Uniqueness criteria for stationary solutions to the Navier-Stokes equations in exterior domains

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Chapter 1

Introduction

Let $n = 2, 3$ and let Ω be an exterior domain in $\mathbb{R}^n$ with smooth boundary. We consider the steady motion of a viscous incompressible fluid in the exterior of an obstacle $\mathbb{R}^n \setminus \Omega$, which is described by the stationary Navier-Stokes equation

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \Omega, \\ \text{div } u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.1}$$

Here $u = (u_1, \cdots, u_n)$ and p denote, respectively, the unknown velocity and pressure of the fluid, while $f = (f_1, \cdots, f_n)$ is a given external force of the form $f = \text{div } F = (\sum_{i=1}^n \partial_i F_{ij})_{j=1}^n$ with $F = (F_{ij})_{i,j=1}^n$. In this thesis we investigate the uniqueness of weak solutions to (1.1). The contents of this thesis are based on the author's research papers [28, 29, 30].

The purpose of this thesis is to establish uniqueness theorems when one of solutions is not necessarily small. Uniqueness criteria for two small solutions are well-known and are not difficult problems in general. The smallness of at least one solution seems to be essential for the uniqueness. To see this, let D be a bounded domain in $\mathbb{R}^3$ with smooth boundary and let us consider the problem

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = f & \text{in } D, \\ \text{div } u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{1.2}$$

The existence of a weak solution u with finite Dirichlet integral $\int_D |\nabla u|^2 \, dx < \infty$ to (1.2) is well-known, see [25, 11, 12]. Furthermore, we can easily verify that the solution u is unique if the external force f is sufficiently small in a sense. On the other hand, it is also known that for large f the problem (1.2) can admit at least two solutions, see [12, VIII, Theorem 2.2]. Hence, we need the smallness of a solution to obtain the uniqueness even in bounded domains. From this observation, it is natural to assume the smallness condition in the uniqueness problem in exterior domains, and the case where only one solution is small is worth investigating.

This thesis is devoted to the study of the uniqueness problem to the stationary Navier-Stokes equation in three-dimensional and two-dimensional exterior domains. In the case $n = 3$, we consider the uniqueness of solutions in the Lorentz space $L^{q,\infty}(\Omega)$ introduced by Kozono-Yamazaki [23]. The linear approximation is the important method in the analysis of the Navier-Stokes equation, and the L^q theory of the Stokes equation

$$\begin{cases} -\Delta u + \nabla p = \operatorname{div} F & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.3}$$

was investigated by several authors [6, 14, 20]. They showed that for every $F \in L^q(\Omega)$ there exists a unique solution of (1.3) in the class $\nabla u \in L^q(\Omega)$ if and only if $3/2 < q < 3$. We cannot expect in general to obtain a solution with $\nabla u \in L^{3/2}(\Omega)$ of (1.3). Indeed, the derivative of the fundamental solution of the Stokes equation decays like $|x|^{-2}$ at infinity. On the other hand, the nonlinear problem (1.1) requires $q = 3/2$ to estimate the nonlinearity. Hence the L^q theory of the linearized problem (1.3) makes no contribution to the nonlinear problem (1.1). By extending the class of solutions to the Lorentz space $L^{q,\infty}(\Omega)$, Kozono-Yamazaki [23] established the linear theory within the class

$$u \in L^{3,\infty}(\Omega) \quad \text{with} \quad \nabla u \in L^{3/2,\infty}(\Omega) \tag{1.4}$$

and, as an application, proved the existence of a solution to (1.1) within that class for small $F \in L^{3/2,\infty}(\Omega)$. We note that (1.1) has a solution u with $\nabla u \in L^{3/2}(\Omega)$ only in a special situation. It was pointed out by Kozono-Sohr [21] and Borchers-Miyakawa [7] that such a solution exists only if

$$\int_{\partial\Omega} (T[u,p] + F) \cdot \nu \, dS = 0 \tag{1.5}$$

where $T[u,p] := (\partial_i u_j + \partial_j u_i - p\delta_{ij})_{i,j=1}^3$ denotes the stress tensor and ν is the outer unit normal to $\partial\Omega$. We need the class (1.4) to ensure the existence for (1.1), see also Section 3.1 for the importance of the class (1.4).

We investigate the uniqueness of solutions in the class of Kozono-Yamazaki [23]. They also proved the uniqueness within the class of solutions that have sufficiently small $L^{3,\infty}$-norm. For the uniqueness results in which only one solution is small, we refer to Galdi [12], Miyakawa [27] and Kozono-Yamazaki [24] although their class of solutions is the Leray class $\nabla u \in L^2(\Omega)$ and is different from ours. Galdi [12] and Miyakawa [27] proved that if u and v are solutions in the Leray class, u satisfies the energy inequality $\|\nabla u\|^2_{L^2(\Omega)} \leq -(F, \nabla u)$ and $\sup_{x\in\Omega} |x||v(x)|$ is small, then $u = v$. Kozono-Yamazaki [24] extended their result and showed the uniqueness under the weaker assumptions that u satisfies the energy inequality and v is small in $L^{3,\infty}(\Omega)$. See also Galdi-Sohr [16], whose argument yields almost the same uniqueness criterion for time-periodic solutions as in [12, 27]. Taniuchi [35] established a uniqueness theorem for time-periodic solutions of the nonstationary Navier-Stokes equation without assuming the energy inequality. Since stationary solutions can be regarded as time-periodic solutions with arbitrary period, he, when restricted to the stationary problem, showed that if $u, v \in L^{3,\infty}(\Omega)$ are solutions, and if u is small in $L^{3,\infty}(\Omega)$ and $u, v \in L^{6,2}(\Omega)$, then $u = v$. Farwig-Taniuchi [10] proved almost the same uniqueness theorem for almost time-periodic solutions.

Our uniqueness results to the three-dimensional exterior problem (1.1) consist of two theorems. Given two solutions u and v of (1.1) in the class of Kozono-Yamazaki [23], we shall show that if u is small in $L^{3,\infty}(\Omega)$ and $u, v \in L^r(\Omega)$ for some $r > 3$, then $u = v$. Furthermore, as the main theorem in our results, it shall be proved that the uniqueness holds under the weaker assumptions that u is small in $L^{3,\infty}(\Omega)$ and $v \in L^3(\Omega) + L^\infty(\Omega)$. Since the constant determining the smallness of u in the main theorem is not greater than the one in the first theorem, the inclusion relation between the two theorems is not apparent although the space $L^3(\Omega)+L^\infty(\Omega)$ contains $L^r(\Omega)$ for all $r \geq 3$, see also Remark 3.3 below. It should be emphasized that we do not assume the energy inequality and our results are not covered by [35, 10]. Furthermore, the space $L^3(\Omega) + L^\infty(\Omega)$ is critical in view of local regularity and allows slow decays of solutions.

For the proof, we employ the duality argument. We consider the equation

$$\begin{cases} -\Delta w + w \cdot \nabla u + v \cdot \nabla w + \nabla \pi = 0 & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

which $w := u - v$ obeys, and its dual equation

$$\begin{cases} -\Delta\psi - \displaystyle\sum_{i=1}^{3} u_i \nabla\psi_i - v\cdot\nabla\psi + \nabla\chi = f & \text{in } \Omega, \\ \text{div } \psi = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

We intend to take ψ and w as test functions, respectively, in the weak form of (1.6) and (1.7) to conclude $w = 0$. However, one cannot take w directly as a test function since $C_0^\infty(\Omega)$ is not dense in the class of Kozono-Yamazaki [23]. To overcome this difficulty, the regularity theory for the Stokes and perturbed Stokes equations shall be established. We regard (1.6) as the Stokes and perturbed Stokes equations, and then the regularity theory, together with the additional regularity of solutions, yields the better regularity of w. By the better regularity of w and the existence result for (1.7), we can employ the duality argument.

The uniqueness of weak solutions to (1.1) in a plane exterior domain Ω is also the main subject in this thesis. The two-dimensional exterior problem is different from the three-dimensional one in many points and possesses peculiar difficulties. One of the main difficulties stems from the so-called Stokes paradox. In the theory of the Navier-Stokes equation, the linear approximation often plays a crucial role, however, the Stokes paradox tells us that the linear theory is not available in the analysis of the two-dimensional exterior problem (1.1) in general. Indeed, it is known that the Stokes equation (1.3) admits a solution only if some compatibility conditions are satisfied, see [14, 20, 12]. For instance, even if $F \in C_0^\infty(\Omega)^{2\times 2}$, (1.3) has a solution only if (1.5) holds, see also [9]. This is not surprising, since the natural solution of the Stokes equation should behave at infinity as the fundamental solution $E(x) = O(\log|x|)$. The condition $(1.3)_4$ makes (1.3) over-determined. Another difficulty is little information about the asymptotic behavior of Leray's solution in spite of important contributions [17, 18, 3], see also Section 4.1. Leray [25] showed the existence of a weak solution u with finite Dirichlet integral to the problem $(1.1)_{1,2,3}$ with $f = 0$, see also [11]. However, it is not known whether his solution of $(1.1)_{1,2,3}$ satisfies $(1.1)_4$ even in a weak sense. This is due to the fact that we cannot control the behavior of the solution u at infinity only from the class $\nabla u \in L^2(\Omega)$. Owing to these difficulties, the general theory of the existence for (1.1) is not established yet.

By introducing the symmetry, Galdi [13], Yamazaki [37] and Pileckas-Russo [32] obtained the existence results for (1.1). Assuming that Ω is symmetric with respect to the coordinate axes x_1 and x_2:

$$(x_1, x_2) \in \Omega \;\Rightarrow\; (x_1, -x_2),\; (-x_1, x_2) \in \Omega \tag{1.8}$$

and $f = (f_1, f_2)$ satisfies the symmetry condition

$$\begin{aligned} f_1(x_1, x_2) &= f_1(x_1, -x_2) = -f_1(-x_1, x_2), \\ f_2(x_1, x_2) &= -f_2(x_1, -x_2) = f_2(-x_1, x_2), \end{aligned} \tag{1.9}$$

Galdi [13] and Pileckas-Russo [32] constructed a weak solution u with $\nabla u \in L^2(\Omega)$ and the same symmetry (1.9). Galdi [13] also proved that, due to the symmetry property (1.9), the symmetric weak solution u satisfies $(1.1)_4$ in the sense of

$$\lim_{r\to\infty} \int_0^{2\pi} |u(r, \theta)|^2 \, d\theta = 0, \tag{1.10}$$

see also Russo [33]. Under the stronger symmetry assumption that Ω satisfies

$$(x_1, x_2) \in \Omega \;\Rightarrow\; (x_1, -x_2),\; (-x_1, x_2),\; (x_2, x_1),\; (-x_2, -x_1) \in \Omega \tag{1.11}$$

and f satisfies

$$f_1(x_2, x_1) = -f_1(-x_2, -x_1) = f_2(x_1, x_2) \tag{1.12}$$

as well as (1.9), Yamazaki [37] showed that if f decays rapidly and is small in a sense, then there exists a weak solution u of (1.1) with $\sup_{x\in\Omega}(|x| + 1)|u(x)|$ small and the same symmetry properties (1.9) and (1.12). To the best of our knowledge, [37] is the only literature that provides the existence result of a symmetric weak solution to (1.1) with specific decay rate.

We consider the uniqueness of weak solutions to (1.1), which are less symmetric than (1.9); to be precise, a weak solution $u = (u_1, u_2)$ is assumed to satisfy the condition that

$$\begin{aligned} &\text{for each } i = 1, 2 \text{ either } u_i(x_1, x_2) = -u_i(x_1, -x_2) \\ &\text{or } u_i(x_1, x_2) = -u_i(-x_1, x_2) \text{ holds.} \end{aligned} \tag{1.13}$$

Note that even (1.13) is enough to ensure (1.10), see [13, 33]. Thus far, there are few results on the uniqueness of weak solutions. Yamazaki [37] proved that his solution is unique in the class of weak solutions with $\sup_{x\in\Omega}(|x| + 1)|u(x)|$ small as well as symmetry (1.9) and (1.12), see also [36]. We shall show that if u and v are weak solutions of (1.1) with finite Dirichlet integral and symmetry (1.13), u satisfies the energy inequality $\|\nabla u\|^2_{L^2(\Omega)} \le (f, u)$ and $\sup_{x\in\Omega}(|x| + 1)|v(x)|$ is small, then $u = v$. Our result seems to be the first uniqueness theorem established under the smallness of only one solution. As an application, our uniqueness theorem, together with the result of Yamazaki [37], describes the asymptotic behavior as $|x| \to \infty$ of some symmetric weak solutions.

The proof of the above uniqueness theorem is based on a density property for the solenoidal vector field and the Hardy inequality for symmetric functions. The important step in the proof is to take solutions as test functions in the weak form of (1.1). However, owing to the lack of information on the class of the nonlinear term $u \cdot \nabla u$, it is not clear how we should take v as a test function. Thanks to the symmetry property of u, the Hardy inequality due to Galdi [13] implies that the term $u \cdot \nabla u$ divided by $|x| + 1$ belongs to $L^1(\Omega)$. Based on this observation, we shall construct an approximate sequence $\{v_n\}_{n=1}^{\infty} \subset C_{0,\sigma}^{\infty}(\Omega)$ of the solution v such that $(|x|+1)v_n \to (|x|+1)v$ weakly $*$ in $L^{\infty}(\Omega)$ as well as $\nabla v_n \to \nabla v$ in $L^2(\Omega)$ as $n \to \infty$. This density property enables us to take the solution v as a test function in the weak form of (1.1).

This thesis is organized as follows. In Chapter 2, we shall collect some function spaces and lemmas needed in this thesis. Chapter 3 is devoted to the investigation of the uniqueness to the three-dimensional exterior problem. We discuss the uniqueness of symmetric weak solutions in two-dimensional exterior domains in Chapter 4.

Chapter 2

Preliminaries

This chapter is devoted to the introduction of some function spaces and their properties used in this thesis. In what follows, we adopt the same symbols for vector and scalar function spaces as long as there is no confusion. For a domain $U \subseteq \mathbb{R}^n$, $n = 2, 3$, the space of smooth functions with compact support in U is denoted by $C_0^\infty(U)$ and $C_{0,\sigma}^\infty(U) := \{\varphi \in C_0^\infty(U);\ \mathrm{div}\,\varphi = 0 \text{ in } U\}$. For $1 \le p \le \infty$, let $L^p(U)$ be the usual Lebesgue space with norm $\|\cdot\|_{p,U}$. We denote by $L^{p,q}(U)$ the Lorentz space over U with norm $\|\cdot\|_{p,q,U}$ for $1 < p < \infty$ and $1 \le q \le \infty$. For the definition of the Lorentz space, see Adams-Fournier [2] and Bergh-Löfström [4]. We use the abbreviation $\|\cdot\|_p = \|\cdot\|_{p,\Omega}$ and $\|\cdot\|_{p,q} = \|\cdot\|_{p,q,\Omega}$ for the exterior domain Ω. Let $W_0^{1,q}(U)$ be the Sobolev space defined by the completion of $C_0^\infty(U)$ in the norm $\|\cdot\|_{1,q,U} := \|\cdot\|_{q,U} + \|\nabla\cdot\|_{q,U}$.

Furthermore, we need the homogeneous Sobolev spaces. For $1 < p < \infty$, let $\dot{H}_p^1(U)$ be the completion of $C_0^\infty(U)$ in the norm $\|\nabla\cdot\|_{p,U}$, and let $\dot{H}_p^{-1}(U)$ be the dual space of $\dot{H}_{p'}^1(U)$ where $1/p + 1/p' = 1$. In particular, in the case $p = 2$, we use the notation $\dot{H}_0^1(U)$ and $\dot{H}_0^{-1}(U)$ instead of $\dot{H}_2^1(U)$ and $\dot{H}_2^{-1}(U)$ respectively. We also define $\dot{H}_{0,\sigma}^1(U)$ by the completion of $C_{0,\sigma}^\infty(U)$ in the norm $\|\nabla\cdot\|_{2,U}$. If U is bounded, then $\dot{H}_{0,\sigma}^1(U) = H_{0,\sigma}^1(U)$ where $H_{0,\sigma}^1(U) := \overline{C_{0,\sigma}^\infty(U)}^{\|\cdot\|_{1,q,U}}$. The dual space of $\dot{H}_{0,\sigma}^1(U)$ is denoted by $\dot{H}_{0,\sigma}^{-1}(U)$. Via real interpolation, we define the space $\dot{H}_{p,q}^1(U)$ with norm $\|\nabla\cdot\|_{p,q,U}$ by

$$\dot{H}_{p,q}^1(U) := (\dot{H}_{p_0}^1(U), \dot{H}_{p_1}^1(U))_{\theta,q}$$

where $1 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ satisfy $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1 \le q \le \infty$. Note that the space $\dot{H}_{p,q}^1(U)$ is independent of the choice of p_0 and p_1 up to equivalent norms. Note also that $C_0^\infty(U)$ is dense in $\dot{H}_{p,q}^1(U)$ for $1 < p < \infty$ and $1 \le q < \infty$, and we define the space $\widehat{H}_{p,\infty}^1(U)$ by the completion of $C_0^\infty(U)$ in the norm $\|\nabla\cdot\|_{p,\infty,U}$. By $\dot{H}_{p,q}^{-1}(U)$ for $1 < q \le \infty$ and $\dot{H}_{p,1}^{-1}(U)$ we denote the dual spaces of $\dot{H}_{p',q'}^1(U)$ and $\widehat{H}_{p',\infty}^1(U)$ respectively.

