \pi_1(X)$ is a *type-preserving* isomorphism which is induced by orientation preserving homeomorphism from S to X. Here *type-preserving* means that f maps the homotopy class of loops around the puncture of S to the same homotopy class of loops around the puncture of X. (X, f) is equivalent to (Y, g) if and only if there exists a conformal map h from X to Y that induces an isomorphism $h_* : \pi_1(X) \to \pi_1(Y)$ such that $g^{-1}h_*f$ is an inner automorphism of $\pi_1(S)$. We denote the equivalence class of (X, f) by $[X, f]$. For short, let X denote $[X, f] \in \mathcal{T}(S)$.

By fixing a pair (α, β) of canonical generators of $\pi_1(S)$, the Teichmüller space $\mathcal{T}(S)$ can be naturally identified with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \Im z > 0\}$ as follows: for any point $\tau \in \mathbb{H}$, we associate it with the point $[X, f] \in \mathcal{T}(S)$ where X is equal to $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ with one point removed, and $f : \pi_1(S) \to \pi_1(X)$ is a type-preserving isomorphism that sends α and β to the images of segments $[0, 1]$ and $[0, \tau]$ in X, respectively.

2.2 Extended mapping class group

The *extended mapping class group* $\mathrm{Mod}(S)$ is the group of isotopy classes of (not necessarily orientation preserving) homeomorphisms of S onto itself. The *outer automorphism* of $\pi_1(S)$ is $\mathrm{Out}(\pi_1(S)) = \mathrm{Aut}(\pi_1(S))/\mathrm{Inn}(\pi_1(S))$, where $\mathrm{Aut}(\pi_1(S))$ and $\mathrm{Inn}(\pi_1(S))$ denote the automorphism group and the inner automorphism group of $\pi_1(S)$, respectively.

We can identify $\mathrm{Mod}(S)$ with $\mathrm{Out}(\pi_1(S))$. In fact, for an arbitrary homeomorphism h of S onto itself, and for any path δ connecting the base point $p \in S$ to $h(p)$, we have an automorphism $(h_\delta)_* \in \mathrm{Aut}(\pi_1(S))$ which is defined by $(h_\delta)_*(\gamma) = \delta^{-1}(h \circ \gamma)\delta$. When we choose another path δ' which connects p to $h(p)$, then $(h_\delta)_*$ is conjugate to $(h_{\delta'})_*$ by $\delta^{-1}\delta'$. The outer automorphism $h_* \in \mathrm{Out}(\pi_1(S))$ is thus well defined independently of the choice of the path δ. Furthermore h is homotopic to the identity if and only if it acts trivially on $\pi_1(S)$. Therefore the map $\mathrm{Mod}(S) \to \mathrm{Out}(\pi_1(S))$ defined as above is a homomorphism (see chapter 8 in [4] for more details).

By fixing a basis of $H_1(S, \mathbb{Z}) \approx \mathbb{Z}^2$, we can also identify $\mathrm{Mod}(S)$ with $GL(2, \mathbb{Z})$ (see for example [4] P.231). In fact, when we fix a pair (α, β) of generators of $H_1(S, \mathbb{Z})$, we identify $h \in \mathrm{Mod}(S)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ if h induces an isomorphism $h_* : H_1(S, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ such that $h_*(\alpha) = a\alpha + c\beta$ and $h_*(\beta) = b\alpha + d\beta$. Then we often identify the following three groups:

$$\mathrm{Out}(\pi_1(S)) \cong \mathrm{Mod}(S) \cong GL(2, \mathbb{Z}).$$

Let $\mathrm{Mod}^+(S)$ (resp. $\mathrm{Mod}^-(S)$) denote the subset of $\mathrm{Mod}(S)$ consisting of isotopy classes of orientation preserving (resp. reversing) homeomorphisms from S to itself. In particular, we can identify $\mathrm{Mod}^+(S)$ with $SL(2, \mathbb{Z})$. An element $\varphi \in \mathrm{Mod}^+(S)$ induces a map $\mathcal{T}(S) \to \mathcal{T}(S)$ given by sending $X = [X, f]$ to $\varphi \cdot X := [X, f \circ \varphi_*^{-1}]$, where $\varphi_* \in \mathrm{Out}(\pi_1(S))$ denotes the element corresponding to φ by the above identification. Let $\bar{S}$ be S with its orientation reversed. Then $\varphi \in \mathrm{Mod}^-(S)$ induces a map $\mathcal{T}(S) \to \mathcal{T}(\bar{S})$ in a similar way.

2.3 Action of $\mathrm{Mod}^+(S)$ on $\mathcal{T}(S)$

The action of $\mathrm{Mod}^+(S)$ on $\mathcal{T}(S)$ can be identified with the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$ by Möbius transformation (see [4], P.346):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(z) = \frac{az-b}{-cz+d}.$$

Note that $\varphi \in \mathrm{Mod}^+(S)$ acts on $\mathcal{T}(S)$ trivially if and only if it corresponds to $\pm I$ under the identification $\mathrm{Mod}^+(S) \cong SL(2,\mathbb{Z})$. A fundamental domain for the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$ is

$$K = \left\{ z \in \mathbb{H} : -\frac{1}{2} \leq \Re z \leq \frac{1}{2}, |z| \geq 1 \right\}.$$

In K, there are only three points i, $e^{i\pi/3}$ and $e^{2i\pi/3}$ that are fixed by some elements of $SL(2,\mathbb{Z})$.

2.4 Quasi-Fuchsian space and Earle slices

The *quasi-Fuchsian space* $\mathcal{QF}(S)$ of S is defined to be the set of conjugacy classes of faithful and discrete representations $\rho : \pi_1(S) \to PSL(2,\mathbb{C})$ such that the image of the loop around the puncture is parabolic and that the image $\Gamma := \rho(\pi_1(S))$ is quasi-Fuchsian. The region of discontinuity Ω of Γ has exactly two simply connected invariant components $\Omega_\pm$. We choose the labeling $\pm$ in such a way that the homotopy basis of Ω_+/Γ induced by the ordered pair $(\rho(\alpha), \rho(\beta))$ is canonical with respect to the orientation of Ω_+.

Earle slices for once punctured torus are defined by the following theorem.

