ON τ-TILTING THEORY

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CONTENTS

INTRODUCTION

Tilting theory has been central in the representation theory of finite dimensional algebras since the early seventies [BGP, APR, BB, HR, Bo]. An important role in tilting theory is played by the notion of tilting modules, which was introduced by Brenner-Butler as a generalization of progenerators in Morita theory, because it is known that the endomorphism algebras of tilting modules is derived equivalent to the original algebra [Ha, Ric]. Thus these algebras share the same homological behavior. Hence it is important to give a classification and a construction of tilting modules for a given algebra.

An effective method to construct tilting modules is given by mutation [RS, Un]. It is an operation to replace an indecomposable direct summand of a given tilting module to get a new one. However, it is known that mutation of tilting modules is often impossible depending on a choice of indecomposable direct summands. One of our aim in this paper is to improve behavior of mutation of tilting modules.

Let Λ be a basic finite dimensional algebra over an algebraically closed field K. The behavior is caused by the following property for almost complete tilting modules: An

almost complete tilting module U can be completed in at most two different ways to a tilting module [RS, Un]. Moreover there are exactly two ways if and only if U is a faithful Λ-module [HU1]. Namely, for every faithful almost complete tilting module U, there exist exactly two indecomposable modules X and X' (called complements) such that $T := X \oplus U$ and $T' := X' \oplus U$ are tilting modules. Thus, by mutation, these tilting modules T and T' are connected to each other. On the other hand, if an almost complete tilting module U for a tilting module $X \oplus U$ is not faithful, then there is no mutation for the indecomposable module X.

Even for a finite dimensional path algebra KQ, where Q is a finite acyclic quiver, not all almost complete tilting modules U are faithful. However, the above property for almost complete tilting modules can be reformulated in terms of the path algebra $\Lambda = KQ$ as follows [IT, Rin]: A Λ-module T is *support tilting* if T is a tilting $(\Lambda/\langle e \rangle)$-module for some idempotent e of Λ. Using the more general class of support tilting modules, it holds for path algebras that almost complete support tilting modules can be completed in exactly two ways to support tilting modules.

The above result for path algebras does not necessarily hold for a finite dimensional algebra. The reason is that there may be sincere modules which are not faithful. We are looking for a generalization of tilting modules where we have such a result, and where at the same time some of the essential properties of tilting modules still hold. It is then natural to try to find a class of modules satisfying the following properties:

(i) The analogs of basic almost complete tilting modules have exactly two complements.

(ii) In the hereditary case the class of modules should coincide with the tilting modules.

(iii) There is a natural connection with torsion pairs in $\mathsf{mod}\,\Lambda$.

(iv) The modules have exactly $|\Lambda|$ non-isomorphic indecomposable direct summands, where $|X|$ denotes the number of nonisomorphic indecomposable direct summands of X.

There is a generalization of tilting modules to tilting modules of finite projective dimension [Ha, Miy]. But it is easy to see that they do not satisfy the required properties. The finite dimensional module category is naturally embedded in the derived category of Λ. The tilting and silting complexes for Λ [Rin, AI, Ai] are also extensions of the tilting modules. An almost complete silting complex has infinitely many complements. But as we shall see, things work well when we restrict to the two-term silting complexes.

In Part 1, we give a class of modules satisfying the properties (i)–(iv) above. Namely, it turns out that a natural class of modules to consider is given as follows. As usual, we denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated right Λ-modules, $\mathsf{proj}\,\Lambda$ the category of finitely generated projective right Λ-modules and τ the AR translation (see section 1.2).

Definition A.   (a) We call M in $\mathsf{mod}\,\Lambda$ *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$.

(b) We call M in $\mathsf{mod}\,\Lambda$ *τ-tilting* (respectively, *almost complete τ-tilting*) if M is τ-rigid and $|M| = |\Lambda|$ (respectively, $|M| = |\Lambda| - 1$).

(c) We call M in $\mathsf{mod}\,\Lambda$ *support τ-tilting* if there exists an idempotent e of Λ such that M is a τ-tilting $(\Lambda/\langle e \rangle)$-module.

Any τ-rigid module is rigid (i.e. $\mathrm{Ext}^1_\Lambda(M, M) = 0$), and the converse holds if the projective dimension is at most one. In particular, any partial tilting module is a τ-rigid module, and any tilting module is a τ-tilting module. Thus we can regard τ-tilting modules as a generalization of tilting modules.

The first main result of this part is the following analog of Bongartz completion for tilting modules.

Theorem B (Theorem 2.10). *Any τ-rigid Λ-module is a direct summand of some τ-tilting Λ-module.*

As indicated above, in order to get our theory to work nicely, we need to consider support τ-tilting modules. It is often convenient to view them, and the τ-rigid modules, as certain pairs of Λ-modules.