We use the intersection and sum spaces. For Banach spaces X_0 and X_1, the intersection and sum spaces of X_0 and X_1 are denoted by $X_0 \cap X_1$ and $X_0 + X_1$, respectively, with norm $\| \cdot \|_{X_0 \cap X_1} := \| \cdot \|_{X_0} + \| \cdot \|_{X_1}$ and $\|u\|_{X_0+X_1} := \inf\{\|u_0\|_{X_0} + \|u_1\|_{X_1} : u = u_0 + u_1\}$.

We also collect some notation. For $R > 0$ we set $B(0,R) := \{x \in \mathbb{R}^n; |x| < R\}$ and $\Omega_R := \Omega \cap B(0,R)$. The exponent p' denotes the Hölder conjugate of p, that is, p' satisfies $1/p + 1/p' = 1$. We denote by C various constants and, in particular, $C = C(\cdot, \cdots, \cdot)$ denotes constants depending only on the quantities in parentheses. By $(\cdot, \cdot)$ we indicate various duality pairings.

Next, we state some properties of function spaces. It should be noted that we need the following properties of the Lorentz space and the homogeneous Sobolev space only in Chapter 3. Hence we assume below that Ω is an exterior domain in $\mathbb{R}^3$ with smooth boundary. It is known that

$$L^{p,q_0}(\Omega) \subset L^{p,q_1}(\Omega) \quad \text{if} \quad q_0 \le q_1, \qquad L^{p,p}(\Omega) = L^p(\Omega)$$

and

$$L^{p,q}(\Omega)^* = L^{p',q'}(\Omega) \quad (1 \le q < \infty)$$

where $L^{p,q}(\Omega)^*$ denotes the dual space of $L^{p,q}(\Omega)$. Note that the embedding $L^{p,q_0}(\Omega) \subset L^{p,q_1}(\Omega)$ $(q_0 \le q_1)$ is continuous. Furthermore, we have

$$L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega) \subset L^{p,q}(\Omega)$$

for $1 < p_0 < p < p_1 < \infty$ and $1 \le q_0, q_1, q \le \infty$. We define the exponent p^* by $1/p^* := 1/p - 1/3$ for $1 < p < 3$. The Hölder inequality in the Lorentz space and the embedding properties of the homogeneous Sobolev space $\dot{H}^1_{p,q}(\Omega)$ are now introduced. For the latter convenience, we add assertions on the intersection spaces, which are immediate consequences of assertions (i).

Lemma 2.1 ([24])**.** (i) *Let* $1 < p_0, p_1 < \infty$ *with* $1/p := 1/p_0 + 1/p_1 < 1$ *and assume* $1 \le q_0, q_1 \le \infty$ *and* $q := \min\{q_0, q_1\}$. *For* $u \in L^{p_0,q_0}(\Omega)$ *and* $v \in L^{p_1,q_1}(\Omega)$, *we have* $u \cdot v \in L^{p,q}(\Omega)$ *with*

$$\|u \cdot v\|_{p,q} \le C\|u\|_{p_0,q_0}\|v\|_{p_1,q_1}$$

where $C = C(p_0, q_0, p_1, q_1)$.
(ii) *Let* $1 < p_0, p_1, p_2 < \infty$ *with* $1/p := 1/p_0 + 1/p_1 < 1$ *and* $1/r := 1/p_0 + 1/p_2 < 1$. *Assume* $1 \le q_0, q_1, q_2 \le \infty$, $q := \min\{q_0, q_1\}$ *and* $s := \min\{q_0, q_2\}$. *For* $u \in L^{p_0,q_0}(\Omega)$ *and* $v \in L^{p_1,q_1}(\Omega) \cap L^{p_2,q_2}(\Omega)$, *we have* $u \cdot v \in L^{p,q}(\Omega) \cap L^{r,s}(\Omega)$ *with*

$$\|u \cdot v\|_{L^{p,q} \cap L^{r,s}} \le \widetilde{C}\|u\|_{p_0,q_0}\|v\|_{L^{p_1,q_1} \cap L^{p_2,q_2}}$$

where $\widetilde{C} = \widetilde{C}(p_0, q_0, p_1, q_1, p_2, q_2)$.

Remark 2.1. We may assume that the constant $\widetilde{C}$ in (ii) is taken as $\widetilde{C} = \max_{i=1,2} C_i$, where $C_i = C(p_0, q_0, p_i, q_i)$ $(i = 1, 2)$ is the constant in (i). Similar assertions are true in Lemmas 2.2(ii) and 3.3 below.

Lemma 2.2 ([23])**.** (i) *Let* $1 < p < 3$ *and* $1 \le q \le \infty$. *We have the embedding* $\dot{H}^1_{p,q}(\Omega) \subset L^{p^*,q}(\Omega)$ *with the estimate*

$$\|u\|_{p^*,q} \le C\|\nabla u\|_{p,q}$$

where $C = C(p, q)$. *Also, the space* $\dot{H}^1_{3,1}(\Omega)$ *is embedded in* $L^\infty(\Omega) \cap C(\Omega)$ *with the estimate*

$$\|u\|_\infty \le \frac{1}{3}\|\nabla u\|_{3,1}$$

and $u(x) \to 0$ *as* $|x| \to \infty$ *for* $u \in \dot{H}^1_{3,1}(\Omega)$.
(ii) *Let* $1 < p_0, p_1 < 3$ *and* $1 \le q_0, q_1 \le \infty$. *We have the embedding* $\dot{H}^1_{p_0,q_0}(\Omega) \cap \dot{H}^1_{p_1,q_1}(\Omega) \subset L^{p_0^*,q_0}(\Omega) \cap L^{p_1^*,q_1}(\Omega)$ *with the estimate*

$$\|u\|_{L^{p_0^*,q_0} \cap L^{p_1^*,q_1}} \le \widetilde{C}\|\nabla u\|_{L^{p_0,q_0} \cap L^{p_1,q_1}}$$

where $\widetilde{C} = \widetilde{C}(p_0, q_0, p_1, q_1)$. *In the case* $(p_i, q_i) = (3, 1)$ $(i = 0, 1)$, *we have only to replace* $L^{p_i^*,q_i}$ *above by* L^∞.

We also need the Bogovski operator in this thesis.

Lemma 2.3 ([5, 8, 12])**.** *Let* D *be a bounded domain in* $\mathbb{R}^n$, $n \ge 2$, *with Lipschitz boundary and* $1 < q < \infty$.
(i) *There exists a linear operator* $B_D : C_0^\infty(D) \to C_0^\infty(D)^n$ *such that*

$$\|\nabla B_D f\|_{q,D} \le C\|f\|_{q,D}$$

with $C = C(n, q, D)$ *independent of* f *and that*

$$\mathit{div}\ B_D f = f \quad \mathit{if} \quad \int_D f\,dx = 0.$$

Furthermore, by continuity, B_D *is uniquely extended to a bounded linear operator from* $L^q(D)$ *to* $W_0^{1,q}(D)^n$.
(ii) *Let* $y \in \mathbb{R}^n$, $t \in \mathbb{R} \setminus \{0\}$ *and*

$$D_t := \{(1 - t)y + tx : x \in D\}.$$

Then the constant $C = C(n, q, D_t)$ *associated with the operator* B_{D_t} *is independent of* y *and* t.

Chapter 3

Uniqueness of weak solutions in three-dimensional exterior domains

3.1 Introduction

Let Ω be an exterior domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. In this chapter, we consider the uniqueness of solutions in the class (1.4) to the stationary Navier-Stokes equation

$$\begin{cases} -\Delta u + u\cdot\nabla u + \nabla p = \operatorname{div} F & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.1}$$

As stated in Chapter 1, (1.4) is the important class of solutions in the existence theory for (3.1). Here we emphasize the importance of the class (1.4) from the viewpoint of the asymptotic behavior. Leray [25] and Fujita [11] proved that for every $F \in L^2(\Omega)$ the problem (3.1) has a solution $u \in \dot{H}^1_{0,\sigma}(\Omega)$, however, they did not provide enough information about the asymptotic behavior of their solutions at infinity. Concerning the existence of a solution with specific decay rate, Galdi-Simader [15] constructed a solution u with

$$\sup_{x\in\Omega}(|x|+1)|u(x)| < \infty \quad \text{and} \quad \nabla u \in L^q(\Omega) \text{ for all } q > \frac{3}{2}$$

provided $\sup_{x\in\Omega}(|x|+1)^2|F(x)|$ is sufficiently small. Novotny-Padula [31] and Borchers-Miyakawa [7] also obtained a solution u with

$$\sup_{x\in\Omega}(|x|+1)|u(x)| + \sup_{x\in\Omega}(|x|+1)^2|\nabla u(x)| < \infty$$

for the data F with $\sup_{x\in\Omega}(|x|+1)^2|F(x)| + \sup_{x\in\Omega}(|x|+1)^3|\nabla F(x)|$ small. It should be noted that we cannot expect a solution of (3.1) to decay faster than $|x|^{-1}$ in general. Indeed, it was shown in [22] that if $u \in L^3(\Omega)$ is a solution of (3.1), then such a solution u must satisfy (1.5). By introducing the Lorentz space, Kozono-Yamazaki [23] generalized the results [15, 31, 7]. They showed that if F is sufficiently small in $L^{3/2,\infty}(\Omega)$, then there exists a solution u in the class (1.4) of (3.1). Notice that the class of Kozono-Yamazaki [23] is consistent with the ones of [15, 31, 7] and is the suitable class from the viewpoint of the asymptotic behavior expected in general.

The results in this chapter are the uniqueness theorems for solutions in the class of Kozono-Yamazaki [23]. Let u and v be solutions in that class of (3.1). We shall show that if u is small in $L^{3,\infty}(\Omega)$ and $u, v \in L^r(\Omega)$ for some $r > 3$, then $u = v$. As the main result in this chapter, it shall also be proved that if u is small in $L^{3,\infty}(\Omega)$ and $v \in L^3(\Omega) + L^\infty(\Omega)$, then $u = v$. Since the constant determining the smallness of u in the main theorem is not greater than the one in the first theorem, the inclusion relation between the two theorems is not apparent although the space $L^3(\Omega) + L^\infty(\Omega)$ contains $L^r(\Omega)$ for all $r \geq 3$, see also Remark 3.3 below. It should be emphasized that we do not assume the energy inequality. Furthermore, the space $L^3(\Omega) + L^\infty(\Omega)$ is critical in view of local regularity and allows slow decays of solutions.

For the proof, we consider the equation (1.6) which the difference $w := u - v$ obeys and its dual equation (1.7). We make use of the duality relation between uniqueness for (1.6) and solvability of (1.7). The important step in the duality argument is to take ψ and w as test functions, respectively, in the weak form of (1.6) and (1.7). However, it is difficult to take w directly as a test function since $C_0^\infty(\Omega)$ is not dense in the class of Kozono-Yamazaki [23]. To get around this difficulty, we shall refine the regularity theory for the Stokes equation (1.3) and then apply it to the solution w of (1.6) assuming the additional regularity of solutions u and v. This is the crucial part in the proof of the first uniqueness theorem. Since our main aim in this chapter is to prove the uniqueness without assuming the additional regularity of u, we need to consider the regularity of solutions to the perturbed Stokes equation

$$\begin{cases} -\Delta w + w\cdot\nabla u + \operatorname{div}\,(\theta_{1,\epsilon}\otimes w) + \nabla\pi = f & \text{in } \Omega, \\ \operatorname{div}\, w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

instead of the Stokes equation (1.3), where $\theta_{1,\epsilon}$ is the L^3-part of $v \in L^3(\Omega) + L^\infty(\Omega)$ with $\|\theta_{1,\epsilon}\|_3 \leq \epsilon$ and $\theta_{2,\epsilon} = v - \theta_{1,\epsilon} \in L^\infty(\Omega)$, and $\theta_{1,\epsilon}\otimes w = (\theta_{1,\epsilon,i}w_j)_{i,j=1}^3$. Such a decomposition of v is actually possible for every $\epsilon > 0$, see Section 3.4. Due to the smallness of u as well as $\theta_{1,\epsilon}$, one can establish the regularity theory for the perturbed Stokes equation (3.2), whose proof is based on the unique solvability of the Stokes equation (1.3) in intersection spaces. We then regard the equation (1.6) as (3.2) with the external force $f = -\mathrm{div}\ (\theta_{2,\epsilon} \otimes w)$, the regular part of $v \cdot \nabla w$, and by the regularity theory for (3.2) we deduce better regularity of w, which enables us to employ the duality argument.

The outline of this chapter is as follows. We shall state our results precisely in Section 3.2. Section 3.3 is devoted to introducing the theory of the Stokes equations in exterior domains. In Section 3.4, we shall prove the solvability of the dual equation, the first uniqueness theorem and the regularity theory for the perturbed Stokes equation mentioned above, and finally we give the proof of our main result.

3.2 Results

Before stating our results, we give the definition of a solution to (3.1).

Definition 3.1. Given $F \in L^{3/2,\infty}(\Omega)$, we say that a pair $\{u, p\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L^{3/2,\infty}(\Omega)$ is a solution of (3.1) if $\{u, p\}$ satisfies

$$\begin{cases} (\nabla u, \nabla \varphi) - (u \cdot \nabla \varphi, u) - (p, \mathrm{div}\ \varphi) = -(F, \nabla \varphi) & \text{for all } \varphi \in C_0^\infty(\Omega), \\ \mathrm{div}\ u = 0 & \text{in } \Omega. \end{cases}$$

As we can see in Lemma 2.2(i), the space $\dot{H}^1_{3/2,\infty}(\Omega)$ is continuously embedded in $L^{3,\infty}(\Omega)$.

Now we state our results.

Theorem 3.1. *Suppose that $\{u, p\}, \{v, q\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L^{3/2,\infty}(\Omega)$ are solutions of* (3.1). *There exists an absolute constant $\delta > 0$ such that if*

$$\|u\|_{3,\infty} \leq \delta$$

and

$$u, v \in L^r(\Omega)$$

for some $r > 3$, then $\{u, p\} = \{v, q\}$.

Remark 3.1. The constant δ is independent of Ω and r. This constant is equal to the one in Lemma 3.4 below.

Remark 3.2. The set of solutions satisfying $u \in \dot{H}^1_{3/2,\infty}(\Omega) \cap L^r(\Omega)$ for some $r > 3$ is not empty. According to Kozono-Yamazaki [23, Main Theorem (3)], if $F \in L^{3/2,\infty}(\Omega) \cap L^{s,\infty}(\Omega)$ for some $3/2 < s < 3$ and F is sufficiently small in $L^{3/2,\infty}(\Omega)$, then there exists a solution such that $u \in \dot{H}^1_{3/2,\infty}(\Omega) \cap \dot{H}^1_{s,\infty}(\Omega)$. Since $\dot{H}^1_{s,\infty}(\Omega) \subset L^{s^*,\infty}(\Omega)$ from Lemma 2.2(i) and $L^{3,\infty}(\Omega) \cap L^{s^*,\infty}(\Omega) \subset L^r(\Omega)$ for $3 < r < s^*$, it follows that $u \in L^r(\Omega)$ for such $r > 3$.

The next theorem is the main result in this chapter.

Theorem 3.2. *Suppose that* $\{u, p\}, \{v, q\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L^{3/2,\infty}(\Omega)$ *are solutions of* (3.1)*. There exists a constant* $\delta = \delta(\Omega) > 0$ *such that if*

$$\|u\|_{3,\infty} \le \delta$$

and

$$v \in L^3(\Omega) + L^\infty(\Omega),$$

then $\{u, p\} = \{v, q\}$.

Remark 3.3. The constant δ in this theorem is selected so that it is not greater than the constants in Lemmas 3.4 and 3.5. Hence, this constant is not greater than the one in Theorem 3.1. From this point of view, it is not clear whether Theorem 3.1 is covered by Theorem 3.2 or not.

Remark 3.4. The set of solutions $u \in \dot{H}^1_{3/2,\infty}(\Omega)$ satisfying $u \in L^3(\Omega) + L^\infty(\Omega)$ is not empty. Indeed, as is stated in Remark 3.2, there exists a solution $u \in \dot{H}^1_{3/2,\infty}(\Omega) \cap L^r(\Omega)$ for some $r > 3$, and the space $L^r(\Omega)$ is contained in $L^3(\Omega) + L^\infty(\Omega)$.

3.3 Stokes equations in exterior domains

In this section, we introduce the theory of the existence, uniqueness and regularity for the Stokes equation in exterior domains. The next lemma concerns the unique solvability of the Stokes equation.