Theorem 2.1 ([3],[8])**.** *Let S be a once punctured torus, and $\mathcal{T}(S)$ the Teichmüller space of S. Then for any $X \in \mathcal{T}(S)$ and an automorphism θ of $\pi_1(S)$ induced by an orientation reversing diffeomorphism of S, there exists a unique representation $\rho : \pi_1(S) \to \Gamma$ in the quasi-Fuchsian space $\mathcal{QF}(S)$, such that*

1. *$\Omega_+/\Gamma = X$ in $\mathcal{T}(S)$, where the marking of Ω_+/Γ is induced from ρ, and*

2. *there exists a conformal map $F \colon \Omega_+ \to \Omega_-$ such that $F \circ \rho(\gamma) = \rho(\theta(\gamma)) \circ F$ for all $\gamma \in \pi_1(S)$.*

Furthermore, if $\theta^2 = id$, then F is a Möbius transformation of order two.

We denote the representation ρ obtained in Theorem 2.1 by $\rho_{\theta,X}$. For a given automorphism θ of $\pi_1(S)$ induced by an orientation reversing diffeomorphism of S, the *Earle slice* $\mathcal{E}_\theta$ of $\mathcal{QF}(S)$ is defined to be the subset

$$\mathcal{E}_\theta := \{\rho_{\theta,X} | X \in \mathcal{T}(S)\}$$

of $\mathcal{QF}(S)$.

3 Classification of Earle slices associated with involutions

Let $\rho \in \mathcal{QF}(S)$ and $\Gamma = \rho(\pi_1(S))$. The Kleinian manifold $M_\Gamma = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is bounded by a pair of Riemann surfaces denoted by X and Y. The orientations of X and Y are induced from that of $\Omega(\Gamma)$. There is a homeomorphism $\psi : S \times [0,1] \to M_\Gamma$ such that $\psi_* = \rho$. After equipping X and Y with markings induced by ψ, we can regard X and Y as elements of $\mathcal{T}(S)$ and $\mathcal{T}(\bar{S})$, respectively. The map $\mathcal{QF}(S) \to \mathcal{T}(S) \times \mathcal{T}(\bar{S})$, $\rho \mapsto (X, Y)$ defined as above has been proved to be a homeomorphism by Bers. So we denote by $Q(X, Y) := \rho \in \mathcal{QF}(S)$ the pre-image of (X, Y).

Next we use this fact to interpret the definition of Earle slice.

Theorem 3.1. *For every automorphism θ of $\pi_1(S)$ induced by an orientation reversing diffeomorphism of S, the Earle slice $\mathcal{E}_\theta$ can be written as*

$$\mathcal{E}_\theta = \{Q(X, \theta \cdot X) | X \in \mathcal{T}(S)\}.$$

Recall that θ induces a map $\mathcal{T}(S) \to \mathcal{T}(\bar{S})$ given by sending $X = [X, f]$ to $\theta \cdot X = [X, f \circ \theta^{-1}]$.

Proof. Let $\rho \in \mathcal{E}_\theta$. By Theorem 2.1, there exists a conformal map $F\colon \Omega_+ \to \Omega_-$, such that $F \circ \rho(\gamma) = \rho(\theta(\gamma)) \circ F$ for all $\gamma \in \pi_1(S)$. The conformal map F induces an orientation preserving conformal map $\underline{F} : \Omega_+/\Gamma \to \Omega_-/\Gamma$ such that $\underline{F}_*(\rho(\gamma)) = \rho(\theta(\gamma))$. We equip Ω_+/Γ and Ω_-/Γ with markings induced by ρ. We regard the marking $\rho : \pi_1(S) \to \Gamma = \pi_1(\Omega_+/\Gamma)$ of Ω_+/Γ as induced from an orientation preserving homeomorphism $S \to \Omega_+/\Gamma$. On the other hand, we regard the marking $\rho : \pi_1(S) = \pi_1(\bar{S}) \to \Gamma = \pi_1(\Omega_-/\Gamma)$ of Ω_-/Γ as induced from an orientation preserving homeomorphism $\bar{S} \to \Omega_-/\Gamma$. Thus one can see that $X = (\Omega_+/\Gamma, \rho) \in \mathcal{T}(S)$ and $Y = (\Omega_-/\Gamma, \rho) \in \mathcal{T}(\bar{S})$.

From the definition of Teichmüller space, we have $X = [\Omega_+/\Gamma, \rho] = [\Omega_-/\Gamma, \underline{F}_* \circ \rho] = [\Omega_-/\Gamma, \rho \circ \theta]$. So

$$\theta \cdot X = \theta \cdot [\Omega_-/\Gamma, \rho \circ \theta] = [\Omega_-/\Gamma, \rho \circ \theta \circ \theta^{-1}] = [\Omega_-/\Gamma, \rho] = Y.$$

Thus we obtain $\rho = Q(X, Y) = Q(X, \theta \cdot X)$. □

Let

$$\mathrm{Inv}^-(S) := \{\varphi \in \mathrm{Mod}^-(S) | \varphi^2 = id\}$$

and

$$\mathrm{Inv}^-(S)_* := \{\varphi_* \in \mathrm{Out}(\pi_1(S)) | \varphi \in \mathrm{Inv}^-(S)\}.$$

Under the identification $\mathrm{Out}(\pi_1(S)) \cong GL(2, \mathbb{Z})$, we have the following bijective correspondence:

$$\mathrm{Inv}^-(S)_* \longleftrightarrow \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in GL(2, \mathbb{Z}) : a^2 + bc = 1 \right\}.$$

In this paper, we focus on Earle slices $\mathcal{E}_\theta$ associated with $\theta \in \mathrm{Inv}^-(S)_*$.

We have the following classification of elements of $\mathrm{Inv}^-(S)_*$, which is deduced from Theorem 3.4 below.

Theorem 3.2. *Let $\theta \in \mathrm{Inv}^-(S)_*$, then exactly one of the following is satisfied:*

1. *There exists a pair (α, β) of canonical generators of $\pi_1(S)$ such that $\theta(\alpha) = \alpha$, $\theta(\beta) = \beta^{-1}$.*

2. *There exists a pair (α, β) of canonical generators of $\pi_1(S)$ such that $\theta(\alpha) = \beta$, $\theta(\beta) = \alpha$.*

Definition 3.3 ([6], [8], [7]). We say that θ is *rectangular* if θ satisfies condition *1* of Theorem 3.2. The corresponding Earle slice $\mathcal{E}_\theta$ is called *rectangular Earle slice.* We say θ is *rhombic* if θ satisfies condition *2* of Theorem 3.2. The corresponding Earle slice $\mathcal{E}_\theta$ is called *rhombic Earle slice.*

An ordered pair (ω_1, ω_2) of $\mathbb{Z}^2$ is *canonical* if $\det(\omega_1, \omega_2) = 1$. We obtain Theorem 3.2 from the bijective correspondence $\mathrm{Out}(\pi_1(S)) \cong GL(2, \mathbb{Z})$ and the following theorem.