Definition C. Let (M, P) be a pair with $M \in \mathsf{mod}\,\Lambda$ and $P \in \mathsf{proj}\,\Lambda$.
(a) We call (M, P) a τ-*rigid* pair if M is τ-rigid and $\mathrm{Hom}_\Lambda(P, M) = 0$.
(b) We call (M, P) a *support τ-tilting* (respectively, *almost complete support τ-tilting*) pair if (M, P) is τ-rigid and $|M| + |P| = |\Lambda|$ (respectively, $|M| + |P| = |\Lambda| - 1$).

These notions are compatible with those in Definition A (see Proposition 2.3 for details). As usual, we say that (M, P) is *basic* if M and P are basic. Similarly we say that (M, P) is a *direct summand* of (M', P') if M is a direct summand of M' and P is a direct summand of P'.

The second main result of this part is the following.

Theorem D (Theorem 2.18). *Let Λ be a finite dimensional K-algebra. Then any basic almost complete support τ-tilting pair for Λ is a direct summand of exactly two basic support τ-tilting pairs.*

These two support τ-tilting pairs are said to be *mutations* of each other. We will define the support τ-tilting quiver $\mathrm{Q}(\mathsf{s}\tau\text{-tilt}\,\Lambda)$ by using mutation (Definition 2.29).

The third main result of this part is to obtain a close connection between support τ-tilting modules and other important objects in tilting theory. The corresponding definitions will be given in section 1.

Theorem E (Theorems 2.7, 3.2, 4.1 and 4.7). *Let Λ be a finite dimensional K-algebra. We have bijections between*
(a) *the set f-torsΛ of functorially finite torsion classes in $\mathsf{mod}\,\Lambda$,*
(b) *the set $\mathsf{s}\tau$-tiltΛ of isomorphism classes of basic support τ-tilting modules,*
(c) *the set 2-siltΛ of isomorphism classes of basic two-term silting complexes for Λ,*
(d) *the set c-tilt$\mathcal{C}$ of isomorphism classes of basic cluster-tilting objects in a 2-CY triangulated category $\mathcal{C}$ if Λ is an associated 2-CY tilted algebra to $\mathcal{C}$.*

By Theorem E, we can regard $\mathsf{s}\tau$-tiltΛ as a partially ordered set by using the inclusion relation of f-torsΛ (i.e. we write $T \geq U$ if $\mathsf{Fac}\,T \supseteq \mathsf{Fac}\,U$). Then we have the following fourth main result, which is an analog of [HU2, Theorem 2.1] and [AI, Theorem 2.35].

Theorem F (Corollary 2.34). *The support τ-tilting quiver $\mathrm{Q}(\mathsf{s}\tau\text{-tilt}\,\Lambda)$ is the Hasse quiver of the partially ordered set $\mathsf{s}\tau$-tiltΛ.*

In Part 2, we classify τ-tilting modules over any Nakayama algebra Λ, and moreover to give an algorithm to construct the Hasse quiver of support τ-tilting Λ-modules. The following theorem is our first main result of this part.

Theorem G (Theorem 8.10, 8.12 and 8.17). *Let Λ be a Nakayama algebra with n simple modules. Assume that the Loewy length of each indecomposable projective Λ-module is at least n. Then there are bijections between*

(1) *the set τ-tilt Λ of isomorphism classes of basic τ-tilting Λ-modules,*
(2) *the set $\mathrm{ps}\tau$-tilt Λ of isomorphism classes of basic proper support τ-tilting Λ-modules,*
(3) *the set $\mathcal{T}(n)$ of triangulations of an n-regular polygon with a puncture,*
(4) *the set $\mathcal{Z}(n)$ of sequences $(a_1, a_2, \cdots, a_n)$ of nonnegative integers with $\sum_{i=1}^{n} a_i = n$.*

We have a similar result for an arbitrary Nakayama algebra (see Theorem 8.22). Theorem G is analogous to known classification results in representation theory: tilting modules for cyclic quivers [BK], torsion pairs in tube categories [BBM], cluster-tilting objects in cluster categories of type A [CCS] and D [Sc], cluster-tilting modules over self-injective algebras and Gorenstein orders of type A and D [Iy].

Next, we study a relationship between support τ-tilting Λ-modules over an algebra Λ (not necessarily Nakayama) and those over the factor algebra $\overline{\Lambda} = \Lambda/\operatorname{soc} Q$, where Q is an indecomposable projective-injective Λ-module. The following theorem is the second main result in this part.

Theorem H (Theorem 9.5 and 9.8)**.** *Let Λ be a basic finite dimensional K-algebra, Q an indecomposable projective-injective Λ-module and $\overline{Q} := Q/\operatorname{soc} Q$. Then all support τ-tilting modules of Λ can be obtained explicitly from those of $\overline{\Lambda}$. Moreover, the Hasse quiver of support τ-tilting modules of Λ can be constructed explicitly from that of $\overline{\Lambda}$.*

As an application, we have the following result for a construction of the Hasse quivers of Nakayama algebras. Since each Nakayama algebra Λ always has an indecomposable projective-injective Λ-module Q and the factor algebra $\overline{\Lambda}$ is a Nakayama algebra again, we can iteratively apply Theorem H.