Lemma 3.1 ([23], [34])**.** *Suppose one of the following holds:*

$$\begin{cases} (p, q) = (3/2, \infty), \\ 3/2 < p < 3, \quad 1 \le q \le \infty, \\ (p, q) = (3, 1). \end{cases}$$

Then for every $f \in \dot{H}^{-1}_{p,q}(\Omega)$ *and* $g \in L^{p,q}(\Omega)$, *there exists a unique solution* $\{u, p\} \in \dot{H}^1_{p,q}(\Omega) \times L^{p,q}(\Omega)$ *of the Stokes equation*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

with the estimate

$$\|\nabla u\|_{p,q} + \|p\|_{p,q} \leq C(\|f\|_{\dot{H}^{-1}_{p,q}} + \|g\|_{p,q})$$

where $C = C(\Omega, p, q)$.

The regularity theory for the Stokes equation in exterior domains was established by Kozono-Yamazaki [23] assuming $1 \leq q_0 \leq q_1 \leq \infty$, and we refine it since we need to consider the case $q_1 \leq q_0$.

Lemma 3.2 ([23, 28])**.** *Let* $1 < p_0 < 3$ *and* $1 \leq q_0 \leq \infty$, *or* $(p_0, q_0) = (3, 1)$. *Assume that one of the following holds:*

$$\begin{cases} (p_1, q_1) = (3/2, \infty), \\ 3/2 < p_1 < 3, \quad 1 \leq q_1 \leq \infty, \\ (p_1, q_1) = (3, 1). \end{cases}$$

Suppose that $\{u, p\} \in \dot{H}^1_{p_0,q_0}(\Omega) \times L^{p_0,q_0}(\Omega)$ *is a solution of the Stokes equation* (3.3) *for* $f \in \dot{H}^{-1}_{p_0,q_0}(\Omega) \cap \dot{H}^{-1}_{p_1,q_1}(\Omega)$ *and* $g \in L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega)$. *Then we have*

$$u \in \dot{H}^1_{p_0,q_0}(\Omega) \cap \dot{H}^1_{p_1,q_1}(\Omega), \qquad p \in L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega)$$

with the estimate

$$\begin{aligned} &\|\nabla u\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} + \|p\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} \\ &\qquad \leq C(\|f\|_{\dot{H}^{-1}_{p_0,q_0} \cap \dot{H}^{-1}_{p_1,q_1}} + \|g\|_{L^{p_0,q_0} \cap L^{p_1,q_1}}) \end{aligned}$$

where $C = C(\Omega, p_0, q_0, p_1, q_1)$.

Proof. According to Lemma 3.1, there exists a unique pair $\{\bar{u}, \bar{p}\} \in \dot{H}^1_{p_1,q_1}(\Omega) \times L^{p_1,q_1}(\Omega)$ satisfying the Stokes equation in the sense of distributions. Set $v := u - \bar{u}$ and $q := p - \bar{p}$. Then the pair $\{v, q\}$ satisfies

$$\begin{cases} -\Delta v + \nabla q = 0 & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \end{cases}$$

in the sense of distributions, and by the interior regularity for the Stokes equation, $\{v, q\}$ is smooth in Ω and satisfies this equation in the classical sense. Let us take $R > 0$ so that $\mathbb{R}^3 \setminus \Omega \subset B(0, R)$ and select a cutoff function $\zeta \in C_0^\infty(\mathbb{R}^3)$ satisfying $0 \le \zeta(x) \le 1$, $\zeta(x) = 1$ for $|x| < R+1$ and $\zeta(x) = 0$ for $|x| > R+2$. Set

$$w := (1-\zeta)v + B_D[\nabla\zeta \cdot v], \quad \pi := (1-\zeta)q$$

where B_D is the Bogovski operator introduced in Lemma 2.3 in the bounded domain $D := \{R < |x| < R+3\}$. Since $v = 0$ on $\partial\Omega$, we have

$$\int_D \nabla\zeta \cdot v \, dx = -\int_{\partial\Omega \cup \partial B(0,R)} v \cdot \nu \, dS = -\int_{\Omega_R} \mathrm{div}\ v \, dx = 0$$

and thus div $w = 0$, where ν is the unit outer normal to $\partial\Omega_R$. Hence the pair $\{w, \pi\}$ is a smooth solution of

$$\begin{cases} -\Delta w + \nabla\pi = F & \text{in } \mathbb{R}^3, \\ \mathrm{div}\ w = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where

$$F = 2\nabla\zeta \cdot \nabla v + (\Delta\zeta)v - (\nabla\zeta)q - \Delta B_D[\nabla\zeta \cdot v].$$

Observe that $w(x) = v(x)$ and $\pi(x) = q(x)$ for sufficiently large $|x|$ and that supp $F \subset D$. By the class of $\{v, q\}$ we find that w and π are tempered distributions. Therefore they can be represented as

$$\begin{aligned} w(x) &= \int_{\mathbb{R}^3} E(x-y)F(y)\, dy + U(x), \\ \pi(x) &= \int_{\mathbb{R}^3} e(x-y) \cdot F(y)\, dy + P(x), \end{aligned}$$

where $\{E, e\}$ is the fundamental solution of the Stokes equation and U and P are the Stokes polynomials. Since supp F is contained in D, it follows that the first terms of w, ∇w and π above behave, respectively, like $E(x) = O(|x|^{-1})$, $\nabla E(x) = O(|x|^{-2})$ and $e(x) = O(|x|^{-2})$ as $|x| \to \infty$. Furthermore, we see $U = 0$ and $P = 0$ because Lemma 2.2 (i) implies that w and π are small at infinity. Consequently, we obtain

$$|v(x)| = O(|x|^{-1}), \quad |\nabla v(x)| + |q(x)| = O(|x|^{-2}) \quad \text{as } |x| \to \infty. \tag{3.4}$$

Next, choose a cutoff function $\eta \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \eta(x) \le 1$, $\eta(x) = 1$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| > 2$, and set $\eta_R(x) := \eta(x/R)$. Then multiplying $-\Delta v + \nabla q = 0$ by

$\eta_R v$ and integrating by parts over Ω yield

$$\begin{aligned} 0 &= \int_\Omega (-\Delta v + \nabla q) \cdot \eta_R v \, dx \\ &= \int_{\Omega_{2R}} \nabla v \cdot (\nabla(\eta_R v)) \; dx - \int_{\Omega_{2R}} q \text{ div } (\eta_R v) \, dx \\ &= \int_\Omega \eta_R |\nabla v|^2 \, dx + \int_{R<|x|<2R} \{\nabla v \cdot (\nabla \eta_R) v - (\nabla \eta_R \cdot v) q\} \; dx. \end{aligned} \tag{3.5}$$

Here we have used the fact that the pair $\{v, q\}$ has the regularity also near the boundary since $\partial\Omega$ is smooth. Since $|\nabla \eta_R| \le C/R$ for some constant $C > 0$ independent of R, it follows from (3.4) that

$$\left| \int_{R<|x|<2R} \{\nabla v \cdot (\nabla \eta_R) v - (\nabla \eta_R \cdot v) q\} \; dx \right| \le \frac{C}{R}$$

for sufficiently large $R > 0$. Letting $R \to \infty$ in (3.5), we derive

$$\int_\Omega |\nabla v|^2 \, dx = 0.$$

Therefore v is a constant in Ω, and by the boundary condition we deduce $v = 0$ in Ω. We also have $\nabla q = 0$ in Ω, and thus $q = 0$ in Ω since q is small at infinity. $\square$

As an immediate consequence of Lemmas 3.1 and 3.2, we obtain the following lemma on the unique solvability of the Stokes equation in intersection spaces, which plays a crucial role in the proof of Lemma 3.5.

Lemma 3.3. *Assume for each $i = 0, 1$ that one of the following holds:*

$$\begin{cases} (p_i, q_i) = (3/2, \infty), \\ 3/2 < p_i < 3, \quad 1 \le q_i \le \infty, \\ (p_i, q_i) = (3, 1). \end{cases}$$

Then for every $f \in \dot{H}^{-1}_{p_0,q_0}(\Omega) \cap \dot{H}^{-1}_{p_1,q_1}(\Omega)$ and $g \in L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega)$, there exists a unique solution

$$\{u, p\} \in (\dot{H}^1_{p_0,q_0}(\Omega) \cap \dot{H}^1_{p_1,q_1}(\Omega)) \times (L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega))$$

of the Stokes equation (3.3) *with the estimate*

$$\begin{aligned} &\|\nabla u\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} + \|p\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} \\ &\qquad \le C(\|f\|_{\dot{H}^{-1}_{p_0,q_0} \cap \dot{H}^{-1}_{p_1,q_1}} + \|g\|_{L^{p_0,q_0} \cap L^{p_1,q_1}}) \end{aligned} \tag{3.6}$$

where $C = C(\Omega, p_0, q_0, p_1, q_1)$.

3.4 Proofs of Theorems 3.1 and 3.2

If $\{u, p\}, \{v, q\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L^{3/2,\infty}(\Omega)$ are solutions of (3.1), then $w := u - v$ and $\pi := p - q$ satisfy $(1.6)_1$ in the sense of distributions, that is,

$$(\nabla w, \nabla \varphi) - \left(\sum_{i=1}^{3} u_i \nabla \varphi_i, w\right) - (v \cdot \nabla \varphi, w) - (\pi, \operatorname{div} \varphi) = 0 \tag{3.7}$$

holds for all $\varphi \in C_0^\infty(\Omega)$. Observe that if $\{w, \pi\} \in \dot{H}^1_{p,q}(\Omega) \times L^{p,q}(\Omega)$ for some $3/2 < p < 3$ and $1 < q \le \infty$ additionally, then by continuity we can take $\varphi \in \dot{H}^1_{p',q'}(\Omega)$ as test functions. Recall that the dual equation of (1.6) is given by (1.7). For given $f \in \dot{H}_0^{-1}(\Omega)$, we say that a pair $\{\psi, \chi\} \in \dot{H}_0^1(\Omega) \times L^2(\Omega)$ is a solution of (1.7) if the pair $\{\psi, \chi\}$ satisfies

$$(\nabla \psi, \nabla \widetilde{\varphi}) - \left(\sum_{i=1}^{3} u_i \nabla \psi_i, \widetilde{\varphi}\right) - (v \cdot \nabla \psi, \widetilde{\varphi}) - (\chi, \operatorname{div} \widetilde{\varphi}) = (f, \widetilde{\varphi}) \tag{3.8}$$

for all $\widetilde{\varphi} \in C_0^\infty(\Omega)$ as well as $\operatorname{div} \psi = 0$. We need only the solvability of (1.7) within the L^2-framework. Notice that we can take $\widetilde{\varphi} \in \dot{H}_0^1(\Omega)$ as test functions.

We shall construct a solution $\{\psi, \chi\} \in \dot{H}^1_{0,\sigma}(\Omega) \times L^2(\Omega)$ of (1.7) assuming $u, v \in L^{3,\infty}(\Omega)$ and $\operatorname{div} v = 0$ in Ω. Observe that for $\psi \in \dot{H}^1_{0,\sigma}(\Omega)$ we have $\sum_{i=1}^3 u_i \nabla \psi_i + v \cdot \nabla \psi \in \dot{H}_0^{-1}(\Omega)$ and

$$(v \cdot \nabla \psi, \psi) = 0, \qquad \left\| \sum_{i=1}^{3} u_i \nabla \psi_i \right\|_{\dot{H}_0^{-1}} \le C_1 C_2 \|u\|_{3,\infty} \|\nabla \psi\|_2 \tag{3.9}$$

where C_1 and C_2 are the constants in Lemmas 2.1(i) and 2.2(i) with $(p_0, q_0, p_1, q_1) = (3, \infty, 2, 2)$ and $(p, q) = (2, 2)$ respectively. Note also that $\dot{H}^1_{0,\sigma}(\Omega)$ is a Hilbert space with inner product $(\nabla \cdot, \nabla \cdot)$ and $\dot{H}^1_{0,\sigma}(\Omega) = \{\psi \in \dot{H}_0^1(\Omega);\ \operatorname{div} \psi = 0 \text{ in } \Omega\}$, see Borchers-Sohr [8, Lemma 4.1].

Lemma 3.4. *Suppose $u, v \in L^{3,\infty}(\Omega)$ and $\operatorname{div} v = 0$ in Ω. There exists an absolute constant $\delta > 0$ such that if $\|u\|_{3,\infty} \le \delta$, then for every $f \in \dot{H}_0^{-1}(\Omega)$ there exists a solution $\{\psi, \chi\} \in \dot{H}^1_{0,\sigma}(\Omega) \times L^2(\Omega)$ of (1.7).*

Proof. We intend to construct a function $\psi \in \dot{H}^1_{0,\sigma}(\Omega)$ satisfying

$$(\nabla \psi, \nabla \varphi) - \left(\sum_{i=1}^{3} u_i \nabla \psi_i, \varphi\right) - (v \cdot \nabla \psi, \varphi) = (f, \varphi) \tag{3.10}$$

for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$ by the Galerkin approximation. Let $\{\varphi_j\}_{j=1}^\infty \subset C^\infty_{0,\sigma}(\Omega)$ be an orthonormal basis of $\dot{H}^1_{0,\sigma}(\Omega)$. For the existence of such an orthonormal basis $\{\varphi_j\}_{j=1}^\infty$, see

Galdi [12, VII, Lemma 2.1]. Fix a positive integer m, and we seek a function ψ_m of the form

$$\psi_m(x) = \sum_{j=1}^{m} \xi_{jm}\varphi_j(x) \tag{3.11}$$

where we hope to select the coefficients ξ_{jm} so that

$$(\nabla\psi_m, \nabla\varphi_j) - \left(\sum_{i=1}^{3} u_i \nabla\psi_{m,i}, \varphi_j\right) - (v\cdot\nabla\psi_m, \varphi_j) = (f, \varphi_j) \tag{3.12}$$

for $j = 1, \dots, m$.

Substituting (3.11) into the system (3.12), we have

$$(I - A - B)\xi = \eta.$$

Here $\xi = (\xi_{jm})_{j=1}^m$, I is the identity matrix and

$$A = (A_{jk})_{j,k=1}^m, \quad A_{jk} = \left(\sum_{i=1}^{3} u_i \nabla\varphi_{k,i}, \varphi_j\right),$$
$$B = (B_{jk})_{j,k=1}^m, \quad B_{jk} = (v\cdot\nabla\varphi_k, \varphi_j), \quad \eta = (\eta_j)_{j=1}^m, \quad \eta_j = (f, \varphi_j).$$

For the existence of a function ψ_m of the form (3.11) satisfying (3.12), it suffices to show that the matrix $I - A - B$ has the inverse. Suppose $(I - A - B)\xi = 0$, and then $\langle (I - A - B)\xi, \xi\rangle = 0$ where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{R}^m$. Now let us take a positive constant $\delta > 0$ so that

$$0 < \delta < \frac{1}{C_1C_2}$$

where C_1 and C_2 are the constants appearing in (3.9) and assume $\|u\|_{3,\infty} \le \delta$. Since

$$\langle A\xi, \xi\rangle = \left(\sum_{i=1}^{3} u_i \nabla\psi_{m,i}, \psi_m\right)$$

and B is a skew-symmetric matrix, we have

$$|\xi|^2 = \left(\sum_{i=1}^{3} u_i \nabla\psi_{m,i}, \psi_m\right) \le C_1C_2\|u\|_{3,\infty}\|\nabla\psi_m\|_2^2 \le C_1C_2\delta|\xi|^2. \tag{3.13}$$

This implies $\xi = 0$. Hence $I - A - B$ has the inverse matrix.

Multiplying (3.12) by ξ_{jm} and summing for $j = 1, \dots, m$, we obtain

$$\|\nabla\psi_m\|_2^2 - \left(\sum_{i=1}^{3} u_i \nabla\psi_{m,i}, \psi_m\right) = (f, \psi_m).$$

Thus the assumption $\|u\|_{3,\infty} \le \delta$ and similar calculations to (3.13) imply

$$\|\nabla \psi_m\|_2 \le C\|f\|_{\dot{H}_0^{-1}}$$

for some appropriate constant $C > 0$. Since $\dot{H}^1_{0,\sigma}(\Omega)$ is a Hilbert space, there exist a subsequence of $\{\psi_m\}_{m=1}^\infty$, which we denote by $\{\psi_m\}_{m=1}^\infty$ itself for simplicity, and $\psi \in \dot{H}^1_{0,\sigma}(\Omega)$ such that $\nabla\psi_m \rightharpoonup \nabla\psi$ weakly in $L^2(\Omega)$ as $m \to \infty$. Letting $m \to \infty$ in (3.12) and recalling that $\{\varphi_j\}_{j=1}^\infty \subset C^\infty_{0,\sigma}(\Omega)$ is an orthonormal basis of $\dot{H}^1_{0,\sigma}(\Omega)$, we derive (3.10) for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$.

