Well-posedness and scattering for nonlinear Schrödinger equations with derivative nonlinearity and higher order KdV type equations

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Chapter 1

Overview

1.1 Introduction

In this article, we study the Cauchy problem of nonlinear dispersive equations with derivative nonlinearity. Specifically, we deal with the system of quadratic derivative nonlinear Schrödinger equations:

$$\begin{cases} (i\partial_t + \alpha\Delta)u = -(\nabla \cdot w)v, \\ (i\partial_t + \beta\Delta)v = -(\nabla \cdot \overline{w})u, \\ (i\partial_t + \gamma\Delta)w = \nabla(u \cdot \overline{v}) \end{cases} \tag{1.1}$$

with α, β, $\gamma \in \mathbb{R}\backslash\{0\}$, the nonlinear Schrödinger equations:

$$(i\partial_t + \Delta)u = \partial_k(\overline{u}^m) \tag{1.2}$$

with $m \in \mathbb{N}$, $m \geq 2$, $1 \leq k \leq d$ and the higher order KdV type equations:

$$\partial_t u + (-1)^{k+1}\partial_x^{2k+1}u + \frac{1}{2}\partial_x(u^2) = 0 \tag{1.3}$$

with $k \in \mathbb{N}$, where u, v, w in (1.1) are $\mathbb{C}^d$ valued, u in (1.2) is $\mathbb{C}$ valued, u in (1.3) is $\mathbb{R}$ valued, d is a dimension of the spatial variable, $\partial_k = \partial/\partial x_k$ and Δ is the usual Laplacian $\Delta = \sum_{k=1}^{d} \partial^2/\partial x_k^2$. The system (1.1) is a model of laser-plasma interaction ([11]). The equation (1.2) is a mathematical model which is studied by Grünrock ([23]). The equation (1.3) is called "Korteweg-de Vries equation" ("KdV equation" for short) when $k = 1$ and "Kawahara equation" when $k = 2$ which are the model of water waves ([42], [52]). We assume that the spatial variable x is in $\mathbb{R}^d$ or $\mathbb{T}^d$ $(= \mathbb{R}^d/(2\pi\mathbb{Z})^d)$.

Our first aim is to obtain the well-posedness (existence of solutions, their uniqueness and their continuous dependence on initial data) for the Cauchy problem of these equations. Well-posedness is a fundamental and important property of differential equations since the existence of solutions and their uniqueness are necessary to justify the equation as a model of physical phenomena and the continuous dependence on initial data is necessary to justify numerical solutions of the equation. We say that a differential equation is locally well-posed (LWP for short) in the Banach space $\mathcal{H}$ if for any initial data $u_0 \in \mathcal{H}$, there exist $T > 0$ and a solution $u \in C([0,T];\mathcal{H})$ of the equation, such solution is unique in the suitable Banach space $\mathcal{X}_T$ which is embedded in $C([0,T];\mathcal{H})$ and the data-to-solution map $u_0 \mapsto u$ is continuous from each ball in $\mathcal{H}$ to $\mathcal{X}_T$. Furthermore if for any $u_0 \in \mathcal{H}$ and $T > 0$, there exists a solution $u \in C([0,T];\mathcal{H})$ and the uniqueness and the continuous of the data-to-solution map above hold, then we say that the differential equation is globally well-posed (GWP for short) in $\mathcal{H}$.

To obtain the well-posedness of a differential equation, we use the iteration argument as below. We consider the general problem:

$$\begin{cases} i\partial_t u + \mathcal{L}u = \mathcal{N}(u), \\ u|_{t=0} = u_0, \end{cases} \tag{1.4}$$

where $\mathcal{L}$ is a linear differential operator given as $\mathcal{L} = p(-i\nabla)$ for a real coefficients polynomial p and $\mathcal{N}(u)$ is a nonlinear term. For example, (1.4) corresponds (1.2) when $p(\xi) = -|\xi|^2, \mathcal{N}(u) = \partial_k(\overline{u}^m)$. By the Duamel's formula, (1.4) can be rewritten as the integral equation:

$$u(t) = e^{it\mathcal{L}}u_0 - i\int_0^t e^{i(t-t')\mathcal{L}}\mathcal{N}(u(t'))dt', \tag{1.5}$$

where $\{e^{it\mathcal{L}}\}_{t\in\mathbb{R}}$ is one-parameter unitary group defined by $\widehat{e^{it\mathcal{L}}f} = e^{itp(\xi)}\widehat{f}$ and $\widehat{\cdot}$ denotes the spatial Fourier transform. We call the second term of R.H.S of (1.5) "Duamel term" and the solution of (1.5) "mild solution". In this article, a solution of Cauchy problem means the mild solution. If we have the estimates

$$||e^{it\mathcal{L}}u_0||_{\mathcal{X}_T} \le C_1||u_0||_{\mathcal{H}}, \quad \left|\left|\int_0^t e^{i(t-t')\mathcal{L}}\mathcal{N}(u(t'))dt'\right|\right|_{\mathcal{X}_T} \le C_2T^{\delta}||u||_{\mathcal{X}_T}^a \tag{1.6}$$

and

$$\left|\left|\int_0^t e^{i(t-t')\mathcal{L}}(\mathcal{N}(u(t')) - \mathcal{N}(v(t')))dt'\right|\right|_{\mathcal{X}_T} \le C_2T^{\delta}(||u||_{\mathcal{X}_T}^{a-1} + ||v||_{\mathcal{X}_T}^{a-1})||u-v||_{\mathcal{X}_T} \tag{1.7}$$

for some C_1, C_2, $\delta > 0$ and $a > 1$, then we obtain the well-posedness of (1.4) in $\mathcal{H}$ with the solution $u \in \mathcal{X}_T$. In fact for given $u_0 \in \mathcal{H}$, we put $\Phi_{u_0}[u](t)$:=(R.H.S of (1.5)) and $\mathcal{X}_{T,u_0} := \{u \in \mathcal{X}_T | \ ||u||_{\mathcal{X}_T} \leq 2C_1||u_0||_{\mathcal{H}}\}$, then Φ_{u_0} is a contraction map on $\mathcal{X}_{T,u_0}$ for $T = (2^{a+1}C_1^{a-1}C_2||u_0||^{a-1})^{-1/\delta}$. Therefore by the Banach fixed point theorem, there exists unique $u \in \mathcal{X}_{T,u_0}$ such that $u(t) = \Phi_{u_0}[u](t)$ on $t \in [0,T]$, namely u is a solution of (1.5). If the estimates (1.6) and (1.7) are proved as $\delta = 0$, we obtain the well-posedness of (1.4) for any $u_0 \in \mathcal{H}$ with $||u_0||_{\mathcal{H}} \leq (2^{a+1}C_1^{a-1}C_2)^{-1/(a-1)}$by the same argument. So the key estimates to obtain the well-posedness are (1.6) and (1.7).

In this article, we always set $\mathcal{H} = H^s$ which is the inhomogeneous Sobolev space with the norm

$$||f||_{H^s} := ||\langle\xi\rangle^s \widehat{f}||_{L^2_\xi} = ||(1+|\xi|)^s \widehat{f}||_{L^2_\xi}$$

or $\mathcal{H} = \dot{H}^s$ which is the homogeneous Sobolev space with the norm

$$||f||_{\dot{H}^s} := |||\xi|^s \widehat{f}||_{L^2_\xi}.$$

The lower regularity s we choose, the more difficult we show the nonlinear estimates such as (1.6) and (1.7). For example, if $s > d/2$ then the estimate

$$||fg||_{H^s} \leq C||f||_{H^s}||g||_{H^s} \tag{1.8}$$

holds. But if $s \leq d/2$ then (1.8) fails generally. Because of such reason, our interest is the well-posedness of differential equations with "low regularity" initial data. One of the target of the regularity for the well-posedness is "scaling critical regularity" which is decided by the invariant transformation for the equation. For example the equation (1.2) is invariant under the following scaling transformation:

$$u_\lambda(t,x) = \lambda^{-1/(m-1)}u(\lambda^{-2}t, \lambda^{-1}x).$$

Since

$$||u_\lambda(0,\cdot)||_{\dot{H}^s} = \lambda^{d/2-1/(m-1)-s}||u(0,\cdot)||_{\dot{H}^s},$$

the scaling critical regularity for (1.2) is $s_c = d/2 - 1/(m-1)$. We note that if $s > s_c$, then large initial data and short time settings is equivalent to small initial data and long time settings. It match the iteration argument above. While if $s < s_c$, then small initial data and short time settings is equivalent to large initial data and long time settings. But the latter settings is more difficult to obtain well-posedness than the former settings. In fact, there are many blow up results for large initial data. Therefore, the well-posedness for $s < s_c$ is not expected. The critical case $s = s_c$ is the most important because if $s = s_c$, the time interval of the solution is not depend on $\dot{H}^{s_c}$-norm of the initial data since $\dot{H}^{s_c}$-norm is invariant by the scaling transformation. Therefore if we obtain LWP for $u_0 \in H^{s_c}$ with $||u_0||_{H^{s_c}} \leq r$ for some $r > 0$, then it is expected that GWP for same u_0 is also obtained.

Next, we explain the function spaces. If we choose $\mathcal{X}_T = C([0,T]; H^s)$, then it is difficult to obtain the estimates (1.6) and (1.7) for (1.1), (1.2) and (1.3) because of the derivative loss arising from the nonlinearity in these equations. To recover the derivative loss, we use the Bourgain space $X_{\mathcal{L}}^{s,b}$ and the spaces $U_{\mathcal{L}}^2 H^s$, $V_{\mathcal{L}}^2 H^s$ which norms depend on the structure of the linear terms of the equation (1.4).

The Bourgain space $X_{\mathcal{L}}^{s,b}$ is introduced by Bourgain to prove the well-posedness of the nonlinear Schrödinger equation and the KdV equation with the periodic initial data ([5]). His method is called "Fourier restriction norm method". The norm of the Bourgain space, which is called "Bourgain norm" depends on the linear terms of the equation. The Bourgain space $X_{\mathcal{L}}^{s,b}$ is the completion of the Schwarz space with respect to the norm defined by

$$||u||_{X_{\mathcal{L}}^{s,b}} := ||\langle \tau - p(\xi)\rangle^b \langle \xi\rangle^s \widetilde{u}||_{L^2_{\tau,\xi}} = ||e^{-it\mathcal{L}}u||_{H_t^b H_x^s},$$

where $\widetilde{\cdot}$ denotes the space-time Fourier transform. The weight function $\langle \tau - p(\xi)\rangle$ is decided by the hypersurface $\{(\tau,\xi) | \tau - p(\xi) = 0\}$ which contains $\operatorname{supp} \widetilde{u_{\mathcal{L}}}$, where $u_{\mathcal{L}} := e^{it\mathcal{L}}u_0$ is a solution of the Linearized equation of (1.4). If $b > 1/2$, then $X_{\mathcal{L}}^{s,b}([0,T])$ is embedded in $C([0,T]; H^s)$ continuously. We note that the function $e^{-it\mathcal{L}}u$ with $u \in X_{\mathcal{L}}^{s,b}$ has the regularity with respect to the time variable and we use it to recover the derivative loss. For example if $\operatorname{supp} \widetilde{u} \subset \{(\tau,\xi) |\ |\tau - p(\xi)| \geq |\xi|^{\epsilon}\}$, then we have

$$||u||_{X_{\mathcal{L}}^{s,b}} \geq C||\langle \xi\rangle^{s+\epsilon b}\widetilde{u}||_{L^2_{\tau,\xi}} = C||u||_{L_t^2 H_x^{s+\epsilon b}},$$

where $C > 0$ is a constant. Therefore, $L_t^2 H_x^{s+\epsilon b}$-norm is controlled by $X_{\mathcal{L}}^{s,b}$-norm. So, there is the derivative gain ϵb with respect to the spatial variable. To obtain (1.6) and (1.7) as the solution space $\mathcal{X}_T = X_{\mathcal{L}}^{s,b}([0,T])$, we use the duality argument. Since $e^{-it\mathcal{L}}\mathcal{N}(u)$ in the Duamel term of (1.5) has the first derivative gain with respect to the time variable arising from the integral and the dual space of $X_{\mathcal{L}}^{s,b-1}$ for the L^2-inner product is $X_{\mathcal{L}}^{-s,1-b}$, we obtain

$$\left|\left|\int_0^t e^{i(t-t')\mathcal{L}}\mathcal{N}(u(t'))dt'\right|\right|_{X_{\mathcal{L}}^{s,b}([0,T])} \leq C' \sup_{||v||_{X_{\mathcal{L}}^{-s,1-b}}=1} \left|(\mathcal{N}(u),v)_{L^2_{tx}}\right|, \tag{1.9}$$

where $C' > 0$ is a constant ([21] Lemma 2.1). We note that if we choose $b > 1/2$, then the derivative gain for the function v in the dual space $X_{\mathcal{L}}^{-s,1-b}$ with $\operatorname{supp}\widetilde{v} \subset \{(\tau,\xi)|\ |\tau - p(\xi)| \geq |\xi|^{\epsilon}\}$ is less than $\epsilon/2$. While if we choose $b < 1/2$, then the derivative gain for the function u in the solution space $X_{\mathcal{L}}^{s,b}$ with $\operatorname{supp}\widetilde{u} \subset \{(\tau,\xi)|\ |\tau - p(\xi)| \geq |\xi|^{\epsilon}\}$ is less than $\epsilon/2$. So, the best choice of the index b to get the derivative gain is $b = 1/2$. But if we choose $b = 1/2$, then $X_{\mathcal{L}}^{s,b}([0,T])$ is not embedded in $C([0,T];H^s)$. To avoid this problem, we use the auxiliary space Y^s which also be introduced by Bourgain. The norm of Y^s is defined by $||u||_{Y^s} := ||\langle\xi\rangle^s\widetilde{u}||_{L_{\xi}^2L_{\tau}^1}$. To obtain the well-posedness of (1.3), we will set the solution space $\mathcal{X}_T = Z_{\mathcal{L}}^s([0,T]) := X_{\mathcal{L}}^{s,1/2}([0,T]) \cap Y^s([0,T])$ in Chapter 6. The norm of $Z_{\mathcal{L}}^s$ is defined by $||u||_{Z_{\mathcal{L}}^s} := ||u||_{X_{\mathcal{L}}^{s,1/2}} + ||u||_{Y^s}$ and $Z_{\mathcal{L}}^s([0,T])$ is embedded in $C([0,T];H^s)$ continuously.

For $1 < p, p' < \infty$ satisfying $1/p + 1/p'$, the space V^p consists of bounded p-variation functions and the space U^p is the dual space of $V^{p'}$ in some sense. The bounded p-variation function is introduced by Wiener ([72]). The first application of the spaces U^p and V^p for differential equations is Tataru's unpublished work for wave maps. The properties of the spaces U^p and V^p are studied by Hadac, Herr and Koch ([25], [26]) to prove the well-posedness of KP-II equation in the scaling critical Sobolev space. Furthermore, Herr, Tataru and Tzvetkov ([31]) applied the spaces U^p and V^p to the 3D quintic nonlinear Schrödinger equation with the periodic setting and proved its well-posedness in the scaling critical Sobolev space. We explain the advantage of U^p and V^p compared with the Bourgain space. One of the key estimate to prove the well-posedness of dispersive equation such as (1.4) is the Strichartz estimate

$$||e^{it\mathcal{L}}\varphi||_{L_t^pL_x^q} \lesssim ||\varphi||_{\dot{H}_x^s}. \tag{1.10}$$

If (1.10) holds, we call (p,q,s) a "admissible exponents". It is known that the Bourgain space $X_{\mathcal{L}}^{s,b}$ is embedded in $L_t^pL_x^q$ continuously if $b > 1/2$ and (p,q,s) are an admissible exponents ([21] Lemma 2.3). Therefore, if we use both the Bourgain space $X_{\mathcal{L}}^{s,b}$ and the Strichartz estimate (1.10), then we have to choose $b > 1/2$. But as mentioned above, $b > 1/2$ is not the best choice of index to get the derivative gain. Because of such reason, it is difficult to imply the well-posedness in the scaling critical Sobolev space by using the Bourgain space. An idea to avoid this difficulty is to use the Besov type Bourgain space $X_{\mathcal{L}}^{s,1/2,1}$ which norm is defined by

$$||u||_{X_{\mathcal{L}}^{s,1/2,1}} := \sum_{M\in 2^{\mathbb{N}\cup\{0\}}} M^{1/2}||\langle\xi\rangle^s\psi_M(\tau - p(\xi))\widetilde{u}||_{L_{\tau,\xi}^2},$$

where $\{\psi_M\}_{M\in 2^{\mathbb{N}}} \subset C^\infty(\mathbb{R})$ satisfies $\operatorname{supp}\psi_M \subset [-2M,-M/2]\cup[M/2,2M]$ for $M \geq 2$, $\operatorname{supp}\psi_1 \subset [-2,2]$ and $\sum_{M\in 2^{\mathbb{N}\cup\{0\}}}\psi_M \equiv 1$. The Besov type Bourgain spaces are introduced by Muramatu and Taoka ([57]) and $X^{s,1/2,1}_{\mathcal{L}}$-norm is stronger than $X^{s,1/2}_{\mathcal{L}}$-norm. We note that the derivative gain for $u \in X^{s,1/2,1}_{\mathcal{L}}$ is same as for $u \in X^{s,1/2}_{\mathcal{L}}$ and $X^{s,1/2,1}_{\mathcal{L}}$ is embedded in $L^p_tL^q_x$ continuously if (p,q,s) are an admissible exponents. So, if we use $X^{s,1/2,1}_{\mathcal{L}}$ as the solution space, we can apply the Strichartz estimate and obtain the best derivative gain. But the norm of the dual space of $X^{s,1/2,1}_{\mathcal{L}}$ is too weak. In fact, the dual space of $X^{s,1/2,1}_{\mathcal{L}}$ in the sense of (1.9) is $X^{-s,1/2,\infty}_{\mathcal{L}}$ whose norm is defined by

$$||u||_{X^{-s,1/2,\infty}_{\mathcal{L}}} := \sup_{M\in 2^{\mathbb{N}\cup\{0\}}} M^{1/2}||\langle\xi\rangle^{-s}\psi_M(\tau-p(\xi))\widetilde{u}||_{L^2_{\tau,\xi}},$$

since the dual space of l^1 for the l^2-inner product is l^∞. For any $s' \in \mathbb{R}$ and $b' < 1/2$, $X^{s',1/2,\infty}_{\mathcal{L}}$-norm is stronger than $X^{s',b'}_{\mathcal{L}}$-norm, but weaker than $X^{s',1/2}$-norm. Therefore, we cannot apply the Strichartz estimate to the dual function because $X^{s',1/2,\infty}_{\mathcal{L}}$ is not embedded in $L^p_tL^q_x$ for any admissible exponents (p,q,s'). This difficulty can be overcome if we use the spaces $U^2_{\mathcal{L}}H^s$ and $V^2_{\mathcal{L}}H^s$ whose norms are defined by

$$||u||_{U^2_{\mathcal{L}}H^s} := ||e^{-it\mathcal{L}}u||_{U^2_tH^s_x},\ ||u||_{V^2_{\mathcal{L}}H^s} := ||e^{-it\mathcal{L}}u||_{V^2_tH^s_x}.$$

Since the dual space of $U^2_{\mathcal{L}}H^s$ in the sense of (1.9) is $V^2_{\mathcal{L}}H^{-s}$ ([25] Remark 2.11, 2.12) and $U^2_{\mathcal{L}}H^s$, $V^2_{\mathcal{L}}H^s$ are embedded in $L^p_tL^q_x$ continuously if (p,q,s) are an admissible exponents with $p>2$ ([25] Proposition 2.19), we can apply the Strichartz estimate also to the dual function. Furthermore since the continuous embeddings

$$\dot{X}^{s,1/2,1}_{\mathcal{L}} \hookrightarrow U^2_{\mathcal{L}}H^s \hookrightarrow V^2_{\mathcal{L}}H^s \hookrightarrow \dot{X}^{s,1/2,\infty}_{\mathcal{L}}$$

hold ([25] Remark 2.17), the derivative gain for $u \in V^2_{\mathcal{L}}H^s$ is same as for $u \in \dot{X}^{s,1/2}_{\mathcal{L}}$, which is the best derivative gain, where $\dot{X}^{s,1/2,1}_{\mathcal{L}}$ and $\dot{X}^{s,1/2,\infty}_{\mathcal{L}}$ are the homogeneous Besov type Bourgain space. Because of such reason, the spaces $U^2_{\mathcal{L}}H^s$ and $V^2_{\mathcal{L}}H^s$ are suitable spaces to obtain the well-posedness in the scaling critical Sobolev space. We will use these spaces and prove the well-posedness of (1.1) and (1.2) in the scaling critical Sobolev space in Chapter 3, 4, 5.

If we obtain GWP for a differential equation, our interest is focused on the asymptotic behavior of the solution naturally. We say that the solution of the equation (1.4) scatters in $\mathcal{H}$ if there exist $u_\pm \in \mathcal{H}$ such that $u(t) - e^{it\mathcal{L}}u_\pm$ is converges to 0 in $\mathcal{H}$ as $t \to \pm\infty$, namely the solution behaves like a solution of the linearized equation of (1.4) asymptotically in time. Our second aim is to obtain the scattering for the solution of the equations (1.1) and (1.2). To obtain the scattering, we will use the fact that $\lim_{t\to\pm\infty} u(t)$ exist for any $u \in V^2$ ([25] Proposition 2.4) in Chapter 3, 5.

Most part of this article are based on the author's papers [32], [33], [34] and [35]. The rest of this article is planned as follows. In Chapter 2, we will introduce the property of the spaces U^2, V^2 referred from [25] and [26]. In Chapter 3, which is based on [33], we will prove the well-posedness and scattering for the system (1.1) and the equation (1.2) for $d \geq 2$, $m = 2$ with the nonperiodic initial data. Furthermore, we also prove the ill-posedness of the system (1.1) in a weak sense. In Chapter 4, which is based on [34], we will prove the well-posedness of the system (1.1) and the equation (1.2) for $d \geq 2$, $m = 2$ with the periodic initial data. In Chapter 5, which is based on [35], we will prove the well-posedness and scattering for the equation (1.2) for $d \geq 1$, $m \geq 3$ with the nonperiodic initial data. In Chapter 6, which is based on [32], We will prove the well-posedness of the equation (1.3) with the periodic initial data. In particular, the well-posedness results in Chapter 3, 4, 5 contain the scaling critical case.

Notation.

- We put $\langle \cdot \rangle := 1 + |\cdot|$.
- We put $\mathbb{Z}^* := \mathbb{Z}\backslash\{0\}$.
- For $\lambda \geq 1$, we put $\mathbb{Z}_\lambda := \{n/\lambda |\ n \in \mathbb{Z}\}$ and $\mathbb{Z}_\lambda^* := \mathbb{Z}_\lambda\backslash\{0\}$.
- We define the integral on $\mathbb{Z}_\lambda^*$ as

$$\int_{\mathbb{Z}_\lambda^*} f(\xi)d\xi := \frac{1}{2\pi\lambda}\sum_{\xi\in\mathbb{Z}_\lambda^*} f(\xi).$$

- For a function f defined on $\mathbb{Z}_\lambda^*$, we put

$$||f||_{l_\xi^2(\lambda)} := \left(\int_{\mathbb{Z}_\lambda^*} |f(\xi)|^2 d\xi\right)^{1/2}.$$

- For functions F and G defined on $\mathbb{R}\times\mathbb{Z}_\lambda^*$, we put

$$F * G(\tau,\xi) := \int_{\mathbb{Z}_\lambda^*\backslash\{\xi\}}\int_{\mathbb{R}} F(\tau-\tau_1,\xi-\xi_1)G(\tau_1,\xi_1)d\tau_1 d\xi_1$$

- We define the integral on $\mathbb{T}_\lambda$ as

$$\int_{\mathbb{T}_\lambda} f(s)ds := \int_0^{2\pi\lambda} f(s)ds.$$

- We define the integral on $\mathbb{T}^d$ as

$$\int_{\mathbb{T}^d} f(x)dx := \int_{[0,2\pi]^d} f(x)dx.$$

- We define the spatial Fourier transform for the function on X as

$$\mathcal{F}_x[f](\xi) = \widehat{f}(\xi) := \int_X f(x)e^{-i\xi\cdot x}dx,\ \xi \in X',$$

  where $(X, X') = (\mathbb{T}_\lambda, \mathbb{Z}_\lambda)$ or $(\mathbb{R}^d, \mathbb{R}^d)$ or $(\mathbb{T}^d, \mathbb{Z}^d)$.

- We define the space time Fourier transform for the function on $\mathbb{R} \times X$ as

$$\mathcal{F}[f](\tau, \xi) = \widetilde{f}(\tau, \xi) := \int_{\mathbb{R}} \int_X f(t, x)e^{-it\tau}e^{-ix\cdot\xi}dxdt,\ \tau \in \mathbb{R}, \xi \in X',$$

  where (X, X') is same as above.

- For $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R}^d)$ (resp. $\dot{H}^s(\mathbb{R}^d)$) as the space of all $\mathcal{S}'(\mathbb{R}^d)$ functions for which the norm

$$||f||_{H^s} := ||\langle\xi\rangle^s \widehat{f}||_{L^2_\xi(\mathbb{R}^d)} \text{ (resp. } |||\xi|^s \widehat{f}||_{L^2_\xi}).$$

- For $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{T}^d)$ (resp. $H^s(\mathbb{T}_\lambda)$) as the space of all $L^2(\mathbb{T}^d)$ (resp. $L^2(\mathbb{T}_\lambda)$) functions for which the norm

$$||f||_{H^s} := ||\langle\xi\rangle^s \widehat{f}||_{l^2_\xi(\mathbb{Z}^d)} \text{ (resp. } ||\langle\xi\rangle^s \widehat{f}||_{l^2_\xi(\lambda)}).$$

- For a Banach space $\mathcal{H}$ and $r > 0$, we put

$$B_r(\mathcal{H}) := \{f \in \mathcal{H}|\ ||f||_{\mathcal{H}} \leq r\}.$$

- We will use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant C and write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$.

- We will use the convention that capital letters denote dyadic numbers, e.g. $N = 2^n$ for $n \in \mathbb{Z}$ and for a dyadic summation we write $\sum_N a_N := \sum_{n\in\mathbb{Z}} a_{2^n}$ and $\sum_{N\geq M} a_N := \sum_{n\in\mathbb{Z},2^n\geq M} a_{2^n}$ for brevity.

- Let $\chi \in C_0^\infty((-2,2))$ be an even, non-negative function such that $\chi(s) = 1$ for $|s| \leq 1$. We define $\psi_0(\xi) := \chi(|\xi|)$ and $\psi_N(\xi) := \psi_0(N^{-1}\xi) - \psi_0(2N^{-1}\xi)$ for $N \in 2^{\mathbb{Z}}$.

- We define frequency and modulation projections

  $$\widehat{P_S u}(\xi) := \chi_S(\xi)\widehat{u}(\xi),\ \widehat{P_N u}(\xi) := \psi_N(\xi)\widehat{u}(\xi),\ \widetilde{Q_M^{\mathcal{L}} u}(\tau,\xi) := \psi_M(\tau - p(\xi))\widetilde{u}(\tau,\xi)$$

  for the set $S \subset \mathbb{Z}^d$ and the dyadic numbers N, M, where χ_S is the characteristic function of S, and $\mathcal{L}$ is a linear differential operator given as $\mathcal{L} = p(-i\nabla)$ for a real coefficients polynomial p. Furthermore, we define $P_0 := Id - \sum_{N\geq 1} P_N$, $Q_{\geq M}^{\mathcal{L}} := \sum_{N \geq M} Q_N^{\mathcal{L}}$ and $Q_{<M}^{\mathcal{L}} := Id - Q_{\geq M}^{\mathcal{L}}$.

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1.2 Review for the Schrödinger equations with derivative nonlinearity

We consider the Cauchy problem of the system of Schrödinger equations:

$$\begin{cases} (i\partial_t + \alpha\Delta)u = -(\nabla \cdot w)v, \quad t \in (0,\infty),\ x \in \mathbb{R}^d \text{ or } \mathbb{T}^d, \\ (i\partial_t + \beta\Delta)v = -(\nabla \cdot \overline{w})u, \quad t \in (0,\infty),\ x \in \mathbb{R}^d \text{ or } \mathbb{T}^d, \\ (i\partial_t + \gamma\Delta)w = \nabla(u \cdot \overline{v}), \quad t \in (0,\infty),\ x \in \mathbb{R}^d \text{ or } \mathbb{T}^d, \\ (u(0,x), v(0,x), w(0,x)) = (u_0(x), v_0(x), w_0(x)), \quad x \in \mathbb{R}^d \text{ or } \mathbb{T}^d, \end{cases} \tag{1.11}$$

and the Cauchy problem of the nonlinear Schrödinger equations:

$$\begin{cases} (i\partial_t + \Delta)u = \partial_k(\overline{u}^m), \quad t \in (0,\infty),\ x \in \mathbb{R}^d \text{ or } \mathbb{T}^d, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d \text{ or } \mathbb{T}^d, \end{cases} \tag{1.12}$$

where α, β, $\gamma \in \mathbb{R}\backslash\{0\}$, $m \in \mathbb{N}$, $m \geq 2$, $1 \leq k \leq d$, $\partial_k = \partial/\partial x_k$, the unknown functions u, v, w in (1.11) are $\mathbb{C}^d$-valued and the unknown function u in (1.12) is $\mathbb{C}$-valued. The system (1.11) was introduced by Colin and Colin in [11] as a model of laser-plasma interaction. (1.12) is a mathematical model dealt by Grünrock ([23]). The scaling critical regularity for (1.11) is $s_c = d/2 - 1$ and for (1.12) is $s_c = d/2 - 1/(m-1)$.

First, we introduce some known results for related Schrödinger equations. The system (1.11) and equation (1.12) have derivative nonlinearity. A derivative loss arising from the nonlinearity makes the problem difficult. In fact, Mizohata ([55]) proved that a necessary condition for the L^2 well-posedness of the problem:

$$\begin{cases} i\partial_t u - \Delta u = b_1(x)\nabla u, \ t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^d \end{cases}$$

is the uniform bound

$$\sup_{x\in\mathbb{R}^d, \omega\in S^{d-1}, R>0} \left| \mathrm{Re} \int_0^R b_1(x+r\omega)\cdot \omega dr \right| < \infty.$$

Furthermore, Christ ([9]) proved that the flow map of the Cauchy problem:

$$\begin{cases} i\partial_t u - \partial_x^2 u = u\partial_x u, \ t \in \mathbb{R}, \ x \in \mathbb{R} \text{ or } \mathbb{T}, \\ u(0,x) = u_0(x), \ x \in \mathbb{R} \text{ or } \mathbb{T}, \end{cases} \tag{1.13}$$

is not continuous on $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$. While, Ozawa ([60]) proved that the local well-posedness of (1.13) in the space of all function $\phi \in H^1(\mathbb{R})$ satisfying the bounded condition

$$\sup_{x\in\mathbb{R}} \left| \int_{-\infty}^x \phi \right| < \infty.$$

Furthermore, he proved that if the initial data ϕ satisfies some condition, then the local solution can be extend globally in time and the solution scatters. For the Cauchy problem of the one dimensional derivative Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\lambda\partial_x(|u|^2 u), \ t \in \mathbb{R}, \ x \in \mathbb{R} \text{ or } \mathbb{T}, \\ u(0,x) = u_0(x), \ x \in \mathbb{R} \text{ or } \mathbb{T}, \end{cases} \tag{1.14}$$

Takaoka ([66]) proved the local well-posedness in $H^s(\mathbb{R})$ for $s \geq 1/2$ by using the gauge transform. This result was extended to global well-posedness ([16], [17], [54], [67]). While, ill-posedness of (1.14) was obtained for $s < 1/2$ ([3], [67]). For the periodic case, Herr ([30]) proved the local well-posedness of (1.14) in $H^s(\mathbb{T})$ for $s \geq$

1/2 by using the gauge transform and Win ([73]) proved the global well-posedness of (1.14) in $H^s(\mathbb{T})$ for $s > 1/2$. For more general problem:

$$\begin{cases} i\partial_t u - \Delta u = P(u, \overline{u}, \nabla u, \nabla \overline{u}),\ t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}^d, \\ P \text{ is a polynomial which has no constant and linear terms,} \end{cases} \tag{1.15}$$

there are many positive results for the well-posedness in the weighted Sobolev space ([1], [2], [7], [8], [48], [65]). Kenig, Ponce and Vega ([48]) also obtained that (1.15) is locally well-posed in H^s (without weight) for large enough s when P has no quadratic terms. For the equation (1.12), Grünrock ([23]) proved the global well-posedness in $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$ when $d = 1$, $m = 2$ and local well-posedness in $H^s(\mathbb{R}^d)$ and $H^s(\mathbb{T}^d)$ for $s > s_c$ when $d \geq 1$, $m + d \geq 4$. We extend his results to the following.

Main theorem 1.1. *Let* $s_c = d/2 - 1/(m-1)$.
(i) *Assume* $d \geq 1$, $m + d \geq 4$. *Then* (1.12) *is globally well-posed for small data in* $\dot{H}^{s_c}(\mathbb{R}^d)$ *(resp.* $H^s(\mathbb{R}^d)$ *for* $s \geq s_c$*). Furthermore, the solution scatters in* $\dot{H}^{s_c}(\mathbb{R}^d)$ *(resp.* $H^s(\mathbb{R}^d)$ *for* $s \geq s_c$*)*
(ii) *Assume* $d \geq 5$, $m = 2$. *Then* (1.12) *is locally well-posed for small data in* $H^{s_c}(\mathbb{T}^d)$.

Next, we introduce some known results for systems of quadratic nonlinear derivative Schrödinger equations. Ikeda, Katayama and Sunagawa ([36]) considered (1.11) for the nonperiodic case with null form nonlinearity and obtained the small data global existence and the scattering in the weighted Sobolev space for the dimension $d \geq 2$ under the condition $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) = 0$. While, Ozawa and Sunagawa ([61]) gave the examples of the quadratic derivative nonlinearity which causes the small data blow up for a system of Schrödinger equations with the nonperiodic setting. As the known result for (1.11), we introduce the work by Colin and Colin ([11]). They proved that the local existence of the solution of (1.11) in $H^s(\mathbb{R}^d)$ for $s > d/2 + 3$. We extend the results by Colin and Colin ([11]) and also prove the well-posedness for the periodic case. We put $\theta := \alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma)$ and $\kappa := (\alpha - \beta)(\alpha - \gamma)(\beta + \gamma)$. Note that if α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ and $\theta \geq 0$, then $\kappa \neq 0$. Our result for (1.11) is following.

Main theorem 1.2. *Let* $s_c = d/2 - 1$.
(i) *We assume that* α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ *satisfy* $\kappa \neq 0$ *if* $d \geq 4$, *and* $\theta > 0$ *if* $d = 2, 3$.

Then (1.11) *is globally well-posed for small data in* $\dot{H}^{s_c}(\mathbb{R}^d)$ *(resp.* $H^s(\mathbb{R}^d)$ *for* $s \geq s_c$*). Furthermore, the solution scatters in* $\dot{H}^{s_c}(\mathbb{R}^d)$ *(resp.* $H^s(\mathbb{R}^d)$ *for* $s \geq s_c$*).*
(ii) *We assume that* $d \geq 5$, $s \geq s_c$ *and* α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ *satisfy* $\theta > 0$ *and* α/β, $\beta/\gamma \in \mathbb{Q}$. *Then* (1.11) *is locally well-posed for small data in* $H^s(\mathbb{T}^d)$.

Furthermore, we obtain following subcritical results:

Main theorem 1.3. *Let* $s_c = d/2 - 1$ *and* α, β, $\gamma \in \mathbb{R}\backslash\{0\}$.
(i) *We assume that* $d \geq 4$, $s > s_c$ *and* α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ *satisfy* $(\alpha - \gamma)(\beta + \gamma) \neq 0$. *Then* (1.11) *is locally well-posed in* $H^s(\mathbb{R}^d)$.
(ii) *We assume that* $d = 2$, 3 *and* $s > s_c$ *if* $\theta > 0$, $s \geq 1$ *if* $\theta \leq 0$ *and* $\kappa \neq 0$, $s > 1$ *if* $\alpha = \beta$. *Then* (1.11) *is locally well-posed in* $H^s(\mathbb{R}^d)$.
(iii) *We assume that* $d = 1$ *and* $s \geq 0$ *if* $\theta > 0$, $s \geq 1$ *if* $\theta = 0$, $s \geq 1/2$ *if* $\theta < 0$ *and* $(\alpha - \gamma)(\beta + \gamma) \neq 0$. *Then* (1.11) *is locally well-posed in* $H^s(\mathbb{R}^d)$.
(iv) *We assume that* $d \geq 1$, $s > \max\{s_c, 0\}$ *and* α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ *satisfy* $\theta > 0$ *and* α/β, $\beta/\gamma \in \mathbb{Q}$. *Then* (1.11) *is locally well-posed in* $H^s(\mathbb{T}^d)$.

System (1.11) has the following conservation quantities (see Proposition 3.30):

$$M(u, v, w) := 2||u||_{L_x^2}^2 + ||v||_{L_x^2}^2 + ||w||_{L_x^2}^2,$$
$$H(u, v, w) := \alpha||\nabla u||_{L_x^2}^2 + \beta||\nabla v||_{L_x^2}^2 + \gamma||\nabla w||_{L_x^2}^2 + 2\mathrm{Re}(w, \nabla(u \cdot \overline{v}))_{L_x^2}.$$

By using the conservation law for M and H, we obtain the following result.

Main theorem 1.4.
(i) *Let* $d = 1$. *We assume that* α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ *satisfy* $\theta > 0$. *For every* $(u_0, v_0, w_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$, *we can extend the local* $L^2(\mathbb{R})$ *solution of* Main theorem 1.3 *globally in time.*
(ii) *We assume that* α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ *have the same sign and satisfy* $\kappa \neq 0$ *if* $d = 2$, 3 *and* $(\alpha - \gamma)(\beta + \gamma) \neq 0$ *if* $d = 1$. *There exists* $r > 0$ *such that for every* $(u_0, v_0, w_0) \in B_r(H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d))$, *we can extend the local* $H^1(\mathbb{R}^d)$ *solution of* Main theorem 1.3 *globally in time.*
(iii) *Let* $d = 1$, 2, 3. *We assume that* α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ *have the same sign and satisfy* $\theta > 0$. *There exists* $r > 0$ *such that for every* $(u_0, v_0, w_0) \in B_r(H^1(\mathbb{T}^d) \times H^1(\mathbb{T}^d) \times H^1(\mathbb{T}^d))$, *we can extend the local* $H^1(\mathbb{T}^d)$ *solution of* Main theorem 1.3 *globally in time.*

While, we obtain the negative result as follows.

Main theorem 1.5. *Let $d \geq 1$ and α, β, $\gamma \in \mathbb{R}\backslash\{0\}$. We assume $s \in \mathbb{R}$ if $(\alpha-\gamma)(\beta+\gamma)=0$, $s<1$ if $\theta=0$, and $s<1/2$ if $\theta<0$. Then the flow map of (1.11) is not C^2 in $H^s(\mathbb{R}^d)$.*

The main tools of our results for the scaling critical case are U^p space and V^p space, which are applied to prove the well-posedness and the scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch ([25], [26]). For the equation (1.12), Grünrock used the Bourgain space to construct the solution and proved the well-posedness in the scaling subcritical Sobolev space. But as mentioned in previous section, the well-posedness of (1.12) in the scaling critical Sobolev space cannot be obtained by using the Bourgain space. To overcome this difficulty, we apply the U^2, V^2 type spaces to construct the solution. Because of such reason, we obtain the well-posedness of (1.12) in the scaling critical Sobolev space. We give the proof of Main theorem 1.1 (i) for $d \geq 2$, $m=2$ in Chapter 3, the proof of (i) for $d \geq 1$, $m \geq 3$ in Chapter 5 and the proof of (ii) in Chapter 4.

While even if we use the U^2, V^2 type spaces, we cannot obtain the well-posedness of (1.11) in the scaling critical Sobolev space generally since there are the cases that the resonance arises from the nonlinear interaction. We study the resonance and the nonresonance for (1.11) and prove that if α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ satisfy $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$, then the resonance never arises (see Lemma 3.16). So, $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$ is the nonresonance condition for (1.11). By applying both the nonresonance condition and U^2, V^2 type spaces, we succeed in proving the well-posedness of (1.11) for the nonperiodic case in the scaling critical Sobolev space. We remark that Oh ([59]) also studied the resonance and the nonresonance for the system of KdV equations and proved that the regularity for the well-posedness of the system under the nonresonance condition is lower than the regularity under the resonance condition. We give the proof of Main theorems 1.2– 1.5 for the nonperiodic case in Chapter 3.

For the periodic case of (1.11), there are also the other difficulty that the number of admissible pairs of the Strichartz estimate for the periodic function is less than the nonperiodic function. To overcome this difficulty, we show the bilinear estimate

$$\begin{aligned}&||P_{N_3}(P_{N_1}u_1 \cdot P_{N_2}u_2)||_{L^2(\mathbb{T}_{|\sigma-1|}\times\mathbb{T}^d)} \\ &\lesssim N_{\min}^s \left(\frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}}\right)^\delta ||P_{N_1}u_1||_{Y^0_{\sigma_1}} ||P_{N_2}u_2||_{Y^0_{\sigma_2}}\end{aligned} \tag{1.16}$$

and use it instead of the Strichartz estimate (see Proposition 4.16). The estimate

(1.16) for the case $N_1 \sim N_3 \gtrsim N_2$ is proved by Wang ([71]). Therefore, we will prove (1.16) for the case $N_1 \sim N_2 \gg N_3$. But the proof is difference from Wang's proof on the several points. We give the proof of Main theorems 1.2– 1.4 for the periodic case in Chapter 4.

1.3 Review for the higher order KdV type equations

We consider the Cauchy problem of the periodic high order KdV type equations;

$$\begin{cases} \partial_t u + (-1)^{k+1}\partial_x^{2k+1} u + \frac{1}{2}\partial_x(u^2) = 0, & (t,x) \in (0,\infty)\times\mathbb{T}, \\ u(0,x) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{1.17}$$

where $k \in \mathbb{N}$ and the unknown function u is real valued.

