The explicit estimation for the argument of the Riemann zeta function on the critical line

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- Supremum of the function $S_1(t)$ on short intervals, preprint at http://arxiv.org/abs/1301.0057.

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Chapter 1

Introduction

In this chapter, we present the Riemann-von Mangoldt formula and some properties of the function $S(t)$ appearing in this formula. Also, we define the functions $S_1(t)$, $S_2(t)$, $\cdots$, $S_m(t)$ which are integrations of $S(t)$, and present order results on those functions.

1.1 The argument of $\zeta(s)$ on the critical line

The Riemann zeta function is an infinite series given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where s is a complex variable written as $s = \sigma + it$. This function admits an analytic continuation as a meromorphic function over all complex plane, and holomorphic except for the point $s = 1$ in this plane. At the point $s = 1$, $\zeta(s)$ have a simple pole with residue 1.

To define the argument of the Riemann zeta function on the critical line, we should consider the number of zeros of $\zeta(s)$ in the bounded region. Let $N(T)$ be the number of non-trivial zeros ($\rho = \beta + i\gamma$) of $\zeta(s)$ in the rectangular area $0 < \sigma < 1,\ 0 < t \leq T$ $(T > 0)$ counted with the multiplicity. The function $N(t)$ can be approximated as follows which is called the Riemann-von Mangoldt formula;

Theorem 1.1.

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right) \quad as \quad T \to \infty,$$

where

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right).$$

Proof. Define $\xi(s)$ by

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

We assume $T > 3$. If T is not equal to any γ, $\xi(s)$ has $2N(T)$ zeros in the rectangle whose vertices are $2 \pm iT$ and $-1 \pm iT$, and does not have zeros on the boundary. Denote boundary of this rectangle by C.

By the argument principle, we get

$$4\pi N(T) = \int_C d\arg\xi(s).$$

Putting $\xi(s) = \frac{s}{2}(s-1)\phi(s)$, we have

$$\log\xi(s) = \log\left|\frac{s}{2}(s-1)\right| + \log|\phi(s)| + i\arg\frac{s}{2}(s-1) + i\arg\phi(s).$$

Thus,

$$\int_C d\arg\xi(s) = \int_C d\arg\frac{s}{2}(s-1) + \int_C d\arg\phi(s).$$

By the argument principle, the first term on the right-hand side is

$$\int_C d\arg\frac{s}{2}(s-1) = \frac{1}{i}\int_C \frac{(\frac{s}{2}(s-1))'}{\frac{s}{2}(s-1)}ds = 4\pi.$$

The second term on the right-hand side is

$$\int_C d\arg\phi(s) = 4\int_L d\arg\phi(s),$$

where L is the broken line made up of the L_1 from 2 to $2+iT$ followed by L_2 from $2+iT$ to $\frac{1}{2}+iT$. Hence

$$\pi N(T) = \pi + \int_L d\arg \pi^{-\frac{s}{2}} + \int_L d\arg \Gamma\left(\frac{s}{2}\right) + \int_L d\arg \zeta(s). \tag{1.1}$$

The second term on the right-hand side of (1.1) is

$$\int_L d\arg \pi^{-\frac{s}{2}} = \int_L d\arg e^{-\frac{t}{2} i \log \pi} = -\int_L d\left(\frac{t}{2}\log \pi\right) = -\frac{T}{2}\log \pi.$$

We apply Stirling's formula

$$\log \Gamma(z+\alpha) = \left(z+\alpha-\frac{1}{2}\right)\log z - z + \log\sqrt{2\pi} + O(|z|^{-1}) \quad (z\to\infty)$$

to the third term on the right-hand side of (1.1) with $z = \frac{iT}{2}$ and $\alpha = \frac{1}{4}$. Then we have

$$\begin{aligned}\int_L d\arg \Gamma\left(\frac{s}{2}\right) &= \int_L d\left\{\Im \log \Gamma\left(\frac{s}{2}\right)\right\} = \Im\left\{\log\Gamma\left(\frac{1}{4}+\frac{iT}{2}\right) - \log\Gamma(1)\right\} \\ &= \frac{T}{2}\log\frac{T}{2} - \frac{\pi}{8} - \frac{T}{2} + O(T^{-1}) \quad (T\to\infty).\end{aligned}$$

Substituting this into (1.1), we have

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + \frac{1}{\pi}\int_L d\arg\zeta(s) + O(T^{-1}).$$

Since $\frac{1}{\pi}\int_L d\arg\zeta(s) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+iT)$, we obtain the result.

□

So, if $T \neq \gamma$, we may define

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right).$$

Rigorously speaking, $S(T)$ is obtained by continuous variation along the segments connecting 2, $2+iT$, and $\frac{1}{2}+iT$, starting with the value zero. Also, if $T=\gamma$, we define

$$S(T) = \frac{1}{2}\{S(T+0)+S(T-0)\}.$$

We have known the classical result that $S(T) = O(\log T)$ obtained by von-Mangoldt (cf. (9.4.2) on p.214 of Titchmarsh [20]). Therefore, $N(T)$ is estimated by

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Also by the mean value theorem, we see easily that

$$N(T+h) - N(T) = O(\log T),$$

where the implied contstant on the right-hand side depends on h, for $h > 0$.

Moreover, it is a classical result of Littlewood [11] that under the Riemann Hypothesis we have

$$S(T) = O\left(\frac{\log T}{\log\log T}\right).$$

1.2 Properties of $S(t)$

We have a corollary to the Riemann-von Mangoldt formula.

Theorem 1.2.
1. *$S(t)$ is a piecewise smooth function with discontinuities at the ordinates of the complex zeros of $\zeta(s)$.*
2. *When t passes through a point of discontinuity, the function $S(t)$ makes a jump equal to the sum of the multiplicities of the zeros of $\zeta(s)$ with this point as ordinate.*
3. *Let γ and γ' be neighboring ordinates of zeros of $\zeta(s)$. Then, $S(t)$ is monotonically decreasing with derivatives*

$$S'(t) = -\frac{1}{2\pi}\log\frac{t}{2\pi} + O\left(\frac{1}{t^2}\right) \quad \textit{and} \quad S''(t) = -\frac{1}{2\pi t} + O\left(\frac{1}{t^3}\right)$$

on every interval (γ, γ').

In this paper, we prove only the last assertion.

Proof. Let $\gamma < t < \gamma'$. Then, the quantity $N(t) = N(\gamma + 0)$ is constant in the entire interval. By (1.1),

$$N(T) = \frac{1}{\pi}\left(\pi - \frac{T}{2}\log\pi + \Im\log\Gamma\left(\frac{1}{4} + \frac{iT}{2}\right) + \pi S(T)\right)$$

$$= \frac{1}{\pi}\left\{\pi - \frac{T}{2}\log\pi + \frac{T}{4}\log\left(\frac{T^2}{4}+\frac{1}{16}\right) - \frac{1}{4}\left(\frac{\pi}{2} - \arctan\frac{1}{2T}\right) - \frac{T}{2}\right.$$
$$\left. - \frac{T}{2}\int_0^\infty \frac{\frac{1}{2}-\{u\}}{\left(u+\frac{1}{4}\right)^2+\left(\frac{T}{2}\right)^2}du + \pi S(T)\right\},$$

where $\{u\}$ is the fractional part of u. Here we put

$$\delta(T) = \frac{T}{4}\log\left(1+\frac{1}{4T^2}\right) + \frac{1}{4}\arctan\frac{1}{2T} - \frac{T}{2}\int_0^\infty \frac{\frac{1}{2}-\{u\}}{\left(u+\frac{1}{4}\right)^2+\left(\frac{T}{2}\right)^2}du.$$

Then, we have

$$S(t) = -\frac{t}{2\pi}\log\frac{t}{2\pi} + \frac{t}{2\pi} - \frac{7}{8} - \delta(t) + N(\gamma+0).$$

Differentiating the above equation, we have

$$S'(t) = -\frac{t}{2\pi}\log\frac{t}{2\pi} - \delta'(t), \quad S''(t) = -\frac{1}{2\pi t} - \delta''(t). \tag{1.2}$$

Here, we put

$$j(t) = \int_0^\infty \frac{\frac{1}{2}-u+[u]}{\left(u+\frac{1}{4}\right)^2+\left(\frac{t}{2}\right)^2}du.$$

So

$$\delta(t) = \frac{t}{4}\log\left(1+\frac{1}{4t^2}\right) + \frac{1}{4}\arctan\frac{1}{2t} - \frac{t}{2}\cdot j(t).$$

The contribution to $\delta^{(k)}(t)$ of the first and second term of the right-hand side is $O(t^{-k-2})$ $(k=0,1,2)$ by the Taylor expansion.

Next, we put $\rho(u) = \frac{1}{2} - u + [u]$ and $\int_0^u \rho(z)dz = \sigma(u)$ to estimate $j(t)$. Then, we have

$$j(t) = \int_0^\infty \frac{1}{\left(u+\frac{1}{4}\right)^2+\left(\frac{t}{2}\right)^2}d\sigma(u) = 2\int_0^\infty \frac{\sigma(u)\left(u+\frac{1}{4}\right)}{\left\{\left(u+\frac{1}{4}\right)^2+\left(\frac{t}{2}\right)^2\right\}^2}du.$$

Using the inequality $0 \le \sigma(u) \le \frac{1}{8}$, we have

$$|j(t)| \le \frac{1}{4}\int_0^\infty \frac{u+\frac{1}{4}}{\left(u+\frac{1}{4}\right)^2+\left(\frac{t}{2}\right)^2}du \ll \int_0^t \frac{u}{t^4}du + \int_t^\infty \frac{1}{u^3}du \ll \frac{1}{t^2},$$

$$|j'(t)| \leq 8\int_0^\infty \frac{\sigma(u)\left(u+\frac{1}{4}\right)^2}{\left(u+\frac{1}{4}\right)^2+\left(\frac{t}{2}\right)^3}du \ll \int_0^t \frac{u^2}{t^6}du + \int_t^\infty \frac{1}{u^4}du \ll \frac{1}{t^3},$$

$$|j''(t)| \leq 64\int_0^\infty \frac{\sigma(u)\left(u+\frac{1}{4}\right)^3}{\left(u+\frac{1}{4}\right)^2+\left(\frac{t}{2}\right)^4}du \ll \int_0^t \frac{u^3}{t^8}du + \int_t^\infty \frac{1}{u^5}du \ll \frac{1}{t^4}.$$

Hence

$$\delta'(t) = -\frac{1}{2}\{j(t)+tj'(t)\} + O\left(\frac{1}{t^2}\right) = O\left(\frac{1}{t^2}\right), \tag{1.3}$$

$$\delta''(t) = -\frac{1}{2}\{2j'(t)+tj''(t)\} + O\left(\frac{1}{t^3}\right) = O\left(\frac{1}{t^3}\right). \tag{1.4}$$

Therefore, we obtain the result by (1.2), (1.3), and (1.4).

1.3 The integral of $S(t)$

For any positive number T, the function $S_1(T)$ is defined by

$$S_1(T) = \int_0^T S(t)dt + C,$$

where C is the constant defined by

$$C = \frac{1}{\pi}\int_{\frac{1}{2}}^\infty \log|\zeta(\sigma)|d\sigma.$$

Classical results of J. E. Littlewood imply that $S_1(T) = O(\log T)$, and under the Riemann Hypothesis $S_1(T) = O\left(\frac{\log T}{(\log\log T)^2}\right)$.