Now we have $-\Delta\psi - \sum_{i=1}^3 u_i \nabla\psi_i - v\cdot\nabla\psi - f \in \dot{H}_0^{-1}(\Omega)$ and

$$\left(-\Delta\psi - \sum_{i=1}^3 u_i\nabla\psi_i - v\cdot\nabla\psi - f, \varphi\right) = 0 \quad \text{for all } \varphi \in C^\infty_{0,\sigma}(\Omega).$$

Hence, according to Borchers-Sohr [8, Theorem 4.2], there exists a unique $\chi \in L^2(\Omega)$ such that

$$-\Delta\psi - \sum_{i=1}^3 u_i\nabla\psi_i - v\cdot\nabla\psi - f = -\nabla\chi$$

in the sense of distributions. Consequently, the pair $\{\psi,\chi\} \in \dot{H}^1_{0,\sigma}(\Omega)\times L^2(\Omega)$ is a solution of (1.7). □

Collecting Lemmas 3.2 and 3.4, we give the proof of Theorem 3.1. In the proof of the theorem, it is an important step to take ψ and w as test functions, respectively, in (3.7) and (3.8). However, we have difficulty in taking $w \in \dot{H}^1_{3/2,\infty}(\Omega)$ as a test function since $C_0^\infty(\Omega)$ is not dense in $\dot{H}^1_{p,\infty}(\Omega)$ for $1 < p < \infty$, even though we have $\{\psi,\chi\} \in \dot{H}^1_{3,1}(\Omega)\times L^{3,1}(\Omega)$. To overcome this difficulty, we use Lemma 3.2 and the additional regularity of u and v.

Proof of Theorem 3.1. It suffices to prove the theorem assuming $u, v \in L^{6(k+1)/(2k+1),2}(\Omega)$ for some positive integer k, since for each $r > 3$ there exists a positive integer k such that $L^{3,\infty}(\Omega)\cap L^r(\Omega) \subset L^{6(k+1)/(2k+1),2}(\Omega)$. Thus we assume $u, v \in L^{6(k+1)/(2k+1),2}(\Omega)$ for some positive integer k instead of $u, v \in L^r(\Omega)$ for some $r > 3$.

Set $w := u - v$ and $\pi := p - q$. Then the pair $\{w,\pi\} \in \dot{H}^1_{3/2,\infty}(\Omega)\times L^{3/2,\infty}(\Omega)$ satisfies (1.6) in the sense of distributions. Given a positive integer k, we set

$$q_\ell := \frac{6(k+1)}{4k+3-\ell}$$

for $\ell = 1,\dots,k$ and then

$$\frac{3}{2} < q_1 < \cdots < q_\ell < q_{\ell+1} < \cdots < q_k = 2.$$

Let us regard (1.6) as the Stokes equation with the external force $-(w \cdot \nabla u + v \cdot \nabla w)$ and write

$$w \cdot \nabla u + v \cdot \nabla w = \mathrm{div}\,(w \otimes u + v \otimes w).$$

Since $w \in L^{6(k+1)/(2k+1),2}(\Omega)$, by Lemmas 2.1(i) and 2.2(i) we have

$$w \otimes u + v \otimes w \in L^{3/2,\infty}(\Omega) \cap L^{q_1,2}(\Omega)$$

and thus

$$w \cdot \nabla u + v \cdot \nabla w \in \dot{H}^{-1}_{3/2,\infty}(\Omega) \cap \dot{H}^{-1}_{q_1,2}(\Omega).$$

Hence Lemma 3.2 implies

$$\{w, \pi\} \in \dot{H}^1_{q_1,2}(\Omega) \times L^{q_1,2}(\Omega).$$

Furthermore, we claim that if $\{w, \pi\} \in \dot{H}^1_{q_\ell,2}(\Omega) \times L^{q_\ell,2}(\Omega)$ is a solution of (1.6), then $\{w, \pi\} \in \dot{H}^1_{q_{\ell+1},2}(\Omega) \times L^{q_{\ell+1},2}(\Omega)$. Indeed, Lemmas 2.1(ii) and 2.2(i) yield

$$w \otimes u + v \otimes w \in L^{q_\ell,2}(\Omega) \cap L^{q_{\ell+1},2}(\Omega),$$

which implies

$$w \cdot \nabla u + v \cdot \nabla w \in \dot{H}^{-1}_{q_\ell,2}(\Omega) \cap \dot{H}^{-1}_{q_{\ell+1},2}(\Omega).$$

Using Lemma 3.2, we obtain

$$\{w, \pi\} \in \dot{H}^1_{q_{\ell+1},2}(\Omega) \times L^{q_{\ell+1},2}(\Omega).$$

Therefore, starting from q_1, we eventually derive

$$\{w, \pi\} \in \dot{H}^1_0(\Omega) \times L^2(\Omega).$$

We take the constant $\delta > 0$ from Lemma 3.4 and assume $\|u\|_{3,\infty} \le \delta$. Then for every $f \in \dot{H}^{-1}_0(\Omega)$ there exists a solution $\{\psi, \chi\} \in \dot{H}^1_0(\Omega) \times L^2(\Omega)$. By continuity we can take $\psi \in \dot{H}^1_0(\Omega)$ as a test function in (3.7) to obtain

$$(\nabla w, \nabla \psi) - \left(\sum_{i=1}^{3} u_i \nabla \psi_i, w \right) - (v \cdot \nabla \psi, w) = 0. \tag{3.14}$$

On the other hand, taking $w \in \dot{H}^1_0(\Omega)$ as a test function in (3.8) yields

$$(\nabla \psi, \nabla w) - \left(\sum_{i=1}^{3} u_i \nabla \psi_i, w \right) - (v \cdot \nabla \psi, w) = (f, w). \tag{3.15}$$

It follows from (3.14) and (3.15) that $w \in \dot{H}^1_0(\Omega)$ satisfies

$$(f, w) = 0 \quad \text{for all } f \in \dot{H}^{-1}_0(\Omega).$$

Consequently, we conclude that $w = 0$ in Ω. We also obtain $\nabla\pi = 0$ in Ω, and $\pi \in L^{3,\infty}(\Omega) \cap L^2(\Omega)$ implies $\pi = 0$ in Ω. $\square$

In the proof of Theorem 3.1, we got around the difficulty mentioned above by assuming the additional regularity of u and v. However, we do not assume the additional regularity of u to prove our main theorem (Theorem 3.2). We thus develop the regularity theory for the perturbed Stokes equation (3.2).

The smallness of $\theta_{1,\epsilon}$ is justified in the following way. We assume $v \in L^3(\Omega) + L^\infty(\Omega)$ in the main theorem and thus there exist some $\omega_1 \in L^3(\Omega)$ and $\omega_2 \in L^\infty(\Omega)$ such that $v = \omega_1 + \omega_2$. Fix such ω_1 and ω_2. Since $C_0^\infty(\Omega)$ is dense in $L^3(\Omega)$ and the embedding $L^3(\Omega) \subset L^{3,\infty}(\Omega)$ is continuous, for each $\epsilon > 0$ we can introduce the mollification $\omega_{1,\epsilon} \in C_0^\infty(\Omega)$ of ω_1, which leads to the decomposition

$$v = \theta_{1,\epsilon} + \theta_{2,\epsilon}, \qquad \theta_{1,\epsilon} \in L^3(\Omega), \qquad \theta_{2,\epsilon} \in L^\infty(\Omega), \tag{3.16}$$

where $\theta_{1,\epsilon} = \omega_1 - \omega_{1,\epsilon}$ with

$$\|\theta_{1,\epsilon}\|_{3,\infty} \le \epsilon \tag{3.17}$$

and $\theta_{2,\epsilon} = \omega_{1,\epsilon} + \omega_2$. Note that $\theta_{2,\epsilon} = v - \theta_{1,\epsilon} \in L^{3,\infty}(\Omega)$. Note furthermore that $v \cdot \nabla w = \operatorname{div}(\theta_{1,\epsilon} \otimes w) + \operatorname{div}(\theta_{2,\epsilon} \otimes w)$.

Now we establish the regularity theory for the perturbed Stokes equation (3.2).

Lemma 3.5. *Let $u, \theta_{1,\epsilon} \in L^{3,\infty}(\Omega)$ with (3.17) and assume for each $i = 0, 1$ that one of the following holds:*

$$\begin{cases} (p_i, q_i) = (3/2, \infty), \\ 3/2 < p_i < 3, \quad 1 \le q_i \le \infty. \end{cases}$$

Suppose that $\{w, \pi\} \in \dot{H}^1_{p_0,q_0}(\Omega) \times L^{p_0,q_0}(\Omega)$ is a solution of (3.2) for $f \in \dot{H}^{-1}_{p_0,q_0}(\Omega) \cap \dot{H}^{-1}_{p_1,q_1}(\Omega)$. There exist constants $\delta = \delta(\Omega, p_0, q_0, p_1, q_1) > 0$ and $\widetilde{\delta} = \widetilde{\delta}(\Omega, p_0, q_0, p_1, q_1) > 0$ such that if

$$\|u\|_{3,\infty} \le \delta, \qquad \epsilon \le \widetilde{\delta},$$

then

$$w \in \dot{H}^1_{p_0,q_0}(\Omega) \cap \dot{H}^1_{p_1,q_1}(\Omega), \qquad \pi \in L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega)$$

with the estimate

$$\|\nabla w\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} + \|\pi\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} \le C\|f\|_{\dot{H}^{-1}_{p_0,q_0} \cap \dot{H}^{-1}_{p_1,q_1}} \tag{3.18}$$

where $C = C(\Omega, p_0, q_0, p_1, q_1)$.

Proof. Set

$$X := (\dot{H}^1_{p_0,q_0}(\Omega) \cap \dot{H}^1_{p_1,q_1}(\Omega)) \times (L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega)).$$

We first show that if $\|u\|_{3,\infty}$ and ϵ are small, then for every $f \in \dot{H}^{-1}_{p_0,q_0}(\Omega) \cap \dot{H}^{-1}_{p_1,q_1}(\Omega)$ there exists a unique solution $\{w, \pi\} \in X$ of (3.2) satisfying the estimate (3.18).

Let

$$S : \{f, g\} \in (\dot{H}^{-1}_{p_0,q_0}(\Omega) \cap \dot{H}^{-1}_{p_1,q_1}(\Omega)) \times (L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega)) \mapsto \{w, \pi\} \in X$$

be the solution operator defined by Lemma 3.3. We intend to construct a solution $\{w, \pi\} \in X$ of

$$\{w, \pi\} = S\{f - w \cdot \nabla u - \text{div } (\theta_{1,\epsilon} \otimes w), 0\}. \tag{3.19}$$

Then such a pair $\{w, \pi\} \in X$ is a solution of (3.2). We shall solve (3.19) by the following successive approximation:

$$\begin{cases} \{w_0, \pi_0\} = S\{f, 0\}, \\ \{w_{j+1}, \pi_{j+1}\} = \{w_0, \pi_0\} - S\{w_j \cdot \nabla u + \text{div } (\theta_{1,\epsilon} \otimes w_j), 0\} \quad (j = 0, 1, \ldots). \end{cases} \tag{3.20}$$

By Lemma 3.3, we obtain

$$\|\nabla w_0\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} + \|\pi_0\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} \le C_0 \|f\|_{\dot{H}^{-1}_{p_0,q_0} \cap \dot{H}^{-1}_{p_1,q_1}},$$

where $C_0 = C(\Omega, p_0, q_0, p_1, q_1)$ is the constant in (3.6). Since $w_j \cdot \nabla u = \text{div } (w_j \otimes u)$, we have

$$\begin{aligned} \|S\{w_j \cdot \nabla u + \text{div } (\theta_{1,\epsilon} \otimes w_j), 0\}\|_X &\le C_0 \|w_j \cdot \nabla u + \text{div } (\theta_{1,\epsilon} \otimes w_j)\|_{\dot{H}^{-1}_{p_0,q_0} \cap \dot{H}^{-1}_{p_1,q_1}} \\ &\le C_0 \|w_j \otimes u + \theta_{1,\epsilon} \otimes w_j\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} \\ &\le C_0 C_1 (\|u\|_{3,\infty} + \|\theta_{1,\epsilon}\|_{3,\infty}) \|w_j\|_{L^{p_0^*,q_0} \cap L^{p_1^*,q_1}} \\ &\le M(\|u\|_{3,\infty} + \epsilon) \|\nabla w_j\|_{L^{p_0,q_0} \cap L^{p_1,q_1}}, \end{aligned} \tag{3.21}$$

with $M = C_0 C_1 C_2$, where the constants $C_1 = \widetilde{C}(3, \infty, p_0^*, q_0, p_1^*, q_1)$ and $C_2 = \widetilde{C}(p_0, q_0, p_1, q_1)$ are those in Lemmas 2.1(ii) and 2.2(ii) respectively. Thus

$$\begin{aligned} \|\{w_{j+1}, \pi_{j+1}\}\|_X &\le \|\{w_0, \pi_0\}\|_X + \|S\{w_j \cdot \nabla u + \text{div } (\theta_{1,\epsilon} \otimes w_j), 0\}\|_X \\ &\le C_0 \|f\|_{\dot{H}^{-1}_{p_0,q_0} \cap \dot{H}^{-1}_{p_1,q_1}} + M(\|u\|_{3,\infty} + \epsilon) \|\nabla w_j\|_{L^{p_0,q_0} \cap L^{p_1,q_1}}. \end{aligned} \tag{3.22}$$

Let us take the constants δ and $\widetilde{\delta}$ so that

$$0 < \delta < \frac{1}{M}, \qquad \widetilde{\delta} = \frac{1}{2}\left(\frac{1}{M} - \delta\right)$$

and assume

$$\|u\|_{3,\infty} \le \delta, \qquad \epsilon \le \widetilde{\delta}.$$

Then

$$M(\|u\|_{3,\infty} + \epsilon) \le \frac{1}{2}(1 + \delta M) < 1,$$

from which together with (3.22) we deduce that

$$\|\{w_j, \pi_j\}\|_X \le \frac{C_0 \|f\|_{\dot{H}^{-1}_{p_0,q_0} \cap \dot{H}^{-1}_{p_1,q_1}}}{1 - M(\|u\|_{3,\infty} + \epsilon)}$$

for all $j = 0, 1, \ldots$. Define $\{v_k, q_k\}_{k=0}^{\infty}$ by

$$v_k := w_k - w_{k-1}, \quad q_k := \pi_k - \pi_{k-1} \quad (v_{-1} = 0,\ q_{-1} = 0).$$

The same calculation as (3.21) yields

$$\begin{aligned}\|\{v_{k+1}, q_{k+1}\}\|_X &= \|S\{-v_k \cdot \nabla u - \operatorname{div}\,(\theta_{1,\epsilon} \otimes v_k), 0\}\|_X \\ &\le M(\|u\|_{3,\infty} + \epsilon)\|\nabla v_k\|_{L^{p_0,q_0} \cap L^{p_1,q_1}}\end{aligned}$$

for all $k = 0, 1, \ldots$, and we inductively have

$$\|\{v_k, q_k\}\|_X \le \{M(\|u\|_{3,\infty} + \epsilon)\}^k \, C_0 \|f\|_{\dot{H}^{-1}_{p_0,q_0} \cap \dot{H}^{-1}_{p_1,q_1}} \quad (k = 0, 1, \ldots).$$

Since $M(\|u\|_{3,\infty} + \epsilon) < 1$ and $\{w_j, \pi_j\} = \{\sum_{k=0}^{j} v_k, \sum_{k=0}^{j} q_k\}$, we can verify that there exist some $w \in \dot{H}^1_{p_0,q_0}(\Omega) \cap \dot{H}^1_{p_1,q_1}(\Omega)$ and $\pi \in L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega)$ such that

$$w_j \to w \quad \text{in } \dot{H}^1_{p_0,q_0}(\Omega) \cap \dot{H}^1_{p_1,q_1}(\Omega), \quad \pi_j \to \pi \quad \text{in } L^{p_0,q_0}(\Omega) \cap L^{p_1,q_1}(\Omega)$$

as $j \to \infty$. In the same way as (3.21), we obtain

$$\begin{aligned}&\|S\{w_j \cdot \nabla u + \operatorname{div}\,(\theta_{1,\epsilon} \otimes w_j), 0\} - S\{w \cdot \nabla u + \operatorname{div}\,(\theta_{1,\epsilon} \otimes w), 0\}\|_X \\ &\le M(\|u\|_{3,\infty} + \epsilon)\|\nabla w_j - \nabla w\|_{L^{p_0,q_0} \cap L^{p_1,q_1}} \\ &\to 0 \qquad (j \to \infty).\end{aligned}$$

Hence letting $j \to \infty$ in $(3.20)_2$, we deduce that the pair $\{w, \pi\} \in X$ is a solution of (3.19).