Theorem 3.4 ([5] P.166, Lemma 5.5). *For $A \in GL(2, \mathbb{Z})$ with $\det(A) = -1$ and $A^2 = E$, exactly one of the following is satisfied:*

1. *There exists a pair* (ω_1, ω_2) *of canonical generators of* $\mathbb{Z}^2$ *such that* $A\omega_1 = \omega_1$ *and* $A\omega_2 = -\omega_2$.

2. *There exists a pair* (ω_1, ω_2) *of canonical generators of* $\mathbb{Z}^2$ *such that* $A\omega_1 = \omega_2$ *and* $A\omega_2 = \omega_1$.

Remark 4. In Theorem 3.4, A is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in Case 1, and is conjugate to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in Case 2.

Throughout this paper, we denote $\theta_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\theta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Lemma 3.5. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$ *be as in Theorem 3.4. Then* A *is conjugate to* θ_1 *by some element of* $SL(2,\mathbb{Z})$ *if and only if both* b *and* c *are even.*

Proof. Let $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$. Since we have

$$B\theta_1 B^{-1} = \begin{pmatrix} * & -2pq \\ -2rs & * \end{pmatrix},$$

$$B\theta_2 B^{-1} = \begin{pmatrix} * & p^2 - q^2 \\ s^2 - r^2 & * \end{pmatrix},$$

the "only if" part is trivial.

We only need to show that either $p^2 - q^2$ or $s^2 - r^2$ is odd. Suppose for contradiction that both $p^2 - q^2$ and $s^2 - r^2$ are even. Then p and q have the same parity, and so do s and r. Then ps and qr have the same parity. This contradicts $ps - qr = 1$. □

4 Stabilizers of Earle slices

A mapping class $\varphi \in \mathrm{Mod}^+(S)$ acts on $\mathcal{QF}(S)$ by sending a representation $Q(X,Y)$ to $\varphi \cdot Q(X,Y) := Q(X,Y) \circ \varphi_*^{-1}$. This action is compatible with the actions of $\mathrm{Mod}^+(S)$ on $\mathcal{T}(S)$ and $\mathcal{T}(\bar{S})$; that is

$$\varphi \cdot Q(X,Y) = Q(\varphi \cdot X, \varphi \cdot Y).$$

In this chapter, we will study the action of $\mathrm{Mod}^+(S)$ on the set of Earle slices of $\mathcal{QF}(S)$. Especially, we will give the stabilizer subgroup of each Earle slice in $\mathrm{Mod}^+(S) \cong SL(2,\mathbb{Z})$.

Lemma 4.1. *Let $\theta, \theta' \in \mathrm{Inv}^-(S)_*$. Then $\mathcal{E}_\theta = \mathcal{E}_{\theta'}$ if and only if $\theta = \pm\theta'$ in $GL(2,\mathbb{Z})$.*

Proof. Suppose that $\mathcal{E}_\theta = \mathcal{E}_{\theta'}$. Then $Q(X, \theta \cdot X) = Q(X, \theta' \cdot X)$ for all $X \in \mathcal{T}(S)$. Then we have $\theta \cdot X = \theta' \cdot X$ and hence $X = \theta^{-1} \cdot \theta' \cdot X$ for all $X \in \mathcal{T}(S)$. From subsection 2.3, we know that an element in $\mathrm{Mod}^+(S)$ acts on $\mathcal{T}(S)$ trivially if and only if it corresponds to $\pm I$. Thus $\theta^{-1} \circ \theta' = \pm I$ in $SL(2,\mathbb{Z})$. That means $\theta = \pm\theta'$ in $GL(2,\mathbb{Z})$.

The converse is trivial. □

Theorem 4.2. *Let $\theta \in \mathrm{Inv}^-(S)_*$ and $r \in \mathrm{Mod}^+(S)$. Then we have the following:*

(a) The set $r(\mathcal{E}_\theta) = \{\rho \circ r_*^{-1} | \rho \in \mathcal{E}_\theta\}$ is equal to $\mathcal{E}_{\theta'}$ where $\theta' = r_* \circ \theta \circ r_*^{-1}$.

(b) $\mathcal{E}_\theta$ is rhombic (resp. rectangular) if and only if $r(\mathcal{E}_\theta)$ is rhombic (resp. rectangular).

Proof. (a) By Theorem 3.1, we have

$$r(\mathcal{E}_\theta) = \{Q(r \cdot X, r \cdot \theta \cdot X) | X \in \mathcal{T}(S)\}.$$

Since r induces an automorphism of $\mathcal{T}(S)$, we obtain

$$\{Q(r \cdot X, r \cdot \theta \cdot X) | X \in \mathcal{T}(S)\} = \{Q(X, r \cdot \theta \cdot r^{-1} \cdot X) | X \in \mathcal{T}(S)\}.$$

For all $X = [X, f] \in \mathcal{T}(S)$, we have

$$\begin{aligned} r \cdot \theta \cdot r^{-1} \cdot X = [X, f \circ r_* \circ \theta^{-1} \circ r_*^{-1}] &= [X, f \circ (r_* \circ \theta \circ r_*^{-1})^{-1}] \\ &= \theta' \cdot X, \end{aligned}$$

thus $r(\mathcal{E}_\theta) = \mathcal{E}_{\theta'}$.

(b) It follows from *(a)* and the definitions of the two kinds of Earle slices. □

By Theorem 3.4, $\theta \in \mathrm{Inv}^-(S)_*$ is conjugate to θ_1 or θ_2 in $GL(2,\mathbb{Z})$ by some element in $SL(2,\mathbb{Z})$. Therefore we have the following corollary.