Theorem I (Algorithm 9.13)**.** *Let Λ be a Nakayama algebra. Then there exists an algorithm to construct the Hasse quiver of support τ-tilting Λ-modules.*

In Part 3, we study τ-rigid modules over algebras with radical square zero, which provide one of the most fundamental classes of algebras in representation theory (*e.g.*, work of Yoshii [Yo] in 1956 and Gabriel [Ga] in 1972). For an algebra Λ with radical square zero, an important role is played by a stable equivalence $F : \underline{\operatorname{mod}}\Lambda \to \underline{\operatorname{mod}} KQ^{\mathrm{s}}$, where Q^{s} is the separated quiver (see Subsection 10.2) of the quiver Q for Λ. In particular, we have the following famous theorem characterizing representation-finiteness.

Theorem J. [Ga, ARS] *Let Λ be a finite dimensional K-algebra with radical square zero. Then the following are equivalent:*

(1) *Λ is representation-finite.*
(2) *The separated quiver for Λ is a disjoint union of Dynkin quivers.*

The following main theorem of this part is an analog of this result for τ-rigid-finiteness. Let Q_0 be the vertex set of a quiver Q. Then $Q_0^{\mathrm{s}} = \{i^+, i^- \mid i \in Q_0\}$ is the vertex set of the separated quiver Q^{s}. A full subquiver of Q^{s} is called a *single subquiver* if, for any $i \in Q_0$, the vertex set contains at most one of i^+ or i^-.

Theorem K (Theorem 11.1)**.** *Let Λ be a finite dimensional K-algebra with radical square zero. Then the following are equivalent:*

(1) *Λ is τ-rigid-finite.*
(2) *Every single subquiver of the separated quiver for Λ is a disjoint union of Dynkin quivers.*

The following result plays a crucial reole in the proof of Theorem K.

Theorem L (Theorem 11.2). *Let X be an indecomposable Λ-module. Let $P_1 \to P_0 \to X \to 0$ be a minimal projective presentation of X. The following are equivalent:*

(1) *X is a τ-rigid Λ-module.*
(2) *FX is a τ-rigid KQ^{s}-module and $\mathsf{add}\, P_0 \cap \mathsf{add}\, P_1 = 0$.*

As an application, we give a positive answer to a question given by Zhang [Zh].

Corollary M (Corollary 12.2). *Let Λ be a finite dimensional K-algebra with radical square zero. If every indecomposable Λ-module is τ-rigid, then Λ is representation-finite.*

For recent results on τ-tilting theory, we refer to [Ad1, Ad2, AAC, BY, Ch, DIJ, IJY, Ja1, MPS, Ma, Miz1, Miz2, Zh].

Notation. Throughout this paper, K is an algebraically closed field. By an algebra we mean a basic and finite dimensional K-algebra and by a module we mean a finitely generated right module. Let $\{e_1, e_2, \cdots, e_n\}$ be a complete set of primitive orthogonal idempotents of an algebra Λ and $\mathcal{E}_\Lambda := \{\sum_{i \in I} e_i \mid I \subset \{1, 2, \cdots, n\}\}$. For each $i \in \{1, 2, \cdots, n\}$, we put $P(i) = e_i\Lambda$, $S(i) = P(i)/\operatorname{rad} P(i)$ and $E(i) = \operatorname{Hom}_K(\Lambda e_i, K)$. For an algebra Λ, we denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated right Λ-modules, $\underline{\mathsf{mod}}\,\Lambda$ the stable category, $\mathsf{proj}\,\Lambda$ the category of finitely generated projective right Λ-modules and $\mathsf{inj}\,\Lambda$ the category of finitely generated injective right Λ-modules. For an Λ-module M, we denote by $\mathsf{add}\, M$ (respectively, $\mathsf{Fac}\, M$, $\mathsf{Sub}\, M$) the category of all direct summands (respectively, factor modules, submodules) of finite direct sums of copies of M. We denote by τ_Λ the Auslander-Reiten translation of Λ and by $\langle e \rangle$ a two-sided ideal of Λ generated by $e \in \Lambda$. For two sets X and Y, we denote by $X \coprod Y$ the disjoint union. We denote by $[i, j]$ the interval $\{i, i+1, \cdots, j-1, j\}$ of integers. Fix an integer $n > 0$. For any integer i, there exist integers j and $1 \le k \le n$ such that $i = nj + k$. Then we let $(i)_n := k$. For integers i, j with $(i)_n \le (j)_n$, we let

$$[i, j]_n := \{(i)_n, (i+1)_n, \cdots, (j-1)_n, (j)_n\}.$$

We call a quiver *Dynkin* (respectively, *Euclidean*) if the underlying graph is one of Dynkin (respectively, Euclidean) graphs of type A, D and E (respectively, $\widetilde{A}, \widetilde{D}$ and $\widetilde{E}$).

