Description of the Dixmier-Douady class in simplicial de Rham complexes

(単体的ド・ラーム複体上におけるディクシミエ・ドゥアディ類の記述)

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Abstract

On the basis of A.L.Carey, D.Crowley, M.K.Murray's work, we exhibit a cocycle in the simplicial de Rham complex which represents the Dixmier-Douady class. We exhibit also the "Chern-Simons form" of the Dixmier-Douady class. After that, we explain that this cocycle coincides with a kind of transgression of the second Chern class when we consider a central extension of the loop group and a connection due to J.Mickelsson and J-L.Brylinski, D.McLaughlin.

Contents

1 Introduction

In [7], Carey, Crowley, Murray proved that when a Lie group G admits a central $U(1)$-extension $1 \to U(1) \to \widehat{G} \to G \to 1$, there exists a characteristic class of principal G-bundle $\phi : Y \to M$ which belongs to a cohomology group $H^2(M, \underline{U(1)}) \cong H^3(M, \mathbb{Z})$. Here $\underline{U(1)}$ stands for a sheaf of continuous $U(1)$-valued functions on M. This class is called the Dixmier-Douady class associated to the central $U(1)$-extension $\widehat{G} \to G$.

On the other hand, for any Lie group G there is a topological space BG called the classifying space such that the characteristic classes of principal G-bundles are in one-to-one correspondence with the cohomology classes in $H^*(BG)$. In general BG is a very huge space so we can not use the usual de Rham theory on it. In order to describe the cocycle of $H^*(BG)$, we will use the following simplicial de Rham complex theory due to Segal [28], Bott, Shulman, Stasheff [3] and Dupont [10].

For any Lie group G, we have a simplicial manifold $\{NG(*)\}$. It is a sequence of manifolds $\{NG(p) = G^p\}_{p=0,1,\cdots}$ together with face maps $\varepsilon_i : NG(p) \to NG(p-1)$ for $i = 0, \cdots, p$ satisfying the relations $\varepsilon_i \varepsilon_j = \varepsilon_{j-1} \varepsilon_i$ for $i < j$ (The standard definition also involves degeneracy maps but we do not need them here). Then the n-th cohomology group of the classifying space BG is isomorphic to the total cohomology of the double complex $\{\Omega^q(NG(p))\}_{p+q=n}$. See [3] [10] [20] for details.

There is also a simplicial manifold $\{PG(*)\}$ for G which plays the role of the total space EG of the universal bundle. Since $H^*(EG)$ is trivial if we pull-back any cocycle on $\Omega^*(NG)$ to $\Omega^*(PG)$, it becomes a exact form so there exist a cochain on $\Omega^{*-1}(PG)$ such that its coboudary coincides to the pull-back of that cocycle. Such a cochain can be called the "Chern-Simons form" of that cocycle.

In [30], the author exhibited some cocycles on $\Omega^*(NU(n))$ which represents the Chern character and the Chern-Simons form of the second Chern class on $\Omega^3(PU(n))$.

In this paper we exhibit a cocycle on $\Omega^*(NG(*))$ which represents the Dixmier-Douady class. It is described as follows. See Theorem 3.3.

Theorem A *The universal Dixmier-Douady class associated to* π *and a section* $\hat{s}$ *is represented by the sum of following* $c_1(\theta)$ *and* $-\left(\frac{-1}{2\pi i}\right) \hat{s}^*(\delta\theta)$:

$$
\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle -d} & & & & \\
c_1(\theta)\in\Omega^2(G) & \xrightarrow{\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*} & \Omega^2(G\times G) & & \\
 & & \uparrow{\scriptstyle d} & & \\
 & & -\left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta\theta)\in\Omega^1(G\times G) & \xrightarrow{\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*-\varepsilon_3^*} & 0
\end{array}
$$

□

As a consequence of our result, we can see that if G is given a discrete topology, the Dixmier-Douady class in $H^3(BG^\delta,\mathbb{R})$ is 0. We can also see if G is simply connected, the Dixmier-Douady class in $H^3(BG,\mathbb{R})$ is not 0 if $\widehat{G}\to G$ is not trivial as a principal $U(1)$-bundle. See Corollary 3.1 and Corollary 3.2.

Such a cocycle is also studied in a general setting by Behrend, Tu, Xu and Laurent-Gengoux [1] [2] [33] [34], and Ginot, Stiénon [12]. They described the cocycle in another way. Our construction is more explicit so that we can observe what kind of influence the section $\hat{s}$ of $\delta\widehat{G}:=\varepsilon_0^*\widehat{G}\otimes(\varepsilon_1^*\widehat{G})^{\otimes -1}\otimes\varepsilon_2^*\widehat{G}$ have on the cocycle. We can also see the relation between such a section $\hat{s}$ and the group structure of $\widehat{G}$.

Furthermore, our construction has an advantage that we can also exhibit the "Chern-Simons form" of the Dixmier-Douady class on $\Omega^*(PG(*))$. It is described as follows. See Theorem 3.6.

Theorem B *The Chern-Simons form of the Dixmier-Douady class is a sum of following* $-c_1(\theta)$ *and* $-\left(\frac{-1}{2\pi i}\right)\bar{s}_\rho^*(\bar{\delta}_\rho\theta)$:

$$
\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle d} & & & & \\
-c_1(\theta)\in\Omega^2(G) & \xrightarrow{\bar{\varepsilon}_0^*-\bar{\varepsilon}_1^*} & \Omega^2(PG(1)) & & \\
 & & \uparrow{\scriptstyle -d} & & \\
 & & -\left(\frac{-1}{2\pi i}\right)\bar{s}_\rho^*(\bar{\delta}_\rho\theta)\in\Omega^1(PG(1)) & \xrightarrow{\bar{\varepsilon}_0^*-\bar{\varepsilon}_1^*+\bar{\varepsilon}_2^*} & \Omega^1(PG(2))
\end{array}
$$

□

As a consequence, we can see that the Dixmier-Douady class is mapped to the first Chern class of $\widehat{G} \to G$ by a kind of the transgression map in the sense of Heitsch and Lawson [13].

One of the important examples of the Lie group which have a non-trivial central $U(1)$-extension is a free loop group of a finite dimensional compact Lie group [5][18][27]. We explain that our cocycle coincides with a kind of transgression of the universal second Chern class when we use the central extension $\widehat{LSU(2)} \to LSU(2)$ and the connection form due to Mickelsson [18] and Brylinski, McLaughlin [5][6]. We consider also the case of semi-direct product $LSU(2) \rtimes S^1$ and construct a cocycle in a certain triple complex. Finally, as a natural development of these theory, we give a short survey of the theory of a central $U(1)$-extension of a Lie groupoid. Given a surjective submersion $\phi : Y \to M$, we obtain the groupoid $Y^{[2]} \rightrightarrows Y$, where $Y^{[2]}$ is the fiber product defined as $Y^{[2]} := \{(y_1, y_2)|\phi(y_1) = \phi(y_2)\}$. A central $U(1)$-extension of the groupoid $Y^{[2]} \rightrightarrows Y$ is called a bundle gerbe over M. Bundle gerbe was invented by Murray in [21]. Murray and Stevenson showed that there is one-to-one correspondence between the isomorphism classes of bundle gerbes over M and the cohomology group $H^3(M, \mathbb{Z})$ [22].

The outline of this paper is as follows. Section 2 is a preliminary. We briefly recall the notion of simplicial manifold NG and the relation with the classifying space BG. In Section 3, we recall the definition of the Dixmier-Douady class and construct a cocycle in $\Omega^*(NG(*))$ and prove the main theorem (Theorem 3.3). We also exhibit the "Chern-Simons form" of the Dixmier-Douady class. In Section 4, we discuss the case of central $U(1)$-extension of the loop group following the idea of Brylinski, McLaughlin [6] and Murray, Stevenson [23][24]. Section 5 is a short survey of the theory of a central $U(1)$-extension of a groupoid.

2 The double complex on simplicial manifold

In this section first we recall the relation between the simplicial manifold NG and the classifying space BG.

As a convention of this paper, a Lie group means a paracompact Lie group modeled on a Hausdorff locally convex topological vector space. For example, we will consider not only the case of a finite dimensional Li group, but also the case of an infinite dimensional loop group, unitary group acting on a Hilbert space. See Section 3.1.

For any Lie group G, we define simplicial manifolds NG, PG and a simplicial G-bundle $\rho : PG \to NG$ as follows:

$NG(p) := \overbrace{G \times \cdots \times G}^{p-times} \ni (g_1, \cdots, g_p)$:
face operators $\varepsilon_i : NG(p) \to NG(p-1)$

$$\varepsilon_i(g_1, \cdots, g_p) = \begin{cases} (g_2, \cdots, g_p) & i = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots, g_p) & i = 1, \cdots, p-1 \\ (g_1, \cdots, g_{p-1}) & i = p. \end{cases}$$

$PG(p) := \overbrace{G \times \cdots \times G}^{p+1-times} \ni (\bar{g}_0, \cdots, \bar{g}_p)$:
face operators $\bar{\varepsilon}_i : PG(p) \to PG(p-1)$

$$\bar{\varepsilon}_i(\bar{g}_0, \cdots, \bar{g}_p) = (\bar{g}_0, \cdots, \bar{g}_{i-1}, \bar{g}_{i+1}, \cdots, \bar{g}_p) \qquad i = 0, 1, \cdots, p.$$

We define $\rho : PG \to NG$ as $\rho(\bar{g}_0, \cdots, \bar{g}_p) := (\bar{g}_0 \bar{g}_1^{-1}, \cdots, \bar{g}_{p-1} \bar{g}_p^{-1})$.

To any simplicial manifold $X = \{X_*\}$, we can associate a topological space $\parallel X \parallel$ called the fat realization. Since any G-bundle $\rho : E \to M$ can be realized as the pull-back of the fat realization of ρ, $\parallel \rho \parallel$ is the universal bundle $EG \to BG$ [28].

Now we construct a double complex associated to a simplicial manifold.

Definition 2.1. For any simplicial manifold $\{X_*\}$ with face operators $\{\varepsilon_*\}$, we define double complex as follows:

$$\Omega^{p,q}(X) \stackrel{\text{def}}{=} \Omega^q(X_p).$$

Derivatives are:

$$d' := \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*, \qquad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).$$

□

For NG and PG the following theorem holds [3][10][20].

Theorem 2.1. *There exist ring isomorphisms*

$$H^*(\Omega^*(NG)) \cong H^*(BG), \qquad H^*(\Omega^*(PG)) \cong H^*(EG).$$

Here $\Omega^*(NG)$ *and* $\Omega^*(PG)$ *mean the total complexes.* □

Remark 2.1. To prove this theorem, they used the property that G is an ANR (absolute neighborhood retract) and the theorem of de Rham on G holds true.

Remark 2.2. The cohomology group of the horizontal complex in the edge $(\Omega^0(NG(p)), d' := \sum_{i=0}^{p+1}(-1)^i\varepsilon_i^*)$ is called the smooth cohomology of G. Note that even when G is given a discrete topology, this complex and cohomology still make sense. Furthermore even the coefficient is changed to $U(1)$, we can define the smooth cohomology. It is denoted by $H^*(G, U(1))$.

For a principal G-bundle $Y \to M$ and an open covering $\{U_\alpha\}$ of M, the transition functions $(g_{\alpha_0\alpha_1}, g_{\alpha_1\alpha_2}, \cdots, g_{\alpha_{p-1}\alpha_p}) : U_{\alpha_0\alpha_1\cdots\alpha_p} \to NG(p)$ induce the cohomology map $H^*(NG) \to H^*_{\check{C}ech-deRham}(M)$. The elements in the image are the characteristic class of Y [20].

Example 2.1. In the case of special orthogonal group $G = SO(2)$, the Euler class $e \in H^2(BSO(2), \mathbb{R})$ is represented by the cocycle below.

$$\begin{array}{ccc} 0 & & \\ \uparrow{\scriptstyle -d} & & \\ \frac{-1}{2\pi i}(\mathrm{Pf}(h^{-1}dh)) \in \Omega^1(SO(2)) & \xrightarrow{\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*} & 0 \end{array}$$

Here Pf is defined as:

$$\mathrm{Pf}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{4\pi i}(a_{12} - a_{21}).$$

When we give a discrete topology to $SO(2)$, the cocycle above vanishes. This means that the Euler class of a flat principal $SO(2)$-bundle is a torsion class. On the other hand, in the case of special linear group $SL(2)$, the Euler class in $H^2(BSL(2), \mathbb{R})$ is represented as the sum of differential forms which belong to $\Omega^1(SL(2))$ and $\Omega^0(SL(2) \times SL(2))$ (See for example [4][10, Chapter 9]). So the Euler class of a flat principal $SL(2)$-bundle is not necessarily a torsion class. You can find an example of a flat principal $SL(2)$-bundle over a closed oriented surface whose genus is g such that its Euler number is $g-1$, for instance in [10, Chapter 9].