Corollary 4.3. *For any $\theta \in \mathrm{Inv}^-(S)_*$, there exists $r \in \mathrm{Mod}^+(S)$ such that $\mathcal{E}_\theta = r(\mathcal{E}_{\theta_1})$ or $\mathcal{E}_\theta = r(\mathcal{E}_{\theta_2})$.*

Theorem 4.4. *For every Earle slice $\mathcal{E}_\theta$, we have*

$$\mathrm{Stab}_{\mathrm{Mod}^+(S)}\mathcal{E}_\theta = \left\{ \pm I, \pm g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g^{-1} \right\},$$

where g is an element of $SL(2,\mathbb{Z})$ such that $\theta = g\theta_1 g^{-1}$ if θ is rectangular, or $\theta = g\theta_2 g^{-1}$ if θ is rhombic.

Proof. If $\mathcal{E}_\theta = r(\mathcal{E}_\theta)$, it follows from Theorem 4.2 that $r(\mathcal{E}_\theta) = \mathcal{E}_{r_*\circ\theta\circ r_*^{-1}}$. Then by Lemma 4.1, we have $\theta = \pm r_* \circ \theta \circ r_*^{-1}$ in $GL(2,\mathbb{Z})$.

We first assume that $\theta = \theta_1$. Let $r_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. By a calculation, we obtain from $\theta = \pm r_* \circ \theta \circ r_*^{-1}$ that $r_* = \pm I, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Similarly, when $\theta = \theta_2$, we obtain the same result $r_* = \pm I, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In general, by Theorem 3.4, θ is conjugate to θ_1 or θ_2 by some element of $SL(2,\mathbb{Z})$. Suppose $\theta = g\theta_i g^{-1}$ for some $g \in SL(2,\mathbb{Z})$, where $i = 1$ or 2. Combining with $r_*\theta r_*^{-1} = \pm\theta$, we obtain

$$(g^{-1}r_*g)\theta_i(g^{-1}r_*g)^{-1} = \pm\theta_i.$$

It follows from the discussion above that $g^{-1}r_*g = \pm I, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus we obtain

$$r_* = \pm I, \pm g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g^{-1}.$$

□

5 Intersection of two Earle slices

Recall that the Teichmüller space $\mathcal{T}(S)$ of once punctured torus S can be identified with the upper half plane $\mathbb{H}$ after fixing a pair of generators of $\pi_1(S)$. Let S_z denote the element in $\mathcal{T}(S)$ corresponding to $z \in \mathbb{H}$. Let

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Note that $C^2 = D^3 = I$ in $SL(2,\mathbb{Z})/\{\pm I\}$. Since $SL(2,\mathbb{Z})$ acts on $\mathbb{H}$, we have $C(i) = i$, $D^{\pm1}(e^{i\pi/3}) = e^{i\pi/3}$. So if we regard C and $D^{\pm1}$ as elements of $\mathrm{Mod}^+(S)$

acting on $\mathcal{T}(S)$, we can say that C fixes $S_i \in \mathcal{T}(S)$ and $D^{\pm 1}$ fixes $S_{e^{i\pi/3}} \in \mathcal{T}(S)$. Conversely $\varphi \in \mathrm{Mod}^+(S) \cong SL(2,\mathbb{Z})$ has a fixed point in $\mathcal{T}(S)$, then φ is conjugate to $\pm C$ or $\pm D$ or $\pm D^{-1}$. In what follows, we will always let C and D be as above.

We first give a necessary and sufficient condition for $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$.

Theorem 5.1. *Let $\theta, \theta' \in \mathrm{Inv}^-(S)_*$ with $\theta \neq \pm\theta'$. Then $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$ if and only if $\theta^{-1}\theta'$ is conjugate in $SL(2,\mathbb{Z})$ to $\pm C$ or $\pm D$ or $\pm D^{-1}$. Moreover, if $\theta^{-1}\theta' = \pm gCg^{-1}$ for some $g \in SL(2,\mathbb{Z})$, then*

$$\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} = \{Q(S_{g(i)}, \theta \cdot S_{g(i)})\}$$

and if $\theta^{-1}\theta'$ is equal to $\pm\, gDg^{-1}$ or $\pm\, gD^{-1}g^{-1}$, then

$$\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} = \{Q(S_{g(e^{i\pi/3})}, \theta \cdot S_{g(e^{i\pi/3})})\}.$$

Proof. If $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$, then there exists $X = [X, f] \in \mathcal{T}(S)$ such that $\theta \cdot X = \theta' \cdot X$ and hence that $X = \theta^{-1} \cdot \theta' \cdot X$. So $\theta^{-1}\theta'$ fixes the point $[X, f]$. By identifying $\mathrm{Mod}^+(S)$ with $SL(2,\mathbb{Z})$, we have that $\theta^{-1}\theta'$ is conjugate to $\pm C$ or $\pm D$ or $\pm D^{-1}$ in $SL(2,\mathbb{Z})$.

When $\theta^{-1}\theta' = \pm gCg^{-1}$ for some $g \in SL(2,\mathbb{Z})$, $\theta^{-1}\theta'$ fixes $S_{g(i)}$. Therefore $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} = \{Q(S_{g(i)}, \theta \cdot S_{g(i)})\}$. Similarly, if $\theta^{-1}\theta' = \pm gDg^{-1}$ or $\pm\, gD^{-1}g^{-1}$, we have $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} = \{Q(S_{g(e^{i\pi/3})}, \theta \cdot S_{g(e^{i\pi/3})})\}$. □

Examples. (1) Since ${\theta_2}^{-1}\theta_1 = C$, we have $\mathcal{E}_{\theta_1} \cap \mathcal{E}_{\theta_2} = \{Q(S_i, \theta_1 \cdot S_i)\}$.

(2) Since $\theta_2 D$ is conjugate to θ_2 by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$, we know that $\theta_2 D$ is in $\mathrm{Inv}^{-1}(S)_*$ and is rhombic. We have $\mathcal{E}_{\theta_2} \cap \mathcal{E}_{\theta_2 D} = \{Q(S_{e^{i\pi/3}}, \theta_2 \cdot S_{e^{i\pi/3}})\}$ since ${\theta_2}^{-1}(\theta_2 D) = D$. Similarly, we have $\mathcal{E}_{\theta_2} \cap \mathcal{E}_{\theta_2 D^{-1}} = \{Q(S_{e^{i\pi/3}}, \theta_2 \cdot S_{e^{i\pi/3}})\}$. Therefore,

$$\mathcal{E}_{\theta_2} \cap \mathcal{E}_{\theta_2 D} \cap \mathcal{E}_{\theta_2 D^{-1}} = \{Q(S_{e^{i\pi/3}}, \theta_2 \cdot S_{e^{i\pi/3}})\}.$$

Theorem 5.2. *Let $\theta, \theta' \in \mathrm{Inv}^-(S)_*$ with $\theta \neq \pm\theta'$ and assume that $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$. Then $\theta^{-1}\theta'$ is conjugate to $\pm D$ or $\pm D^{-1}$ in $SL(2,\mathbb{Z})$ if and only if both θ and θ' are rhombic.*

Proof. First, we assume that $\theta^{-1}\theta'$ is conjugate to $\pm D$ in $SL(2,\mathbb{Z})$. The proof for $\pm D^{-1}$ is similar. Since $\theta \in \mathrm{Inv}^-(S)_*$, we can let $\theta = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in GL(2,\mathbb{Z})$ with $a^2 + bc = 1$.