When $k = 1$, the equation (1.17) is called "KdV equation". We first introduce some known results for the KdV equation. In [5], Bourgain introduced a new method called "Fourier restriction norm method" and proved that the KdV equation is locally well-posed in $L^2(\mathbb{T})$. In [47], Kenig, Ponce and Vega refined a bilinear estimate used in the Fourier restriction norm method, and proved that the KdV equation is locally well-posed in $H^s(\mathbb{T})$ for $s \geq -1/2$. In [18], by using the local well-posedness result and the almost conservation law, Colliander, Keel, Staffilani, Takaoka and Tao obtained that the KdV equation is the globally well-posed in $H^s(\mathbb{T})$ for $s \geq -1/2$. Their method is called "I-method". On the other hand, In [10], Christ, Colliander and Tao proved that the KdV equation is ill-posed in $H^s(\mathbb{T})$ for $-2 < s < -1/2$ in weak sense. Local well-posedness of the non-periodic KdV equation also was studied by many people before Bourgain's work ([4], [20], [41], [43], [44]) and after Bourgain's work ([18], [24], [45], [47], [49], [50], [58], [69]).

Next, we introduce some known results for the fifth order KdV type equations

$$\partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \partial_x(u^2) = 0 \tag{1.18}$$

and

$$\partial_t u - \partial_x^5 u - 30u^2\partial_x u + 20\partial_x u\partial_x^2 u + 10u\partial_x^3 u = 0. \tag{1.19}$$

Especially, (1.18) is called "Kawahara equation". Local well-posedness of these equations are studied for non-periodic case. For the known results of the non-periodic Kawahara equation, see [6], [19], [38], [70] and the equation (1.19), see [53], [63].

Now, we introduce the high order nonlinear dispersive equations related to (1.17). In [46], Kenig, Ponce and Vega studied the high order nonlinear dispersive equations

$$\partial_t u + \partial_x^{2k+1} u + P(u, \partial_x u, \cdots, \partial_x^{2k} u) = 0, \tag{1.20}$$

where P is a polynomial without constant and linear terms. They proved that (1.20) is LWP in $L^2(|x|^m dx) \cap H^s(\mathbb{R})$, where $s > 0$ and $m \in \mathbb{Z}^+$ are sufficiently large. In [62], Pilod proved that (6.4) with

$$P(u, \partial_x u, \cdots, \partial_x^{2k} u) = \sum_{0 \le k_1 + k_2 \le 2k} a_{k_1,k_2} \partial_x^{k_1} u \partial_x^{k_2} u$$

is locally well-posed in $H^s(\mathbb{R}) \cap H^{s-2k}(x^2 dx)$ for $s \in \mathbb{N}$ and $s > 2k + 1/4$. He also proved some ill-posedness results for (1.20).

We return to introduce the known result for (1.17) for general $k \in \mathbb{N}$. In [22], Gorsky and Himonas proved that (1.17) is locally well-posed in $H^s(\mathbb{T})$ for $s \ge -1/2$ by an argument similar to [47]. Our result is an extension of the result by Gorsky and Himonas as follows.

Main theorem 1.6. *Let $k \in \mathbb{N}$. If $s \ge -k/2$ then* (1.17) *is locally well-posed in $H^s(\mathbb{T})$.*

Remark 1.1. *After our work, Kato* ([40]) *extended the result in* Main theorem 1.6 *for $k = 2$ to local well-posedness in $H^s(\mathbb{T})$ for $s \ge -3/2$ and global well-posedness in $H^s(\mathbb{T})$ for $s \ge -1$.*

A bilinear estimate plays an important role to prove well-posedness of (1.17). Gorsky and Himonas derived the following bilinear estimate for $s \ge -1/2$:

$$||\partial_x(uv)||_{X^{s,-1/2}} \le C||u||_{X^{s,1/2}}||v||_{X^{s,1/2}}, \tag{1.21}$$

where

$$||u||_{X^{s,b}} := ||\langle \tau - \xi^{2k+1} \rangle^b \langle \xi \rangle^s \widetilde{u}||_{l_\xi^2 L_\tau^2}.$$

But as mentioned in [22], the estimate (1.21) with $s < -1/2$ has been open problem. We extend (1.21) to prove Main theorem 1.6 as follows.

Theorem 1.2. *Let $k \in \mathbb{N}$. For $s \ge -k/2$, the bilinear estimate* (1.21) *holds.*

On the other hand, we also obtain negative result for $s < -k/2$.

Theorem 1.3. *Let $k \in \mathbb{N}$. For any $s < -k/2$, the bilinear estimate* (1.21) *fails.*

Remark 1.4. *By* Theorems 1.2, 1.3*,* $s = -k/2$ *is optimal regularity for the bilinear estimate* (1.21)*. But this does not imply ill-posedness of* (1.17) *for* $s < -k/2$*.*

The bilinear estimate (1.21) for $s = -k/2$ can be written as

$$\begin{aligned} &||\langle\xi\rangle^{1-k/2}\langle\tau - \xi^{2k+1}\rangle^{-1/2}\widetilde{u} * \widetilde{v}||_{l_\xi^2 L_\tau^2} \\ \lesssim\ &||\langle\xi\rangle^{-k/2}\langle\tau - \xi^{2k+1}\rangle^{1/2}\widetilde{u}||_{l_\xi^2 L_\tau^2}||\langle\xi\rangle^{-k/2}\langle\tau - \xi^{2k+1}\rangle^{1/2}\widetilde{v}||_{l_\xi^2 L_\tau^2}, \end{aligned}$$

where

$$\widetilde{u} * \widetilde{v}(\tau, \xi) = \frac{1}{2\pi}\sum_{\xi=\xi_1+\xi_2}\int_{\tau=\tau_1+\tau_2}\widetilde{u}(\tau_1, \xi_1)\widetilde{v}(\tau_2, \xi_2)d\tau_1.$$

We note that the most difficult region to prove this estimate is $|\xi_1| \sim |\xi_2| \gg |\xi|$. Gorsky and Himonas used the estimate

$$|\xi^{2k+1} - \xi_1^{2k+1} - \xi_2^{2k+1}| \gtrsim |\xi\xi_1\xi_2| \cdot |\xi|^{2k-2} \qquad (1.22)$$

to prove (1.21). On the other hand, we use the refined estimate

$$|\xi^{2k+1} - \xi_1^{2k+1} - \xi_2^{2k+1}| \sim |\xi\xi_1\xi_2| \max\{|\xi|, |\xi_1|, |\xi_2|\}^{2k-2},$$

which is better estimate than (1.22) in the region $|\xi_1| \sim |\xi_2| \gg |\xi|$ (see Lemma 6.8). Because of such reason, we could improve the bilinear estimate. We give the proof of Main theorem 1.6 in Chapter 6.

Chapter 2

Function spaces

In this chapter, we define the U^p space and the V^p space, and introduce the properties of these spaces which are proved in [25] and [26]. These spaces will be used in Chapter 3, 4, 5. Throughout this section let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$.

We define the set of finite partitions $\mathcal{Z}$ as

$$\mathcal{Z} := \left\{ \{t_k\}_{k=0}^{K} | K \in \mathbb{N}, -\infty < t_0 < t_1 < \cdots < t_K \leq \infty \right\}$$

and $t_K = \infty$, we put $v(t_K) := 0$ for all functions $v : \mathbb{R} \to \mathcal{H}$.

Definition 2.1. *Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^{K} \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset \mathcal{H}$ with $\sum_{k=0}^{K-1} ||\phi_k||_{\mathcal{H}}^p = 1$ we call the function $a : \mathbb{R} \to \mathcal{H}$ given by*

$$a(t) = \sum_{k=1}^{K} \mathbf{1}_{[t_{k-1}, t_k)}(t) \phi_{k-1}$$

a "U^p-atom". Furthermore, we define the atomic space

$$U^p(\mathbb{R}; \mathcal{H}) := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \middle| a_j : U^p\text{-atom},\ \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with the norm

$$||u||_{U^p(\mathbb{R};\mathcal{H})} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \middle| u = \sum_{j=1}^{\infty} \lambda_j a_j,\ a_j : U^p\text{-atom},\ \lambda_j \in \mathbb{C} \right\}.$$

Definition 2.2. *Let $1 \leq p < \infty$. We define the space of the bounded p-variation*

$$V^p(\mathbb{R}; \mathcal{H}) := \{ v : \mathbb{R} \to \mathcal{H} | \ ||v||_{V^p(\mathbb{R};\mathcal{H})} < \infty \}$$

with the norm

$$||v||_{V^p(\mathbb{R};\mathcal{H})} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^{K} ||v(t_k) - v(t_{k-1})||_{\mathcal{H}}^p \right)^{1/p}.$$

Likewise, let $V^p_{-,rc}(\mathbb{R};\mathcal{H})$ *denote the closed subspace of all right-continuous functions* $v \in V^p(\mathbb{R};\mathcal{H})$ *with* $\lim_{t\to-\infty} v(t) = 0$*, endowed with the same norm* $||\cdot||_{V^p(\mathbb{R};\mathcal{H})}$.

Proposition 2.3 ([25] Proposition 2,2, 2.4, Corollary 2.6)**.** *Let* $1 \le p < q < \infty$.
(i) $U^p(\mathbb{R};\mathcal{H})$, $V^p(\mathbb{R};\mathcal{H})$ *and* $V^p_{-,rc}(\mathbb{R};\mathcal{H})$ *are Banach spaces.*
(ii) *Every* $u \in U^p(\mathbb{R};\mathcal{H})$ *is right-continuous as* $u : \mathbb{R} \to L^2$.
(iii) *For Every* $u \in U^p(\mathbb{R};\mathcal{H})$, $\lim_{t\to-\infty} u(t) = 0$ *and* $\lim_{t\to\infty} u(t)$ *exists in* L^2.
(iv) *For Every* $v \in V^p(\mathbb{R};\mathcal{H})$, $\lim_{t\to-\infty} v(t)$ *and* $\lim_{t\to\infty} v(t)$ *exist in* L^2.
(v) *The embeddings* $U^p(\mathbb{R};\mathcal{H}) \hookrightarrow V^p_{-,rc}(\mathbb{R};\mathcal{H}) \hookrightarrow U^q(\mathbb{R};\mathcal{H}) \hookrightarrow L^\infty_t(\mathbb{R};\mathcal{H})$ *are continuous.*

Theorem 2.4 ([25] Proposition 2.10, Remark 2.12)**.** *Let* $1 < p < \infty$ *and* $1/p + 1/p' = 1$*. If* $u \in V^1_{-,rc}(\mathbb{R};\mathcal{H})$ *be absolutely continuous on every compact intervals, then*

$$||u||_{U^p(\mathbb{R};\mathcal{H})} = \sup_{v \in V^{p'}(\mathbb{R};\mathcal{H}), ||v||_{V^{p'}(\mathbb{R};\mathcal{H})}=1} \left| \int_{-\infty}^{\infty} (u'(t), v(t))_{\mathcal{H}} dt \right|.$$

Definition 2.5. *Let* $1 \le p < \infty$*. For the operator* $\mathcal{L}$ *given as* $\mathcal{L} = p(-i\nabla)$ *for a real coefficients polynomial* p*, we define*

$$U^p_{\mathcal{L}}\mathcal{H} := \{u : \mathbb{R} \to \mathcal{H} |\ e^{-it\mathcal{L}} u \in U^p(\mathbb{R};\mathcal{H})\}$$

with the norm $||u||_{U^p_{\mathcal{L}}\mathcal{H}} := ||e^{-it\mathcal{L}} u||_{U^p(\mathbb{R};\mathcal{H})}$,

$$V^p_{\mathcal{L}}\mathcal{H} := \{v : \mathbb{R} \to \mathcal{H} |\ e^{-it\mathcal{L}} v \in V^p_{-,rc}(\mathbb{R};\mathcal{H})\}$$

with the norm $||v||_{V^p_{\mathcal{L}}\mathcal{H}} := ||e^{-it\mathcal{L}} v||_{V^p(\mathbb{R};\mathcal{H})}$.

Remark 2.6. *We note that* $||\overline{u}||_{U^p_{\mathcal{L}}\mathcal{H}} = ||u||_{U^p_{-\mathcal{L}}\mathcal{H}}$ *and* $||\overline{v}||_{V^p_{\mathcal{L}}\mathcal{H}} = ||v||_{V^p_{-\mathcal{L}}\mathcal{H}}$.

Proposition 2.7 ([25] Corollary 2.18)**.** *Let* $1 < p < \infty$*. We have*

$$||Q^{\mathcal{L}}_M u||_{L^p_t(\mathbb{R};\mathcal{H})} \lesssim M^{-1/p} ||u||_{V^p_{\mathcal{L}}\mathcal{H}},\ ||Q^{\mathcal{L}}_{\ge M} u||_{L^p_t(\mathbb{R};\mathcal{H})} \lesssim M^{-1/p} ||u||_{V^p_{\mathcal{L}}\mathcal{H}}, \tag{2.1}$$

$$||Q^{\mathcal{L}}_{<M} u||_{V^p_{\mathcal{L}}\mathcal{H}} \lesssim ||u||_{V^p_{\mathcal{L}}\mathcal{H}},\quad ||Q^{\mathcal{L}}_{\ge M} u||_{V^p_{\mathcal{L}}\mathcal{H}} \lesssim ||u||_{V^p_{\mathcal{L}}\mathcal{H}}, \tag{2.2}$$

$$||Q^{\mathcal{L}}_{<M} u||_{U^p_{\mathcal{L}}\mathcal{H}} \lesssim ||u||_{U^p_{\mathcal{L}}\mathcal{H}},\quad ||Q^{\mathcal{L}}_{\ge M} u||_{U^p_{\mathcal{L}}\mathcal{H}} \lesssim ||u||_{U^p_{\mathcal{L}}\mathcal{H}}. \tag{2.3}$$

Proposition 2.8 ([25] Proposition 2.19)**.** *Let*

$$T_0 : \mathcal{H} \times \cdots \times \mathcal{H} \to L^1_{loc}$$

be a m-linear operator and $I \subset \mathbb{R}$ be an interval. Assume that for some $1 \leq p, q < \infty$

$$||T_0(e^{it\mathcal{L}_1}\phi_1, \cdots, e^{it\mathcal{L}_m}\phi_m)||_{L^p_t(I:L^q_x)} \lesssim \prod_{i=1}^{m} ||\phi_i||_{\mathcal{H}}.$$

Then, there exists $T : U^p_{\mathcal{L}_1}\mathcal{H} \times \cdots \times U^p_{\mathcal{L}_m}\mathcal{H} \to L^p_t(I; L^q_x)$ satisfying

$$||T(u_1, \cdots, u_m)||_{L^p_t(I;L^q_x)} \lesssim \prod_{i=1}^{m} ||u_i||_{U^p_{\mathcal{L}_i}\mathcal{H}}$$

such that $T(u_1, \cdots, u_m)(t)(x) = T_0(u_1(t), \cdots, u_m(t))(x)$ a.e.

Proposition 2.9 ([25] Proposition 2.20)**.** *Let $q > 1$, E be a Banach space and $T : U^q_{\mathcal{L}}\mathcal{H} \to E$ be a bounded, linear operator with $||Tu||_E \leq C_q||u||_{U^q_{\mathcal{L}}\mathcal{H}}$ for all $u \in U^q_{\mathcal{L}}\mathcal{H}$. In addition, assume that for some $1 \leq p < q$ there exists $C_p \in (0, C_q]$ such that the estimate $||Tu||_E \leq C_p||u||_{U^p_{\mathcal{L}}\mathcal{H}}$ holds true for all $u \in U^p_{\mathcal{L}}\mathcal{H}$. Then, T satisfies the estimate*

$$||Tu||_E \lesssim C_p \left(1 + \ln \frac{C_q}{C_p}\right) ||u||_{V^p_{\mathcal{L}}\mathcal{H}}, \quad u \in V^p_{\mathcal{L}}\mathcal{H},$$

where implicit constant depends only on p and q.

Chapter 3

System of quadratic derivative nonlinear Schrödinger equations on $\mathbb{R}^d$

3.1 Review for results

We consider the Cauchy problem of the system of Schrödinger equations:

$$\begin{cases} (i\partial_t + \alpha\Delta)u = -(\nabla \cdot w)v, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \\ (i\partial_t + \beta\Delta)v = -(\nabla \cdot \overline{w})u, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \\ (i\partial_t + \gamma\Delta)w = \nabla(u \cdot \overline{v}), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \\ (u(0,x), v(0,x), w(0,x)) = (u_0(x), v_0(x), w_0(x)), \quad x \in \mathbb{R}^d \end{cases} \tag{3.1}$$

where α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ and the unknown functions u, v, w are d-dimensional complex vector valued. The system (3.1) was introduced by Colin and Colin in [11] as a model of laser-plasma interaction. (3.1) is invariant under the following scaling transformation:

$$A_\lambda(t,x) = \lambda^{-1}A(\lambda^{-2}t, \lambda^{-1}x) \ \ (A = (u,v,w)), \tag{3.2}$$

and the scaling critical regularity is $s_c = d/2 - 1$. The aim of this chapter is to prove the well-posedness and the scattering of (3.1) in the scaling critical Sobolev space.

First, we introduce some known results for related problems. The system (3.1) has quadratic nonlinear terms which contains a derivative. A derivative loss arising from the nonlinearity makes the problem difficult. In fact, Mizohata ([55]) proved that a necessary condition for the L^2 well-posedness of the problem:

$$\begin{cases} i\partial_t u - \Delta u = b_1(x)\nabla u, \ t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^d \end{cases}$$

is the uniform bound

$$\sup_{x\in\mathbb{R}^d, \omega\in S^{d-1}, R>0} \left| \mathrm{Re} \int_0^R b_1(x+r\omega)\cdot\omega dr \right| < \infty.$$

Furthermore, Christ ([9]) proved that the flow map of the Cauchy problem:

$$\begin{cases} i\partial_t u - \partial_x^2 u = u\partial_x u, \ t \in \mathbb{R}, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \ x \in \mathbb{R} \end{cases} \tag{3.3}$$

is not continuous on $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. While, there are positive results for the Cauchy problem:

$$\begin{cases} i\partial_t u - \Delta u = \overline{u}(\nabla \cdot \overline{u}), \ t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^d. \end{cases} \tag{3.4}$$

Grünrock ([23]) proved that (3.4) is globally well-posed in $L^2(\mathbb{R})$ for $d = 1$ and locally well-posed in $H^s(\mathbb{R}^d)$ for $d \geq 2$ and $s > s_c$ $(= d/2 - 1)$. For more general problem:

$$\begin{cases} i\partial_t u - \Delta u = P(u, \overline{u}, \nabla u, \nabla \overline{u}), \ t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^d, \\ P \text{ is a polynomial which has no constant and linear terms,} \end{cases} \tag{3.5}$$

there are many positive results for the well-posedness in the weighted Sobolev space ([1], [2], [7], [8], [48], [65]). Kenig, Ponce and Vega ([48]) also obtained that (3.5) is locally well-posed in $H^s(\mathbb{R}^d)$ (without weight) for large enough s when P has no quadratic terms.

The Benjamin–Ono equation:

$$\partial_t u + H\partial_x^2 u = u\partial_x u, \ (t,x) \in \mathbb{R} \times \mathbb{R} \tag{3.6}$$

is also related to the quadratic derivative nonlinear Schrödinger equation. It is known that the flow map of (3.6) is not uniformly continuous on $H^s(\mathbb{R})$ for $s > 0$ ([51]). But the Benjamin–Ono equation has better structure than the equation

(3.3). Actually, Tao ([68]) proved that (3.6) is globally well-posed in H^1 by using the gauge transform. Furthermore, Ionescu and Kenig ([37]) proved that (3.6) is globally well-posed in $H^s_r(\mathbb{R})$ for $s \geq 0$, where $H^s_r(\mathbb{R})$ is the Banach space of the all real valued function $f \in H^s(\mathbb{R})$.

Next, we introduce some known results for systems of quadratic nonlinear derivative Schrödinger equations. Ikeda, Katayama and Sunagawa ([36]) considered (3.1) with null form nonlinearity and obtained the small data global existence and the scattering in the weighted Sobolev space for the dimension $d \geq 2$ under the condition $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) = 0$. While, Ozawa and Sunagawa ([61]) gave the examples of the quadratic derivative nonlinearity which causes the small data blow up for a system of Schrödinger equations. As the known result for (3.1), we introduce the work by Colin and Colin ([11]). They proved that the local existence of the solution of (3.1) in $H^s(\mathbb{R}^d)$ for $s > d/2 + 3$. There are also some known results for a system of Schrödinger equations with no derivative nonlinearity ([12], [13], [14], [28], [29]). Our results are an extension of the results by Colin and Colin ([11]) and Grünrock ([23]).

Now, we give the main results in this chapter. To begin with, we define the function spaces to construct the solution.

Definition 3.1. *Let* s, $\sigma \in \mathbb{R}$.
(i) *We define* $\dot{Z}^s_\sigma := \{u \in C(\mathbb{R}; \dot{H}^s(\mathbb{R}^d)) \cap U^2_{\sigma\Delta}L^2 |\ ||u||_{\dot{Z}^s_\sigma} < \infty\}$ *with the norm*

$$||u||_{\dot{Z}^s_\sigma} := \left(\sum_N N^{2s} ||P_N u||^2_{U^2_{\sigma\Delta}L^2}\right)^{1/2}.$$

(ii) *We define* $Z^s_\sigma := \{u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap U^2_{\sigma\Delta}L^2 |\ ||u||_{Z^s_\sigma} < \infty\}$ *with the norm*

$$||u||_{Z^s_\sigma} := ||u||_{\dot{Z}^0_\sigma} + ||u||_{\dot{Z}^s_\sigma}.$$

(iii) *We define* $\dot{Y}^s_\sigma := \{u \in C(\mathbb{R}; \dot{H}^s(\mathbb{R}^d)) \cap V^2_{\sigma\Delta}L^2 |\ ||u||_{\dot{Y}^s_\sigma} < \infty\}$ *with the norm*

$$||u||_{\dot{Y}^s_\sigma} := \left(\sum_N N^{2s} ||P_N u||^2_{V^2_{\sigma\Delta}L^2}\right)^{1/2}.$$

(iv) *We define* $Y^s_\sigma := \{u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap V^2_{\sigma\Delta}L^2 |\ ||u||_{Y^s_\sigma} < \infty\}$ *with the norm*

$$||u||_{Y^s_\sigma} := ||u||_{\dot{Y}^0_\sigma} + ||u||_{\dot{Y}^s_\sigma}.$$

Remark 3.2 ([25] Remark 2.23)**.** *Let E be a Banach space of continuous functions $f : \mathbb{R} \to \mathcal{H}$, for some Hilbert space $\mathcal{H}$. We also consider the corresponding restriction space to the interval $I \subset \mathbb{R}$ by*

$$E(I) = \{u \in C(I, \mathcal{H}) | \exists v \in E \ s.t. \ v(t) = u(t), \ t \in I\}$$

endowed with the norm $||u||_{E(I)} = \inf\{||v||_E | v(t) = u(t), \ t \in I\}$. Obviously, $E(I)$ is also a Banach space.

For an interval $I \subset \mathbb{R}$, We define $\dot{X}^s(I) := \dot{Z}^s_\alpha(I) \times \dot{Z}^s_\beta(I) \times \dot{Z}^s_\gamma(I)$ and $X^s(I) := Z^s_\alpha(I) \times Z^s_\beta(I) \times Z^s_\gamma(I)$. Furthermore, we put $\theta := \alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma)$ and $\kappa := (\alpha - \beta)(\alpha - \gamma)(\beta + \gamma)$. Note that if α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ and $\theta \geq 0$, then $\kappa \neq 0$.

Theorem 3.3. *Let $s_c = d/2 - 1$.*
(i) *We assume that α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ satisfy $\kappa \neq 0$ if $d \geq 4$, and $\theta > 0$ if $d = 2, 3$. Then* (3.1) *is globally well-posed for small data in $\dot{H}^{s_c}(\mathbb{R}^d)$. More precisely, there exists $r > 0$ such that for all initial data $(u_0, v_0, w_0) \in B_r(\dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d))$, there exists a solution*

$$(u, v, w) \in \dot{X}^{s_c}_r([0, \infty)) \subset C([0, \infty); \dot{H}^{s_c}(\mathbb{R}^d))$$

of the system (3.1) *on $(0, \infty)$. Such solution is unique in $\dot{X}^{s_c}_r([0, \infty))$ which is a closed subset of $\dot{X}^{s_c}([0, \infty))$* (see (3.49))*. Moreover, the flow map*

$$S_+ : B_r(\dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d)) \ni (u_0, v_0, w_0) \mapsto (u, v, w) \in \dot{X}^{s_c}([0, \infty))$$

is Lipschitz continuous.
(ii) *The statement in* (i) *remains valid if we replace the space $\dot{H}^{s_c}(\mathbb{R}^d)$, $\dot{X}^{s_c}([0, \infty))$ and $\dot{X}^{s_c}_r([0, \infty))$ by $H^s(\mathbb{R}^d)$, $X^s([0, \infty))$ and $X^s_r([0, \infty))$ for $s \geq s_c$.*

Remark 3.4. *Due to the time reversibility of the system* (3.1)*, the above theorems also hold in corresponding intervals $(-\infty, 0)$. We denote the flow map with respect to $(-\infty, 0)$ by S_-.*

Corollary 3.5. *$s_c = d/2 - 1$.*
(i) *We assume that α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ satisfy $\kappa \neq 0$ if $d \geq 4$, and $\theta > 0$ if $d = 2, 3$. Let $r > 0$ be as in* Theorem 3.3. *For every $(u_0, v_0, w_0) \in B_r(\dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d))$, there exists $(u_\pm, v_\pm, w_\pm) \in \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d)$ such that*

$$S_\pm(u_0, v_0, w_0) - (e^{it\alpha\Delta}u_\pm, e^{it\beta\Delta}v_\pm, e^{it\gamma\Delta}w_\pm) \to 0$$
$$\text{in } \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d) \text{ as } t \to \pm\infty.$$

(ii) *The statement in* (i) *remains valid if we replace the space* $\dot{H}^{s_c}(\mathbb{R}^d)$ *by* $H^s(\mathbb{R}^d)$ *for* $s \geq s_c$.

Theorem 3.6. *Let* $s_c = d/2 - 1$ *and* $\alpha,\ \beta,\ \gamma \in \mathbb{R}\backslash\{0\}$.
(i) *Let* $d \geq 4$. *We assume* $(\alpha-\gamma)(\beta+\gamma) \neq 0$ *and* $s > s_c$. *Then* (3.1) *is locally well-posed in* $H^s(\mathbb{R}^d)$. *More precisely, for any* $r > 0$ *and for all initial data* $(u_0, v_0, w_0) \in B_r(H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d))$, *there exist* $T = T(r) > 0$ *and a solution*

$$(u, v, w) \in X^s([0,T]) \subset C\left([0,T]; H^s(\mathbb{R}^d)\right)$$

of the system (3.1) *on* $(0, T]$. *Such solution is unique in* $X^s_r([0,T])$ *which is a closed subset of* $X^s([0,T])$. *Moreover, the flow map*

$$S_+ : B_r(H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)) \ni (u_0, v_0, w_0) \mapsto (u, v, w) \in X^s([0,T])$$

is Lipschitz continuous.
(ii) *Let* $d = 2,\ 3$. *We assume* $s > s_c$ *if* $\theta > 0$, $s \geq 1$ *if* $\theta \leq 0$ *and* $\kappa \neq 0$, *and* $s > 1$ *if* $\alpha = \beta$. *Then the statement in* (i) *remains valid.*
(iii) *Let* $d = 1$. *We assume* $s \geq 0$ *if* $\theta > 0$, $s \geq 1$ *if* $\theta = 0$, *and* $s \geq 1/2$ *if* $\theta < 0$ *and* $(\alpha-\gamma)(\beta+\gamma) \neq 0$. *Then the statement in* (i) *remains valid.*

Remark 3.7. *For the case* $d = 1$, $1 > s \geq 1/2$, $\theta < 0$ *and* $(\alpha-\gamma)(\beta+\gamma) \neq 0$, *we prove the well-posedness as* $X^s([0,T]) = X^{s,b}_\alpha([0,T]) \times X^{s,b}_\beta([0,T]) \times X^{s,b}_\gamma([0,T])$, *where* $X^{s,b}_\sigma$ *denotes the standard Bourgain space which is the completion of the Schwarz space with respect to the norm* $||u||_{X^{s,b}_\sigma} := ||\langle\xi\rangle^s\langle\tau+\sigma\xi^2\rangle^b\widetilde{u}||_{L^2_{\tau\xi}}$ *(see* Section 3.8 *).*

System (3.1) has the following conservation quantities (see Proposition 3.30):

$$M(u,v,w) := 2||u||^2_{L^2_x} + ||v||^2_{L^2_x} + ||w||^2_{L^2_x},$$
$$H(u,v,w) := \alpha||\nabla u||^2_{L^2_x} + \beta||\nabla v||^2_{L^2_x} + \gamma||\nabla w||^2_{L^2_x} + 2\mathrm{Re}(w, \nabla(u\cdot\overline{v}))_{L^2_x}.$$

By using the conservation law for M and H, we obtain the following result.

Theorem 3.8.
(i) *Let* $d = 1$ *and assume that* $\alpha,\ \beta,\ \gamma \in \mathbb{R}\backslash\{0\}$ *satisfy* $\theta > 0$. *For every* $(u_0, v_0, w_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$, *we can extend the local* L^2 *solution of* Theorem 3.6 *globally in time.*
(ii) *We assume that* $\alpha,\ \beta,\ \gamma \in \mathbb{R}\backslash\{0\}$ *have the same sign and satisfy* $\kappa \neq 0$ *if* $d = 2,\ 3$ *and* $(\alpha-\gamma)(\beta+\gamma) \neq 0$ *if* $d = 1$. *There exists* $r > 0$ *such that for every* $(u_0, v_0, w_0) \in B_r(H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d))$, *we can extend the local* H^1 *solution of* Theorem 3.6 *globally in time.*

While, we obtain the negative result as follows.

Theorem 3.9. *Let $d \geq 1$ and α, β, $\gamma \in \mathbb{R}\backslash\{0\}$. We assume $s \in \mathbb{R}$ if $(\alpha-\gamma)(\beta+\gamma) = 0$, $s < 1$ if $\theta = 0$, and $s < 1/2$ if $\theta < 0$. Then the flow map of* (3.1) *is not C^2 in $H^s(\mathbb{R}^d)$.*

Furthermore, for the equation (3.4), we obtain the following result.

Theorem 3.10. *Let $d \geq 2$ and $s_c = d/2 - 1$. Then, the equation* (3.4) *is globally well-posed for small data in $\dot{H}^{s_c}(\mathbb{R}^d)$* (resp. *$H^s(\mathbb{R}^d)$ for $s \geq s_c$) and the solution converges to a free solution in $\dot{H}^{s_c}(\mathbb{R}^d)$* (resp. *$H^s(\mathbb{R}^d)$ for $s \geq s_c$) asymptotically in time.*

Remark 3.11. *The results by Grünrock* ([23]) *are not contained the critical case $s = s_c$ and global property of the solution. In this sense,* Theorem 3.10 *is the extension of the results by Grünrock* ([23]).

The main tools of our results are U^p space and V^p space which are applied to prove the well-posedness and scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch ([25], [26]). After their work, U^p space and V^p space are used to prove the well-posedness of the 3D periodic quintic nonlinear Schrödinger equation at the scaling critical regularity by Herr, Tataru and Tzvetkov ([31]) and to prove the well-posedness and the scattering of the quadratic Klein-Gordon system at the scaling critical regularity by Schottdorf ([64]).

The rest of this chapter is planned as follows. In Sections 2, 3 and 4, we will give the bilinear and trilinear estimates which will be used to prove the well-posedness. In Section 5, we will give the proof of the well-posedness and the scattering (Theorems 3.3, 3.6, 3.10 and Corollary 3.5). In Section 6, we will give the a priori estimates and show Theorem 3.8. In Section 7, we will give the proof of C^2-ill-posedness (Theorem 3.9). In Section 8, we will give the proof of the bilinear estimates for the standard 1-dimensional Bourgain norm under the condition $(\alpha - \gamma)(\beta + \gamma) \neq 0$ and $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) \neq 0$.

3.2 Strichartz and bilinear Strichartz estimates

In this section, implicit constants in $\ll$ actually depend on σ_1, σ_2. First, we give the Strichartz estimate for the Schrödinger equation.

Proposition 3.12 (Strichartz estimate)**.** *Let $\sigma \in \mathbb{R}\backslash\{0\}$ and (p,q) be an admissible pair of exponents for the Schrödinger equation, i.e. $2 \le q \le 2d/(d-2)$ ($2 \le q < \infty$ if $d=2$, $2 \le q \le \infty$ if $d=1$), $2/p = d(1/2-1/q)$. Then, we have*

$$||e^{it\sigma\Delta}\varphi||_{L_t^pL_x^q} \lesssim ||\varphi||_{L_x^2}$$

for any $\varphi \in L^2(\mathbb{R}^d)$.

By Proposition 2.8 and 3.12, we have following:

Corollary 3.13. *Let $\sigma \in \mathbb{R}\backslash\{0\}$ and (p,q) be an admissible pair of exponents for the Schrödinger equation, i.e. $2 \le q \le 2d/(d-2)$ ($2 \le q < \infty$ if $d=2$, $2 \le q \le \infty$ if $d=1$), $2/p = d(1/2-1/q)$. Then, we have*

$$||u||_{L_t^pL_x^q} \lesssim ||u||_{U^p_{\sigma\Delta}}, \quad u \in U^p_{\sigma\Delta}L^2, \tag{3.7}$$

$$||u||_{L_t^pL_x^q} \lesssim ||u||_{V^{\widetilde{p}}_{\sigma\Delta}}, \quad u \in V^{\widetilde{p}}_{\sigma\Delta}L^2,\ (1 \le \widetilde{p} < p). \tag{3.8}$$

Next, we show the bilinear Stricharz estimate.

Lemma 3.14. *Let $d \in \mathbb{N}$, $s_c = d/2-1$, $b > 1/2$ and σ_1, $\sigma_2 \in \mathbb{R}\backslash\{0\}$. For any dyadic numbers L, $H \in 2^{\mathbb{Z}}$ with $L \ll H$, we have*

$$||P_Hu_1P_Lu_2||_{L^2_{tx}} \lesssim L^{s_c}\left(\frac{L}{H}\right)^{1/2}||P_Hu_1||_{X^{0,b}_{\sigma_1}}||P_Lu_2||_{X^{0,b}_{\sigma_2}}, \tag{3.9}$$

where $||u||_{X^{0,b}_{\sigma}} := ||\langle\tau+\sigma|\xi|^2\rangle^b\widetilde{u}||_{L^2_{\tau\xi}}$.

Proof. For the case $d=2$ and $(\sigma_1,\sigma_2)=(1,\pm1)$, the estimate (3.9) is proved by Colliander, Delort, Kenig, and Staffilani ([15], Lemma 1). The proof for general case as following is similar to their argument.

We put $g_1(\tau_1,\xi_1) := \langle\tau_1+\sigma_1|\xi_1|^2\rangle^b\widetilde{P_Hu_1}(\tau_1,\xi_1)$, $g_2(\tau_2,\xi_2) := \langle\tau_2+\sigma_2|\xi_2|^2\rangle^b\widetilde{P_Lu_2}(\tau_2,\xi_2)$ and $A_N := \{\xi \in \mathbb{R}^d | N/2 \le |\xi| \le 2N\}$ for a dyadic number N. By the Plancherel's theorem and the duality argument, it is enough to prove the estimate

$$\begin{aligned}I &:= \left|\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{A_L}\int_{A_H} f(\tau_1+\tau_2,\xi_1+\xi_2)\frac{g_1(\tau_1,\xi_1)}{\langle\tau_1+\sigma_1|\xi_1|^2\rangle^b}\frac{g_2(\tau_2,\xi_2)}{\langle\tau_2+\sigma_2|\xi_2|^2\rangle^b}d\xi_1d\xi_2d\tau_1d\tau_2\right|\\ &\lesssim \frac{L^{(d-1)/2}}{H^{1/2}}||f||_{L^2_{\tau\xi}}||g_1||_{L^2_{\tau\xi}}||g_2||_{L^2_{\tau\xi}}\end{aligned}$$

for $f \in L^2_{\tau\xi}$. We change the variables $(\tau_1,\tau_2) \mapsto (\theta_1,\theta_2)$ as $\theta_i = \tau_i+\sigma_i|\xi_i|^2$ $(i=1,2)$ and put

$$F(\theta_1,\theta_2,\xi_1,\xi_2) := f(\theta_1+\theta_2-\sigma_1|\xi_1|^2-\sigma_2|\xi_2|^2,\xi_1+\xi_2),$$

$$G_i(\theta_i,\xi_i) := g_i(\theta_i-\sigma_i|\xi_i|^2,\xi_i),\quad (i=1,2).$$

Then, we have

$$I \leq \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\langle\theta_1\rangle^b\langle\theta_2\rangle^b}\left(\int_{A_L}\int_{A_H} |F(\theta_1,\theta_2,\xi_1,\xi_2)G_1(\theta_1,\xi_1)G_2(\theta_2,\xi_2)|d\xi_1 d\xi_2\right)d\theta_1 d\theta_2$$
$$\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\langle\theta_1\rangle^b\langle\theta_2\rangle^b}\left(\int_{A_L}\int_{A_H} |F(\theta_1,\theta_2,\xi_1,\xi_2)|^2 d\xi_1 d\xi_2\right)^{1/2} ||G_1(\theta_1,\cdot)||_{L^2_\xi}||G_2(\theta_2,\cdot)||_{L^2_\xi} d\theta_1 d\theta_2$$

by the Cauchy-Schwarz inequality. For $1 \leq j \leq d$, we put

$$A_H^j := \{\xi_1 = (\xi_1^{(1)},\cdots,\xi_1^{(d)}) \in \mathbb{R}^d |\ H/2 \leq |\xi_1| \leq 2H,\ |\xi_1^{(j)}| \geq H/(2\sqrt{d})\}$$

and

$$K_j(\theta_1,\theta_2) := \int_{A_L}\int_{A_H^j} |F(\theta_1,\theta_2,\xi_1,\xi_2)|^2 d\xi_1 d\xi_2.$$

We consider only the estimate for K_1. The estimates for other K_j are obtained by the same way.

Assume $d \geq 2$. By changing the variables $(\xi_1,\xi_2) = (\xi_1^{(1)},\cdots,\xi_1^{(d)},\xi_2^{(1)},\cdots,\xi_2^{(d)}) \mapsto (\mu,\nu,\eta)$ as

$$\begin{cases} \mu = \theta_1 + \theta_2 - \sigma_1|\xi_1|^2 - \sigma_2|\xi_2|^2 \in \mathbb{R}, \\ \nu = \xi_1 + \xi_2 \in \mathbb{R}^d, \\ \eta = (\xi_2^{(2)}\cdots,\xi_2^{(d)}) \in \mathbb{R}^{d-1}, \end{cases} \tag{3.10}$$

we have

$$d\mu d\nu d\eta = 2|\sigma_1\xi_1^{(1)} - \sigma_2\xi_2^{(1)}|d\xi_1 d\xi_2$$

and

$$F(\theta_1,\theta_2,\xi_1,\xi_2) = f(\mu,\nu).$$

We note that $|\sigma_1\xi_1^{(1)} - \sigma_2\xi_2^{(1)}| \sim H$ for any $(\xi_1,\xi_2) \in A_H^1 \times A_L$ with $L \ll H$. Furthermore, $\xi_2 \in A_L$ implies that $\eta \in [-2L,2L]^{d-1}$. Therefore, we obtain

$$K_1(\theta_1,\theta_2) \lesssim \frac{1}{H}\int_{[-2L,2L]^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}} |f(\mu,\nu)|^2 d\mu d\nu d\eta \sim \frac{L^{d-1}}{H}||f||^2_{L^2_{\tau\xi}}.$$

As a result, we have

$$I \lesssim \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\langle\theta_1\rangle^b\langle\theta_1\rangle^b}\left(\sum_{j=1}^d K_j(\theta_1,\theta_2)\right)^{1/2} ||G_1(\theta_1,\cdot)||_{L^2_\xi}||G_2(\theta_2,\cdot)||_{L^2_\xi} d\theta_1 d\theta_2$$
$$\lesssim \frac{L^{(d-1)/2}}{H^{1/2}}||f||_{L^2_{\tau\xi}}||g_1||_{L^2_{\tau\xi}}||g_2||_{L^2_{\tau\xi}}$$

by the Cauchy-Schwarz inequality and changing the variables $(\theta_1,\theta_2) \mapsto (\tau_1,\tau_2)$ as $\theta_i = \tau_i + \sigma_i|\xi_i|^2$ $(i = 1,2)$.