3 Dixmier-Douady class on the double complex

3.1 Definition of the Dixmier-Douady class

To begin with, we recall the definition of a central extension of a group.

Definition 3.1. For any group G, its subgroup is called the center of G when it consists of the element of G that is commutative with any element in G. Given two groups N, G, if we can construct the group $\widehat{G}$ such that it has the normal subgroup $\bar{N}$ which is isomorphic to N and $\widehat{G}/\bar{N}$ is isomorphic to G, then $\widehat{G}$ is called a extension of G by N.
When N is abelian and the center of $\widehat{G}$ contains N, $\widehat{G}$ is called a central N-extension of G. $\square$

Next, we recall the definition of the Dixmier-Douady class, following [7]. Let $\phi : Y \to M$ be a principal G-bundle and $\{U_\alpha\}$ a Leray covering of M. When G has a central $U(1)$-extension $\pi : \widehat{G} \to G$, the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ lift to $\widehat{G}$. i.e. there exist continuous maps $\hat{g}_{\alpha\beta} : U_{\alpha\beta} \to \widehat{G}$ such that $\pi \circ \hat{g}_{\alpha\beta} = g_{\alpha\beta}$. This is because each $U_{\alpha\beta}$ is contractible so the pull-back of π by $g_{\alpha\beta}$ has a global section. Now the $U(1)$-valued functions $c_{\alpha\beta\gamma}$ on $U_{\alpha\beta\gamma}$ are defined as $(\hat{g}_{\beta\gamma}(\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma})^{-1}\hat{g}_{\alpha\beta}) \cdot c_{\alpha\beta\gamma} := \hat{g}_{\beta\gamma}\hat{g}_{\alpha\gamma}^{-1}\hat{g}_{\alpha\beta} \in g_{\beta\gamma}^*\widehat{G} \otimes (g_{\alpha\gamma}^*\widehat{G})^{\otimes -1} \otimes g_{\alpha\beta}^*\widehat{G}$. Then it is easily seen that $\{c_{\alpha\beta\gamma}\}$ is a $U(1)$-valued Čech-cocycle on M and hence defines a cohomology class in $H^2(M, \underline{U(1)}) \cong H^3(M, \mathbb{Z})$. This class is called the Dixmier-Douady class of Y.

Remark 3.1. Let $s_{\alpha\beta\gamma}$ be a section of $\widehat{G}_{\alpha\beta\gamma} := g_{\beta\gamma}^*\widehat{G} \otimes (g_{\alpha\gamma}^*\widehat{G})^{\otimes -1} \otimes g_{\alpha\beta}^*\widehat{G}$ such that $\delta s_{\alpha\beta\gamma} := s_{\beta\gamma\delta} \otimes s_{\alpha\gamma\delta}^{\otimes -1} \otimes s_{\alpha\beta\delta} \otimes s_{\alpha\beta\gamma}^{\otimes -1} = 1$. This condition makes sence since $\widehat{G}_{\beta\gamma\delta} \otimes \widehat{G}_{\alpha\gamma\delta}^{\otimes -1} \otimes \widehat{G}_{\alpha\beta\delta} \otimes \widehat{G}_{\alpha\beta\gamma}$ is canonically trivial. Then we can define a $U(1)$-valued Čech-cocycle $c_{\alpha\beta\gamma}^s$ on M by the equation $s_{\alpha\beta\gamma} \cdot c_{\alpha\beta\gamma}^s = \hat{g}_{\beta\gamma}\hat{g}_{\alpha\gamma}^{-1}\hat{g}_{\alpha\beta}$.

The cohomology class $[c^s_{\alpha\beta\gamma}] \in H^2(M, \underline{U(1)}) \cong H^3(M, \mathbb{Z})$ can be also called the Dixmier-Douady class of Y.

Example 3.1. Recall that the complex spin group $Spin^{\mathbb{C}}(n)$ is defined as $Spin^{\mathbb{C}}(n) := Spin(n) \times_{\mathbb{Z}_2} U(1)$. When we consider the central $U(1)$-extension $1 \to U(1) \to Spin^{\mathbb{C}}(n) \to SO(n) \to 1$, the Dixmier-Douady class of the $Spin^{\mathbb{C}}(n)$-bundle coincides with the third integral Stiefel-Whitney class $w_3(TM)$. Let B denote the Bockstein map and $w_2(TM)$ the second Stiefel-Whitney class. Then $w_3(TM) = Bw_2(TM)$ hence $w_3(TM)$ is a 2-torsion class.

To obtain a non-torsion class, G must be infinite dimensional (cf. for example [5] Ch.4 p.166) and we require also G to have a partition of unity so that we can consider a connection form on the $U(1)$-bundle over G. A good example which satisfies such a condition is the loop group of a finite dimensional compact Lie group [5] [27].

Another important example is the restricted unitary group $U_{res}(H)$ [5] [27]. Here H is an infinite-dimensional, separable Hilbert space with an orthogonal decomposition $H = H_+ \oplus H_-$. This group consists of the unitary operator of H such that with block decomposition $\binom{AB}{CD}$, B and C are Hilbert-Schmidt operators (We can also see that these groups are ANR and the theorem of de Rham holds on them [17][26]).

Let $U(H)$ denote the group of unitary operators on H endowed with the strong operator topology and let $PU(H) = U(H)/U(1)$ be the projective unitary group with the quotient topology. Here $U(1)$ consists of scalar multiples of the identity operator on H of norm equal to 1. The definition of the Dixmier-Douady class above is valid for the central extension $U(1) \to U(H) \to PU(H)$ and we obtain the Dixmier-Douady class for each principal $PU(H)$-bundle. It is well-known that for any topological space M, the cohomology group $H^3(M, \mathbb{Z})$ is isomorphic to $[M, BPU(H)]$ which is the set of homotopy classes of continuous maps from M to $BPU(H)$. So there is one-to-one correspondence between the set of isomorphism classes of principal $PU(H)$-bundles over M and the cohomology group $H^3(M, \mathbb{Z})$. The corresponding element in $H^3(M, \mathbb{Z})$ is the Dixmier-Douady class of each principal $PU(H)$-bundle.
For $g \in U(H)$, let $Ad(g)$ denote the automorphism $T \to gTg^{-1}$ of $\mathcal{K}$ which is the C^*-algebra of compact operators on H. Ad is a continuous homomorphism of $U(H)$ onto $\mathrm{Aut}(\mathcal{K})$ with kernel $U(1)$ where $\mathrm{Aut}(\mathcal{K})$ is given

the point-norm topology. Under this homomorphism we can identify $PU(H)$ with $\mathrm{Aut}(\mathcal{K})$. Since $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$, the set of isomorphism classes of locally trivial bundles over M with fiber $\mathcal{K}$ and the structure group $\mathrm{Aut}(\mathcal{K})$ forms a group under the tensor product. The inverse is the conjugate bundle. Then the following theorem holds.

Theorem 3.1 (Dixmier-Douady [9])**.** *The group of isomorphism classes of locally trivial bundles over M with fiber $\mathcal{K}$ and the structure group $\mathrm{Aut}(\mathcal{K})$ is isomorphic to $H^3(M, \mathbb{Z})$.* □

3.2 Construction of the cocycle

Let $\pi : \widehat{G} \to G$ be a central $U(1)$-extension of a Lie group G. Following [6] [7], we recognize it as a $U(1)$-bundle. Using the face operators $\{\varepsilon_i\} : NG(2) \to NG(1) = G$, we can construct a $U(1)$-bundle over $NG(2) = G \times G$ as $\delta\widehat{G} := {\varepsilon_0}^*\widehat{G} \otimes ({\varepsilon_1}^*\widehat{G})^{\otimes -1} \otimes {\varepsilon_2}^*\widehat{G}$. Here we define the tensor product $S \otimes T$ of $U(1)$-bundles S and T over M by

$$S \otimes T := \bigcup_{x \in M} (S_x \times T_x/(s,t) \sim (sz, tz^{-1}), (z \in U(1)).$$

Lemma 3.1. *$\delta\widehat{G} \to G \times G$ is a trivial bundle.*

Proof. We can construct a bundle isomorphism $f : {\varepsilon_0}^*\widehat{G} \otimes {\varepsilon_2}^*\widehat{G} \to {\varepsilon_1}^*\widehat{G}$ as follows. First we define f to be the map sending $[((g_1, g_2), \hat{g}_2), ((g_1, g_2), \hat{g}_1)]$ such that $\pi(\hat{g}_2) = g_2, \pi(\hat{g}_1) = g_1$ to $((g_1, g_2), \hat{g}_1\hat{g}_2)$. Then we have the inverse f^{-1} which sends $((g_1, g_2), \hat{g})$ such that $\pi(\hat{g}) = g_1g_2$ to $[((g_1, g_2), \hat{g}_2), ((g_1, g_2), \hat{g}\hat{g}_2^{-1})]$ such that $\pi(\hat{g}_2) = g_2$ □

Remark 3.2. $\delta(\delta\widehat{G})$ is canonically isomorphic to $G \times G \times G \times U(1)$ because $\varepsilon_i\varepsilon_j = \varepsilon_{j-1}\varepsilon_i$ for $i < j$.

For any connection θ on $\widehat{G}$, there is an induced connection $\delta\theta$ on $\delta\widehat{G}$ [5, Brylinski].

Proposition 3.1. *Let $c_1(\theta)$ denote the first Chern form of $\widehat{G}$ i.e. the 2-form on G which hits $\left(\frac{-1}{2\pi i}\right) d\theta \in \Omega^2(\widehat{G})$ by π^*, and $\hat{s}$ any global section of $\delta\widehat{G}$. Then the following equation holds.*

$$(\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*)c_1(\theta) = \left(\frac{-1}{2\pi i}\right) d(\hat{s}^*(\delta\theta)) \quad \in \Omega^2(NG(2)).$$

Proof. Choose an open cover $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ of G such that there exist local sections $\eta_\lambda : V_\lambda \to \widehat{G}$ of π. Then $\{\varepsilon_0^{-1}(V_\lambda)\cap\varepsilon_1^{-1}(V_{\lambda'})\cap\varepsilon_2^{-1}(V_{\lambda''})\}_{\lambda,\lambda',\lambda''\in\Lambda}$ is an open cover of $G\times G$ and we have induced local sections $\varepsilon_0^*\eta_\lambda\otimes(\varepsilon_1^*\eta_{\lambda'})^{\otimes -1}\otimes\varepsilon_2^*\eta_{\lambda''}$ on this covering.

If we pull back $\delta\theta$ by these sections, the induced form on $\varepsilon_0^{-1}(V_\lambda)\cap\varepsilon_1^{-1}(V_{\lambda'})\cap\varepsilon_2^{-1}(V_{\lambda''})$ is $\varepsilon_0^*(\eta_\lambda^*\theta)-\varepsilon_1^*(\eta_{\lambda'}^*\theta)+\varepsilon_2^*(\eta_{\lambda''}^*\theta)$. We restrict $(\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*)c_1(\theta)$ on $\varepsilon_0^{-1}(V_\lambda)\cap\varepsilon_1^{-1}(V_{\lambda'})\cap\varepsilon_2^{-1}(V_{\lambda''})$ then it is equal to $\left(\frac{-1}{2\pi i}\right)d(\varepsilon_0^*(\eta_\lambda^*\theta)-\varepsilon_1^*(\eta_{\lambda'}^*\theta)+\varepsilon_2^*(\eta_{\lambda''}^*\theta))$, because $c_1(\theta)=\sum\left(\frac{-1}{2\pi i}\right)d(\eta_\lambda^*\theta)$.