We next prove that if $\{w, \pi\} \in X$ is a solution of (3.2), then the estimate (3.18) as well as uniqueness holds. Since $\{w, \pi\} \in X$ satisfies

$$\begin{cases} -\Delta w + \nabla \pi = f - w \cdot \nabla u - \operatorname{div}\,(\theta_{1,\epsilon} \otimes w) & \text{in } \Omega, \\ \qquad \operatorname{div}\, w = 0 & \text{in } \Omega, \end{cases}$$

in the sense of distributions, Lemma 3.3 and the estimate as in (3.21) give

$$\begin{aligned}\|\{w,\pi\}\|_X &\le C_0\|f - w\cdot\nabla u - \mathrm{div}\,(\theta_{1,\epsilon}\otimes w)\|_{\dot H^{-1}_{p_0,q_0}\cap \dot H^{-1}_{p_1,q_1}}\\ &\le C_0\|f\|_{\dot H^{-1}_{p_0,q_0}\cap \dot H^{-1}_{p_1,q_1}} + M(\|u\|_{3,\infty}+\epsilon)\|\nabla w\|_{L^{p_0,q_0}\cap L^{p_1,q_1}}.\end{aligned}$$

It follows from $M(\|u\|_{3,\infty}+\epsilon) < 1$ that the estimate (3.18) holds for some constant $C = C(\Omega, p_0, q_0, p_1, q_1)$.

Finally, we prove the assertion of the present lemma. Suppose that $\{w,\pi\} \in \dot H^1_{p_0,q_0}(\Omega)\times L^{p_0,q_0}(\Omega)$ is a solution of (3.2) for $f \in \dot H^{-1}_{p_0,q_0}(\Omega)\cap \dot H^{-1}_{p_1,q_1}(\Omega)$. According to the arguments above, there exists a unique solution $\{\bar w, \bar\pi\} \in X$ of (3.2) provided $\|u\|_{3,\infty} \le \delta$ and $\epsilon \le \widetilde{\delta}$. Set $W := w - \bar w$ and $\varPi := \pi - \bar\pi$. Then $\{W, \varPi\}$ satisfies

$$\begin{cases} -\Delta W + \nabla \varPi = -W\cdot\nabla u - \mathrm{div}\,(\theta_{1,\epsilon}\otimes W) & \text{in } \Omega,\\ \quad\mathrm{div}\, W = 0 & \text{in } \Omega,\end{cases}$$

in the sense of distributions, and Lemma 3.1 and calculating in the same way as (3.21) yield

$$\begin{aligned}\|\nabla W\|_{p_0,q_0} + \|\varPi\|_{p_0,q_0} &\le C_3\|W\cdot\nabla u + \mathrm{div}\,(\theta_{1,\epsilon}\otimes W)\|_{\dot H^{-1}_{p_0,q_0}}\\ &\le C_3C_4C_5(\|u\|_{3,\infty}+\epsilon)\|\nabla W\|_{p_0,q_0},\end{aligned}$$

where $C_3 = C(\Omega, p_0, q_0)$, $C_4 = C(3,\infty,p_0^*,q_0)$ and $C_5 = C(p_0,q_0)$ are the constants, respectively, in Lemmas 3.1, 2.1(i) and 2.2(i). In view of Remark 2.1, we have $C_3C_4C_5(\|u\|_{3,\infty}+\epsilon) \le M(\|u\|_{3,\infty}+\epsilon) < 1$ under the assumptions $\|u\|_{3,\infty} \le \delta$ and $\epsilon \le \widetilde{\delta}$, and we obtain $W = 0$ and $\varPi = 0$. Consequently, we deduce $w \in \dot H^1_{p_1,q_1}(\Omega)$ and $\pi \in L^{p_1,q_1}(\Omega)$. □

Remark 3.5. The proof of Lemma 3.5 does not work when $(p_i, q_i) = (3,1)$ $(i = 0,1)$. This stems from the observation that the term $w\cdot\nabla u + \mathrm{div}\,(\theta_{1,\epsilon}\otimes w)$ does not always belong to $\dot H^{-1}_{3,1}(\Omega)$ for $w \in \dot H^1_{3,1}(\Omega)$. Indeed, it can be verified that $w\cdot\nabla u + \mathrm{div}\,(\theta_{1,\epsilon}\otimes w) \in \dot H^{-1}_{3,\infty}(\Omega)$ for $w \in \dot H^1_{3,1}(\Omega)$, however, it does not imply $w\cdot\nabla u + \mathrm{div}\,(\theta_{1,\epsilon}\otimes w) \in \dot H^{-1}_{3,1}(\Omega)$

Based on Lemmas 3.4 and 3.5, we give the proof of the main theorem.

Proof of Theorem 3.2. Set $w := u - v$ and $\pi := p - q$. Then the pair $\{w,\pi\} \in \dot H^1_{3/2,\infty}(\Omega)\times L^{3/2,\infty}(\Omega)$ satisfies (1.6) in the sense of distributions. For small $\epsilon > 0$ to be determined later, we decompose $v \in L^3(\Omega) + L^\infty(\Omega)$ as in (3.16) so that (3.17) holds. Let us regard (1.6) as the equation (3.2) with the external force $-\mathrm{div}\,(\theta_{2,\epsilon}\otimes w)$. By Lemmas 2.1(i) and 2.2(i), we see that

$$\theta_{2,\epsilon}\otimes w \in L^{3/2,\infty}(\Omega)\cap L^{3,\infty}(\Omega) \subset L^2(\Omega)$$

and thus we have

$$\mathrm{div}\,(\theta_{2,\epsilon}\otimes w) \in \dot H^{-1}_{3/2,\infty}(\Omega)\cap \dot H^{-1}_0(\Omega).$$

Therefore Lemma 3.5 implies

$$\{w, \pi\} \in \dot{H}_0^1(\Omega) \times L^2(\Omega) \tag{3.23}$$

if $\|u\|_{3,\infty} \leq \delta_0$ and $\epsilon \leq \widetilde{\delta_0}$, where δ_0 and $\widetilde{\delta_0}$ are the constants in Lemma 3.5 with $(p_0, q_0, p_1, q_1) = (3/2, \infty, 2, 2)$.

Let δ_1 be the absolute constant in Lemma 3.4. Now we take the constant δ so that

$$0 < \delta \leq \min\{\delta_0, \delta_1\}$$

and assume

$$\|u\|_{3,\infty} \leq \delta.$$

Furthermore, we take $\epsilon > 0$ so that $\epsilon \leq \widetilde{\delta_0}$. Then (3.23) holds and for every $f \in \dot{H}_0^{-1}(\Omega)$ there exists a solution $\{\psi, \chi\} \in \dot{H}_0^1(\Omega) \times L^2(\Omega)$ of (1.7). By continuity, we can take $\psi \in \dot{H}_0^1(\Omega)$ as a test function in (3.7) to obtain (3.14). On the other hand, taking $w \in \dot{H}_0^1(\Omega)$ as a test function in (3.8) yields (3.15). Then it follows from (3.14) and (3.15) that $w \in \dot{H}_0^1(\Omega)$ satisfies

$$(f, w) = 0 \quad \text{for all } f \in \dot{H}_0^{-1}(\Omega).$$

Consequently, we conclude that $w = 0$ in Ω. We also obtain $\nabla\pi = 0$ in Ω, and $\pi \in L^{3,\infty}(\Omega) \cap L^2(\Omega)$ implies $\pi = 0$ in Ω. This completes the proof of the main theorem. $\square$

Chapter 4

Uniqueness of symmetric weak solutions in plane exterior domains

4.1 Introduction

Let Ω be an exterior domain in $\mathbb{R}^2$ with Lipschitz boundary. In this chapter, we study the uniqueness of symmetric weak solutions to the stationary Navier-Stokes equation

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \Omega, \\ \text{div } u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{4.1}$$

Leray [25] and Fujita [11] proved the existence of weak solutions $u \in \dot{H}^1_{0,\sigma}(\Omega)$ to $(4.1)_{1,2,3}$, however, it is not known whether their solutions satisfy $(4.1)_4$ even in a weak sense. Indeed, we cannot control the behavior of the solution u at infinity only from the information $\nabla u \in L^2(\Omega)$. Such a difficulty is not found in the three-dimensional exterior problem. Later on, Gilbarg-Weinberger [17, 18] and Amick [3] showed in the case $f = 0$ that there exists some vector $\bar{u} \in \mathbb{R}^2$ such that a weak solution $u \in \dot{H}^1_{0,\sigma}(\Omega)$ of $(4.1)_{1,2,3}$ converges to $\bar{u}$ at infinity in the sense of

$$\lim_{r \to \infty} \int_0^{2\pi} |u(r,\theta) - \bar{u}|^2 \, d\theta = 0.$$

Unfortunately, we do not know whether $\bar{u} = 0$ or not, and this is a well-known open problem. Owing to the difficulties peculiar to the two-dimensional exterior problem (4.1), the Stokes paradox in particular, we know very little about the problem and the existence results for (4.1) are obtained only under the symmetry assumption.

Galdi [13] and Pileckas-Russo [32] obtained the existence results concerning (4.1) by introducing the symmetry. We note that the inhomogeneous boundary condition $u = u_*$ on $\partial\Omega$, instead of $(4.1)_3$, is considered in [13, 32] and [37] below, however, we restrict our attention to the problem (4.1). Assuming the symmetry conditions (1.8) and (1.9) on Ω and $f \in \dot{H}^{-1}_{0,\sigma}(\Omega)$ respectively, Galdi [13] and Pileckas-Russo [32] showed that the problem (4.1) admits at least one weak solution $u \in \dot{H}^1_{0,\sigma}(\Omega)$ with the same symmetry (1.9). Due to the symmetry property (1.9), their solutions satisfy $(4.1)_4$ in the sense of (1.10), see [13, 33]. Under the stronger symmetry assumptions (1.11) on Ω, (1.9) and (1.12) on f, Yamazaki [37] showed that if f decays rapidly and is small in a sense, then there exists a solution u of (4.1) with $\sup_{x\in\Omega}(|x| + 1)|u(x)|$ small and the same symmetry properties (1.9) and (1.12).

The purpose of this chapter is to show the uniqueness of weak solutions to (4.1) with symmetry (1.13), which is weaker than (1.9) and also ensures the decay of solutions at infinity in the sense of (1.10). We shall show that if $u, v \in \dot{H}^1_{0,\sigma}(\Omega)$ are weak solutions of (4.1) with symmetry (1.13), u satisfies the energy inequality $\|\nabla u\|_2^2 \le (f, u)$ and $\sup_{x\in\Omega}(|x| + 1)|v(x)|$ is small, then $u = v$. As an application, our uniqueness theorem, together with the result of Yamazaki [37], describes the asymptotic behavior as $|x| \to \infty$ of some symmetric weak solutions. Since we consider the homogeneous boundary condition $(4.1)_3$ and in this case it is easy to verify that the symmetric weak solution constructed by Pileckas-Russo [32] fulfills the energy inequality, we can give information on the asymptotic behavior of their solution such as $|u(x)| = O(|x|^{-1})$ at infinity provided that f satisfies the conditons imposed by [37].

For the proof of our uniqueness theorem, a density property for the solenoidal vector field, together with the Hardy inequality for symmetric functions, plays a crucial role. We shall prove that a function $\psi \in \dot{H}^1_{0,\sigma}(\Omega)$ with $\sup_{x\in\Omega}(|x| + 1)|\psi(x)| < \infty$ can be taken as a test function in the weak form of (4.1). In two-dimensional exterior domains, we have great difficulty in taking a class of test functions larger than $C^\infty_{0,\sigma}(\Omega)$, while it is relatively easy in n-dimensional exterior domains, $n \ge 3$, as we can see in [27]. This is due to the lack of information on the class of the nonlinear term $u \cdot \nabla u$. However, thanks to the symmetry property of u, the Hardy inequality due to Galdi [13] (see Lemma 4.5 below) implies that the term $u \cdot \nabla u$ divided by $|x| + 1$ belongs to $L^1(\Omega)$. With these observations in mind, we shall construct an approximate sequence $\{\psi_n\}_{n=1}^\infty \subset C^\infty_{0,\sigma}(\Omega)$ such that $(|x| + 1)\psi_n \to (|x| + 1)\psi$ weakly $*$ in $L^\infty(\Omega)$ as well as $\nabla\psi_n \to \nabla\psi$ in $L^2(\Omega)$ as $n \to \infty$. This density property enables us to take the solution v as a test function in the weak form of (4.1).

This chapter is organized as follows. In Section 4.2, we shall state the main result on the uniqueness of symmetric weak solutions. After introducing the result of Yamazaki [37] precisely, we shall provide a corollary on the asymptotic behavior of a symmetric weak solution. Section 4.3 is devoted to the proof of the density property mentioned above. The proof of our uniqueness theorem shall be given in Section 4.4.

4.2 Results

In this chapter, we need some symmetry as well as the function spaces introduced in Chapter 2. We say that Ω is a symmetric exterior domain if Ω satisfies the condition (1.8). The subspace of $\dot{H}^1_{0,\sigma}(\Omega)$ consisting of functions with the symmetry property (1.13) is denoted by $\dot{H}^{1,S}_{0,\sigma}(\Omega)$.

Our definition of a symmetric weak solution to (4.1) is as follows.

Definition 4.1. Let Ω be a symmetric exterior domain. Given $f \in \dot{H}^{-1}_{0,\sigma}(\Omega)$, a function $u \in \dot{H}^{1,S}_{0,\sigma}(\Omega)$ is called a symmetric weak solution of (4.1) if u satisfies

$$(\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi) = (f, \varphi) \quad \text{for all } \varphi \in C^\infty_{0,\sigma}(\Omega). \tag{4.2}$$

Remark 4.1. If $u \in \dot{H}^{1,S}_{0,\sigma}(\Omega)$, then u automatically satisfies (1.10), see Galdi [13, Lemma 3.2] and Russo [33, Theorem 5].

Remark 4.2. Our definition of a symmetric weak solution is different from that in [13, 32]. Let $C^{\infty,S}_{0,\sigma}(\Omega) := \{\varphi \in C^\infty_{0,\sigma}(\Omega);\ \varphi \text{ satisfies (1.9)}\}$ and define H^S by the completion of $C^{\infty,S}_{0,\sigma}(\Omega)$ in the norm $\|\nabla \cdot\|_2$. In their definition, for $f \in (H^S)^*$, a function $u \in H^S$ is a symmetric weak solution of (4.1) if u satisfies the weak form (4.2) for all $\varphi \in C^{\infty,S}_{0,\sigma}(\Omega)$. Here $(H^S)^*$ denotes the dual space of H^S. Notice that $H^S = \{u \in \dot{H}^1_{0,\sigma}(\Omega);\ u \text{ satisfies (1.9)}\} \subset \dot{H}^{1,S}_{0,\sigma}(\Omega)$ and $\dot{H}^{-1}_{0,\sigma}(\Omega) \subset (H^S)^*$. We can verify that their solutions satisfy (4.2) for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$. Indeed, for $\varphi \in C^\infty_{0,\sigma}(\Omega)$ we set the function $\varphi^S = (\varphi^S_1, \varphi^S_2) \in C^{\infty,S}_{0,\sigma}(\Omega)$ by

$$\varphi^S_1(x_1, x_2) := \frac{1}{4}(\varphi_1(x_1, x_2) + \varphi_1(x_1, -x_2) - \varphi_1(-x_1, x_2) - \varphi_1(-x_1, -x_2)),$$
$$\varphi^S_2(x_1, x_2) := \frac{1}{4}(\varphi_2(x_1, x_2) - \varphi_2(x_1, -x_2) + \varphi_2(-x_1, x_2) - \varphi_2(-x_1, -x_2)).$$

Suppose $f \in \dot{H}^{-1}_{0,\sigma}(\Omega)$ with the symmetry property (1.9) and $u \in H^S$ is a symmetric weak solution in the sense of [13, 32]. By the symmetry property, direct calculations yield $(\nabla u, \nabla(\varphi - \varphi^S)) = (u \cdot \nabla u, \varphi - \varphi^S) = (f, \varphi - \varphi^S) = 0$. Hence

$$(\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi) = (\nabla u, \nabla \varphi^S) + (u \cdot \nabla u, \varphi^S) = (f, \varphi^S) = (f, \varphi)$$

for all $\varphi \in C_{0,\sigma}^{\infty}(\Omega)$.

Now we are in a position to state our main result on the uniqueness of symmetric weak solutions.

Theorem 4.1. *Let Ω be a symmetric exterior domain with Lipschitz boundary. Suppose $u, v \in \dot{H}_{0,\sigma}^{1,S}(\Omega)$, having the same symmetry property in* (1.13)*, are symmetric weak solutions of* (4.1)*. There exists a constant $\delta = \delta(\Omega)$ such that if u satisfies the energy inequality*

$$\|\nabla u\|_2^2 \leq (f, u)$$

and

$$\sup_{x \in \Omega}(|x| + 1)|v(x)| \leq \delta,$$

then $u = v$.