Next, we introduce the functions $S_2(T)$, $S_3(T)$, $\cdots$ similarly to the case of $S_1(T)$. When $T \neq \gamma$, we put

$$S_0(T) = S(T)$$

and

$$S_m(T) = \int_0^T S_{m-1}(t)dt + C_m$$

for any integer $m \geq 1$, where C_m's are the constants which are defined by, for any integer $k \geq 1$,

$$C_{2k-1} = \frac{1}{\pi}(-1)^{k-1} \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{(2k-1)-times} \log|\zeta(\sigma)|(d\sigma)^{2k-1},$$

and

$$C_{2k} = (-1)^{k-1} \underbrace{\int_{\frac{1}{2}}^{1} \int_{\sigma}^{1} \cdots \int_{\sigma}^{1}}_{2k-times} (d\sigma)^{2k} = \frac{(-1)^{k-1}}{(2k)!2^{2k}}.$$

When $T = \gamma$, we put

$$S_m(T) = \frac{1}{2}\{S_m(T+0) + S_m(T-0)\}.$$

A. Fujii [3] proved

$$S_m(T) \ll \frac{T^{m-1}}{\log T} \tag{1.5}$$

for any integer $m \geq 2$. To obtain this result, we introduce two functions $I_m(T)$ and $N_{h,2r}(T)$.

When $T \neq \gamma$, we define for any integer $k \geq 1$

$$I_{2k-1}(T) = \frac{1}{\pi}(-1)^{k-1}\Re\left\{\underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{(2k-1)-times} \log\zeta(\sigma+iT)(d\sigma)^{2k-1}\right\}$$

and

$$I_{2k}(T) = \frac{1}{\pi}(-1)^{k}\Im\left\{\underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{2k-times} \log\zeta(\sigma+iT)(d\sigma)^{2k}\right\}.$$

Also, when $T = \gamma$, we define for $m \geq 1$

$$I_m(T) = \frac{1}{2}\{I_m(T+0) + I_m(T-0)\}.$$

Then, $I_m(T)$ can be expressed as a single integral of the following form (cf. Lemma 2 in Fujii [3]): for any integer $m \geq 1$

$$I_m(T) = -\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\frac{1}{2}}^{\infty}\left(\sigma - \frac{1}{2}\right)^m \frac{\zeta'}{\zeta}(\sigma + iT)d\sigma\right\}.$$

By A. Fujii [3], for any integer $m \geq 1$, we have

$$I_m(T) \ll_m \log T, \tag{1.6}$$

where $\ll_m$ means that the constant of the upper bound depend on m.

For $h \geq 1$ and $r \geq 1$,

$$N_{h,2r}(T) = \underbrace{\int_0^T \cdots \int_0^t}_{h-times} \underbrace{\int_{\frac{1}{2}}^1 \int_\sigma^1 \cdots \int_\sigma^1}_{2r-times} N(\sigma, t)(d\sigma)^{2r}(dt)^h,$$

and for $h = 0$ and $r \geq 1$,

$$N_{0,2r}(T) = \underbrace{\int_{\frac{1}{2}}^1 \int_\sigma^1 \cdots \int_\sigma^1}_{2r-times} N(\sigma, T)(d\sigma)^{2r},$$

where $N(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ for $\sigma \geq \frac{1}{2}$ and $T \geq T_0$ such that $\beta > \sigma$ and $0 < \gamma < T$ when $T \neq \gamma$. When $t = \gamma$, we put

$$N(\sigma, T) = \frac{1}{2}\{N(\sigma, T + 0) + N(\sigma, T - 0)\}.$$

Then, by A. Fujii [3] we have following results;

$$S_1(T) = I_1(T)$$

and for any integer $m \geq 2$

$$S_m(T) = I_m(T) + 2\sum_{\substack{h+2r=m \\ h\geq 0, r\geq 1}} (-1)^{r-1} N_{h,2r}(T). \tag{1.7}$$

Also, by (1.6) we get

$$S_m(T) = 2\sum_{\substack{h+2r=m \\ h\geq 0, r\geq 1}} (-1)^{r-1} N_{h,2r}(T) + O(\log T)$$

for any integer $m \geq 2$.

To obtain (1.5), we estimate the first term on the right-hand side of the above inequality. We apply Selberg's density theorem (cf. Theorem 1 of Selberg [15]); for $T > T_0$ and some positive constant C,

$$N(\sigma, T) \ll T \log T \cdot e^{-C\left(\sigma - \frac{1}{2}\right) \log T}$$

uniformly for $\sigma \geq \frac{1}{2}$. Then we have

$$\underbrace{\int_{\frac{1}{2}}^{1} \int_{\sigma}^{1} \cdots \int_{\sigma}^{1}}_{2r-times} N(\sigma, T)(d\sigma)^{2r} \ll \frac{T}{(\log T)^{2r-1}}.$$

Hence

$$N_{h,2r} \ll \frac{T^{h+1}}{(\log T)^{2r-1}}$$

and

$$2 \sum_{\substack{h+2r=m \\ h \geq 0, r \geq 1}} (-1)^{r-1} N_{h,2r}(T) \ll \frac{T^{m-1}}{\log T}.$$

So, we have

$$S_m(T) = O\left(\frac{T^{m-1}}{\log T}\right) + O(\log T) = O\left(\frac{T^{m-1}}{\log T}\right).$$

Thus we obtain (1.5).

Under the Riemann Hypothesis, the second term on the right-hand side of (1.7) vanishes. Therefore, when we estimate $S_m(T)$ under the Riemann Hypothesis, we should consider $S_m(T) = I_m(T)$ for $m \geq 1$.

Concerning $S_m(T)$ for $m \geq 2$, Littlewood [11] have shown under the Riemann Hypothesis that

$$S_m(T) = O\left(\frac{\log T}{(\log \log T)^{m+1}}\right).$$

Thus,

$$S(t) = \begin{cases} O(\log t) & \textit{unconditionally}, \\ O\left(\frac{\log t}{\log \log t}\right) & \textit{assuming R.H.} \end{cases}$$

$$S_1(t) = \begin{cases} O(\log t) & \textit{unconditionally}, \\ O\left(\frac{\log t}{(\log\log t)^2}\right) & \textit{assuming R.H.} \end{cases}$$

and

$$S_m(t) = \begin{cases} O\left(\frac{t^{m-1}}{\log t}\right) & \textit{unconditionally}, \\ O\left(\frac{\log t}{(\log\log t)^{m+1}}\right) & \textit{assuming R.H.} \end{cases}$$

for $m \geq 2$.

For the functions $S(t), S_1(t), \cdots, S_m(t)$ defined in this chapter, we present explicit upper bounds on the order of them in Chapter 2. Also in Chapter 3, we present some Ω-results, and especially those of functions $S(t)$ and $S_1(t)$ especially.

Chapter 2

Explicit upper bounds

In this chapter, we introduce explicit upper bounds for order estimations of the functions $S(t)$, $S_1(t)$, $\cdots$, $S_m(t)$. The following explicit upper bounds for $S(t)$ and $S_1(t)$ are obtained by A. Fujii. The author obtained the bounds for $S_m(t)$ by generalizing Fujii's result for $S_1(t)$.

2.1 Explicit upper bounds of $S(t)$

For explicit upper bounds of $|S(t)|$, we have known the following result.

Theorem 2.1. (cf. Theorem 1 of Chapter 3 in [8])

$$|S(t)| < \begin{cases} 8\log t & \textit{unconditionally}, \\ 61\frac{\log t}{\log\log t} & \textit{assuming R.H.} \end{cases}$$

for $t > t_0$.

To prove Theorem 2.1, we introduce some lemmas and notations.

For $x \geq 2$ and $n \in \mathbb{Z}_{>0}$, we denote

$$\hat{\Lambda}_x(n) = \begin{cases} \Lambda(n) & \text{for } \ 1 \leq n \leq x, \\ \Lambda(n) \cdot \frac{\left(\log\frac{x^3}{n}\right)^2 - 2\left(\log\frac{x^2}{n}\right)^2}{2(\log x)^2} & \text{for } \ x \leq n \leq x^2, \\ \Lambda(n) \cdot \frac{\left(\log\frac{x^3}{n}\right)^2}{2(\log x)^2} & \text{for } \ x^2 \leq n \leq x^3, \end{cases}$$

with

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $2 \leq x \leq t^2$. We denote

$$\sigma_{x,t} = \frac{1}{2} + 2\max\left(\left|\beta_1 - \frac{1}{2}\right|, \frac{1}{\log x}\right),$$

where $\rho = \beta_1 + i\gamma$ is non-trivial zeros of $\zeta(s)$ such that $|t - \gamma| \leq \frac{x^{3|\beta_1 - \frac{1}{2}|}}{\log x}$. Also, we put

$$r(x,t) = \sum_{n \leq x^3} \frac{\hat{\Lambda}_x(n)}{n^{\sigma_{x,t} + it}}.$$

Using these notations, we state the following lemma.

Lemma 2.1. (cf. Theorem 1 of Chapter 2 in [8])
For $e^{16} < x \leq t^2$

$$S(t) = -\frac{1}{\pi} \sum_{n \leq x^3} \frac{\hat{\Lambda}_x(n)}{n^{\sigma_{x,t}}} \cdot \frac{\sin(t \log n)}{\log n} + 15\left(\sigma_{x,t} - \frac{1}{2}\right)(|r(x,t)| + \log|t|).$$

In this thesis, the proof of this lemma is omitted.

Proof of Theorem 2.1. We take $x = \sqrt{\log t}$ in Lemma 2.1. Since $\hat{\Lambda}_x(n) \leq \log n$, we have

$$\begin{aligned} |S(t)| &\leq 15\left\{\sum_{n \leq x^3} \frac{\log n}{n^{\frac{1}{2}}} + \left(\sigma_{x,t} - \frac{1}{2}\right)\log t\right\} \\ &\leq 15\left\{12x^{\frac{3}{2}}\log x + \left(\sigma_{x,t} - \frac{1}{2}\right)\log t\right\} \\ &< 15\left(\sigma_{\sqrt{\log t},t} - \frac{1}{2}\right)\log t + (\log t)^{\frac{4}{5}}. \end{aligned}$$

Since $\sigma_{x,t} - \frac{1}{2} \leq 2$, we obtain

$$|S(t)| < 7.5\log t + (\log t)^{\frac{4}{5}} < 8\log t.$$

On the other hand, since $\sigma_{x,t} - \frac{1}{2} = \frac{2}{\log x}$ under the Riemann Hypothesis, we obtain

$$|S(t)| < 60\frac{\log t}{\log\log t} + (\log t)^{\frac{4}{5}} < 61\frac{\log t}{\log\log t}.$$

□

The first explicit upper bound of $S(t)$ is the inequality

$$|S(t)| \leq a\log t + b\log\log t + c$$

for $t \geq t_0$, which was given by von Mangoldt [22]. After von Mangoldt, several mathematicians improved the values of a, b, c and t_0. The table of values of a, b, c, and t_0 is as follows;

	a	b	c	t_0
Von Mangoldt [22] (1905)	0.432	1.917	12.204	28.588
Grossmann [7] (1913)	0.291	1.787	6.137	50
Bäcklund [1] (1914)	0.275	0.979	7.446	200
Bäcklund [2] (1918)	0.137	0.443	4.35	200
Rosser [13] (1941)	0.137	0.443	1.588	1467
Trudgian [18] (2012)	0.17	0	1.998	e
Trudgian [19] (2012)	0.111	0.275	2.450	e

So, the latest result on the upper bound of $S(t)$ is

$$|S(t)| \leq 0.111\log t + 0.275\log\log t + 2.450$$

for $t \geq e$, due to Trudgian [19] .

Also, there exists a recent result on the explicit upper bound for $S(t)$ under the Riemann Hypothesis. Under the Riemann Hypothesis, it was shown that

$$|S(t)| \leq 0.83\frac{\log t}{\log\log t}$$

for $t > t_0$ in Fujii [5].

2.2 Explicit upper bounds of $S_1(t)$

The classical result on explicit upper bounds for $S_1(t)$ is the following theorem.