For $d = 1$, we obtain the same result by changing the variables $(\xi_1, \xi_2) \mapsto (\mu, \nu)$ as $\mu = \theta_1 + \theta_2 - \sigma_1|\xi_1|^2 - \sigma_2|\xi_2|^2$, $\nu = \xi_1 + \xi_2$ instead of (3.10). □

Corollary 3.15. *Let $d \in \mathbb{N}$, $s_c = d/2 - 1$ and σ_1, $\sigma_2 \in \mathbb{R}\backslash\{0\}$.*
(i) *If $d \geq 2$, then for any dyadic numbers L, $H \in 2^{\mathbb{Z}}$ with $L \ll H$, we have*

$$||P_H u_1 P_L u_2||_{L^2_{tx}} \lesssim L^{s_c} \left(\frac{L}{H}\right)^{1/2} ||P_H u_1||_{U^2_{\sigma_1\Delta}L^2} ||P_L u_2||_{U^2_{\sigma_2\Delta}L^2}, \tag{3.11}$$

$$||P_H u_1 P_L u_2||_{L^2_{tx}} \lesssim L^{s_c} \left(\frac{L}{H}\right)^{1/2} \left(1 + \ln \frac{H}{L}\right)^2 ||P_H u_1||_{V^2_{\sigma_1\Delta}L^2} ||P_L u_2||_{V^2_{\sigma_2\Delta}L^2}. \tag{3.12}$$

(ii) *If $d = 1$, then for any dyadic numbers L, $H \in 2^{\mathbb{Z}}$ with $L \ll H$, we have*

$$||P_H u_1 P_L u_2||_{L^2([0,1]\times\mathbb{R})} \lesssim \frac{1}{H^{1/2}} ||P_H u_1||_{U^2_{\sigma_1\Delta}L^2} ||P_L u_2||_{U^2_{\sigma_2\Delta}L^2}, \tag{3.13}$$

$$||P_H u_1 P_L u_2||_{L^2([0,1]\times\mathbb{R})} \lesssim \min\left\{L^{1/6}, \frac{(1 + \ln H)^2}{H^{1/2}}\right\} ||P_H u_1||_{V^2_{\sigma_1\Delta}L^2} ||P_L u_2||_{V^2_{\sigma_2\Delta}L^2}. \tag{3.14}$$

Proof. To obtain (3.11) and (3.13), we use the argument of the proof of Corollary 2.21 (27) in [25]. Let ϕ_1, $\phi_2 \in L^2(\mathbb{R}^d)$ and define $\phi_j^\lambda(x) := \phi_j(\lambda x)$ $(j = 1, 2)$ for $\lambda \in \mathbb{R}$. By using the rescaling $(t, x) \mapsto (\lambda^2 t, \lambda x)$, we have

$$\begin{aligned}
&||P_H(e^{it\sigma_1\Delta}\phi_1)P_L(e^{it\sigma_2\Delta}\phi_2)||_{L^2([-T,T]\times\mathbb{R}^d)} \\
&= \lambda^{s_c+2}||P_{\lambda H}(e^{it\sigma_1\Delta}\phi_1^\lambda)P_{\lambda L}(e^{it\sigma_2\Delta}\phi_2^\lambda)||_{L^2([-\lambda^{-2}T,\lambda^{-2}T]\times\mathbb{R}^d)}.
\end{aligned}$$

Therefore by putting $\lambda = \sqrt{T}$ and Lemma 3.14, we have

$$\begin{aligned}
&||P_H(e^{it\sigma_1\Delta}\phi_1)P_L(e^{it\sigma_2\Delta}\phi_2)||_{L^2([-T,T]\times\mathbb{R}^d)} \\
&\lesssim \sqrt{T}^{2(s_c+1)} L^{s_c} \left(\frac{L}{H}\right)^{1/2} ||P_{\sqrt{T}H}\phi_1^{\sqrt{T}}||_{L^2_x} ||P_{\sqrt{T}L}\phi_2^{\sqrt{T}}||_{L^2_x} \\
&= L^{s_c} \left(\frac{L}{H}\right)^{1/2} ||P_H\phi_1||_{L^2_x} ||P_L\phi_2||_{L^2_x}.
\end{aligned}$$

Let $T \to \infty$, then we obtain

$$||P_H(e^{it\sigma_1\Delta}\phi_1)P_L(e^{it\sigma_2\Delta}\phi_2)||_{L^2_{tx}} \lesssim L^{s_c} \left(\frac{L}{H}\right)^{1/2} ||P_H\phi_1||_{L^2_x} ||P_L\phi_2||_{L^2_x}$$

and (3.11), (3.13) follow from proposition 2.8.

To obtain (3.12) and (3.14), we first prove the U^4 estimate for $d \geq 2$ and U^8 estimate for $d = 1$. Assume $d \geq 2$. By the Cauchy-Schwarz inequality, the Sobolev embedding $\dot{W}^{s_c,2d/(d-1)}(\mathbb{R}^d) \hookrightarrow L^{2d}(\mathbb{R}^d)$ and (3.7), we have

$$\begin{aligned}||P_H u_1 P_L u_2||_{L^2_{tx}} &\lesssim L^{s_c}||P_H u_1||_{L^4_t L^{2d/(d-1)}_x}||P_L u_2||_{L^4_t L^{2d/(d-1)}_x} \\ &\lesssim L^{s_c}||P_H u_1||_{U^4_{\sigma_1\Delta}L^2}||P_L u_2||_{U^4_{\sigma_2\Delta}L^2}\end{aligned} \tag{3.15}$$

for any dyadic numbers L, $H \in 2^{\mathbb{Z}}$. While if $d = 1$, then by the Hölder's inequality and (3.7), we have

$$\begin{aligned}||P_H u_1 P_L u_2||_{L^2([0,1]\times\mathbb{R})} &\leq ||\mathbf{1}_{[0,1)}||_{L^4_t}||P_H u_1||_{L^8_t L^4_x}||P_L u_2||_{L^8_t L^4_x} \\ &\lesssim ||P_H u_1||_{U^8_{\sigma_1\Delta}L^2}||P_L u_2||_{U^8_{\sigma_2\Delta}L^2}\end{aligned} \tag{3.16}$$

for any dyadic numbers L, $H \in 2^{\mathbb{Z}}$. We use the interpolation between (3.11) and (3.15) via Proposition 2.9. Then, we get (3.12) by the same argument of the proof of Corollary 2.21 (28) in [25]. The estimate (3.14) follows from

$$\begin{aligned}||P_H u_1 P_L u_2||_{L^2([0,1]\times\mathbb{R})} &\leq ||\mathbf{1}_{[0,1)}||_{L^3_t}L^{1/6}||P_H u_1||_{L^{12}_t L^3_x}||P_L u_2||_{L^{12}_t L^3_x} \\ &\lesssim L^{1/6}||P_H u_1||_{V^2_{\sigma_1\Delta}L^2}||P_L u_2||_{V^2_{\sigma_2\Delta}L^2}.\end{aligned} \tag{3.17}$$

and the interpolation between (3.13) and (3.16), where we used the Hölder's inequality, the Sobolev embedding $\dot{W}^{1/6,3}(\mathbb{R}) \hookrightarrow L^6(\mathbb{R})$ and (3.8) to obtain (3.17). □

3.3 Time global estimates

In this and next section, implicit constants in $\ll$ actually depend on σ_1, σ_2, σ_3.

Lemma 3.16. *Let $d \in \mathbb{N}$. We assume that σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)(\sigma_3+\sigma_1) \neq 0$ and (τ_1,ξ_1), (τ_2,ξ_2), $(\tau_3,\xi_3) \in \mathbb{R}\times\mathbb{R}^d$ satisfy $\tau_1+\tau_2+\tau_3 = 0$, $\xi_1+\xi_2+\xi_3 = 0$.*
(i) *If there exist $1 \leq i, j \leq 3$ such that $|\xi_i| \ll |\xi_j|$, then we have*

$$\max_{1\leq j\leq 3}|\tau_j + \sigma_j|\xi_j|^2| \gtrsim \max_{1\leq j\leq 3}|\xi_j|^2. \tag{3.18}$$

(ii) *If $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$, then we have* (3.18).

Proof. By the triangle inequality and the completing the square, we have

$$\begin{aligned}M_0 :&= \max_{1\leq j\leq 3}|\tau_j + \sigma_j|\xi_j|^2| \\ &\gtrsim |\sigma_1|\xi_1|^2 + \sigma_2|\xi_2|^2 + \sigma_3|\xi_3|^2| \\ &= |(\sigma_1+\sigma_3)|\xi_1|^2 + 2\sigma_3\xi_1\cdot\xi_2 + (\sigma_2+\sigma_3)|\xi_2|^2| \\ &= |\sigma_1+\sigma_3|\left|\left|\xi_1 + \frac{\sigma_3}{\sigma_1+\sigma_3}\xi_2\right|^2 + \frac{\sigma_1\sigma_2\sigma_3}{(\sigma_1+\sigma_3)^2}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}+\frac{1}{\sigma_3}\right)|\xi_2|^2\right|.\end{aligned} \tag{3.19}$$

We first prove (i). By the symmetry, we can assume $|\xi_1| \sim |\xi_3| \gtrsim |\xi_2|$. If $|\xi_1| \gg |\xi_2|$, then we have $M_0 \gtrsim |\xi_1|^2 \sim \max_{1\le j\le 3}|\xi_j|^2$ by (3.19). Next, we prove (ii). By the symmetry, we can assume $|\xi_1| \sim |\xi_2| \gtrsim |\xi_3|$. If $\sigma_1\sigma_2\sigma_3(1/\sigma_1+1/\sigma_2+1/\sigma_3)>0$, then we have $M_0 \gtrsim |\xi_2|^2 \sim \max_{1\le j\le 3}|\xi_j|^2$ by (3.19). □

In the following Propositions and Corollaries in this and next section, we assume $P_{N_1}u_1 \in V^2_{\sigma_1\Delta}L^2$, $P_{N_2}u_2 \in V^2_{\sigma_2\Delta}L^2$ and $P_{N_3}u_3 \in V^2_{\sigma_3\Delta}L^2$ for each N_1, N_2, $N_3 \in 2^{\mathbb{Z}}$. Propositions 3.17, 3.18 and its proofs are based on Proposition 3.1 in [25].

Proposition 3.17. *Let $d\ge 2$, $s_c = d/2-1$, $0<T\le\infty$ and σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)(\sigma_3+\sigma_1)\neq 0$. For any dyadic numbers N_2, $N_3\in 2^{\mathbb{Z}}$ with $N_2\sim N_3$, we have*

$$\begin{aligned}&\left|\sum_{N_1\ll N_2} N_{\max}\int_0^T\int_{\mathbb{R}^d}(P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)dxdt\right|\\ &\lesssim \left(\sum_{N_1\ll N_2} N_1^{2s_c}||P_{N_1}u_1||^2_{V^2_{\sigma_1\Delta}L^2}\right)^{1/2}||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2}||P_{N_3}u_3||_{V^2_{\sigma_3\Delta}L^2},\end{aligned} \tag{3.20}$$

where $N_{\max} := \max\limits_{1\le j\le 3} N_j$.

Proof. We define $f_{j,N_j,T} := \mathbf{1}_{[0,T)}P_{N_j}u_j$ $(j=1,2,3)$. For sufficiently large constant C, we put $M := C^{-1}N_{\max}^2$ and decompose $Id = Q^{\sigma_j\Delta}_{<M}+Q^{\sigma_j\Delta}_{\ge M}$ $(j=1,2,3)$. We divide the integrals on the left-hand side of (3.20) into eight piece of the form

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q_1^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_3^{\sigma_3\Delta}f_{3,N_3,T})dxdt \tag{3.21}$$

with $Q_j^{\sigma_j\Delta}\in\{Q^{\sigma_j\Delta}_{\ge M}, Q^{\sigma_j\Delta}_{<M}\}$ $(j=1,2,3)$. By the Plancherel's theorem, we have

$$(3.21) = c\int_{\tau_1+\tau_2+\tau_3=0}\int_{\xi_1+\xi_2+\xi_3=0}\prod_{j=1}^3\mathcal{F}[Q_j^{\sigma_j\Delta}f_{j,N_j,T}](\tau_j,\xi_j),$$

where c is a constant. Therefore, Lemma 3.16 (i) implies that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q^{\sigma_1\Delta}_{<M}f_{1,N_1,T})(Q^{\sigma_2\Delta}_{<M}f_{2,N_2,T})(Q^{\sigma_3\Delta}_{<M}f_{3,N_3,T})dxdt = 0$$

when $N_1\ll N_2$. So, let us now consider the case that $Q_j^{\sigma_j\Delta} = Q^{\sigma_j\Delta}_{\ge M}$ for some $1\le j\le 3$.

First, we consider the case $Q_1^{\sigma_1\Delta} = Q_{\geq M}^{\sigma_1\Delta}$. By the Hölder's inequality and the Sobolev embedding $\dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\left|\sum_{N_1 \ll N_2} N_{\max} \int_{\mathbb{R}}\int_{\mathbb{R}^d} (Q_{\geq M}^{\sigma_1\Delta} f_{1,N_1,T})(Q_2^{\sigma_2\Delta} f_{2,N_2,T})(Q_3^{\sigma_3\Delta} f_{3,N_3,T}) dxdt\right| \\
&\lesssim \left|\left|\sum_{N_1 \ll N_2} N_{\max} |\nabla|^{s_c} Q_{\geq M}^{\sigma_1\Delta} f_{1,N_1,T}\right|\right|_{L^2_{tx}} ||Q_2^{\sigma_2\Delta} f_{2,N_2,T}||_{L_t^4 L_x^{2d/(d-1)}} ||Q_3^{\sigma_3\Delta} f_{3,N_3,T}||_{L_t^4 L_x^{2d/(d-1)}}.
\end{aligned} \tag{3.22}$$

Furthermore, by the L^2 orthogonality and (2.1) with $p = 2$, we have

$$\left|\left|\sum_{N_1 \ll N_2} N_{\max} |\nabla|^{s_c} Q_{\geq M}^{\sigma_1\Delta} f_{1,N_1,T}\right|\right|_{L^2_{tx}} \lesssim \left(\sum_{N_1 \ll N_2} N_{\max}^2 N_1^{2s_c} M^{-1} ||f_{1,N_1,T}||^2_{V^2_{\sigma_1\Delta} L^2}\right)^{1/2}$$

While by (3.8) and (2.2), we have

$$||Q_2^{\sigma_2\Delta} f_{2,N_2,T}||_{L_t^4 L_x^{2d/(d-1)}} \lesssim ||f_{2,N_2,T}||_{V^2_{\sigma_2\Delta} L^2}, \ ||Q_3^{\sigma_3\Delta} f_{3,N_3,T}||_{L_t^4 L_x^{2d/(d-1)}} \lesssim ||f_{3,N_3,T}||_{V^2_{\sigma_3\Delta} L^2}.$$

Therefore, we obtain

$$\begin{aligned}
&\left|\sum_{N_1 \ll N_2} N_{\max} \int_{\mathbb{R}}\int_{\mathbb{R}^d} (Q_{\geq M}^{\sigma_1\Delta} f_{1,N_1,T})(Q_2^{\sigma_2\Delta} f_{2,N_2,T})(Q_3^{\sigma_3\Delta} f_{3,N_3,T}) dxdt\right| \\
&\lesssim \left(\sum_{N_1 \ll N_2} N_1^{2s_c} ||P_{N_1} u_1||^2_{V^2_{\sigma_1\Delta} L^2}\right)^{1/2} ||P_{N_2} u_2||_{V^2_{\sigma_2\Delta} L^2} ||P_{N_3} u_3||_{V^2_{\sigma_3\Delta} L^2},
\end{aligned}$$

since $M \sim N_{\max}^2$ and $||\mathbf{1}_{[0,T)} f||_{V^2_{\sigma\Delta} L^2} \lesssim ||f||_{V^2_{\sigma\Delta} L^2}$ for any $\sigma \in \mathbb{R}$ and any $T \in (0, \infty]$.

Next, we consider the case $Q_3^{\sigma_3\Delta} = Q_{\geq M}^{\sigma_3\Delta}$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|\sum_{N_1 \ll N_2} N_{\max} \int_{\mathbb{R}}\int_{\mathbb{R}^d} (Q_1^{\sigma_1\Delta} f_{1,N_1,T})(Q_2^{\sigma_2\Delta} f_{2,N_2,T})(Q_{\geq M}^{\sigma_3\Delta} f_{3,N_3,T}) dxdt\right| \\
&\leq \sum_{N_1 \ll N_2} N_{\max} ||(Q_1^{\sigma_1\Delta} f_{1,N_1,T})(Q_2^{\sigma_2\Delta} f_{2,N_2,T})||_{L^2_{tx}} ||Q_{\geq M}^{\sigma_3\Delta} f_{3,N_3,T}||_{L^2_{tx}}.
\end{aligned}$$

Furthermore, by (2.1) with $p = 2$, we have

$$||Q_{\geq M}^{\sigma_3\Delta} f_{3,N_3,T}||_{L^2_{tx}} \lesssim M^{-1/2} ||f_{3,N_3,T}||_{V^2_{\sigma_3\Delta} L^2}. \tag{3.23}$$

While by (3.12), (2.2) and the Cauchy-Schwarz inequality for the dyadic sum, we have

$$\begin{aligned}
&\sum_{N_1\ll N_2} ||(Q_1^{\sigma_1\Delta} f_{1,N_1,T})(Q_2^{\sigma_2\Delta} f_{2,N_2,T})||_{L^2_{tx}} \\
&\lesssim \sum_{N_1\ll N_2} N_1^{s_c}\left(\frac{N_1}{N_2}\right)^{1/4} ||f_{1,N_1,T}||_{V^2_{\sigma_1\Delta}L^2}||f_{2,N_2,T}||_{V^2_{\sigma_2\Delta}L^2} \\
&\lesssim \left(\sum_{N_1\ll N_2} N_1^{2s_c}||f_{1,N_1,T}||^2_{V^2_{\sigma_1\Delta}L^2}\right)^{1/2} ||f_{2,N_2,T}||_{V^2_{\sigma_2\Delta}L^2}.
\end{aligned} \tag{3.24}$$

Therefore, we obtain

$$\begin{aligned}
&\left|\sum_{N_1\ll N_2} N_{\max}\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q_1^{\sigma_1\Delta} f_{1,N_1,T})(Q_2^{\sigma_2\Delta} f_{2,N_2,T})(Q_{\geq M}^{\sigma_3\Delta} f_{3,N_3,T})dxdt\right| \\
&\lesssim \left(\sum_{N_1\ll N_2} N_1^{2s_c}||P_{N_1}u_1||^2_{V^2_{\sigma_1\Delta}L^2}\right)^{1/2} ||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2}||P_{N_3}u_3||_{V^2_{\sigma_3\Delta}L^2},
\end{aligned}$$

since $M\sim N_{\max}^2$ and $||\mathbf{1}_{[0,T)}f||_{V^2_{\sigma\Delta}L^2}\lesssim ||f||_{V^2_{\sigma\Delta}L^2}$ for any $\sigma\in\mathbb{R}$ and any $T\in(0,\infty]$.

For the case $Q_2^{\sigma_2\Delta}=Q_{\geq M}^{\sigma_2\Delta}$ is proved in exactly same way as the case $Q_3^{\sigma_3\Delta}=Q_{\geq M}^{\sigma_3\Delta}$. □

Proposition 3.18. *Let $d\geq 2$, $s_c=d/2-1$, $s\geq 0$, $0<T\leq\infty$ and σ_1, σ_2, $\sigma_3\in\mathbb{R}\backslash\{0\}$ satisfy $(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)(\sigma_3+\sigma_1)\neq 0$. For any dyadic numbers N_1, $N_2\in 2^{\mathbb{Z}}$ with $N_1\sim N_2$, we have*

$$\begin{aligned}
&\left(\sum_{N_3\ll N_2} N_3^{2s}\sup_{||u_3||_{V^2_{\sigma_3\Delta}L^2}=1}\left|N_{\max}\int_0^T\int_{\mathbb{R}^d}(P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)dxdt\right|^2\right)^{1/2} \\
&\lesssim N_1^{s_c}||P_{N_1}u_1||_{V^2_{\sigma_1\Delta}L^2}N_2^{s}||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2},
\end{aligned} \tag{3.25}$$

where $N_{\max}:=\max\limits_{1\leq j\leq 3}N_j$.

Proof. We define $f_{j,N_j,T}:=\mathbf{1}_{[0,T)}P_{N_j}u_j$ $(j=1,2,3)$. For sufficiently large constant C, we put $M:=C^{-1}N_{\max}^2$ and decompose $Id=Q_{<M}^{\sigma_j\Delta}+Q_{\geq M}^{\sigma_j\Delta}$ $(j=1,2,3)$. We divide the integrals on the left-hand side of (3.25) into eight piece of the form

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q_1^{\sigma_1\Delta} f_{1,N_1,T})(Q_2^{\sigma_2\Delta} f_{2,N_2,T})(Q_3^{\sigma_3\Delta} f_{3,N_3,T})dxdt$$

with $Q_j^{\sigma_j\Delta}\in\{Q_{\geq M}^{\sigma_j\Delta},Q_{<M}^{\sigma_j\Delta}\}$ $(j=1,2,3)$. By the same argument of the proof of Proposition 3.17, we consider only the case that $Q_j^{\sigma_j\Delta}=Q_{\geq M}^{\sigma_j\Delta}$ for some $1\leq j\leq 3$.

First, we consider the case $Q_1^{\sigma_1\Delta} = Q_{\geq M}^{\sigma_1\Delta}$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q_{\geq M}^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_3^{\sigma_3\Delta}f_{3,N_3,T})dxdt\right|\\&\leq ||Q_{\geq M}^{\sigma_1\Delta}f_{1,N_1,T}||_{L^2_{tx}}||(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_3^{\sigma_3\Delta}f_{3,N_3,T})||_{L^2_{tx}}.\end{aligned}$$

Furthermore by (2.1) with $p = 2$, we have

$$||Q_{\geq M}^{\sigma_1\Delta}f_{1,N_1,T}||_{L^2_{tx}} \lesssim M^{-1/2}||f_{1,N_1,T}||_{V^2_{\sigma_1\Delta}L^2}. \tag{3.26}$$

While by (3.12) and (2.2), we have

$$\begin{aligned}&||(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_3^{\sigma_3\Delta}f_{3,N_3,T})||_{L^2_{tx}}\\&\lesssim N_3^{s_c}\left(\frac{N_3}{N_2}\right)^{1/4}||f_{2,N_2,T}||_{V^2_{\sigma_2\Delta}L^2}||f_{3,N_3,T}||_{V^2_{\sigma_3\Delta}L^2}\end{aligned}\tag{3.27}$$

when $N_3 \ll N_2$. Therefore, we obtain

$$\begin{aligned}&\sum_{N_3\ll N_2}N_3^{2s}\sup_{||u_3||_{V^2_{\sigma_3\Delta}L^2}=1}\left|N_{\max}\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q_{\geq M}^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_3^{\sigma_3\Delta}f_{3,N_3,T})dxdt\right|^2\\&\lesssim N_1^{2s_c}||P_{N_1}u_1||^2_{V^2_{\sigma_1\Delta}L^2}N_2^{2s}||P_{N_2}u_2||^2_{V^2_{\sigma_2\Delta}L^2}\end{aligned}$$

by $M \sim N_{\max}^2$, $N_1 \sim N_2$ and $||\mathbf{1}_{[0,T)}f||_{V^2_{\sigma\Delta}L^2} \lesssim ||f||_{V^2_{\sigma\Delta}L^2}$ for any $\sigma \in \mathbb{R}$ and any $T \in (0,\infty]$.

Next, we consider the case $Q_3^{\sigma_3\Delta} = Q_{\geq M}^{\sigma_3\Delta}$. We define $\widetilde{P}_{N_3} = P_{N_3/2} + P_{N_3} + P_{2N_3}$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q_1^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_{\geq M}^{\sigma_3\Delta}f_{3,N_3,T})dxdt\right|\\&\lesssim ||\widetilde{P}_{N_3}((Q_1^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T}))||_{L^2_{tx}}||Q_{\geq M}^{\sigma_3\Delta}f_{3,N_3,T}||_{L^2_{tx}}\end{aligned}$$

since $P_{N_3} = \widetilde{P}_{N_3}P_{N_3}$. Furthermore, by (2.1) with $p = 2$, we have

$$||Q_{\geq M}^{\sigma_3\Delta}f_{3,N_3,T}||_{L^2_{tx}} \lesssim M^{-1/2}||f_{3,N_3,T}||_{V^2_{\sigma_3\Delta}L^2}. \tag{3.28}$$

Therefore, we obtain

$$\begin{aligned}&\sum_{N_3\ll N_2}N_3^{2s}\sup_{||u_3||_{V^2_{\sigma_3\Delta}L^2}=1}\left|N_{\max}\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q_1^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_{\geq M}^{\sigma_3\Delta}f_{3,N_3,T})dxdt\right|^2\\&\lesssim \sum_{N_3\ll N_2}N_3^{2s}N_{\max}^2M^{-1}||\widetilde{P}_{N_3}((Q_1^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T}))||^2_{L^2_{tx}}\\&\lesssim N_2^{2s}||(Q_1^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T})||^2_{L^2_{tx}}\\&\lesssim N_1^{2s_c}||P_{N_1}u_1||^2_{V^2_{\sigma_1\Delta}L^2}N_2^{2s}||P_{N_2}u_2||^2_{V^2_{\sigma_2\Delta}L^2}\end{aligned}\tag{3.29}$$

by $M \sim N_{\max}^2$, $N_1 \sim N_2$, L^2-orthogonality, (3.15), the embedding $V^2_{-,rc} \hookrightarrow U^4$, (2.2) and $||\mathbf{1}_{[0,T)} f||_{V^2_{\sigma\Delta}L^2} \lesssim ||f||_{V^2_{\sigma\Delta}L^2}$ for any $\sigma \in \mathbb{R}$ and any $T \in (0, \infty]$.

For the case $Q_2^{\sigma_2\Delta} = Q_{\geq M}^{\sigma_2\Delta}$ is proved in exactly same way as the case $Q_1^{\sigma_1\Delta} = Q_{\geq M}^{\sigma_1\Delta}$. □

Proposition 3.19. *Let $s_c = d/2 - 1$ and , $0 < T \leq \infty$.*
(i) *Let $d \geq 4$. For any σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ and any dyadic numbers N_1, N_2, $N_3 \in 2^{\mathbb{Z}}$, we have*

$$\begin{aligned}&\left| N_{\max} \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right| \\ &\lesssim N_{\max}^{s_c} ||P_{N_1} u_1||_{V^2_{\sigma_1\Delta}L^2} ||P_{N_2} u_2||_{V^2_{\sigma_2\Delta}L^2} ||P_{N_3} u_3||_{V^2_{\sigma_3\Delta}L^2},\end{aligned} \tag{3.30}$$

where $N_{\max} := \max\limits_{1\leq j\leq 3} N_j$.
(ii) *Let $d = 2$, 3 and $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$. For any dyadic numbers N_1, N_2, $N_3 \in 2^{\mathbb{Z}}$, we have* (3.30).

Proof. First, we consider the case $d \geq 4$. By the Hölder's inequality, the Sobolev embedding $\dot{W}^{s_c-1, 6d/(3d-4)}(\mathbb{R}^d) \hookrightarrow L^{3d/4}(\mathbb{R}^d)$ and (3.8), we have

$$\begin{aligned}(\text{L.H.S of (3.30)}) &\lesssim N_{\max} ||P_{N_1 u_1}||_{L_t^3 L_x^{6d/(3d-4)}} ||P_{N_2 u_2}||_{L_t^3 L_x^{6d/(3d-4)}} |||\nabla|^{s_c-1} P_{N_3 u_3}||_{L_t^3 L_x^{6d/(3d-4)}} \\ &\lesssim N_{\max}^{s_c} ||P_{N_1 u_1}||_{V^2_{\sigma_1\Delta}L^2} ||P_{N_2 u_2}||_{V^2_{\sigma_2\Delta}L^2} ||P_{N_3 u_3}||_{V^2_{\sigma_3\Delta}L^2}.\end{aligned}$$

Next, we consider the case $d = 2$, 3 and $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$. We define $f_{j,N_j,T} := \mathbf{1}_{[0,T)} P_{N_j} u_j$ $(j = 1,2,3)$. For sufficiently large constant C, we put $M := C^{-1} N_{\max}^2$ and decompose $Id = Q_{<M}^{\sigma_j\Delta} + Q_{\geq M}^{\sigma_j\Delta}$ $(j = 1,2,3)$. We divide the integral on the left-hand side of (3.30) into eight piece of the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_1^{\sigma_1\Delta} f_{1,N_1,T})(Q_2^{\sigma_2\Delta} f_{2,N_2,T})(Q_3^{\sigma_3\Delta} f_{3,N_3,T}) dx dt$$

with $Q_j^{\sigma_j\Delta} \in \{Q_{\geq M}^{\sigma_j\Delta}, Q_{<M}^{\sigma_j\Delta}\}$ $(j = 1,2,3)$. Since $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$, Lemma 3.16 (ii) implies that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_{<M}^{\sigma_1\Delta} f_{1,N_1,T})(Q_{<M}^{\sigma_2\Delta} f_{2,N_2,T})(Q_{<M}^{\sigma_3\Delta} f_{3,N_3,T}) dx dt = 0$$

for any N_1, N_2, $N_3 \in 2^{\mathbb{Z}}$. So, let us now consider the case that $Q_j^{\sigma_j\Delta} = Q_{\geq M}^{\sigma_j\Delta}$ for some $1 \leq j \leq 3$. We consider only for the case $Q_1^{\sigma_1\Delta} = Q_{\geq M}^{\sigma_1\Delta}$ since for the other cases is same manner.

By the Cauchy-Schwarz inequality, we have

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q_{\geq M}^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_3^{\sigma_3\Delta}f_{3,N_3,T})dxdt\right|$$
$$\leq ||Q_{\geq M}^{\sigma_1\Delta}f_{1,N_1,T}||_{L^2_{tx}}||(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_3^{\sigma_3\Delta}f_{3,N_3,T})||_{L^2_{tx}}.$$

Furthermore by (2.1) with $p=2$, we have

$$||Q_{\geq M}^{\sigma_1\Delta}f_{1,N_1,T}||_{L^2_{tx}}\lesssim M^{-1/2}||f_{1,N_1,T}||_{V^2_{\sigma_1\Delta}L^2}. \tag{3.31}$$

While by (3.15), the embedding $V^2_{-,rc}\hookrightarrow U^4$ and (2.2), we have

$$||(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_3^{\sigma_3\Delta}f_{3,N_3,T})||_{L^2_{tx}}\lesssim N_{\max}^{s_c}||f_{2,N_2,T}||_{V^2_{\sigma_2\Delta}L^2}||f_{3,N_3,T}||_{V^2_{\sigma_3\Delta}L^2}. \tag{3.32}$$

Therefore, we obtain

$$\left|N_{\max}\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q_{\geq M}^{\sigma_1\Delta}f_{1,N_1,T})(Q_2^{\sigma_2\Delta}f_{2,N_2,T})(Q_3^{\sigma_3\Delta}f_{3,N_3,T})dxdt\right|$$
$$\lesssim N_{\max}^{s_c}||P_{N_1}u_1||_{V^2_{\sigma_1\Delta}L^2}||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2}||P_{N_3}u_3||_{V^2_{\sigma_3\Delta}L^2},$$

since $M\sim N_{\max}^2$ and $||\mathbf{1}_{[0,T)}f||_{V^2_{\sigma\Delta}L^2}\lesssim ||f||_{V^2_{\sigma\Delta}L^2}$ for any $\sigma\in\mathbb{R}$ and any $T\in(0,\infty]$. □

Proposition 3.18 and Proposition 3.19 imply the following:

Corollary 3.20. *Let σ_1, σ_2, $\sigma_3\in\mathbb{R}\backslash\{0\}$ satisfy $(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)(\sigma_3+\sigma_1)\neq 0$ if $d\geq 4$, and $\sigma_1\sigma_2\sigma_3(1/\sigma_1+1/\sigma_2+1/\sigma_3)>0$ if $d=2$, 3. Then the estimate* (3.25) *holds if we replace $\sum_{N_3\ll N_2}$ by $\sum_{N_3\lesssim N_2}$.*

3.4 Time local estimates

Proposition 3.21. *Let $s>s_c$ $(=d/2-1)$, $0<T<\infty$ if $d\geq 2$ and $s\geq 0$, $T=1$ if $d=1$. We assume σ_1, σ_2, $\sigma_3\in\mathbb{R}\backslash\{0\}$ satisfy $(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)(\sigma_3+\sigma_1)\neq 0$. For any dyadic numbers N_2, $N_3\in 2^{\mathbb{Z}}$ with $N_2\sim N_3$, we have*

$$\begin{aligned}&\left|\sum_{N_1\ll N_2}N_{\max}\int_0^T\int_{\mathbb{R}^d}(P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)dxdt\right|\\ &\lesssim T^{\delta}\left(\sum_{N_1\ll N_2}(N_1\vee 1)^{2s}||P_{N_1}u_1||^2_{V^2_{\sigma_1\Delta}L^2}\right)^{1/2}||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2}||P_{N_3}u_3||_{V^2_{\sigma_3\Delta}L^2}\end{aligned} \tag{3.33}$$

for some $\delta>0$, where $N_{\max}:=\max\limits_{1\leq j\leq 3}N_j$.

Proof. First, we assume $d \geq 2$. We choose $\delta > 0$ satisfying $\delta < (s - s_c)/2$ and $\delta \ll 1$. In the proof of proposition 3.17, for L.H.S of (3.22), we use the Sobolev embedding $\dot{H}^{s_c+2\delta} \hookrightarrow L^{d/(1-2\delta)}$ instead of $\dot{H}^{s_c} \hookrightarrow L^d$. Then we have

$$\begin{aligned}
&\left| \sum_{N_1 \ll N_2} N_{\max} \int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_{\geq M}^{\sigma_1 \Delta} f_{1,N_1,T})(Q_2^{\sigma_2 \Delta} f_{2,N_2,T})(Q_3^{\sigma_3 \Delta} f_{3,N_3,T}) dxdt \right| \\
&\lesssim ||\mathbf{1}_{[0,T)}||_{L_t^{1/\delta}} \left|\left| \sum_{N_1 \ll N_2} N_{\max} |\nabla|^{s_c+2\delta} Q_{\geq M}^{\sigma_1 \Delta} f_{1,N_1,T} \right|\right|_{L_{tx}^2} ||Q_2^{\sigma_2 \Delta} f_{2,N_2,T}||_{L_t^p L_x^q} ||Q_3^{\sigma_3 \Delta} f_{3,N_3,T}||_{L_t^p L_x^q} \\
&\leq T^{\delta} \left|\left| \sum_{N_1 \ll N_2} N_{\max} \langle \nabla \rangle^s Q_{\geq M}^{\sigma_1 \Delta} f_{1,N_1,T} \right|\right|_{L_{tx}^2} ||Q_2^{\sigma_2 \Delta} f_{2,N_2,T}||_{L_t^p L_x^q} ||Q_3^{\sigma_3 \Delta} f_{3,N_3,T}||_{L_t^p L_x^q}
\end{aligned}$$

with $(p, q) = (4/(1-2\delta), 2d/(d-1+2\delta))$ which is the admissible pair of the Strichartz estimate. Furthermore for L.H.S of (3.23), we use the Hölder's inequality and (2.1) with $p = 2/(1 - 2\delta)$ instead of $p = 2$. Then we have

$$||Q_{\geq M}^{\sigma_3 \Delta} f_{3,N_3,T}||_{L_{tx}^2} \leq ||\mathbf{1}_{[0,T)}||_{L_t^{1/\delta}} ||Q_{\geq M}^{\sigma_3 \Delta} f_{3,N_3,T}||_{L_t^{2/(1-2\delta)} L_x^2} \lesssim T^{\delta} M^{-(1-2\delta)/2} ||f_{3,N_3,T}||_{V_{\sigma_3 \Delta}^2 L^2}.$$

For the other part, by the same way of the proof of proposition 3.17, we obtain (3.33).

Next, we assume $d = 1$. In the proof of proposition 3.17, for L.H.S of (3.22), we use the Hölder's inequality as follows:

$$\begin{aligned}
&\left| \sum_{N_1 \ll N_2} N_{\max} \int_{\mathbb{R}} \int_{\mathbb{R}} (Q_{\geq M}^{\sigma_1 \Delta} f_{1,N_1,T})(Q_2^{\sigma_2 \Delta} f_{2,N_2,T})(Q_3^{\sigma_3 \Delta} f_{3,N_3,T}) dxdt \right| \\
&\lesssim ||\mathbf{1}_{[0,T)}||_{L_t^4} \left|\left| \sum_{N_1 \ll N_2} N_{\max} Q_{\geq M}^{\sigma_1 \Delta} f_{1,N_1,T} \right|\right|_{L_{tx}^2} ||Q_2^{\sigma_2 \Delta} f_{2,N_2,T}||_{L_t^8 L_x^4} ||Q_3^{\sigma_3 \Delta} f_{3,N_3,T}||_{L_t^8 L_x^4}.
\end{aligned}$$

We note that $(8, 4)$ is the admissible pair of the Strichartz estimate for $d = 1$. Furthermore for the first inequality in (3.24), we use (3.14) instead of (3.12). For the other part, by the same way of the proof of proposition 3.17, we obtain (3.33) with $T = 1$. □

Proposition 3.22. *Let $s > s_c$ $(= d/2 - 1)$, $0 < T < \infty$ if $d \geq 2$ and $s \geq 0$, $T = 1$ if $d = 1$. We assume σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$. For any dyadic numbers N_1, $N_2 \in 2^{\mathbb{Z}}$ with $N_1 \sim N_2$, we have*

$$\begin{aligned}
&\left(\sum_{N_3 \ll N_2} N_3^{2s} \sup_{||u_3||_{V_{\sigma_3 \Delta}^2 L^2} = 1} \left| N_{\max} \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dxdt \right|^2 \right)^{1/2} \\
&\lesssim T^{\delta} (N_1 \vee 1)^s ||P_{N_1} u_1||_{V_{\sigma_1 \Delta}^2 L^2} (N_2 \vee 1)^s ||P_{N_2} u_2||_{V_{\sigma_2 \Delta}^2 L^2}.
\end{aligned} \tag{3.34}$$

for some $\delta > 0$*, where* $N_{\max} := \max_{1\le j\le 3} N_j$.

Proof. First, we assume $d \geq 2$. We choose $\delta > 0$ satisfying $\delta < (s-s_c)/2$ and $\delta \ll 1$. In the proof of proposition 3.18, for L.H.S of (3.26) and (3.28), we use the Hölder's inequality and (2.1) with $p = 2/(1-2\delta)$ instead of $p = 2$. Then we have

$$||Q^{\sigma_1\Delta}_{\geq M} f_{1,N_1,T}||_{L^2_{tx}} \leq ||\mathbf{1}_{[0,T)}||_{L^{1/\delta}_t} ||Q^{\sigma_1\Delta}_{\geq M} f_{1,N_1,T}||_{L^{2/(1-2\delta)}_t L^2_x} \lesssim T^\delta M^{-(1-2\delta)/2} ||f_{1,N_1,T}||_{V^2_{\sigma_1\Delta}L^2},$$
$$||Q^{\sigma_3\Delta}_{\geq M} f_{3,N_3,T}||_{L^2_{tx}} \leq ||\mathbf{1}_{[0,T)}||_{L^{1/\delta}_t} ||Q^{\sigma_3\Delta}_{\geq M} f_{3,N_3,T}||_{L^{2/(1-2\delta)}_t L^2_x} \lesssim T^\delta M^{-(1-2\delta)/2} ||f_{3,N_3,T}||_{V^2_{\sigma_3\Delta}L^2}.$$

For the other part, by the same way of the proof of proposition 3.18, we obtain (3.34).

Next, we assume $d = 1$. In the proof of proposition 3.18, for L.H.S of (3.27), we use (3.14) instead of (3.12) and for the third inequality in (3.29), we use (3.16) and $V^2_{-,rc} \hookrightarrow U^8$ instead of (3.15) and $V^2_{-,rc} \hookrightarrow U^4$. For the other part, by the same way of the proof of proposition 3.18, we obtain (3.34) with $T = 1$. □

Proposition 3.23.
(i) *Let* $d \geq 4$*,* $s > s_c$ *and* $0 < T < \infty$*. For any* $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\backslash\{0\}$*, any dyadic numbers* $N_1, N_2, N_3 \in 2^{\mathbb{Z}}$ *and* $1 \leq j \leq 3$*, we have*

$$\begin{aligned}&\left| N_j \int_0^T \int_{\mathbb{R}^d} (P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)dxdt \right| \\ &\lesssim T^\delta (N_j \vee 1)^s ||P_{N_1}u_1||_{V^2_{\sigma_1\Delta}L^2} ||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2} ||P_{N_3}u_3||_{V^2_{\sigma_3\Delta}L^2}.\end{aligned} \tag{3.35}$$

for some $\delta > 0$.
(ii) *Let* $d = 1, 2, 3$*,* $s \geq 1$*,* $0 < T < \infty$*. For any* $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\backslash\{0\}$*, any dyadic numbers* $N_1, N_2, N_3 \in 2^{\mathbb{Z}}$ *and* $1 \leq j \leq 3$ *, we have* (3.35).
(iii) *Let* $s > s_c$*,* $0 < T < \infty$ *if* $d = 2, 3$ *and* $s \geq 0$*,* $T = 1$ *if* $d = 1$*. We assume* $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\backslash\{0\}$ *satisfy* $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$*. For any dyadic numbers* $N_1, N_2, N_3 \in 2^{\mathbb{Z}}$ *with* $N_1 \sim N_2 \sim N_3$ *and* $1 \leq j \leq 3$*, we have* (3.35).

Proof. By symmetry, it is enough to prove for $j = 3$. We choose $\delta > 0$ satisfying $\delta < (s - s_c)/2$ and $\delta \ll 1$.

First, we consider the case $d \geq 4$. By the Hölder's inequality and the Sobolev embedding $\dot{W}^{s_c+2\delta-1,6d/(3d-4+12\delta)}(\mathbb{R}^d) \hookrightarrow L^{3d/4}(\mathbb{R}^d)$, we have

$$\begin{aligned}&\left| \int_0^T \int_{\mathbb{R}^d} (P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)dxdt \right| \\ &\lesssim ||\mathbf{1}_{[0,T)}||_{L^{1/\delta}_t} ||P_{N_1}u||_{L^3_t L^{6d/(3d-4)}_x} ||P_{N_2}u_2||_{L^3_t L^{6d/(3d-4)}_x} |||\nabla|^{s_c+2\delta-1}(P_{N_3}u_3)||_{L^p_t L^q_x}\end{aligned}$$

with $(p,q) = (3/(1-3\delta), 6d/(3d-4+12\delta)$ which is the admissible pair of the Strichartz estimate. Therefore we obtain (3.35) by (3.8).