Remark 4.3. The existence of a symmetric weak solution was proved by Galdi [13] and Pileckas-Russo [32]. It was shown in [32] that for every $f \in \dot{H}_{0,\sigma}^{-1}(\Omega)$ satisfying (1.9) there exists a symmetric weak solution $u \in H^S$ of (4.1), see also Remark 4.2. Since we consider the boundary condition $u = 0$ on $\partial\Omega$, we can easily verify that the solution constructed by Pileckas-Russo [32] satisfies the energy inequality. Yamazaki [37] obtained a symmetric weak solution v with $\sup_{x\in\Omega}(|x| + 1)|v(x)|$ small. For the details of [37], see below.

Remark 4.4. The assumption on the symmetry of weak solutions is closely related to the decay rate of v. If v decays faster, that is, $\sup_{x\in\Omega}(|x| + 1)^{\alpha}|v(x)|$ is sufficiently small for some $\alpha > 1$, then we can prove the uniqueness without symmetry, see Remark 4.11.

Remark 4.5. Our uniqueness theorem is also valid even if we replace Ω by $\mathbb{R}^2$. This is based on the fact that the Hardy inequality for symmetric functions introduced in Lemma 4.5 below holds even in $\mathbb{R}^2$. For the existence of a symmetric weak solution v with $\sup_{x\in\mathbb{R}^2}(|x| + 1)|v(x)|$ small, see Yamazaki [36].

Remark 4.6. The same type of uniqueness theorems without symmetry in n-dimensional exterior domains, $n \geq 3$, are well known [12, 27, 24].

We apply our result to deduce the asymptotic behavior of a symmetric weak solution. To this end, we need the following existence result due to Yamazaki [37]. Let Ω be an exterior domain with $C^{2+\mu}$-boundary, $\mu > 0$, satisfying (1.11). Take $R > 0$ so that $\partial\Omega \subset B(0, R)$. For $q \in [1, \infty)$ and $\alpha > 0$, we denote by $\chi(q, \alpha)$ the set of locally integrable functions f on Ω such that

$$\|f\|_{\chi(q,\alpha)} := \frac{R^{\alpha-2/q}}{\pi^{1/q}}\|f\|_{q,\Omega_R} + \sup_{r\geq R} \frac{r^{\alpha-2/q}}{\pi^{1/q}}\|f\|_{L^q(r\leq|x|\leq 4r)} < \infty.$$

Then $\chi(q,\alpha)$ is a Banach space and is independent of the choice of R up to equivalent norms. Note also that $\chi(q,\alpha) \subset \chi(s,\alpha)$ if $1 \le s < q < \infty$ and that $\chi(q,\alpha) \subset L^q(\Omega)$ if $\alpha > 2/q$. We especially need the case $q > 2$ and $\alpha + 1 \in [2,3]$. In such a case, $\chi(q,\alpha+1) \subset L^r(\Omega)$ for all $r \in (1,q]$ and, furthermore, the space $\chi(q,\alpha+1)$ describes the decay of L^q-norm in detail.

Assume that the external force $f = (f_1, f_2)$ is represented as

$$\begin{aligned} f_1(x) &= \frac{\partial F}{\partial x_1}(x) + \frac{\partial G}{\partial x_2}(x) + \frac{\partial H}{\partial x_2}(x), \\ f_2(x) &= -\frac{\partial F}{\partial x_2}(x) + \frac{\partial G}{\partial x_1}(x) - \frac{\partial H}{\partial x_1}(x) \end{aligned} \tag{4.3}$$

with scalar-valued functions $F(x)$, $G(x)$ and $H(x)$ satisfying the symmetry conditions

$$\begin{cases} F(x_1,x_2) = F(x_1,-x_2) = F(-x_1,x_2), \\ F(x_1,x_2) = -F(x_2,x_1) = -F(-x_2,-x_1), \end{cases} \tag{4.4}$$

$$\begin{cases} G(x_1,x_2) = -G(x_1,-x_2) = -G(-x_1,x_2), \\ G(x_1,x_2) = G(x_2,x_1) = G(-x_2,-x_1), \end{cases} \tag{4.5}$$

$$\begin{cases} H(x_1,x_2) = -H(x_1,-x_2) = -H(-x_1,x_2), \\ H(x_1,x_2) = -H(x_2,x_1) = -H(-x_2,-x_1). \end{cases} \tag{4.6}$$

Notice that f satisfies the symmetry properties (1.9) and (1.12). Yamazaki [37] proved that for $q > 2$ and $\alpha \in [1,2]$ there exists a constant $\beta = \beta(\Omega, q, \alpha)$ such that if

$$\|F\|_{\chi(q,\alpha+1)} + \|G\|_{\chi(q,\alpha+1)} + \|H\|_{\chi(q,\alpha+1)} \le \beta, \tag{4.7}$$

then the problem (4.1) admits a unique solution u with $\nabla u \in \chi(q,\alpha+1)$ and $\sup_{x\in\Omega}(|x|+1)^\alpha |u(x)| < \infty$ subject to the estimate

$$\begin{aligned} &\sup_{x\in\Omega}(|x|+1)^\alpha |u(x)| + \|\nabla u\|_{\chi(q,\alpha+1)} \\ &\qquad\qquad \le \gamma(\|F\|_{\chi(q,\alpha+1)} + \|G\|_{\chi(q,\alpha+1)} + \|H\|_{\chi(q,\alpha+1)}) \end{aligned} \tag{4.8}$$

with $\gamma = \gamma(\Omega, q, \alpha)$. The solution u also satisfies the symmetry properties (1.9) and (1.12).

Observe that we may assume the external force f is given in the form (4.3) without loss of generality. Indeed, if $f = \operatorname{div} \Psi = \left(\sum_{i=1}^2 \partial_i \Psi_{ij}\right)_{j=1,2}$ with $\Psi = \{\Psi_{ij}\}_{i,j=1,2}$, then we put

$$\begin{aligned} \Phi &= \frac{1}{2}(\Psi_{11} + \Psi_{22}), \quad F = \frac{1}{2}(\Psi_{11} - \Psi_{22}), \\ G &= \frac{1}{2}(\Psi_{12} + \Psi_{21}), \quad H = \frac{1}{2}(-\Psi_{12} + \Psi_{21}), \end{aligned}$$

to deduce that f is represented as (4.3) by absorbing the term $\nabla\Phi$ into ∇p. Note also that, by the properties of $\chi(q,\alpha+1)$, we have $F, G, H, \nabla u \in L^r(\Omega)$ for every $r \in (1,q]$. Since $q > 2$, it follows that $f \in \dot{H}^{-1}_{0,\sigma}(\Omega)$ and $u \in \dot{H}^{1,S}_{0,\sigma}(\Omega)$ in particular.

As a consequence of Theorem 4.1 and the result of Yamazaki [37] mentioned above, we derive the following assertion.

Corollary 4.1. *Let $q \in (2,\infty)$. Assume that Ω is an exterior domain with $C^{2+\mu}$-boundary, $\mu > 0$, satisfying* (1.11) *and the external force f is given in the form* (4.3) *with F, G and H satisfying the conditions* (4.4), (4.5) *and* (4.6) *respectively. Suppose $u \in \dot{H}^{1,S}_{0,\sigma}(\Omega)$ is a symmetric weak solution of* (4.1) *with the energy inequality $\|\nabla u\|_2^2 \le (f,u)$. If*

$$\|F\|_{\chi(q,2)} + \|G\|_{\chi(q,2)} + \|H\|_{\chi(q,2)} \le \min\{\beta, \gamma^{-1}\delta\}$$

where $\beta = \beta(\Omega,q,1)$, $\gamma = \gamma(\Omega,q,1)$ and $\delta = \delta(\Omega)$ are the constants, respectively, in (4.7), (4.8) *and Theorem* 4.1, *then*

$$(|x|+1)|u(x)| \in L^\infty(\Omega) \quad \text{and} \quad \nabla u \in L^r(\Omega) \quad \text{for every} \quad r \in (1,q].$$

4.3 Density property

In this section we prove the density property for the solenoidal vector field. The main result in this section is the following proposition.

Proposition 4.1. *Let Ω be an exterior domain with Lipschitz boundary. For every $v \in \dot{H}^1_{0,\sigma}(\Omega)$ with*

$$\sup_{x\in\Omega}(|x|+1)|v(x)| < \infty,$$

there exists a sequence $\{v_n\}_{n=1}^\infty \subset C^\infty_{0,\sigma}(\Omega)$ such that

$$\begin{aligned} &\nabla v_n \to \nabla v && \text{in } L^2(\Omega), \\ &(|x|+1)v_n \to (|x|+1)v && \text{weakly } * \text{ in } L^\infty(\Omega) \end{aligned}$$

as $n \to \infty$.

For the proof of this proposition, we show the corresponding density property in the whole plane $\mathbb{R}^2$ and bounded domains. Based on the analysis in $\mathbb{R}^2$ and bounded domains, we can prove the density property in exterior domains above. It should be emphasized that we need no symmetry in this section.

We first show the density property in the whole plane $\mathbb{R}^2$.

Lemma 4.1. *For every $v \in \dot{H}^1_{0,\sigma}(\mathbb{R}^2)$ with*

$$\sup_{x\in\mathbb{R}^2}(|x|+1)|v(x)| < \infty,$$

there exists a sequence $\{v_n\}_{n=1}^\infty \subset C^\infty_{0,\sigma}(\mathbb{R}^2)$ such that

$$\begin{aligned} &\nabla v_n \to \nabla v && \text{in } L^2(\mathbb{R}^2), \\ &(|x|+1)v_n \to (|x|+1)v && \text{weakly } * \text{ in } L^\infty(\mathbb{R}^2) \end{aligned}$$

as $n \to \infty$.

Proof. Choose a cutoff function $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $0 \le \psi(x) \le 1$, $\psi(x) = 1$ for $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge 2$, and set $\psi_m(x) := \psi(x/m)$. We put

$$v_\epsilon := J_\epsilon * v, \qquad v_{\epsilon,m} := \psi_m v_\epsilon - B_m[\nabla\psi_m \cdot v_\epsilon],$$

where J_ϵ is the Friedrichs mollifier and B_m is the operator introduced in Lemma 2.3 for the bounded domain $E_m := \{m/2 < |x| < 3m\}$. Since div $v_\epsilon = 0$ in $\mathbb{R}^2$ and $\int_{E_m} \nabla\psi_m \cdot v_\epsilon \, dx = 0$, we can verify that $v_{\epsilon,m} \in C^\infty_{0,\sigma}(\mathbb{R}^2)$ for all $m = 1, 2, \dots$ and $\epsilon > 0$.

Let $M := \sup_{x\in\mathbb{R}^2}(|x|+1)|v(x)|$. We show the estimate

$$\sup_{x\in\mathbb{R}^2}(|x|+1)|v_{\epsilon,m}(x)| \le CM \tag{4.9}$$

with C independent of m and ϵ. We have

$$(|x|+1)|v_\epsilon| \le I_1 + I_2$$

where

$$\begin{aligned} I_1(x) &:= \int_{\mathbb{R}^2} J_\epsilon(x-y)\big||x|-|y|\big||v(y)|\, dy, \\ I_2(x) &:= \int_{\mathbb{R}^2} J_\epsilon(x-y)(|y|+1)|v(y)|\, dy. \end{aligned}$$

We see

$$I_2 \le \sup_{y\in\mathbb{R}^2}(|y|+1)|v(y)| \int_{\mathbb{R}^2} J_\epsilon(x-y)\, dy = M.$$

Recall that $J_\epsilon(x-y) = 0$ for $|x-y| \ge \epsilon$, and for $|x-y| \le \epsilon \le 1$ there holds

$$\big||x|-|y|\big| \le |x-y| \le \epsilon \le 1.$$

Hence

$$I_1 \le \sup_{y\in\mathbb{R}^2}|v(y)| \int_{\mathbb{R}^2} J_\epsilon(x-y)\, dy \le M.$$

Therefore we obtain the estimate

$$(|x|+1)|v_\epsilon| \le 2M. \tag{4.10}$$

On the other hand, since supp $B_m[\nabla\psi_m \cdot v_\epsilon]$ is contained in E_m, it follows from the Gagliardo-Nirenberg inequality and the Sobolev embedding that

$$\begin{aligned}(|x|+1)|B_m[\nabla\psi_m \cdot v_\epsilon]| &\le Cm|B_m[\nabla\psi_m \cdot v_\epsilon]| \\ &\le Cm\|B_m[\nabla\psi_m \cdot v_\epsilon]\|_{4,E_m}^{1/2}\|\nabla B_m[\nabla\psi_m \cdot v_\epsilon]\|_{4,E_m}^{1/2} \\ &\le Cm\|\nabla B_m[\nabla\psi_m \cdot v_\epsilon]\|_{4/3,E_m}^{1/2}\|\nabla\psi_m \cdot v_\epsilon\|_{4,E_m}^{1/2} \\ &\le Cm\|\nabla\psi_m \cdot v_\epsilon\|_{4/3,E_m}^{1/2}\|\nabla\psi_m \cdot v_\epsilon\|_{4,E_m}^{1/2}.\end{aligned}$$

Note that the constants C above are independent of m and ϵ, due to Lemma 2.3(ii). Since $|v_\epsilon| \le CM/m$ on E_m and $|\nabla\psi_m| \le C/m$ for some constants C independent of m and ϵ, direct calculations yield

$$\|\nabla\psi_m \cdot v_\epsilon\|_{4/3,E_m}^{1/2} \le CM^{1/2}m^{-1/4}, \quad \|\nabla\psi_m \cdot v_\epsilon\|_{4,E_m}^{1/2} \le CM^{1/2}m^{-3/4}.$$

Thus we derive

$$(|x|+1)|B_m[\nabla\psi_m \cdot v_\epsilon]| \le CM \tag{4.11}$$

with C independent of m and ϵ. The uniform estimate (4.9) follows from (4.10) and (4.11).

Next, in view of supp $\nabla\psi_m \subset E_m$ and $|v_\epsilon| \le CM/m$ on E_m, we have

$$\begin{aligned}\|\nabla v_{\epsilon,m} - \nabla v_\epsilon\|_{2,\mathbb{R}^2} &\le \|(\psi_m-1)\nabla v_\epsilon\|_{2,\mathbb{R}^2} + \|(\nabla\psi_m)v_\epsilon\|_{2,E_m} + \|\nabla B_m[\nabla\psi_m \cdot v_\epsilon]\|_{2,E_m} \\ &\le \|(\psi_m-1)\nabla v_\epsilon\|_{2,\mathbb{R}^2} + \|(\nabla\psi_m)v_\epsilon\|_{2,E_m} + C\|\nabla\psi_m \cdot v_\epsilon\|_{2,E_m} \\ &\le \|(\psi_m-1)\nabla v_\epsilon\|_{2,\mathbb{R}^2} + CMm^{-1} \\ &\to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

Here we have used Lemma 2.3(ii). Furthermore, the class of ∇v implies

$$\nabla v_\epsilon \to \nabla v \quad \text{in } L^2(\mathbb{R}^2) \quad \text{as} \quad \epsilon \downarrow 0.$$

From the arguments above, v is an accumulation point of the two-parameters family $\{v_{\epsilon,m}\}_{\epsilon>0,m\in\mathbb{N}} \subset C_{0,\sigma}^\infty(\mathbb{R}^2)$ in $\dot{H}_{0,\sigma}^1(\mathbb{R}^2)$, that is, we can take a subsequence $\{v_{\epsilon_j,m_j}\}_{j=1}^\infty$ such that $\|\nabla v_{\epsilon_j,m_j} - \nabla v\|_{2,\mathbb{R}^2} \le \frac{1}{j}$. We conclude from the uniform estimate (4.9) that there exists a subsequence $\{v_n\}_{n=1}^\infty \subset \{v_{\epsilon_j,m_j}\}_{j=1}^\infty$ satisfying the desired density property. □

In order to establish the density property in a bounded domain D, we need two lemmas. We first construct an approximate sequence when D is star-shaped, by following the argument due to Masuda [26, Proposition 1]. Recall that D is star-shaped with respect to some point $x \in D$ if $\lambda^{-1}(\overline{D} - x) \subset D - x$ for all $\lambda > 1$. By a translation we may assume that $\lambda^{-1}\overline{D} \subset D$ for all $\lambda > 1$ if D is star-shaped.

Lemma 4.2. *Let D be a star-shaped bounded domain with Lipschitz boundary. For every $v \in H^1_{0,\sigma}(D) \cap L^\infty(D)$, there exists a sequence $\{v_n\}_{n=1}^\infty \subset C^\infty_{0,\sigma}(D)$ such that*

$$\begin{aligned} &\nabla v_n \to \nabla v && \text{in } L^2(D), \\ &(|x|+1)v_n \to (|x|+1)v && \text{weakly } * \text{ in } L^\infty(D) \end{aligned}$$

as $n \to \infty$.