Theorem 2.2. (cf. Theorem 2 of Chapter 3 in [8])

$$|S_1(t)| < \begin{cases} 1.2\log t & \textit{unconditionally}, \\ 40\frac{\log t}{(\log\log t)^2} & \textit{assuming R.H.} \end{cases}$$

for $t > t_0$.

We apply the following Lemma 2.2 to the proof of Theorem 2.2. But the proof of Lemma 2.2 is omitted.

Lemma 2.2. (cf. Theorem 2 of Chapter 2 in [8])
For $e^{16} < x \leq t^2$

$$\begin{aligned} S_1(t) + C =& \frac{1}{\pi}\sum_{n\leq x^3}\frac{\hat{\Lambda}_x(n)}{n^{\sigma_{x,t}}}\cdot\frac{\cos(t\log n)}{(\log n)^2}\left\{1+\left(\sigma_{x,t}-\frac{1}{2}\right)\log n\right\} \\ &+ 5\theta\left(\sigma_{x,t}-\frac{1}{2}\right)^2(|r(x,t)|+\log|t|), \end{aligned}$$

where $|\theta \leq 1|$ *and* $C = \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty}\log|\zeta(\sigma)|d\sigma$.

Proof of Theorem 2.2. We apply Lemma 2.2 with $x = \sqrt{\log t}$. Then,

$$\begin{aligned} |S_1(t)+C| &\leq \frac{1}{\pi}\sum_{n\leq x^3}\frac{1}{n^{\frac{1}{2}}\log n} + \left(\frac{1}{2\pi}+\frac{9}{8}\right)\sum_{2\leq n\leq x^3}\frac{1}{n^{\frac{1}{2}}} + \frac{9}{8}\cdot\left(\frac{1}{2}\right)^2\cdot 4 \\ &\qquad + \frac{9}{4}\left(\sigma_{\sqrt{\log t},t}-\frac{1}{2}\right)^2\log t \\ &\leq \frac{9}{4}\left(\sigma_{\sqrt{\log t},t}-\frac{1}{2}\right)^2\log t + \frac{7}{2}(\log t)^{\frac{3}{4}}. \end{aligned}$$

Since $\sigma_{x,t} - \frac{1}{2} \leq \frac{1}{2}$, we obtain

$$|S_1(t)| \leq \frac{9}{4}\cdot\frac{1}{4}\log t + \frac{7}{2}(\log t)^{\frac{3}{4}} + |C| < 1.2\log t.$$

Also, since $\sigma_{x,t} - \frac{1}{2} = \frac{2}{\log x}$ under the Riemann Hypothesis, we obtain

$$|S_1(t)| \leq \frac{9}{4} \cdot \frac{16 \log t}{(\log \log t)^2} + \frac{7}{2}(\log t)^{\frac{3}{4}} + |C| < 40 \frac{\log t}{(\log \log t)^2}.$$

□

Moreover in Fujii [6], it was shown under the Riemann Hypothesis that

$$|S_1(t)| \leq 0.51 \frac{\log t}{(\log \log t)^2} \tag{2.1}$$

for $t > t_0$.

2.3 Explicit upper bounds of $S_m(t)$

The author obtained explicit upper bounds of $S_m(t)$ by generalizing techniques in the proof of Fujii's result (2.1).

Theorem 2.3.
Under the Riemann Hypothesis for any integer $m \geq 1$, if m is odd,

$$\begin{aligned}|S_m(t)| \leq \frac{\log t}{(\log \log t)^{m+1}} \cdot \frac{1}{2\pi m!} \Bigg\{ & \frac{1}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)} \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left(\frac{1}{e} + \frac{1}{2^{j+1} e^2} \right) \\ & + \frac{1}{m+1} \cdot \frac{\frac{1}{e}\left(1 + \frac{1}{e}\right)}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)} + \frac{1}{m(m+1)} \cdot \frac{1}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)} \Bigg\} \\ & + O\left(\frac{\log t}{(\log \log t)^{m+2}} \right),\end{aligned}$$

abd if m is even,

$$\begin{aligned}|S_m(t)| \leq \frac{\log t}{(\log \log t)^{m+1}} \cdot \frac{1}{2\pi m!} \Bigg\{ & \frac{1}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)} \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left(\frac{1}{e} + \frac{1}{2^{j+1} e^2} \right) \\ & + \frac{1}{m+1} \cdot \frac{\frac{1}{e}\left(1 + \frac{1}{e}\right)}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)} + \frac{\pi}{2} \cdot \frac{1}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)} \Bigg\} + O\left(\frac{\log t}{(\log \log t)^{m+2}} \right).\end{aligned}$$

This is Theorem 1 of author's result in [23]. It is to be stressed that the argument when the number of integration is odd is different from that when the number of integration is even.

The basic policy of the proof of this theorem is based on A. Fujii [6]. In the case when m is odd, we can directly generalize the proof of A. Fujii [6]. In the case when m is even, it is an extension of the method of A. Fujii [5].

The table of values of the constant part that is, the quantity in the curly parentheses for $m = 1, 2, \cdots$ is as follows;

m	constant part
1	$0.5090250\cdots$
2	$0.6002287\cdots$
3	$0.3426155\cdots$
4	$0.3509932\cdots$
5	$0.3254150\cdots$
6	$0.3235654\cdots$
7	$0.3216216\cdots$
8	$0.3210078\cdots$
9	$0.3206855\cdots$
10	$0.3205262\cdots$
$\vdots$	$\vdots$

To prove Theorem 2.3, we introduce some lemmas and some notatinos. Let $s = \sigma + it$. We suppose that $\sigma \geq \frac{1}{2}$ and $t \geq 2$. Let X be a positive number satisfying $4 \leq X \leq t^2$. Also, we put

$$\sigma_1 = \frac{1}{2} + \frac{1}{\log X}$$

and

$$\Lambda_X(n) = \begin{cases} \Lambda(n) & \text{for } \ 1 \leq n \leq X, \\ \Lambda(n)\frac{\log \frac{X^2}{n}}{\log X} & \text{for } \ X \leq n \leq X^2, \end{cases}$$

with

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using these notations, we state the following lemma.

Lemma 2.3.
Let $t \geq 2$, $X > 0$ such that $4 \leq X \leq t^2$. For $\sigma \geq \sigma_1 = \frac{1}{2} + \frac{1}{\log X}$,

$$\frac{\zeta'}{\zeta}(\sigma + it) = -\sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma+it}} - \frac{\left(1 + X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)\omega'} \Re\left(\sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right)$$
$$+ \frac{\left(1 + X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)\omega'} \cdot \frac{1}{2}\log t + O\left(X^{\frac{1}{2}-\sigma}\right),$$

where $|\omega| \leq 1, -1 \leq \omega' \leq 1$.

This has been proved in Fujii [6]. Moreover, we will use the following two lemmas.

Lemma 2.4. (cf. 2.12.7 of Titchmarsh[20])

$$\frac{\zeta'}{\zeta}(s) = \log 2\pi - 1 - \frac{E}{2} - \frac{1}{s-1} - \frac{1}{2} \cdot \frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$
$$= \log 2\pi - 1 - \frac{E}{2} - \frac{1}{s-1} - \frac{1}{2}\log\left(\frac{s}{2} + 1\right) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$
$$+ O\left(\frac{1}{|s|}\right)$$

where E is the Euler constant and ρ runs through zeros of $\zeta(s)$.

Lemma 2.5. (Lemma 1 of Selberg [14])
For $X > 1$, $s \neq 1$, $s \neq -2q$ $(q = 1, 2, 3, \cdots)$, $s \neq \rho$,

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n<X^2} \frac{\Lambda_X(n)}{n^s} + \frac{X^{2(1-s)} - X^{1-s}}{(1-s)^2 \log X} + \frac{1}{\log X}\sum_{q=1}^{\infty} \frac{X^{-2q-s} - X^{-2(2q+s)}}{(2q+s)^2}$$
$$+ \frac{1}{\log X}\sum_{\rho} \frac{X^{\rho-s} - X^{2(\rho-s)}}{(s-\rho)^2}.$$

By Lemma 2.4, we have

$$\Re\frac{\zeta'}{\zeta}(\sigma_1+it) = -\frac{1}{2}\log t + \sum_\gamma \frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2} + O(1). \tag{2.2}$$

Since for $\sigma_1 \le \sigma$

$$\begin{aligned}\frac{1}{\log X}\left|\sum_\rho \frac{X^{\rho-s}-X^{2(\rho-s)}}{(s-\rho)^2}\right| &\le \frac{X^{\frac{1}{2}-\sigma}}{\log X}\sum_\gamma \frac{1+X^{\frac{1}{2}-\sigma}}{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2}\\ &\le \left(1+X^{\frac{1}{2}-\sigma}\right)X^{\frac{1}{2}-\sigma}\sum_\gamma \frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2},\end{aligned}$$

we have

$$\frac{1}{\log X}\sum_\rho \frac{X^{\rho-s}-X^{2(\rho-s)}}{(s-\rho)^2} = \left(1+X^{\frac{1}{2}-\sigma}\right)X^{\frac{1}{2}-\sigma}\cdot\omega\sum_\gamma \frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2},$$

where $|\omega| \le 1$. Since for $\sigma \ge \frac{1}{2}$ and $X \le t^2$

$$\left|\frac{X^{2(1-s)}-X^{1-s}}{(1-s)^2\log X}\right| \ll \frac{X^{2(1-\sigma)}}{t^2\log X} \le \frac{X^{\frac{1}{2}-\sigma}}{\log X},$$

we have for $\sigma_1 \le \sigma$

$$\begin{aligned}\frac{\zeta'}{\zeta}(\sigma+it) = &-\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma+it}} + O\left(\frac{X^{\frac{1}{2}-\sigma}}{\log X}\right)\\ &+\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}\sum_\gamma \frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}\end{aligned}$$

by Lemma 2.5. Especially,

$$\begin{aligned}\Re\frac{\zeta'}{\zeta}(\sigma_1+it) = &\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right) + O\left(\frac{1}{\log X}\right)\\ &+\left(1+\frac{1}{e}\right)\frac{1}{e}\omega'\sum_\gamma \frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2},\end{aligned} \tag{2.3}$$

where $-1 \le \omega' \le 1$.

Hence by (2.2) and (2.3), we get

$$\sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} = \frac{1}{1 - \frac{1}{e}\left(1 + \frac{1}{e}\right)\omega'} \cdot \frac{1}{2}\log t + O\left(\left|\sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) \tag{2.4}$$

This relation will be used in the following proof of Theorem 2.3.

Proof of Theorem 2.3. If m is odd, we have

$$\begin{aligned} I_m(t) &= \frac{i^{m+1}}{\pi m!}\Im\left\{ i\left\{ \int_{\sigma_1}^{\infty} \left(\sigma - \frac{1}{2}\right)^m \frac{\zeta'}{\zeta}(\sigma+it)d\sigma + \frac{\left(\sigma_1 - \frac{1}{2}\right)^{m+1}}{m+1} \cdot \frac{\zeta'}{\zeta}(\sigma_1+it) \right.\right. \\ &\quad \left.\left. - \int_{\frac{1}{2}}^{\sigma_1} \left(\sigma - \frac{1}{2}\right)^m \left\{ \frac{\zeta'}{\zeta}(\sigma_1+it) - \frac{\zeta'}{\zeta}(\sigma+it) \right\} d\sigma \right\}\right\} \\ &= \frac{i^{m+1}}{\pi m!}\Im\left\{ i(J_1 + J_2 + J_3) \right\}, \end{aligned} \tag{2.5}$$

say.