Suppose first that $\theta^{-1}\theta' = \pm D$. Then we have $\theta' = \pm\theta D = \pm \begin{pmatrix} a+b & -a \\ c-a & -c \end{pmatrix}$. Since $\theta' \in \mathrm{Inv}^-(S)_*$, we obtain $a+b-c=0$. Suppose for contradiction that θ is rectangular. It follows immediately from Lemma 3.5, both b and c are even. By $a+b-c=0$, a is also even. Hence $a^2+bc \neq 1$, which is a contradiction. Thus θ is rhombic. The proof for θ' is parallel.

In general, $\theta' = \pm\theta g D g^{-1}$ for some $g \in SL(2,\mathbb{Z})$. Then we have $g^{-1}\theta' g = \pm g^{-1}\theta g D$. One can see that $g^{-1}\theta g$ and $g^{-1}\theta' g$ are rhombic from the discussion of $\theta^{-1}\theta' = \pm D$. Therefore, both θ and θ' are rhombic.

Next we will show the necessity. By Theorem 5.1, $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$ implies that $\theta^{-1}\theta'$ is conjugate to $\pm C$ or $\pm D$ or $\pm D^{-1}$. Since θ and θ' are rhombic, by taking a conjugation if necessary, we may assume $\theta = \theta_2$ and $\theta' = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2+bc = 1$. Then we have $\theta_2^{-1}\theta' = \begin{pmatrix} c & -a \\ a & b \end{pmatrix}$. Suppose for contradiction that $\theta_2^{-1}\theta'$ is conjugate to $\pm C$. Then $\mathrm{Tr}\,\theta_2^{-1}\theta' = b+c = 0$. It follows from $a^2+bc=1$ that $a^2-b^2=1$. Therefore $a=\pm 1$ and $b=c=0$. So $\theta' = \pm\theta_1$. It contradicts the assumption that θ' is rhombic. Thus $\theta^{-1}\theta'$ is conjugate to $\pm D$ or $\pm D^{-1}$. $\square$

Next we will study the case that both θ and θ' are rectangular.

Theorem 5.3. *If $\theta, \theta' \in \mathrm{Inv}^-(S)_*$ are rectangular and $\theta \neq \pm\theta'$, then $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} = \emptyset$.*

Proof. Suppose that $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$, $\theta \neq \pm\theta'$, and θ is rectangular. We will show that θ' is not rectangular.

By Theorem 5.1 and Theorem 5.2, we know that $\theta^{-1}\theta'$ is conjugate to $\pm C$ in $SL(2,\mathbb{Z})$. Let $\theta^{-1}\theta' = gCg^{-1}$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. We first suppose that $\theta = \theta_1$. Then $\theta' = \pm\theta_1 g C g^{-1} = \pm \begin{pmatrix} -bd-ac & b^2+a^2 \\ d^2+c^2 & -bd-ac \end{pmatrix}$. By Lemma 3.5, one can see that θ' is not rectangular. In fact, if both b^2+a^2 and d^2+c^2 are even, then a and b have the same parity, so do d and c. Therefore ad and bc have the same parity. Thus $ad-bc$ is even. It contradicts the fact that $g \in SL(2,\mathbb{Z})$.

In general, $\theta = h\theta_1 h^{-1}$ for some $h \in SL(2,\mathbb{Z})$. Then we have $\theta' = \pm h\theta_1 h^{-1} g C g^{-1}$, that is $h^{-1}\theta' h = \pm\theta_1(h^{-1}g)C(h^{-1}g)^{-1}$. From the discussion above, we know that $h^{-1}\theta' h$ is not rectangular and hence θ' is not rectangular. $\square$

From Theorem 5.1, Theorem 5.2 and Theorem 5.3, we obtain the following corollary.

Corollary 5.4. *Let $\theta, \theta' \in \operatorname{Inv}^-(S)_*$ with $\theta \neq \pm\theta'$ and assume that $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$. Then $\theta^{-1}\theta'$ is conjugate to $\pm C$ in $SL(2,\mathbb{Z})$ if and only if one of θ and θ' is rectangular and the other is rhombic.*

In the following, we will study for a given Earle slice, how many Earle slices intersect with it.

Theorem 5.5. *For any rhombic (resp. rectangular) Earle slice $\mathcal{E}_\theta$, there exists a unique rectangular (resp. rhombic) Earle slice $\mathcal{E}_{\theta'}$, such that $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$.*

Proof. We only consider the case of rhombic Earle slice. The proof for rectangular Earle slice is similar.

First, we show the existence. From Example (1), we see that for the rhombic Earle slice $\mathcal{E}_{\theta_2}$, there exists the rectangular Earle slice $\mathcal{E}_{\theta_1}$ such that $\mathcal{E}_{\theta_1} \cap \mathcal{E}_{\theta_2} \neq \emptyset$.

In general, for any rhombic Earle slice $\mathcal{E}_{h\theta_2 h^{-1}}$ with $h \in SL(2,\mathbb{Z})$, there exists a rectangular Earle slice $\mathcal{E}_{h\theta_1 h^{-1}}$ such that $\mathcal{E}_{h\theta_2 h^{-1}} \cap \mathcal{E}_{h\theta_1 h^{-1}} \neq \emptyset$, since $(h\theta_2^{-1}h^{-1})(h\theta_1 h^{-1}) = hCh^{-1}$.

Next, we will show the uniqueness. By Corollary 5.4, if one rhombic Earle slice $\mathcal{E}_\theta$ intersects with one rectangular Earle slice $\mathcal{E}_{\theta'}$, we have $\theta^{-1}\theta' = \pm gCg^{-1}$ for some $g \in SL(2,\mathbb{Z})$.