Part 1. τ-tilting theory

This part is based on the paper [AIR] (joint work with O. Iyama and I. Reiten).

1. Background and preliminary results

In this section we give some background material on each of the 4 topics involved in our main results. This concerns the relationship between tilting modules and functorially finite subcategories and some results on τ-rigid and τ-tilting modules, including new basic results about them which will be useful in the next section. Further we recall known results on silting complexes, and on cluster-tilting objects in 2-CY triangulated categories.

1.1. **Torsion pairs and tilting modules.** Let Λ be a finite dimensional K-algebra. For a subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$, we let

$$\mathcal{C}^{\perp} := \{X \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}_{\Lambda}(\mathcal{C}, X) = 0\},$$
$$\mathcal{C}^{\perp_1} := \{X \in \mathsf{mod}\,\Lambda \mid \mathrm{Ext}^1_{\Lambda}(\mathcal{C}, X) = 0\}.$$

Dually we define $^{\perp}\mathcal{C}$ and $^{\perp_1}\mathcal{C}$. We call T in $\mathsf{mod}\,\Lambda$ a *partial tilting module* if $\mathrm{pd}_{\Lambda} T \leq 1$ and $\mathrm{Ext}^1_{\Lambda}(T, T) = 0$. A partial tilting module is called a *tilting module* if there is an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ with T_0 and T_1 in $\mathsf{add}\,T$. Then any tilting module satisfies $|T| = |\Lambda|$. Moreover it is known that for any partial tilting module T, there is a tilting module U such that $T \in \mathsf{add}\,U$ and $\mathsf{Fac}\,U = T^{\perp_1}$, called the *Bongartz completion* of T. Hence a partial tilting module T is a tilting module if and only if $|T| = |\Lambda|$. Dually T in $\mathsf{mod}\,\Lambda$ is a *(partial) cotilting module* if DT is a (partial) tilting Λ^{op}-module.

On the other hand, we say that a full subcategory $\mathcal{T}$ of $\mathsf{mod}\,\Lambda$ is a *torsion class* (respectively, *torsionfree class*) if it is closed under factor modules (respectively, submodules) and extensions. A pair $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. In this case $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsionfree class. Conversely, any torsion class $\mathcal{T}$ (respectively, torsionfree class $\mathcal{F}$) gives rise to a torsion pair $(\mathcal{T}, \mathcal{F})$.

We say that $X \in \mathcal{T}$ is Ext-*projective* (respectively, Ext-*injective*) if $\mathrm{Ext}^1_{\Lambda}(X, \mathcal{T}) = 0$ (respectively, $\mathrm{Ext}^1_{\Lambda}(\mathcal{T}, X) = 0$). We denote by $P(\mathcal{T})$ the direct sum of one copy of each of the indecomposable Ext-projective objects in $\mathcal{T}$ up to isomorphism. Similarly we denote by $I(\mathcal{F})$ the direct sum of one copy of each of the indecomposable Ext-injective objects in $\mathcal{F}$ up to isomorphism.

We first recall the following relevant result on torsion pairs and tilting modules.

Proposition 1.1. [AS, Ho, Sma] *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathsf{mod}\,\Lambda$. Then the following conditions are equivalent.*

(a) *$\mathcal{T}$ is functorially finite.*
(b) *$\mathcal{F}$ is functorially finite.*
(c) *$\mathcal{T} = \mathsf{Fac}\,X$ for some X in $\mathsf{mod}\,\Lambda$.*
(d) *$\mathcal{F} = \mathsf{Sub}\,Y$ for some Y in $\mathsf{mod}\,\Lambda$.*
(e) *$P(\mathcal{T})$ is a tilting $(\Lambda/\operatorname{ann}\mathcal{T})$-module.*
(f) *$I(\mathcal{F})$ is a cotilting $(\Lambda/\operatorname{ann}\mathcal{F})$-module.*
(g) *$\mathcal{T} = \mathsf{Fac}\,P(\mathcal{T})$.*
(h) *$\mathcal{F} = \mathsf{Sub}\,I(\mathcal{F})$.*

Proof. The conditions (a), (b), (c), (d), (e) and (f) are equivalent by [Sma, Theorem].

(g)$\Rightarrow$(c) is clear.

(e)$\Rightarrow$(g) There exists an exact sequence $0 \to \Lambda/\operatorname{ann}\mathcal{T} \xrightarrow{a} T^0 \to T^1 \to 0$ with $T^0, T^1 \in \mathsf{add}\,P(\mathcal{T})$. For any $X \in \mathcal{T}$, we take a surjection $f : (\Lambda/\operatorname{ann}\mathcal{T})^{\ell} \to X$. It follows from $\mathrm{Ext}^1_{\Lambda}(T^{1\ell}, X) = 0$ that f factors through $a^{\ell} : (\Lambda/\operatorname{ann}\mathcal{T})^{\ell} \to T^{0\ell}$. Thus $X \in \mathsf{Fac}\,P(\mathcal{T})$.