First, we estimate J_1. By Lemma 2.3,

$$\begin{aligned} J_1 &= \int_{\sigma_1}^{\infty} \left(\sigma - \frac{1}{2}\right)^m \left\{ -\sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma+it}} - \frac{\left(1 + X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1 - \frac{1}{e}\left(1+\frac{1}{e}\right)\omega'} \Re\left(\sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right) \right. \\ &\quad \left. + \frac{\left(1 + X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1 - \frac{1}{e}\left(1+\frac{1}{e}\right)\omega'} \cdot \frac{1}{2}\log t + O\left(X^{\frac{1}{2}-\sigma}\right) \right\} d\sigma \\ &= -\int_{\sigma_1}^{\infty} \left(\sigma - \frac{1}{2}\right)^m \sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma+it}} d\sigma + \eta_1(t), \end{aligned}$$

say. Then, by integration by parts repeatedly

$$J_1 = -\sum_{j=0}^{m} \left(\frac{m!}{(m-j)!} \left(\sigma_1 - \frac{1}{2}\right)^{m-j} \sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma_1+it}(\log n)^{j+1}} \right) + \eta_1(t). \tag{2.6}$$

And we have

$$|\eta_1(t)| = \left| \int_{\sigma_1}^{\infty} \left(\sigma - \frac{1}{2}\right)^m \frac{\left(1 + X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1 - \frac{1}{e}\left(1+\frac{1}{e}\right)\omega'} d\sigma \right| \cdot \left| -\Re\left(\sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right) + \frac{1}{2}\log t \right|$$

$$
\begin{aligned}
&+ O\left\{\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m X^{\frac{1}{2}-\sigma}d\sigma\right\} \\
&\leq \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\left|\frac{1}{2}\log t-\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right)\right| \\
&\cdot\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\left(1+X^{\frac{1}{2}-\sigma}\right)X^{\frac{1}{2}-\sigma}d\sigma+O\left\{\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m X^{\frac{1}{2}-\sigma}d\sigma\right\} \\
&\leq \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right) \\
&+ O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) \\
&= \eta_2(t)+O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right), \qquad (2.7)
\end{aligned}
$$

say, since by partial integration

$$
\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\left(1+X^{\frac{1}{2}-\sigma}\right)X^{\frac{1}{2}-\sigma}d\sigma=\frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right).
$$

Next, applying Lemma 2.3 to J_2, we get

$$
\begin{aligned}
J_2 &= \frac{1}{(m+1)(\log X)^{m+1}}\cdot\left\{\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}-\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right)\right. \\
&\left.+\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(X^{\frac{1}{2}-\sigma_1}\right)\right\} \\
&= \frac{1}{(m+1)(\log X)^{m+1}}\cdot\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left\{\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right\} \\
&= \eta_3(t)+O\left\{\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right\}, \qquad (2.8)
\end{aligned}
$$

say.

Next, we estimate J_3. By Lemma 2.4, we have

$$
\Im(iJ_3)=\Re(J_3)=-\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\Re\left\{\frac{\zeta'}{\zeta}(\sigma_1+it)-\frac{\zeta'}{\zeta}(\sigma+it)\right\}d\sigma
$$

$$
\begin{aligned}
&= -\sum_{\gamma} \frac{1}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} \cdot \\
&\int_{\frac{1}{2}}^{\sigma_1} \left(\sigma - \frac{1}{2}\right)^m \frac{(\sigma_1 - \sigma)\left\{(t-\gamma)^2 - \left(\sigma_1 - \frac{1}{2}\right)\left(\sigma - \frac{1}{2}\right)\right\}}{\left(\sigma - \frac{1}{2}\right)^2 + (t-\gamma)^2} d\sigma \\
&+ O\left(\frac{1}{t(\log X)^{m+1}}\right) \\
&= -\sum_{\gamma} \frac{1}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} \cdot K(\gamma) + O\left(\frac{1}{(\log X)^{m+1}}\right), \qquad (2.9)
\end{aligned}
$$

say, where γ is the imaginary part of $\rho = \beta + i\gamma$.

If $t = \gamma$,

$$
\begin{aligned}
K(\gamma) &= -\int_{\frac{1}{2}}^{\sigma_1} \left(\sigma - \frac{1}{2}\right)^{m-1} \left(\sigma_1 - \frac{1}{2}\right)(\sigma_1 - \sigma)\, d\sigma \\
&= -\frac{1}{m(m+1)} \left(\sigma_1 - \frac{1}{2}\right)^{m+2}. \qquad (2.10)
\end{aligned}
$$

If $t \neq \gamma$, by putting $\sigma - \frac{1}{2} = v$, $\sigma_1 - \frac{1}{2} = \frac{1}{\log X} = \Delta$ and $|t - \gamma| = B$, we get

$$
\begin{aligned}
K(\gamma) &= \int_0^{\Delta} v^m \frac{(\Delta - v)(B^2 - \Delta v)}{v^2 + B^2} dv \\
&= \int_0^{\Delta} \left\{ v^m \Delta - (B^2 + \Delta) v^{m-1} + \frac{B^2(B^2 + \Delta^2) v^{m-1}}{v^2 + B^2} \right\} dv \\
&= \frac{\Delta^{m+2}}{m+1} - \frac{(B^2 + \Delta^2)\Delta^m}{m} + \int_0^{\Delta} \frac{(B^2 + \Delta^2) v^{m-1}}{\left(\frac{v}{B}\right)^2 + 1} dv.
\end{aligned}
$$

Putting $\frac{v}{B} = u$, we have

$$
\begin{aligned}
K(\gamma) &= \frac{\Delta^{m+2}}{m+1} - \frac{(B^2 + \Delta^2)\Delta^m}{m} + (B^2 + \Delta^2) \int_0^{\frac{\Delta}{B}} \frac{(uB)^{m-1} B}{1 + u^2} du \\
&= \frac{\Delta^{m+2}}{m+1} - \frac{(B^2 + \Delta^2)\Delta^m}{m} \\
&\quad + (B^2 + \Delta^2) B^m i^{m+1} \left\{ \sum_{j=1}^{\frac{m-1}{2}} \frac{(-1)^{j-1}}{2j-1} \left(\frac{\Delta}{B}\right)^{2j-1} - \arctan\left(\frac{\Delta}{B}\right) \right\}
\end{aligned}
$$

$$= \Delta^{m+2}\left\{\frac{1}{m+1} - \frac{1}{m}\cdot\frac{B^2}{\Delta^2} - \frac{1}{m}\right.$$
$$\left. + \left(\frac{B^{m+2}}{\Delta^{m+2}} + \frac{B^m}{\Delta^m}\right) i^{m+1}\left\{\sum_{j=1}^{\frac{m-1}{2}} \frac{(-1)^{j-1}}{2j-1}\left(\frac{\Delta}{B}\right)^{2j-1} - \arctan\left(\frac{\Delta}{B}\right)\right\}\right\}.$$

Putting $y = \frac{\Delta}{B}$, we get

$$K(\gamma) = \Delta^{m+2}\left\{-i^{m+1}\left(\frac{1}{y^{m+2}} + \frac{1}{y^m}\right)\arctan y - \frac{1}{my^2}\right.$$
$$\left. + i^{m+1}\left(\frac{1}{y^{m+2}} + \frac{1}{y^m}\right)\sum_{j=1}^{\frac{m-1}{2}} \frac{(-1)^{j-1}}{2j-1} y^{2j-1} - \frac{1}{m(m+1)}\right\}$$
$$= \Delta^{m+2}\left(g(y) - \frac{1}{m(m+1)}\right). \tag{2.11}$$

When y tends to 0, $g(y)$ is convergent to $\frac{2}{m(m+2)}$ since

$$g(y) = \frac{2}{m(m+2)} - \frac{2}{(m+2)(m+4)}y^2 + \frac{2}{(m+4)(m+6)}y^4 - \cdots.$$

When y tends to infinity, $g(y)$ tends to 0. Hence for $y > 0$, we get $g'(y) < 0$. Hence

$$0 \le g(y) \le \frac{2}{m(m+2)},$$

so that

$$-\frac{1}{m(m+1)} \le g(y) - \frac{1}{m(m+1)} \le \frac{1}{(m+1)(m+2)}. \tag{2.12}$$

Therefore by (2.11) and (2.12), we obtain

$$-\frac{\Delta^{m+2}}{m(m+1)} \le K(\gamma) \le \frac{\Delta^{m+2}}{(m+1)(m+2)},$$

so that

$$-\frac{1}{m(m+1)}\left(\sigma_1 - \frac{1}{2}\right)^{m+2} \le K(\gamma) \le \frac{1}{(m+1)(m+2)}\left(\sigma_1 - \frac{1}{2}\right)^{m+2}.$$

Hence

$$-\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}K(\gamma)$$
$$\leq\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+2}}{m(m+1)}\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2} \tag{2.13}$$

and

$$-\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}K(\gamma)$$
$$\geq-\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+2}}{(m+1)(m+2)}\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}. \tag{2.14}$$

By (2.4), (2.13) and (2.14), we have

$$-\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}K(\gamma)$$
$$\leq\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+1}}{m(m+1)}\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right)\right\} \tag{2.15}$$

and

$$-\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}K(\gamma)$$
$$\geq-\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+1}}{(m+1)(m+2)}\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right)\right\}. \tag{2.16}$$

Hence by (2.9), (2.10), (2.15) and (2.16), if $m\equiv 1 \pmod 4$,

$$i^{m+1}\Im(iJ_3)\leq\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(\log X)^{m+1}}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t$$
$$+O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right)$$

$$= \eta_4(t) + O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) \tag{2.17}$$

and

$$\begin{aligned} i^{m+1}\Im(iJ_3) &\geq -\frac{1}{m(m+1)}\cdot\frac{1}{(\log X)^{m+1}}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t \\ &\quad + O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) \\ &= -\eta_5(t) + O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) \end{aligned} \tag{2.18}$$

since $-i^{m+1} = 1$ and $i^{m+1} = -1$. And if $m \equiv 3 \pmod 4$,

$$\begin{aligned} i^{m+1}\Im(iJ_3) &\leq \frac{1}{m(m+1)}\cdot\frac{1}{(\log X)^{m+1}}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t \\ &\quad + O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) \\ &= \eta_5(t) + O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} i^{m+1}\Im(iJ_3) &\geq -\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(\log X)^{m+1}}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t \\ &\quad + O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) \\ &= -\eta_4(t) + O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) \end{aligned} \tag{2.20}$$

since $-i^{m+1} = -1$ and $i^{m+1} = 1$.