Suppose first that $\theta = \theta_2$. Let $\theta' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $a = -d$ and $a^2 + bc = 1$. We also have $b + c = \operatorname{Tr}(\theta_2^{-1}\theta') = \operatorname{Tr}(\pm gCg^{-1}) = \pm \operatorname{Tr} C = 0$. Thus one can see that $a = -d = \pm 1$ and $b = c = 0$. It follows that $\theta' = \pm\theta_1$. Therefore $\mathcal{E}_{\theta_1}$ is the unique rectangular Earle slice which satisfies the desired property.

In general, $\theta = h\theta_2 h^{-1}$ for some $h \in SL(2,\mathbb{Z})$. Then we have $\theta^{-1}\theta' = h\theta_2 h^{-1}\theta' = gCg^{-1}$ and so $\theta_2(h^{-1}\theta' h) = \pm(h^{-1}g)C(h^{-1}g)^{-1}$. It follows from the discussion above that $h^{-1}\theta' h = \theta_1$, that is $\theta' = h\theta_1 h^{-1}$. Thus for any rhombic Earle slice $\mathcal{E}_{h\theta_2 h^{-1}}$ with $h \in SL(2,\mathbb{Z})$, there exists a unique rectangular Earle slice $\mathcal{E}_{h\theta_1 h^{-1}}$ such that $\mathcal{E}_{h\theta_2 h^{-1}} \cap \mathcal{E}_{h\theta_1 h^{-1}} \neq \emptyset$. □

Theorem 5.6. *For any rhombic Earle slice $\mathcal{E}_\theta$, there exist exactly four distinct rhombic Earle slices $\mathcal{E}_{\varphi_i}$, $i = 1, 2, 3, 4$ such that $\mathcal{E}_{\varphi_i} \cap \mathcal{E}_\theta \neq \emptyset$. Furthermore, the four Earle slices form two pairs, $\{\mathcal{E}_{\varphi_1}, \mathcal{E}_{\varphi_2}\}$ and $\{\mathcal{E}_{\varphi_3}, \mathcal{E}_{\varphi_4}\}$ such that:*

$$\mathcal{E}_\theta \cap \mathcal{E}_{\varphi_1} = \mathcal{E}_\theta \cap \mathcal{E}_{\varphi_2};$$

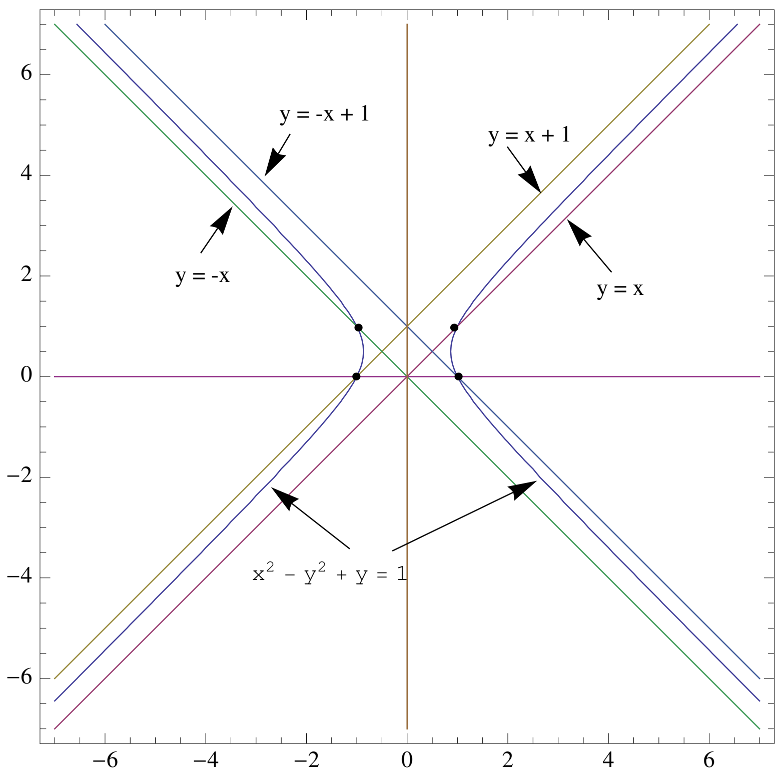

Figure 1: The curves defined by $x^2 - y^2 + y = 1$ and the integer points.

$$\mathcal{E}_\theta \cap \mathcal{E}_{\varphi_3} = \mathcal{E}_\theta \cap \mathcal{E}_{\varphi_4};$$

$$\mathcal{E}_\theta \cap \mathcal{E}_{\varphi_1} \cap \mathcal{E}_{\varphi_2} \cap \mathcal{E}_{\varphi_3} \cap \mathcal{E}_{\varphi_4} = \emptyset.$$

Proof. By Theorem 5.2, if two rhombic Earle slices $\mathcal{E}_\theta$ and $\mathcal{E}_{\theta'}$ intersect, then $\theta^{-1}\theta'$ is conjugate to $\pm D$ or $\pm D^{-1}$ in $SL(2,\mathbb{Z})$.

Suppose first that $\theta = \theta_2$. Let $\mathcal{E}_{\theta'}$ be rhombic Earle slice such that $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$. One may assume that $\theta' = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 + bc = 1$. Since $\theta_2^{-1}\theta' = \begin{pmatrix} c & -a \\ a & b \end{pmatrix}$ is conjugate to $\pm D$ or $\pm D^{-1}$, we have $\mathrm{Tr}(\theta_2^{-1}\theta') = b + c = 1$ or -1. By substituting this to the equation $a^2 + bc = 1$, we obtain $a^2 - b^2 + b = 1$ or $a^2 - b^2 - b = 1$.

Next, we will show how to find all integer solutions to $a^2 - b^2 + b = 1$. It is equivalent to find all integer solutions to the equation $x^2 - y^2 + y = 1$. Since the distance between two different integer numbers cannot be less than 1, one can see that the part of the curve $x^2 - y^2 + y = 1$, which lies between the straight lines $y = x$ and $y = x + 1$, has no integer solution (see Figure 1). So does for the part which lies between the straight lines $y = -x$ and $y = -x + 1$. Then one can see from Figure 1 that all integer solutions to $x^2 - y^2 + y = 1$ are $(x, y) = (1, 0)$, $(-1, 0)$, $(1, 1)$ and

$(-1,1)$. Therefore, all integer solutions for $a^2 - b^2 + b = 1$ are $a = \pm 1, b = 0$ and $a = \pm 1, b = 1$. Since $b + c = 1$ in this case, we obtain that $c = 1$ when $b = 0$, and $c = 0$ when $b = 1$.