Second, we consider the case $d = 1, 2, 3$ and $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\backslash\{0\}$ are arbitary. By the Hölder's inequality and (3.8), we have

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^d} (P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)dxdt\right| \\
&\lesssim ||\mathbf{1}_{[0,T)}||_{L_t^{4/(4-d)}} ||P_{N_1}u_1||_{L_t^{12/d}L_x^3} ||P_{N_2}u_2||_{L_t^{12/d}L_x^3} ||P_{N_3}u_3||_{L_t^{12/d}L_x^3} \\
&\lesssim T^{1-d/4} ||P_{N_1}u_1||_{V^2_{\sigma_1\Delta}L^2} ||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2} ||P_{N_3}u_3||_{V^2_{\sigma_3\Delta}L^2}
\end{aligned}$$

and obtain (3.35) as $\delta = 1 - d/4$ for $s \geq 1$.

Third, we consider the case $d = 2, 3$ and $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$. In the proof of proposition 3.19, for L.H.S of (3.31), we use the Hölder's inequality and (2.1) with $p = 2/(1-2\delta)$ instead of $p = 2$. Then we have

$$||Q^{\sigma_1\Delta}_{\geq M} f_{1,N_1,T}||_{L^2_{tx}} \leq ||\mathbf{1}_{[0,T)}||_{L_t^{1/\delta}} ||Q^{\sigma_1\Delta}_{\geq M} f_{1,N_1,T}||_{L_t^{2/(1-2\delta)}L_x^2} \lesssim T^\delta M^{-(1-2\delta)/2} ||f_{1,N_1,T}||_{V^2_{\sigma_1\Delta}L^2}.$$

For the other part, by the same way of the proof of proposition 3.19, we obtain (3.35).

Finally, we consider the case $d = 1$ and $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$. In the proof of proposition 3.19, for L.H.S of (3.32), we use (3.16) and $V^2_{-,rc} \hookrightarrow U^8$ instead of (3.15) and $V^2_{-,rc} \hookrightarrow U^4$. For the other part, by the same way of the proof of proposition 3.19, we obtain (3.35) with $T = 1$. □

Proposition 3.22 and Proposition 3.23 imply the following:

Corollary 3.24. *Let $0 < T < \infty$ if $d \geq 2$ and $T = 1$ if $d = 1$. We assume σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)(\sigma_3+\sigma_1) \neq 0$.*
(i) *Let $s > s_c$ if $d \geq 4$, and $s \geq 1$ if $d = 1, 2, 3$. Then the estimate* (3.34) *holds if we replace $\sum_{N_3 \ll N_2}$ by $\sum_{N_3 \lesssim N_2}$.*
(ii) *Let $s > s_c$ if $d = 2, 3$ and $s \geq 0$ if $d = 1$. We assume σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1\sigma_2\sigma_3(1/\sigma_1+1/\sigma_2+1/\sigma_3) > 0$. Then the estimate* (3.34) *holds if we replace $\sum_{N_3 \ll N_2}$ by $\sum_{N_3 \lesssim N_2}$.*

Let (i,j,k) is one of the permutation of $(1,2,3)$. If $\sigma_i + \sigma_j = 0$, then Proposition 3.16 (i) fails only for the case $|\xi_k| \ll |\xi_i| \sim |\xi_j|$. We obtain following estimates for the case $|\xi_k| \ll |\xi_i| \sim |\xi_j|$.

Corollary 3.25. *Let $s > s_c$ if $d \geq 4$, and $s > 1$ if $d = 2, 3$.*
(i) *We assume σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_2 + \sigma_3 = 0$ and $(\sigma_1 + \sigma_2)(\sigma_3 + \sigma_1) \neq 0$. Then for any $0 < T < \infty$, and any dyadic numbers N_2, $N_3 \in 2^{\mathbb{Z}}$ with $N_2 \sim N_3$, we have*

$$\begin{aligned}&\left|\sum_{N_1 \ll N_2} N_1 \int_0^T \int_{\mathbb{R}^d} (P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)dxdt\right| \\ &\lesssim T^{\delta} \left(\sum_{N_1 \ll N_2} (N_1 \vee 1)^{2s} ||P_{N_1}u_1||^2_{V^2_{\sigma_1 \Delta}L^2}\right)^{1/2} ||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2} ||P_{N_3}u_3||_{V^2_{\sigma_3\Delta}L^2}\end{aligned} \tag{3.36}$$

(ii) *We assume σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1 + \sigma_2 = 0$ and $(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$. Then for any $0 < T < \infty$, and any dyadic numbers N_1, $N_2 \in 2^{\mathbb{Z}}$ with $N_1 \sim N_2$, we have*

$$\begin{aligned}&\left(\sum_{N_3 \lesssim N_2} N_3^{2s} \sup_{||u_3||_{V^2_{\sigma_3\Delta}L^2}=1} \left|N_3 \int_0^T \int_{\mathbb{R}^d} (P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)dxdt\right|^2\right)^{1/2} \\ &\lesssim T^{\delta} (N_1 \vee 1)^s ||P_{N_1}u_1||_{V^2_{\sigma_1\Delta}L^2} (N_2 \vee 1)^s ||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2}.\end{aligned} \tag{3.37}$$

for some $\delta > 0$.

Proof. By the Hölder's inequality, $V^2_{-,rc}(\mathbb{R}; L^2) \hookrightarrow L^{\infty}(\mathbb{R}; L^2)$ and (3.12), we have

$$\begin{aligned}&\left|N_1 \int_0^T \int_{\mathbb{R}^d} (P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)dxdt\right| \\ &\leq N_1 ||\mathbf{1}_{[0,T)}||_{L^2_t} ||(P_{N_1}u_1)(P_{N_2}u_2)||_{L^2_{tx}} ||P_{N_3}u_3||_{L^{\infty}_t L^2_x} \\ &\lesssim T^{1/2} N_1^{s_c+1} \left(\frac{N_1}{N_2}\right)^{1/2} ||P_{N_1}u_1||_{V^2_{\sigma_1\Delta}L^2} ||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2} ||P_{N_3}u_3||_{V^2_{\sigma_3\Delta}L^2}\end{aligned} \tag{3.38}$$

for $N_1 \ll N_2$. We use (3.38) for the summation for $N_1 < 1$ and use (3.35) with $j = 1$ for the summation for $1 \leq N_1 \ll N_2$. Then, we obtain (3.36) by the Cauchy-Schwarz inequality for the dyadic sum.

The estimate (3.37) is obtained by using (3.35) with $j = 3$. □

3.5 Proof of the well-posedness and the scattering

In this section, we prove Theorems 3.3, 3.6, 3.10 and Corollary 3.5. We define the map $\Phi(u,v,w) = (\Phi^{(1)}_{T,\alpha,u_0}(w,v), \Phi^{(1)}_{T,\beta,v_0}(\overline{w},v), \Phi^{(2)}_{T,\gamma,w_0}(u,\overline{v}))$ as

$$\Phi^{(1)}_{T,\sigma,\varphi}(f,g)(t) := e^{it\sigma\Delta}\varphi - I^{(1)}_{T,\sigma}(f,g)(t),$$
$$\Phi^{(2)}_{T,\sigma,\varphi}(f,g)(t) := e^{it\sigma\Delta}\varphi + I^{(2)}_{T,\sigma}(f,g)(t),$$

where

$$I^{(1)}_{T,\sigma}(f,g)(t) := \int_0^t \mathbf{1}_{[0,T)}(t')e^{i(t-t')\sigma\Delta}(\nabla\cdot f(t'))g(t')dt',$$
$$I^{(2)}_{T,\sigma}(f,g)(t) := \int_0^t \mathbf{1}_{[0,T)}(t')e^{i(t-t')\sigma\Delta}\nabla(f(t')\cdot g(t'))dt'.$$

To prove the existence of the solution of (3.1), we prove that Φ is a contraction map on a closed subset of $\dot{Z}^s_\alpha([0,T])\times\dot{Z}^s_\beta([0,T])\times\dot{Z}^s_\gamma([0,T])$ or $Z^s_\alpha([0,T])\times Z^s_\beta([0,T])\times Z^s_\gamma([0,T])$. Key estimates are the followings:

Proposition 3.26. *We assume that α, β, $\gamma\in\mathbb{R}\backslash\{0\}$ satisfy the condition in* Theorem 3.3. *Then for $s_c = d/2-1$ and any $0 < T \le \infty$, we have*

$$||I^{(1)}_{T,\alpha}(w,v)||_{\dot{Z}^{s_c}_\alpha} \lesssim ||w||_{\dot{Y}^{s_c}_\gamma}||v||_{\dot{Y}^{s_c}_\beta}, \tag{3.39}$$
$$||I^{(1)}_{T,\beta}(\overline{w},u)||_{\dot{Z}^{s_c}_\beta} \lesssim ||w||_{\dot{Y}^{s_c}_\gamma}||u||_{\dot{Y}^{s_c}_\alpha}, \tag{3.40}$$
$$||I^{(2)}_{T,\gamma}(u,\overline{v})||_{\dot{Z}^{s_c}_\gamma} \lesssim ||u||_{\dot{Y}^{s_c}_\alpha}||v||_{\dot{Y}^{s_c}_\beta}. \tag{3.41}$$

Proof. We prove only (3.41) since (3.39) and (3.40) are proved by the same way. We show the estimate

$$||I^{(2)}_{T,\gamma}(u,\overline{v})||_{\dot{Z}^{s}_\gamma} \lesssim ||u||_{\dot{Y}^{s_c}_\alpha}||v||_{\dot{Y}^{s}_\beta} + ||u||_{\dot{Y}^{s}_\alpha}||v||_{\dot{Y}^{s_c}_\beta} \tag{3.42}$$

for $s\ge 0$. (3.41) follows from (3.42) as $s = s_c$. We put $(u_1,u_2) := (u,\overline{v})$ and $(\sigma_1,\sigma_2,\sigma_3) := (\alpha,-\beta,-\gamma)$. To obtain (3.42), we use the argument of the proof of

Theorem 3.2 in [25]. We define

$$J_1 := \left\| \sum_{N_2} \sum_{N_1 \ll N_2} I^{(2)}_{T,-\sigma_3}(P_{N_1}u_1, P_{N_2}u_2) \right\|_{\dot{Z}^s_{-\sigma_3}},$$

$$J_2 := \left\| \sum_{N_2} \sum_{N_1 \sim N_2} I^{(2)}_{T,-\sigma_3}(P_{N_1}u_1, P_{N_2}u_2) \right\|_{\dot{Z}^s_{-\sigma_3}},$$

$$J_3 := \left\| \sum_{N_1} \sum_{N_2 \ll N_1} I^{(2)}_{T,-\sigma_3}(P_{N_1}u_1, P_{N_2}u_2) \right\|_{\dot{Z}^s_{-\sigma_3}},$$

where implicit constants in $\ll$ actually depend on σ_1, σ_2, σ_3.

First, we prove the estimate for J_1. By Theorem 2.4, we have

$$J_1 \leq \left\{ \sum_{N_3} N_3^{2s} \left(\sum_{N_2 \sim N_3} \left\| e^{it\sigma_3\Delta} P_{N_3} \sum_{N_1 \ll N_2} I^{(2)}_{T,-\sigma_3}(P_{N_1}u_1, P_{N_2}u_2) \right\|_{U^2(\mathbb{R};L^2)} \right)^2 \right\}^{1/2}$$

$$= \left\{ \sum_{N_3} N_3^{2s} \left(\sum_{N_2 \sim N_3} \sup_{||u_3||_{V^2_{\sigma_3\Delta}L^2}=1} \left| \sum_{N_1 \ll N_2} N_3 \int_0^T \int_{\mathbb{R}^d} (P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3) dx dt \right| \right)^2 \right\}^{1/2}.$$

Therefore by Proposition 3.17, we have

$$J_1 \lesssim \left\{ \sum_{N_3} N_3^{2s} \left(\sum_{N_2 \sim N_3} \left(\sum_{N_1 \ll N_2} N_1^{2s_c} ||P_{N_1}u_1||^2_{V^2_{\sigma_1}L^2} \right)^{1/2} ||P_{N_2}u_2||_{V^2_{\sigma_2\Delta}L^2} \right)^2 \right\}^{1/2}$$

$$\lesssim \left(\sum_{N_1} N_1^{2s_c} ||P_{N_1}u_1||^2_{V^2_{\sigma_1\Delta}L^2} \right)^{1/2} \left(\sum_{N_2} N_2^{2s} ||P_{N_2}u_2||^2_{V^2_{\sigma_2\Delta}L^2} \right)^{1/2}$$

$$= ||u_1||_{\dot{Y}^{s_c}_{\sigma_1}} ||u_2||_{\dot{Y}^s_{\sigma_2}}.$$

Second, we prove the estimate for J_2. By Theorem 2.4, we have

$$J_2 \leq \sum_{N_2} \sum_{N_1 \sim N_2} \left(\sum_{N_3 \lesssim N_2} N_3^{2s} ||e^{it\sigma_3\Delta} P_{N_3} I^{(2)}_{T,-\sigma_3}(P_{N_1}u_1, P_{N_2}u_2)||^2_{U^2(\mathbb{R};L^2)} \right)^{1/2}$$

$$= \sum_{N_2} \sum_{N_1 \sim N_2} \left(\sum_{N_3 \lesssim N_2} N_3^{2s} \sup_{||u_3||_{V^2_{\sigma_3\Delta}L^2}=1} \left| \int_0^T \int_{\mathbb{R}^d} (P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3) dx dt \right|^2 \right)^{1/2}.$$

Therefore by Corollary 3.20 and Cauchy-Schwarz inequality for dyadic sum, we have

$$\begin{aligned}
J_2 &\lesssim \sum_{N_2} \sum_{N_1 \sim N_2} N_1^{s_c} ||P_{N_1} u_1||_{V^2_{\sigma_1 \Delta} L^2} N_2^s ||P_{N_2} u_2||_{V^2_{\sigma_2 \Delta} L^2} \\
&\lesssim \left(\sum_{N_1} N_1^{2s_c} ||P_{N_1} u_1||^2_{V^2_{\sigma_1 \Delta} L^2} \right)^{1/2} \left(\sum_{N_2} N_2^{2s} ||P_{N_2} u_2||^2_{V^2_{\sigma_2 \Delta} L^2} \right)^{1/2} \\
&= ||u_1||_{\dot{Y}^{s_c}_{\sigma_1}} ||u_2||_{\dot{Y}^s_{\sigma_2}}.
\end{aligned}$$

Finally, we prove the estimate for J_3. By the same manner as for J_1, we have

$$J_3 \lesssim ||u_1||_{\dot{Y}^s_{\sigma_1}} ||u_2||_{\dot{Y}^{s_c}_{\sigma_2}}.$$

Therefore, we obtain (3.42) since $||u_1||_{\dot{Y}^s_{\sigma_1}} = ||u||_{\dot{Y}^s_\alpha}$ and $||u_2||_{\dot{Y}^{s_c}_{\sigma_2}} = ||v||_{\dot{Y}^{s_c}_\beta}$. □

Corollary 3.27. *We assume that α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ satisfy the condition in* Theorem 3.3. *Then for $s \geq s_c$ $(= d/2 - 1)$ and any $0 < T \leq \infty$, we have*

$$||I^{(1)}_{T,\alpha}(w, v)||_{Z^s_\alpha} \lesssim ||w||_{Y^s_\gamma} ||v||_{Y^s_\beta}, \tag{3.43}$$

$$||I^{(1)}_{T,\beta}(\overline{w}, u)||_{Z^s_\beta} \lesssim ||w||_{Y^s_\gamma} ||u||_{Y^s_\alpha}, \tag{3.44}$$

$$||I^{(2)}_{T,\gamma}(u, \overline{v})||_{Z^s_\gamma} \lesssim ||u||_{Y^s_\alpha} ||v||_{Y^s_\beta}. \tag{3.45}$$

Proof. We prove only (3.45) since (3.43) and (3.44) are proved by the same way. By (3.42), we have

$$\begin{aligned}
||I^{(2)}_{T,\gamma}(u, \overline{v})||_{Z^s_\gamma} &= ||I^{(2)}_{T,\gamma}(u, \overline{v})||_{\dot{Z}^0_\gamma} + ||I^{(2)}_{T,\gamma}(u, \overline{v})||_{\dot{Z}^s_\gamma} \\
&\lesssim ||u||_{\dot{Y}^{s_c}_\alpha} ||v||_{\dot{Y}^0_\beta} + ||u||_{\dot{Y}^0_\alpha} ||v||_{\dot{Y}^{s_c}_\beta} + ||u||_{\dot{Y}^{s_c}_\alpha} ||v||_{\dot{Y}^s_\beta} + ||u||_{\dot{Y}^s_\alpha} ||v||_{\dot{Y}^{s_c}_\beta}.
\end{aligned}$$

We decompose $u = P_0 u + (Id - P_0)u$ and $v = P_0 v + (Id - P_0)v$. Since

$$\begin{aligned}
&||P_0 u||_{\dot{Y}^{s_c}_\alpha} \lesssim ||P_0 u||_{\dot{Y}^0_\alpha}, \ ||(Id - P_0)u||_{\dot{Y}^{s_c}_\alpha} \lesssim ||(Id - P_0)u||_{\dot{Y}^s_\alpha}, \\
&||P_0 v||_{\dot{Y}^{s_c}_\beta} \lesssim ||P_0 v||_{\dot{Y}^0_\beta}, \ ||(Id - P_0)v||_{\dot{Y}^{s_c}_\beta} \lesssim ||(Id - P_0)v||_{\dot{Y}^s_\beta}
\end{aligned}$$

for $s \geq s_c$, we obtain (3.45). □

Proposition 3.28.

(i) *Let $d \geq 2$. We assume that α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ and $s \in \mathbb{R}$ satisfy the condition in* Theorem 3.6. *Then there exists $\delta > 0$ such that for any $0 < T < \infty$, we have*

$$||I^{(1)}_{T,\alpha}(w, v)||_{Z^s_\alpha} \lesssim T^\delta ||w||_{Z^s_\gamma} ||v||_{Z^s_\beta}, \tag{3.46}$$

$$||I^{(1)}_{T,\beta}(\overline{w}, u)||_{Z^s_\beta} \lesssim T^\delta ||w||_{Z^s_\gamma} ||u||_{Z^s_\alpha}, \tag{3.47}$$

$$||I^{(2)}_{T,\gamma}(u, \overline{v})||_{Z^s_\gamma} \lesssim T^\delta ||u||_{Z^s_\alpha} ||v||_{Z^s_\beta}. \tag{3.48}$$

(ii) *Let* $d = 1$. *We assume that* $\alpha,\ \beta,\ \gamma \in \mathbb{R}\backslash\{0\}$ *and* $s \in \mathbb{R}$ *satisfy the condition in* Theorem 3.6 *except the case* $1 > s \geq 1/2$, $\theta < 0$ *and* $(\alpha - \gamma)(\beta + \gamma) \neq 0$. *Then we have* (3.46)–(3.48) *with* $T = 1$.

Proof. We obtain (3.46)–(3.48) by using Proposition 3.21 and Corollary 3.24 if $\alpha \neq \beta$, using Corollary 3.25 if $d \geq 2$ and $\alpha = \beta$ instead of Proposition 3.17 and Corollary 3.20 in the proof of Proposition 3.26. □

Proof of Theorem 3.3. We prove only the homogeneous case. The inhomogeneous case is also proved by the same way. For $r > 0$, we define

$$\dot{X}^s_r(I) := \left\{ (u, v, w) \in \dot{X}^s(I) \,\middle|\, ||u||_{\dot{Z}^s_\alpha(I)},\ ||v||_{\dot{Z}^s_\beta(I)},\ ||w||_{\dot{Z}^s_\gamma(I)} \leq 2r \right\} \tag{3.49}$$

which is a closed subset of $\dot{X}^s(I)$. Let $(u_0,\ v_0,\ w_0) \in B_r(\dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d))$ be given. For $(u, v, w) \in \dot{X}^{s_c}_r([0, \infty))$, we have

$$\begin{aligned}
||\Phi^{(1)}_{T,\alpha,u_0}(w, v)||_{\dot{Z}^{s_c}_\alpha([0,\infty))} &\leq ||u_0||_{\dot{H}^{s_c}} + C||w||_{\dot{Z}^{s_c}_\gamma([0,\infty))}||v||_{\dot{Z}^{s_c}_\beta([0,\infty))} \leq r(1 + 4Cr),\\
||\Phi^{(1)}_{T,\beta,v_0}(\overline{w}, u)||_{\dot{Z}^{s_c}_\beta([0,\infty))} &\leq ||v_0||_{\dot{H}^{s_c}} + C||w||_{\dot{Z}^{s_c}_\gamma([0,\infty))}||u||_{\dot{Z}^{s_c}_\alpha([0,\infty))} \leq r(1 + 4Cr),\\
||\Phi^{(2)}_{T,\gamma,w_0}(u, \overline{v})||_{\dot{Z}^{s_c}_\gamma([0,\infty))} &\leq ||w_0||_{\dot{H}^{s_c}} + C||u||_{\dot{Z}^{s_c}_\alpha([0,\infty))}||v||_{\dot{Z}^{s_c}_\beta([0,\infty))} \leq r(1 + 4Cr)
\end{aligned}$$

and

$$\begin{aligned}
||\Phi^{(1)}_{T,\alpha,u_0}(w_1, v_1) - \Phi^{(1)}_{T,\alpha,u_0}(w_2, v_2)||_{\dot{Z}^{s_c}_\alpha([0,\infty))} &\leq 2Cr \left(||w_1 - w_2||_{\dot{Z}^{s_c}_\gamma([0,\infty))} + ||v_1 - v_2||_{\dot{Z}^{s_c}_\beta([0,\infty))} \right),\\
||\Phi^{(1)}_{T,\beta,v_0}(\overline{w_1}, u_1) - \Phi^{(1)}_{T,\beta,v_0}(\overline{w_2}, u_2)||_{\dot{Z}^{s_c}_\beta([0,\infty))} &\leq 2Cr \left(||w_1 - w_2||_{\dot{Z}^{s_c}_\gamma([0,\infty))} + ||u_1 - u_2||_{\dot{Z}^{s_c}_\alpha([0,\infty))} \right),\\
||\Phi^{(2)}_{T,\gamma,w_0}(u_1, \overline{v_1}) - \Phi^{(1)}_{T,\gamma,w_0}(u_2, \overline{v_2})||_{\dot{Z}^{s_c}_\gamma([0,\infty))} &\leq 2Cr \left(||u_1 - u_2||_{\dot{Z}^{s_c}_\alpha([0,\infty))} + ||v_1 - v_2||_{\dot{Z}^{s_c}_\beta([0,\infty))} \right)
\end{aligned}$$

by Proposition 3.26 and

$$||e^{i\sigma t\Delta}\varphi||_{\dot{Z}^{s_c}_\sigma([0,\infty))} \leq ||\mathbf{1}_{[0,\infty)}e^{i\sigma t\Delta}\varphi||_{\dot{Z}^{s_c}_\sigma} \leq ||\varphi||_{\dot{H}^{s_c}},$$

where C is an implicit constant in (3.39)–(3.41). Therefore if we choose r satisfying

$$r < (4C)^{-1},$$

then Φ is a contraction map on $\dot{X}^{s_c}_r([0, \infty))$. This implies the existence of the solution of the system (3.1) and the uniqueness in the ball $\dot{X}^{s_c}_r([0, \infty))$. The Lipschitz continuously of the flow map is also proved by similar argument. □

Theorem 3.6 except the case $d = 1$, $1 > s \geq 1/2$, $\theta < 0$ and $(\alpha - \gamma)(\beta + \gamma) \neq 0$ is proved by the same way for the proof of Theorem 3.3.

Remark 3.29. *For $d = 1$ and $s > s_c$ (in particular $s \geq 0$), we can assume the H^s-norm of the initial data is small enough by the scaling* (3.2) *with large λ since $s_c < 0$.*

Proof of Corollary 3.5. We prove only the homogeneous case. The inhomogeneous case is also proved by the same way. By Proposition 3.26, the global solution $(u, v, w) \in \dot{X}^{s_c}([0, \infty))$ of (3.1) which was constructed in Theorem 3.3 satisfies

$$\begin{aligned} &N^{s_c}(e^{-it\alpha\Delta}P_N I^{(1)}_{\infty,\alpha}(w, v),\ e^{-it\beta\Delta}P_N I_{\infty,\beta}(\overline{w}, u),\ e^{-it\gamma\Delta}P_N I^{(2)}_{\infty,\gamma}(u, \overline{v})) \\ &\in V^2(\mathbb{R}; L^2) \times V^2(\mathbb{R}; L^2) \times V^2(\mathbb{R}; L^2) \end{aligned}$$

for each $N \in 2^{\mathbb{Z}}$. This implies that

$$(u_+, v_+, w_+) := \lim_{t\to\infty}(u_0 - e^{-it\alpha\Delta}I^{(1)}_{\infty,\alpha}(w, v),\ v_0 - e^{-it\beta\Delta}I_{\infty,\beta}(\overline{w}, u),\ w_0 + e^{-it\gamma\Delta}I^{(2)}_{\infty,\gamma}(u, \overline{v}))$$

exists in $\dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d)$ by Proposition 2.3 (iv). Then we obtain

$$(u, v, w) - (e^{it\alpha\Delta}u_+, e^{it\beta\Delta}v_+, e^{it\gamma\Delta}w_+) \to 0$$

in $\dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d)$ as $t \to \infty$. □

Theorem 3.10 is proved by using the estimate (3.39) and (3.43) for $(\alpha, \beta, \gamma) = (-1, 1, 1)$.

3.6 A priori estimates

In this section, we prove Theorem 3.8. We define

$$\begin{aligned} M(u, v, w) &:= 2||u||^2_{L^2_x} + ||v||^2_{L^2_x} + ||w||^2_{L^2_x} \\ H(u, v, w) &:= \alpha||\nabla u||^2_{L^2_x} + \beta||\nabla v||^2_{L^2_x} + \gamma||\nabla w||^2_{L^2_x} + 2\mathrm{Re}(w, \nabla(u \cdot \overline{v}))_{L^2_x} \end{aligned}$$

and put $M_0 := M(u_0, v_0, w_0)$, $H_0 := H(u_0, v_0, w_0)$.

Proposition 3.30. *For the smooth solution (u, v, w) of the system* (3.1)*, we have*

$$M(u, v, w) = M_0,\ H(u, v, w) = H_0$$

Proof. For the system

$$(i\partial_t + \alpha\Delta)u = -(\nabla \cdot w)v \tag{3.50}$$

$$(i\partial_t + \beta\Delta)v = -(\nabla \cdot \overline{w})u \tag{3.51}$$

$$(i\partial_t + \gamma\Delta)w = \nabla(u \cdot \overline{w}), \tag{3.52}$$

We have the conservation law for M by calculating

$$\mathrm{Im}\int_{\mathbb{R}^d}\{(-2u\times\overline{(3.50)})+(\overline{v}\times(3.51))+(\overline{w}\times(3.52))\}dx$$

and for H by calculating

$$\mathrm{Re}\int_{\mathbb{R}^d}\{(\partial_t u\times\overline{(3.50)})+(\partial_t\overline{v}\times(3.51))+(\partial_t\overline{w}\times(3.52))\}dx.$$

□

The following a priori estimates imply Theorem 3.8.

Proposition 3.31. *We assume α, β and γ have the same sign and put*

$$\rho_{max}:=\max\{|\alpha|,|\beta|,|\gamma|\},\ \ \rho_{min}:=\min\{|\alpha|,|\beta|,|\gamma|\}.$$

(i) *Let $d=1$, 2. For the data $(u_0,v_0,w_0)\in H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$ satisfying*

$$M_0^{1-d/4}\ll\rho_{min}, \tag{3.53}$$

there exists $C>0$ such that for the solution $(u,v,w)\in\big(C([0,T];H^1(\mathbb{R}^d))\big)^3$ of (3.1)*, the following estimate holds:*

$$\sup_{0\le t\le T}\Big(||\nabla u(t)||_{L_x^2}^2+||\nabla v(t)||_{L_x^2}^2+||\nabla w(t)||_{L_x^2}^2\Big)\le\frac{H_0+CM_0^{1-d/4}}{\rho_{min}-CM_0^{1-d/4}}. \tag{3.54}$$

(ii) *Let $d=3$. If the data $(u_0,v_0,w_0)\in H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$ satisfies*

$$||\nabla u_0||_{L_x^2}^2+||\nabla v_0||_{L_x^2}^2+||\nabla w_0||_{L_x^2}^2<\epsilon^2/\rho_{max} \tag{3.55}$$

for some ϵ with $0<\epsilon\ll 1$, then for the solution $(u,v,w)\in\big(C([0,T];H^1(\mathbb{R}^d))\big)^3$ of (3.1)*, the following estimate holds:*

$$\sup_{0\le t\le T}\Big(||\nabla u(t)||_{L_x^2}^2+||\nabla v(t)||_{L_x^2}^2+||\nabla w(t)||_{L_x^2}^2\Big)<3\epsilon^2/\rho_{\min}. \tag{3.56}$$

Proof. We put

$$F=F(t):=||\nabla u(t)||_{L_x^2}^2+||\nabla v(t)||_{L_x^2}^2+||\nabla w(t)||_{L_x^2}^2.$$

Since α, β and γ are same sign, we have

$$F\le\frac{1}{\rho_{min}}(H(u,v,w)+2|((\nabla\cdot w),(u\cdot\overline{v}))_{L_x^2}|).$$

By the Cauchy-Schwarz inequality and the Gagliardo-Nirenberg inequality we have

$$\begin{aligned}|((\nabla\cdot w),(u\cdot\overline{v}))_{L^2_x}|) &\le ||\nabla\cdot w||_{L^2_x}||u||_{L^4_x}||v||_{L^4_x}\\ &\lesssim ||\nabla\cdot w||_{L^2_x}||u||_{L^2_x}^{1-d/4}||\nabla u||_{L^2_x}^{d/4}||v||_{L^2_x}^{1-d/4}||\nabla v||_{L^2_x}^{d/4}\\ &\lesssim M(u,v,w)^{1-d/4}F^{(d+2)/4}\end{aligned}$$

for $d\le 4$. Therefore, by using Proposition 3.30, we obtain

$$F\le \frac{1}{\rho_{min}}\left(H_0+CM_0^{1-d/4}F^{(d+2)/4}\right) \tag{3.57}$$

for some constant $C>0$. For $d\le 2$ we have $F^{(d+2)/4}\le 1+F$ because of $(d+2)/4\le 1$. Therefore if (3.53) holds, then the estimate (3.54) follows from (3.57).

By the same argument as above, we obtain

$$H_0\le \rho_{max}F(0)+2|((\nabla\cdot w(0)),(u(0)\cdot\overline{v(0)}))_{L^2_x}|\le \rho_{max}F(0)+CM_0^{1-d/4}F(0)^{(d+2)/4}$$

for some constant $C>0$ and $d\le 4$. Therefore if (3.55) holds for some ϵ with $0<\epsilon\ll 1$, we have

$$H_0<\epsilon^2(1+CM_0^{1-d/4}\rho_{max}^{-(d+2)/4}\epsilon^{(d-2)/2}).$$

By choosing ϵ sufficiently small, we have $H_0<2\epsilon^2$ for $d=3$ (and also $d=4$). Therefore the estimate

$$F\le \frac{1}{\rho_{min}}\left(2\epsilon^2+CM_0^{1-d/4}F^{(d+2)/4}\right) \tag{3.58}$$

follows from (3.57). If there exists $t_0\in[0,T]$ such that $F(t_0)<4\epsilon^2/\rho_{min}$ for sufficiently small ϵ, then we have $F(t_0)<3\epsilon^2/\rho_{min}$ by (3.58). Since $F(0)<\epsilon^2/\rho_{min}<4\epsilon^2/\rho_{min}$ and $F(t)$ is continuous with respect to t, we obtain (3.56). □

3.7 C^2-ill-posedness

In this section, we prove Theorem 3.9. We rewrite Theorem 3.9 as follows:

Theorem 3.32. *Let $d\ge 1$, $0<T\ll 1$ and $\alpha,\ \beta,\ \gamma\in\mathbb{R}\backslash\{0\}$. We assume $s\in\mathbb{R}$ if $(\alpha-\gamma)(\beta+\gamma)=0$, $s<1$ if $\alpha\beta\gamma(1/\alpha-1/\beta-1/\gamma)=0$, and $s<1/2$ if $\alpha\beta\gamma(1/\alpha-1/\beta-1/\gamma)<0$. Then for every $C>0$ there exist $f,\ g\in H^s(\mathbb{R}^d)$ such that*

$$\sup_{0\le t\le T}\left|\left|\int_0^t e^{i(t-t')\gamma\Delta}\nabla((e^{it'\alpha\Delta}f)(\overline{e^{it'\beta\Delta}g}))dt'\right|\right|_{H^s}\ge C||f||_{H^s}||g||_{H^s}. \tag{3.59}$$

Proof. We prove only for $d = 1$. For $d \geq 2$, it is enough to replace D_1, D_2 and D by $D_1 \times [0,1]^{d-1}$, $D_2 \times [0,1]^{d-1}$ and $D \times [1/2,1]^{d-1}$ in the following argument. We use the argument of the proof of Theorem 1 in [56]. For the sets D_1, $D_2 \subset \mathbb{R}$, we define the functions f, $g \in H^s(\mathbb{R})$ as

$$\widehat{f}(\xi) = \mathbf{1}_{D_1}(\xi),\ \widehat{g}(\xi) = \mathbf{1}_{D_2}(\xi).$$

First, we consider the case $(\alpha - \gamma)(\beta + \gamma) = 0$. We assume $\alpha - \gamma = 0$. (For the case $\beta + \gamma = 0$ is proved by similar argument.) We put $M := -(\beta + \gamma)/2\gamma$, then we have

$$\alpha|\xi_1|^2 - \beta|\xi - \xi_1|^2 - \gamma|\xi|^2 = 2\gamma\{\xi_1 - M(\xi - \xi_1)\}(\xi - \xi_1).$$

For $N \gg 1$, we define the sets D_1, D_2 and $D \subset \mathbb{R}$ as

$$D_1 := [N,\ N + N^{-1}],\ D_2 := [N^{-1},\ 2N^{-1}],\ D := [N + 3N^{-1}/2,\ N + 2N^{-1}]$$

Then, we have

$$||f||_{H^s} \sim N^{s-1/2},\ ||g||_{H^s} \sim N^{-1/2},\ |(\widehat{f} * \widehat{g})(\xi)| \gtrsim N^{-1}\mathbf{1}_D(\xi)$$

and

$$\int_0^t e^{-it'(\alpha|\xi_1|^2 - \beta|\xi-\xi_1|^2 - \gamma|\xi|^2)} dt' \sim t$$

for any $\xi \in D_1$ satisfying $\xi - \xi_1 \in D_2$ and $0 \leq t \ll 1$. This implies

$$\sup_{0 \leq t \leq T} \left|\left| \int_0^t e^{i(t-t')\gamma\Delta} \nabla((e^{it'\alpha\Delta} f)(\overline{e^{it'\beta\Delta} g})) dt' \right|\right|_{H^s} \gtrsim N^{s-1/2}$$

Therefore we obtain (3.59) because $s - 1/2 > s - 1$ for any $s \in \mathbb{R}$.

Second, we consider the case $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) = 0$. We put $M := \gamma/(\alpha - \gamma)$, then $M \neq -1$ since $\alpha \neq 0$ and we have

$$\alpha|\xi_1|^2 - \beta|\xi - \xi_1|^2 - \gamma|\xi|^2 = (\alpha - \gamma)|\xi_1 - M(\xi - \xi_1)|^2.$$

For $N \gg 1$, we define the sets D_1, D_2 and $D \subset \mathbb{R}$ as

$$D_1 := [N,\ N+1],\ D_2 := [N/M,\ N/M + 1/|M|],\ D := [(1+1/M)N + 1/2,\ (1+1/M)N + 1].$$

Then, we have

$$||f||_{H^s} \sim N^s,\ ||g||_{H^s} \sim N^s,\ |(\widehat{f} * \widehat{g})(\xi)| \gtrsim \mathbf{1}_D(\xi)$$

and

$$\int_0^t e^{-it'(\alpha|\xi_1|^2-\beta|\xi-\xi_1|^2-\gamma|\xi|^2)}dt' \sim t$$

for any $\xi \in D_1$ satisfying $\xi - \xi_1 \in D_2$ and $0 \le t \ll 1$. This implies

$$\sup_{0\le t\le T}\left|\left|\int_0^t e^{i(t-t')\gamma\Delta}\nabla((e^{it'\alpha\Delta}f)(\overline{e^{it'\beta\Delta}g}))dt'\right|\right|_{H^s} \gtrsim N^{s+1}$$

Therefore we obtain (3.59) because $s+1 > 2s$ for any $s < 1$.

Finally, we consider the case $(\alpha-\gamma)(\beta+\gamma) \neq 0$ and $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) < 0$. We put

$$M_\pm := \frac{\gamma}{\alpha-\gamma} \pm \frac{1}{\alpha-\gamma}\sqrt{-\alpha\beta\gamma\left(\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}\right)},$$

then $M_\pm \in \mathbb{R}$ and $M_+ \neq M_-$ since $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) < 0$, and we have

$$\alpha|\xi_1|^2 - \beta|\xi-\xi_1|^2 - \gamma|\xi|^2 = (\alpha+\gamma)\{\xi_1 - M_+(\xi-\xi_1)\}\{\xi_1 - M_-(\xi-\xi_1)\}.$$

Because $M_+ \neq M_-$, at least one of M_+ and M_- is not equal to -1. We can assume $M_+ \neq -1$ without loss of generality. For $N \gg 1$, we define the sets D_1, D_2 and $D \subset \mathbb{R}$ as

$$D_1 := [N,\ N+N^{-1}],\ D_2 := [N/M_+,\ N/M_+ + N^{-1}/|M_+|],$$
$$D := [(1+1/M_+)N + N^{-1}/2,\ (1+1/M_+)N + N^{-1}].$$

Then, we have

$$||f||_{H^s} \sim N^{s-1/2},\ ||g||_{H^s} \sim N^{s-1/2},\ |(\widehat{f} * \widehat{g})(\xi)| \gtrsim N^{-1}\mathbf{1}_D(\xi)$$

and

$$\int_0^t e^{-it'(\alpha|\xi_1|^2-\beta|\xi-\xi_1|^2-\gamma|\xi|^2)}dt' \sim t$$

for any $\xi \in D_1$ satisfying $\xi - \xi_1 \in D_2$ and $0 \le t \ll 1$. This implies

$$\sup_{0\le t\le T}\left|\left|\int_0^t e^{i(t-t')\gamma\Delta}\nabla((e^{it'\alpha\Delta}f)(\overline{e^{it'\beta\Delta}g}))dt'\right|\right|_{H^s} \gtrsim N^{s-1/2}.$$

Therefore we obtain (3.59) because $s - 1/2 > 2s - 1$ for any $s < 1/2$. □

3.8 Bilinear estimates for 1D Bourgain norm

In this section, we give the bilinear estimates for the standard 1-dimensional Bourgain norm under the condition $(\alpha-\gamma)(\beta+\gamma) \neq 0$ and $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) \neq 0$. Which estimates imply the well-posedness of (3.1) for $1 > s \ge 1/2$ as the solution (u, v, w) be in the Bourgain space $X^s_\alpha([0,T]) \times X^s_\beta([0,T]) \times X^s_\gamma([0,T])$.

Lemma 3.33. *Let σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $(\sigma_2+\sigma_3)(\sigma_3+\sigma_1) \neq 0$ and (τ_1,ξ_1), (τ_2,ξ_2), $(\tau_3,\xi_3) \in \mathbb{R}\times\mathbb{R}$ satisfy $\tau_1+\tau_2+\tau_3 = 0$, $\xi_1+\xi_2+\xi_3 = 0$. If there exist $1 \leq i,j \leq 3$ such that $|\xi_i| \ll |\xi_j|$, then we have*

$$\max_{1\leq j\leq 3} |\tau_j + \sigma_j \xi_j^2| \gtrsim \xi_3^2.$$

Proof. For the case $\sigma_1+\sigma_2 \neq 0$, proof was complete in Lemma 3.16. We assume $\sigma_1+\sigma_2 = 0$. Then we have

$$\begin{aligned} M_0 :=& \max\{|\tau_1+\sigma_1\xi_1^2|, |\tau_2+\sigma_2\xi_2^2|, |\tau_3+\sigma_3\xi_3^2|\} \\ &\gtrsim |\sigma_1\xi_1^2+\sigma_2\xi_2^2+\sigma_3\xi_3^2| \\ &= |\xi_3||(\sigma_1+\sigma_3)\xi_3+2\sigma_1\xi_2| \\ &= |\xi_3||(\sigma_2+\sigma_3)\xi_3+2\sigma_2\xi_1| \end{aligned}$$

by the triangle inequality. Therefore if $|\xi_i| \ll |\xi_j|$ for some $1 \leq i,j \leq 3$, then we have $M_0 \gtrsim \xi_3^2$. □

Lemma 3.34. *We assume σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $\theta := \sigma_1\sigma_2\sigma_3(1/\sigma_1+1/\sigma_2+1/\sigma_3) \neq 0$. For any $(\tau,\xi) \in \mathbb{R}\times\mathbb{R}$ with $|\xi| \geq 1$ and $b > 1/2$, we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{d\tau_1 d\xi_1}{\langle\tau_1+\sigma_1\xi_1^2\rangle^{2b}\langle\tau-\tau_1+\sigma_2(\xi-\xi_1)^2\rangle^{2b}} \lesssim \langle(\sigma_1+\sigma_2)(\tau-\sigma_3\xi^2)+\theta\xi^2\rangle^{-1/2} \quad (3.60)$$

and

$$\int_{|\xi_1|\gg|\xi-\xi_1| \text{ or } |\xi_1|\ll|\xi-\xi_1|}\int_{\mathbb{R}} \frac{d\tau_1 d\xi_1}{\langle\tau_1+\sigma_1\xi_1^2\rangle^{2b}\langle\tau-\tau_1+\sigma_2(\xi-\xi_1)^2\rangle^{2b}} \lesssim \langle\xi\rangle^{-1}, \quad (3.61)$$

where implicit constants in $\ll$ actually depend on σ_1, σ_2.