Therefore by (2.5), (2.7), (2.8), (2.17), (2.18), (2.19) and (2.20), we obtain

$$I_m(t) = \frac{1}{\pi m!}\left\{-i^{m+1}\sum_{j=0}^{m}\left(\frac{m!}{(m-j)!}\left(\sigma_1-\frac{1}{2}\right)^{m-j}\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}(\log n)^{j+1}}\right)\right.$$

$$
\begin{aligned}
&+ O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right)\Bigg\} \\
&+ \frac{1}{\pi m!}\cdot\Xi(t), \qquad (2.21)
\end{aligned}
$$

where $\Xi(t)$ satisfies the following inequalities. If $m \equiv 1 \pmod 4$,

$$
\begin{aligned}
\Xi(t) &\le \eta_2(t) - \eta_3(t) + \eta_4(t) \\
&= \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right) \\
&\quad - \frac{1}{m+1}\cdot\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}} \\
&\quad + \frac{1}{(m+1)(m+2)}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}},
\end{aligned}
$$

and

$$
\begin{aligned}
\Xi(t) &\ge -\eta_2(t) - \eta_3(t) - \eta_5(t) \\
&= -\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right) \\
&\quad - \frac{1}{m+1}\cdot\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}} \\
&\quad - \frac{1}{m(m+1)}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}},
\end{aligned}
$$

and if $m \equiv 3 \pmod 4$,

$$
\begin{aligned}
\Xi(t) &\le \eta_2(t) + \eta_3(t) + \eta_5(t) \\
&= \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right) \\
&\quad + \frac{1}{m+1}\cdot\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}} \\
&\quad + \frac{1}{m(m+1)}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}},
\end{aligned}
$$

and

$$\begin{aligned}
\Xi(t) &\geq -\eta_2(t) - \eta_3(t) - \eta_4(t) \\
&= -\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \cdot \frac{1}{2}\log t \cdot \frac{1}{(\log X)^{m+1}} \left(\sum_{j=0}^{m} \frac{m!}{(m-j)!} \left(\frac{1}{e} + \frac{1}{2^{j+1}e^2} \right) \right) \\
&\quad - \frac{1}{m+1} \cdot \frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'} \cdot \frac{1}{2}\log t \cdot \frac{1}{(\log X)^{m+1}} \\
&\quad - \frac{1}{(m+1)(m+2)} \cdot \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'} \cdot \frac{1}{2}\log t \cdot \frac{1}{(\log X)^{m+1}}.
\end{aligned}$$

In (2.21), we have

$$\left| \sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma_1+it}} \right| \leq \sum_{n<X} \frac{\Lambda(n)}{n^{\frac{1}{2}}} + \sum_{X\leq n\leq X^2} \frac{\Lambda(n)\log\frac{X^2}{n}}{n^{\frac{1}{2}}} \cdot \frac{1}{\log X} \ll \frac{X}{\log X}. \tag{2.22}$$

Hence the second term on the right-hand side of (2.21) is $\ll \frac{X}{(\log X)^{m+2}}$. Similarly, since

$$\begin{aligned}
\left| \sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma_1+it}(\log n)^{j+1}} \right| &\leq \sum_{n<X} \frac{\Lambda(n)}{n^{\frac{1}{2}}(\log n)^{j+1}} + \sum_{X\leq n\leq X^2} \frac{\Lambda(n)\log\frac{X^2}{n}}{n^{\frac{1}{2}}(\log n)^{j+1}} \cdot \frac{1}{\log X} \\
&\ll \frac{X}{(\log X)^{j+2}},
\end{aligned}$$

we estimate that the first term on the right-hand side of (2.21) is $\ll \frac{X}{(\log X)^{m+2}}$.

Therefore, taking $X = \log t$, we obtain

$$\begin{aligned}
|I_m(t)| &= \frac{1}{\pi m!}\Xi(t) + O\left(\frac{\log t}{(\log\log t)^{m+2}} \right) \\
&= \frac{\log t}{(\log\log t)^{m+1}} \cdot \frac{1}{2\pi m!} \left\{ \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left(\frac{1}{e} + \frac{1}{2^{j+1}e^2} \right) \right. \\
&\quad \left. + \frac{1}{m+1} \cdot \frac{\frac{1}{e}\left(1+\frac{1}{e}\right)}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} + \frac{1}{m(m+1)} \cdot \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \right\} \\
&\quad + O\left(\frac{\log t}{(\log\log t)^{m+2}} \right).
\end{aligned}$$

This is the first part of the theorem.

If m is even, we get similarly

$$\begin{aligned} I_m(t) &= \frac{-i^m}{\pi m!}\Im\left\{\left\{\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\frac{\zeta'}{\zeta}(\sigma+it)d\sigma+\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+1}}{m+1}\cdot\frac{\zeta'}{\zeta}(\sigma_1+it)\right.\right. \\ &\quad \left.\left.-\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\left\{\frac{\zeta'}{\zeta}(\sigma_1+it)-\frac{\zeta'}{\zeta}(\sigma+it)\right\}d\sigma\right\}\right\} \\ &= \frac{-i^m}{\pi m!}\Im\left\{(J_1+J_2+J_3)\right\}, \end{aligned} \tag{2.23}$$

say. By Lemma 2.3 and (2.22), we have

$$\begin{aligned} J_1 &= -\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma+it}}d\sigma+\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m O\left(X^{\frac{1}{2}-\sigma}\right)d\sigma \\ &\quad +\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\left\{-\frac{\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right)\right. \\ &\qquad \left.+\frac{\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\right\}d\sigma \\ &= -\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma+it}}d\sigma+O\left\{\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m X^{\frac{1}{2}-\sigma}d\sigma\right\}+\eta_1'(t) \\ &= \sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\sigma_1-\frac{1}{2}\right)^{m-j}\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}(\log n)^{j+1}} \\ &\qquad +O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right)+\eta_1'(t) \\ &\ll \frac{X}{(\log X)^{m+2}}+\eta_1'(t), \end{aligned} \tag{2.24}$$

say, and

$$\begin{aligned} J_2 &= \frac{1}{(m+1)(\log X)^{m+1}}\left\{\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}-\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right)\right. \\ &\quad \left.+\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(X^{\frac{1}{2}-\sigma_1}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(m+1)(\log X)^{m+1}} \cdot \frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'} \cdot \frac{1}{2}\log t \\
&\qquad + O\left\{\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right\} \\
&\ll \eta_3'(t) + \frac{X}{(\log X)^{m+2}}, \qquad (2.25)
\end{aligned}
$$

say. Similarly to $\eta_1(t)$, we have

$$
|\eta_1'(t)| \le \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \cdot \frac{1}{2}\log t \cdot \frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right).
$$

Finally, we estimate J_3. By Stirling's formula, we get

$$
\begin{aligned}
\left|\frac{\Gamma'}{\Gamma}\left(\frac{\sigma_1+it}{2}+1\right)\right| &= \left|\frac{i}{2}\log\frac{it}{2}+\left(\frac{\sigma_1+it+1}{2}\right)\frac{1}{t}-\frac{i}{2}+O\left(\frac{1}{t}\right)\right| \\
&\le \frac{1}{2}\log t + O\left(\frac{1}{t}\right). \qquad (2.26)
\end{aligned}
$$

Also $\left|\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+it}{2}+1\right)\right|$ is estimated similarly.

Hence by (2.26) and Lemma 2.4, we have

$$
\begin{aligned}
&\left|\Im\left\{\frac{\zeta'}{\zeta}(\sigma_1+it)-\frac{\zeta'}{\zeta}(\sigma+it)\right\}\right| \\
&\qquad \le \Im\left\{\sum_\rho\left(\frac{1}{\sigma_1+it-\rho}+\frac{1}{\rho}\right)-\sum_\rho\left(\frac{1}{\sigma+it-\rho}+\frac{1}{\rho}\right)\right\}+O\left(\frac{1}{t}\right) \\
&\qquad = \sum_\gamma \frac{(t-\gamma)\left\{\left(\sigma-\frac{1}{2}\right)^2-\left(\sigma_1-\frac{1}{2}\right)^2\right\}}{\left\{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}\left\{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}}+O\left(\frac{1}{t}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|\Im(J_3)| \\
&\quad \le \left|\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\sum_\gamma\frac{(t-\gamma)\left\{\left(\sigma-\frac{1}{2}\right)^2-\left(\sigma_1-\frac{1}{2}\right)^2\right\}}{\left\{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}\left\{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}}d\sigma\right|
\end{aligned}
$$

$$+\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\cdot O\left(\frac{1}{t}\right)d\sigma.$$

If $t=\gamma$, the first term on the right-hand side of the above inequality is 0. If $t\neq\gamma$, since $\sigma<\sigma_1$, we have

$$\begin{aligned}
&\left|\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\left\{\sum_\gamma\frac{(t-\gamma)\left\{\left(\sigma-\frac{1}{2}\right)^2-\left(\sigma_1-\frac{1}{2}\right)^2\right\}}{\left\{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}\left\{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}}\right\}d\sigma\right|\\
&<\sum_\gamma\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}\int_{\frac{1}{2}}^{\sigma_1}\frac{|t-\gamma|}{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2}d\sigma\\
&\leq\sum_\gamma\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}\int_{\frac{1}{2}}^{\infty}\frac{|t-\gamma|}{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2}d\sigma\\
&\leq\frac{\pi}{2}\left(\sigma_1-\frac{1}{2}\right)^{m+1}\sum_\gamma\frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}.
\end{aligned}$$

Applying (2.4) and (2.22), and taking $X=\log t$ lastly, the right-hand side of the above inequality is

$$\begin{aligned}
&\leq\frac{\pi}{2}\left(\sigma_1-\frac{1}{2}\right)^{m+1}\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right)\right\}\\
&=\frac{\pi}{4}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{\log t}{(\log X)^{m+1}}+O\left(\frac{X}{(\log X)^{m+2}}\right)\\
&\leq\frac{\pi}{4}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{\log t}{(\log\log t)^{m+1}}+O\left(\frac{\log t}{(\log\log t)^{m+2}}\right).
\end{aligned}\tag{2.27}$$

Also,

$$\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\cdot O\left(\frac{1}{t}\right)d\sigma=O\left(\frac{1}{t(\log X)^{m+1}}\right).\tag{2.28}$$

By (2.27) and (2.28),

$$|\Im(J_3)|\leq\frac{\pi}{4}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{\log t}{(\log\log t)^{m+1}}$$

$$+ O\left(\frac{1}{t(\log\log t)^{m+1}}\right) + O\left(\frac{\log t}{(\log\log t)^{m+2}}\right). \quad (2.29)$$

Therefore, we obtain by (2.23), (2.24), (2.25), (2.29), $\eta_1'(t)$ and $\eta_3'(t)$

$$\begin{aligned}|S_m(t)| \leq \frac{1}{2\pi m!} \cdot \frac{\log t}{(\log\log t)^{m+1}} \Bigg\{ & \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left(\frac{1}{e} + \frac{1}{2^{j+1}e^2}\right) \\ & + \frac{1}{m+1} \cdot \frac{\left(1+\frac{1}{e}\right)\frac{1}{e}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} + \frac{\pi}{2} \cdot \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \Bigg\} \\ & + O\left(\frac{\log t}{(\log\log t)^{m+2}}\right).\end{aligned}$$

□

Chapter 3

Ω-results

In this chapter, we show some Ω-result for functions $S(t)$ and $S_1(t)$. Also, we show the result on the supremum in short intervals for $S(t)$, and obtain the same type of result for $S_1(t)$. Moreover, we give an improved Ω-result of Selberg's result on $S_1(t)$ which is deduced from the result on the supremum for $S_1(t)$.

3.1 Some Ω-results for $S_m(t)$

Let $g(x)$ be a non-negative function for sufficiently large value of t. The notation $f(x) = \Omega(g(x))$ as $t \to \infty$ means that there exists an infinite series $\{x_j\}$, $j = 1, 2, \cdots \to \infty$, such that $f(x_j) > cg(x_j)$, where c is some constant. Similarly, $f(x) = \Omega_\pm(g(x))$ means that there exist two series $\{x_j\}$ and $\{y_j\}$, $j = 1, 2, \cdots \to \infty$, such that $f(x_j) > cg(x_j)$ and $f(y_j) < -cg(y_j)$.

In 1946, Selberg [15] proved that

$$S(t) = \begin{cases} \Omega_\pm\left(\frac{(\log t)^{\frac{1}{3}}}{(\log\log t)^{\frac{7}{3}}}\right) & \textit{unconditionally}, \\ \Omega_\pm\left(\left(\frac{\log t}{\log\log t}\right)^{\frac{1}{2}}\right) & \textit{assuming R.H.} \end{cases}$$

In 1986, Tsang [21] proved that

$$S(t) = \Omega_\pm\left(\left(\frac{\log t}{\log\log t}\right)^{\frac{1}{3}}\right) \tag{3.1}$$

by improving the methods of the above results of Selberg.