Also

$$d(\varepsilon_0^*(\eta_\lambda^*\theta)-\varepsilon_1^*(\eta_{\lambda'}^*\theta)+\varepsilon_2^*(\eta_{\lambda''}^*\theta)) = d(\hat{s}^*(\delta\theta))|_{\varepsilon_0^{-1}(V_\lambda)\cap\varepsilon_1^{-1}(V_{\lambda'})\cap\varepsilon_2^{-1}(V_{\lambda''})}.$$

Since $\delta\theta$ is a connection form. This completes the proof. □

Proposition 3.2. *We take a section $\hat{s}$ on $\delta\widehat{G}$ such that $\delta\hat{s} := \varepsilon_0^*\hat{s}\otimes(\varepsilon_1^*\hat{s})^{\otimes-1}\otimes\varepsilon_2^*\hat{s}\otimes(\varepsilon_3^*\hat{s})^{\otimes-1} = 1$ on $\delta(\delta\widehat{G})$. Then for the face operators $\{\varepsilon_i\}_{i=0,1,2,3} : NG(3)\to NG(2)$, we have*

$$(\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*-\varepsilon_3^*)(\hat{s}^*(\delta\theta)) = 0.$$

Proof. We consider the $U(1)$-bundle $\delta(\delta\widehat{G})$ over $NG(3)=G\times G\times G$ and the induced connection $\delta(\delta\theta)$ on it. Composing $\{\varepsilon_i\} : NG(3)\to NG(2)$ and $\{\varepsilon_i\} : NG(2)\to G$, we define maps $\{r_i\}_{i=0,1,\cdots,5} : NG(3)\to G$ as follows.

$$r_0=\varepsilon_0\circ\varepsilon_1=\varepsilon_0\circ\varepsilon_0\,,\quad r_1=\varepsilon_0\circ\varepsilon_2=\varepsilon_1\circ\varepsilon_0\,,\quad r_2=\varepsilon_0\circ\varepsilon_3=\varepsilon_2\circ\varepsilon_0$$

$$r_3=\varepsilon_1\circ\varepsilon_2=\varepsilon_1\circ\varepsilon_1\,,\quad r_4=\varepsilon_1\circ\varepsilon_3=\varepsilon_2\circ\varepsilon_1\,,\quad r_5=\varepsilon_2\circ\varepsilon_3=\varepsilon_2\circ\varepsilon_2.$$

Then $\{\bigcap r_i^{-1}(V_{\lambda^{(i)}})\}$ is a covering of $NG(3)$. Since each $\bigcap r_i^{-1}(V_{\lambda^{(i)}})$ is equal to

$$\varepsilon_0^{-1}(\varepsilon_0^{-1}(V_\lambda)\cap\varepsilon_1^{-1}(V_{\lambda'})\cap\varepsilon_2^{-1}(V_{\lambda''}))\cap\varepsilon_1^{-1}(\varepsilon_0^{-1}(V_\lambda)\cap\varepsilon_1^{-1}(V_{\lambda^{(3)}})\cap\varepsilon_2^{-1}(V_{\lambda^{(4)}}))$$

$$\cap\varepsilon_2^{-1}(\varepsilon_0^{-1}(V_{\lambda'})\cap\varepsilon_1^{-1}(V_{\lambda^{(3)}})\cap\varepsilon_2^{-1}(V_{\lambda^{(5)}}))\cap\varepsilon_3^{-1}(\varepsilon_0^{-1}(V_{\lambda''})\cap\varepsilon_1^{-1}(V_{\lambda^{(4)}})\cap\varepsilon_2^{-1}(V_{\lambda^{(5)}})).$$

We have the following induced local sections on it.

$$\varepsilon_0^*(\varepsilon_0^*\eta_\lambda\otimes(\varepsilon_1^*\eta_{\lambda'})^{\otimes-1}\otimes\varepsilon_2^*\eta_{\lambda''})\otimes\varepsilon_1^*(\varepsilon_0^*\eta_\lambda\otimes(\varepsilon_1^*\eta_{\lambda^{(3)}})^{\otimes-1}\otimes\varepsilon_2^*\eta_{\lambda^{(4)}})^{\otimes-1}$$

$$\otimes\varepsilon_2^*(\varepsilon_0^*\eta_{\lambda'}\otimes(\varepsilon_1^*\eta_{\lambda^{(3)}})^{\otimes-1}\otimes\varepsilon_2^*\eta_{\lambda^{(5)}})\otimes\varepsilon_3^*(\varepsilon_0^*\eta_{\lambda''}\otimes(\varepsilon_1^*\eta_{\lambda^{(4)}})^{\otimes-1}\otimes\varepsilon_2^*\eta_{\lambda^{(5)}})^{\otimes-1}.$$

From direct computations we can check that this is equal to canonical section 1 on $\delta(\delta\widehat{G})$ and the pull-back of $\delta(\delta\theta)$ by this section is equal to 0. This means that $\delta(\delta\theta)$ is the Maurer-Cartan connection. Hence if we pull back $\delta(\delta\theta)$ by the induced section $\delta\hat{s}$, it is also equal to 0 and this pull-back is nothing but $(\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^* - \varepsilon_3^*)(\hat{s}^*(\delta\theta))$. □

The propositions above give the cocycle $c_1(\theta) - \left(\frac{-1}{2\pi i}\right) \hat{s}^*(\delta\theta) \in \Omega^3(NG)$ described in the following diagram.

$$
\begin{array}{ccccc}
0 & & & & \\
\Big\uparrow{\scriptstyle -d} & & & & \\
c_1(\theta) \in \Omega^2(G) & \xrightarrow{\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*} & \Omega^2(G \times G) & & \\
 & & \Big\uparrow{\scriptstyle d} & & \\
 & & -\left(\frac{-1}{2\pi i}\right) \hat{s}^*(\delta\theta) \in \Omega^1(G \times G) & \xrightarrow{\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*-\varepsilon_3^*} & 0
\end{array}
$$

Then we can show:

Proposition 3.3. *The cohomology class* $[c_1(\theta) - \left(\frac{-1}{2\pi i}\right) \hat{s}^*(\delta\theta)] \in H^3(\Omega(NG))$ *does not depend on* θ.

Proof. Suppose θ_0 and θ_1 are two connections on $\widehat{G}$. Consider the $U(1)$-bundle $\widehat{G} \times [0,1] \to G \times [0,1]$ and the connection form $t\theta_0 + (1-t)\theta_1$ on it. Then we obtain the cocycle

$$c_1(t\theta_0 + (1-t)\theta_1) - \left(\frac{-1}{2\pi i}\right) \hat{s}^*(\delta(t\theta_0 + (1-t)\theta_1))$$

on $\Omega^3(NG \times [0,1])$. Let $i_0 : NG \times \{0\} \to NG \times [0,1]$ and $i_1 : NG \times \{1\} \to NG \times [0,1]$ be the natural inclusion map. When we identify $NG \times \{0\}$ with $NG \times \{1\}$, $(i_0^*)^{-1} i_1^* : H(\Omega^*(NG \times \{0\})) \to H(\Omega^*(NG \times \{1\}))$ is the identity map. Hence $[c_1(\theta_0) - \left(\frac{-1}{2\pi i}\right) \hat{s}^*(\delta\theta_0)] = [c_1(\theta_1) - \left(\frac{-1}{2\pi i}\right) \hat{s}^*(\delta\theta_1)]$. □

Now we consider what happens if we change the section $\hat{s}$. There is a natural section $\hat{s}_{nt}$ of $\delta\widehat{G}$ defined as;

$$\hat{s}_{nt}(g_1, g_2) := [((g_1, g_2), \hat{g}_2), ((g_1, g_2), \hat{g}_1\hat{g}_2)^{\otimes -1}, ((g_1, g_2), \hat{g}_1)].$$

Then any other section $\hat{s}$ such that $\delta\hat{s} = 1$ can be represented by $\hat{s} = \hat{s}_{nt} \cdot \varphi$ where φ is a $U(1)$-valued smooth function on $G \times G$ which satisfies $\delta\varphi = 1$. If we pull back $\delta\theta$ by $\hat{s}$, the equation $\hat{s}^*(\delta\theta) = \hat{s}^*_{nt}(\delta\theta) + d\log\varphi$ holds. If there exists a $U(1)$-valued smooth function φ' on G which satisfies $\delta\varphi' = \varphi$, the cohomology class $[-\left(\frac{-1}{2\pi i}\right) d\log\varphi]$ is equal to 0 in $H^3(\Omega(NG))$. So we have the following proposition.

Proposition 3.4. *Up to the cohomology class in the $U(1)$-valued smooth cohomology $H^2(G, U(1))$, the cohomology class $[c_1(\theta) - \left(\frac{-1}{2\pi i}\right) \hat{s}^*(\delta\theta)]$ is decided uniquely by the central $U(1)$-extension $\widehat{G} \to G$.* □

Next we discuss about the relation between the section $\hat{s}$ and the multiplication of $\widehat{G}$. Using the section $\hat{s}$, we can define another multiplication $m : \widehat{G} \times \widehat{G} \to \widehat{G}$ of $\widehat{G}$ by:

$$\hat{s}(g_1, g_2) =: [((g_1, g_2), \hat{g}_2), ((g_1, g_2), m(\hat{g}_1, \hat{g}_2))^{\otimes -1}, ((g_1, g_2), \hat{g}_1)].$$

Since $\hat{s}(g_1, g_2)$ is equal to $\hat{s}_{nt} \cdot \varphi$, we can see that $m(\hat{g}_1, \hat{g}_2) = \hat{g}_1\hat{g}_2(\varphi(g_1, g_2))^{-1}$. When $\widehat{G}$ is given this new structure, $\hat{s}$ is of course a natural section of $\delta\widehat{G}$. We say that $f : \widehat{G} \to (\widehat{G}, m)$ is an isomorphism between the central $U(1)$-extensions if f is a group isomorphism and $\pi(\hat{g}) = \pi(f(\hat{g}))$, $f(\hat{g}z) = f(\hat{g})z$ holds for any $\hat{g} \in \widehat{G}$ and $z \in U(1)$. Then the theorem below holds.

Theorem 3.2. *Let $\hat{s}$ be a section of $\delta\widehat{G}$ defined by $\hat{s} := \hat{s}_{nt} \cdot \varphi$ for a $U(1)$-valued smooth function on $G \times G$ which satisfies $\delta\varphi = 1$. When we reconstruct the the multiplication m of $\widehat{G}$ such that $\hat{s}$ becomes a natural section of $\delta\widehat{G}$, $(\widehat{G}, m)$ is isomorphic to $\widehat{G}$ if and only if $[\varphi] \in H^2(G, U(1))$ is 0.*

Proof. Assume that there exists a $U(1)$-valued smooth function φ' on G which satisfies $\varphi(g_1, g_2) = \delta\varphi'(g_1, g_2) := \varphi'(g_2) \cdot (\varphi'(g_1g_2))^{-1} \cdot \varphi'(g_1)$. We define a map $f : \widehat{G} \to \widehat{G}$ by $f(\hat{g}) := \hat{g} \cdot \varphi'(g)$. Then

$$m(f(\hat{g}_1), f(\hat{g}_2)) = f(\hat{g}_1)f(\hat{g}_2)(\varphi(g_1, g_2))^{-1} = \hat{g}_1\varphi'(g_1)\hat{g}_2\varphi'(g_2)(\varphi(g_1, g_2))^{-1}$$

is equal to

$$f(\hat{g}_1\hat{g}_2) = \hat{g}_1\hat{g}_2\varphi'(g_1g_2)$$

and $\pi(\hat{g}) = \pi(f(\hat{g}))$, $f(\hat{g}z) = f(\hat{g})z$. Moreover f has the inverse map $f^{-1}(\hat{g}) := \hat{g} \cdot (\varphi'(g))^{-1}$ hence f is an isomorphism from $\widehat{G}$ to $(\widehat{G}, m)$.

Conversely, assume that there exists an isomorphism f from $\widehat{G}$ to $(\widehat{G}, m)$. Since $\pi(\hat{g}) = \pi(f(\hat{g}))$ and $f(\hat{g}z) = f(\hat{g})z$, we can define a $U(1)$-valued map

φ' on G by $f(\hat{g}) =: \hat{g} \cdot \varphi'(g)$. Now $m(f(\hat{g}_1), f(\hat{g}_2)) = f(\hat{g}_1\hat{g}_2)$ induces the equation $\varphi(g_1, g_2) = \varphi'(g_2) \cdot (\varphi'(g_1g_2))^{-1} \cdot \varphi'(g_1)$.

$\square$

Remark 3.3. Let $H^{(n)}$ denote the separable Hilbert space $L^2(S^1; \mathbb{C}^n)$ of square-summable $\mathbb{C}^n$-valued functions on the circle. The diffeomorphism $f : S^1 \to S^1$ acts on functions $\{\xi : S^1 \to \mathbb{C}^n\} \in H^{(n)}$ by $(f \cdot \xi)(t) := \xi(f^{-1}(t)) \cdot |(f^{-1})'(t)|^{1/2}$. It is known that the inclusion $\mathrm{Diff}^+S^1 \hookrightarrow U_{res}(H)$ induces the discrete topology on $\mathrm{Diff}^+\mathrm{S}^1$ (See [27]). So the cohomology class in $H^2(U_{res}(H), U(1))$ induces the cohomology class in $H^2(\mathrm{Diff}^{+\delta}S^1, U(1))$. This fact may suggest some relationship between the Dixmier-Douady class and the characteristic classes of flat S^1-bundles.