Dually (h) is also equivalent to the other conditions. $\square$

There is also a tilting quiver associated with the (classical) tilting modules. The vertices are the isomorphism classes of basic tilting modules. Let $X \oplus U$ and $Y \oplus U$ be basic tilting modules, where X and $Y \not\simeq X$ are indecomposable. Then it is known that there is some exact sequence $0 \to X \xrightarrow{f} U' \xrightarrow{g} Y \to 0$, where $f : X \to U'$ is a minimal left ($\mathsf{add}\,U$)-approximation and $g : U' \to Y$ is a minimal right ($\mathsf{add}\,U$)-approximation. We say that $Y \oplus U$ is a left mutation of $X \oplus U$. Then we draw an arrow $X \oplus U \to Y \oplus U$, so that we get a quiver for the tilting modules. On the other hand, the set of basic tilting modules has

a natural partial order given by $T \geq U$ if and only if $\mathsf{Fac}\,T \supseteq \mathsf{Fac}\,U$, and we can consider the associated Hasse quiver. These two quivers coincide [HU2, Theorem 2.1].

1.2. τ-**tilting modules.** Let Λ be a finite dimensional K-algebra. We have dualities

$$D := \mathrm{Hom}_K(-, K) : \mathsf{mod}\,\Lambda \leftrightarrow \mathsf{mod}\,\Lambda^{\mathrm{op}} \quad \text{and} \quad (-)^* := \mathrm{Hom}_\Lambda(-, \Lambda) : \mathsf{proj}\,\Lambda \leftrightarrow \mathsf{proj}\,\Lambda^{\mathrm{op}}$$

which induce equivalences

$$\nu := D(-)^* : \mathsf{proj}\,\Lambda \to \mathsf{inj}\,\Lambda \quad \text{and} \quad \nu^{-1} := (-)^* D : \mathsf{inj}\,\Lambda \to \mathsf{proj}\,\Lambda$$

called *Nakayama functors*. For X in $\mathsf{mod}\,\Lambda$ with a minimal projective presentation

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \longrightarrow 0,$$

we define $\mathrm{Tr}\,X$ in $\mathsf{mod}\,\Lambda^{\mathrm{op}}$ and τX in $\mathsf{mod}\,\Lambda$ by exact sequences

$$P_0^* \xrightarrow{d_1^*} P_1^* \longrightarrow \mathrm{Tr}\,X \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \tau X \longrightarrow \nu P_1 \xrightarrow{\nu d_1} \nu P_0.$$

Then Tr and τ give bijections between the isomorphism classes of indecomposable non-projective Λ-modules, the isomorphism classes of indecomposable non-projective Λ^{op}-modules and the isomorphism classes of indecomposable non-injective Λ-modules. We denote by $\underline{\mathsf{mod}}\,\Lambda$ the *stable category* modulo projectives and by $\overline{\mathsf{mod}}\,\Lambda$ the *costable category* modulo injectives. Then Tr gives the *Auslander-Bridger transpose duality*

$$\mathrm{Tr} : \underline{\mathsf{mod}}\,\Lambda \leftrightarrow \underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$$

and τ gives the *AR translations*

$$\tau = D\,\mathrm{Tr} : \underline{\mathsf{mod}}\,\Lambda \to \overline{\mathsf{mod}}\,\Lambda \quad \text{and} \quad \tau^{-1} = \mathrm{Tr}\,D : \overline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda.$$

We have a functorial isomorphism

$$\underline{\mathrm{Hom}}_\Lambda(X, Y) \simeq D\,\mathrm{Ext}^1_\Lambda(Y, \tau X)$$

for any X and Y in $\mathsf{mod}\,\Lambda$ called *AR duality*. In particular, if M is τ-rigid, then we have $\mathrm{Ext}^1_\Lambda(M, M) = 0$ (i.e. M is rigid) by AR duality. More precisely, we have the following result, which we often use in this paper.

Proposition 1.2. *For X and Y in $\mathsf{mod}\,\Lambda$, we have the following.*
  (a) *[AS, Proposition 5.8] $\mathrm{Hom}_\Lambda(X, \tau Y) = 0$ if and only if $\mathrm{Ext}^1_\Lambda(Y, \mathsf{Fac}\,X) = 0$.*
  (b) *[AS, Theorem 5.10] If X is τ-rigid, then $\mathsf{Fac}\,X$ is a functorially finite torsion class and $X \in \mathsf{add}\,P(\mathsf{Fac}\,X)$.*
  (c) *If $\mathcal{T}$ is a torsion class in $\mathsf{mod}\,\Lambda$, then $P(\mathcal{T})$ is a τ-rigid Λ-module.*

Proof. (c) Since $T := P(\mathcal{T})$ is Ext-projective in $\mathcal{T}$, we have $\mathrm{Ext}^1_\Lambda(T, \mathsf{Fac}\,T) = 0$. This implies that $\mathrm{Hom}_\Lambda(T, \tau T) = 0$ by (a). $\qquad\square$

We have the following direct consequence (see also [Sk, ASS]).