Proof. Let $\tilde{v}$ be the zero extension of v, that is, $\tilde{v}(x) = v(x)$ if $x \in D$ and $\tilde{v}(x) = 0$ if $x \in \mathbb{R}^2 \setminus D$. For $\lambda > 1$ and small $\epsilon > 0$, we set

$$v_\lambda(x) := \tilde{v}(\lambda x), \quad v_{\lambda,\epsilon} := J_\epsilon * v_\lambda$$

where J_ϵ is the Friedrichs mollifier. It follows from $\operatorname{supp} v_\lambda \subset \lambda^{-1}\overline{D} \subset D$ that $\operatorname{supp} v_{\lambda,\epsilon} \subset D$. Thus $v_{\lambda,\epsilon} \in C^\infty_{0,\sigma}(D)$ for all $\lambda > 1$ and small $\epsilon > 0$. We can also verify that $\nabla v_{\lambda,\epsilon} \to \nabla v_\lambda$ in $L^2(D)$ as $\epsilon \downarrow 0$. Since $C^\infty_{0,\sigma}(D)$ is dense in $H^1_{0,\sigma}(D)$, for each $\kappa > 0$ there exists a function $\psi \in C^\infty_{0,\sigma}(D)$ such that $\|\nabla v - \nabla\psi\|_{2,D} = \|\nabla v_\lambda - \nabla\psi_\lambda\|_{2,D} < \kappa/3$. In addition, by the uniform continuity of $\nabla\psi$, there holds $\|\nabla\psi_\lambda - \nabla\psi\|_{2,D} < \kappa/3$ provided $1 < \lambda \le 1+\delta$ for sufficiently small $\delta > 0$. Hence, for $1 < \lambda \le 1 + \delta$, we deduce

$$\begin{aligned} \|\nabla v_\lambda - \nabla v\|_{2,D} &\le \|\nabla v_\lambda - \nabla\psi_\lambda\|_{2,D} + \|\nabla\psi_\lambda - \nabla\psi\|_{2,D} + \|\nabla\psi - \nabla v\|_{2,D} \\ &< \frac{\kappa}{3} + \frac{\kappa}{3} + \frac{\kappa}{3} \\ &= \kappa. \end{aligned}$$

Furthermore, we have $\|v_\lambda\|_{\infty,\mathbb{R}^2} = \|v\|_{\infty,D}$, which gives the estimate

$$\|v_{\lambda,\epsilon}\|_{\infty,D} \le \|v\|_{\infty,D}.$$

Since v is an accumulation point of the family $\{v_{\lambda,\epsilon}\}_{\lambda>1,\epsilon>0} \subset C^\infty_{0,\sigma}(D)$ in $H^1_{0,\sigma}(D)$ and the uniform estimate above holds, we can take a subsequence $\{v_n\}_{n=1}^\infty$ such that

$$\nabla v_n \to \nabla v \quad \text{in } L^2(D) \quad \text{as} \quad n \to \infty \quad \text{and} \quad \|v_n\|_{\infty,D} \le \|v\|_{\infty,D}.$$

The estimate yields

$$\|(|x|+1)v_n\|_{\infty,D} \le C\|v\|_{\infty,D} \tag{4.12}$$

with $C = C(D)$. For $\varphi \in C_0^\infty(D)$ we employ the Poincaré inequality to obtain

$$\begin{aligned}((|x|+1)v_n - (|x|+1)v, \varphi) &\le \|v_n - v\|_{2,D}\|(|x|+1)\varphi\|_{2,D} \\ &\le C\|\nabla v_n - \nabla v\|_{2,D}\|(|x|+1)\varphi\|_{2,D} \\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned} \tag{4.13}$$

Since $C_0^\infty(D)$ is dense in $L^1(D)$, it follows from (4.12) and (4.13) that

$$(|x|+1)v_n \to (|x|+1)v \quad \text{weakly} \ * \ \text{in } L^\infty(D)$$

as $n \to \infty$. The proof is complete. □

Next, we employ a localization procedure which is similar to Abe-Giga [1, Lemma 6.2].

Lemma 4.3. *Let D be a bounded domain with Lipschitz boundary. Suppose $\{G_m\}_{m=1}^N$ is an open covering of $\overline{D}$ and $D_m := D \cap G_m$. Then there exists a family of bounded linear operators $\{T_m\}_{m=1}^N$ from $H_{0,\sigma}^1(D) \cap L^\infty(D)$ to $H_{0,\sigma}^1(D_m) \cap L^\infty(D_m)$ satisfying $v = \sum_{m=1}^N T_m v$.*

Proof. Following Abe-Giga [1, Lemma 6.2], we give the proof by induction with respect to N. If $N = 1$, the assertion is obvious.

Assume that the assertion is valid for N. Set

$$U := \bigcup_{m=2}^{N+1} D_m, \quad V := \bigcup_{m=2}^{N+1} G_m, \quad E := D_1 \cap U.$$

Then $D = D_1 \cup U$ and $\{G_1, V\}$ is a covering of $\overline{D}$. Let $\{\xi_1, \xi_2\}$ be a smooth partition of unity of D associated with $\{G_1, V\}$, that is, $0 \le \xi_i \le 1$ $(i = 1, 2)$, supp $\xi_1 \subset G_1$, supp $\xi_2 \subset V$ and $\xi_1 + \xi_2 = 1$ on D. For $v \in H_{0,\sigma}^1(D) \cap L^\infty(D)$ we define the operator T_1 by

$$T_1 v := \xi_1 v - B_E[\nabla \xi_1 \cdot v]$$

where B_E is the Bogovski operator defined by Lemma 2.3 for E. In the case where E is the union of disjoint Lipschitz domains, for instance, E_1 and E_2, we have only to replace the term $B_E[\nabla \xi_1 \cdot v]$ above by $\sum_{i=1}^2 B_{E_i}[\nabla \xi_1 \cdot v]$. Since $\nabla \xi_1 = 0$ in $D_1 \setminus E$, we have

$$\int_E \nabla \xi_1 \cdot v \, dx = \int_{D_1} \nabla \xi_1 \cdot v \, dx = 0.$$

Hence Lemma 2.3(i) and $\nabla \xi_1 \cdot v \in L^\infty(E)$ imply div $T_1 v = 0$ in D_1 and $B_E[\nabla \xi_1 \cdot v] \in W_0^{1,q}(E)$ for all $1 < q < \infty$. Using the Sobolev embedding, Lemma 2.3(i) and the Poincaré inequality, for $q > 2$ we obtain

$$\|B_E[\nabla \xi_1 \cdot v]\|_{\infty,E} \le C\|B_E[\nabla \xi_1 \cdot v]\|_{1,q,E} \le C\|\nabla \xi_1 \cdot v\|_{q,E} \le C\|v\|_{\infty,D}. \tag{4.14}$$

This estimate, together with $\|T_1 v\|_{1,2,D_1} \le C\|v\|_{1,2,D}$, shows that T_1 is a bounded linear operator from $H^1_{0,\sigma}(D) \cap L^\infty(D)$ to $H^1_{0,\sigma}(D_1) \cap L^\infty(D_1)$. On the other hand, we put

$$T_U v := \xi_2 v - B_E[\nabla\xi_2 \cdot v].$$

The same argument as above yields that T_U is a bounded linear operator from $H^1_{0,\sigma}(D) \cap L^\infty(D)$ to $H^1_{0,\sigma}(U) \cap L^\infty(U)$. Furthermore

$$v = T_1 v + T_U v.$$

Since U is covered by $\{G_m\}_{m=2}^{N+1}$, by the induction assumption there exists a family of bounded linear operators $\{\widehat{T}_m\}_{m=2}^{N+1}$ from $H^1_{0,\sigma}(U)\cap L^\infty(U)$ to $H^1_{0,\sigma}(D_m)\cap L^\infty(D_m)$ satisfying $u = \sum_{m=2}^{N+1} \widehat{T}_m u$ for $u \in H^1_{0,\sigma}(U) \cap L^\infty(U)$. Setting

$$T_1 := T_1, \quad T_m := \widehat{T}_m \cdot T_U \quad (m = 2, \dots, N+1),$$

we conclude that $\{T_m\}_{m=1}^{N+1}$ is a family of bounded linear operators from $H^1_{0,\sigma}(D) \cap L^\infty(D)$ to $H^1_{0,\sigma}(D_m) \cap L^\infty(D_m)$ satisfying $v = \sum_{m=1}^{N+1} T_m v$. $\square$

Collecting Lemmas 4.2 and 4.3, we can construct an approximate sequence in general bounded domains.

Lemma 4.4. *The assertion in Lemma* 4.2 *is also valid when D is a bounded domain with Lipschitz boundary.*

Proof. It is well known that, by the assumption on the boundary ∂D, there exists an open covering $\{G_m\}_{m=1}^N$ of $\overline{D}$ such that $D_m = D \cap G_m$ $(m = 1, \dots, N)$ are star-shaped bounded domains with Lipschitz boundary with respect to some open balls in D_m. Let $\{T_m\}_{m=1}^N$ be the family of bounded linear operators introduced in Lemma 4.3. For $m = 1, \dots, N$, we put $v_m := T_m v$. Then $v_m \in H^1_{0,\sigma}(D_m) \cap L^\infty(D_m)$ and $v = \sum_{m=1}^N v_m$. Since D_m are star-shaped, for each $m = 1, \dots, N$ we can take by the proof of Lemma 4.2 a sequence $\{v_{m,n}\}_{n=1}^\infty \subset C^\infty_{0,\sigma}(D_m)$ such that

$$\begin{aligned} &\nabla v_{m,n} \to \nabla v_m \quad \text{in } L^2(D_m) \ \text{ as } \ n \to \infty, \\ &\|(|x|+1)v_{m,n}\|_{\infty,D_m} \le C\|v_m\|_{\infty,D_m} \end{aligned}$$

with C independent of n. We denote the zero extension of $v_{m,n}$ to $D \setminus D_m$ by $v_{m,n}$ itself for simplicity, and set $v_n := \sum_{m=1}^N v_{m,n}$. Then we derive

$$\|\nabla v_n - \nabla v\|_{2,D} \le \sum_{m=1}^N \|\nabla v_{m,n} - \nabla v_m\|_{2,D_m} \to 0 \quad \text{as} \ \ n \to \infty.$$

Since T_m is a bounded linear operator from $H^1_{0,\sigma}(D) \cap L^\infty(D)$ to $H^1_{0,\sigma}(D_m) \cap L^\infty(D_m)$, we have

$$
\begin{aligned}
\|(|x|+1)v_n\|_{\infty,D} &\leq \sum_{m=1}^{N} \|(|x|+1)v_{m,n}\|_{\infty,D_m} \\
&\leq \sum_{m=1}^{N} C\|v_m\|_{\infty,D_m} \\
&\leq C\|v\|_{H^1_{0,\sigma}(D)\cap L^\infty(D)}
\end{aligned}
$$

with C independent of n. This estimate, together with the same calculation as (4.13) and the density property of $C_0^\infty(D)$ in $L^1(D)$, yields

$$
(|x|+1)v_n \to (|x|+1)v \quad \text{weakly } * \text{ in } L^\infty(D)
$$

as $n \to \infty$, and the result follows. □

Using Lemmas 4.1 and 4.4, we can prove Proposition 4.1. For the proof, we follow Kozono-Sohr [19, Theorem 2].

Proof of Proposition 4.1. Let $M := \sup_{x\in\Omega}(|x|+1)|v(x)|$ and take $R > 0$ so that $\partial\Omega \subset B(0,R)$. We define a function $\tilde{v}$ by the zero extension of v. Then $\tilde{v} \in \dot{H}^1_{0,\sigma}(\mathbb{R}^2)$ with $\sup_{x\in\mathbb{R}^2}(|x|+1)|\tilde{v}(x)| = M$. In view of Lemma 4.1, there exists a sequence $\{\tilde{v}_n\}_{n=1}^\infty \subset C^\infty_{0,\sigma}(\mathbb{R}^2)$ such that

$$
\begin{aligned}
&\nabla\tilde{v}_n \to \nabla\tilde{v} && \text{in } L^2(\mathbb{R}^2), \\
&(|x|+1)\tilde{v}_n \to (|x|+1)\tilde{v} && \text{weakly } * \text{ in } L^\infty(\mathbb{R}^2)
\end{aligned}
\tag{4.15}
$$

as $n \to \infty$. In addition, we observe that $\nabla\tilde{v}_n \to \nabla\tilde{v}$ in $L^2(\mathbb{R}^2)$ implies $\tilde{v}_n \to \tilde{v}$ in $L^2(\Omega_R)$. Indeed, by the definition of $\tilde{v}$ and the construction of $\tilde{v}_n$ in the proof of Lemma 4.1, we may assume $\tilde{v}_n - \tilde{v} = 0$ in some open ball contained in $\mathbb{R}^2 \setminus \Omega$. Hence we employ the Poincaré inequality to deduce

$$
\|\tilde{v}_n - \tilde{v}\|_{2,\Omega_R} \leq C\|\nabla\tilde{v}_n - \nabla\tilde{v}\|_{2,\Omega_R} \to 0 \quad \text{as} \ \ n \to \infty. \tag{4.16}
$$

Let $\zeta \in C^\infty(\mathbb{R}^2)$ be a cutoff function such that $0 \leq \zeta(x) \leq 1$, $\zeta(x) = 1$ for $|x| \geq R$ and $\zeta(x) = 0$ in the neighbourhood of $\partial\Omega$. Put $w_n := B_{\Omega_R}[\nabla\zeta \cdot \tilde{v}_n]$ and $w := B_{\Omega_R}[\nabla\zeta \cdot \tilde{v}]$ where B_{Ω_R} is the operator defined by Lemma 2.3 for Ω_R. Since $\nabla\zeta \cdot \tilde{v}_n \in C_0^\infty(\Omega_R)$ and $\int_{\Omega_R} \nabla\zeta \cdot \tilde{v}_n \, dx = 0$, we deduce $w_n \in C_0^\infty(\Omega_R)$ and div $w_n = \nabla\zeta \cdot \tilde{v}_n$ in Ω_R. Similarly, it follows from $\nabla\zeta \cdot \tilde{v} \in L^\infty(\Omega_R)$ and $\int_{\Omega_R} \nabla\zeta \cdot \tilde{v}\, dx = 0$ that $w \in W_0^{1,q}(\Omega_R)$ $(1 < q < \infty)$ satisfies div $w = \nabla\zeta \cdot \tilde{v}$ in Ω_R. Furthermore, by Lemma 2.3(i) and (4.16), we obtain

$$\|\nabla w_n - \nabla w\|_{2,\Omega_R} \leq C\|\nabla\zeta \cdot (\tilde{v}_n - \tilde{v})\|_{2,\Omega_R} \to 0 \tag{4.17}$$

as $n \to \infty$. From the proof of Lemma 4.1, we may assume $\|\tilde{v}_n\|_{\infty,\mathbb{R}^2} \leq CM$ with C independent of n. Thus the same calculation as (4.14) yields

$$\|(|x|+1)w_n\|_{\infty,\Omega_R} \leq CM$$

with $C = C(R)$. This estimate, together with the same calculation as (4.13) and the density property of $C_0^\infty(\Omega_R)$ in $L^1(\Omega_R)$, leads us to

$$(|x|+1)w_n \to (|x|+1)w \quad \text{weakly } * \text{ in } L^\infty(\Omega_R) \quad (n \to \infty). \tag{4.18}$$

We also set $u := (1-\zeta)\tilde{v} + w$. Then $u \in H^1_{0,\sigma}(\Omega_R) \cap L^\infty(\Omega_R)$, and hence, according to Lemma 4.4, we can take a sequence $\{u_n\}_{n=1}^\infty \subset C^\infty_{0,\sigma}(\Omega_R)$ such that

$$\begin{aligned} &\nabla u_n \to \nabla u && \text{in } L^2(\Omega_R), \\ &(|x|+1)u_n \to (|x|+1)u && \text{weakly } * \text{ in } L^\infty(\Omega_R) \end{aligned} \tag{4.19}$$

as $n \to \infty$.

Now we define the sequence $\{v_n\}_{n=1}^\infty$ by

$$v_n := \zeta\tilde{v}_n - \tilde{w}_n + \tilde{u}_n$$

where $\tilde{w}_n$ and $\tilde{u}_n$ denote the zero extension of w_n and u_n respectively. Then $v_n \in C^\infty_{0,\sigma}(\Omega)$ for all $n = 1, 2, \ldots$. Since $v(x) = \zeta(x)\tilde{v}(x) - w(x) + u(x)$ for $x \in \Omega$, the properties (4.15), (4.16), (4.17), (4.18) and (4.19) yield

$$\begin{aligned} &\nabla v_n \to \nabla v && \text{in } L^2(\Omega), \\ &(|x|+1)v_n \to (|x|+1)v && \text{weakly } * \text{ in } L^\infty(\Omega) \end{aligned}$$

as $n \to \infty$. $\square$

Remark 4.7. In the case $\sup_{x\in\Omega}(|x|+1)^\alpha|v(x)| < \infty$ with $\alpha > 1$, we can prove similarly the existence of a sequence $\{v_n\}_{n=1}^\infty \subset C^\infty_{0,\sigma}(\Omega)$ such that $\nabla v_n \to \nabla v$ in $L^2(\Omega)$ and $(|x|+1)^\alpha v_n \to (|x|+1)^\alpha v$ weakly $*$ in $L^\infty(\Omega)$ as $n \to \infty$.