3.3 Main results

We fix any section $\hat{s}$ of $\delta\widehat{G}$ which satisfies $\delta s = 1$. Since $g^*_{\beta\gamma}\widehat{G} \otimes (g^*_{\alpha\gamma}\widehat{G})^{\otimes -1} \otimes g^*_{\alpha\beta}\widehat{G}$ is the pull-back of $\delta\widehat{G}$ by $(g_{\alpha\beta}, g_{\beta\gamma}) : U_{\alpha\beta\gamma} \to G \times G$, there is an induced section of $g^*_{\beta\gamma}\widehat{G} \otimes (g^*_{\alpha\gamma}\widehat{G})^{\otimes -1} \otimes g^*_{\alpha\beta}\widehat{G}$. So we can define the Dixmier-Douady class by using this section.

Now we are ready to state the main theorem.

Definition 3.2. We call the sum of $c_1(\theta) \in \Omega^2(NG(1))$ and $-\left(\frac{-1}{2\pi i}\right)\hat{s}^*(\delta\theta) \in \Omega^1(NG(2))$ the simplicial Dixmier-Douady cocycle associated to π and $\hat{s}$.

Theorem 3.3. *The simplicial Dixmier-Douady cocycle represents the universal Dixmier-Douady class associated to π and a section $\hat{s}$.*

Proof. We show that the $[C_{2,1} + C_{1,2}]$ described in the diagram below is equal to $[\{\left(\frac{-1}{2\pi i}\right) d\log c_{\alpha\beta\gamma}\}]$ as a Čech-de Rham cohomology class of $M = \bigcup U_\alpha$.

$$\begin{array}{ccc} C_{2,1} \in \prod \Omega^2(U_{\alpha\beta}) & & \\ \uparrow{\scriptstyle -d} & & \\ \prod \Omega^1(U_{\alpha\beta}) & \xrightarrow{\check{\delta}} & C_{1,2} \in \prod \Omega^1(U_{\alpha\beta\gamma}). \end{array}$$

Here $C_{2,1}$ and $C_{1,2}$ are Čech-de Rham cocycles defined by

$$C_{2,1} = \{(g^*_{\alpha\beta}c_1(\theta))\}, \qquad C_{1,2} = \left\{-\left(\frac{-1}{2\pi i}\right)(g_{\alpha\beta}, g_{\beta\gamma})^*\hat{s}^*(\delta\theta)\right\}.$$

Since $g_{\alpha\beta}^* c_1(\theta) = \hat{g}_{\alpha\beta}^* \pi^*(c_1(\theta)) = d\left(\frac{-1}{2\pi i}\right)\hat{g}_{\alpha\beta}^*\theta$, we can see

$$[C_{2,1} + C_{1,2}] = [\check{\delta}\{\left(\frac{-1}{2\pi i}\right)\hat{g}_{\alpha\beta}^*\theta\} + C_{1,2}].$$

By definition $(\hat{s} \circ (g_{\alpha\beta}, g_{\beta\gamma}))(p) \cdot c_{\alpha\beta\gamma}(p) = (\hat{g}_{\beta\gamma} \otimes \hat{g}_{\alpha\gamma}^{\otimes -1} \otimes \hat{g}_{\alpha\beta})(p)$ for any $p \in U_{\alpha\beta\gamma}$. Hence $(g_{\alpha\beta}, g_{\beta\gamma})^*\hat{s}^*(\delta\theta) + d\log c_{\alpha\beta\gamma} = \check{\delta}\{\hat{g}_{\alpha\beta}^*\theta\}$.

□

Corollary 3.1. *If the principal G-bundle over M is flat, then its Dixmier-Douady class is 0 in $H^3(M,\mathbb{R})$.*

Proof. This is because the cocycle in Theorem 3.3 vanishes when G is given a discrete topology. □

Corollary 3.2. *If the first Chern class of $\pi : \widehat{G} \to G$ is not 0 in $H^2(G,\mathbb{R})$, the corresponding Dixmier-Douady class of the universal G-bundle is not 0. Especially, if G is simply connected and $\pi : \widehat{G} \to G$ is not trivial as a principal $U(1)$-bundle, then the corresponding Dixmier-Douady class of the universal G-bundle is not 0.*

Proof. In that situation, any differential form $x \in \Omega^1(NG(1))$ does not hit $c_1(\theta) \in \Omega^2(NG(1))$ by $d : \Omega^1(NG(1)) \to \Omega^2(NG(1))$. □

3.4 Another description

On the other hand, there is a simplicial manifold $N\widehat{G}$ and face operators $\hat{\varepsilon}_i$ of it. Using this, Behrend and Xu described the cocycle which represents the Dixmier-Douady class in another way.

Proposition 3.5 ([1][2])**.** *Let $\widehat{G} \times \widehat{G} \to G \times G$ be a product $(U(1) \times U(1))$-bundle. Then the 1-form $(\hat{\varepsilon}_0^* - \hat{\varepsilon}_1^* + \hat{\varepsilon}_2^*)\theta$ on $\widehat{G} \times \widehat{G}$ is horizontal and $(U(1) \times U(1))$-invariant, hence there exists the 1-form χ on $G \times G$ which satisfies $(\pi \times \pi)^*\chi = (\hat{\varepsilon}_0^* - \hat{\varepsilon}_1^* + \hat{\varepsilon}_2^*)\theta$.*

Proof. For example, see [12, G.Ginot, M.Stiénon]. □

Behrend and Xu proved the theorem below in [2].

Theorem 3.4 ([1][2])**.** *The cohomology class $[c_1(\theta) - \left(\frac{-1}{2\pi i}\right)\chi] \in H^3(\Omega(NG))$ represents the universal Dixmier-Douady class.*

Now we show our cocycle in Section 3.2 satisfies the required condition in Proposition 3.5 when we choose a natural section $s_{nt} : G \times G \to \delta\widehat{G}$.

Theorem 3.5. *The equation* $(\pi \times \pi)^* s_{nt}^*(\delta\theta) = (\hat{\varepsilon}_0^* - \hat{\varepsilon}_1^* + \hat{\varepsilon}_2^*)\theta$ *holds.*

Proof. Choose an open covering $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ of G such that all the intersections of open sets in $\mathcal{V}$ are contractible and there exist local sections $\eta_\lambda : V_\lambda \to \widehat{G}$ of π. Then $\{\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda'}) \cap \varepsilon_2^{-1}(V_{\lambda''})\}_{\lambda,\lambda',\lambda'' \in \Lambda}$ is an open cover of $G \times G$ and there are the induced local sections $\varepsilon_0^*\eta_\lambda \otimes (\varepsilon_1^*\eta_{\lambda'})^{\otimes -1} \otimes \varepsilon_2^*\eta_{\lambda''}$ on this covering.

If we pull back $\delta\theta$ by these sections, the induced form on $\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda'}) \cap \varepsilon_2^{-1}(V_{\lambda''})$ is $\varepsilon_0^*(\eta_\lambda^*\theta) - \varepsilon_1^*(\eta_{\lambda'}^*\theta) + \varepsilon_2^*(\eta_{\lambda''}^*\theta)$.

We define $U(1)$-valued functions $\tau_{\lambda\lambda'\lambda''}$ on $\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda'}) \cap \varepsilon_2^{-1}(V_{\lambda''})$ by

$$(\varepsilon_0^*\eta_\lambda \otimes (\varepsilon_1^*\eta_{\lambda'})^{\otimes -1} \otimes \varepsilon_2^*\eta_{\lambda''}) \cdot \tau_{\lambda\lambda'\lambda''} = s_{nt}.$$

Then $\varepsilon_0^*(\eta_\lambda^*\theta) - \varepsilon_1^*(\eta_{\lambda'}^*\theta) + \varepsilon_2^*(\eta_{\lambda''}^*\theta) + \tau_{\lambda\lambda'\lambda''}^{-1} d\tau_{\lambda\lambda'\lambda''}$ is equal to $s_{nt}^*\delta\theta$ hence we obtain $(\pi \times \pi)^* s_{nt}^*\delta\theta = (\pi \times \pi)^*(\varepsilon_0^*(\eta_\lambda^*\theta) - \varepsilon_1^*(\eta_{\lambda'}^*\theta) + \varepsilon_2^*(\eta_{\lambda''}^*\theta)) + (\pi \times \pi)^*\tau_{\lambda\lambda'\lambda''}^{-1} d\tau_{\lambda\lambda'\lambda''}$.

Let $\tilde{\varphi}_\lambda : \pi^{-1}(V_\lambda) \to V_\lambda \times U(1)$ be a local trivialization of π. We put $\varphi_\lambda := \mathrm{pr}_2 \circ \tilde{\varphi}_\lambda : \pi^{-1}(V_\lambda) \to U(1)$. For any $\hat{g} \in \pi^{-1}(V_\lambda)$ the equation $\hat{g} = \eta_\lambda \circ \pi(\hat{g}) \cdot \varphi_\lambda(\hat{g})$ holds so we can see

$$\hat{\varepsilon}_i^*\theta = \hat{\varepsilon}_i^*(\pi^*(\eta_\lambda^*\theta)) + \hat{\varepsilon}_i^*\varphi_\lambda^{-1} d\varphi_\lambda = (\pi \times \pi)^*\varepsilon_i^*(\eta_\lambda^*\theta) + \hat{\varepsilon}_i^*\varphi_\lambda^{-1} d\varphi_\lambda$$

on $\hat{\varepsilon}_i^{-1}(\pi^{-1}(V_\lambda)) = (\pi \times \pi)^{-1}(\varepsilon_i^{-1}(V_\lambda))$.

Therefore on $(\pi \times \pi)^{-1}(\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_{\lambda'}) \cap \varepsilon_2^{-1}(V_{\lambda''}))$ there is a differential form $\hat{\varepsilon}_0^*\theta - \hat{\varepsilon}_1^*\theta + \hat{\varepsilon}_2^*\theta = (\pi \times \pi)^*(\varepsilon_0^*(\eta_\lambda^*\theta) - \varepsilon_1^*(\eta_{\lambda'}^*\theta) + \varepsilon_2^*(\eta_{\lambda''}^*\theta)) + \hat{\varepsilon}_0^*\varphi_\lambda^{-1} d\varphi_\lambda - \hat{\varepsilon}_1^*\varphi_{\lambda'}^{-1} d\varphi_{\lambda'} + \hat{\varepsilon}_2^*\varphi_{\lambda''}^{-1} d\varphi_{\lambda''}$.

Since $\hat{\varepsilon}_i = (\eta_\lambda \circ \pi \circ \hat{\varepsilon}_i) \cdot \varphi_\lambda \circ \hat{\varepsilon}_i = (\eta_\lambda \circ \varepsilon_i \circ (\pi \times \pi)) \cdot \varphi_\lambda \circ \hat{\varepsilon}_i$, we can see that $\hat{\varepsilon}_0 \otimes \hat{\varepsilon}_1^{\otimes -1} \otimes \hat{\varepsilon}_2 : \widehat{G} \times \widehat{G} \to \delta\widehat{G}$ is equal to

$$((\varepsilon_0^*\eta_\lambda \otimes (\varepsilon_1^*\eta_{\lambda'})^{\otimes -1} \otimes \varepsilon_2^*\eta_{\lambda''}) \circ (\pi \times \pi)) \cdot (\varphi_\lambda \circ \hat{\varepsilon}_0)(\varphi_{\lambda'} \circ \hat{\varepsilon}_1)^{-1}(\varphi_{\lambda''} \circ \hat{\varepsilon}_2).$$

We have $\tau_{\lambda\lambda'\lambda''} \circ (\pi \times \pi) = (\varphi_\lambda \circ \hat{\varepsilon}_0)(\varphi_{\lambda'} \circ \hat{\varepsilon}_1)^{-1}(\varphi_{\lambda''} \circ \hat{\varepsilon}_2)$ because $s_{nt} \circ (\pi \times \pi) = \hat{\varepsilon}_0 \otimes \hat{\varepsilon}_1^{\otimes -1} \otimes \hat{\varepsilon}_2$, so it follows that $(\hat{\varepsilon}_0^* - \hat{\varepsilon}_1^* + \hat{\varepsilon}_2^*)\theta = (\pi \times \pi)^* s_{nt}^*\delta\theta$. This completes the proof.