Proof. We put $I(\tau,\xi)$:=(L.H.S of (3.60)). By Lemma 2.3, (2.8) in [47], we have

$$I(\tau,\xi) \lesssim \int_{\mathbb{R}} \frac{d\xi_1}{\langle\sigma_1\xi_1^2+\sigma_2(\xi-\xi_1)^2+\sigma_3\xi^2+(\tau-\sigma_3\xi^2)\rangle^{2b}}.$$

We change the variable $\xi_1 \mapsto \mu$ as $\mu = \sigma_1\xi_1^2+\sigma_2(\xi-\xi_1)^2+\sigma_3\xi^2$, then we have

$$d\mu = 2|\sigma_1\xi_1-\sigma_2(\xi-\xi_1)|d\xi_1 \sim \left|(\sigma_1+\sigma_2)\mu-\theta\xi^2\right|^{1/2} d\xi_1.$$

Therefore if $\sigma_1+\sigma_2 = 0$, we obtain

$$I(\tau,\xi) \lesssim \frac{1}{|\xi|}\int_{\mathbb{R}} \frac{d\mu}{\langle\mu+(\tau-\sigma_3\xi^2)\rangle^{2b}} \lesssim \langle\xi\rangle^{-1}$$

for $b > 1/2$ since $\theta \neq 0$ and $|\xi| \geq 1$. While if $\sigma_1 + \sigma_2 \neq 0$, we obtain

$$I(\tau, \xi) \lesssim \int_{\mathbb{R}} \frac{d\mu}{\langle \mu + (\tau - \sigma_3 \xi^2) \rangle^{2b} \, |(\sigma_1 + \sigma_2)\mu - \theta \xi^2|^{1/2}} \lesssim \langle (\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) + \theta \xi^2 \rangle^{-1/2}$$

for $b > 1/2$ by Lemma 2.3, (2.9) in [47]. The estimate (3.61) follows from

$$d\mu = 2|\sigma_1 \xi_1 - \sigma_2(\xi - \xi_1)| d\xi_1 \sim \max\{|\xi_1|, |\xi - \xi_1|\} d\xi_1 \sim |\xi| d\xi_1$$

when $|\xi_1| \gg |\xi - \xi_1|$ or $|\xi_1| \ll |\xi - \xi_1|$. □

Proposition 3.35. *We assume $d = 1$ and σ_1, σ_2, $\sigma_3 \in \mathbb{R} \backslash \{0\}$ satisfy $(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_3) \neq 0$ and $\theta := \sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) \neq 0$. Then for $3/4 \geq b > 1/2$ and $1 > s \geq 1/2$, we have*

$$||(\partial_x u_3) u_2||_{X^{s,b-1}_{-\sigma_1}} \lesssim ||u_3||_{X^{s,b}_{\sigma_3}} ||u_2||_{X^{s,b}_{\sigma_2}}, \tag{3.62}$$

$$||\partial_x (u_1 u_2)||_{X^{s,b-1}_{-\sigma_3}} \lesssim ||u_1||_{X^{s,b}_{\sigma_1}} ||u_2||_{X^{s,b}_{\sigma_2}}, \tag{3.63}$$

where

$$||u||_{X^{s,b}_{\sigma}} := ||\langle \xi \rangle^s \langle \tau + \sigma \xi^2 \rangle^b \widetilde{u}||_{L^2_{\tau\xi}}.$$

Proof. We prove only (3.63) since the proof of (3.62) is similar. By the Cauchy-Schwarz inequality, we have

$$||\partial_x (u_1 u_2)||_{X^{s,b-1}_{-\sigma_3}} \lesssim ||I||_{L^\infty_{\tau\xi}} ||u_1||_{X^{s,b}_{\sigma_1}} ||u_2||_{X^{s,b}_{\sigma_2}},$$

where

$$I(\tau, \xi) := \left(\frac{\langle \xi \rangle^{2s} |\xi|^2}{\langle \tau - \sigma_3 \xi^2 \rangle^{2(1-b)}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \tau_1 + \sigma_1 \xi_1^2 \rangle^{2b} \langle \tau - \tau_1 + \sigma_2 (\xi - \xi_1)^2 \rangle^{2b}} d\tau_1 d\xi_1 \right)^{1/2}.$$

It is enough to prove $I(\tau, \xi) \lesssim 1$ for $|\xi| \geq 1$. For fixed $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}$, we divide $\mathbb{R} \times \mathbb{R}$ into three regions S_1, S_2, S_3 as

$$S_1 := \{(\tau_1, \xi_1) \in \mathbb{R} \times \mathbb{R} | \ |\xi| \ll |\xi_1|\}$$
$$S_2 := \{(\tau_1, \xi_1) \in \mathbb{R} \times \mathbb{R} | \ |\xi| \gtrsim |\xi_1|, \ \max\{|\tau_1 + \sigma_1 \xi_1^2|, |\tau - \tau_1 + \sigma_2 (\xi - \xi_1)^2|\} \gtrsim \xi^2\}$$
$$S_3 := \{(\tau_1, \xi_1) \in \mathbb{R} \times \mathbb{R} | \ |\xi| \gtrsim |\xi_1|, \ \max\{|\tau_1 + \sigma_1 \xi_1^2|, |\tau - \tau_1 + \sigma_2 (\xi - \xi_1)^2|\} \ll \xi^2\}$$

First, we consider the region S_1. For any $(\tau_1, \xi_1) \in S_1$, we have

$$\langle \xi \rangle^{2s} |\xi|^2 \langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s} \lesssim \langle \xi \rangle^{2-2s}$$

because $|\xi| \ll |\xi_1| \sim |\xi - \xi_1|$. Therefore, we have

$$I(\tau,\xi) \lesssim \left(\frac{\langle \xi \rangle^{2-2s}}{\langle \tau - \sigma_3 \xi^2 \rangle^{2(1-b)} \langle (\sigma_1+\sigma_2)(\tau - \sigma_3\xi^2) + \theta \xi^2 \rangle^{1/2}} \right)^{1/2}$$

for $b > 1/2$ by (3.60). Because $\theta \neq 0$,

$$\xi^2 = \frac{1}{\theta} \left\{ (\sigma_1+\sigma_2)(\tau - \sigma_3\xi^2) + \theta\xi^2 - (\sigma_1+\sigma_2)(\tau - \sigma_3\xi^2) \right\}.$$

Therefore we obtain

$$\begin{aligned} I(\tau,\xi) \lesssim & \left(\frac{1}{\langle \tau - \sigma_3 \xi^2 \rangle^{s-(2b-1)} \langle (\sigma_1+\sigma_2)(\tau - \sigma_3\xi^2) + \theta \xi^2 \rangle^{1/2}} \right. \\ & \qquad \left. + \frac{1}{\langle \tau - \sigma_3 \xi^2 \rangle^{2(1-b)} \langle (\sigma_1+\sigma_2)(\tau - \sigma_3\xi^2) + \theta \xi^2 \rangle^{s-1/2}} \right)^{1/2} \\ \lesssim & \, 1 \end{aligned}$$

for $3/4 \geq b > 1/2$ and $1 > s \geq 1/2$.

Second, we consider the region S_2. We assume $|\tau - \tau_1 + \sigma_2(\xi - \xi_1)^2| \gtrsim \xi^2$ ($\gtrsim |\xi|^{1/b}|\xi_1|^{1-1/2b}|\xi - \xi_1|^{1-1/2b}$) since for the case $|\tau_1 + \sigma_1\xi_1^2| \gtrsim \xi^2$ is same argument. Then, we have

$$I(\tau,\xi) \lesssim \left(\frac{\langle \xi \rangle^{2s}}{\langle \tau - \sigma_3 \xi^2 \rangle^{2(1-b)}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{1-2b-2s} \langle \xi - \xi_1 \rangle^{1-2b-2s}}{\langle \tau_1 + \sigma_1 \xi_1^2 \rangle^{2b}} d\tau_1 d\xi_1 \right)^{1/2}.$$

Because

$$\int_{\mathbb{R}} \frac{d\tau_1}{\langle \tau_1 + \sigma_1 \xi_1^2 \rangle^{2b}} \lesssim 1$$

for $b > 1/2$, we obtain

$$\begin{aligned} I(\tau,\xi) &\lesssim \left(\frac{\langle \xi \rangle^{2s}}{\langle \tau - \sigma_3 \xi^2 \rangle^{2(1-b)}} \int_{\mathbb{R}} \frac{d\xi_1}{\langle \xi_1 \rangle^{2s+2b-1} \langle \xi - \xi_1 \rangle^{2s+2b-1}} \right)^{1/2} \\ &\lesssim \left(\frac{1}{\langle \tau - \sigma_3 \xi^2 \rangle^{2(1-b)} \langle \xi \rangle^{2b-1}} \right)^{1/2} \\ &\lesssim 1 \end{aligned}$$

for $1 \geq b > 1/2$ and $s \geq 1/2$ by Lemma 2.3, (2.8) in [47].

Finally, we consider the region S_3. To begin with, we consider the case $|\tau - \sigma_3\xi^2| \gtrsim \xi^2$. Then we have

$$\frac{\langle \xi \rangle^{2s} |\xi|^2}{\langle \tau - \sigma_3 \xi^2 \rangle^{2(1-b)}} \langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s} \lesssim \begin{cases} \langle \xi \rangle^{4b-2-2s} & \text{if } \ |\xi_1| \sim |\xi - \xi_1| \\ \langle \xi \rangle^{4b-2} & \text{if } \ |\xi_1| \gg |\xi - \xi_1| \text{ or } |\xi_1| \ll |\xi - \xi_1| \end{cases}$$

since $|\xi| \sim \max\{|\xi_1|, |\xi - \xi_1|\}$ for any $(\tau, \xi) \in S_3$. Therefore we obtain

$$I(\tau, \xi) \lesssim \left(\frac{\langle \xi \rangle^{4b-2-2s}}{\langle (\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) + \theta \xi^2 \rangle^{1/2}} + \langle \xi \rangle^{4b-3} \right)^{1/2} \lesssim 1$$

for $3/4 \geq b > 1/2$ and $s \geq 1/2$ by (3.60) and (3.61). Next, we consider the case $|\tau - \sigma_3 \xi^2| \ll \xi^2$. Because $(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_3) \neq 0$ and

$$\max\{|\tau_1 + \sigma_1 \xi_1^2|, |\tau - \tau_1 + \sigma_2 (\xi - \xi_1)^2|, |\tau - \sigma_3 \xi^2|\} \ll \xi^2,$$

we have $|\xi| \sim |\xi - \xi_1| \sim |\xi_1|$ by Lemma 3.33. Therefore, we have

$$I(\tau, \xi) \lesssim \left(\frac{\langle \xi \rangle^{2-2s}}{\langle \tau - \sigma_3 \xi^2 \rangle^{2(1-b)} \langle (\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) + \theta \xi^2 \rangle^{1/2}} \right)^{1/2}$$

for $b > 1/2$ by (3.60). Because $\theta \neq 0$ and $|\tau - \sigma_3 \xi^2| \ll \xi^2$, we have

$$I(\tau, \xi) \lesssim \left(\frac{\langle \xi \rangle^{1-2s}}{\langle \tau - \sigma_3 \xi^2 \rangle^{2(1-b)}} \right)^{1/2} \lesssim 1$$

for $1 \geq b > 1/2$ and $s \geq 1/2$. □

Corollary 3.36. *We assume $d = 1$ and α, β, $\gamma \in \mathbb{R} \backslash \{0\}$ satisfy $(\alpha - \gamma)(\beta + \gamma) \neq 0$ and $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) \neq 0$. Then for $3/4 \geq b > 1/2$ and $1 > s \geq 1/2$, we have*

$$||(\partial_x w) v||_{X_\alpha^{s,b-1}} \lesssim ||w||_{X_\gamma^{s,b}} ||v||_{X_\beta^{s,b}}, \tag{3.64}$$

$$||(\partial_x \overline{w}) u||_{X_\beta^{s,b-1}} \lesssim ||w||_{X_\gamma^{s,b}} ||u||_{X_\alpha^{s,b}}, \tag{3.65}$$

$$||\partial_x (u \overline{v})||_{X_\gamma^{s,b-1}} \lesssim ||u||_{X_\alpha^{s,b}} ||v||_{X_\beta^{s,b}}, . \tag{3.66}$$

Proof. (3.64) follows from (3.62) with $(u_2, u_3) = (v, w)$ and $(\sigma_1, \sigma_2, \sigma_3) = (-\alpha, \beta, \gamma)$. (3.65) follows from (3.62) with $(u_2, u_3) = (u, \overline{w})$ and $(\sigma_1, \sigma_2, \sigma_3) = (\alpha, -\beta, -\gamma)$. (3.66) follows from (3.63) with $(u_1, u_2) = (u, \overline{v})$ and $(\sigma_1, \sigma_2, \sigma_3) = (\alpha, -\beta, -\gamma)$. □

Theorem 3.6 (iii) under the condition $1 > s \geq 1/2$, $\theta = \alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) < 0$ and $(\alpha - \gamma)(\beta + \gamma) \neq 0$ follows from Lemma 2.1 in [21] and Corollary 3.36.

Chapter 4

System of quadratic derivative nonlinear Schrödinger equations on $\mathbb{T}^d$

4.1 Review for results

We consider the Cauchy problem of the system of Schrödinger equations:

$$\begin{cases} (i\partial_t + \alpha\Delta)u = -(\nabla \cdot w)v, \quad (t,x) \in (0,\infty) \times \mathbb{T}^d, \\ (i\partial_t + \beta\Delta)v = -(\nabla \cdot \overline{w})u, \quad (t,x) \in (0,\infty) \times \mathbb{T}^d, \\ (i\partial_t + \gamma\Delta)w = \nabla(u \cdot \overline{v}), \quad (t,x) \in (0,\infty) \times \mathbb{T}^d, \\ (u(0,x), v(0,x), w(0,x)) = (u_0(x), v_0(x), w_0(x)), \quad x \in \mathbb{T}^d, \end{cases} \tag{4.1}$$

where α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ and the unknown functions u, v, w are d-dimensional complex vector valued. The system (4.1) was introduced by Colin and Colin in [11] as a model of laser-plasma interaction. (4.1) is invariant under the following scaling transformation:

$$A_\lambda(t,x) = \lambda^{-1}A(\lambda^{-2}t, \lambda^{-1}x) \ \ (A = (u,v,w)),$$

and the scaling critical regularity is $s_c = d/2 - 1$. The aim of this chapter is to prove the well-posedness of (4.1) in the scaling critical Sobolev space.

First, we introduce some known results for related problems. The system (4.1) has quadratic nonlinear terms which contains a derivative. A derivative loss arising from the nonlinearity makes the problem difficult. In fact, Christ ([9]) proved that the flow map of the Cauchy problem:

$$\begin{cases} i\partial_t u - \partial_x^2 u = u\partial_x u, \ t \in \mathbb{R}, \ x \in \mathbb{T}, \\ u(0,x) = u_0(x), \ x \in \mathbb{T} \end{cases}$$

is not continuous on $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$. While, there are positive results for the Cauchy problem:

$$\begin{cases} i\partial_t u - \Delta u = \overline{u}(\nabla \cdot \overline{u}), \ t \in \mathbb{R}, \ x \in \mathbb{T}^d, \\ u(0,x) = u_0(x), \ x \in \mathbb{T}^d. \end{cases} \tag{4.2}$$

Grünrock ([23]) proved that (4.2) is globally well-posed in $L^2(\mathbb{T})$ for $d = 1$ and locally well-posed in $H^s(\mathbb{T}^d)$ for $d \geq 2$ and $s > s_c$ $(= d/2 - 1)$. For the Cauchy problem of the one dimensional derivative Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\lambda \partial_x(|u|^2 u), \ t \in \mathbb{R}, \ x \in \mathbb{T}, \\ u(0,x) = u_0(x), \ x \in \mathbb{T}, \end{cases}$$

Herr ([30]) proved the local well-posedness in $H^s(\mathbb{T})$ for $s \geq 1/2$ by using the gauge transform and Win ([73]) proved the global well-posedness in $H^s(\mathbb{T})$ for $s > 1/2$. For the nonperiodic case, there are many results for the well-posedness of the nonlinear Schrödinger equations with derivative nonlinearity ([1], [2], [3], [7], [8], [16], [17], [48], [54], [60], [65], [66], [67]).

Next, we introduce some known results for (4.1). For the nonperiodic case, Colin and Colin ([11]) proved the local existence of the solution of (4.1) in $H^s(\mathbb{R}^d)$ for $s > d/2 + 3$. We proved that (4.1) for the nonperiodic case is globally well-posed and the solution scatters for small data in $H^{s_c}(\mathbb{R}^d)$ under the condition $(\alpha - \beta)(\alpha - \gamma)(\beta + \gamma) \neq 0$ if $d \geq 4$ and $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$ if $d = 2$, 3 in Chapter 3. We also obtained some well-posedness results at the subcritical regularity under the other condition for α, β and γ. But there are no well-posedness result of (4.1) for the periodic case.

Now, we give the main results in the present chapter. To begin with, we define the function spaces to construct the solution.

Definition 4.1. *Let* s, $\sigma \in \mathbb{R}$.
(i) *We define* Z^s_σ *as the space of all functions* $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ *such that for every* $\xi \in \mathbb{Z}^d$ *the map* $t \mapsto e^{it\sigma|\xi|^2}\widehat{u(t)}(\xi)$ *is in* $U^2(\mathbb{R};\mathbb{C})$, *and for which the norm*

$$||u||_{Z^s_\sigma} := \left(\sum_{\xi \in \mathbb{Z}^d} \langle \xi \rangle^{2s} ||e^{it\sigma|\xi|^2}\widehat{u(t)}(\xi)||^2_{U^2(\mathbb{R};\mathbb{C})} \right)^{1/2}$$

is finite.
(ii) *We define Y^s_σ as the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that for every $\xi \in \mathbb{Z}^d$ the map $t \mapsto e^{it\sigma|\xi|^2}\widehat{u(t)}(\xi)$ is in $V^2_{-,rc}(\mathbb{R};\mathbb{C})$, and for which the norm*

$$||u||_{Y^s_\sigma} := \left(\sum_{\xi\in\mathbb{Z}^d} \langle\xi\rangle^{2s} ||e^{it\sigma|\xi|^2}\widehat{u(t)}(\xi)||^2_{V^2(\mathbb{R};\mathbb{C})} \right)^{1/2}$$

is finite.

Remark 4.2 ([25] Remark 2.23)**.** *Let E be a Banach space of continuous functions $f : \mathbb{R} \to \mathcal{H}$, for some Hilbert space $\mathcal{H}$. We also consider the corresponding restriction space to the interval $I \subset \mathbb{R}$ by*

$$E(I) = \{u \in C(I, \mathcal{H}) | \exists v \in E \ s.t. \ v(t) = u(t), \ t \in I\}$$

endowed with the norm $||u||_{E(I)} = \inf\{||v||_E | v(t) = u(t), \ t \in I\}$. Obviously, $E(I)$ is also a Banach space.

The spaces Z^s_σ and Y^s_σ satisfy following properties.

Proposition 4.3 ([31] Proposition 2.8, Corollary 2.9)**.** *The embeddings*

$$U^2_{\sigma\Delta}H^s \hookrightarrow Z^s_\sigma \hookrightarrow Y^s_\sigma \hookrightarrow V^2_{\sigma\Delta}H^s$$

are continuous. Furthermore if $\mathbb{Z}^d = \cup C_k$ be a partition of $\mathbb{Z}^d$, then

$$\left(\sum_k ||P_{C_k}u||^2_{V^2_{\sigma\Delta}H^s} \right)^{1/2} \lesssim ||u||_{Y^s_\sigma}. \tag{4.3}$$

For an interval $I \subset \mathbb{R}$, we define $X^s(I) := Z^s_\alpha(I) \times Z^s_\beta(I) \times Z^s_\gamma(I)$. Our results are followings.

Theorem 4.4. *Let $s_c = d/2-1$. We assume that α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ satisfy $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$ and α/β, $\beta/\gamma \in \mathbb{Q}$.*
(i) *If $d \geq 5$, then (4.1) is locally well-posed for small data in $H^{s_c}(\mathbb{T}^d)$. More precisely, there exists $r > 0$ such that for all initial data $(u_0, v_0, w_0) \in B_r(H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d))$, there exist $T = T(r) > 0$ and a solution*

$$(u, v, w) \in X^{s_c}_r([0, T)) \subset C\left([0, T); H^{s_c}(\mathbb{T}^d)\right)$$

of the system (4.1) *on* $(0,T)$. *Such solution is unique in* $X_r^{s_c}([0,T))$ *which is a closed subset of* $X^{s_c}([0,T))$ (see (4.26)). *Moreover, the flow map*

$$S : B_r(H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d) \times H^{s_c}(\mathbb{T}^d)) \ni (u_0, v_0, w_0) \mapsto (u, v, w) \in X^{s_c}([0,T))$$

is Lipschitz continuous.
(ii) *If* $d \geq 1$ *and* $s > \max\{s_c, 0\}$, *then* (4.1) *is locally well-posed in* $H^s(\mathbb{T}^d)$. *More precisely, the statement in* (i) *holds for any* $r > 0$ *if we replace* s_c *by* s.

Remark 4.5. $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$ *is the nonresonance condition for* (4.1). *Oh* ([59]) *also studied the resonance and the nonresonance for the system of KdV equations. He proved that if the coefficient of the linear term of the system satisfies the nonresonance condition, then the well-posedness of the system is obtained at lower regularity than the regularity for the coefficient satisfying the resonance condition.*

We recall the following conservation quantities given in Chapter 3 (see Proposition 3.30).

$$M(u,v,w) := 2||u||_{L_x^2}^2 + ||v||_{L_x^2}^2 + ||w||_{L_x^2}^2,$$
$$H(u,v,w) := \alpha||\nabla u||_{L_x^2}^2 + \beta||\nabla v||_{L_x^2}^2 + \gamma||\nabla w||_{L_x^2}^2 + 2\mathrm{Re}(w, \nabla(u \cdot \overline{v}))_{L_x^2}.$$

By using the conservation law for H, we obtain the following result.

Theorem 4.6. *Let* $d = 1,\ 2,\ 3$. *We assume that* $\alpha,\ \beta,\ \gamma \in \mathbb{R}\backslash\{0\}$ *have the same sign and satisfy* $\alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma) > 0$ *and* $\alpha/\beta,\ \beta/\gamma \in \mathbb{Q}$. *There exists* $r > 0$ *such that for every* $(u_0, v_0, w_0) \in B_r(H^1(\mathbb{T}^d) \times H^1(\mathbb{T}^d) \times H^1(\mathbb{T}^d))$, *we can extend the local* H^1 *solution of* Theorem 4.4 *globally in time.*

Remark 4.7. Theorem 4.6 *follows from the a priori estimate which is obtained by the conservation law for* H. *Proof of the a priori estimate is the same as the nonperiodic case* (see Proposition 3.31).

Furthermore, for the equation (4.2), we obtain the following result.

Theorem 4.8. *Let* $d \geq 5$ *and* $s_c = d/2 - 1$. *Then, the equation* (4.2) *is locally well-posed for small data in* H^{s_c}.

Remark 4.9. *The results by Grünrock* ([23]) *do not contain the critical case* $s = s_c$
.

The main tools of our results are U^p space and V^p space which are applied to prove the well-posedness and the scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch ([25], [26]). For the periodic case, U^p space and V^p space are used to prove the well-posedness of the power type nonlinear Schrödinger equations at the scaling critical regularity by Herr, Tataru and Tzvetkov ([31]) and Wang ([71]). To obtain the well-posedness of (4.1), we show the following bilinear estimate.

Proposition 4.10. *Let $s > 0$ if $d = 1$, $s > s_c$ $(= d/2 - 1)$ if $d = 2, 3, 4$, $s \geq s_c$ if $d \geq 5$ and σ_1, $\sigma_2 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1 + \sigma_2 \neq 0$ and $\sigma_1/\sigma_2 = m_1/m_2$ for some m_1, $m_2 \in \mathbb{Z}\backslash\{0\}$. For any dyadic numbers N_1, N_2, $N_3 \geq 1$, we have*

$$\begin{aligned} &||P_{N_3}(P_{N_1}u_1 \cdot P_{N_2}u_2)||_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)} \\ &\lesssim N_{\min}^s \left(\frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}}\right)^{\delta} ||P_{N_1}u_1||_{Y^0_{\sigma_1}} ||P_{N_2}u_2||_{Y^0_{\sigma_2}} \end{aligned} \tag{4.4}$$

for some $\delta > 0$, where $\sigma := \sigma_1/m_1 = \sigma_2/m_2$, $N_{\max} := \max_{1\leq j\leq 3} N_j$, $N_{\min} := \min_{1\leq j\leq 3} N_j$.

Wang ([71]) proved (4.10) for the case $N_1 \sim N_3 \gtrsim N_2$. Therefore, Proposition 4.10 is the extension of his estimate. To prove Proposition 4.10, we will show the new bilinear estimate (Proposition 4.16) which is the estimate (4.4) for the case $N_1 \sim N_2 \gg N_3$.

The rest of this chapter is planned as follows. In Section 2, we will give the L^4-Strichartz estimates on torus and the bilinear estimates. In Section 3, we will give the trilinear estimates. In Section 4, we will give the proof of the well-posedness (Theorems 4.4, 4.8).

4.2 Strichartz and bilinear Strichartz estimates

In this section, we introduce some L^4-Strichartz estimates on torus proved in [5], [31], [71] and the bilinear estimate proved in [71]. Furthermore, we show the new bilinear estimate (Proposition 4.16) to obtain Proposition 4.10.

For a dyadic number $N \geq 1$, we define $\mathcal{C}_N$ as the collection of disjoint cubes $C \subset \mathbb{Z}^d$ of side-length N with arbitrary center and orientation. Furthermore for dyadic numbers $N \geq 1$ and $M \geq 1$, we define $\mathcal{R}_M(N)$ as the collection of all sets of the form

$$\left(\xi_0 + [-N, N]^d\right) \cap \{\xi \in \mathbb{Z}^d | \ |a \cdot \xi - A| \leq M\}$$

with some $\xi_0 \in \mathbb{Z}^d$, $a \in \mathbb{R}^d$, $|a| = 1$ and $A \in \mathbb{R}$.

Proposition 4.11 ([5], [31], [71]). *Let $\sigma \in \mathbb{R}$, $m \in \mathbb{Z}\backslash\{0\}$.*
(i) *For any dyadic number $N \geq 1$ and $s \geq 0$ if $d = 1$, $s > d/4 - 1/2$ if $d = 2$, 3, $s \geq d/4 - 1/2$ if $d \geq 4$, we have*

$$||P_N e^{it\sigma\Delta}\varphi||_{L^4(\mathbb{T}_{|m/\sigma|}\times\mathbb{T}^d)} \lesssim N^s ||P_N\varphi||_{L^2(\mathbb{T}^d)}. \tag{4.5}$$

(ii) *For any $C \in \mathcal{C}_N$ with $N \geq 1$ and $s \geq 0$ if $d = 1$, $s > d/4 - 1/2$ if $d = 2$, 3, $s \geq d/4 - 1/2$ if $d \geq 4$, we have*

$$||P_C e^{it\sigma\Delta}\varphi||_{L^4(\mathbb{T}_{|m/\sigma|}\times\mathbb{T}^d)} \lesssim N^s ||P_C\varphi||_{L^2(\mathbb{T}^d)}. \tag{4.6}$$

(iii) *For any $R \in \mathcal{R}_M(N)$ with $N \geq M \geq 1$ and $s > 0$ if $d = 1$, $s > d/4 - 1/2$ if $d = 2$, 3, 4, $s \geq d/4 - 1/2$ if $d \geq 5$, we have*

$$||P_R e^{it\sigma\Delta}\varphi||_{L^4(\mathbb{T}_{|m/\sigma|}\times\mathbb{T}^d)} \lesssim N^s \left(\frac{M}{N}\right)^{\delta} ||P_R\varphi||_{L^2(\mathbb{T}^d)} \tag{4.7}$$

for some $\delta > 0$.

Remark 4.12. *Implicit constants in the estimates (4.5)–(4.7) depend on m and σ.*

By Propositions 2.8 and 4.11, we have following:

Corollary 4.13. *Let $\sigma \in \mathbb{R}$, $m \in \mathbb{Z}\backslash\{0\}$ and $s \geq 0$ if $d = 1$, $s > d/4 - 1/2$ if $d = 2$, 3, $s \geq d/4 - 1/2$ if $d \geq 4$. For any dyadic number $N \geq 1$ and $C \in \mathcal{C}_N$, we have*

$$||P_N u||_{L^4(\mathbb{T}_{|m/\sigma|}\times\mathbb{T}^d)} \lesssim N^s ||P_N u||_{U^4_{\sigma\Delta}L^2}, \tag{4.8}$$

$$||P_C u||_{L^4(\mathbb{T}_{|m/\sigma|}\times\mathbb{T}^d)} \lesssim N^s ||P_C u||_{U^4_{\sigma\Delta}L^2}. \tag{4.9}$$

Proposition 4.14 ([71] Proposition 4.2). *Let $s > 0$ if $d = 1$, $s > s_c$ $(= d/2 - 1)$ if $d = 2, 3, 4$, $s \geq s_c$ if $d \geq 5$ and σ_1, $\sigma_2 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1/\sigma_2 = m_1/m_2$ for some m_1, $m_2 \in \mathbb{Z}\backslash\{0\}$. For any dyadic numbers H and L with $H \geq L \geq 1$, we have*

$$||P_H u_1 \cdot P_L u_2||_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)} \lesssim L^s \left(\frac{L}{H} + \frac{1}{L}\right)^{\delta} ||P_H u_1||_{Y^0_{\sigma_1}} ||P_L u_2||_{Y^0_{\sigma_2}} \tag{4.10}$$

for some $\delta > 0$, where $\sigma := \sigma_1/m_1 = \sigma_2/m_2$.

Remark 4.15. *Wang proved* (4.10) *only for $d \geq 5$. To obtain* (4.10) *for $1 \leq d \leq 4$, we choose $p = q = 4$ and use* (4.7) *as above in the proof of* [[71] Proposition 4.2] *for $k = 1$, $n \geq 5$. The other parts are the same way.*

We get the following bilinear estimate.

Proposition 4.16. *Let* $s > 0$ *if* $d = 1$, $s > s_c$ $(= d/2 - 1)$ *if* $d = 2, 3, 4$, $s \geq s_c$ *if* $d \geq 5$ *and* σ_1, $\sigma_2 \in \mathbb{R}\backslash\{0\}$ *satisfy* $\sigma_1 + \sigma_2 \neq 0$ *and* $\sigma_1/\sigma_2 = m_1/m_2$ *for some* m_1, $m_2 \in \mathbb{Z}\backslash\{0\}$. *For any dyadic numbers* L, H, H' *with* $H \sim H' \gg L \geq 1$, *we have*

$$||P_L(P_H u_1 \cdot P_{H'} u_2)||_{L^2(\mathbb{T}_{|\sigma^{-1}|} \times \mathbb{T}^d)} \lesssim L^s \left(\frac{L}{H} + \frac{1}{L}\right)^{\delta} ||P_H u_1||_{Y^0_{\sigma_1}} ||P_{H'} u_2||_{Y^0_{\sigma_2}} \tag{4.11}$$

for some $\delta > 0$, *where* $\sigma := \sigma_1/m_1 = \sigma_2/m_2$.

Proof. We decompose $P_H u_1 = \sum_{C_1 \in \mathcal{C}_L} P_{C_1} P_H u_1$. For fixed $C_1 \in \mathcal{C}_L$, let ξ_0 be the center of C_1. Since $\xi_1 \in C_1$ and $|\xi_1 + \xi_2| \leq 2L$ imply $|\xi_2 + \xi_0| \leq 3L$, we obtain

$$||P_L(P_{C_1} P_H u_1 \cdot P_{H'} u_2)||_{L^2(\mathbb{T}_{|\sigma^{-1}|} \times \mathbb{T}^d)} \leq ||P_{C_1} P_H u_1 \cdot P_{C_2(C_1)} P_{H'} u_2||_{L^2(\mathbb{T}_{|\sigma|} \times \mathbb{T}^d)},$$

where $C_2(C_1) := \{\xi_2 \in \mathbb{Z}^d | \ |\xi_2 + \xi_0| \leq 3L\}$. If we prove

$$\begin{aligned} &||P_{C_1} P_H u_1 \cdot P_{C_2(C_1)} P_{H'} u_2||_{L^2(\mathbb{T}_{|\sigma^{-1}|} \times \mathbb{T}^d)} \\ &\lesssim L^s \left(\frac{L}{H} + \frac{1}{L}\right)^{\delta} ||P_{C_1} P_H u_1||_{V^2_{\sigma_1 \Delta} L^2} ||P_{C_2(C_1)} P_{H'} \phi_2||_{V^2_{\sigma_2 \Delta} L^2}, \end{aligned} \tag{4.12}$$

then we obtain

$$\begin{aligned} &||P_L(P_H u_1 \cdot P_{H'} u_2)||_{L^2(\mathbb{T}_{|\sigma^{-1}|} \times \mathbb{T}^d)} \\ &\lesssim \sum_{C_1 \in \mathcal{C}_L} L^s \left(\frac{L}{H} + \frac{1}{L}\right)^{\delta} ||P_{C_1} P_H u_1||_{V^2_{\sigma_1 \Delta} L^2} ||P_{C_2(C_1)} P_{H'} u_2||_{V^2_{\sigma_2 \Delta} L^2} \\ &\lesssim L^s \left(\frac{L}{H} + \frac{1}{L}\right)^{\delta} \left(\sum_{C_1 \in \mathcal{C}_L} ||P_{C_1} P_H u_1||^2_{V^2_{\sigma_1 \Delta} L^2}\right)^{1/2} \left(\sum_{C_1 \in \mathcal{C}_L} ||P_{C_2(C_1)} P_{H'} u_2||^2_{V^2_{\sigma_2 \Delta} L^2}\right)^{1/2} \end{aligned}$$

and the proof is complete by (4.3). The estimate (4.12) follows by interpolation between

$$\begin{aligned} &||P_{C_1} P_H u_1 \cdot P_{C_2(C_1)} P_{H'} u_2||_{L^2(\mathbb{T}_{|\sigma^{-1}|} \times \mathbb{T}^d)} \\ &\lesssim L^s ||P_{C_1} P_H u_1||_{U^4_{\sigma_1 \Delta} L^2} ||P_{C_2(C_1)} P_{H'} u_2||_{U^4_{\sigma_2 \Delta} L^2} \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} &||P_{C_1} P_H u_1 \cdot P_{C_2(C_1)} P_{H'} u_2||_{L^2(\mathbb{T}_{|\sigma^{-1}|} \times \mathbb{T}^d)} \\ &\lesssim L^s \left(\frac{L}{H} + \frac{1}{L}\right)^{\delta'} ||P_{C_1} P_H u_1||_{U^2_{\sigma_1 \Delta} L^2} ||P_{C_2(C_1)} P_{H'} u_2||_{U^2_{\sigma_2 \Delta} L^2} \end{aligned} \tag{4.14}$$

via Proposition 2.9. The estimate (4.13) is proved by the Cauchy-Schwartz inequality and (4.9). While the estimate (4.14) follows from the estimate

$$\begin{aligned}
&||P_{C_1}P_H(e^{it\sigma_1}\phi_1)\cdot P_{C_2(C_1)}P_{H'}(e^{it\sigma_2}\phi_2)||_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)} \\
&\lesssim L^s\left(\frac{L}{H}+\frac{1}{L}\right)^{\delta'}||P_{C_1}P_H\phi_1||_{L^2(\mathbb{T}^d)}||P_{C_2(C_1)}P_{H'}\phi_2||_{L^2(\mathbb{T}^d)}.
\end{aligned} \tag{4.15}$$

and Proposition 2.8.

Now, we prove the estimate (4.15). Put $u_j = e^{it\sigma_j\Delta}\phi_j$ $(j = 1,2)$. We note that u_1u_2 is periodic function with period $2\pi|\sigma^{-1}|$ with respect to t since $\sigma_1/\sigma_2 = m_1/m_2 \in \mathbb{Q}$. We partition $C_1 = \cup_k R_{1,k}$ and $C_2(C_1) = \cup_l R_{2,l}$ into almost disjoint strips as

$$\begin{aligned}
R_{1,k} &= \{\xi_1 \in C_1|\xi_1\cdot\xi_0 \in [|\xi_0|Mk, |\xi_0|M(k+1)]\},\ k\sim H/M,\\
R_{2,l} &= \{\xi_2 \in C_2(C_1)|\xi_2\cdot\xi_0 \in [-|\xi_0|M(l+1), -|\xi_0|Ml\},\ l\sim H/M,
\end{aligned}$$

where $M = \max\{L^2/H, 1\}$. The condition for k and l as above follows from $|\xi_0| \sim H \gg L$. To obtain the almost orthogonality for the summation

$$P_{C_1}P_Hu_1\cdot P_{C_2(C_1)}P_{H'}u_2 = \sum_k\sum_l P_{R_{1,k}}P_Hu_1\cdot P_{R_{2,l}}P_{H'}u_2,$$

we use the argument in [[31] Proposition 3.5]. Since $L^2 \lesssim M^2k \sim M^2l$, we have

$$|\xi_1|^2 = \frac{|\xi_1\cdot\xi_0|^2}{|\xi_0|^2} + |\xi_1-\xi_0|^2 - \frac{|(\xi_1-\xi_0)\cdot\xi_0|^2}{|\xi_0|^2} = M^2k^2 + O(M^2k)$$

and

$$|\xi_2|^2 = \frac{|\xi_2\cdot\xi_0|^2}{|\xi_0|^2} + |\xi_2+\xi_0|^2 - \frac{|(\xi_2+\xi_0)\cdot\xi_0|^2}{|\xi_0|^2} = M^2l^2 + O(M^2k)$$

for any $\xi_1 \in R_{1,k}$ and $\xi_2 \in R_{2,l}$. More precisely, there exist the constants A_1 ,$A_2 > 0$ which do not depend on k and l, such that $\xi_1 \in R_{1,k}$ and $\xi_2 \in R_{2,l}$ satisfy

$$M^2k^2 \le |\xi_1|^2 \le M^2k^2 + A_1M^2k,\ M^2l^2 \le |\xi_2|^2 \le M^2l^2 + A_2M^2k.$$

Furthermore, $\sigma_1k^2 + \sigma_2l^2 \neq 0$ because

$$|\sigma_1|\xi_1|^2 + \sigma_2|\xi_2|^2| = |(\sigma_1+\sigma_2)|\xi_1|^2 - \sigma_2(\xi_1-\xi_2)\cdot(\xi_1+\xi_2)| \sim H^2.$$

Therefore, the expression $P_{R_{1,k}}P_Hu_1\cdot P_{R_{2,l}}P_{H'}u_2$ are localized at time frequency $M^2(\sigma_1k^2+\sigma_2l^2) + O(M^2k)$. This implies the almost orthogonality:

$$||P_{C_1}P_Hu_1\cdot P_{C_2(C_1)}P_{H'}u_2||^2_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)} \lesssim \sum_k\sum_l ||P_{R_{1,k}}P_Hu_1\cdot P_{R_{2,l}}P_{H'}u_2||^2_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)}.$$

By the Cauchy-Schwartz inequality and (4.7), we have

$$\begin{aligned}||P_{R_{1,k}}P_H u_1 \cdot P_{R_{2,l}}P_{H'}u_2||_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)} &\lesssim ||P_{R_{1,k}}P_H u_1||_{L^4(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)}||P_{R_{2,l}}P_{H'}u_2||_{L^4(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)} \\ &\lesssim L^s\left(\frac{M}{L}\right)^{\delta'}||P_{R_{1,k}}P_H\phi_1||_{L^2(\mathbb{T}^d)}||P_{R_{2,l}}P_{H'}\phi_2||_{L^2(\mathbb{T}^d)}\end{aligned}$$

for some $\delta' > 0$ and any $s > 0$ if $d = 1$, $s > s_c$ if $d = 2, 3, 4$ and $s \geq s_c$ if $d \geq 5$ since $R_{1,k} \in \mathcal{R}_M(L)$, $R_{2,l} \in \mathcal{R}_M(3L)$. Therefore, we obtain (4.15) by the L^2-orthogonality and $M \leq L^2/H + 1$. □

Remark 4.17. Proposition 4.10 *is implied from* Propositions 4.14 *and* 4.16*.*

To deal with large data at the scaling subcritical regularity, we show the following.

Proposition 4.18. *Let $d \geq 1$, $s > s_0 := \max\{s_c, 0\}$ and σ_1, $\sigma_2 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1+\sigma_2 \neq 0$ and $\sigma_1/\sigma_2 = m_1/m_2$ for some m_1, $m_2 \in \mathbb{Z}\backslash\{0\}$. For any dyadic numbers N_1, N_2, $N_3 \geq 1$ and $0 < T \leq 2\pi|\sigma^{-1}|$, we have*

$$\begin{aligned}&||P_{N_3}(P_{N_1}u_1 \cdot P_{N_2}u_2)||_{L^2([0,T)\times\mathbb{T}^d)} \\ &\lesssim T^{\epsilon}N_{\min}^s\left(\frac{N_{\min}}{N_{\max}}+\frac{1}{N_{\min}}\right)^{\delta}||P_{N_1}u_1||_{Y^0_{\sigma_1}}||P_{N_2}u_2||_{Y^0_{\sigma_2}}\end{aligned} \tag{4.16}$$

for some $\delta > 0$ and $\epsilon > 0$, where $\sigma := \sigma_1/m_1 = \sigma_2/m_2$, $N_{\max} := \max\limits_{1\leq j\leq 3} N_j$, $N_{\min} := \min\limits_{1\leq j\leq 3} N_j$.