Similarly, we know that all integer solutions to $a^2 - b^2 - b = 1$ are $a = \pm 1, b = 0$ and $a = \pm 1, b = -1$. By $b + c = -1$, one can see that $c = -1$ when $b = 0$, and $c = 0$ when $b = -1$.

From the above arguments, we obtain eight solutions $\pm\psi_i$ $(i = 1, 2, 3, 4)$ for θ', where

$$\psi_1 := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \psi_2 := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \psi_3 := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \psi_4 := \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then θ' can only be one of $\pm\psi_i$, where $i = 1, 2, 3, 4$. Since $\mathcal{E}_\phi = \mathcal{E}_{-\phi}$ for any $\phi \in \mathrm{Inv}^-(S)_*$, there exist exactly four distinct rhombic Earle slices intersect with $\mathcal{E}_{\theta_2}$.

Furthermore, we have

$$\theta_2^{-1}\psi_1 = D, \quad \theta_2^{-1}\psi_2 = D^{-1}, \quad \theta_2^{-1}\psi_3 = CDC^{-1}, \quad \theta_2^{-1}\psi_4 = CD^{-1}C^{-1}.$$

Therefore, one sees from Theorem 5.1 that they form two pairs, $\{\mathcal{E}_{\psi_1}, \mathcal{E}_{\psi_2}\}$ and $\{\mathcal{E}_{\psi_3}, \mathcal{E}_{\psi_4}\}$ which satisfy the desired properties.

In general, we suppose that $\theta = h\theta_2 h^{-1}$ with $h \in SL(2, \mathbb{Z})$. Let $\mathcal{E}_{\theta'}$ be rhombic Earle slice such that $\mathcal{E}_\theta \cap \mathcal{E}_{\theta'} \neq \emptyset$. Since $\theta^{-1}\theta' = h\theta_2 h^{-1}\theta'$ is conjugate to $\pm D$ or $\pm D^{-1}$, $\theta_2(h^{-1}\theta' h)$ is also conjugate to $\pm D$ or $\pm D^{-1}$. It follows from the above discussion that $h^{-1}\theta' h = \pm\psi_i$, $i = 1, 2, 3$ *or* 4. Thus $\theta' = \pm h\psi_i h^{-1}$, $i = 1, 2, 3$ *or* 4. So for any rhombic Earle slice $\mathcal{E}_{h\theta_2 h^{-1}}$, there exist exactly four rhombic Earle slices $\mathcal{E}_{h\psi_i h^{-1}}$, $i = 1, 2, 3, 4$ that satisfy the condition. □

Remark 5. In our new notation, Example (2) can be written as

$$\mathcal{E}_{\theta_2} \cap \mathcal{E}_{\psi_1} \cap \mathcal{E}_{\psi_2} = \{Q(S_{e^{i\pi/3}}, \theta_2 \cdot S_{e^{i\pi/3}})\}.$$

Since $C(e^{i\pi/3}) = e^{2i\pi/3}$, we have

$$\mathcal{E}_{\theta_2} \cap \mathcal{E}_{\psi_3} \cap \mathcal{E}_{\psi_4} = \{Q(S_{e^{2i\pi/3}}, \theta_2 \cdot S_{e^{2i\pi/3}})\}.$$

Theorem 5.7. *The union of all rhombic Earle slices is connected.*

Proof. It suffices to show that any rhombic Earle slice $\mathcal{E}_{g\theta_2 g^{-1}}$, $g \in SL(2, \mathbb{Z})$, connects with $\mathcal{E}_{\theta_2}$ through a finite sequence of rhombic Earle slices.

From Theorem 5.6, we know that there exist exactly four distinct rhombic Earle slices $\mathcal{E}_{g\psi_i g^{-1}}$, $i = 1, 2, 3, 4$ such that $\mathcal{E}_{g\psi_i g^{-1}} \cap \mathcal{E}_{g\theta_2 g^{-1}} \neq \emptyset$. Let $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. By calculating, we obtain that

$$\psi_1 = T\theta_2 T^{-1}, \quad \psi_2 = S\theta_2 S^{-1}, \quad \psi_3 = S^{-1}\theta_2 S, \quad \psi_4 = T^{-1}\theta_2 T.$$

It is known that $SL(2, \mathbb{Z})$ is generated by S and T. Therefore $g = s_1 s_2 \ldots s_n$ for some $n \in \mathbb{Z}^+$ with $s_j \in \{S, S^{-1}, T, T^{-1}\}$. Let $g_k = s_1 s_2 \ldots s_k$ for $1 \leqslant k \leqslant n$ and $g_0 = I$. Then

$$g_k \theta_2 g_k^{-1} = g_{k-1} s_k \theta_2 s_k^{-1} g_{k-1}^{-1} = g_{k-1} \psi_i g_{k-1}^{-1},$$

where $i = 1, 2, 3$ *or* 4, which depends on s_k. It follows from the proof of Theorem 5.6 that

$$\mathcal{E}_{g_k \theta_2 g_k^{-1}} \cap \mathcal{E}_{g_{k-1}\theta_2 g_{k-1}^{-1}} = \mathcal{E}_{g_{k-1}\psi_i g_{k-1}^{-1}} \cap \mathcal{E}_{g_{k-1}\theta_2 g_{k-1}^{-1}} = g_{k-1}(\mathcal{E}_{\psi_i} \cap \mathcal{E}_{\theta_2}) \neq \emptyset.$$

Therefore $\mathcal{E}_{g\theta_2 g^{-1}}$ connects with $\mathcal{E}_{\theta_2}$ through a finite sequence of rhombic Earle slices. □

Remark 6. From the relationship between ψ_i and θ_2, where $i = 1, 2, 3, 4$, we have

$$\mathcal{E}_{\psi_1} = T(\mathcal{E}_{\theta_2}), \quad \mathcal{E}_{\psi_2} = S(\mathcal{E}_{\theta_2}), \quad \mathcal{E}_{\psi_3} = S^{-1}(\mathcal{E}_{\theta_2}), \quad \mathcal{E}_{\psi_4} = T^{-1}(\mathcal{E}_{\theta_2}).$$

Combining Theorem 5.4 and Theorem 5.7, we obtain the following.