For $S_1(t)$, Selberg [15] proved

$$S_1(t) = \Omega_{\pm}\left(\frac{(\log t)^{\frac{1}{3}}}{(\log\log t)^{\frac{10}{3}}}\right). \tag{3.2}$$

The above inequality in the Ω_+ case was improved to the following result using another method also given in Selberg [15];

$$S_1(t) = \Omega_{+}\left(\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{4}}\right).$$

Also, Tsang [21] proved for $S_1(t)$ that

$$S_1(t) = \begin{cases} \Omega_{+}\left(\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{9}{4}}}\right) & \textit{unconditionally}, \\ \Omega_{-}\left(\frac{(\log t)^{\frac{1}{3}}}{(\log\log t)^{\frac{4}{3}}}\right) & \textit{unconditionally}, \\ \Omega_{\pm}\left(\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{3}{2}}}\right) & \textit{assuming R.H.} \end{cases} \tag{3.3}$$

3.2 Results for the supremum

For the supremum on $S(t)$, in 1986 Tsang [21] improved the methods of [15] to obtain the following inequalities;

$$\sup_{T\le t\le 2T}(\pm S(t)) \ge A\left(\frac{\log T}{\log\log T}\right)^{a},$$

where $A > 0$ is an absolute constant and the value of a is equal to $\frac{1}{2}$ if the Riemann Hypothesis is true and equal to $\frac{1}{3}$ otherwise.

Also, more explicitly, in 2005 Karatsuba and Korolev [8] established the following result; Let $0 < \epsilon < \frac{1}{10^3}$, $T \ge T_0(\epsilon) > 0$, and $H = T^{\frac{27}{82}+\epsilon}$. Then

$$\sup_{T-H\le t\le T+2H}(\pm S(t)) \ge \frac{\epsilon^{\frac{5}{4}}}{1000}\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}. \tag{3.4}$$

Moreover, Korolev [10] proved the following result; Let $T > T_0 > 0$ and $\frac{\log T}{(\log\log T)^{\frac{3}{2}}} < H < T$. Then

$$\sup_{T-H\le t\le T+2H}(\pm S(t)) \ge \frac{1}{90\pi}\left(\frac{\log H}{\log\log H}\right)^{\frac{1}{2}} \quad \textit{assuming R.H.} \tag{3.5}$$

Next, we consider $S(t+h)-S(t)$ for small positive values of h. This measures the variation of $S(t)$ over short intervals. By Selberg, it is known that there exists $c>0$ such that when $T\to\infty$,

$$\sup_{t\in[T,2T]} \pm\{S(t+h)-S(t)\} \ge c(h\log T)^{\frac{1}{2}} \quad \textit{assuming R.H.}$$

for any $h\in\left[\frac{1}{\log T},\frac{1}{\log\log T}\right]$. Also, without assuming the Riemann Hypothesis, it is known by Tsang that there exists $c>0$ such that when $T\to\infty$,

$$\sup_{t\in[T,2T]} \pm\{S(t+h)-S(t)\} \ge c(h\log T)^{\frac{1}{3}} \quad \textit{unconditionally}$$

for any $h\in\left[\frac{1}{\log T},\frac{1}{\log\log T}\right]$.

For $S_1(t+h)-S_1(t)$ Tsang proved the following result; there exists $c>0$ such that when $T\to\infty$,

$$\sup_{t\in[T,2T]} \pm\{S_1(t+h)-S_1(t)\} \ge \begin{cases} ch\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}} & \textit{unconditionally} \\ ch\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{2}} & \textit{assuming R.H.} \end{cases}$$

for any $h\in\left[0,\frac{1}{\log\log T}\right]$.

Here, (3.1) is easily obtained by applying above result directly since

$$\begin{aligned} c\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}} &\le \frac{1}{h}\left\{\pm(S_1(t+h)-S_1(t))\right\} = \frac{1}{h}\int_t^{t+h}\pm S(u)du \\ &\le \sup_{u\in[t,t+h]}\pm S(u). \end{aligned}$$

3.3 A generalization of $S(t)$

We define for $\sigma \in [\frac{1}{2}, 1]$,

$$S(\sigma, t) = \frac{1}{\pi} \Im \log \zeta(\sigma + it).$$

Then, we see easily that $S(t) = S(\frac{1}{2}, t)$.

We have known few results for the function $S(\sigma, t)$. One of the results is as follows; there exists a positive constant c such that when $T \to \infty$

$$\sup_{t \in [T, 2T]} \pm S(\sigma, t) \geq \begin{cases} c \left(\frac{\log T}{\log \log T} \right)^{\frac{1}{3}} & for\ \frac{1}{2} \leq \sigma \leq \frac{1}{2} + \left(\frac{\log \log T}{\log T} \right)^{\frac{1}{3}}, \\ c \left(\frac{\left(\sigma - \frac{1}{2}\right) \log T}{\log \log T} \right)^{\frac{1}{2}} & for\ \frac{1}{2} + \left(\frac{\log \log T}{\log T} \right)^{\frac{1}{3}} \leq \sigma \leq \frac{1}{2} + \frac{1}{\log \log T}. \end{cases}$$

(3.1) is a particular case of the above result. This result can be compared with a result of Montgomery (Theorem 1 of [12]) which says that; for fixed $\sigma > \frac{1}{2}$

$$S(\sigma, t) = \Omega_{\pm} \left(\frac{\left(\sigma - \frac{1}{2}\right)^{\frac{1}{2}} (\log t)^{1-\sigma}}{(\log \log t)^{\sigma}} \right).$$

3.4 An explicit supremum for $S_1(t)$

The author obtained the following results for $S_1(t)$ by using techniques in the proof of the result (3.4) of Karatsuba and Kolorev.

Theorem 3.1.
Let $0 < \epsilon < \frac{1}{10^3}$, $T \geq T_0(\epsilon) > 0$, and $H = T^{\frac{27}{82} + \epsilon}$. Then

$$\sup_{T - H \leq t \leq T + 2H} (\pm S_1(t)) \geq \frac{\epsilon}{4000\pi} \left(\frac{(\log T)^{\frac{1}{3}}}{(\log \log T)^{\frac{5}{3}}} \right).$$

This is Theorem 1 of author's result in [24]. This can be proven similarly to (3.4). However, lemmas to apply for proof of this theorem is different from (3.4).

Lemma 3.1.
Let $f(z)$ be a function taking real values on the real line, analytic on the strip $|\Im z| \leq 1$, and satisfying the inequality $|f(z)| \leq c(|z|+1)^{-(1+\alpha)}$, $c > 0$, $\alpha > 0$, on this strip. Then, the formula

$$\begin{aligned}\int_{-\infty}^{\infty} f(u)S_1(t+u)du =& \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\frac{1}{2}}(\log n)^2}\Re\left(\frac{1}{n^{it}}\hat{f}(\log n)\right) - C\hat{f}(0)\\ &+2\left\{\sum_{\beta>\frac{1}{2}}\int_{\frac{1}{2}}^{\beta}\int_0^{\beta-\sigma}\Re f(\gamma-t-ix)dxd\sigma\right.\\ &\left.-\int_{\frac{1}{2}}^{1}\int_0^{1-\sigma}\Re f(-t-ix)dxd\sigma\right\},\end{aligned}$$

where $\hat{f}(x)$ is given by the formula

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(u)e^{-ixu}du,$$

holds for any t, where the summation in the last sum is taken over all complex zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ to the right of the critical line, and where

$$C = \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty}\log|\zeta(\sigma)|d\sigma.$$

This is Lemma 1 of author's result in [24]. The proof of this lemma is an analogue of the proof of Theorem 2 of Chapter 3 in Karatsuba and Korolev [8].

Outline of the proof of Lemma 3.1. Put $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$. We set $\psi(z) = f(i(\sigma - x) - t)$ and take $X \geq 2(|t| + 10)$ such that the distance from the ordinate of any zero of $\zeta(s)$ to X is not less than $c(\log X)^{-1}$, where c is a positive absolute constant.

Let Γ be the boundary of the rectangle with the vertices $\sigma \pm iX$, $\frac{3}{2} \pm iX$, and let a horizontal cut be drawn from the line $\Re s = \sigma$ inside this rectangle to each zero $\rho = \beta + i\gamma$ and also to the point $z = 1$. Then the functions $\log\zeta(z)$ and $\psi(z)$ are analytic inside Γ.

By the residue theorem, the following equality holds:

$$\begin{aligned}
0 &= \int_{\Gamma} \psi(z) \log \zeta(z) dz \\
&= \left(\int_{\frac{3}{2}-iX}^{\frac{3}{2}+iX} - \int_{\sigma+iX}^{\frac{3}{2}+iX} - \int_{\sigma-iX}^{\sigma+iX} + \int_{\sigma-iX}^{\frac{3}{2}-iX} \right) \psi(z) \log \zeta(z) dz \\
&= I_1 - I_2 - I_3 + I_4,
\end{aligned}$$

say. Then, we have

$$I_1 = i \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma+it} \log n} \hat{f}(\log n) + O\left(\frac{1}{X^{\alpha}}\right)$$

since for $\alpha = \frac{3}{2} - \sigma$

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi\left(\frac{3}{2} + iu\right) \log \zeta\left(\frac{3}{2} + iu\right) du &= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\frac{3}{2}} \log n} \int_{-\infty}^{\infty} \frac{1}{n^{iu}} f(u - t - i\alpha) du \\
&= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma+it} \log n} \hat{f}(\log n).
\end{aligned}$$

Also,

$$\begin{aligned}
I_2 &= O\left(\frac{(\log X)^2}{X^{(1+\alpha)}}\right), \\
I_4 &= O\left(\frac{(\log X)^2}{X^{(1+\alpha)}}\right)
\end{aligned}$$

as in p. 461 of Karatsuba and Korolev [8].

And

$$\begin{aligned}
\lim_{X \to \infty} I_3 &= i \int_{-\infty}^{\infty} f(u) \log \zeta(\sigma + i(t+u)) du \\
&= i \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}+it} \log n} \cdot \hat{f}(\log n) \\
&\quad + 2\pi i \left(\sum_{\beta > \sigma} \int_0^{\beta-\sigma} f(\gamma - t - ix) dx - \int_0^{1-\sigma} f(-t - ix) dx \right)
\end{aligned}$$

as in pp. $461-462$ of Karatsuba and Korolev [8]. Then we get for $\sigma \geq \frac{1}{2}$

$$\int_{-\infty}^{\infty} f(u)\log\zeta(\sigma+i(t+u))du = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\sigma+it}\log n}\hat{f}(\log n)$$
$$+2\pi\left(\sum_{\beta>\sigma}\int_0^{\beta-\sigma} f(\gamma-t-ix)dx - K(\sigma)\int_0^{1-\sigma} f(-t-ix)dx\right),$$

where

$$K(\sigma) = \begin{cases} 1 & \text{for } \frac{1}{2}\leq\sigma\leq 1, \\ 0 & \text{for } \sigma>1. \end{cases}$$

Here, applying

$$S_1(t) = \frac{1}{\pi}\int_{\frac{1}{2}}^{\frac{3}{2}} \log|\zeta(\sigma+it)|d\sigma + C$$

in Selberg [15] and integrating in σ over the interval $[\frac{1}{2}, \frac{3}{2}]$, we have

$$\int_{-\infty}^{\infty}\int_{\frac{1}{2}}^{\frac{3}{2}} f(u)\log|\zeta(\sigma+i(t+u))|d\sigma du$$
$$= \pi\int_{-\infty}^{\infty} S_1(t+u)f(u)du + \pi\int_{-\infty}^{\infty} f(u)Cdu$$
$$= \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\frac{1}{2}}(\log n)^2}\Re\left(\frac{1}{n^{it}}\hat{f}(\log n)\right)$$
$$+2\pi\left(\int_{\frac{1}{2}}^{\frac{3}{2}}\sum_{\beta>\sigma}\int_0^{\beta-\sigma}\Re f(\gamma-t-ix)dxd\sigma - \int_{\frac{1}{2}}^{1}\int_0^{1-\sigma}\Re f(-t-ix)dxd\sigma\right).$$