Proof. We first prove the case $N_1 \sim N_2 \gg N_3$. By Corollary 4.1 in [71], we have

$$||P_C u||_{L^p(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)} \lesssim N^{\max\{d/2-(d+2)/p,a\}}||P_C u||_{U^p_{\sigma_i\Delta}L^2}\ (i = 1, 2)$$

for any $p > 4$, $a > 0$ and $C \in \mathcal{C}_N$ with $N \geq 1$. Therefore, for any $\epsilon'' > 0$, there exists $\epsilon' > 0$ such that

$$\begin{aligned}||P_C u||_{L^4([0,T)\times\mathbb{T}^d)} &\lesssim ||1||_{L^{2/\epsilon'}([0,T)\times\mathbb{T}^d)}||P_C u||_{L^{4/(1-2\epsilon')}(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)} \\ &\lesssim T^{\epsilon'/2}N^{s_0/2+\epsilon''}||P_C u||_{U^4_{\sigma_i\Delta}L^2}\ (i = 1, 2)\end{aligned} \tag{4.17}$$

for any $0 < T \leq 2\pi|\sigma^{-1}|$ since $U^4_{\sigma\Delta}L^2 \hookrightarrow U^p_{\sigma\Delta}L^2$ for $p > 4$. For the L.H.S of (4.13) with $H = N_1$, $H' = N_2$, $L = N_3$, we use (4.17) with $\epsilon'' = (s - s_0)/2$, then we have

$$\begin{aligned}&||P_{C_1}P_{N_1}u_1 \cdot P_{C_2(C_1)}P_{N_2}u_2||_{L^2([0,T)\times\mathbb{T}^d)} \\ &\lesssim T^{\epsilon'}N_3^s||P_{C_1}P_{N_1}u_1||_{U^4_{\sigma_1\Delta}L^2}||P_{C_2(C_1)}P_{N_2}u_2||_{U^4_{\sigma_2\Delta}L^2}.\end{aligned} \tag{4.18}$$

While by (4.14) with $H = N_1$, $H' = N_2$, $L = N_3$, we have

$$\begin{aligned} &||P_{C_1}P_{N_1}u_1 \cdot P_{C_2(C_1)}P_{N_2}u_2||_{L^2([0,T)\times\mathbb{T}^d)} \\ &\lesssim N_3^s\left(\frac{N_3}{N_1}+\frac{1}{N_3}\right)^{\delta'} ||P_{C_1}P_{N_1}u_1||_{U^2_{\sigma_1\Delta}L^2}||P_{C_2(C_1)}P_{N_2}u_2||_{U^2_{\sigma_2\Delta}L^2} \end{aligned} \tag{4.19}$$

for any $s > s_0$, $0 < T \leq 2\pi|\sigma^{-1}|$ and some $\delta' > 0$. By interpolation between (4.18) and (4.19) via Proposition 2.9, we obtain

$$\begin{aligned} &||P_{C_1}P_{N_1}u_1 \cdot P_{C_2(C_1)}P_{N_2}u_2||_{L^2([0,T)\times\mathbb{T}^d)} \\ &\lesssim T^{\epsilon}N_3^s\left(\frac{N_3}{N_1}+\frac{1}{N_3}\right)^{\delta} ||P_{C_1}P_{N_1}u_1||_{U^2_{\sigma_1\Delta}L^2}||P_{C_2(C_1)}P_{N_2}u_2||_{U^2_{\sigma_2\Delta}L^2} \end{aligned} \tag{4.20}$$

for some $\delta > 0$ and $\epsilon > 0$. Therefore, we have (4.16) by the same argument of the proof of Proposition 4.16.

For the case $N_1 \sim N_3 \gtrsim N_2$, we also obtain (4.16) by using above argument in the proof of [[71] Proposition 4.2]. □

4.3 Trilinear estimates

In this section, we give the trilinear estimates. We recall the modulation estimate given in Chapter 3 (see Lemma 4.19).

Lemma 4.19. *Let $d \in \mathbb{N}$. We assume that σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$ and (τ_1, ξ_1), (τ_2, ξ_2), $(\tau_3, \xi_3) \in \mathbb{R}\times\mathbb{R}^d$ satisfy $\tau_1+\tau_2+\tau_3 = 0$, $\xi_1+\xi_2+\xi_3 = 0$. Then we have*

$$\max_{1\leq j\leq 3}|\tau_j + \sigma_j|\xi_j|^2| \gtrsim \max_{1\leq j\leq 3}|\xi_j|^2.$$

Proposition 4.20. *Let $s > 0$ if $d = 1$, $s > s_c$ $(= d/2-1)$ if $d = 2,3,4$, $s \geq s_c$ if $d \geq 5$ and σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1\sigma_2\sigma_3(1/\sigma_1+1/\sigma_2+1/\sigma_3) > 0$ and $\sigma_1/\sigma_2 = m_1/m_2$, $\sigma_2/\sigma_3 = m_2/m_3$ for some m_1, m_2, $m_3 \in \mathbb{Z}\backslash\{0\}$. For $0 < T \leq 2\pi|\sigma^{-1}|$, any dyadic numbers N_1, N_2, $N_3 \geq 1$ and $P_{N_j}u_j \in V^2_{\sigma_j}L^2$ $(j = 1,2,3)$ with $\max_{1\leq j\leq 3}|\xi_j| \neq 0$ for $\xi_j \in \operatorname{supp}\widehat{u_j(t)}$, we have*

$$\left|N_{\max}\int_0^T\int_{\mathbb{T}^d}\left(\prod_{j=1}^3 P_{N_j}u_j\right)dxdt\right| \lesssim N_{\min}^s\left(\frac{N_{\min}}{N_{\max}}+\frac{1}{N_{\min}}\right)^{\delta}\prod_{j=1}^3||P_{N_j}u_j||_{Y^0_{\sigma_j}} \tag{4.21}$$

for some $\delta > 0$, where $\sigma := \sigma_1/m_1 = \sigma_2/m_2 = \sigma_3/m_3$, $N_{\max} = \max_{1\leq j\leq 3} N_j$, $N_{\min} = \min_{1\leq j\leq 3} N_j$.

Proof. We define $u_{j,T} := \mathbf{1}_{[0,T)}u_j$ $(j = 1, 2, 3)$. Furthermore for sufficiently large constant C, we put $M := C^{-1}N_{\max}^2$ and decompose $Id = Q_{<M}^{\sigma_j\Delta} + Q_{\geq M}^{\sigma_j\Delta}$ $(j = 1, 2, 3)$. We divide the integrals on the left-hand side of (4.21) into eight piece of the form

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d}\left(\prod_{j=1}^{3} Q_j^{\sigma_j\Delta}P_{N_j}u_{j,T}\right)dxdt \tag{4.22}$$

with $Q_j^{\sigma_j\Delta} \in \{Q_{\geq M}^{\sigma_j\Delta}, Q_{<M}^{\sigma_j\Delta}\}$ $(j = 1, 2, 3)$. By the Plancherel's theorem, we have

$$(4.22) = c\int_{\tau_1+\tau_2+\tau_3=0}\sum_{\xi_1+\xi_2+\xi_3=0}\prod_{j=1}^{3}\mathcal{F}[Q_j^{\sigma_j\Delta}P_{N_j}u_{j,T}](\tau_j, \xi_j),$$

where c is a constant. Therefore, Lemma 4.19 and $\max_{1\leq j\leq 3}|\xi_j| \neq 0$ imply that

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d}\left(\prod_{j=1}^{3} Q_{<M}^{\sigma_j\Delta}P_{N_j}u_{j,T}\right)dxdt = 0.$$

So, let us now consider the case that $Q_j^{\sigma_j\Delta} = Q_{\geq M}^{\sigma_j\Delta}$ for some $1 \leq j \leq 3$.

We consider only for the case $Q_3^{\sigma_3\Delta} = Q_{\geq M}^{\sigma_3\Delta}$ since the other cases are similar. By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^d}(Q_1^{\sigma_1\Delta}P_{N_1}u_1)(Q_2^{\sigma_2\Delta}P_{N_2}u_2)(Q_{\geq M}^{\sigma_3\Delta}P_{N_3}u_3)dxdt\right|\\ &\lesssim ||\widetilde{P}_{N_3}(Q_1^{\sigma_1\Delta}P_{N_1}u_1 \cdot Q_2^{\sigma_2\Delta}P_{N_2}u_2)||_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)}||Q_{\geq M}^{\sigma_3\Delta}P_{N_3}u_3||_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)}\end{aligned}$$

since $P_{N_3} = \widetilde{P}_{N_3}P_{N_3}$, where $\widetilde{P}_{N_3} = P_{N_3/2} + P_{N_3} + P_{2N_3}$. Furthermore by (4.4) and (2.2), we have

$$\begin{aligned}&||\widetilde{P}_{N_3}(Q_1^{\sigma_1\Delta}P_{N_1}u_1 \cdot Q_2^{\sigma_2\Delta}P_{N_2}u_2)||_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)}\\ &\lesssim N_{\min}^s\left(\frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}}\right)^{\delta}||P_{N_1}u_1||_{Y_{\sigma_1}^0}||P_{N_2}u_2||_{Y_{\sigma_2}^0}.\end{aligned}$$

While by (2.1) with $p = 2$, $M \sim N_{\max}^2$ and $Y_{\sigma_3}^0 \hookrightarrow V_{\sigma_3}^2L^2$, we have

$$||Q_{\geq M}^{\sigma_3}P_{N_3}u_3||_{L^2(\mathbb{T}_{|\sigma^{-1}|}\times\mathbb{T}^d)} \lesssim ||Q_{\geq M}^{\sigma_3\Delta}P_{N_3}u_3||_{L^2(\mathbb{R}\times\mathbb{T}^d)} \lesssim N_{\max}^{-1}||P_{N_3}u_3||_{Y_{\sigma_3}^0}.$$

□

By using (4.16) instead of (4.4) in the proof of Proposition 4.20, we get following.

Proposition 4.21. *Let $d \geq 1$, $s > \max\{s_c, 0\}$ and σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1\sigma_2\sigma_3(1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$ and $\sigma_1/\sigma_2 = m_1/m_2$, $\sigma_2/\sigma_3 = m_2/m_3$ for some m_1, m_2, $m_3 \in \mathbb{Z}\backslash\{0\}$. For $0 < T \leq 2\pi|\sigma^{-1}|$, any dyadic numbers N_1, N_2, $N_3 \geq 1$ and $P_{N_j}u_j \in V^2_{\sigma_j}L^2$ $(j = 1, 2, 3)$ with $\max_{1\leq j\leq 3}|\xi_j| \neq 0$ for $\xi_j \in \operatorname{supp}\widehat{u_j(t)}$, we have*

$$\left| N_{\max} \int_0^T \int_{\mathbb{T}^d} \left(\prod_{j=1}^3 P_{N_j}u_j \right) dxdt \right| \lesssim T^{\epsilon} N_{\min}^s \left(\frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}} \right)^{\delta} \prod_{j=1}^3 ||P_{N_j}u_j||_{Y^0_{\sigma_j}}$$

for some $\delta > 0$ and $\epsilon > 0$, where $\sigma := \sigma_1/m_1 = \sigma_2/m_2 = \sigma_3/m_3$, $N_{\max} = \max_{1\leq j\leq 3} N_j$, $N_{\min} = \min_{1\leq j\leq 3} N_j$.

4.4 Proof of the well-posedness

In this section, we prove Theorems 4.4, 4.8. First, we give the duality for the spaces $Z^s_{\sigma}([0,T))$ and $Y^{-s}_{\sigma}([0,T))$ as follows.

Proposition 4.22 ([31] Proposition 2.10)**.** *For $f \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{T}^d))$ and $\sigma \in \mathbb{R}\backslash\{0\}$, we define*

$$I_{\sigma}[f](t) := \int_0^t e^{i(t-t')\sigma\Delta} f(t')dt'.$$

for $t \geq 0$ and $I_{\sigma}[f](t) = 0$ for $t < 0$. Then for $s \geq 0$, $T > 0$, $\sigma \in \mathbb{R}\backslash\{0\}$ and $f \in L^1([0,T); H^s(\mathbb{T}^d))$, we have $I_{\sigma}[f] \in Z^s_{\sigma}([0,T))$ and

$$||I_{\sigma}[f]||_{Z^s_{\sigma}([0,T))} \leq \sup_{v \in Y^{-s}_{\sigma}([0,T)), ||v||_{Y^{-s}_{\sigma}}=1} \left| \int_0^T \int_{\mathbb{T}^d} f(t,x)\overline{v(t,x)}dt \right|.$$

Next, we define the map

$$\Phi(u,v,w) = (\Phi^{(1)}_{\alpha,u_0}(w,v), \Phi^{(1)}_{\beta,v_0}(\overline{w},v), \Phi^{(2)}_{\gamma,w_0}(u,\overline{v}))$$

as

$$\Phi^{(1)}_{\sigma,\varphi}(f,g)(t) := e^{it\sigma\Delta}\varphi - I^{(1)}_{\sigma}(f,g)(t),$$
$$\Phi^{(2)}_{\sigma,\varphi}(f,g)(t) := e^{it\sigma\Delta}\varphi + I^{(2)}_{\sigma}(f,g)(t),$$

where

$$I^{(1)}_{\sigma}(f,g)(t) := \int_0^t \mathbf{1}_{[0,\infty)}(t') e^{i(t-t')\sigma\Delta}(\nabla \cdot f(t'))g(t')dt',$$
$$I^{(2)}_{\sigma}(f,g)(t) := \int_0^t \mathbf{1}_{[0,\infty)}(t') e^{i(t-t')\sigma\Delta}\nabla(f(t') \cdot g(t'))dt'.$$

To obtain the well-posedness of (4.1), we prove that Φ is a contraction map on a closed subset of $Z^s_{\alpha}([0,T))\times Z^s_{\beta}([0,T))\times Z^s_{\gamma}([0,T))$. We consider only for small data since for large data at the scaling subcritical regularity is similar argument. Key estimates are the followings:

Proposition 4.23. *We assume that $s>0$ if $d=1$, $s>s_c$ $(=d/2-1)$ if $d=2,3,4$, $s\geq s_c$ if $d\geq 5$ and α, β, $\gamma\in\mathbb{R}\backslash\{0\}$ satisfy $\alpha\beta\gamma(1/\alpha-1/\beta-1/\gamma)>0$ and α/β, $\beta/\gamma\in\mathbb{Q}$. Then for any $0<T\ll 1$, such that*

$$||I^{(1)}_{\alpha}(w,v)||_{Z^s_{\alpha}([0,T))}\lesssim ||w||_{Y^s_{\gamma}([0,T))}||v||_{Y^s_{\beta}([0,T))}, \tag{4.23}$$

$$||I^{(1)}_{\beta}(\overline{w},u)||_{Z^s_{\beta}([0,T))}\lesssim ||w||_{Y^s_{\gamma}([0,T))}||u||_{Y^s_{\alpha}([0,T))}, \tag{4.24}$$

$$||I^{(2)}_{\gamma}(u,\overline{v})||_{Z^s_{\gamma}([0,T))}\lesssim ||u||_{Y^s_{\alpha}([0,T))}||v||_{Y^s_{\beta}([0,T))}. \tag{4.25}$$

Proof. We prove only (4.25) since (4.23) and (4.24) are proved by the same way. Let $(u_1,u_2):=(u,\overline{v})$ and $(\sigma_1,\sigma_2,\sigma_3):=(\alpha,-\beta,-\gamma)$. Since σ_1/σ_2, $\sigma_2/\sigma_3\in\mathbb{Q}$, there exist m_1, m_2, $m_3\in\mathbb{Z}\backslash\{0\}$ such that $\sigma_1/\sigma_2=m_1/m_2$, $\sigma_2/\sigma_3=m_2/m_3$. We choose $T>0$ satisfying $T\leq 2\pi|\sigma^{-1}|$, where $\sigma:=\sigma_1/m_1=\sigma_2/m_2=\sigma_3/m_3$. We define

$$S_j:=\{(N_1,N_2,N_3)|N_{\max}\sim N_{\mathrm{med}}\gtrsim N_{\min}\geq 1,\ N_{\min}=N_j\}\ (j=1,2,3)$$

and $S:=\bigcup_{j=1}^3 S_j$, where $(N_{\max},N_{\mathrm{med}},N_{\min})$ be one of the permutation of (N_1,N_2,N_3) such that $N_{\max}\geq N_{\mathrm{med}}\geq N_{\min}$. By Proposition 4.22, the Plancherel's theorem ,(4.21), we have

$$\begin{aligned}\left|\left|I^{(2)}_{-\sigma_3}(u_1,u_2)\right|\right|_{Z^s_{-\sigma_3}([0,T))}&\lesssim \sup_{||u_3||_{Y^{-s}_{\sigma_3}}=1}\left|\int_0^T\int_{\mathbb{T}^d}u_1u_2(\nabla\cdot u_3)dxdt\right|\\ &\leq \sup_{||u_3||_{Y^{-s}_{\sigma_3}}=1}\sum_S\left|N_3\int_0^T\int_{\mathbb{T}^d}P_{N_1}u_1P_{N2}u_2P_{N_3}u_3dxdt\right|\\ &\leq \sup_{||u_3||_{Y^{-s}_{\sigma_3}}=1}\sum_S N^s_{\min}\left(\frac{N_{\min}}{N_{\max}}+\frac{1}{N_{\min}}\right)^{\delta}\prod_{j=1}^3||P_{N_j}u_j||_{Y^0_{\sigma_j}}\end{aligned}$$

since

$$\left|\int_0^T\int_{\mathbb{T}^d}u_1u_2(\nabla\cdot u_3)dxdt\right|=0$$

if $\max_{1\leq j\leq 3}|\xi_j| = 0$ for $\xi_j \in \operatorname{supp}\widehat{u_j(t)}$. Furthermore, we have

$$\begin{aligned}
&\sum_{S_1} N_{\min}^s \left(\frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}}\right)^\delta \prod_{j=1}^3 ||P_{N_j}u_j||_{Y^0_{\sigma_j}} \\
&\sim \sum_{N_2}\sum_{N_3\sim N_2}\sum_{N_1\lesssim N_2} N_3^s N_1^s \left(\frac{N_1}{N_2} + \frac{1}{N_1}\right)^\delta ||P_{N_1}u_1||_{Y^0_{\sigma_1}}||P_{N_2}u_2||_{Y^0_{\sigma_2}}||P_{N_3}u_3||_{Y^{-s}_{\sigma_3}} \\
&\leq ||u_1||_{Y^s_{\sigma_1}}||u_2||_{Y^s_{\sigma_2}}||u_3||_{Y^{-s}_{\sigma_3}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{S_3} N_{\min}^s \left(\frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}}\right)^\delta \prod_{j=1}^3 ||P_{N_j}u_j||_{Y^0_{\sigma_j}} \\
&\sim \sum_{N_1}\sum_{N_2\sim N_1}\sum_{N_3\lesssim N_2} N_3^{2s} \left(\frac{N_3}{N_2} + \frac{1}{N_3}\right)^\delta ||P_{N_1}u_1||_{Y^0_{\sigma_1}}||P_{N_2}u_2||_{Y^0_{\sigma_2}}||P_{N_3}u_3||_{Y^{-s}_{\sigma_3}} \\
&\leq ||u_1||_{Y^s_{\sigma_1}}||u_2||_{Y^s_{\sigma_2}}||u_3||_{Y^{-s}_{\sigma_3}}
\end{aligned}$$

by the Cauchy-Schwartz inequality for the dyadic sum. By the same way as the estimate for the summation of S_1, we have

$$\sum_{S_2} N_{\min}^s \left(\frac{N_{\min}}{N_{\max}} + \frac{1}{N_{\min}}\right)^\delta \prod_{j=1}^3 ||P_{N_j}u_j||_{Y^0_{\sigma_j}} \lesssim ||u_1||_{Y^s_{\sigma_1}}||u_2||_{Y^s_{\sigma_2}}||u_3||_{Y^{-s}_{\sigma_3}}.$$

Therefore, we obtain (4.25) since $||u_1||_{Y^s_{\sigma_1}} = ||u||_{Y^s_\alpha}$ and $||u_2||_{Y^s_{\sigma_2}} = ||v||_{Y^s_\beta}$. □

Proof of Theorem 4.4. For $r > 0$, we define

$$X^s_r(I) := \left\{(u,v,w) \in X^s(I) \,\middle|\, ||u||_{Z^s_\alpha(I)},\ ||v||_{Z^s_\beta(I)},\ ||w||_{Z^s_\gamma(I)} \leq 2r\right\} \tag{4.26}$$

which is a closed subset of $X^s(I)$. Let $s > 0$ if $d = 1$, $s > s_c$ $(= d/2-1)$ if $d = 2, 3, 4$, $s \geq s_c$ if $d \geq 5$, $(u_0, v_0, w_0) \in B_r(H^s(\mathbb{T}^d)\times H^s(\mathbb{T}^d)\times H^s(\mathbb{T}^d))$ be given and σ be given in the proof of Proposition 4.23. For $0 < T \leq 2\pi|\sigma^{-1}|$ and $(u,v,w) \in X^s_r([0,T])$, we have

$$\begin{aligned}
||\Phi^{(1)}_{\alpha,u_0}(w,v)||_{Z^s_\alpha([0,T))} &\leq ||u_0||_{H^s} + C||w||_{Z^s_\gamma([0,T))}||v||_{Z^s_\beta([0,T))} \leq r(1+4Cr),\\
||\Phi^{(1)}_{\beta,v_0}(\overline{w},u)||_{Z^s_\beta([0,T))} &\leq ||v_0||_{H^s} + C||w||_{Z^s_\gamma([0,T))}||u||_{Z^s_\alpha([0,T))} \leq r(1+4Cr),\\
||\Phi^{(2)}_{\gamma,w_0}(u,\overline{v})||_{Z^s_\gamma([0,T))} &\leq ||w_0||_{H^s} + C||u||_{Z^s_\alpha([0,T))}||v||_{Z^s_\beta([0,T))} \leq r(1+4Cr)
\end{aligned}$$

and

$$\begin{aligned}
||\Phi^{(1)}_{\alpha,u_0}(w_1,v_1) - \Phi^{(1)}_{\alpha,u_0}(w_2,v_2)||_{Z^s_\alpha([0,T))} &\leq 2Cr\left(||w_1-w_2||_{Z^s_\gamma([0,T))} + ||v_1-v_2||_{Z^s_\beta([0,T))}\right),\\
||\Phi^{(1)}_{\beta,v_0}(\overline{w_1},u_1) - \Phi^{(1)}_{\beta,v_0}(\overline{w_2},u_2)||_{Z^s_\beta([0,T))} &\leq 2Cr\left(||w_1-w_2||_{Z^s_\gamma([0,T))} + ||u_1-u_2||_{Z^s_\alpha([0,T))}\right),\\
||\Phi^{(2)}_{\gamma,w_0}(u_1,\overline{v_1}) - \Phi^{(2)}_{\gamma,w_0}(u_2,\overline{v_2})||_{Z^s_\gamma([0,T))} &\leq 2Cr\left(||u_1-u_2||_{Z^s_\alpha([0,T))} + ||v_1-v_2||_{Z^s_\beta([0,T))}\right)
\end{aligned}$$

by Proposition 4.23 and

$$||e^{i\sigma t\Delta}\varphi||_{Z^s_\sigma([0,T))} \le ||\mathbf{1}_{[0,T)}e^{i\sigma t\Delta}\varphi||_{Z^s_\sigma} \le ||\varphi||_{H^s},$$

where C is an implicit constant in (4.23)–(4.25). Therefore if we choose r satisfying

$$r < (4C)^{-1},$$

then Φ is a contraction map on $X^s_r([0,T))$. This implies the existence of the solution of the system (4.1) and the uniqueness in the ball $X^s_r([0,\infty))$. The Lipschitz continuously of the flow map is also proved by similar argument. □

Theorem 4.8 is proved by using the estimate (4.23) for $(\alpha,\beta,\gamma) = (-1,1,1)$.

Chapter 5

Nonlinear Schrödinger equations with a derivative nonlinearity

5.1 Review for results

We consider the Cauchy problem of the nonlinear Schrödinger equations:

$$\begin{cases} (i\partial_t + \Delta)u = \partial_k(\overline{u}^m), \quad (t,x) \in (0,\infty)\times\mathbb{R}^d \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d \end{cases} \tag{5.1}$$

where $m \in \mathbb{N}$, $m \geq 2$, $1 \leq k \leq d$, $\partial_k = \partial/\partial x_k$ and the unknown function u is $\mathbb{C}$-valued. (5.1) is invariant under the following scaling transformation:

$$u_\lambda(t,x) = \lambda^{-1/(m-1)}u(\lambda^{-2}t, \lambda^{-1}x),$$

and the scaling critical regularity is $s_c = d/2 - 1/(m-1)$. The aim of this chapter is to prove the well-posedness and the scattering for the solution of (5.1) in the scaling critical Sobolev space.

First, we introduce some known results for related problems. The nonlinear term in (5.1) contains a derivative. A derivative loss arising from the nonlinearity makes the problem difficult. In fact, Mizohata ([55]) proved that a necessary condition for the L^2 well-posedness of the problem:

$$\begin{cases} i\partial_t u - \Delta u = b_1(x)\cdot\nabla u, \ t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^d \end{cases}$$

is the uniform bound

$$\sup_{x\in\mathbb{R}^d, \omega\in S^{d-1}, R>0} \left| \mathrm{Re}\int_0^R b_1(x+r\omega)\cdot\omega dr \right| < \infty.$$

Furthermore, Christ ([9]) proved that the flow map of the Cauchy problem:

$$\begin{cases} i\partial_t u - \partial_x^2 u = u\partial_x u, \ t \in \mathbb{R}, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \ x \in \mathbb{R} \end{cases} \tag{5.2}$$

is not continuous on $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. While, Ozawa ([60]) proved that the local well-posedness of (5.2) in the space of all function $\phi \in H^1(\mathbb{R})$ satisfying the bounded condition

$$\sup_{x\in\mathbb{R}} \left| \int_{-\infty}^{x} \phi \right| < \infty.$$

Furthermore, he proved that if the initial data ϕ satisfies some condition, then the local solution can be extend globally in time and the solution scatters. For the Cauchy problem of the one dimensional derivative Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\lambda\partial_x(|u|^2 u), \ t \in \mathbb{R}, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}, \end{cases} \tag{5.3}$$

Takaoka ([66]) proved the local well-posedness in $H^s(\mathbb{R})$ for $s \geq 1/2$ by using the gauge transform. This result was extended to global well-posedness ([16], [17], [54], [67]). While, ill-posedness of (5.3) was obtained for $s < 1/2$ ([3], [67]). Hao ([27]) considered the Cauchy problem:

$$\begin{cases} i\partial_t u - \partial_x^2 u + i\lambda|u|^k\partial_x u, \ t \in \mathbb{R}, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \ x \in \mathbb{R} \end{cases}$$

for $k \geq 5$ and obtained local well-posedness in $H^{1/2}(\mathbb{R})$. For more general problem:

$$\begin{cases} i\partial_t u - \Delta u = P(u, \overline{u}, \nabla u, \nabla \overline{u}), \ t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^d, \\ P \text{ is a polynomial which has no constant and linear terms,} \end{cases} \tag{5.4}$$

there are many positive results for the well-posedness in the weighted Sobolev space ([1], [2], [7], [8], [48], [65]). Kenig, Ponce and Vega ([48]) also obtained that (5.4) is locally well-posed in $H^s(\mathbb{R}^d)$ (without weight) for large enough s when P has no quadratic terms.

The Benjamin–Ono equation:

$$\partial_t u + H\partial_x^2 u = u\partial_x u, \ (t,x) \in \mathbb{R} \times \mathbb{R} \tag{5.5}$$

is also related to the quadratic derivative nonlinear Schrödinger equation. It is known that the flow map of (5.5) is not uniformly continuous on $H^s(\mathbb{R})$ for $s > 0$ ([51]). But the Benjamin–Ono equation has better structure than the equation (5.2). Actually, Tao ([68]) proved that (5.5) is globally well-posed in $H^1(\mathbb{R})$ by using the gauge transform. Furthermore, Ionescu and Kenig ([37]) proved that (5.5) is globally well-posed in $H_r^s(\mathbb{R})$ for $s \geq 0$, where $H_r^s(\mathbb{R})$ is the Banach space of the all real valued function $f \in H^s(\mathbb{R})$.

Next, we introduce some known results for (5.1). Grünrock ([23]) proved that (5.1) is locally well-posed in $L^2(\mathbb{R})$ when $d = 1$, $m = 2$ and in $H^s(\mathbb{R}^d)$ for $s > s_c$ when $d \geq 1$, $m + d \geq 4$. We proved that (5.1) with $d \geq 2$, $m = 2$ is globally well-posed for small data in $H^{s_c}(\mathbb{R}^d)$ (also in $\dot{H}^{s_c}(\mathbb{R}^d)$) and the solution scatters in Chapter 3. The results are an extension of the results by Grünrock ([23]) for $d \geq 2$, $m = 2$. The main results in this chapter are an extension of the results by Grünrock ([23]) for $d \geq 1$, $m \geq 3$.

Now, we give the main results in the present chapter. To begin with, we define the function spaces to construct the solution.

Definition 5.1. *Let* $s \in \mathbb{R}$.
(i) *We define* $\dot{Z}^s := \{u \in C(\mathbb{R}; \dot{H}^s(\mathbb{R}^d)) \cap U_\Delta^2 L^2 |\ ||u||_{\dot{Z}^s} < \infty\}$ *with the norm*

$$||u||_{\dot{Z}^s} := \left(\sum_N N^{2s} ||P_N u||_{U_\Delta^2 L^2}^2 \right)^{1/2}.$$

(ii) *We define* $Z^s := \{u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap U_\Delta^2 L^2 |\ ||u||_{Z^s} < \infty\}$ *with the norm*

$$||u||_{Z^s} := ||u||_{\dot{Z}^0} + ||u||_{\dot{Z}^s}.$$

(iii) *We define* $\dot{Y}^s := \{u \in C(\mathbb{R}; \dot{H}^s(\mathbb{R}^d)) \cap V_\Delta^2 L^2 |\ ||u||_{\dot{Y}^s} < \infty\}$ *with the norm*

$$||u||_{\dot{Y}^s} := \left(\sum_N N^{2s} ||P_N u||_{V_\Delta^2 L^2}^2 \right)^{1/2}.$$

(iv) *We define* $Y^s := \{u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap V_\Delta^2 L^2 |\ ||u||_{Y^s} < \infty\}$ *with the norm*

$$||u||_{Y^s} := ||u||_{\dot{Y}^0} + ||u||_{\dot{Y}^s}.$$

Remark 5.2 ([25] Remark 2.23). *Let* E *be a Banach space of continuous functions* $f : \mathbb{R} \to \mathcal{H}$*, for some Hilbert space* $\mathcal{H}$*. We also consider the corresponding restriction space to the interval* $I \subset \mathbb{R}$ *by*

$$E(I) = \{u \in C(I, \mathcal{H}) | \exists v \in E \ s.t.\ v(t) = u(t),\ t \in I\}$$

endowed with the norm $||u||_{E(I)} = \inf\{||v||_E | v(t) = u(t),\ t \in I\}$. *Obviously,* $E(I)$ *is also a Banach space.*

Our results are followings.

Theorem 5.3. *Let* $d \geq 1$, $m \geq 3$ *and* $s_c = d/2 - 1/(m-1)$.
(i) *The equation* (5.1) *is globally well-posed for small data in* $\dot{H}^{s_c}(\mathbb{R}^d)$. *More precisely, there exists* $r > 0$ *such that for all initial data* $u_0 \in B_r(\dot{H}^{s_c}(\mathbb{R}^d))$, *there exists a solution*

$$u \in \dot{Z}_r^{s_c}([0,\infty)) \subset C([0,\infty); \dot{H}^{s_c}(\mathbb{R}^d))$$

of (5.1) *on* $(0,\infty)$. *Such solution is unique in* $\dot{Z}_r^{s_c}([0,\infty))$ *which is a closed subset of* $\dot{Z}^{s_c}([0,\infty))$ (see (5.19)). *Moreover, the flow map*

$$S_+ : B_r(\dot{H}^{s_c}(\mathbb{R}^d)) \ni u_0 \mapsto u \in \dot{Z}^{s_c}([0,\infty))$$

is Lipschitz continuous.
(ii) *The statement in* (i) *remains valid if we replace the space* $\dot{H}^{s_c}(\mathbb{R}^d)$, $\dot{Z}^{s_c}([0,\infty))$ *and* $\dot{Z}_r^{s_c}([0,\infty))$ *by* $H^s(\mathbb{R}^d)$, $Z^s([0,\infty))$ *and* $Z_r^s([0,\infty))$ *for* $s \geq s_c$.

Remark 5.4. *Due to the time reversibility of the system (5.1), the above theorems also hold in corresponding intervals* $(-\infty, 0)$. *We denote the flow map with* $t \in (-\infty, 0)$ *by* S_-.

Corollary 5.5. *Let* $d \geq 1$, $m \geq 3$ *and* $s_c = d/2 - 1/(m-1)$.
(i) *Let* $r > 0$ *be as in* Theorem 5.3. *For every* $u_0 \in B_r(\dot{H}^{s_c}(\mathbb{R}^d))$, *there exists* $u_\pm \in \dot{H}^{s_c}$ *such that*

$$S_\pm(u_0) - e^{it\Delta}u_\pm \to 0 \text{ in } \dot{H}^{s_c}(\mathbb{R}^d) \text{ as } t \to \pm\infty.$$

(ii) *The statement in* (i) *remains valid if we replace the space* $\dot{H}^{s_c}(\mathbb{R}^d)$ *by* $H^s(\mathbb{R}^d)$ *for* $s \geq s_c$.

The main tools of our results are U^p space and V^p space which are applied to prove the well-posedness and the scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch ([25], [26]).

The rest of this chapter is planned as follows. In Section 2, we will give the multilinear estimates which are main estimates in this chapter. In Section 3, we will give the proof of the well-posedness and the scattering (Theorems 5.3 and Corollary 5.5).

5.2 Strichartz and multilinear estimates

In this section, we give the Strichartz estimate and prove multilinear estimates which will be used to prove the well-posedness. First, we recall the Strichartz estimate given in Chapter 3 (see Proposition 3.12, Corollary 3.13).

Proposition 5.6 (Strichartz estimate). *Let (p,q) be an admissible pair of exponents for the Schrödinger equation, i.e. $2 \le q \le 2d/(d-2)$ ($2 \le q < \infty$ if $d=2$, $2 \le q \le \infty$ if $d=1$), $2/p = d(1/2 - 1/q)$. Then, we have*

$$||e^{it\Delta}\varphi||_{L_t^p L_x^q} \lesssim ||\varphi||_{L_x^2}$$

for any $\varphi \in L^2(\mathbb{R}^d)$.

Corollary 5.7. *Let (p,q) be an admissible pair of exponents for the Schrödinger equation, i.e. $2 \le q \le 2d/(d-2)$ ($2 \le q < \infty$ if $d=2$, $2 \le q \le \infty$ if $d=1$), $2/p = d(1/2 - 1/q)$. Then, we have*

$$||u||_{L_t^p L_x^q} \lesssim ||u||_{U_\Delta^p}, \quad u \in U_\Delta^p L^2. \tag{5.6}$$

Next, we prove the multilinear estimate as follows.

Lemma 5.8. *Let $d \ge 1$, $m \ge 2$, $s_c = d/2 - 1/(m-1)$ and $b > 1/2$. For any dyadic numbers $N_1 \gg N_2 \ge \cdots \ge N_m$, we have*

$$\left|\left|\prod_{j=1}^m P_{N_j} u_j\right|\right|_{L_{tx}^2} \lesssim ||P_{N_1}u_1||_{X^{0,b}} \prod_{j=2}^m \left(\frac{N_j}{N_1}\right)^{1/2(m-1)} N_j^{s_c} ||P_{N_j}u_j||_{X^{0,b}}, \tag{5.7}$$

where $||u||_{X^{0,b}} := ||\langle\tau + |\xi|^2\rangle^b \widetilde{u}||_{L_{\tau\xi}^2}$.

Proof. For the case $d=2$ and $m=2$, the estimate (5.7) is proved by Colliander, Delort, Kenig, and Staffilani ([15] Lemma 1). The proof for general case as following is similar to their argument.

We put $g_j(\tau_j,\xi_j) := \langle\tau_j + |\xi_j|^2\rangle^b \widetilde{P_{N_j}u_j}(\tau_j,\xi_j)$ ($j = 1,\cdots,m$) and $A_N := \{\xi \in \mathbb{R}^d | N/2 \le |\xi| \le 2N\}$ for a dyadic number N. By the Plancherel's theorem and the duality argument, it is enough to prove the estimate

$$\begin{aligned} I &:= \left| \int_{\mathbb{R}^m} \int_{\prod_{j=1}^m A_{N_j}} g_0\left(\sum_{j=1}^m \tau_j, \sum_{j=1}^m \xi_j\right) \prod_{j=1}^m \frac{g_j(\tau_j,\xi_j)}{\langle\tau_j + |\xi_j|^2\rangle^b} d\xi_* d\tau_* \right| \\ &\lesssim \left(\prod_{j=2}^m \left(\frac{N_j}{N_1}\right)^{1/2(m-1)} N_j^{s_c}\right) \prod_{j=0}^m ||g_j||_{L_{\tau\xi}^2} \end{aligned}$$

for $g_0 \in L^2_{\tau\xi}$, where $\xi_* = (\xi_1, \cdots, \xi_m)$, $\tau_* = (\tau_1, \cdots, \tau_m)$. We change the variables $\tau_* \mapsto \theta_* = (\theta_1, \cdots, \theta_m)$ as $\theta_j = \tau_j + |\xi_j|^2$ $(j = 1, \cdots, m)$ and put

$$G_0(\theta_*, \xi_*) := g_0\left(\sum_{j=1}^{m}(\theta_j - |\xi_j|^2), \sum_{j=1}^{m}\xi_j\right),$$

$$G_j(\theta_j, \xi_j) := g_j(\theta_j - |\xi_j|^2, \xi_j)\ (j = 1, \cdots, m).$$

Then, we have

$$\begin{aligned} I &\leq \int_{\mathbb{R}^m}\left(\prod_{j=1}^{m}\frac{1}{\langle\theta_j\rangle^b}\right)\left(\int_{\prod_{j=1}^{m}A_{N_j}}\left|G_0(\theta_*, \xi_*)\prod_{j=1}^{m}G_j(\theta_j, \xi_j)\right|d\xi_*\right)d\theta_* \\ &\lesssim \int_{\mathbb{R}^m}\left(\prod_{j=1}^{m}\frac{1}{\langle\theta_j\rangle^b}\right)\left(\int_{\prod_{j=1}^{m}A_{N_j}}|G_0(\theta_*, \xi_*)|^2 d\xi_*\right)^{1/2}\prod_{j=1}^{m}||G_j(\theta_j, \cdot)||_{L^2_\xi}d\theta_* \end{aligned}$$

by the Cauchy-Schwartz inequality. For $1 \leq k \leq d$, we put

$$A^k_{N_1} := \{\xi_1 = (\xi_1^{(1)}, \cdots, \xi_1^{(d)}) \in \mathbb{R}^d |\ N_1/2 \leq |\xi_1| \leq 2N_1,\ |\xi_1^{(k)}| \geq N_1/(2\sqrt{d})\}$$

and

$$J_k(\theta_*) := \left(\int_{A^k_{N_1}\times\prod_{j=2}^{m}A_{N_j}}|G_0(\theta_*, \xi_*)|^2 d\xi_*\right).$$

We consider only the estimate for J_1. The estimates for other J_k are obtained by the same way.

Assume $d \geq 2$. By changing the variables $(\xi_1, \xi_2) = (\xi_1^{(1)}, \cdots, \xi_1^{(d)}, \xi_2^{(1)}, \cdots, \xi_2^{(d)}) \mapsto (\mu, \nu, \eta)$ as

$$\begin{cases} \mu = \sum_{j=1}^{m}(\theta_j - |\xi_j|^2) \in \mathbb{R}, \\ \nu = \sum_{j=1}^{m}\xi_j \in \mathbb{R}^d, \\ \eta = (\xi_2^{(2)} \cdots, \xi_2^{(d)}) \in \mathbb{R}^{d-1}, \end{cases} \tag{5.8}$$

we have

$$d\mu d\nu d\eta = 2|\xi_1^{(1)} - \xi_2^{(1)}|d\xi_1 d\xi_2$$

and

$$G_0(\theta_*, \xi_*) = g_0(\mu, \nu).$$

We note that $|\xi_1^{(1)} - \xi_2^{(1)}| \sim N_1$ for any $(\xi_1, \xi_2) \in A^1_{N_1} \times A_{N_2}$ with $N_1 \gg N_2$. Furthermore, $\xi_2 \in A_{N_2}$ implies that $\eta \in [-2N_2, 2N_2]^{d-1}$. Therefore, we obtain

$$\begin{aligned} J_1(\theta_*) &\lesssim \int_{\prod_{j=3}^{m}A_{N_j}}\left(\int_{[-2N_2,2N_2]^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}}|g_0(\mu, \nu)|^2\frac{1}{N_1}d\mu d\nu d\eta\right)d\xi_3\cdots d\xi_m \\ &\sim \frac{N_2^{d-1}}{N_1}\left(\prod_{j=3}^{m}N_j^d\right)||g_0||^2_{L^2_{\tau\xi}} \leq \left(\prod_{j=2}^{m}\left(\frac{N_j}{N_1}\right)^{1/(m-1)}N_j^{d-2/(m-1)}\right)||g_0||^2_{L^2_{\tau\xi}} \end{aligned}$$

since $N_2 \geq N_j$ for $3 \leq j \leq m$. As a result, we have

$$
\begin{aligned}
I &\lesssim \int_{\mathbb{R}^m} \left(\prod_{j=1}^{m} \frac{1}{\langle \theta_j \rangle^b} \right) \left(\sum_{k=1}^{d} J_k(\theta_*) \right)^{1/2} \prod_{j=1}^{m} ||G_j(\theta_j, \cdot)||_{L^2_\xi} d\theta_* \\
&\lesssim \left(\prod_{j=2}^{m} \left(\frac{N_j}{N_1} \right)^{1/2(m-1)} N_j^{s_c} \right) \prod_{j=0}^{m} ||g_j||_{L^2_{\tau\xi}}
\end{aligned}
$$

by the Cauchy-Schwartz inequality and changing the variables $\theta_* \mapsto \tau_*$ as $\theta_j = \tau_j + |\xi_j|^2$ $(j = 1, \cdots, m)$.