Proposition 1.3. *Any τ-rigid Λ-module M satisfies $|M| \leq |\Lambda|$.*

Proof. By Proposition 1.2(b) we have $|M| \leq |P(\mathsf{Fac}\,M)|$. By Proposition 1.1(e), we have $|P(\mathsf{Fac}\,M)| = |\Lambda/\mathrm{ann}\,M|$. Since $|\Lambda/\mathrm{ann}\,M| \leq |\Lambda|$, we have the assertion. $\qquad\square$

As an immediate consequence, if τ-rigid Λ-modules M and N satisfy $M \in \mathsf{add}\,N$ and $|M| \geq |\Lambda|$, then $\mathsf{add}\,M = \mathsf{add}\,N$.

Finally we note the following relationship between τ-tilting modules and classical notions.

Proposition 1.4. [ASS, VIII.5.1]
  (a) *Any faithful τ-rigid Λ-module is a partial tilting Λ-module.*
  (b) *Any faithful τ-tilting Λ-module is a tilting Λ-module.*

1.3. **Silting complexes.** Let Λ be a finite dimensional K-algebra and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ be the category of bounded complexes of finitely generated projective Λ-modules. We recall the definition of silting complexes and mutations.

Definition 1.5. [AI, Ai, BRT, KV] Let $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.
  (a) We call P *presilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, P[i]) = 0$ for any $i > 0$.
  (b) We call P *silting* if it is presilting and satisfies $\mathsf{thick}\, P = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$, where $\mathsf{thick}\, P$ is the smallest full subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ which contains P and is closed under cones, $[\pm 1]$, direct summands and isomorphisms.

We denote by $\mathrm{silt}\,\Lambda$ the set of isomorphism classes of basic silting complexes for Λ.

The following result is important.

Proposition 1.6. [AI, Theorem 2.27, Corollary 2.28]
  (a) *For any $P \in \mathrm{silt}\,\Lambda$, we have $|P| = |\Lambda|$.*
  (b) *Let $P = \bigoplus_{i=1}^{n} P_n$ be a basic silting complex for Λ with P_i indecomposable. Then $P_1, \cdots, P_n$ give a basis of the Grothendieck group $K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda))$.*

We call a presilting complex P for Λ *almost complete silting* if $|P| = |\Lambda| - 1$. There is a similar type of mutation as for tilting modules.

Definition-Proposition 1.7. [AI, Theorem 2.31] *Let $P = X \oplus Q$ be a basic silting complex with X indecomposable. We consider a triangle*

$$X \xrightarrow{\ f\ } Q' \longrightarrow Y \longrightarrow X[1]$$

with a minimal left $(\mathsf{add}\, Q)$-approximation f of X. Then the left mutation of P with respect to X is $\mu_X^-(P) := Y \oplus Q$. Dually we define the right mutation $\mu_X^+(P)$ of P with respect to X.[1] Then the left mutation and the right mutation of P are also basic silting complexes.

There is the following partial order on the set $\mathrm{silt}\,\Lambda$.

Definition-Proposition 1.8. [AI, Theorem 2.11, Proposition 2.14] *For $P, Q \in \mathrm{silt}\,\Lambda$, we write*

$$P \geq Q$$

if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, Q[i]) = 0$ for any $i > 0$, which is equivalent to $P^{\perp > 0} \supseteq Q^{\perp > 0}$ where $P^{\perp > 0}$ is a subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ consisting of the X satisfying $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, X[i]) = 0$ for any $i > 0$. Then we have a partial order on $\mathrm{silt}\,\Lambda$.

We define the *silting quiver* $\mathrm{Q}(\mathrm{silt}\,\Lambda)$ of Λ as follows:
  • The set of vertices is $\mathrm{silt}\,\Lambda$.
  • We draw an arrow from P to Q if Q is a left mutation of P.

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[1]These notations μ^- and μ^+ are the opposite of those in [AI]. They are easy to remember since they are the same direction as τ^{-1} and τ.

Then the silting quiver gives the Hasse quiver of the partially ordered set silt Λ by [AI, Theorem 2.35], similar to the situation for tilting modules. We shall later restrict to two-term silting complexes to get exactly two complements for almost complete silting complexes.

1.4. **Cluster-tilting objects.** Let $\mathcal{C}$ be a K-linear Hom-finite Krull-Schmidt triangulated category. Assume that $\mathcal{C}$ is *2-Calabi-Yau* (*2-CY* for short) i.e. there exists a functorial isomorphism $D\operatorname{Ext}_{\mathcal{C}}^1(X,Y) \simeq \operatorname{Ext}_{\mathcal{C}}^1(Y,X)$. An important class of objects in these categories are the cluster-tilting objects. We recall the definition of these and related objects.