□

3.5 “Chern-Simons form”

As mentioned in Section 2, PG plays the role of the universal G-bundle and NG, the classifying space BG. So, the pull-back of the cocycle in Definition 3.1 to $\Omega^*(PG)$ by $\rho : PG \to NG$ should be a coboundary of a cochain on PG. In this section we shall exhibit an explicit form of the cochain, which can be called Chern-Simons form for the Dixmier-Douady class.
Recall $PG(1) = G \times G$ and $\rho : PG(1) \to NG$ is defined as $\rho(\bar{g}_0, \bar{g}_1) = \bar{g}_0\bar{g}_1^{-1}$. Then we consider the $U(1)$-bundle $\bar{\delta}_\rho\widehat{G} := \bar{\varepsilon}_0^*\widehat{G} \otimes \rho^*\widehat{G} \otimes (\bar{\varepsilon}_1^*\widehat{G})^{\otimes -1}$ over $G \times G$ and the induced connection $\bar{\delta}_\rho\theta$ on it. We can check that $\bar{\delta}_\rho\widehat{G}$ is a trivial bundle by using the same argument in Lemma 3.1, and we take a section $\bar{s}_\rho$of it as

$$\bar{s}_\rho(\bar{g}_0, \bar{g}_1) := [((\bar{g}_0, \bar{g}_1), \hat{\bar{g}}_1), ((\bar{g}_0, \bar{g}_1), \hat{\bar{g}}_0\hat{\bar{g}}_1^{-1}), ((\bar{g}_0, \bar{g}_1), \hat{\bar{g}}_0)^{\otimes -1}].$$

Theorem 3.6. *The cochain* $-c_1(\theta) - \left(\frac{-1}{2\pi i}\right) \bar{s}_\rho^*(\bar{\delta}_\rho\theta) \in \Omega^2(PG)$ *is a Chern-Simons form of* $c_1(\theta) - \left(\frac{-1}{2\pi i}\right) \hat{s}_{nt}^*(\delta\theta) \in \Omega^3(NG)$ *i.e. the following equation holds.*

$$\rho^*(c_1(\theta) - \left(\frac{-1}{2\pi i}\right) \hat{s}_{nt}^*(\delta\theta)) = (d' + d'')(-c_1(\theta) - \left(\frac{-1}{2\pi i}\right) \bar{s}_\rho^*(\bar{\delta}_\rho\theta)).$$

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle d} & & & & \\
-c_1(\theta) \in \Omega^2(G) & \xrightarrow{\bar{\varepsilon}_0^* - \bar{\varepsilon}_1^*} & \Omega^2(PG(1)) & & \\
 & & \uparrow{\scriptstyle -d} & & \\
 & & -\left(\frac{-1}{2\pi i}\right) \bar{s}_\rho^*(\bar{\delta}_\rho\theta) \in \Omega^1(PG(1)) & \xrightarrow{\bar{\varepsilon}_0^* - \bar{\varepsilon}_1^* + \bar{\varepsilon}_2^*} & \Omega^1(PG(2))
\end{array}$$

Proof. Repeating the same argument as that in Proposition 3.1, we can see $(\bar{\varepsilon}_0^* + \rho^* - \bar{\varepsilon}_1^*)((c_1(\theta)) = \left(\frac{-1}{2\pi i}\right) d(\bar{s}_\rho^*(\bar{\delta}_\rho\theta)) \in \Omega^2(PG(1))$. Because $(\varepsilon_0, \varepsilon_1, \varepsilon_2) \circ \rho = (\rho \circ \bar{\varepsilon_0}, \rho \circ \bar{\varepsilon_1}, \rho \circ \bar{\varepsilon_2})$, we can see that $(\bar{\varepsilon_0}^*\bar{\delta}_\rho\widehat{G}) \otimes (\bar{\varepsilon_1}^*\bar{\delta}_\rho\widehat{G})^{\otimes -1} \otimes (\bar{\varepsilon}_2^*\bar{\delta}_\rho\widehat{G})$ is $\rho^*(\delta\widehat{G})$. Hence $(\bar{\varepsilon}_0^* - \bar{\varepsilon}_1^* + \bar{\varepsilon}_2^*)\bar{s}_\rho^*(\bar{\delta}_\rho\theta) = \rho^*(\hat{s}_{nt}^*(\delta\theta))$. □

By restricting the Chern-Simons form on $\Omega^*(PG)$ to the edge $\Omega^*(PG(0))$, we obtain a cocycle on $\Omega^*(G)$. So there is an induced map of the cohomology class $H^*(BG) \cong H(\Omega^*(NG)) \to H^{*-1}(G)$. This map coincides with the transgression map for the universal bundle $EG \to BG$ in the sense of

Heitsch and Lawson in [13]. Hence as a corollary of Theorem 3.6, we obtain an alternative proof of the following proposition from [7, Theorem 4.1] [29, Theorem 4.1].

Proposition 3.6. *The transgression map of the universal bundle $EG \to BG$ maps the Dixmier-Douady class to the negative of the first Chern class of $\pi : \widehat{G} \to G$.* □

4 The String class

Using the idea of Brylinski, McLaughlin [6] and Murray, Stevenson [23][24], we discuss the case of central $U(1)$-extension of a loop group.

4.1 In the case of special unitary group

It is known that the second Chern class $c_2 \in H^4(BSU(2))$ of the universal $SU(2)$-bundle $ESU(2) \to BSU(2)$ is represented in $\Omega^4(NSU(2))$ as the sum of following differential forms $C_{1,3}$ and $C_{2,2}$ (see for example [15] or [30]):

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle -d} & & & & \\
C_{1,3} \in \Omega^3(SU(2)) & \xrightarrow{\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*} & \Omega^3(SU(2) \times SU(2)) & & \\
& & \uparrow{\scriptstyle d} & & \\
& & C_{2,2} \in \Omega^2(SU(2) \times SU(2)) & \xrightarrow{\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^* - \varepsilon_3^*} & 0
\end{array}$$

$$C_{1,3} = \left(\frac{-1}{2\pi i}\right)^2 \frac{-1}{6}\mathrm{tr}(h^{-1}dh)^3, \qquad C_{2,2} = \left(\frac{-1}{2\pi i}\right)^2 \frac{1}{2}\mathrm{tr}(h_2^{-1}h_1^{-1}dh_1 dh_2).$$

Pulling back this cocycle by the evaluation map

$$ev : LSU(2) \times S^1 \to SU(2), (\gamma, z) \mapsto \gamma(z)$$

and integrating along the circle, we obtain the cocycle in $\Omega^3(NLSU(2))$. Here $LSU(2)$ is the free loop group of $SU(2)$ and the map $\int_{S^1} ev^*$ is also called the transgression map.

Now we pose the following problem. Is there corresponding central extension of $LSU(2)$ and connection form on it such that the Dixmier-Douady class in $\Omega^3(NLSU(2))$ constructed previous section coincides with $\int_{S^1} ev^*(C_{1,3} + C_{2,2})$? In this section, we explain that the central extension and the connection form constructed by Mickelsson and Brylinski, McLaughlin in [5] [6] [18] meet such a condition.

To begin with, we recall the definition of the $U(1)$-bundle $\pi : Q(\nu) \to LSU(2)$ and the multiplication $m : Q(\nu) \times Q(\nu) \to Q(\nu)$ in [5] [6]. We fix any based point $x_0 \in SU(2)$ and denote $\gamma_0 \in LSU(2)$ the constant loop at x_0. For any $\gamma \in LSU(2)$, we consider all paths $\sigma_\gamma : [0,1] \to LSU(2)$ that satisfies $\sigma_\gamma(0) = \gamma_0$ and $\sigma_\gamma(1) = \gamma$. Then the equivalence relation $\sim$ on $\{\sigma_\gamma\} \times S^1$ is defined as follows:

$$(\sigma_\gamma, z) \sim (\sigma'_\gamma, z') \Leftrightarrow z = z' \cdot \exp\left(\int_{I^2 \times S^1} 2\pi i F^*\nu\right).$$

Here $F : I^2 \times S^1 \to SU(2)$ is any homotopy map that satisfies

$$F(0,t,z) = \sigma_\gamma(t)(z), \quad F(1,t,z) = \sigma'_\gamma(t)(z)$$

and

$$\nu = C_{1,3} = \left(\frac{-1}{2\pi i}\right)^2 \frac{-1}{6} \mathrm{tr}(h^{-1}dh)^3.$$

It is well known $\nu \in \Omega^3(SU(2))$ is a closed, integral form hence this relation is well-defined. Now the fiber $\pi^{-1}(\gamma)$ of $Q(\nu)$ is defined as the quotient space $\{\sigma_\gamma\} \times S^1/\sim$.
We can adapt the same construction for any closed integral 3-form on $SU(2)$. Let η,η' be such forms and suppose there is a 2-form β with $d\beta = \eta' - \eta$. Then the isomorphism from $Q(\eta)$ to $Q(\eta')$ is constructed as:

$$[(\sigma_\gamma, z)]_\eta \mapsto [(\sigma_\gamma, z \cdot \exp\left(\int_{I^1 \times S^1} 2\pi i \sigma_\gamma^* \beta\right))]_{\eta'}.$$

Here we regard σ_γ as a map from $[0,1] \times S^1$ to $SU(2)$.

For the face operators $\{\varepsilon_i\} : SU(2) \times SU(2) \to SU(2)$ (we use the same notation for the face operators $LSU(2) \times LSU(2) \to LSU(2)$), we can check that $\varepsilon_0^* Q(\nu) \otimes \varepsilon_1^* Q(\nu)^{\otimes -1} \otimes \varepsilon_2^* Q(\nu)$ is isomorphic to $Q(\varepsilon_0^*\nu - \varepsilon_1^*\nu + \varepsilon_2^*\nu) =$

$Q(-dC_{2,2})$ over $LSU(2)\times LSU(2)$. The isomorphism from $Q(0)$ to $Q(-dC_{2,2})$ is given by

$$[(\sigma_{\gamma_1},\sigma_{\gamma_2},z)]_0 \mapsto [(\sigma_{\gamma_1},\sigma_{\gamma_2},z\cdot\exp\left(\int_{I^1\times S^1} 2\pi i(\sigma_{\gamma_1},\sigma_{\gamma_2})^*C_{2,2}\right))]_{-dC_{2,2}}.$$

Now we can define a section s_L of $\varepsilon_0^*Q(\nu)\otimes\varepsilon_1^*Q(\nu)^{\otimes-1}\otimes\varepsilon_2^*Q(\nu)$ over $LSU(2)\times LSU(2)$ by:

$$s_L(\gamma_1,\gamma_2) := [(\sigma_{\gamma_1},\sigma_{\gamma_2},\exp\left(\int_{I^1\times S^1} 2\pi i(\sigma_{\gamma_1},\sigma_{\gamma_2})^*C_{2,2}\right))]_{-dC_{2,2}}.$$

The multiplication $m : Q(\nu)\times Q(\nu)\to Q(\nu)$ is defined by the following equation

$$s_L(\gamma_1,\gamma_2) = ([\sigma_{\gamma_1},z_1]_{\varepsilon_0^*\nu})\otimes((\gamma_1\gamma_2),m([\sigma_{\gamma_1},z_1]_\nu,[\sigma_{\gamma_2},z_2]_\nu))^{\otimes-1}\otimes([\sigma_{\gamma_2},z_2]_{\varepsilon_2^*\nu}).$$

Next we recall how Brylinski and McLaughlin constructed the connection on $Q(\nu)$. Let denote $P_1SU(2)$ the space of paths on $SU(2)$ which starts from based point x_0 and $f : P_1SU(2)\to SU(2)$ a map that is defined by $f(\gamma)=\gamma(1)$. It is well known that f is a fibration. Then we define the 2-form ω on $P_1SU(2)$ as:

$$\omega_\gamma(u,v) = \int_0^1 \nu\left(\frac{d\gamma}{dt},u(t),v(t)\right)dt.$$

Note that $d\omega = f^*\nu$ holds. Let $\mathcal{U}=\{U_\iota\}$ be an open covering of $SU(2)$. Since $SU(2)$ is simply connected, we can take $\mathcal{U}$ such that each U_ι is contractible and $\{LU_\iota\}$ is an open covering of $LSU(2)$. For example, we take $\mathcal{U}=\{U_x := SU(2)-\{x\}|x\in SU(2)\}$.
Now we quote the lemma from [6].

Lemma 4.1 (Brylinski, McLaughlin [6])**.** *(1) There exists a line bundle L over each $f^{-1}(U_\iota)$ with a fiberwise connection such that its first Chern form is equal to $\omega|_{f^{-1}(U_\iota)}$. This line bundle is called the pseudo-line bundle.*
(2) There exists a connection ∇ on each pseudo-line bundle L such that its first Chern form R satisfies the condition that $R-\omega|_{f^{-1}(U_\iota)}$ is basic.