For $d = 1$, we obtain the same result by changing the variables $(\xi_1, \xi_2) \mapsto (\mu, \nu)$ as $\mu = \sum_{j=1}^{m} (\theta_j - |\xi_j|^2)$, $\nu = \sum_{j=1}^{m} \xi_j$ instead of (5.8). □

Corollary 5.9. *Let $m \geq 2$, $m + d \geq 4$ and $s_c = d/2 - 1/(m-1)$. For any dyadic numbers $N_1 \gg N_2 \geq \cdots \geq N_m$ and $0 < \delta < 1/2(m-1)$, we have*

$$
\left|\left| \prod_{j=1}^{m} P_{N_j} u_j \right|\right|_{L^2_{tx}} \lesssim ||P_{N_1} u_1||_{U^2_\Delta} \prod_{j=2}^{m} \left(\frac{N_j}{N_1} \right)^{1/2(m-1)} N_j^{s_c} ||P_{N_j} u_j||_{U^2_\Delta L^2}, \tag{5.9}
$$

$$
\left|\left| \prod_{j=1}^{m} P_{N_j} u_j \right|\right|_{L^2_{tx}} \lesssim ||P_{N_1} u_1||_{V^2_\Delta} \prod_{j=2}^{m} \left(\frac{N_j}{N_1} \right)^{\delta} N_j^{s_c} ||P_{N_j} u_j||_{V^2_\Delta L^2}. \tag{5.10}
$$

Proof. To obtain (5.9), we use the argument of the proof of Corollary 2.21 (27) in [25]. Let $\phi_1, \cdots, \phi_m \in L^2(\mathbb{R}^d)$ and define $\phi_j^\lambda(x) := \phi_j(\lambda x)$ $(j = 1, \cdots, m)$ for $\lambda \in \mathbb{R}$. By using the rescaling $(t, x) \mapsto (\lambda^2 t, \lambda x)$, we have

$$
\left|\left| \prod_{j=1}^{m} P_{N_j}(e^{it\Delta}\phi_j) \right|\right|_{L^2([-T,T]\times\mathbb{R}^d)} = \lambda^{d/2+1} \left|\left| \prod_{j=1}^{m} P_{\lambda N_j}(e^{it\Delta}\phi_j^\lambda) \right|\right|_{L^2([-\lambda^{-2}T,\lambda^{-2}T]\times\mathbb{R}^d)}.
$$

Therefore by putting $\lambda = \sqrt{T}$ and (5.7), we have

$$
\begin{aligned}
&\left|\left| \prod_{j=1}^{m} P_{N_j}(e^{it\Delta}\phi_j) \right|\right|_{L^2([-T,T]\times\mathbb{R}^d)} \\
&\lesssim \sqrt{T}^{md/2} ||P_{\sqrt{T}N_1}\phi_1^{\sqrt{T}}||_{L^2_x} \prod_{j=2}^{m} \left(\frac{N_j}{N_1} \right)^{1/2(m-1)} N_j^{s_c} ||P_{\sqrt{T}N_j}\phi_j^{\sqrt{T}}||_{L^2_x} \\
&= ||P_{N_1}\phi_1||_{L^2_x} \prod_{j=2}^{m} \left(\frac{N_j}{N_1} \right)^{1/2(m-1)} N_j^{s_c} ||P_{N_j}\phi_j||_{L^2_x}.
\end{aligned}
$$

Let $T \to \infty$, then we obtain

$$
\left|\left| \prod_{j=1}^{m} P_{N_j}(e^{it\Delta}\phi_j) \right|\right|_{L^2_{tx}} \lesssim ||P_{N_1}\phi_1||_{L^2_x} \prod_{j=2}^{m} \left(\frac{N_j}{N_1} \right)^{1/2(m-1)} N_j^{s_c} ||P_{N_j}\phi_j||_{L^2_x}
$$

and (5.9) follows from proposition 2.8.

To obtain (5.10), we first prove the U^{2m} estimate. By the Cauchy-Schwartz inequality, the Sobolev embedding $\dot{W}^{s_c,2md/(md-2)}(\mathbb{R}^d) \hookrightarrow L^{m(m-1)d}(\mathbb{R}^d)$ (which holds when $m \geq 2$, $m+d \geq 4$) and (5.6), we have

$$\begin{aligned} \left|\left|\prod_{j=1}^{m} P_{N_j} u_j\right|\right|_{L^2_{tx}} &\lesssim ||P_{N_1}u_1||_{L_t^{2m}L_x^{2md/(md-2)}} \prod_{j=2}^{m} N_j^{s_c}||P_{N_j}u_j||_{L_t^{2m}L_x^{2md/(md-2)}} \\ &\lesssim ||P_{N_1}u_1||_{U^{2m}_{\Delta}L^2} \prod_{j=2}^{m} N_j^{s_c}||P_{N_j}u_j||_{U^{2m}_{\Delta}L^2} \end{aligned} \tag{5.11}$$

for any dyadic numbers $N_1, \cdots, N_m \in 2^{\mathbb{Z}}$. We use the interpolation between (5.9) and (5.11) via Proposition 2.9. Then, we get (5.10) by the same argument of the proof of Corollary 2.21 (28) in [25]. □

Lemma 5.10. *We assume that* (τ_0, ξ_0), (τ_1, ξ_1), $\cdots$, $(\tau_m, \xi_m) \in \mathbb{R} \times \mathbb{R}^d$ *satisfy* $\sum_{j=0}^{d} \tau_j = 0$ *and* $\sum_{j=0}^{d} \xi_j = 0$. *Then, we have*

$$\max_{0 \leq j \leq m} |\tau_j + |\xi_j|^2| \geq \frac{1}{m+1} \max_{0 \leq j \leq m} |\xi_j|^2. \tag{5.12}$$

Proof. By the triangle inequality, we obtain (5.12). □

The following propositions will be used to prove the key estimate for the well-posedness in the next section.

Proposition 5.11. *Let* $d \geq 1$, $m \geq 3$, $s_c = d/2 - 1/(m-1)$ *and* $0 < T \leq \infty$. *For a dyadic number* $N_1 \in 2^{\mathbb{Z}}$, *we define the set* $S(N_1)$ *as*

$$S(N_1) := \{(N_2, \cdots, N_m) \in (2^{\mathbb{Z}})^{m-1} | N_1 \gg N_2 \geq \cdots \geq N_m\}.$$

If $N_0 \sim N_1$, *then we have*

$$\begin{aligned} &\left| \sum_{S(N_1)} \int_0^T \int_{\mathbb{R}^d} \left(N_0 \prod_{j=0}^{m} P_{N_j} u_j \right) dxdt \right| \\ &\lesssim ||P_{N_0}u_0||_{V^2_{\Delta}L^2} ||P_{N_1}u_1||_{V^2_{\Delta}L^2} \prod_{j=2}^{m} ||u_j||_{\dot{Y}^{s_c}}. \end{aligned} \tag{5.13}$$

Proof. We define $u_{j,N_j,T} := \mathbf{1}_{[0,T)} P_{N_j} u_j$ $(j = 1, \cdots, m)$ and put $M := N_0^2/4(m+1)$. We decompose $Id = Q^{\Delta}_{<M} + Q^{\Delta}_{\geq M}$. We divide the integrals on the left-hand side of (5.13) into 2^{m+1} piece of the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 \prod_{j=0}^{m} Q_j^{\Delta} u_{j,N_j,T} \right) dxdt \tag{5.14}$$

with $Q_j^{\Delta} \in \{Q_{\geq M}^{\Delta}, Q_{<M}^{\Delta}\}$ $(j = 0, \cdots, m)$. By the Plancherel's theorem, we have

$$(5.14) = c \int_{\sum_{j=0}^{m} \tau_j = 0} \int_{\sum_{j=0}^{m} \xi_j = 0} N_0 \prod_{j=0}^{m} \mathcal{F}[Q_j^{\Delta} u_{j,N_j,T}](\tau_j, \xi_j),$$

where c is a constant. Therefore, Lemma 5.10 implies that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 \prod_{j=0}^{m} Q_{<M}^{\Delta} u_{j,N_j,T} \right) dxdt = 0.$$

So, let us now consider the case that $Q_j^{\Delta} = Q_{\geq M}^{\Delta}$ for some $0 \leq j \leq m$.

First, we consider the case $Q_0^{\Delta} = Q_{\geq M}^{\Delta}$. By the Cauchy-Schwartz inequality, we have

$$\begin{aligned} &\left| \sum_{S(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 Q_{\geq M}^{\Delta} u_{0,N_0,T} \prod_{j=1}^{m} Q_j^{\Delta} u_{j,N_j,T} \right) dxdt \right| \\ &\leq \sum_{S(N_1)} N_0 ||Q_{\geq M}^{\Delta} u_{0,N_0,T}||_{L_{tx}^2} \left|\left| \prod_{j=1}^{m} Q_j^{\Delta} u_{j,N_j,T} \right|\right|_{L_{tx}^2}. \end{aligned}$$

Furthermore by (2.1) with $p = 2$ and $M \sim N_0^2$, we have

$$||Q_{\geq M}^{\Delta} u_{0,N_0,T}||_{L_{tx}^2} \lesssim N_0^{-1} ||u_{0,N_0,T}||_{V_{\Delta}^2 L^2}.$$

While by (5.10), (2.2) and the Cauchy-Schwartz inequality for the dyadic sum, we have

$$\begin{aligned} \sum_{S(N_1)} \left|\left| \prod_{j=1}^{m} Q_j^{\Delta} u_{j,N_j,T} \right|\right|_{L_{tx}^2} &\lesssim ||u_{1,N_1,T}||_{V_{\Delta}^2 L^2} \sum_{S(N_1)} \prod_{j=2}^{m} \left(\frac{N_j}{N_1} \right)^{\delta} N_j^{s_c} ||u_{j,N_j,T}||_{V_{\Delta}^2 L^2} \\ &\lesssim ||u_{1,N_1,T}||_{V_{\Delta}^2 L^2} \prod_{j=2}^{m} \left(\sum_{N_j \leq N_1} N_j^{2s_c} ||u_{j,N_j,T}||_{V_{\Delta}^2 L^2}^2 \right)^{1/2}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\left| \sum_{S(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 Q_{\geq M}^{\Delta} u_{0,N_0,T} \prod_{j=1}^{m} Q_j^{\Delta} u_{j,N_j,T} \right) dxdt \right| \\ &\lesssim ||P_{N_0} u_0||_{V_{\Delta}^2 L^2} ||P_{N_1} u_1||_{V_{\Delta}^2 L^2} \prod_{j=2}^{m} ||u_j||_{\dot{Y}^{s_c}} \end{aligned}$$

since $||\mathbf{1}_{[0,T)} u||_{V_{\Delta}^2 L^2} \lesssim ||u||_{V_{\Delta}^2 L^2}$ for any $T \in (0, \infty]$. For the case $Q_1^{\Delta} = Q_{\geq M}^{\Delta}$ is proved in same way.

Next, we consider the case $Q_k^\Delta = Q_{\geq M}^\Delta$ for some $2 \leq k \leq m$. By the Hölder's inequality, we have

$$\begin{aligned}
&\left| \sum_{S(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 Q_{\geq M}^\Delta u_{k,N_k,T} \prod_{\substack{j=0 \\ j\neq k}}^m Q_j^\Delta u_{j,N_j,T} \right) dxdt \right| \\
&\lesssim N_0 ||Q_0^\Delta u_{0,N_0,T}||_{L_t^4 L_x^{2d/(d-1)}} ||Q_1^\Delta u_{1,N_1,T}||_{L_t^4 L_x^{2d/(d-1)}} \\
&\quad \times \left|\left| \sum_{N_k} Q_{\geq M}^\Delta u_{k,N_k,T} \right|\right|_{L_t^2 L_x^{(m-1)d}} \prod_{\substack{j=2 \\ j\neq k}}^m \left|\left| \sum_{N_j} Q_j^\Delta u_{j,N_j,T} \right|\right|_{L_t^\infty L_x^{(m-1)d}}.
\end{aligned}$$

By (5.6), the embedding $V_\Delta^2 L^2 \hookrightarrow U_\Delta^4 L^2$ and (2.2), we have

$$||Q_0^\Delta u_{0,N_0,T}||_{L_t^4 L_x^{2d/(d-1)}} ||Q_1^\Delta u_{1,N_1,T}||_{L_t^4 L_x^{2d/(d-1)}} \lesssim ||u_{0,N_0,T}||_{V_\Delta^2 L^2} ||u_{1,N_1,T}||_{V_\Delta^2 L^2}.$$

While by the Sobolev embedding $\dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^{(m-1)d}(\mathbb{R}^d)$, L^2 orthogonality and (2.1) with $p = 2$, we have

$$\begin{aligned}
\left|\left| \sum_{N_k} Q_{\geq M}^\Delta u_{k,N_k,T} \right|\right|_{L_t^2 L_x^{(m-1)d}} &\lesssim \left(\sum_{N_k} N_k^{2s_c} ||Q_{\geq M}^\Delta u_{k,N_k,T}||_{L_{tx}^2}^2 \right)^{1/2} \\
&\lesssim N_0^{-1} \left(\sum_{N_k} N_k^{2s_c} ||u_{k,N_k,T}||_{V_\Delta^2 L^2}^2 \right)^{1/2}
\end{aligned}$$

since $M \sim N_0^2$. Furthermore by the Sobolev embedding $\dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^{(m-1)d}(\mathbb{R}^d)$, L^2 orthogonality, $V_\Delta^2 L^2 \hookrightarrow L^\infty(\mathbb{R}; L^2)$ and (2.2), we have

$$\begin{aligned}
\left|\left| \sum_{N_j} Q_j^\Delta u_{j,N_j,T} \right|\right|_{L_t^\infty L_x^{(m-1)d}} &\lesssim \left(\sum_{N_j} N_j^{2s_c} ||Q_j^\Delta u_{j,N_j,T}||_{L_t^\infty L_x^2}^2 \right)^{1/2} \\
&\lesssim \left(\sum_{N_j} N_j^{2s_c} ||u_{j,N_j,T}||_{V_\Delta^2 L^2}^2 \right)^{1/2}.
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
&\left| \sum_{S(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 Q_{\geq M}^\Delta u_{k,N_k,T} \prod_{\substack{j=0 \\ j\neq k}}^m Q_j^\Delta u_{j,N_j,T} \right) dxdt \right| \\
&\lesssim ||P_{N_0} u_0||_{V_\Delta^2 L^2} ||P_{N_1} u_1||_{V_\Delta^2 L^2} \prod_{j=2}^m ||u_j||_{\dot{Y}^{s_c}}
\end{aligned}$$

since $||\mathbf{1}_{[0,T)} u||_{V_\Delta^2 L^2} \lesssim ||u||_{V_\Delta^2 L^2}$ for any $T \in (0, \infty]$. □

Proposition 5.12. *Let $d \geq 1$, $m \geq 3$, $s_c = d/2 - 1/(m-1)$ and $0 < T \leq \infty$. For a dyadic number $N_2 \in 2^{\mathbb{Z}}$, we define the set $S_*(N_2)$ as*

$$S_*(N_2) := \{(N_3, \cdots, N_m) \in (2^{\mathbb{Z}})^{m-2} | N_2 \geq N_3 \geq \cdots \geq N_m\}.$$

If $N_0 \lesssim N_1 \sim N_2$, then we have

$$\begin{aligned} &\left| \sum_{S_*(N_2)} \int_0^T \int_{\mathbb{R}^d} \left(N_0 \prod_{j=0}^m P_{N_j} u_j \right) dxdt \right| \\ &\lesssim \frac{N_0}{N_1} ||P_{N_0} u_0||_{V^2_\Delta L^2} ||P_{N_1} u_1||_{V^2_\Delta L^2} N_2^{s_c} ||P_{N_2} u_2||_{V^2_\Delta L^2} \prod_{j=3}^m ||u_j||_{\dot{Y}^{s_c}}. \end{aligned} \tag{5.15}$$

Proof. We define $u_{j,N_j,T} := \mathbf{1}_{[0,T)} P_{N_j} u_j$ $(j = 1, \cdots, m)$ and put $M := N_1^2/4(m+1)$. We decompose $Id = Q^\Delta_{<M} + Q^\Delta_{\geq M}$. We divide the integrals on the left-hand side of (5.15) into 2^{m+1} piece of the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 \prod_{j=0}^m Q_j^\Delta u_{j,N_j,T} \right) dxdt \tag{5.16}$$

with $Q_j^\Delta \in \{Q^\Delta_{\geq M}, Q^\Delta_{<M}\}$ $(j = 0, \cdots, m)$. By the Plancherel's theorem, we have

$$(5.16) = c \int_{\sum_{j=0}^m \tau_j = 0} \int_{\sum_{j=0}^m \xi_j = 0} N_0 \prod_{j=0}^m \mathcal{F}[Q_j^\Delta u_{j,N_j,T}](\tau_j, \xi_j),$$

where c is a constant. Therefore, Lemma 5.10 implies that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 \prod_{j=0}^m Q^\Delta_{<M} u_{j,N_j,T} \right) dxdt = 0.$$

So, let us now consider the case that $Q_j^\Delta = Q^\Delta_{\geq M}$ for some $0 \leq j \leq m$.

We consider only for the case $Q_0^\Delta = Q^\Delta_{\geq M}$ since the case $Q_1^\Delta = Q^\Delta_{\geq M}$ is similar argument and the cases $Q_k^\Delta = Q^\Delta_{\geq M}$ $(k = 2, \cdots, m)$ are similar to the argument in the proof of Proposition 5.11. By the Hölder's inequality and we have

$$\begin{aligned} &\left| \sum_{S_*(N_2)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 Q^\Delta_{\geq M} u_{0,N_0,T} \prod_{j=1}^m Q_j^\Delta u_{j,N_j,T} \right) dxdt \right| \\ &\lesssim N_0 ||Q^\Delta_{\geq M} u_{0,N_0,T}||_{L_t^2 L_x^{(m-1)d}} ||Q_1^\Delta u_{1,N_1,T}||_{L_t^4 L_x^{2d/(d-1)}} ||Q_2^\Delta u_{2,N_2,T}||_{L_t^4 L_x^{2d/(d-1)}} \\ &\quad \times \prod_{j=3}^m \left\| \sum_{N_j} Q_j^\Delta u_{j,N_j,T} \right\|_{L_t^\infty L_x^{(m-1)d}}. \end{aligned}$$

By the Sobolev embedding $\dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^{(m-1)d}(\mathbb{R}^d)$ and (2.1) with $p=2$, we have

$$\begin{aligned} ||Q^{\Delta}_{\geq M} u_{0,N_0,T}||_{L^2_t L^{(m-1)d}_x} &\lesssim N_0^{s_c} ||Q^{\Delta}_{\geq M} u_{0,N_0,T}||_{L^2_{tx}} \\ &\lesssim N_1^{-1} N_2^{s_c} ||u_{0,N_0,T}||_{V^2_\Delta L^2} \end{aligned}$$

since $M \sim N_1^2$ and $N_0 \lesssim N_2$. While by (5.6), the embedding $V^2_\Delta L^2 \hookrightarrow U^4_\Delta L^2$ and (2.2), we have

$$||Q^{\Delta}_1 u_{1,N_1,T}||_{L^4_t L^{2d/(d-1)}_x} ||Q^{\Delta}_2 u_{2,N_2,T}||_{L^4_t L^{2d/(d-1)}_x} \lesssim ||u_{1,N_1,T}||_{V^2_\Delta L^2} ||u_{2,N_2,T}||_{V^2_\Delta L^2}.$$

Furthermore by the Sobolev embedding $\dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^{(m-1)d}(\mathbb{R}^d)$, L^2 orthogonality, $V^2_\Delta L^2 \hookrightarrow L^\infty(\mathbb{R}; L^2)$ and (2.2), we have

$$\begin{aligned} \left|\left| \sum_{N_j} Q^{\Delta}_j u_{j,N_j,T} \right|\right|_{L^\infty_t L^{(m-1)d}_x} &\lesssim \left(\sum_{N_j} N_j^{2s_c} ||Q^{\Delta}_j u_{j,N_j,T}||^2_{L^\infty_t L^2_x} \right)^{1/2} \\ &\lesssim \left(\sum_{N_j} N_j^{2s_c} ||u_{j,N_j,T}||^2_{V^2_\Delta L^2} \right)^{1/2}. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} &\left| \sum_{S_*(N_2)} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(N_0 Q^{\Delta}_{\geq M} u_{0,N_0,T} \prod_{j=1}^{m} Q^{\Delta}_j u_{j,N_j,T} \right) dx dt \right| \\ &\lesssim \frac{N_0}{N_1} ||P_{N_0} u_0||_{V^2_\Delta L^2} ||P_{N_1} u_1||_{V^2_\Delta L^2} N_2^{s_c} ||P_{N_2} u_2||_{V^2_\Delta L^2} \prod_{j=2}^{m} ||u_j||_{\dot{Y}^{s_c}} \end{aligned}$$

since $||\mathbf{1}_{[0,T)} u||_{V^2_\Delta L^2} \lesssim ||u||_{V^2_\Delta L^2}$ for any $T \in (0, \infty]$. □

5.3 Proof of the well-posedness and the scattering

In this section, we prove Theorem 5.3 and Corollary 5.5. We define the map $\Phi_{T,\varphi}$ as

$$\Phi_{T,\varphi}(u)(t) := e^{it\Delta}\varphi - i I_T(u, \cdots, u)(t),$$

where

$$I_T(u_1, \cdots u_m)(t) := \int_0^t \mathbf{1}_{[0,T)}(t') e^{i(t-t')\Delta} \partial_k \left(\prod_{j=1}^{m} \overline{u_j(t')} \right) dt'.$$

To prove the well-posedness of (5.1), we prove that $\Phi_{T,\varphi}$ is a contraction map on a closed subset of $\dot{Z}^s([0,T])$ or $Z^s([0,T])$. Key estimate is the following:

Proposition 5.13. *We assume $d \geq 1$, $m \geq 3$. Then for $s_c = d/2 - 1/(m-1)$ and any $0 < T \leq \infty$, we have*

$$||I_T(u_1, \cdots u_m)||_{\dot{Z}^{s_c}} \lesssim \prod_{j=1}^{m} ||u_j||_{\dot{Y}^{s_c}}. \tag{5.17}$$

Proof. We show the estimate

$$||I_T(u_1, \cdots u_m)||_{\dot{Z}^s} \lesssim \sum_{k=1}^{m} \left(||u_k||_{\dot{Y}^s} \prod_{\substack{j=1 \\ j \neq k}}^{m} ||u_j||_{\dot{Y}^{s_c}} \right) \tag{5.18}$$

for $s \geq 0$. (5.17) follows from (5.18) with $s = s_c$. We decompose

$$I_T(u_1, \cdots u_m) = \sum_{N_1, \cdots, N_m} I_T(P_{N_1} u_1, \cdots P_{N_m} u_m).$$

By symmetry, it is enough to consider the summation for $N_1 \geq \cdots \geq N_m$. We put

$$S_1 := \{(N_1, \cdots, N_m) \in (2^{\mathbb{Z}})^m | N_1 \gg N_2 \geq \cdots \geq N_m\}$$
$$S_2 := \{(N_1, \cdots, N_m) \in (2^{\mathbb{Z}})^m | N_1 \sim N_2 \geq \cdots \geq N_m\}$$

and

$$J_k := \left|\left| \sum_{S_k} I_T(P_{N_1} u_1, \cdots P_{N_m} u_m) \right|\right|_{\dot{Z}^s} \quad (k = 1, 2).$$

First, we prove the estimate for J_1. By Theorem 2.4 and the Plancherel's theorem, we have

$$\begin{aligned}
J_1 &\leq \left\{ \sum_{N_0} N_0^{2s} \left|\left| e^{-it\Delta} P_{N_0} \sum_{S_1} I_T(P_{N_1} u_1, \cdots P_{N_m} u_m) \right|\right|^2_{U^2(\mathbb{R}; L^2)} \right\}^{1/2} \\
&\lesssim \left\{ \sum_{N_0} N_0^{2s} \sum_{N_1 \sim N_0} \left(\sup_{||u_0||_{V^2_\Delta L^2} = 1} \left| \sum_{S(N_1)} \int_0^T \int_{\mathbb{R}^d} \left(N_0 \prod_{j=0}^{m} P_{N_j} u_j \right) dx dt \right| \right)^2 \right\}^{1/2}.
\end{aligned}$$

Therefore by Proposition 5.11, we have

$$\begin{aligned}
J_1 &\lesssim \left\{ \sum_{N_0} N_0^{2s} \sum_{N_1 \sim N_0} \left(\sup_{||u_0||_{V^2_\Delta L^2} = 1} ||P_{N_0} u_0||_{V^2_\Delta L^2} ||P_{N_1} u_1||_{V^2_\Delta L^2} \prod_{j=2}^{m} ||u_j||_{\dot{Y}^{s_c}} \right)^2 \right\}^{1/2} \\
&\lesssim \left(\sum_{N_1} N_1^{2s} ||P_{N_1} u_1||^2_{V^2_\Delta L^2} \right)^{1/2} \prod_{j=2}^{m} ||u_j||_{\dot{Y}^{s_c}} \\
&= ||u_1||_{\dot{Y}^s} \prod_{j=2}^{m} ||u_j||_{\dot{Y}^{s_c}}.
\end{aligned}$$

Next, we prove the estimate for J_2. By Theorem 2.4 and the Plancherel's theorem, we have

$$\begin{aligned} J_2 &\leq \sum_{N_1}\sum_{N_2\sim N_1}\left(\sum_{N_0} N_0^{2s}\left\|e^{-it\Delta}P_{N_0}\sum_{S_*(N_2)} I_T(P_{N_1}u_1,\cdots P_{N_m}u_m)\right\|_{U^2(\mathbb{R};L^2)}^2\right)^{1/2} \\ &= \sum_{N_1}\sum_{N_2\sim N_1}\left(\sum_{N_0\lesssim N_1} N_0^{2s}\sup_{\|u_0\|_{V^2_\Delta L^2}=1}\left|\sum_{S_*(N_2)}\int_0^T\int_{\mathbb{R}^d}\left(N_0\prod_{j=0}^m P_{N_j}u_j\right)dxdt\right|^2\right)^{1/2}. \end{aligned}$$

Therefore by Proposition 5.12 and Cauchy-Schwartz inequality for dyadic sum, we have

$$\begin{aligned} J_2 &\lesssim \sum_{N_1}\sum_{N_2\sim N_1}\left(\sum_{N_0\lesssim N_1} N_0^{2s}\left(\frac{N_0}{N_1}\|P_{N_1}u_1\|_{V^2_\Delta L^2}N_2^{s_c}\|P_{N_2}u_2\|_{V^2_\Delta L^2}\prod_{j=3}^m\|u_j\|_{\dot{Y}^{s_c}}\right)^2\right)^{1/2} \\ &\lesssim \left(\sum_{N_1} N_1^{2s}\|P_{N_1}u_1\|_{V^2_\Delta L^2}^2\right)^{1/2}\left(\sum_{N_2} N_2^{2s_c}\|P_{N_2}u_2\|_{V^2_\Delta L^2}^2\right)^{1/2}\prod_{j=3}^m\|u_j\|_{\dot{Y}^{s_c}} \\ &= \|u_1\|_{\dot{Y}^s}\prod_{j=2}^m\|u_j\|_{\dot{Y}^{s_c}}. \end{aligned}$$

□

The estimates (5.18) with $s=0$ and with $s=s_c$ imply the following.

Corollary 5.14. *We assume $d\geq 1$, $m\geq 3$. Then for $s\geq s_c$ $(=d/2-1/(m-1))$ and any $0<T\leq\infty$, we have*

$$\|I_T(u_1,\cdots u_m)\|_{Z^s}\lesssim\prod_{j=1}^m\|u_j\|_{Y^s}.$$

Proof of Theorem 5.3. We prove only the homogeneous case. The inhomogeneous case is also proved by the same way. For $r>0$, we define

$$\dot{Z}^s_r(I) := \left\{u\in\dot{Z}^s(I)\,\middle|\,\|u\|_{\dot{Z}^s(I)}\leq 2r\right\} \tag{5.19}$$

which is a closed subset of $\dot{Z}^s(I)$. Let $u_0\in B_r(\dot{H}^{s_c}(\mathbb{R}^d))$ be given. For $u\in\dot{Z}^{s_c}_r([0,\infty))$, we have

$$\|\Phi_{T,u_0}(u)\|_{\dot{Z}^{s_c}([0,\infty))}\leq\|u_0\|_{\dot{H}^{s_c}}+C\|u\|^m_{\dot{Z}^{s_c}([0,\infty))}\leq r(1+2^mCr^{m-1})$$

and

$$
\begin{aligned}
||\Phi_{T,u_0}(u) - \Phi_{T,u_0}(v)||_{\dot{Z}^{s_c}([0,\infty))} &\leq C(||u||_{\dot{Z}^{s_c}([0,\infty))} + ||v||_{\dot{Z}^{s_c}([0,\infty))})^{m-1}||u-v||_{\dot{Z}^{s_c}([0,\infty))} \\
&\leq 4^{m-1}Cr^{m-1}||u-v||_{\dot{Z}^{s_c}([0,\infty))}
\end{aligned}
$$

by Proposition 5.13 and

$$
||e^{it\Delta}\varphi||_{\dot{Z}^{s_c}([0,\infty))} \leq ||\mathbf{1}_{[0,\infty)}e^{it\Delta}\varphi||_{\dot{Z}^{s_c}_{\sigma}} \leq ||\varphi||_{\dot{H}^{s_c}},
$$

where C is an implicit constant in (5.17). Therefore if we choose r satisfying

$$
r < (4^{m-1}C)^{-1/(m-1)},
$$

then Φ_{T,u_0} is a contraction map on $\dot{Z}^{s_c}_r([0,\infty))$. This implies the existence of the solution of (5.1) and the uniqueness in the ball $\dot{Z}^{s_c}_r([0,\infty))$. The Lipschitz continuously of the flow map is also proved by similar argument. □

Proof of Corollary 5.5. We prove only the homogeneous case. The inhomogeneous case is also proved by the same way. By Proposition 5.13, the global solution $u \in \dot{Z}^{s_c}([0,\infty))$ of (5.1) which was constructed in Theorem 5.3 satisfies

$$
N^{s_c}e^{-it\Delta}P_N I_\infty(u,\cdots,u) \in V^2(\mathbb{R};L^2)
$$

for each $N \in 2^{\mathbb{Z}}$. This implies that

$$
u_+ := \lim_{t\to\infty}(u_0 - e^{-it\Delta}I_\infty(u,\cdots,u)(t))
$$

exists in $\dot{H}^{s_c}(\mathbb{R}^d)$ by Proposition 2.3. (iv). Then we obtain

$$
u - e^{it\Delta}u_+ \to 0
$$

in $\dot{H}^{s_c}(\mathbb{R}^d)$ as $t \to \infty$. □

Chapter 6

Higher order KdV type equations

6.1 Review for results

We consider the Cauchy problem of the periodic high order KdV type equations;

$$\begin{cases} \partial_t u + (-1)^{k+1}\partial_x^{2k+1}u + \dfrac{1}{2}\partial_x(u^2) = 0, & (t,x) \in (0,\infty)\times\mathbb{T}, \\ u(0,x) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{6.1}$$

where $k \in \mathbb{N}$ and the unknown function u is real valued. The aim of this paper is to prove the local well-posedness (LWP for short) of (6.1) with low regularity initial data.

When $k = 1$, the equation (6.1) is called "KdV equation". We first introduce some known results for the KdV equation. In [5], Bourgain introduced a new method called "Fourier restriction norm method" and proved that the KdV equation is LWP in $L^2(\mathbb{T})$. In [47], Kenig, Ponce and Vega refined a bilinear estimate used in the Fourier restriction norm method, and proved that the KdV equation is LWP in $H^s(\mathbb{T})$ for $s \geq -1/2$. In [18], by using the local well-posedness result and the almost conservation low, Colliander, Keel, Staffilani, Takaoka and Tao obtained that the KdV equation is the globally well-posed in $H^s(\mathbb{T})$ for $s \geq -1/2$. Their method is called "I-method". On the other hand, In [10], Christ, Colliander and Tao proved that the KdV equation is ill-posed. More precisely, the data-to-solution map is not uniformly continuous on $H^s(\mathbb{T})$ for $-2 < s < -1/2$. LWP of the non-periodic KdV equation also was studied by many people before Bourgain's work ([4], [20], [41], [43], [44]) and after Bourgain's work ([18], [24], [45], [47], [49], [50], [58], [69]).

Next, we introduce some known results for the fifth order KdV type equations

$$\partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \partial_x(u^2) = 0 \tag{6.2}$$

and

$$\partial_t u - \partial_x^5 u - 30u^2\partial_x u + 20\partial_x u \partial_x^2 u + 10u\partial_x^3 u = 0. \tag{6.3}$$

Especially, (6.2) is called "Kawahara equation". LWP of these equations are studied for non-periodic case. For the known results of the non-periodic Kawahara equation (6.2), see [6], [19], [38], [70] and the non-periodic fifth order KdV equation (6.3), see [39], [53], [63].

We return to introduce the known result for (6.1) for general $k \in \mathbb{N}$. Recently in [22], Gorsky and Himonas have proved that (6.1) is LWP in $H^s(\mathbb{T})$ for $s \geq -1/2$ by an argument similar to [47]. The main result in the present paper is an extension of the result by Gorsky and Himonas.

Finally, we introduce the high order nonlinear dispersive equations related to (6.1). In [46], Kenig, Ponce and Vega studied the high order nonlinear dispersive equations

$$\partial_t u + \partial_x^{2k+1} u + P(u, \partial_x u, \cdots, \partial_x^{2k} u) = 0, \tag{6.4}$$

where P is a polynomial without constant and linear terms. They proved that (6.4) is LWP in $L^2(|x|^m dx) \cap H^s(\mathbb{R})$, where $s > 0$ and $m \in \mathbb{Z}^+$ are sufficiently large. In [62], Pilod proved that (6.4) with

$$P(u, \partial_x u, \cdots, \partial_x^{2k} u) = \sum_{0 \leq k_1 + k_2 \leq 2k} a_{k_1,k_2} \partial_x^{k_1} u \partial_x^{k_2} u$$

is LWP in $H^s(\mathbb{R}) \cap H^{s-2k}(x^2 dx)$ for $s \in \mathbb{N}$ and $s > 2k + 1/4$. He also proved some ill-posed results for (6.4).

Without loss of generality, we can assume that $\widehat{u}_0(0) = 0$ by the following transform:

$$u \mapsto u - \frac{1}{2\pi}\int_{\mathbb{T}} u_0(x)dx.$$

Therefore, we have only to prove LWP of (6.1) in $\widetilde{H}^s(\mathbb{T})$, where

$$\widetilde{H}^s(\mathbb{T}) := \{f \in H^s(\mathbb{T}) |\ \widehat{f}(0) = 0\},$$

and $\widehat{u}$ is the Fourier transform of u with respect to x. We note that $\widetilde{H}^s(\mathbb{T})$ is a Banach space with respect to the norm

$$||f||_{\widetilde{H}^s(\mathbb{T})} := |||\xi|^s \widehat{f}||_{l^2_\xi}.$$

Next, we define the Bourgain spaces $Z^s(\lambda)$ and $Z^s_{[0,T]}(\lambda)$, where λ is the scale parameter. We will use the scaling property (6.22) in the proof of Theorem 6.2.

Definition 6.1. *Let $\lambda \geq 1$, $\mathbb{T}_\lambda := \mathbb{R}/(2\pi\lambda)\mathbb{Z}$ and X_λ be the space of all $F : \mathbb{R} \times \mathbb{T}_\lambda \to \mathbb{R}$ such that $F(\cdot, x) \in \mathcal{S}(\mathbb{R})$ for all $x \in \mathbb{T}_\lambda$, the map $x \mapsto F(\cdot, x)$ is C^∞ and $\widetilde{F}(\tau, 0) = 0$ for all $\tau \in \mathbb{R}$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space and $\widetilde{F}$ is the Fourier transform of F with respect to x and t.*
(i) *For $s \in \mathbb{R}$, we define the function space $Z^s(\lambda)$ as the completion of X_λ with respect to the norm*

$$||u||_{Z^s(\lambda)} := ||u||_{X^s(\lambda)} + ||u||_{Y^s(\lambda)},$$

where

$$||u||_{X^s(\lambda)} := ||\langle \tau - \xi^{2k+1}\rangle^{1/2}|\xi|^s\widetilde{u}||_{l^2_\xi(\lambda)L^2_\tau}, \ ||u||_{Y^s(\lambda)} := |||\xi|^s\widetilde{u}||_{l^2_\xi(\lambda)L^1_\tau}.$$

(ii) *We define the function space $Z^s_{[0,T]}(\lambda)$ and the norm $||\cdot||_{Z^s_{[0,T]}(\lambda)}$ as*

$$Z^s_{[0,T]}(\lambda) := \{u|_{[0,T]} \ |u \in Z^s(\lambda)\},$$

$$||u||_{Z^s_{[0,T]}(\lambda)} := \inf\{||v||_{Z^s(\lambda)} \ |v \in Z^s(\lambda), \ v(t) = u(t) \text{ on } [0, T]\}.$$

We omit to write down "(λ)" when $\lambda = 1$.

The main result in the present paper is the following theorem:

Theorem 6.2. *Let $k \in \mathbb{N}$. If $s \geq -k/2$ then (6.1) is LWP in $H^s(\mathbb{T})$. More particularly, for all $r \geq 1$ and $u_0 \in B_r(\widetilde{H}^s(\mathbb{T}))$, there exist $T = T(r) > 0$ and a solution $u \in Z^s_{[0,T]} \cap C([0, T]; \widetilde{H}^s(\mathbb{T}))$ of (6.1). Such solution u is unique in a closed subset of $Z^s_{[0,T]}$. Moreover, the map $u_0 \mapsto u$ from $B_r(\widetilde{H}^s(\mathbb{T}))$ into $Z^s_{[0,T]}$ is Lipschitz continuous.*

Remark 6.3. *After our work, Kato ([40]) extended the result in Theorem 6.2 for $k = 2$ to LWP in $H^s(\mathbb{T})$ for $s \geq -3/2$ and GWP in $H^s(\mathbb{T})$ for $s \geq -1$.*

Bilinear estimate plays an important role to prove LWP of (6.1). Gorsky and Himonas derived the following bilinear estimate for $s \geq -1/2$;

$$||\mathcal{F}^{-1}[\langle \tau - \xi^{2k+1}\rangle^{-1}\widetilde{\partial_x(uv)}]||_{Z^s} \leq C||u||_{Z^s}||v||_{Z^s}. \tag{6.5}$$

But as mentioned in [22], the estimate (6.5) with $s < -1/2$ has been open problem. We extend (6.5) to prove Theorem 6.2 and obtain the following bilinear estimate;

Theorem 6.4. *Let $k \in \mathbb{N}$ and $\lambda \geq 1$. For $s \geq -k/2$, there exist a positive constant C_0 and ϵ satisfying $0 < \epsilon < 2k + s - 1/2$ such that the following bilinear estimate holds;*

$$||\mathcal{F}^{-1}[\langle \tau - \xi^{2k+1}\rangle^{-1}\widetilde{\partial_x(uv)}]||_{Z^s(\lambda)} \leq C_0\lambda^\epsilon||u||_{Z^s(\lambda)}||v||_{Z^s(\lambda)}, \tag{6.6}$$

where C_0 does not depend on λ.

On the other hand, we also obtain negative result for $s < -k/2$.

Theorem 6.5. *Let $k \in \mathbb{N}$. For any $s < -k/2$, the bilinear* (6.6) *with $\lambda = 1$ fails.*

Remark 6.6. *By* Theorem 6.4 *and* 6.5*, $s = -k/2$ is optimal regularity for the bilinear estimate* (6.6)*. But this does not imply ill-posedness of* (6.1) *for $s < -k/2$.*

The bilinear estimate (6.12) below with $\lambda = 1$ can be written

$$\begin{aligned} &\| |\xi|^{1-k/2} \langle \tau - \xi^{2k+1} \rangle^{-1/2} \widetilde{u} * \widetilde{v} \|_{l^2_\xi L^2_\tau} \\ &\lesssim \| |\xi|^{-k/2} \langle \tau - \xi^{2k+1} \rangle^{1/2} \widetilde{u} \|_{l^2_\xi L^2_\tau} \| |\xi|^{-k/2} \langle \tau - \xi^{2k+1} \rangle^{1/2} \widetilde{v} \|_{l^2_\xi L^2_\tau}, \end{aligned}$$

where

$$\widetilde{u} * \widetilde{v}(\tau, \xi) = \frac{1}{2\pi} \sum_{\xi = \xi_1 + \xi_2} \int_{\tau = \tau_1 + \tau_2} \widetilde{u}(\tau_1, \xi_1) \widetilde{v}(\tau_2, \xi_2) d\tau_1.$$

We note that the most difficult region to prove this estimate is $|\xi_1| \sim |\xi_2| \gg |\xi|$. Gorsky and Himonas used the estimate

$$|\xi^{2k+1} - \xi_1^{2k+1} - \xi_2^{2k+1}| \gtrsim |\xi \xi_1 \xi_2| \cdot |\xi|^{2k-2} \tag{6.7}$$

to prove (6.5). On the other hand, we use the refined estimate

$$|\xi^{2k+1} - \xi_1^{2k+1} - \xi_2^{2k+1}| \sim |\xi \xi_1 \xi_2| \max\{|\xi|, |\xi_1|, |\xi_2|\}^{2k-2},$$

which is better estimate than (6.7) in the region $|\xi_1| \sim |\xi_2| \gg |\xi|$. Because of such reason, we could improve the bilinear estimate.

The rest of this chapter is planned as follows. In Section 2, we will prepare to prove the bilinear estimate. In Section 3, we will prove the bilinear estimate and give a counterexample. In Section 4, we will prove the well-posedness (Theorem 6.2).

6.2 Preliminary

In this section, we prepare to prove the bilinear estimate.

Lemma 6.7. *Let $k \in \mathbb{N}$. If $p, q, r \in \mathbb{R}$ satisfy $p+q+r = 0$ and $p^{2k+1}+q^{2k+1}+r^{2k+1} = 0$ then at least one of p, q and r is equal to* 0.