Definition 1.9. (a) We call T in $\mathcal{C}$ *rigid* if $\operatorname{Hom}_{\mathcal{C}}(T, T[1]) = 0$.
   (b) We call T in $\mathcal{C}$ *cluster-tilting* if $\operatorname{add} T = \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(T, X[1]) = 0\}$.
   (c) We call T in $\mathcal{C}$ *maximal rigid* if it is rigid and maximal with respect to this property, that is, $\operatorname{add} T = \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(T \oplus X, (T \oplus X)[1]) = 0\}$.

We denote by c-tilt$\mathcal{C}$ the set of isomorphism classes of basic cluster-tilting objects in $\mathcal{C}$. In this setting, there are also mutations of cluster-tilting objects defined via approximations, which we recall [BMRRT, IY].

Definition-Proposition 1.10. [IY, Theorem 5.3] *Let $T = X \oplus U$ be a basic cluster-tilting object in $\mathcal{C}$ and X indecomposable in $\mathcal{C}$. We consider the triangle*

$$X \xrightarrow{\;f\;} U' \longrightarrow Y \longrightarrow X[1]$$

with a minimal left $(\operatorname{add} U)$-approximation f of X. Let $\mu_X^-(T) := Y \oplus U$. Dually we define $\mu_X^+(T)$. A different feature in this case is that we have $\mu_X^-(T) \simeq \mu_X^+(T)$. This is a basic cluster-tilting object which as before we call the mutation of T with respect to X.

In this case we get just a graph rather than a quiver. We define the *cluster-tilting graph* $\mathrm{G}(\text{c-tilt}\mathcal{C})$ of $\mathcal{C}$ as follows:

- The set of vertices is c-tilt$\mathcal{C}$.
- We draw an edge between T and U if U is a mutation of T.

Note that U is a mutation of T if and only if T and U have all but one indecomposable direct summand in common [IY, Theorem 5.3] (see Corollary 4.5(a)).

2. Support τ-tilting modules

Our aim in this section is to develop a basic theory of support τ-tilting modules over any finite dimensional K-algebra. We start with discussing some basic properties of τ-rigid modules and connections between τ-rigid modules and functorially finite torsion classes (Theorem 2.7). As an application, we introduce Bongartz completion of τ-rigid modules (Theorem 2.10). Then we give characterizations of τ-tilting modules (Theorem 2.12). We also give left-right duality of τ-rigid modules (Theorem 2.14). Further we prove our main result which states that an almost complete support τ-tilting module has exactly two complements (Theorem 2.18). As an application, we introduce mutation of support τ-tilting modules. We show that mutation gives the Hasse quiver of the partially ordered set of support τ-tilting modules (Theorem 2.33).

2.1. **Basic properties of τ-rigid modules.** When T is a Λ-module with I an ideal contained in $\operatorname{ann} T$, we investigate the relationship between T being τ-rigid as a Λ-module and as a (Λ/I)-module. We have the following.

Lemma 2.1. *Let Λ be a finite dimensional algebra, and I an ideal in Λ. Let M and N be (Λ/I)-modules. Then we have the following.*

(a) *If $\operatorname{Hom}_\Lambda(N, \tau M) = 0$, then $\operatorname{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.*

(b) *Assume $I = \langle e \rangle$ for an idempotent e in Λ. Then $\operatorname{Hom}_\Lambda(N, \tau M) = 0$ if and only if $\operatorname{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.*

Proof. Note that we have a natural inclusion $\operatorname{Ext}^1_{\Lambda/I}(M, N) \to \operatorname{Ext}^1_\Lambda(M, N)$. This is an isomorphism if $I = \langle e \rangle$ for an idempotent e since $\mathsf{mod}(\Lambda/\langle e \rangle)$ is closed under extensions in $\mathsf{mod}\,\Lambda$.

(a) Assume $\operatorname{Hom}_\Lambda(N, \tau M) = 0$. Then by Proposition 1.2, we have $\operatorname{Ext}^1_\Lambda(M, \mathsf{Fac}\,N) = 0$. By the above observation, we have $\operatorname{Ext}^1_{\Lambda/I}(M, \mathsf{Fac}\,N) = 0$. By Proposition 1.2 again, we have $\operatorname{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.

(b) Assume that $I = \langle e \rangle$ and $\operatorname{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$. By Proposition 1.2, we have $\operatorname{Ext}^1_{\Lambda/I}(M, \mathsf{Fac}\,N) = 0$. By the above observation, we have $\operatorname{Ext}^1_\Lambda(M, \mathsf{Fac}\,N) = 0$. By Proposition 1.2 again, we have $\operatorname{Hom}_\Lambda(N, \tau M) = 0$. $\square$

Recall that M in $\mathsf{mod}\,\Lambda$ is *sincere* if every simple Λ-module appears as a composition factor in M. This is equivalent to the fact that there does not exist a non-zero idempotent e of Λ which annihilates M.