Let K be a 2-form on U_ι which satisfies $f^*K = 2\pi i(R-\omega|_{f^{-1}(U_\iota)})$. Then the 1-form θ_ι on LU_ι is defined by $\theta_\iota := \int_{S^1} ev^*K$. It is easy to see $\left(\frac{-1}{2\pi i}\right)d\theta_\iota =$

$(\int_{S^1} ev^*\nu)|_{LU_\iota}$.

There is a section s_ι on LU_ι defined by $s_\iota(\gamma) := [\sigma_\gamma, H_{\sigma_\gamma}(L,\nabla)]$. Here $H_{\sigma_\gamma}(L,\nabla)$ is the holonomy of (L,∇) along the loop $\sigma_\gamma : S^1 \to f^{-1}(U_\iota)$. We also have the corresponding local trivialization $\varphi_\iota : \pi^{-1}(U_\iota) \to U_\iota \times U(1)$.

Above all, we have the connection form θ on $Q(\nu)$ defined by $\theta|_{\pi^{-1}(U_\iota)} := \pi^*\theta_\iota + d\log(\mathrm{pr}_2 \circ \varphi_\iota)$. Its first Chern form $c_1(\theta)$ is $\int_{S^1} ev^*\nu$ and $d\delta\theta$ is equal to

$$(-2\pi i)\cdot \int_{S^1} ev^*((\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*)\nu) = (-2\pi i)\cdot \pi^* \left(-d\int_{S^1} ev^* C_{2,2}\right).$$

Hence $\delta\theta + (-2\pi i)\cdot\pi^* \int_{S^1} ev^* C_{2,2}$ is a flat connection on $\delta Q(\nu)$. Since $LSU(2)$ is simply connected, it is a trivial connection so

$$s_L^*(\delta\theta + (-2\pi i)\cdot \pi^* \int_{S^1} ev^* C_{2,2}) = 0.$$

So as a reformulation of the Brylinski and McLaughlin's result in [6], we obtain the proposition below.

Proposition 4.1. *Let $(Q(\nu),\theta)$ be a $U(1)$-bundle on $LSU(2)$ with connection and s_L be a global section of $\delta Q(\nu)$ constructed above. Then the cocycle $c_1(\theta) - \left(\frac{-1}{2\pi i}\right) s_L{}^*(\delta\theta)$ on $\Omega^3(NLSU(2))$ is equal to $\int_{S^1} ev^*(C_{1,3}+C_{2,2})$, i.e. the map $\int_{S^1} ev^*$ sends the second Chern class $c_2 \in H^4(BSU(2))$ to the Dixmier-Douady class (associated to $Q(\nu)$) in $H^3(BLSU(2))$.* □

Remark 4.1. We explain what happens if we adapt this construction to the loop group of the unitary group. In the case of unitary group $U(2)$, the second Chern class is represented as the sum of following $C^U_{1,3}$ and $C^U_{2,2}$ described in the diagram below (see [30]):

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle -d} & & & & \\
C^U_{1,3} \in \Omega^3(U(2)) & \xrightarrow{\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*} & \Omega^3(U(2)\times U(2)) & & \\
& & \uparrow{\scriptstyle d} & & \\
& & C^U_{2,2} \in \Omega^2(U(2)\times U(2)) & \xrightarrow{\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*-\varepsilon_3^*} & 0
\end{array}$$

$$C^U_{1,3} = \left(\frac{-1}{2\pi i}\right)^2 \frac{-1}{6}\mathrm{tr}(h^{-1}dh)^3$$

$$C^U_{2,2} = \left(\frac{-1}{2\pi i}\right)^2 \frac{1}{2}\mathrm{tr}(h_2^{-1}h_1^{-1}dh_1dh_2) - \left(\frac{-1}{2\pi i}\right)^2 \frac{1}{2}\mathrm{tr}(h_1^{-1}dh_1)\mathrm{tr}(h_2^{-1}dh_2).$$

We recognize $U(2)$ as a semi-direct group $SU(2) \rtimes U(1)$. Let denote by $\Omega U(1)$ the based loop group of $U(1)$. Then any element γ in $LU(2)$ is decomposed as $\gamma = (\gamma_1, \gamma_2, z) \in LSU(2) \rtimes (\Omega U(1) \rtimes U(1))$. Each connected component of $LU(2)$ is parametrized by the mapping degree of γ_2. We write $\Omega U(1)_n$,$LU(2)_n$ the connected component which includes a based loop γ_2 whose mapping degree is n. We can see $\pi_1(LU(2)_0) = \pi_1(LSU(2)) \oplus \pi_1(LU(1)_0) = \pi_1(LU(1)_0) = \pi_1(\Omega U(1)_0) \oplus \pi_1(U(1)) \cong \mathbb{Z}$. There is a homeomorphism from $\Omega U(1)_0$ to $\Omega U(1)_n$ defined by $\gamma \mapsto \gamma \cdot (e^{is} \mapsto e^{ins})$ for any n so $\pi_1(LU(2)_n)$ is also isomorphic to $\mathbb{Z}$. The generator ψ_n of $\pi_1(LU(2)_n) \cong H_1(LU(2)_n)$ is the map defined as

$$\psi_n(e^{it}) := \left(e^{is} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & e^{i(ns+t)} \end{pmatrix}\right).$$

Hence any cycle $a \in Z_1(LU(2)_n)$ can be written as $m\psi_n + \partial\varrho$ for some 2-chain ϱ.

Since $LU(2)$ is not simply connected we need the differential character k to construct a principal $U(1)$-bundle over $LU(2)$. Differential character is a homomorphism from $Z_1(LU(2))$ to $U(1)$ such that there exists a specific 2-form ω satisfying $k(\partial\varrho) = \exp(\int_\varrho 2\pi i\omega)$ for any 2-singular chains ϱ of $LU(2)$ ([8] see also [24]).

We set $\Phi := \int_{S^1} ev^* C^U_{1,3}$. If we define k as

$$k(a) := \exp(\int_\varrho 2\pi i\Phi)$$

this is well-defined since Φ is integral and we obtain the $U(1)$-bundle Q^U over $LU(2)$ by using this differential character k instead of $\exp(\int_{I^2\times S^1} 2\pi i F^*\nu)$ in Section 4.1. But unfortunately $k(a_1a_2)$ is not equal to $\exp(\int_{(\varrho_1,\varrho_2)} 2\pi i\varepsilon_1^*\Phi)$ in general. So in this way we can not obtain a section s^U_L of δQ^U nor a multiplication $m^U : Q^U \times Q^U \to Q^U$.

4.2 In the case of semi-direct product

In this section we deal with the semi-direct $LG \rtimes S^1$ for $G = SU(2)$.
First we define a bisimplicial manifold $NLG(*) \rtimes NS^1(*)$. A bisimplicial manifold is a sequence of manifolds with horizontal and vertical face and degeneracy operators which commute with each other. A bisimplicial map is a sequence of maps commuting with horizontal and vertical face and degeneracy operators. We define $NLG(*) \rtimes NS^1(*)$ as follows:

$$NLG(p) \rtimes NS^1(q) := \overbrace{LG \times \cdots \times LG}^{p-times} \times \overbrace{S^1 \times \cdots \times S^1}^{q-times}.$$

Horizontal face operators $\varepsilon_i^{LG} : NLG(p) \rtimes NS^1(q) \to NLG(p-1) \rtimes NS^1(q)$ are the same with the face operators of $NLG(p)$.

Vertical face operators $\varepsilon_i^{S^1} : NLG(p) \rtimes NS^1(q) \to NLG(p) \rtimes NS^1(q-1)$ are defined by

$$\varepsilon_i^{S^1}(\vec{\gamma}, z_1, \cdots, z_q) = \begin{cases} (\vec{\gamma}, z_2, \cdots, z_q) & i = 0 \\ (\vec{\gamma}, z_1, \cdots, z_i z_{i+1}, \cdots, z_q) & i = 1, \cdots, q-1 \\ (\vec{\gamma} z_q, z_1, \cdots, z_{q-1}) & i = q. \end{cases}$$

Here $\vec{\gamma} = (\gamma_1, \cdots, \gamma_p)$.

We define a bisimplicial map $\rho_{\rtimes} : PLG(p) \times PS^1(q) \to NLG(p) \rtimes NS^1(q)$ by

$$\rho_{\rtimes}(\vec{\gamma}, z_1, \cdots, z_{q+1}) = (\rho(\vec{\gamma}) z_{q+1}, \rho(z_1, \cdots, z_{q+1})).$$

Now we fix a semi-direct product operator $\cdot_{\rtimes}$of $LG \rtimes S^1$ as $(\gamma, z) \cdot_{\rtimes} (\gamma', z') := (\gamma \cdot (\gamma' z), zz')$, then $LG \rtimes S^1$ acts on $PLG(p) \times PS^1(q)$ by right as $(\vec{\gamma}, \vec{z}) \cdot (\gamma, z) = (\vec{\gamma} \cdot z^{-1}(\gamma), \vec{z} z)$. Since $\rho_{\rtimes}(\vec{\gamma}, \vec{z}) = \rho_{\rtimes}((\vec{\gamma}, \vec{z}) \cdot (\gamma, z))$, one can see that $\rho_{\rtimes}$ is a principal $(LG \rtimes S^1)$-bundle. $\| PLG(*) \times PS^1(*) \|$ is $ELG \times ES^1$ and $\| NLG(*) \rtimes NS^1(*) \|$ is homeomorphic to $(ELG \times ES^1)/(LG \rtimes S^1)$, so $\| NLG(*) \rtimes NS^1(*) \|$ is a model of $B(LG \rtimes S^1)$.

Definition 4.1. For a bisimplicial manifold $NLG(*) \rtimes NS^1(*)$, we have a triple complex as follows:

$$\Omega^{p,q,r}(NLG(*) \rtimes NS^1(*)) \stackrel{\text{def}}{=} \Omega^r(NLG(p) \rtimes NS^1(q)).$$

Derivatives are:

$$d' = \sum_{i=0}^{p+1} (-1)^i (\varepsilon_i^{LG})^*, \qquad d'' = \sum_{i=0}^{q+1} (-1)^i (\varepsilon_i^{S^1})^* \times (-1)^p.$$

$$d''' = (-1)^{p+q} \times \text{the exterior differential on } \Omega^*(NLG(p) \rtimes NS^1(q)).$$

□

The following proposition can be proved by adapting the same argument in the proof of Theorem 2.1 (See [32]).

Proposition 4.2. *There exists an isomorphism*

$$H(\Omega^*(NLG \rtimes NS^1) \cong H^*(B(LG \rtimes S^1)).$$

Here $\Omega^*(NLG \rtimes NS^1)$ *means the total complex.* □

Now we want to construct the cocycle in $\Omega^3(NLG \rtimes NS^1)$ which coincides with $c_1(\theta) - \left(\frac{-1}{2\pi i}\right) {s_L}^*(\delta\theta)$ when it is restricted to $\Omega^3(NLG)$.

To do this, it suffices to construct the differential form τ on $\Omega^1(LG \rtimes S^1)$ such that $d\tau = (-\varepsilon_0^{S^1*} + \varepsilon_1^{S^1*})c_1(\theta)$ and $(\varepsilon_0^{LG*} - \varepsilon_1^{LG*} + \varepsilon_2^{LG*})\tau = (\varepsilon_0^{S^1*} - \varepsilon_1^{S^1*})\left(\frac{-1}{2\pi i}\right) {s_L}^*(\delta\theta)$ and $(-\varepsilon_0^{S^1*} + \varepsilon_1^{S^1*} - \varepsilon_2^{S^1*})\tau = 0$. We consider the trivial $U(1)$-bundle $(\varepsilon_0^{S^1*}Q)^{\otimes -1} \otimes \varepsilon_1^{S^1*}Q$ and the induced connection form $\delta_\rtimes\theta$ on it. We define the section $s_\rtimes : LG \rtimes S^1 \to (\varepsilon_0^{S^1*}Q)^{\otimes -1} \otimes \varepsilon_1^{S^1*}Q$ as $s_\rtimes(\gamma, z) := (\hat{\gamma}, z)^{\otimes -1} \otimes (\hat{\gamma}z, z)$ and set $\tau := \left(\frac{-1}{2\pi i}\right) {s_\rtimes}^*(\delta_\rtimes\theta)$ then we can see that τ satisfies the required conditions.

5 Appendix: A central $U(1)$-extension of a groupoid

This section is a short survey of the theory of a central $U(1)$-extension of a Lie groupoid.

At first we recall the definition of Lie groupoids following [19].