Corollary 5.8. *The union of all Earle slices is connected.*

6 Trace coordinates

Let $\widetilde{\mathcal{QF}}(S)$ be the set of conjugacy classes of type-preserving, faithful and irreducible representations ρ of $\pi_1(S)$ to $SL(2, \mathbb{C})$ such that the images are discrete and quasi-Fuchsian group. Then $\widetilde{\mathcal{QF}}(S)$ is a natural covering of $\mathcal{QF}(S)$. We denote the preimage of $\mathcal{E}_\theta \in \mathcal{QF}(S)$ by in $\widetilde{\mathcal{QF}}(S)$ by $\widetilde{\mathcal{E}}_\theta$. Fix a pair (α, β) of generators of $\pi_1(S)$ and let $\mu : \widetilde{\mathcal{QF}}(S) \to \mathbb{C}^3$ be the map which sends $[\rho]$ to $(\operatorname{Tr}\rho(\alpha), \operatorname{Tr}\rho(\beta), \operatorname{Tr}\rho(\alpha\beta))$. Then μ is an embedding of $\widetilde{\mathcal{QF}}(S)$ into $\{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 = xyz\} - \{(0, 0, 0)\}$(see for example [2]). Thus we often regard $\widetilde{\mathcal{QF}}(S)$ as a subset of $\mathbb{C}^3$.

For $[\rho] \in \widetilde{\mathcal{QF}}(S)$, let $x = \operatorname{Tr}(\rho(\alpha))$, $y = \operatorname{Tr}(\rho(\beta))$ and $z = \operatorname{Tr}(\rho(\alpha\beta))$. In [7], Komori has proved that $\rho \in \mathcal{E}_{\theta_1}$ if and only if $\operatorname{Tr}(\rho(\alpha\beta)) = \operatorname{Tr}(\rho(\alpha\beta^{-1}))$. On the other hand, Komori and Series have showed in [8] that $\rho \in \mathcal{E}_{\theta_2}$ if and only if $\mathrm{Tr}(\rho(\alpha)) = \mathrm{Tr}(\rho(\beta))$. We thus have the following theorem.

Theorem 6.1 ([8], [7]).

$$\begin{aligned} \widetilde{\mathcal{E}}_{\theta_1} &= \widetilde{\mathcal{QF}}(S) \cap \{(x,y,z) \in \mathbb{C}^3 : xy = 2z\}. \\ \widetilde{\mathcal{E}}_{\theta_2} &= \widetilde{\mathcal{QF}}(S) \cap \{(x,y,z) \in \mathbb{C}^3 : x = y\}. \end{aligned}$$

Therefore, we have

$$\widetilde{\mathcal{E}}_{\theta_1} \cap \widetilde{\mathcal{E}}_{\theta_2} = \{(2\sqrt{2}, 2\sqrt{2}, 4), (-2\sqrt{2}, -2\sqrt{2}, 4)\}.$$

Note that the two points correspond to the same representation into $PSL(2,\mathbb{C})$.

Next, we show that

$$\widetilde{\mathcal{E}}_{\psi_1} = \widetilde{\mathcal{QF}}(S) \cap \{(x,y,z) \in \mathbb{C}^3 : x = z\},$$

$$\widetilde{\mathcal{E}}_{\psi_2} = \widetilde{\mathcal{QF}}(S) \cap \{(x,y,z) \in \mathbb{C}^3 : y = z\},$$

where $\psi_1, \psi_2 \in \mathrm{Inv}^-(S)_*$ are associated with $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \in GL(2,\mathbb{Z})$, respectively (see the proof of Theorem 5.6). In fact, recall that θ_2 is an automorphism of $\pi_1(S)$ such that $\theta_2(\alpha) = \beta$ and $\theta_2(\beta) = \alpha$. Let $\tau \in \mathrm{Aut}(\pi_1(S))$ such that $\tau(\alpha) = \alpha$, and $\tau(\beta) = \alpha\beta$.(Note that the class of $\mathrm{Out}(\pi_1(\mathrm{S}))$ represented by τ corresponds to $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.) Then $\tau\theta_2\tau^{-1} \in \mathrm{Aut}(\pi_1(S))$ satisfies $(\tau\theta_2\tau^{-1})(\alpha) = \alpha\beta$ and $(\tau\theta_2\tau^{-1})(\beta) = \beta^{-1}$. One can see that the class of $\mathrm{Out}(\pi_1(S))$ represented by $\tau\theta_2\tau^{-1}$ corresponds to $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \in GL(2,\mathbb{Z})$. Therefore we can write $\psi_1 = \tau\theta_2\tau^{-1}$. For a pair $(\tau(\alpha), \tau(\beta))$ of canonical generators of $\pi_1(S)$, we have $\psi_1(\tau(\alpha)) = \tau(\beta)$ and $\psi_1(\tau(\beta)) = \tau(\alpha)$. From [8], we know that $[\rho] \in \widetilde{\mathcal{E}}_{\psi_1}$ if and only if $\mathrm{Tr}(\rho(\tau(\alpha))) = \mathrm{Tr}(\rho(\tau(\beta)))$, which means $\mathrm{Tr}(\rho(\alpha)) = \mathrm{Tr}(\rho(\alpha\beta))$. Thus we obtain the expression of $\widetilde{\mathcal{E}}_{\psi_1}$ as above. Similarly, we have the expression of $\widetilde{\mathcal{E}}_{\psi_2}$.

A little calculation shows that

$$\widetilde{\mathcal{E}}_{\theta_2} \cap \widetilde{\mathcal{E}}_{\psi_1} \cap \widetilde{\mathcal{E}}_{\psi_2} = \{(3,3,3)\}.$$

In the same way, we can obtain

$$\widetilde{\mathcal{E}}_{\psi_3} = \widetilde{\mathcal{QF}}(S) \cap \{(x,y,z) \in \mathbb{C}^3 : xy = z + x\},$$

$$\widetilde{\mathcal{E}}_{\psi_4} = \widetilde{\mathcal{QF}}(S) \cap \{(x, y, z) \in \mathbb{C}^3 : xy = z + y\}.$$

By a calculation, one can see that

$$\widetilde{\mathcal{E}}_{\theta_2} \cap \widetilde{\mathcal{E}}_{\psi_3} \cap \widetilde{\mathcal{E}}_{\psi_4} = \{(3, 3, 6)\}.$$

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