Proof. We can assume $|p| \geq |q| \geq |r|$ and $p \geq 0$, $q \leq 0$, $r \leq 0$ without loss of generality. Since

$$p^{2k+1} = -(q^{2k+1} + r^{2k+1}) = -(q+r) \sum_{j=0}^{2k} q^{2k-j} (-r)^j = p \sum_{j=0}^{2k} (-1)^j (-q)^{2k-j} (-r)^j$$

and

$$p^{2k+1} = p(-q-r)^{2k} = p\sum_{j=0}^{2k}\binom{2k}{j}(-q)^{2k-j}(-r)^j,$$

we have

$$pqr\sum_{j=1}^{2k-1}\left\{\binom{2k}{j} - (-1)^j\right\}(-q)^{2k-1-j}(-r)^{j-1} = 0. \tag{6.8}$$

While if $pqr \neq 0$, then we have

$$pqr\sum_{j=1}^{2k-1}\left\{\binom{2k}{j} - (-1)^j\right\}(-q)^{2k-1-j}(-r)^{j-1} > 0$$

since $p \geq 0$, $-q \geq 0$, $-r \geq 0$ and

$$\binom{2k}{j} - (-1)^j > 0$$

for $1 \leq j \leq 2k-1$. This contradicts the equation (6.8). Therefore we obtain $pqr = 0$. □

Nobu Kishimoto pointed out Lemma 6.8 and a proof of it to the author. The proof of Lemma 6.8 below is simpler than his.

Lemma 6.8. *If $p, q, r \in \mathbb{R}\backslash\{0\}$ satisfy $p+q+r=0$ then*

$$|p^{2k+1} + q^{2k+1} + r^{2k+1}| \sim |pqr| \max\{|p|, |q|, |r|\}^{2k-2}.$$

Proof. We can assume $|p| \geq |q| \geq |r|$ without loss of generality. By an elementary calculation, we have

$$\begin{aligned} p^{2k+1} + q^{2k+1} + r^{2k+1} &= p^{2k+1} + q^{2k+1} - (p+q)^{2k+1} \\ &= -\sum_{j=1}^{2k}\binom{2k+1}{j}p^{2k+1-j}q^j \\ &= -pq(p+q)\sum_{j=1}^{2k-1}\left(\sum_{l=1}^{j}(-1)^{j+l}\binom{2k+1}{l}\right)p^{2k-1-j}q^{j-1} \\ &= pqr \cdot p^{2k-2}Q(\beta), \end{aligned}$$

where

$$Q(\beta) = \sum_{j=1}^{2k-1}\alpha_j\beta^{j-1}, \quad \alpha_j := \sum_{l=1}^{j}(-1)^{j+l}\binom{2k+1}{l}, \quad \beta := \frac{q}{p}.$$

We note that $-1 < \beta \leq -1/2$ and $Q(-1) = 2k+1$. Furthermore, we have

$$|Q(\beta)| = \frac{|p^{2k+1} + q^{2k+1} + r^{2k+1}|}{|pqr| \cdot |p|^{2k-2}} \neq 0$$

for $-1 < \beta \leq -1/2$ from Lemma 6.7. This implies

$$\inf_{-1<\beta\leq -1/2} |Q(\beta)| > 0.$$

Therefore, we obtain

$$|p^{2k+1} + q^{2k+1} + r^{2k+1}| = |pqr| \cdot |p|^{2k-2} |Q(\beta)| \sim |pqr| \max\{|p|, |q|, |r|\}^{2k-2}. \qquad \square$$

Lemma 6.9. *If $l, m, n \in \mathbb{Z}^*_\lambda$ satisfy $l + m + n = 0$, then we have*

$$|l^{2k+1} + m^{2k+1} + n^{2k+1}| \gtrsim \frac{1}{\lambda} \max\{|l|, |m|, |n|\}^{2k}. \tag{6.9}$$

Proof. We can assume $|l| \geq |m| \geq |n|$ without loss of generality. Since

$$|l| \leq |m| + |n| \leq 2|m|$$

and

$$|n| \geq \frac{1}{\lambda},$$

we obtain

$$|lmn| \geq \frac{1}{2\lambda} |l|^2 \sim \frac{1}{\lambda} \max\{|l|, |m|, |n|\}^2.$$

Therefore, we have (6.9) by Lemma 6.8. $\square$

Lemma 6.10. *Let $a \in \mathbb{R}$. For any δ and δ' satisfying $0 \leq \delta' < \delta < 1$, we have*

$$\int_{\mathbb{R}} \frac{1}{\langle \theta \rangle \langle \theta + a \rangle^{1-\delta'}} d\theta \lesssim \frac{1}{\langle a \rangle^{\delta - \delta'}},$$

where the implicit constant depends only on δ and δ'.

For the proof of Lemma 6.10, see Lemma 4.2 in [21].

Lemma 6.11. *Let $\lambda \geq 1$. For any δ satisfying $1/m < \delta$, we have*

$$\int_{\mathbb{Z}^*_\lambda} \frac{1}{\langle \lambda^{-1} P(\xi) \rangle^{\delta}} d\xi \lesssim \lambda^{\delta(m+1)-1},$$

where $P(x)$ is a polynomial of the form

$$P(x) := \sum_{j=0}^{m} c_j x^j \tag{6.10}$$

with $c_0, \cdots, c_m \in \mathbb{R}$, c_m does not depend on λ and the implicit constant depends only on c_m.

Proof. Let $\gamma_1, \cdots, \gamma_m \in \mathbb{C}$ be the roots of the equation

$$P(x) = 0.$$

Then we have

$$P(\xi) = c_m \prod_{j=1}^{m} (\xi - \gamma_j).$$

We put

$$C_j := \{\xi \in \mathbb{Z}^*_\lambda \mid |\xi - \gamma_j| \le 1\}, \ C := \bigcup_{j=1}^{m} C_j, \ D := \mathbb{Z}^*_\lambda \backslash C.$$

Since $\# C_j \lesssim \lambda$, we have

$$\int_C \frac{1}{\langle \lambda^{-1} P(\xi) \rangle^\delta} d\xi \le \int_C 1 d\xi = \frac{1}{2\pi\lambda} \# C \le \frac{1}{2\pi\lambda} \sum_{j=1}^{m} \# C_j \lesssim 1.$$

On the other hand, since

$$1 \le |\xi - \gamma_j|$$

for any $\xi \in D$ and $1 \le j \le m$, we have

$$\prod_{j=1}^{m} \langle \xi - \gamma_j \rangle \sim \prod_{j=1}^{m} |\xi - \gamma_j| \le \lambda \left\langle \lambda^{-1} \prod_{j=1}^{m} (\xi - \gamma_j) \right\rangle.$$

Therefore, we have

$$\int_D \frac{1}{\langle \lambda^{-1} P(\xi) \rangle^\delta} d\xi \lesssim \lambda^\delta \int_D \frac{1}{\prod_{j=1}^{m} \langle \xi - \gamma_j \rangle^\delta} d\xi \le \lambda^\delta \prod_{j=1}^{m} \left(\int_D \frac{1}{\langle \xi - \gamma_j \rangle^{\delta m}} d\xi \right)^{1/m}$$

by Holder's inequality. Since

$$\int_D \frac{1}{\langle \xi - \gamma_j \rangle^{\delta m}} dn \le \frac{1}{2\pi\lambda} \sum_{\xi \in \mathbb{Z}^*} \frac{\lambda^{\delta m}}{\langle \xi - \lambda\gamma_j \rangle^{\delta m}} \lesssim \lambda^{\delta m - 1}$$

by $1/m < \delta$, we obtain

$$\int_D \frac{1}{\langle \lambda^{-1} P(\xi) \rangle^\delta} d\xi \lesssim \lambda^{\delta(m+1)-1}. \qquad \square$$

6.3 Proof of the bilinear estimate

In this section, we give the proofs of Theorem 6.4 and Theorem 6.5.

Proof of Theorem 6.4.

For $s > -k/2$, we note that

$$\begin{aligned}|\xi^s(\widetilde{u} * \widetilde{v})| &\lesssim |\xi|^{-k/2}\left(|(\xi^{s+k/2}\widetilde{u}) * \widetilde{v}| + |\widetilde{u} * (\xi^{s+k/2}\widetilde{v})|\right) \\ &\lesssim \lambda^{s+k/2}|\xi|^{-k/2}|(\xi^{s+k/2}\widetilde{u}) * (\xi^{s+k/2}\widetilde{v})|\end{aligned}$$

by the triangle inequality and $\lambda^{-1} \leq |\xi|$ for all $\xi \in \mathbb{Z}_\lambda^*$. Thus, we only need to prove the bilinear estimate (6.6) for the case $s = -k/2$.

We put

$$\widetilde{f}(\tau, \xi) := \langle \tau - \xi^{2k+1} \rangle^{1/2} |\xi|^{-k/2} \widetilde{u}(\tau, \xi), \ \widetilde{g}(\tau, \xi) := \langle \tau - \xi^{2k+1} \rangle^{1/2} |\xi|^{-k/2} \widetilde{v}(\tau, \xi),$$

then we have

$$||u||_{X^{-k/2}(\lambda)} = ||\widetilde{f}||_{l_\xi^2(\lambda) L_\tau^2}, \ ||v||_{X^{-k/2}(\lambda)} = ||\widetilde{g}||_{l_\xi^2(\lambda) L_\tau^2}.$$

Furthermore, we put

$$\xi_2 := \xi - \xi_1, \ \tau_2 := \tau - \tau_1, \ \sigma := \tau - \xi^{2k+1}, \ \sigma_1 := \tau_1 - \xi_1^{2k+1}, \ \sigma_2 := \tau_2 - \xi_2^{2k+1},$$

and divide the set $(\mathbb{R} \times \mathbb{Z}_\lambda^*)^2$ into

$$\begin{aligned}A_0 &:= \{(\tau, \xi, \tau_1, \xi_1) \in (\mathbb{R} \times \mathbb{Z}_\lambda^*)^2 | M = |\sigma|, \ \xi \neq \xi_1\}, \\ A_1 &:= \{(\tau, \xi, \tau_1, \xi_1) \in (\mathbb{R} \times \mathbb{Z}_\lambda^*)^2 | M = |\sigma_1|, \ \xi \neq \xi_1\}, \\ A_2 &:= \{(\tau, \xi, \tau_1, \xi_1) \in (\mathbb{R} \times \mathbb{Z}_\lambda^*)^2 | M = |\sigma_2|, \ \xi \neq \xi_1\},\end{aligned}$$

where $M = M(\tau, \xi, \tau_1, \xi_1) := \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$. Since

$$M \gtrsim |\sigma - \sigma_1 - \sigma_2| = |(-\xi)^{2k+1} + \xi_1^{2k+1} + \xi_2^{2k+1}|,$$

we have

$$|\xi\xi_1\xi_2| \lesssim M$$

when $k = 1$ by Lemma 6.8 and

$$|\xi|^{1-k/2} \lesssim \lambda^{k/2-1}, \ |\xi_1|^{k/2} \lesssim (\lambda M)^{1/4}, \ |\xi_2|^{k/2} \lesssim (\lambda M)^{1/4}$$

when $k \geq 2$ by $\lambda^{-1} \leq |\xi|$ for all $\xi \in \mathbb{Z}_\lambda^*$ and Lemma 6.9. Thus we have

$$|\xi|^{1-k/2} |\xi_1|^{k/2} |\xi_2|^{k/2} \lesssim \lambda^{(k-1)/2} M^{1/2} \tag{6.11}$$

for any $k \in \mathbb{N}$. We define χ_Ω as the characteristic function of a set Ω.

Step 1. (Estimate for the norm $||\cdot||_{X^{-k/2}(\lambda)}$)
We prove the estimate

$$||\mathcal{F}^{-1}[\langle\tau-\xi^{2k+1}\rangle^{-1}\widetilde{\partial_x(uv)}]||_{X^{-k/2}(\lambda)} \lesssim \lambda^{\epsilon}||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)}. \tag{6.12}$$

for some ϵ satisfying $0<\epsilon<(3k-1)/2$. From (6.11), we have

$$\begin{aligned}
&||\mathcal{F}^{-1}[\langle\tau-\xi^{2k+1}\rangle^{-1}\widetilde{\partial_x(uv)}]||_{X^{-k/2}(\lambda)} \\
&= \left|\left|\int_{\mathbb{Z}_\lambda^*\backslash\{\xi\}}\int_{\mathbb{R}}\frac{|\xi|^{1-k/2}|\xi_1|^{k/2}|\xi_2|^{k/2}}{\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\widetilde{f}(\tau_1,\xi_1)\widetilde{g}(\tau_2,\xi_2)d\tau_1 d\xi_1\right|\right|_{l_\xi^2(\lambda)L_\tau^2} \\
&\lesssim \sum_{j=0}^{2}||J_j||_{l_\xi^2(\lambda)L_\tau^2},
\end{aligned}$$

where

$$J_j=\int_{\mathbb{Z}_\lambda^*}\int_{\mathbb{R}}\frac{\lambda^{(k-1)/2}M^{1/2}\chi_{A_j}}{\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\widetilde{f}(\tau_1,\xi_1)\widetilde{g}(\tau_2,\xi_2)d\tau_1 d\xi_1.$$

By symmetry, we only need to consider the estimate for J_0 and J_1.

Estimate for J_0

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
||J_0||_{l_\xi^2(\lambda)L_\tau^2} &\le \sup_{\xi,\sigma}(K_0)^{1/2}||\widetilde{f}||_{l_\xi^2(\lambda)L_\tau^2}||\widetilde{g}||_{l_\xi^2(\lambda)L_\tau^2} \\
&= \sup_{\xi,\sigma}(K_0)^{1/2}||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)},
\end{aligned} \tag{6.13}$$

where

$$K_0=\int_{\mathbb{Z}_\lambda^*}\int_{\mathbb{R}}\frac{\lambda^{k-1}M\chi_{A_0}}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle}d\sigma_1 d\xi_1 \sim \int_{\mathbb{Z}_\lambda^*}\int_{\mathbb{R}}\frac{\lambda^{k-1}\chi_{A_0}}{\langle\sigma_1\rangle\langle\sigma_2\rangle}d\sigma_1 d\xi_1.$$

Let ξ and σ be fixed. We define a_1 and $P_1(\xi_1)$ as

$$a_1:=\sigma_1+\sigma_2=\sigma+\xi^{2k+1}-\xi_1^{2k+1}-\xi_2^{2k+1}=:\xi P_1(\xi_1).$$

Since $P_1(x)$ is of the form of (6.10) with $m=2k$, from Lemma 6.10, Lemma 6.11 and $|\xi|\ge\lambda^{-1}$, we have

$$\int_{\mathbb{Z}_\lambda^*}\int_{\mathbb{R}}\frac{1}{\langle\sigma_1\rangle\langle\sigma_2\rangle}d\sigma_1 d\xi_1 \lesssim \int_{\mathbb{Z}_\lambda^*}\frac{1}{\langle a_1\rangle^{\delta}}d\xi_1 \le \int_{\mathbb{Z}_\lambda^*}\frac{1}{\langle\lambda^{-1}P_1(\xi_1)\rangle^{\delta}}d\xi_1 \lesssim \lambda^{\delta(2k+1)-1}$$

for any δ satisfying $1/2k<\delta<1$. This implies

$$K_0\lesssim\lambda^{\delta(2k+1)+k-2}.$$

Therefore, we obtain

$$||J_0||_{l^2_\xi(\lambda)L^2_\tau} \lesssim \lambda^{\delta(k+1/2)+k/2-1}||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)} \tag{6.14}$$

by (6.13).

<u>Estimate for J_1</u>
By using the duality and the Cauchy-Schwartz inequality twice, we have

$$\begin{aligned}||J_1||_{l^2_\xi(\lambda)L^2_\tau} &\lesssim \sup_{\xi_1,\sigma_1}(K_1)^{1/2}||\widetilde{f}||_{l^2_{\xi_1}(\lambda)L^2_{\tau_1}}||\widetilde{g}||_{l^2_{\xi_1}(\lambda)L^2_{\tau_1}} \\ &= \sup_{\xi_1,\sigma_1}(K_1)^{1/2}||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)},\end{aligned} \tag{6.15}$$

where

$$K_1 = \int_{\mathbb{Z}^*_\lambda}\int_{\mathbb{R}} \frac{\lambda^{k-1}M\chi_{A_1}}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle}d\sigma d\xi \sim \int_{\mathbb{Z}^*_\lambda}\int_{\mathbb{R}} \frac{\lambda^{k-1}\chi_{A_1}}{\langle\sigma\rangle\langle\sigma_2\rangle}d\sigma d\xi.$$

Let ξ_1 and σ_1 be fixed. We define a and $P(\xi)$ as

$$a := \sigma - \sigma_2 = \sigma_1 - \xi^{2k+1} + \xi_1^{2k+1} + \xi_2^{2k+1} =: \xi_1 P(\xi).$$

Since $P(x)$ is of the form of (6.10) with $m = 2k$, from Lemma 6.10, Lemma 6.11 and $|\xi_1| \geq \lambda^{-1}$, we have

$$\int_{\mathbb{Z}^*_\lambda}\int_{\mathbb{R}} \frac{1}{\langle\sigma\rangle\langle\sigma_2\rangle}d\sigma d\xi \lesssim \int_{\mathbb{Z}^*_\lambda} \frac{1}{\langle a\rangle^\delta}d\xi \leq \int_{\mathbb{Z}^*_\lambda} \frac{1}{\langle\lambda^{-1}P(\xi)\rangle^\delta}d\xi \lesssim \lambda^{\delta(2k+1)-1}$$

for any δ satisfying $1/2k < \delta < 1$. This implies

$$K_1 \lesssim \lambda^{\delta(2k+1)+k-2}.$$

Therefore, we obtain

$$||J_1||_{l^2_\xi(\lambda)L^2_\tau} \lesssim \lambda^{\delta(k+1/2)+k/2-1}||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)} \tag{6.16}$$

by (6.15).

Putting $\epsilon := \delta\,(k+1/2) + k/2 - 1$, we obtain the estimate (6.12) by (6.14) and (6.16), where ϵ satisfies $0 < \epsilon < (3k-1)/2$ since $1/2k < \delta < 1$.

Step 2. (Estimate for the norm $||\cdot||_{Y^{-k/2}}$)
We prove the estimate

$$||\mathcal{F}^{-1}[\langle\tau - \xi^{2k+1}\rangle^{-1}\widetilde{\partial_x(uv)}]||_{Y^{-k/2}(\lambda)} \lesssim \lambda^\epsilon||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)} \tag{6.17}$$

for some ϵ satisfying $0 < \epsilon < (3k-1)/2$. From (6.11), we have

$$\begin{aligned}
&||\mathcal{F}^{-1}[\langle \tau - \xi^{2k+1}\rangle^{-1}\widetilde{\partial_x(uv)}]||_{Y^{-k/2}(\lambda)} \\
&= \left|\left| \int_{\mathbb{Z}_\lambda^* \setminus \{\xi\}} \int_{\mathbb{R}} \frac{|\xi|^{1-k/2}|\xi_1|^{k/2}|\xi_2|^{k/2}}{\langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \widetilde{f}(\tau_1,\xi_1)\widetilde{g}(\tau_2,\xi_2)d\tau_1 d\xi_1 \right|\right|_{l_\xi^2(\lambda)L_\tau^1} \\
&\lesssim \sum_{j=0}^{2} ||I_j||_{l_\xi^2(\lambda)},
\end{aligned}$$

where

$$I_j = \int_{\mathbb{R}}\int_{\mathbb{Z}_\lambda^*}\int_{\mathbb{R}} \frac{\lambda^{(k-1)/2}M^{1/2}\chi_{A_j}}{\langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}|\widetilde{f}(\tau_1,\xi_1)||\widetilde{g}(\tau_2,\xi_2)|d\tau_1 d\xi_1 d\tau.$$

By symmetry, we only need to consider the estimate for I_0 and I_1.

Estimate for I_0

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
||I_0||_{l_\xi^2(\lambda)} &\le \sup_\xi (L_0)^{1/2}||\widetilde{f}||_{l_\xi^2(\lambda)L_\tau^2}||\widetilde{g}||_{l_\xi^2(\lambda)L_\tau^2} \\
&= \sup_\xi (L_0)^{1/2}||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)},
\end{aligned} \tag{6.18}$$

where

$$L_0 = \int_{\mathbb{R}}\int_{\mathbb{Z}_\lambda^*}\int_{\mathbb{R}} \frac{\lambda^{k-1}M\chi_{A_0}}{\langle\sigma\rangle^2\langle\sigma_1\rangle\langle\sigma_2\rangle}d\sigma_1 d\xi_1 d\sigma \sim \int_{\mathbb{R}}\int_{\mathbb{Z}_\lambda^*}\int_{\mathbb{R}} \frac{\lambda^{k-1}\chi_{A_0}}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle}d\sigma_1 d\xi_1 d\sigma.$$

Let ξ be fixed and we define a_1 and $P_1(\xi_1)$ as

$$a_1 := -\xi^{2k+1} + \xi_1^{2k+1} + \xi_2^{2k+1} =: \xi P_1(\xi_1).$$

Since $\sigma_2 = \sigma - (\sigma_1 + a_1)$ and $P_1(x)$ is of the form of (6.10) with $m = 2k$, from Lemma 6.10, Lemma 6.11 and $|\xi| \ge \lambda^{-1}$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_\lambda^*}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{\langle\sigma\rangle\langle\sigma_2\rangle}d\sigma\right)\frac{1}{\langle\sigma_1\rangle}d\sigma_1 d\xi_1 \lesssim \int_{\mathbb{Z}_\lambda^*}\int_{\mathbb{R}} \frac{1}{\langle\sigma_1 + a_1\rangle^{1-\delta'}\langle\sigma_1\rangle}d\sigma_1 d\xi_1 \\
&\lesssim \int_{\mathbb{Z}_\lambda^*} \frac{1}{\langle a_1\rangle^{\delta-\delta'}}d\xi_1 \le \int_{\mathbb{Z}_\lambda^*} \frac{1}{\langle\lambda^{-1}P_1(\xi_1)\rangle^{\delta-\delta'}}d\xi_1 \lesssim \lambda^{(\delta-\delta')(2k+1)-1}
\end{aligned}$$

for any δ and δ' satisfying $0 < \delta' < \delta < 1$ and $1/2k < \delta - \delta'$. This implies

$$L_0 \lesssim \lambda^{(\delta-\delta')(2k+1)+k-2}.$$

Therefore, we obtain

$$||I_0||_{l_\xi^2(\lambda)} \lesssim \lambda^{(\delta-\delta')(k+1/2)+k/2-1}||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)} \tag{6.19}$$

by (6.18).

Estimate for I_1

Let $\gamma > 0$. By the Cauchy-Schwartz inequality, we have

$$||I_1||_{l^2_\xi(\lambda)} \lesssim \left|\left|\int_{\mathbb{Z}^*_\lambda}\int_{\mathbb{R}} \frac{\lambda^{(k-1)/2}M^{1/2}\chi_{A_1}}{\langle\sigma\rangle^{1/2-\gamma}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}|\widetilde{f}(\tau_1,\xi_1)||\widetilde{g}(\tau_2,\xi_2)|d\tau_1 d\xi_1\right|\right|_{l^2_\xi(\lambda)L^2_\tau}.$$

By using the duality and the Cauchy-Schwartz inequality twice, we have

$$\begin{aligned} ||I_1||_{l^2_\xi(\lambda)} &\lesssim \sup_{\xi_1,\sigma_1}(L_1)^{1/2}||\widetilde{f}||_{l^2_{\xi_1}(\lambda)L^2_{\tau_1}}||\widetilde{g}||_{l^2_{\xi_1}(\lambda)L^2_{\tau_1}} \\ &= \sup_{\xi_1,\sigma_1}(L_1)^{1/2}||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)}, \end{aligned} \tag{6.20}$$

where

$$L_1 = \int_{\mathbb{Z}^*_\lambda}\int_{\mathbb{R}} \frac{\lambda^{k-1}M\chi_{A_1}}{\langle\sigma\rangle^{1-2\gamma}\langle\sigma_1\rangle\langle\sigma_2\rangle}d\sigma d\xi \sim \int_{\mathbb{Z}^*_\lambda}\int_{\mathbb{R}} \frac{\lambda^{k-1}\chi_{A_1}}{\langle\sigma\rangle^{1-2\gamma}\langle\sigma_2\rangle}d\sigma d\xi.$$

Let ξ_1 and σ_1 be fixed. We define a and $P(\xi)$ as

$$a := \sigma - \sigma_2 = \sigma_1 - \xi^{2k+1} + \xi_1^{2k+1} + \xi_2^{2k+1} =: \xi_1 P(\xi).$$

Since $P(x)$ is of the form of (6.10) with $m = 2k$, from Lemma 6.10, Lemma 6.11 and $|\xi_1| \geq \lambda^{-1}$, we have

$$\int_{\mathbb{Z}^*_\lambda}\int_{\mathbb{R}} \frac{1}{\langle\sigma\rangle^{1-2\gamma}\langle\sigma_2\rangle}d\sigma d\xi \lesssim \int_{\mathbb{Z}^*_\lambda}\frac{1}{\langle a\rangle^{\delta-2\gamma}}d\xi \leq \int_{\mathbb{Z}^*_\lambda}\frac{1}{\langle\lambda^{-1}P(\xi)\rangle^{\delta-2\gamma}}d\xi \lesssim \lambda^{(\delta-2\gamma)(2k+1)-1}$$

for all δ satisfying $1/2k < \delta - 2\gamma < 1$. This implies

$$L_1 \lesssim \lambda^{(\delta-2\gamma)(2k+1)+k-2}.$$

Therefore, we obtain

$$||I_1||_{l^2_\xi(\lambda)} \lesssim \lambda^{(\delta-2\gamma)(k+1/2)+k/2-1}||u||_{X^{-k/2}(\lambda)}||v||_{X^{-k/2}(\lambda)} \tag{6.21}$$

by (6.20).

Putting $\gamma = \delta'/2$ and $\epsilon := (\delta - \delta')(k + 1/2) + k/2 - 1$, we obtain the estimate (6.17) by (6.19) and (6.21), where ϵ satisfies $0 < \epsilon < (3k-1)/2$ since $1/2k < \delta - \delta' < 1$. □

Next, we prove Theorem 6.5.

Proof of Theorem 6.5.

Let $N \gg 1$, and we put

$$\widetilde{u}_N(\tau,\xi) := \big(\chi_{\{N\}}(\xi) + \chi_{\{N\}}(-\xi)\big)\,\chi_{[-1,1]}(\tau-\xi^{2k+1}),$$
$$\widetilde{v}_N(\tau,\xi) := \big(\chi_{\{1-N\}}(\xi) + \chi_{\{1-N\}}(-\xi)\big)\,\chi_{[-1,1]}(\tau-\xi^{2k+1}),$$

where

$$\chi_{\{c\}}(\xi) := \begin{cases} 1 & (\xi = c) \\ 0 & (\xi \neq c) \end{cases}, \quad \chi_{[-1,1]}(\sigma) := \begin{cases} 1 & (|\sigma| \leq 1) \\ 0 & (|\sigma| > 1) \end{cases}.$$

for $c \in \mathbb{Z}$. Then we have

$$||u_N||_{Z^s} \sim ||v_N||_{Z^s} \sim N^s(||\chi_{[-1,1]}||_{L^2_\sigma} + ||\chi_{[-1,1]}||_{L^1_\sigma}) \sim N^s.$$

We put

$$A_1(\xi_1,\xi_2) := \chi_{\{N\}}(\xi_1)\chi_{\{1-N\}}(\xi_2),\ A_2(\xi_1,\xi_2) := \chi_{\{N\}}(\xi_1)\chi_{\{1-N\}}(-\xi_2),$$
$$A_3(\xi_1,\xi_2) := \chi_{\{N\}}(-\xi_1)\chi_{\{1-N\}}(\xi_2),\ A_4(\xi_1,\xi_2) := \chi_{\{N\}}(-\xi_1)\chi_{\{1-N\}}(-\xi_2).$$

Then we have

$$\begin{aligned}&||\mathcal{F}^{-1}[\langle\tau-\xi^{2k+1}\rangle^{-1}\partial_x\widetilde{(u_N v_N)}]||_{X^s} \\ &= \left|\left|\sum_{j=1}^{4}\int_{\mathbb{Z}^*}|\xi|^{s+1}A_j(\xi_1,\xi_2)\left(\int_{\mathbb{R}}\langle\sigma\rangle^{-1/2}\chi_{[-1,1]}(\sigma_1)\chi_{[-1,1]}(\sigma_2)d\sigma_1\right)d\xi_1\right|\right|_{l^2_\xi L^2_\sigma}.\end{aligned}$$

From $\sigma_2 = \xi^{2k+1} - \xi_1^{2k+1} - \xi_2^{2k+1} + \sigma - \sigma_1$ and Lemma 6.8, we have

$$\langle\sigma\rangle \sim |\xi^{2k+1} - \xi_1^{2k+1} - \xi_2^{2k+1}| \sim |\xi\xi_1\xi_2|\max\{|\xi|,|\xi_1|,|\xi_2|\}^{2k-2}$$

when there exists σ_1 satisfying $|\sigma_1| \leq 1$ and $|\sigma_2| \leq 1$. This implies

$$\int_{\mathbb{R}}\langle\sigma\rangle^{-1/2}\chi_{[-1,1]}(\sigma_1)\chi_{[-1,1]}(\sigma_2)d\sigma_1 \gtrsim |\xi\xi_1\xi_2|^{-1/2}\max\{|\xi|,|\xi_1|,|\xi_2|\}^{-(k-1)}.$$

As a result, we obtain

$$\begin{aligned}&||\mathcal{F}^{-1}[\langle\tau-\xi^{2k+1}\rangle^{-1}\partial_x\widetilde{(u_N v_N)}]||_{X^s} \\ &\gtrsim \left|\left|\sum_{j=1}^{4}\int_{\mathbb{Z}^*}|\xi|^{s+1}A_j(n_1,n_2)|\xi\xi_1\xi_2|^{-1/2}\max\{|\xi|,|\xi_1|,|\xi_2|\}^{-(k-1)}d\xi_1\right|\right|_{l^2_\xi} \\ &\sim \big|\big|N^{-k}(\chi_{\{1\}}+\chi_{\{-1\}}) + N^{s+1}N^{-k-1/2}(\chi_{\{2N-1\}}+\chi_{\{-2N+1\}})\big|\big|_{l^2_\xi} \\ &\gtrsim N^{-k}.\end{aligned}$$

By above discussion, if the bilinear estimate (6.6) holds, then we have

$$\begin{aligned} N^{-k} &\lesssim ||\mathcal{F}^{-1}[\langle \tau - \xi^{2k+1} \rangle^{-1} \partial_x \widetilde{(u_N v_N)}]||_{X^s} \leq ||\mathcal{F}^{-1}[\langle \tau - \xi^{2k+1} \rangle^{-1} \partial_x \widetilde{(u_N v_N)}]||_{Z^s} \\ &\lesssim ||u_N||_{Z^s} ||v_N||_{Z^s} \sim N^{2s}, \end{aligned}$$

which contradicts the assumption $s < -k/2$. □

6.4 Proof of local well-posedness

In this section, we prove LWP of (6.1) by using the bilinear estimate. First, we consider the scaling property. (6.1) with initial data $u_0 \in \widetilde{H}^s(\mathbb{T})$ is invariant under the following scaling;

$$u_\lambda(t, x) = \lambda^{-2k} u(\lambda^{-2k-1} t, \lambda^{-1} x), \quad u_{\lambda 0}(x) = \lambda^{-2k} u_0(\lambda^{-1} x).$$

More precisely, if u satisfies (6.1) on $[0, T] \times \mathbb{T}$ with initial data $u_0 \in \widetilde{H}^s(\mathbb{T})$, then u_λ satisfies the same equation on $[0, \lambda^{2k+1} T] \times \mathbb{T}_\lambda$ with initial data $u_{\lambda 0} \in \widetilde{H}^s(\mathbb{T}_\lambda)$.

Proposition 6.12. *For $s \in \mathbb{R}$, and $\lambda \geq 1$, we have*

$$||u_{\lambda 0}||_{\widetilde{H}^s(\mathbb{T}_\lambda)} = \lambda^{-2k-s+1/2} ||u_0||_{\widetilde{H}^s(\mathbb{T})}, \tag{6.22}$$

where

$$||f||_{\widetilde{H}^s(\mathbb{T}_\lambda)} := |||\xi|^s \widehat{f}||_{l^2_\xi(\lambda)}.$$

Proof. We note that

$$\left(\frac{1}{2\pi\lambda} \sum_{\xi \in \mathbb{Z}^*_\lambda} |f(\lambda\xi)|^2 \right)^{1/2} = \lambda^{-1/2} \left(\frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}^*} |f(\xi)|^2 \right)^{1/2}$$

for all function f defined on $\mathbb{T}$. This implies

$$||f(\lambda\xi)||_{l^2_\xi(\lambda)} = \lambda^{-1/2} ||f||_{l^2_\xi}. \tag{6.23}$$

Since $\widehat{u}_{\lambda 0}(\xi) = \lambda^{-2k+1} \widehat{u}_0(\lambda\xi)$ and

$$|||\xi|^s \lambda^{-2k+1} \widehat{u}_0(\lambda\xi)||_{l^2_\xi(\lambda)} = \lambda^{-1/2} |||\lambda^{-1}\xi|^s \lambda^{-2k+1} \widehat{u}_0||_{l^2_\xi}$$

by (6.23), we obtain (6.22). □

Next, we give the linear estimates for the equation (6.1).

Proposition 6.13. *Let $\lambda \geq 1$. We put*

$$(U_\lambda(t)f)(x) := \int_{\mathbb{Z}_\lambda^*} \exp(it\xi^{2k+1} + ix\xi)\widehat{f}(\xi)d\xi$$

and

$$(U_\lambda *_R F)(t) := \int_0^t U_\lambda(t-t')F(t')dt'.$$

For $s \in \mathbb{R}$, there exist positive constants C_1 and C_2 such that

$$||\psi U_\lambda(\cdot)f||_{Z^s(\lambda)} \leq C_1||f||_{\widetilde{H}^s(\mathbb{T}_\lambda)},$$
$$||\psi(U_\lambda *_R F)||_{Z^s(\lambda)} \leq C_2||\mathcal{F}^{-1}[\langle \tau - \xi^{2k+1}\rangle^{-1}\widetilde{F}]||_{Z^s(\lambda)},$$

where C_1 and C_2 do not depend on λ and ψ is a cut-off function such that $\psi \in C^\infty(\mathbb{R})$, $\psi(t) = 1$ on $[-1,1]$ and $\mathrm{supp}\ \psi \subset [-2,2]$.

For the proof of Proposition 6.13, see Lemma 2.1 in [21].

Proof of Theorem 6.2.
For $u_0 \in B_r(\widetilde{H}^s(\mathbb{T}))$, we choose $\lambda > 1$ such that $\lambda^{2k+s-1/2-\epsilon} \geq 4C_0C_1C_2r$, where C_1 and C_2 appeared in Proposition 6.13, and C_0 and ϵ appeared in Theorem 6.4. Then we have $||u_{\lambda 0}||_{\widetilde{H}^s(\mathbb{T}_\lambda)} \leq 1/(4C_0C_1C_2\lambda^\epsilon) =: R_\lambda$ from Proposition 6.12. Therefore, we assume that $u_{\lambda 0} \in B_{R_\lambda}(\widetilde{H}^s(\mathbb{T}_\lambda))$.

We define the map $\Phi_{u_0,\lambda}$ as

$$\Phi_{u_0,\lambda}[u](t) := \psi(t)U_\lambda(t)u_{\lambda 0} - \frac{\psi(t)}{2}\int_0^t U_\lambda(t-t')\partial_x(u(t')^2)dt',$$

where ψ is a cut function satisfying the assumption in Proposition 6.13. To prove the existence of the solution of (6.1), we first prove that $\Phi_{u_0,\lambda}$ is a contraction map on $B_{2C_1R_\lambda}(Z^s(\lambda))$.

Step 1. (Existence)
For any $u \in B_{2C_1R_\lambda}(Z^s(\lambda))$, from Proposition 6.13 and Theorem 6.4, we have

$$\begin{aligned}||\Phi_{u_0,\lambda}[u]||_{Z^s(\lambda)} &\leq ||\psi U_\lambda(\cdot)u_{\lambda 0}||_{Z^s(\lambda)} + \frac{1}{2}||\psi\left(U_\lambda *_R (\partial_x(u^2))\right)||_{Z^s(\lambda)} \\ &\leq C_1R_\lambda + \frac{1}{2}C_2C_0\lambda^\epsilon||u||^2_{Z^s(\lambda)} \\ &\leq C_1R_\lambda + (2C_0C_1C_2\lambda^\epsilon R_\lambda)C_1R_\lambda \\ &\leq 2C_1R_\lambda.\end{aligned}$$

Therefore, we obtain $\Phi_{u_0,\lambda}[u] \in B_{2C_1R_\lambda}(Z^s(\lambda))$.

For any u, $v \in B_{2C_1R_\lambda}(Z^s(\lambda))$, from Proposition 6.13 and Theorem 6.4, we have

$$\begin{aligned}
||\Phi_{u_0,\lambda}[u] - \Phi_{u_0,\lambda}[v]||_{Z^s(\lambda)} &\le \frac{1}{2}||\psi\left(U_\lambda *_R \left(\partial_x(u^2 - v^2)\right)\right)||_{Z^s(\lambda)} \\
&\le \frac{1}{2}C_2C_0\lambda^\epsilon||u+v||_{Z^s(\lambda)}||u-v||_{Z^s(\lambda)} \\
&\le 2C_0C_1C_2\lambda^\epsilon R_\lambda||u-v||_{Z^s(\lambda)} \\
&= \frac{1}{2}||u-v||_{Z^s(\lambda)}.
\end{aligned}$$

This implies that $\Phi_{u_0,\lambda}$ is a contraction map.

By above discussion and applying Banach's fixed point theorem, there exists $u'_\lambda \in B_{2C_1R_\lambda}(Z^s(\lambda))$, such that

$$u'_\lambda(t) = \psi(t)U_\lambda(t)u_{\lambda 0} - \frac{\psi(t)}{2}\int_0^t U_\lambda(t-t')\partial_x(u'_\lambda(t')^2)dt'. \tag{6.24}$$

Especially, u'_λ satisfies the integral equation

$$u'_\lambda(t) = U_\lambda(t)u_{\lambda 0} - \frac{1}{2}\int_0^t U_\lambda(t-t')\partial_x(u'_\lambda(t')^2)dt'. \tag{6.25}$$

in the time interval $[0,1]$. Therefore, $u_\lambda = u'_\lambda|_{[0,1]} \in B_{2C_1R_\lambda}(Z^s_{[0,1]}(\lambda))$ is the time local solution of (6.1).

Next, we prove that u_λ is unique in $B_{2C_1R_\lambda}(Z^s_{[0,1]}(\lambda))$.

Step 2. (Uniqueness)
We assume that u_λ, $v_\lambda \in B_{2C_1R_\lambda}(Z^s_{[0,1]}(\lambda))$ satisfy (6.25) on $[0,1]$. For any u'_λ, $v'_\lambda \in Z^{s,1/2}(\lambda)$ such that $u'_\lambda(t) = u_\lambda(t)$, $v'_\lambda(t) = v_\lambda(t)$ on $[0,1]$, we have

$$\begin{aligned}
||u_\lambda - v_\lambda||_{Z^s_{[0,1]}(\lambda)} &\le \frac{1}{2}||\psi\left(U_\lambda *_R \left(\partial_x(u_\lambda^2 - v_\lambda^2)\right)\right)||_{Z^s_{[0,1]}(\lambda)} \\
&\le \frac{1}{2}||\psi\left(U_\lambda *_R \left(\partial_x((u'_\lambda)^2 - (v'_\lambda)^2)\right)\right)||_{Z^s(\lambda)} \\
&\le \frac{1}{2}C_2C_0\lambda^\epsilon||u'_\lambda + v'_\lambda||_{Z^s(\lambda)}||u'_\lambda - v'_\lambda||_{Z^s(\lambda)}
\end{aligned}$$

from Proposition 6.13 and Theorem 6.4. Therefore we obtain

$$\begin{aligned}
||u_\lambda - v_\lambda||_{Z^s_{[0,1]}(\lambda)} &\le \frac{1}{2}C_2C_0\lambda^\epsilon||u_\lambda + v_\lambda||_{Z^s_{[0,1]}(\lambda)}||u_\lambda - v_\lambda||_{Z^s_{[0,1]}(\lambda)} \\
&\le 2C_0C_1C_2\lambda^\epsilon R_\lambda||u_\lambda - v_\lambda||_{Z^s_{[0,1]}(\lambda)} \\
&= \frac{1}{2}||u_\lambda - v_\lambda||_{Z^s_{[0,1]}(\lambda)}.
\end{aligned}$$

Thus, $u_\lambda(t) = v_\lambda(t)$ on $[0, 1]$.

Finally, we prove that the data-to-solution map is Lipschitz continuous.

Step 3. (Lipschitz continuity of the data-to-solution map)
Let u'_λ (resp. v'_λ) $\in B_{2C_1R_\lambda}(Z^{s,1/2}(\lambda))$ be the solution of (6.24) with initial data $u_{\lambda 0}$ (resp. $v_{\lambda 0}$) $\in B_{R_\lambda}(\widetilde{H}^s(\mathbb{T}_\lambda))$ obtained in Step 1. Then we have

$$||u'_\lambda - v'_\lambda||_{Z^s(\lambda)} \leq C_1||u_{\lambda 0} - v_{\lambda 0}||_{\widetilde{H}^s(\mathbb{T}_\lambda)} + \frac{1}{2}||u'_\lambda - v'_\lambda||_{Z^s(\lambda)}.$$

in the same manner as above. Therefore, we obtain

$$||u'_\lambda - v'_\lambda||_{Z^s(\lambda)} \leq 2C_1||u_{\lambda 0} - v_{\lambda 0}||_{\widetilde{H}^s(\mathbb{T}_\lambda)}.$$

This implies that the data-to-solution map is Lipschitz continuous. □

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