Definition 5.1. A Lie groupoid Γ_1 over a manifold Γ_0 is a pair (Γ_1, Γ_0) equipped with following differentiable maps:
(i) surjections $s, t : \Gamma_1 \to \Gamma_0$ called the source and target maps respectively;
(ii) $m : \Gamma_2 \to \Gamma_1$ called multiplication, where $\Gamma_2 := \{(x_1, x_2) \in \Gamma_1 \times \Gamma_1 | \ t(x_1) = s(x_2)\}$;
(iii) an injection $e : \Gamma_0 \to \Gamma_1$ called identities;
(iv) $\iota : \Gamma_1 \to \Gamma_1$ called inversion.
These maps must satisfy:

(1) (associative law) $m(m(x_1,x_2),x_3) = m(x_1, m(x_2,x_3))$ if one is defined, so is the other;
(2) (identities) for each $x \in \Gamma_1, (e(s(x)),x) \in \Gamma_2, (x, e(t(x))) \in \Gamma_2$ and $m(e(s(x)),x) = m(x, e(t(x))) = x$;
(3) (inverses) for each $x \in \Gamma_1, (x,\iota(x)) \in \Gamma_2, (\iota(x),x) \in \Gamma_2, m(x,\iota(x)) = e(s(x))$, and $m(\iota(x),x) = e(t(x))$.

In this paper we denote a Lie groupoid by $\Gamma_1 \rightrightarrows \Gamma_0$.

Example 5.1. Suppose that G is a Lie group acting on a manifold M by left. Then we have a groupoid $\Gamma_1 = G \times M$, $\Gamma_0 = M$. The source map s is defined as $s(g,u) = u$ and the target map t is defined as $t(g,u) = gu$. This groupoid $M \rtimes G \rightrightarrows M$ is often called an action groupoid and denoted by $M//G$.

Example 5.2. Suppose that M is a manifold and $\{U_\alpha\}$ is a covering of M. Then we have a groupoid $\Gamma_1 = \coprod(U_\alpha \cap U_\beta)$, $\Gamma_0 = \coprod U_\alpha$. The source map s is an inclusion map into U_α and the target map t is an inclusion map into U_β.

5.1 Double complex and central $U(1)$-extension

Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a Lie groupoid and denote by s, t, m the source and target maps, and the multiplication of it respectively. Then we can define a simplicial manifold $N\Gamma$ as follows:

$$N\Gamma(p) := \{(x_1, \cdots, x_p) \in \overbrace{\Gamma_1 \times \cdots \times \Gamma_1}^{p-times} \mid t(x_j) = s(x_{j+1}) \;\; j = 1, \cdots, p-1\}$$

face operators $\varepsilon_i : N\Gamma(p) \to N\Gamma(p-1)$

$$\varepsilon_i(x_1, \cdots, x_p) = \begin{cases} (x_2, \cdots, x_p) & i = 0 \\ (x_1, \cdots, m(x_i, x_{i+1}), \cdots, x_p) & i = 1, \cdots, p-1 \\ (x_1, \cdots, x_{p-1}) & i = p. \end{cases}$$

The double complex $\Omega^{*,*}(N\Gamma)$ is also defined in a similar way.

Example 5.3. In the case of an action groupoid $M \rtimes G \rightrightarrows M$ for a smooth manifold M and a compact Lie group G which acts on M, $H(\Omega^*(N\Gamma))$ is isomorphic to the Borel model of the equivariant cohomology $H^*_G(M) := H^*(EG \times_G M)$ (see for example [11]).

Example 5.4. In the case of the groupoid $\coprod(U_\alpha \cap U_\beta) \rightrightarrows \coprod U_\alpha$ for a good covering $\{U_\alpha\}$ in Example 5.2, $H(\Omega^*(N\Gamma))$ is isomorphic to $H^*(M)$.

Now we recall the notion of a central $U(1)$-extension of a groupoid in [2] [33]. A central $U(1)$-extension of a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ consists of a morphism of Lie groupoids

$$\begin{array}{ccc} \widehat{\Gamma}_1 & \xrightarrow{\pi} & \Gamma_1 \\ \downdownarrows & & \downdownarrows \\ \Gamma_0 & \xrightarrow{\text{id}} & \Gamma_0 \end{array}$$

and a right $U(1)$-action on $\widehat{\Gamma}_1$, making $\pi : \widehat{\Gamma}_1 \to \Gamma_1$ a principal $U(1)$-bundle. For any $z_1, z_2 \in U(1)$ and $(\hat{x_1}, \hat{x_2}) \in N\widehat{\Gamma}(2) := \{(\hat{y_1}, \hat{y_2}) \in \widehat{\Gamma}_1 \times \widehat{\Gamma}_1 | t(\hat{y_1}) = s(\hat{y_2})\}$, the equation $\hat{m}(\hat{x_1}z_1, \hat{x_2}z_2) = \hat{m}(\hat{x_1}, \hat{x_2})z_1z_2$ holds.

Note that there is a section $\hat{s}_{st}$ of $\delta\widehat{\Gamma}_1$ defined as

$$\hat{s}_{st}(x_1, x_2) := [((x_1, x_2), \hat{x}_2), ((x_1, x_2), \hat{m}(\hat{x}_1, \hat{x}_2))^{\otimes -1}, ((x_1, x_2), \hat{x}_1)].$$

Furthermore, because of the associative law of $\Gamma_1 \rightrightarrows \Gamma_0$, $\delta(\delta\widehat{\Gamma}_1))$ is canonically isomorphic to the product bundle and $\delta\hat{s}_{st} = 1$ holds.

Let $\widehat{\Gamma}_1 \to \Gamma_1 \rightrightarrows \Gamma_0$ be a central $U(1)$-extension of a groupoid and θ be a connection form of the $U(1)$-bundle $\widehat{\Gamma}_1 \to \Gamma_1$. Then we can use the same argument in Section 3.2 and obtain the cocycle on $\Omega^*(N\Gamma(*))$. In [1][2] and related papers, they call θ a pseudo-connection of a central $U(1)$-extension of a groupoid $\widehat{\Gamma}_1 \to \Gamma_1 \rightrightarrows \Gamma_0$ and when $-\left(\frac{-1}{2\pi i}\right) \hat{s}_{st}^*(\delta\theta) \in \Omega^1(N\Gamma(2))$ vanishes they call θ a connection of $\widehat{\Gamma}_1 \to \Gamma_1 \rightrightarrows \Gamma_0$. If the horizontal complex $\Omega^1(N\Gamma(1)) \xrightarrow{d'} \Omega^1(N\Gamma(2)) \xrightarrow{d'} \Omega^1(N\Gamma(3))$ is exact, a connection of $\widehat{\Gamma}_1 \to \Gamma_1 \rightrightarrows \Gamma_0$ exists.

5.2 Bundle gerbes

5.2.1 The definition and basic properties

In this section, we recall the definition of bundle gerbes and some basic properties of them.

Definition 5.2 (Murray-Stevenson, [21][22])**.** Given a surjective submersion $\phi : Y \to M$, we obtain the groupoid $Y^{[2]} \rightrightarrows Y$ where $Y^{[2]}$ is the fiber product defined as $Y^{[2]} := \{(y_1, y_2)|\phi(y_1) = \phi(y_2)\}$. The source and target maps are defined as $s(y_1, y_2) = y_2, t(y_1, y_2) = y_1$ respectively.

A bundle gerbe over M is a pair of $\phi : Y \to M$, a principal $U(1)$-bundle $\widehat{Y^{[2]}}$ over $Y^{[2]}$ and a section $\hat{s}$ of $\delta\widehat{Y^{[2]}}$ which satisfies $\delta\hat{s} = 1$.

Remark 5.1. Without the assumption of the existence of $\hat{s}$, $\delta\widehat{Y^{[2]}}$ is not necessarily trivial. By using $\hat{s}$, we can construct a multiplication $\hat{m} : \widehat{Y^{[2]}} \times \widehat{Y^{[2]}} \to \widehat{Y^{[2]}}$ such that $\hat{s}$ is a natural section of $\delta\widehat{Y^{[2]}}$. Hence we can recognize bundle gerbe as a kind of a central $U(1)$-extension of a Lie groupoid.

Bundle gerbe was invented by Murray in [21]. It is often denoted by $\mathcal{G}$. Here we recall the classification theory of bundle gerbe due to Murray and Stevenson.

Remark 5.2. In the case that the surjective submersion is given by $\coprod U_\alpha \to M$ and groupoid is $\coprod(U_\alpha \cap U_\beta) \rightrightarrows \coprod U_\alpha$ for a good covering $\{U_\alpha\}$ in Example 5.2, the bundle gerbe $(\widehat{\Gamma}_1 \to \coprod(U_\alpha \cap U_\beta) \rightrightarrows \coprod U_\alpha, \hat{s})$ is called Hitchin-Chatterjee gerbe data ([14]).

Definition 5.3 ([21][22])**.** The bundle gerbe $(\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y, \hat{s})$ is called trivial if there exists a principal $U(1)$-bundle R over Y and a section $v : Y^{[2]} \to \delta R^{\otimes -1} \otimes \widehat{Y}^{[2]}$ such that $\delta v = \hat{s}$. Such a pair (R, v) is called a trivialization of the bundle gerbe $(\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y, \hat{s})$.

Definition 5.4 ([21][22])**.** Bundle gerbes $(\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y, \hat{s})$ and $(\widehat{Y'}^{[2]} \to Y'^{[2]} \rightrightarrows Y', \hat{s}')$ are stably isomorphic if there exists following date:
(i) a surjective submersion $W \to M$;
(ii) smooth maps $\phi : W \to Y$ and $\phi' : W \to Y'$ which are compatible with projections onto M;
(iii) a trivialization of $(\phi^*(\widehat{Y}^{[2]})^{\otimes -1} \otimes \phi'^*\widehat{Y'}^{[2]} \to W^{[2]} \rightrightarrows W, \phi^*\hat{s}^{\otimes -1} \otimes \phi'^* s')$.

Definition 5.5 ([21])**.** We define the product $\mathcal{G} \otimes \mathcal{G}'$ of bundle gerbes $\mathcal{G} = (\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y, \hat{s})$ and $\mathcal{G}' = (\widehat{Y'}^{[2]} \to Y'^{[2]} \rightrightarrows Y', \hat{s}')$ as

$$(\widehat{Y}^{[2]} \otimes \widehat{Y'}^{[2]} \to Y^{[2]} \times_{(\pi,\pi')} Y'^{[2]} \rightrightarrows Y \times_{(\pi,\pi')} Y', \hat{s} \otimes \hat{s}').$$

Here $Y \times_{(\pi,\pi')} Y'$ is defined as $Y \times_{(\pi,\pi')} Y' := \{(y,y') \in Y \times Y' | \pi(y) = \pi'(y')\}$. The inverse $\mathcal{G}^{-1}$ of $\mathcal{G} = (\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y, \hat{s})$ is $((\widehat{Y}^{[2]})^{\otimes -1} \to Y^{[2]} \rightrightarrows Y, \hat{s}^{\otimes -1})$.

Then the following theorem holds true.

Theorem 5.1 ([21][22]). *The isomorphism classes of bundle gerbes over M are parametrized by $H^3(M, \mathbb{Z})$.* □

Proof. We construct the characteristic class in $H^3(M,\mathbb{Z})$. Let $\{U_\alpha\}$ be a Leray covering of M and $s_\alpha : U_\alpha \to Y|_{U_\alpha}$ local sections of ϕ. Then there is an induced section $\psi_{\alpha\beta} : U_{\alpha\beta} : U_\alpha \cap U_\beta \to (s_\alpha, s_\beta)^* \widehat{Y}^{[2]}$. Now a $U(1)$-valued function $g_{\alpha\beta\gamma}$ on $U_{\alpha\beta\gamma}$ is defined as $((s_\alpha, s_\beta, s_\gamma)^* \hat{s}) \cdot g_{\alpha\beta\gamma} := \psi_{\alpha\beta} \otimes \psi_{\beta\gamma} \otimes \psi_{\gamma\alpha}$. Then it is easily seen that $\{g_{\alpha\beta\gamma}\}$ is a $U(1)$-valued Čech-cocycle on M and define a cohomology class in $H^2(M, \underline{U(1)}) \cong H^3(M,\mathbb{Z})$. This class is called the Dixmier-Douady class of bundle gerbe $\mathcal{G} = (\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y, \hat{s})$. We denote it $D(\mathcal{G})$. We can check that $D(\mathcal{G} \otimes \mathcal{G}') = D(\mathcal{G}) + D(\mathcal{G}')$ and if $\mathcal{G}$ is trivial then $D(\mathcal{G})$ is the trivial class in $H^3(M,\mathbb{Z})$. Therefore $\mathcal{G} \mapsto D(\mathcal{G})$ is well-defined monomorphism. Finally we check the surjectivity of this map. Given any $U(1)$-valued Čech-cocycle $\{g_{\alpha\beta\gamma}\}$ of M, we can construct the bundle gerbe $\mathcal{G}$ by $Y := \coprod U_\alpha$, $\widehat{Y}^{[2]} := Y^{[2]} \times U(1)$ and $\hat{s} := \{g_{\alpha\beta\gamma}\}$.