Remark 4.8. This proposition is also valid even if Ω is an exterior domain in $\mathbb{R}^n$ with $n \geq 3$. Indeed, we can easily verify that Lemmas 4.1, 4.2, 4.3 and 4.4 are valid even in $\mathbb{R}^n$ and $D \subset \mathbb{R}^n$, and the proof of Proposition 4.1 still holds for $\Omega \subset \mathbb{R}^n$.

4.4 Proof of Theorem 4.1

In this section we give the proof of our main result. If $u, v \in \dot{H}^{1,S}_{0,\sigma}(\Omega)$ are symmetric weak solutions of (4.1), then u and v satisfy

$$(\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi) = (f, \varphi) \quad \text{for all } \varphi \in C^\infty_{0,\sigma}(\Omega) \tag{4.20}$$

and

$$(\nabla v, \nabla \widetilde{\varphi}) + (v \cdot \nabla v, \widetilde{\varphi}) = (f, \widetilde{\varphi}) \quad \text{for all } \widetilde{\varphi} \in C^\infty_{0,\sigma}(\Omega) \tag{4.21}$$

respectively. We take u and v as test functions, respectively, in (4.21) and (4.20). Notice that we have almost no information on the class of the nonlinear term $u \cdot \nabla u$. The assumption $\sup_{x\in\Omega}(|x| + 1)|v(x)| < \infty$ and Proposition 4.1 play an important role to overcome this difficulty and we also need the Hardy inequality for symmetric functions, which is due to Galdi [13, Lemma 3.1].

Lemma 4.5 ([13])**.** *Let Ω be a symmetric exterior domain with locally Lipschitz boundary and assume that $u \in \dot{H}^1_0(\Omega)$ satisfies the symmetry property* (1.13)*. Then there exists a constant $C = C(\Omega)$ such that*

$$\int_\Omega \frac{|u(x)|^2}{|x|^2}\, dx \le C\|\nabla u\|_2^2.$$

Remark 4.9. If Ω and u are not symmetric, then there holds

$$\int_\Omega \frac{|u(x)|^2}{|x|^2(1 + |\log|x||)^2}\, dx \le C\|\nabla u\|_2^2. \tag{4.22}$$

With the aid of this lemma, we can take u and v as test functions. We also prove that the weak solution v satisfies the energy equality.

Lemma 4.6. *Let Ω be a symmetric exterior domain with Lipschitz boundary. Suppose $u, v \in \dot{H}^{1,S}_{0,\sigma}(\Omega)$ are symmetric weak solutions of* (4.1) *with* $\sup_{x\in\Omega}(|x| + 1)|v(x)| < \infty$*. Then we have*

$$(\nabla u, \nabla v) + (u \cdot \nabla u, v) = (f, v), \tag{4.23}$$

$$(\nabla v, \nabla u) - (v \cdot \nabla u, v) = (f, u). \tag{4.24}$$

In addition, v satisfies the energy equality

$$\|\nabla v\|_2^2 = (f, v). \tag{4.25}$$

Proof. According to Proposition 4.1, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset C_{0,\sigma}^{\infty}(\Omega)$ such that $\nabla v_n \to \nabla v$ in $L^2(\Omega)$ and $(|x|+1)v_n \to (|x|+1)v$ weakly $*$ in $L^{\infty}(\Omega)$ as $n \to \infty$. We substitute v_n for φ in (4.20) to obtain

$$(\nabla u, \nabla v_n) + (u \cdot \nabla u, v_n) = (f, v_n), \tag{4.26}$$

and we write

$$(u \cdot \nabla u, v_n) = \left(\frac{u}{|x|+1} \cdot \nabla u, (|x|+1)v_n \right).$$

By Lemma 4.5 we see that

$$\left\| \frac{u}{|x|+1} \cdot \nabla u \right\|_1 \leq \left\| \frac{u}{|x|+1} \right\|_2 \|\nabla u\|_2 \leq C\|\nabla u\|_2^2,$$

which together with the property of v_n yields

$$(u \cdot \nabla u, v_n) \to (u \cdot \nabla u, v) \quad \text{as} \ \ n \to \infty.$$

Hence we derive (4.23) by letting $n \to \infty$ in (4.26). On the other hand, $v \in L^4(\Omega)$ in particular and by the class of u we can take a sequence $\{u_n\}_{n=1}^{\infty} \subset C_{0,\sigma}^{\infty}(\Omega)$ such that $\nabla u_n \to \nabla u$ in $L^2(\Omega)$ as $n \to \infty$. We insert u_n into $\widetilde{\varphi}$ in (4.21) and integrate the second term by parts to get

$$(\nabla v, \nabla u_n) - (v \cdot \nabla u_n, v) = (f, u_n).$$

Since

$$|(v \cdot \nabla u_n, v)| \leq \|v\|_4^2 \|\nabla u_n\|_2,$$

we obtain (4.24) by passing to the limit $n \to \infty$.

Next, we show the energy equality. Since $v \in \dot{H}_{0,\sigma}^1(\Omega) \cap L^4(\Omega)$ and $C_{0,\sigma}^{\infty}(\Omega)$ is dense in $\dot{H}_{0,\sigma}^1(\Omega) \cap L^4(\Omega)$ ([19, Theorem 2]), there exists a sequence $\{\tilde{v}_n\}_{n=1}^{\infty} \subset C_{0,\sigma}^{\infty}(\Omega)$ such that $\nabla \tilde{v}_n \to \nabla v$ in $L^2(\Omega)$ and $\tilde{v}_n \to v$ in $L^4(\Omega)$ as $n \to \infty$. An integration by parts gives

$$(v \cdot \nabla v, \tilde{v}_n) = -(v \cdot \nabla \tilde{v}_n, v).$$

By the estimates

$$|(v \cdot \nabla v, \tilde{v}_n)| \leq \|v\|_4 \|\nabla v\|_2 \|\tilde{v}_n\|_4, \quad |(v \cdot \nabla \tilde{v}_n, v)| \leq \|v\|_4^2 \|\nabla \tilde{v}_n\|_2,$$

we deduce

$$(v \cdot \nabla v, \tilde{v}_n) \to (v \cdot \nabla v, v), \quad -(v \cdot \nabla \tilde{v}_n, v) \to -(v \cdot \nabla v, v)$$

as $n \to \infty$. Therefore

$$(v \cdot \nabla v, v) = 0.$$

Taking $\tilde{v}_n$ as a test function in (4.21) and then letting $n \to \infty$, we derive the energy equality (4.25). $\square$

Remark 4.10. As we can see in the proof, we can prove this lemma without the symmetry of v. We need the symmetry property of v to apply Lemma 4.5 in the proof of Theorem 4.1 below.

Remark 4.11. If $\sup_{x\in\Omega}(|x|+1)^\alpha|v(x)| < \infty$ with $\alpha > 1$, we can prove (4.23), (4.24) and (4.25) without assuming any symmetry. With the aid of Remark 4.7, we use the inequality (4.11), instead of Lemma 4.5, to take v as a test function in (4.20). The similar argument to the proof of Theorem 4.1 below yields Remark 4.4.

Following the argument due to Miyakawa [27], we give the proof of Theorem 4.1.

Proof of Theorem 4.1. Put $w := u - v$. We first show that

$$(w \cdot \nabla v, v) = 0. \tag{4.27}$$

We apply Proposition 4.1 to take a sequence $\{v_n\}_{n=1}^\infty \subset C_{0,\sigma}^\infty(\Omega)$ such that $\nabla v_n \to \nabla v$ in $L^2(\Omega)$ and $(|x|+1)v_n \to (|x|+1)v$ weakly $*$ in $L^\infty(\Omega)$ as $n \to \infty$. By an integration by parts, we have

$$(w \cdot \nabla v, v_n) = -(w \cdot \nabla v_n, v). \tag{4.28}$$

Since w satisfies the symmetry property (1.13), the same calculation as the proof of (4.23) yields $(w \cdot \nabla v, v_n) \to (w \cdot \nabla v, v)$ as $n \to \infty$. On the other hand, by Lemma 4.5 we see

$$\begin{aligned}
|(w \cdot \nabla v_n, v)| &= \left| \left(\frac{w}{|x|+1} \cdot \nabla v_n, (|x|+1)v \right) \right| \\
&\le \sup_{x\in\Omega}(|x|+1)|v(x)| \left\| \frac{w}{|x|+1} \cdot \nabla v_n \right\|_1 \\
&\le C \sup_{x\in\Omega}(|x|+1)|v(x)| \|\nabla w\|_2 \|\nabla v_n\|_2,
\end{aligned}$$

which implies $-(w \cdot \nabla v_n, v) \to -(w \cdot \nabla v, v)$ as $n \to \infty$. Hence passing to the limit $n \to \infty$ in (4.28), we obtain (4.27).

According to Lemma 4.6, we have

$$(\nabla u, \nabla v) = -(u \cdot \nabla u, v) + (f, v) \tag{4.29}$$

and

$$(\nabla v, \nabla u) = (v \cdot \nabla u, v) + (f, u). \tag{4.30}$$

It follows from (4.27), (4.29) and (4.30) that

$$2(\nabla u, \nabla v) = -(w \cdot \nabla w, v) + (f, u) + (f, v).$$

Thus the energy inequality $\|\nabla u\|_2^2 \leq (f, u)$, the energy equality (4.25) and Lemma 4.5 lead us to

$$\begin{aligned}
\|\nabla w\|_2^2 &= \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - 2(\nabla u, \nabla v) \\
&\leq (w \cdot \nabla w, v) \\
&\leq \sup_{x \in \Omega}(|x|+1)|v(x)| \left\| \frac{w}{|x|+1} \cdot \nabla w \right\|_1 \\
&\leq C\delta \|\nabla w\|_2^2,
\end{aligned}$$

where $C = C(\Omega)$ is the constant in Lemma 4.5. Now we take the constant δ so that

$$0 < \delta < \frac{1}{C}.$$

Then we derive

$$\|\nabla w\|_2 = 0.$$

Consequently, w is a constant in Ω, and by the boundary condition we conclude $w = 0$ in Ω. This completes the proof of Theorem 4.1. $\square$

Bibliography

[1] K. Abe, Y, Giga, *Analyticity of the Stokes semigroup in spaces of bounded functions*, Acta Math, **211** (2013), 1-46.

[2] R.A. Adams, J.J.F. Fournier, *Sobolev Spaces*, Second edition, Academic Press, Amsterdam, 2003.

[3] C. J. Amick, *On Leray's problem of steady Navier-Stokes flow past a body in the plane*, Acta Math, **161** (1988), 71-130.

[4] J. Bergh, J. Löfström, *Interpolation Spaces*, Springer, Berlin, 1976.

[5] M. E. Bogovski, *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, Soviet Math. Dokl. **20** (1979), 1094-1098.

[6] W. Borchers, T. Miyakawa, *Algebraic L^2 decay for Navier-Stokes flows in exterior domains*, Acta Math. **165** (1990), 189-227.

[7] W. Borchers, T. Miyakawa, *On stability of exterior stationary Navier-Stokes flows*, Acta Math, **174** (1995), 311-382.

[8] W. Borchers, H. Sohr, *On the equations rot $v = g$ and div $u = f$ with zero boundary conditions*, Hokkaido Math. J. **19** (1990), 67-87.

[9] I-D. Chang and R. Finn, *On the solutions of a class of equations occurring in continuum mechanics, with application to the Stokes paradox*, Arch. Rational Mech. Anal. **7** (1961), 388—401.

[10] R. Farwig, Y. Taniuchi, *Uniqueness of almost periodic-in-time solutions to Navier-Stokes equations in unbounded domains*, J. Evol. Equ. **11** (2011), 485-508.

[11] H. Fujita, *On the existence and regularity of the steady-state solutions of the Navier-Stokes equation.* J. Fac. Sci. Univ. Tokyo, Sec. I **9** (1961), 59-102.

[12] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, Vol. I: Linearized Steady problems, Vol. II: Nonlinear Steady Problems, Springer, New York, 1994.

[13] G. P. Galdi, *Stationary Navier-Stokes problem in a two-dimensional exterior domain*, Handbook of Differential Equations, Stationary partial differential equations, Vol. I., M. Chipot and P. Quittner, eds., North-Holland, Amsterdam, 2004, pp. 71-155.

[14] G. P. Galdi, C. G. Simader, *Existence, uniqueness and L^q estimates for the Stokes problem in an exterior domain*, Arch. Rational Mech. Anal. **112** (1990), 291-318.

[15] G. P. Galdi, C. G. Simader, *New Estimates for the Stationary Stokes Problem in Exterior Domains with Applications to Navier-Stokes Problem*, Differential and Integral Equations, **7** (1994), 847-861.

[16] G.P. Galdi, H. Sohr, *Existence and uniqueness of time-periodic physically reasonable Navier-Stokes flow past a body*, Arch. Ration. Mech. Anal. **172** (2004), 363-406.

[17] D. Gilbarg, H. F. Weinberger, *Asymptotic properties of Leray's solution of the stationary two-dimensional Navier-Stokes equations*, Russian Math. Surv. **29** (1974), 109-123.

[18] D. Gilbarg, H. F. Weinberger, *Asymptotic properties of steady plane solutions of the Navier-Stokes equations with bounded Dirichlet integral*, Ann. Scuola Norm. Sup. Pisa **5** (1978), 381-404.

[19] H. Kozono, H. Sohr, *Density properties for solenoidal vector fields, with applications to the Navier-Stokes equations in exterior domains*, J. Math. Soc. Japan **44** (1992), 307-330.

[20] H. Kozono, H. Sohr, *On a new class of generalized solutions for the Stokes equations in exterior domains*, Ann. Scuola Norm. Sup. Pisa **19** (1992), 155-181.

[21] H. Kozono, H. Sohr, *On stationary Navier-Stokes equations in unbounded domains*, Ricerche Mat, **42** (1993), 69-86.

[22] H. Kozono, H. Sohr, M. Yamazaki, *Representation formula, net force and energy relation to the stationary Navier-Stokes equations in* 3*-dimensional exterior domains*, Kyushu J. Math. **51** (1997), 239-260.

[23] H. Kozono, M. Yamazaki, *Exterior problem for the stationary Navier-Stokes equations in the Lorentz space*, Math. Ann. **310** (1998), 279-305.

[24] H. Kozono, M. Yamazaki, *Uniqueness criterion of weak solutions to the stationary Navier-Stokes equations in exterior domains*, Nonlinear Anal. Ser. A: Theory Methods **38** (1999), 959-970.

[25] J. Leray, *Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique*, J. Math. Pures Appl. **12** (1933), 1-82.

[26] K. Masuda, *Weak solutions of the Navier-Stokes equations*, Tohoku Math. J. **36** (1984), 623-646.

[27] T. Miyakawa, *On uniqueness of steady Navier-Stokes flows in an exterior domain*, Adv. Math. Sci. Appl. **5** (1995), 411-420.

[28] T. Nakatsuka, *On uniqueness of stationary solutions to the Navier-Stokes equations in exterior domains*, Nonlinear Anal. Ser. A: Theory Methods **75** (2012), 3457-3464.

[29] T. Nakatsuka, *Uniqueness of steady Navier-Stokes flows in exterior domains*, Funkcial. Ekvac, **56** (2013), 323–337. (URL http://www.math.kobe-u.ac.jp/∼fe)

[30] T. Nakatsuka, *On uniqueness of symmetric Navier-Stokes flows around a body in the plane*, arXiv:1310.5506.

[31] A. Novotny, M. Padula, *Note on decay of solutions of steady Navier-Stokes equations in 3-D exterior domains*, Differential and Integral Equations, **8** (1995), 1833-1842.

[32] K. Pileckas, R. Russo, *On the existence of vanishing at infinity symmetric solutions to the plane stationary exterior Navier-Stokes problem*, Math. Ann. **352** (2012), 643-658.

[33] A. Russo, *A note on the exterior two-dimensional steady-state Navier-Stokes problem*, J. Math. Fluid Mech. **11** (2009), 407-414.

[34] Y. Shibata, M. Yamazaki, *Uniform estimates in the velocity at infinity for stationary solutions to the Navier-Stokes exterior problem*, Japan. J. Math. **31** (2005), 225-279.

[35] Y. Taniuchi, *On the uniqueness of time-periodic solutions to the Navier-Stokes equations in unbounded domains*, Math. Z. **261** (2009), 597-615.

[36] M. Yamazaki, *The stationary Navier-Stokes equation on the whole plane with external force with antisymmetry*, Ann. Univ. Ferrara, **55** (2009), 407-423.

[37] M. Yamazaki, *Unique existence of stationary solutions to the two-dimensional Navier-Stokes equations on exterior domains*, Mathematical Analysis on the Navier-Stokes Equations and Related Topics, Past and Future-In memory of Professor Tetsuro Miyakawa, Gakuto International Series in Mathematical Sciences and Applications, Vol. **35**, Gakkōtosho, Tokyo, 2011, pp. 220-241.