Therefore

$$\int_{-\infty}^{\infty} S_1(t+u)f(u)du$$
$$= \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\frac{1}{2}}(\log n)^2}\Re\left(\frac{1}{n^{it}}\hat{f}(\log n)\right) - C\hat{f}(0)$$
$$+2\left(\int_{\frac{1}{2}}^{\frac{3}{2}}\sum_{\beta>\sigma}\int_0^{\beta-\sigma}\Re f(\gamma-t-ix)dxd\sigma - \int_{\frac{1}{2}}^{1}\int_0^{1-\sigma}\Re f(-t-ix)dxd\sigma\right)$$

$$= \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\frac{1}{2}}(\log n)^2}\Re\left(\frac{1}{n^{it}}\hat{f}(\log n)\right) - C\hat{f}(0)$$
$$+ 2\left(\sum_{\beta>\frac{1}{2}}\int_{\frac{1}{2}}^{\beta}\int_0^{\beta-\sigma}\Re f(\gamma-t-ix)dxd\sigma - \int_{\frac{1}{2}}^{1}\int_0^{1-\sigma}\Re f(-t-ix)dxd\sigma\right).$$

□

Lemma 3.2.
For any sufficiently large positive values of H, t, and τ with $\tau < \log t$ and $H < t$,

$$\int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau}\left(\frac{\sin u}{u}\right)^2 S_1\left(t+\frac{2u}{\tau}\right)du = W(t) + R(t) + O\left(\frac{\log t}{\tau H}\right) + O(1),$$

where

$$W(t) = \sum_{p\le e^{\tau}}\frac{\cos(t\log p)}{p^{\frac{1}{2}}\log p}\left(1-\frac{\log p}{\tau}\right),$$
$$R(t) = \tau\sum_{\beta>\frac{1}{2}}\int_{\frac{1}{2}}^{\beta}\int_0^{\beta-\sigma}\Re\left(\frac{\sin\frac{\tau}{2}(\gamma-t-ix)}{\frac{\tau}{2}(\gamma-t-ix)}\right)^2 dxd\sigma.$$

This is Lemma 2 of author's result in [24]. The proof of this lemma is an analogue of the proof of Lemma 2 of Korolev [10].

Proof. Put $f(z) = \left(\frac{\sin\frac{\tau z}{2}}{\frac{z}{2}}\right)^2$. By

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-xui}f(u)du = 2\pi\tau\max\left(0, 1-\left|\frac{x}{\tau}\right|\right),$$

we get

$$\hat{f}(\log n) = \begin{cases} 2\pi\tau\left(1-\frac{\log n}{\tau}\right) & (1\le n\le e^{\tau}) \\ 0 & (n > e^{\tau}) \end{cases}.$$

Then, by Lemma 3.1 we have

$$\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\tau u}{2}}{\frac{\tau u}{2}}\right)^2 S_1(t+u)du = \frac{1}{\pi}\sum_{n\le e^{\tau}}\frac{\Lambda(n)}{n^{\frac{1}{2}}(\log n)^2}\cdot\frac{2\pi}{\tau}\left(1-\frac{\log n}{\tau}\right)\cos(t\log n)$$

$$+ 2\left\{\sum_{\beta>\frac{1}{2}}\int_{\frac{1}{2}}^{\beta}\int_{0}^{\beta-\sigma}\Re\left(\frac{\sin\frac{\tau}{2}(\gamma-t-i\xi)}{\frac{\tau}{2}(\gamma-t-i\xi)}\right)^2 d\xi d\sigma\right.$$
$$\left.-\int_{\frac{1}{2}}^{1}\int_{0}^{1-\sigma}\Re\left(\frac{\sin\frac{\tau}{2}(\gamma-t-i\xi)}{\frac{\tau}{2}(\gamma-t-i\xi)}\right)^2 d\xi d\sigma\right\}$$
$$- C\hat{f}(0) \tag{3.6}$$

Since for $0 \le \xi \le 1-\sigma$

$$\left|\frac{\sin\frac{\tau}{2}(t+i\xi)}{\frac{1}{2}(t+i\xi)}\right|^2 < \frac{1}{10}$$

as in p. 117 of Korolev [10], the second term on the right-hand side of (3.6) is $O(1)$. Also, in the first term on the right-hand side of (3.6), we single out the terms corresponding to the $n = p$ in the sum and estimate the remainder terms. Then, we have

$$\sum_{2\le k}\sum_{p^k\le e^\tau}\frac{\Lambda(p^{\frac{k}{2}})\cos(t\log p^k)}{p^{\frac{k}{2}}(\log p^k)^2}\cdot\frac{2}{\tau}\left(1-\frac{\log p^k}{\tau}\right) < \sum_{2\le k}\sum_{p^k\le e^\tau}\frac{\log p}{p^{\frac{k}{2}}(\log p^k)^2}\cdot\frac{2}{\tau} \ll \frac{1}{\tau}.$$

Hence

$$\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\tau u}{2}}{\frac{\tau u}{2}}\right)^2 S_1(t+u)du = \frac{2}{\tau}\sum_{p\le e^\tau}\frac{\cos(t\log p)}{p^{\frac{1}{2}}\log p}\left(1-\frac{\log p}{\tau}\right) - C\cdot\frac{2\pi}{\tau}$$
$$+ 2\sum_{\beta>\frac{1}{2}}\int_{\frac{1}{2}}^{\beta}\int_{0}^{\beta-\sigma}\Re\left(\frac{\sin\frac{\tau}{2}(\gamma-t-i\xi)}{\frac{\tau}{2}(\gamma-t-i\xi)}\right)d\xi d\sigma$$
$$+ O\left(\frac{1}{\tau}\right). \tag{3.7}$$

Put $v = \frac{\tau u}{2}$. Then the left-hand side of the above is equal to

$$\left(\int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} + \int_{-\infty}^{-\frac{1}{2}H\tau} + \int_{\frac{1}{2}H\tau}^{\infty}\right)\left(\frac{\sin v}{v}\right)^2 S_1\left(t+\frac{2v}{\tau}\right)\frac{2}{\tau}dv.$$

Since $S_1(t) = O(\log t)$, we have

$$\left|\left(\int_{-\infty}^{-\frac{1}{2}H\tau} + \int_{\frac{1}{2}H\tau}^{\infty}\right)\left(\frac{\sin v}{v}\right)^2 S_1\left(t+\frac{2v}{\tau}\right)dv\right| \ll \frac{1}{\tau}\int_{H}^{\infty}\log(t+v')\frac{1}{v'^2}dv'$$

$$\ll \frac{1}{\tau}\left\{\int_H^t \frac{\log t}{v'^2}dv' + \int_t^\infty \frac{\log v'}{v'^2}dv'\right\}$$
$$\ll \frac{1}{\tau}\left(\frac{\log t}{H} + \frac{\log t}{t}\right) \ll \frac{\log t}{\tau H}.$$

Inserting these estimates into (3.7) and dividing by $\frac{2}{\tau}$ the both sides, we obtain the result.

□

Lemma 3.3.
Let ϵ with $0 < \epsilon < \frac{1}{1000}$ be fixed. Let $T \geq T_0(\epsilon) > 0$, $H = T^{\frac{27}{82}+\epsilon}$ and k be an integer such that $k \geq k_0(\epsilon) > 1$, let $m = 2k+1$, $\tau = 2\log\log H$, and $m\tau < \frac{1}{10}\epsilon \log T$. Then the function $R(t)$ defined by Lemma 3.2 satisfies the inequality

$$\int_T^{T+H} |R(t)|^m dt < H\left\{25^m + (\log T)^3\left(\frac{50\tau m^2}{\epsilon^3 \log T}\right)^m\right\}.$$

This is Lemma 3 of author's result in [24]. The proof of this lemma is an analogue of the proof of Lemma 4 of Chapter 3 in Karatsuba and Korolev [8].

Proof. We put

$$L_k = \int_T^{T+H} |R(t)|^{2k+1} dt$$

and note the inequality

$$\left|\Re\left(\frac{\sin(x-yi)}{x-yi}\right)^2\right| < \frac{8ye^{2y}}{1+x^2+y^2}$$

for any x, $y \in \mathbb{R}$, $y \geq 0$ similarly to pp. $476-477$ of Karatsuba and Korolev [8]. Then,

$$|R(t)| \leq \tau\left|\sum_{\substack{\gamma \\ \beta>\frac{1}{2}}} \int_{\frac{1}{2}}^{\beta}\int_0^{\beta-\sigma} \Re\left(\frac{\sin\frac{\tau}{2}(\gamma-t-\xi i)}{\frac{\tau}{2}(\gamma-t-\xi i)}\right)^2 d\xi d\sigma\right|$$
$$< 4\tau^2 \sum_{\substack{\gamma \\ \beta>\frac{1}{2}}} \int_{\frac{1}{2}}^{\beta}\int_0^{\beta-\frac{1}{2}} \frac{\xi e^{\tau(\beta-\frac{1}{2})}}{1+\left\{\frac{\tau}{2}(\gamma-t)\right\}^2 + \left(\frac{\tau}{2}\left(\beta-\frac{1}{2}\right)\right)^2} d\xi d\sigma$$

$$= 8 \sum_{\substack{\gamma \\ \beta > \frac{1}{2}}} \left(\beta - \frac{1}{2}\right)^3 \frac{e^{\tau\left(\beta - \frac{1}{2}\right)}}{\left(\frac{2}{\tau}\right)^2 + (\gamma - t)^2 + \left(\beta - \frac{1}{2}\right)^2}.$$

We split the last sum into two sums. The first sum Σ_1 is the sum of the terms satisfying $|\gamma - t| > (\log T)^2$, and the second sum Σ_2 is the sum of the other terms.

Here, we denote by θ_t the largest difference of the form $\beta - \frac{1}{2}$ for zeros $\rho = \beta + i\gamma$ in the rectangle $\frac{1}{2} < \beta \leq 1$, $|\gamma - t| \leq (\log T)^2$. Also, we denote by θ'_t the supremum of the form $\beta - \frac{1}{2}$ for zeros $\rho = \beta + i\gamma$ in the rectangle $\frac{1}{2} < \beta \leq 1$, $|\gamma - t| > (\log T)^2$.