□

There is a practical method to calculate the Dixmier-Douady class in $H^3(M,\mathbb{R})$. To explain this, we quote the following basic proposition from [23].

Proposition 5.1 ([21]). *The complex*

$$0 \to \Omega^*(M) \xrightarrow{\phi^*} \Omega^*(Y) \xrightarrow{d'} \Omega^*(Y^{[2]}) \xrightarrow{d'} \Omega^*(Y^{[3]}) \xrightarrow{d'} \cdots$$

is exact.

Since the complex $\Omega^1(Y^{[2]}) \xrightarrow{d'} \Omega^1(Y^{[3]}) \xrightarrow{d'} \Omega^1(Y^{[4]})$ is exact hence there exists a connection θ of principal $U(1)$-bundle $\widehat{Y}^{[2]} \to Y^{[2]}$ such that $\hat{s}^*\theta = 0$. We call this a connection of bundle gerbe $\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y$. Let $\theta \in \Omega^1(Y^{[2]})$ be any connection form of bundle gerbe $\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y$. Then there exists a 2-form H on Y which satisfies $\mathrm{pr}_2^* H - \mathrm{pr}_1^* H = c_1(\theta)$ because $\Omega^2(Y) \xrightarrow{d'} \Omega^2(Y^{[2]}) \xrightarrow{d'} \Omega^2(Y^{[3]})$ is exact. This 2-form is called a curving of the bundle gerbe. Furthermore, there exists a closed 3-form D on M such that $\phi^* D = dH$ since $0 \to \Omega^3(M) \xrightarrow{\phi^*} \Omega^3(Y) \xrightarrow{d'} \Omega^3(Y^{[2]})$ is also exact. The cohomology class $[D]$ does not depend on the choice of connection and curving, and coincides with the Dixmier-Douady class of $(\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y, \hat{s})$ in $H^3(M, \mathbb{R})$.

In the case of a central $U(1)$-extension of group G, the Dixmier-Douady class of a principal G-bundle is a torsion class if G is a finite dimensional Lie group. In the case of bundle gerbe, like the bundle gerbe $(\coprod U_{\alpha\beta} \times U(1) \to \coprod U_{\alpha\beta} \rightrightarrows \coprod U_\alpha, \hat{s} := \{g_{\alpha\beta\gamma}\})$ in the proof of Theorem 5.1, there are some bundle gerbes whose Dixmier-Douady class is not torsion class even though their submersion has a finite dimensional fiber.

In general, the following theorem holds.

Theorem 5.2 (Murray-Stevenson, [25])**.** *Let $(\widehat{Y}^{[2]} \to Y^{[2]} \rightrightarrows Y, \hat{s})$ be a bundle gerbe over a simply connected manifold M with connected, finite dimensional fiber F of submersion $\phi : Y \to M$. Then its Dixmier-Douady class is a torsion class.* □

We can check the necessity of the conditions in Theorem 5.2 by considering the examples of bundle gerbes given in the next section.

5.2.2 Examples of bundle gerbes

Example 5.5. Let $\pi : M \to \Sigma_g$ be an oriented S^1-bundle over a closed oriented surface whose genus is g. It is well-known that $H^3(M, \mathbb{Z}) \cong \mathbb{Z}$. Here we show how to construct the bundle gerbe whose Dixmier-Douady class is the generator of $H^3(M, \mathbb{Z})$.

We take an open ball $D^2 \in \Sigma_g$ and a point $p \in D^2$. Then $\pi^{-1}(D^2) \approx D^2 \times U(1)$ and $\pi^{-1}(\Sigma_g \backslash \{p\}) \approx (\Sigma_g \backslash \{p\}) \times U(1)$ because their first Chern classes are 0. For convenience we set $V_1 := \pi^{-1}(D^2)$ and $V_2 := \pi^{-1}(\Sigma_g \backslash \{p\})$. Let denote Y the disjoint union $V_1 \sqcup V_2$ and define a surjective submersion $\phi : Y \to M$ as an inclusion. Then the fiber product $Y^{[2]}$ is $(V_1 \times V_1) \sqcup d(V_1 \cap V_2) \sqcup d(V_2 \cap V_1) \sqcup (V_2 \times V_2)$ where $d(V_1 \cap V_2)$ is the space of the diagonal elements $\{(u, u) | u \in V_1 \cap V_2)\} \subset (V_1 \cap V_2) \times (V_1 \cap V_2)$.

Since $d(V_1 \cap V_2)$ is homotopic to $S^1 \times S^1$ and there is the principal $U(1)$-bundle P over $d(V_1 \cap V_2)$ whose first Chern class c_1 is the generator of $H^2(d(V_1 \cap V_2), \mathbb{Z}) \cong H^2(S^1 \times S^1, \mathbb{Z}) \cong \mathbb{Z}$.

We define a principal $U(1)$-bundle Q over $Y^{[2]}$ as the disjoint union of P over $d(V_2 \cap V_1)$ and $P^{\otimes -1}$ over $d(V_1 \cap V_2)$, and a product bundle on $(V_1 \times$

$V_1) \sqcup (V_2 \times V_2)$. Then δQ over $Y^{[3]}$ is canonically isomorphic to $Y^{[3]} \times U(1)$ so we take a section as $\hat{s} = 1$.

Proposition 5.2. *The Dixmier-Douady class of the bundle gerbe* $(Q \to Y^{[2]} \rightrightarrows Y, \hat{s})$ *is the generator of* $H^3(M, \mathbb{Z}) \cong \mathbb{Z}$.

Proof. Let θ be a connection of bundle gerbe $(Q \to Y^{[2]} \rightrightarrows Y, \hat{s})$, i.e. θ is a connection form of the principal $U(1)$-bundle Q which satisfies $\hat{s}^*(\delta\theta) = 0$. Then there is a 2-form H on Y which satisfies $\mathrm{pr}_2^* H - \mathrm{pr}_1^* H = c_1(\theta)$. There is also the closed 3-form D on E which satisfies $\phi^* D = dH$. Then $[D]$ represents the Dixmier-Douady class of $Q \to Y^{[2]} \rightrightarrows Y$ with $\mathbb{R}$-coefficients.

Now $c_1(\theta)$ is the generator of $H^2(d(V_1 \cap V_2), \mathbb{R})$, and the map $H^2(d(V_1 \cap V_2), \mathbb{R}) \ni c_1(\theta) \mapsto [D] \in H^3(M, \mathbb{R})$ is nothing but the connecting homomorphism in the Mayer-Vietoris sequence of (V_1, V_2) on the de Rham cohomology, so $[D]$ represents the generator of $H^3(M, \mathbb{R})$. This completes the proof. □

Example 5.6. There is an important example of bundle gerbes so-called lifting bundle gerbe defined as follows. Let $\widehat{G} \to G$ be a central $U(1)$-extension of a Lie group G and $\phi : Y \to M$ be a principal G-bundle. We define a map $\zeta : Y^{[2]} \to G$ as $y_1 \zeta(y_1, y_2) = y_2$. Then $(\zeta^* \widehat{G} \to Y^{[2]} \rightrightarrows Y, \hat{s}_{nt})$ is a bundle gerbe. The Dixmier-Douady class of the lifting bundle gerbe coincides with the Dixmier-Douady class of $\phi : Y \to M$.

Remark 5.3. We take M as in Example 5.5 then we can construct the principal $PU(H)$-bundle over M whose Dixmier-Douady class is the generator of $H^3(M, \mathbb{Z})$ using the bundle gerbe in Example 5.5.

First we make trivial principal $PU(H)$-bundles over V_1 and V_2. We denote them by R_1 and R_2. Since $U(H) \to PU(H)$ is a model of the universal $U(1)$-bundle, there is a continuous map $\phi_{12} : V_1 \cap V_2 \to PU(H)$ such that the first Chern class of $\phi_{12}^* U(H)$ is the generator of $H^2(V_1 \cap V_2, \mathbb{Z})$. We also take ϕ_{21} as the inverse valued map of ϕ_{12}. Now by gluing R_1 and R_2 by ϕ_{12}, we obtain a principal $PU(H)$-bundle $\rho : R_1 \cup_{\phi_{12}} R_2 \to M$.

Proposition 5.3. *The Dixmier-Douady class of the principal* $PU(H)$*-bundle* $R_1 \cup_{\phi_{12}} R_2$ *is the generator of* $H^3(M, \mathbb{Z}) \cong \mathbb{Z}$.

Proof. For convenience we write $R := R_1 \cup_{\phi_{12}} R_2$. Then there is a map $\zeta : R^{[2]} \to PU(H)$ which is defined by $r_1 \cdot \zeta(r_1, r_2) = r_2$ for $(r_1, r_2) \in R^{[2]}$. By pulling back $U(H) \to PU(H)$ on $R^{[2]}$ by ζ, we obtain the lifting bundle gerbe $\zeta^* U(H) \to R^{[2]} \rightrightarrows R$.

Now we show that $\zeta^* U(H) \to R^{[2]} \rightrightarrows R$ is stably isomorphic to $Q \to Y^{[2]} \rightrightarrows Y$ in Example 5.5. We define the surjective submersion

$$f : W := (V_1 \sqcup V_2) \times PU(H) \to M$$

as the projection into the first factor. There are also natural projections $f_1 : (V_1 \sqcup V_2) \times PU(H) \to R$ and $f_2 : (V_1 \sqcup V_2) \times PU(H) \to (V_1 \sqcup V_2)$ which satisfy $f = \rho \circ f_1 = i \circ f_2$. Then $f_1^*(\zeta^* U(H))^{\otimes -1} \otimes f_2^* Q$ is canonically trivial since the diagram below is commutative.

$$\begin{array}{ccc} W^{[2]} & \xrightarrow{f_1} & R^{[2]} \\ {\scriptstyle f_2}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\ (V_1 \sqcup V_2)^{[2]} & \xrightarrow{\{\phi_{ij}\}} & PU(H) \end{array}$$

The statement of the proposition follows from this.

□

We give an example of bundle gerbes whose section $\hat{s}$ is not trivial. This construction is given by Johnson in [16] and Murray, Stevenson in [25].

Example 5.7. We can construct the bundle gerbe over the torus $T^3 = S^1 \times S^1 \times S^1$ whose Dixmier-Douady class is the generator of $H^3(T^3, \mathbb{Z}) \cong \mathbb{Z}$ in the following way.

We set $Y := \mathbb{R}^3$ and define the submersion $\phi : Y \to T^3$ by $t \to \exp(2\pi i t)$. We write an element of Y as $\vec{x} = (x_1, x_2, x_3)$. Then $(\vec{x}, \vec{y}) \in Y^{[2]}$ means $\vec{x} - \vec{y} \in \mathbb{Z}^3$. We take a principal $U(1)$-bundle Q over $Y^{[2]}$ as a product $U(1)$-bundle and define the section $\hat{s}$ of δQ by $\hat{s}(\vec{x}, \vec{y}, \vec{z}) := \exp(2\pi i \gamma(\vec{x}, \vec{y}, \vec{z}))$ where γ is defined by $\gamma(\vec{x}, \vec{y}, \vec{z}) := (y_1 - z_1)(x_2 - y_2)x_3$. Then we can check that $\delta \hat{s} = 1$.

There is a projection map $(\vec{x}, \vec{y}, \vec{z}) \mapsto \vec{x}$ and we have $\mathbb{R}^3$-valued differential 1-form $d\vec{x}$ on $Y^{[3]}$. Similarly $d\vec{y}$ and $d\vec{z}$ are defined. Since $\vec{x} - \vec{y} \in \mathbb{Z}^3$ and $\vec{y} - \vec{z} \in \mathbb{Z}^3$, the equation $d\vec{x} = d\vec{y} = d\vec{z}$ holds. Note that each dx_i are pull-backs of $\frac{1}{2\pi} d\theta_i \in \Omega^1(S^1 \times S^1 \times S^1)$ by ϕ where $d\theta_i$ is the volume form of i-th S^1. We define the connection θ and the curving H as $\theta := -2\pi i (x_1 - y_1) x_2 dx_3$, $H := -x_1 dx_2 \wedge dx_3$. Then $dH = dx_1 \wedge dx_2 \wedge dx_3$ so the Dixmier-Douady class is $[\frac{1}{8\pi^3} d\theta_1 \wedge d\theta_2 \wedge d\theta_3] \in H^3(T^3, \mathbb{R})$.

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