As in p. 478 of Karatsuba and Korolev [8], we apply the estimation related to $\sigma_{x,t}$ and the result $N(t+1) - N(t) < 18 \log t$ which is obtained by the Riemann-von Mangoldt formula and $|S(t)| < 8 \log t$ for $t \geq t_0 > 0$. Then we take $x = (\log T)^{\frac{1}{2}}$, and we have

$$\begin{aligned}
\Sigma_1 &< \left(\beta - \frac{1}{2}\right) \sum_{|\gamma - t| > (\log T)^2} \frac{2e^{\frac{\tau}{2}}}{(\gamma - t)^2} < \frac{2}{3}\theta'_t \log T \sum_{|\gamma - t| > (\log T)^2} \frac{1}{n^2} \sum_{n < |\gamma - t| \leq n+1} 1 \\
&< \frac{2}{3}\theta'_t \log T \cdot 36 \sum_{|\gamma - t| > (\log T)^2} \frac{\log T + \log n}{n^2} < 25\theta'_t
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_2 &< 8\theta^3 e^{\tau\theta} \sum_{|\gamma - t| \leq (\log T)^2} \frac{1}{\left(\frac{2}{\tau}\right)^2 + (\gamma - t)^2 + \left(\beta - \frac{1}{2}\right)^2} \\
&< 8\theta^3 e^{\tau\theta} \sum_{\rho} \frac{1}{(\sigma_{x,t} - \beta)^2 + (\gamma - t)^2} < 8\theta^3 e^{\tau\theta} \frac{13}{5} \cdot \frac{1}{\sigma_{x,t} - \frac{1}{2}} \log T \\
&\leq 8\theta^3 e^{\tau\theta} \frac{13}{5} \cdot \frac{5\tau}{39} \log T = 8\theta^3 e^{\tau\theta} \cdot \frac{\tau}{3} \log T.
\end{aligned}$$

From the definitions of θ_t and θ'_t, we get $\theta_t < \frac{1}{2}$ and $\theta'_t < \frac{1}{2}$. Hence, we have

$$|R(t)| < 25 \left(\theta'_t + \frac{7}{2}\theta_t^3 e^{\tau\theta_t} \tau \log T\right) < \frac{25}{2} \left(1 + \frac{7}{2}\theta_t^2 e^{\tau\theta_t} \tau \log T\right).$$

Hence

$$L_k < \left(\frac{25}{2}\right)^m \int_T^{T+H} \left(1 + \frac{7}{2}\theta_t^2 e^{\tau\theta_t} \tau \log T\right)^m dt.$$

This integrand is the same as that in p. 479 of Karatsuba and Korolev [8]. Hence for the estimation of the last integral we can apply the same method as in pp. $480-481$ of Karatsuba and Korolev [8]. Along that way, we have

$$\begin{aligned} L_k &< 25^m H \left\{ 1 + \frac{24}{5} \cdot \frac{1}{m} (\log T)^3 (2m)! \left(\frac{7}{2} \tau \log T \right)^m \left(\frac{\epsilon}{10} \log T \right)^{-2m} \right\} \\ &< 25^m H \left\{ 1 + (\log T)^3 \left(\frac{2m^2 \tau}{\epsilon^3 \log T} \right)^m \right\} \\ &< H \left(25^m + (\log T)^3 \left(\frac{50 m^2 \tau}{\epsilon^3 \log T} \right)^m \right). \end{aligned}$$

□

Lemma 3.4.
Let $T \geq T_0 > 0$, $e^2 < H < T$, $2 < \tau < \log H$, and k be an integer such that $k \geq k_0 > 1$ and $(2k \log k)^2 < e^{\frac{4}{5}\tau}$. Then

$$\int_T^{T+H} W(t)^{2k} dt > \left(\frac{1}{5\sqrt{10e}} \cdot \frac{k^{\frac{1}{2}}}{\log k} \right)^{2k} H - e^{3k\tau}, \tag{3.8}$$

$$\left| \int_T^{T+H} W(t)^{2k+1} dt \right| < e^{3k\tau + \frac{3}{2}\tau}. \tag{3.9}$$

This is Lemma 4 of author's result in [24]. The proof of this lemma is an analogue of the proof of Lemma 3 of Chapter 3 in Karatsuba and Korolev [8]. But in Karatsuba and Korolev [8], the function $W(t)$ is defined by

$$W(t) = -\sum_{p \leq e^\tau} \frac{\sin(t \log p)}{p^{\frac{1}{2}}} \left(1 - \frac{\log p}{\tau} \right),$$

which are defferent from the definition in this thesis.

Proof. As in pp. $474-475$ of Karatsuba and Korolev [8], we can write

$$\int_T^{T+H} W(t)^{2k} dt = I_k = \binom{2k}{k} \frac{H}{2^{2k}} \Sigma + \theta e^{3k\tau},$$

where

$$\Sigma = \sum_{\substack{p_1 \cdots p_k = q_1 \cdots q_k \\ p_1, \cdots, q_k \leq e^\tau}} f(p_1)^2 \cdots f(p_k)^2, \quad f(p) = \frac{1}{p^{\frac{1}{2}} \log p} \left(1 - \frac{\log p}{\tau} \right).$$

Then,

$$\begin{aligned}\Sigma &\geq k! \sum_{\substack{p_1,\cdots,p_k \text{ are distinct} \\ p_1,\cdots,p_k \leq e^\tau}} f(p_1)^2 \cdots f(p_k)^2 \\ &\geq k! \sum_{p_1 \leq e^\tau} f(p_1)^2 \sum_{\substack{p_2 \leq e^\tau \\ p_1 \neq p_2}} f(p_2)^2 \cdots \sum_{\substack{p_k \leq e^\tau \\ p_1,\cdots,p_{k-1} \neq p_k}} f(p_k)^2.\end{aligned}$$

Since $\frac{d}{dp} f(p)^2 < 0$, $f(p)^2$ is monotonically decreasing for $p \geq 2$. Also, since the $(k-1)$th prime does not exceed $2k \log k$, the inner sum of the above inequality is greater than the same sum over $2k \log k < p_k < e^{\frac{4}{5}\tau}$. Hence the inner sum over p_k is greater than

$$\left(\frac{1}{5}\right)^2 \sum_{2k \log k < p \leq e^{\frac{4}{5}\tau}} \frac{1}{p(\log p)^2}.$$

For $(2k \log k)^2 \leq e^{\frac{4}{5}\tau}$, since

$$\begin{aligned}\sum_{U < p \leq U^2} \frac{1}{p(\log p)^2} &\geq \frac{1}{4(\log U)^2} \sum_{U < p \leq U^2} \frac{1}{p} \\ &= \frac{1}{4(\log U)^2} (\log \log U^2 - \log \log U + o(1)) > \frac{1}{8(\log U)^2},\end{aligned}$$

the sum over p_k is greater than $\frac{1}{10}\left(\frac{1}{5}\right)^2 \frac{1}{(\log k)^2}$. Also, the same lower bound holds for the sums over $p_1, p_2, \cdots, p_{k-1}$. Therefore, we see

$$\Sigma \geq k! \left(\frac{1}{250(\log k)^2}\right)^k \geq \sqrt{2\pi k} \left(\frac{1}{5\sqrt{10e}} \cdot \frac{k^{\frac{1}{2}}}{\log k}\right)^{2k}.$$

So,

$$I_k > H \left(\frac{1}{5\sqrt{10e}} \cdot \frac{k^{\frac{1}{2}}}{\log k}\right)^{2k} - e^{3k\tau}.$$

This is the first part of Lemma 3.4. The second part is as same as that proved in [8].

□

The following lemma is a special case of Lemma 4 of Tsang [21], and following type is written in Lemma 1 of Chapter 3 in Karatsuba and Korolev [8].

Lemma 3.5.
Let $H > 0$ and $M > 0$, let $k \geq 1$ be an integer, and let $W(t)$, $R(t)$ be real functions which satisfy the conditions

$$1) \quad \int_T^{T+H} |W(t)|^{2k} dt \geq HM^{2k},$$

$$2) \quad \left| \int_T^{T+H} W(t)^{2k+1} dt \right| \leq \frac{1}{2} HM^{2k+1},$$

$$3) \quad \int_T^{T+H} |R(t)|^{2k+1} dt < H \left(\frac{M}{2} \right)^{2k+1}.$$

Then

$$\max_{T \leq t \leq T+H} \pm(W(t) + R(t)) \geq \frac{1}{8} M.$$

The proof of this lemma is omitted in this thesis.

Proof of Theorem 3.1 Put $\tau = 2 \log \log H$. Consider the right-hand side of the inequality in the statement of Lemma 3.3. We see easily that

$$\frac{50\tau m^2}{\epsilon^3 \log T} < \frac{500k^2}{\epsilon^3} \cdot \frac{\log \log T}{\log T} \leq \frac{k^{\frac{1}{2}}}{\log k} \cdot \frac{500k^{\frac{3}{2}}}{\epsilon^3} \cdot \frac{(\log \log T)^2}{\log T} = \frac{k^{\frac{1}{2}}}{\log k} \delta,$$

say.

Here, putting $k = \left[\frac{\epsilon^2}{1000} \left(\frac{(\log T)^{\frac{2}{3}}}{(\log \log T)^{\frac{4}{3}}} \right) \right]$, we have $\delta < \frac{1}{60}$, $(2k \log k)^2 < e^{\frac{4}{5}\tau}$ and $e^{3k\tau} < H^{\frac{1}{2}}$. Hence, we can apply Lemma 3.3 and Lemma 3.4. Then we have

$$\int_T^{T+H} W(t)^{2k} dt > HM^{2k},$$

$$\left| \int_T^{T+H} W(t)^{2k+1} dt \right| < \frac{1}{2} HM^{2k+1},$$

$$\int_T^{T+H} |R(t)|^{2k+1} dt < H \left(\frac{M}{2} \right)^{2k+1},$$

with $M = \frac{k^{\frac{1}{2}}}{30\log k}$. Thus, we see that $W(t)$ and $R(t)$ satisfy the conditions of Lemma 3.5 with $M = \frac{k^{\frac{1}{2}}}{30\log k}$. Hence there are two points t_0 and t_1 such that

$$W(t_0) + R(t_0) \geq \frac{M}{8}, \quad W(t_1) + R(t_1) \leq -\frac{M}{8}$$

in the interval $T \leq t \leq T + H$. By Lemma 3.2, we have

$$\int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left(\frac{\sin u}{u}\right)^2 S_1\left(t_0 + \frac{2u}{\tau}\right) du \geq \frac{M}{8} + O\left(\frac{\log t_0}{\tau H}\right),$$
$$\int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left(\frac{\sin u}{u}\right)^2 S_1\left(t_1 + \frac{2u}{\tau}\right) du \leq -\frac{M}{8} + O\left(\frac{\log t_1}{\tau H}\right).$$

Here, putting

$$M_0 = \sup_{T-H\leq t\leq T+2H} S_1(t), \quad M_1 = \inf_{T-H\leq t\leq T+2H} S_1(t),$$

we have

$$\int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left(\frac{\sin u}{u}\right)^2 S_1\left(t_0 + \frac{2u}{\tau}\right) du < M_0 \int_{-\infty}^{\infty} \left(\frac{\sin u}{u}\right)^2 = \frac{\pi}{2} M_0 \quad (M_0 > 0),$$
$$\int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left(\frac{\sin u}{u}\right)^2 S_1\left(t_1 + \frac{2u}{\tau}\right) du > M_1 \int_{-\infty}^{\infty} \left(\frac{\sin u}{u}\right)^2 = \frac{\pi}{2} M_1 \quad (M_1 < 0).$$

Therefore, we obtain for $r = 0, 1$

$$(-1)^r M_r > \frac{2}{\pi} \cdot \frac{M}{8} + O\left(\frac{\log t_r}{\tau H}\right) > \frac{1}{4\pi} \cdot \frac{k^{\frac{1}{2}}}{30\log k} > \frac{\epsilon}{4000\pi}\left(\frac{(\log T)^{\frac{1}{3}}}{(\log\log T)^{\frac{5}{3}}}\right).$$

Thus, we obtain the result.

Moreover, we see immediately the following theorem from Theorem 3.1.

Corollary 3.1.

$$S_1(t) = \Omega_{\pm}\left(\frac{(\log t)^{\frac{1}{3}}}{(\log\log t)^{\frac{5}{3}}}\right).$$

This is Theorem 2 of author's result in [24]. This is an improvement on Selberg's result (3.2). The technique of Karatsuba and Korolev [8] can obtain the Ω_+ and the Ω_- estimates simultaneously. Our method is based on the idea of Karatsuba and Korolev [8], so our method can treat the Ω_+ and the Ω_- estimates simultaneously. This should be compared with Tsang's paper, in which the result of the Ω_+ case is different from that of the Ω_- case. Our Ω-result can improve the Selberg's Ω-result, but cannot improve the Tsang's Ω-result.

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