Proposition 2.2.   (a) *τ-tilting modules are precisely sincere support τ-tilting modules.*

(b) *Tilting modules are precisely faithful support τ-tilting modules.*

(c) *Any τ-tilting (respectively, τ-rigid) Λ-module T is a tilting (respectively, partial tilting) $(\Lambda/\operatorname{ann} T)$-module.*

Proof. (a) Clearly sincere support τ-tilting modules are τ-tilting. Conversely, if a τ-tilting Λ-module T is not sincere, then there exists a non-zero idempotent e of Λ such that T is a $(\Lambda/\langle e \rangle)$-module. Since T is τ-rigid as a $(\Lambda/\langle e \rangle)$-module by Lemma 2.1(a), we have $|T| = |\Lambda| > |\Lambda/\langle e \rangle|$, a contradiction to Proposition 1.3.

(b) Clearly tilting modules are faithful τ-tilting. Conversely, any faithful support τ-tilting module T is partial tilting by Proposition 1.4 and satisfies $|T| = |\Lambda|$. Thus T is tilting.

(c) By Lemma 2.1(a), we know that T is a faithful τ-tilting (respectively, τ-rigid) $(\Lambda/\operatorname{ann} T)$-module. Thus the assertion follows from (b) (respectively, Proposition 1.4). $\square$

Immediately we have the following basic observation, which will be used frequently in this paper.

Proposition 2.3. *Let (M, P) be a pair with $M \in \mathsf{mod}\,\Lambda$ and $P \in \mathsf{proj}\,\Lambda$. Let e be an idempotent of Λ such that $\mathsf{add}\,P = \mathsf{add}\,e\Lambda$.*

(a) *(M, P) is a τ-rigid (respectively, support τ-tilting, almost complete support τ-tilting) pair for Λ if and only if M is a τ-rigid (respectively, τ-tilting, almost complete τ-tilting) $(\Lambda/\langle e \rangle)$-module.*

(b) *If (M, P) and (M, Q) are support τ-tilting pairs for Λ, then $\mathsf{add}\,P = \mathsf{add}\,Q$. In other words, M determines P and e uniquely.*

Proof. (a) The assertions follow from Lemma 2.1 and the equation $|\Lambda/\langle e \rangle| = |\Lambda| - |P|$.

(b) This is a consequence of Proposition 2.2(a). $\square$

The following observations are useful.

Proposition 2.4. *Let X be in $\mathsf{mod}\,\Lambda$ with a minimal projective presentation $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0$.*

(a) *For Y in $\mathsf{mod}\,\Lambda$, we have an exact sequence*

$$0 \to \mathrm{Hom}_\Lambda(Y, \tau X) \to D\,\mathrm{Hom}_\Lambda(P_1, Y) \xrightarrow{D(d_1, Y)} D\,\mathrm{Hom}_\Lambda(P_0, Y) \xrightarrow{D(d_0, Y)} D\,\mathrm{Hom}_\Lambda(X, Y) \to 0.$$

(b) *$\mathrm{Hom}_\Lambda(Y, \tau X) = 0$ if and only if the map $\mathrm{Hom}_\Lambda(P_0, Y) \xrightarrow{(d_1, Y)} \mathrm{Hom}_\Lambda(P_1, Y)$ is surjective.*

(c) *X is τ-rigid if and only if the map $\mathrm{Hom}_\Lambda(P_0, X) \xrightarrow{(d_1, X)} \mathrm{Hom}_\Lambda(P_1, X)$ is surjective.*

Proof. (a) We have an exact sequence $0 \to \tau X \to \nu P_1 \xrightarrow{\nu d_1} \nu P_0$. Applying $\mathrm{Hom}_\Lambda(Y, -)$, we have a commutative diagram of exact sequences:

$$
\begin{array}{ccccc}
0 \to \mathrm{Hom}_\Lambda(Y, \tau X) \longrightarrow & \mathrm{Hom}_\Lambda(Y, \nu P_1) & \xrightarrow{(Y, \nu d_1)} & \mathrm{Hom}_\Lambda(Y, \nu P_0) & \\
& \downarrow \wr & & \downarrow \wr & \\
& D\,\mathrm{Hom}_\Lambda(P_1, Y) \xrightarrow{D(d_1, Y)} & D\,\mathrm{Hom}_\Lambda(P_0, Y) & \xrightarrow{D(d_0, Y)} & D\,\mathrm{Hom}_\Lambda(X, Y) \to 0.
\end{array}
$$

Thus the assertion follows.

(b)(c) Immediate from (a). □

We have the following standard observation (cf. [HU2, DeKe]).

Proposition 2.5. *Let X be in $\mathsf{mod}\,\Lambda$ with a minimal projective presentation $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0$. If X is τ-rigid, then P_0 and P_1 have no non-zero direct summands in common.*

Proof. We only have to show that any morphism $s : P_1 \to P_0$ is in the radical. By Proposition 2.4(c), there exists $t : P_0 \to X$ such that $d_0 s = t d_1$. Since P_0 is projective, there exists $u : P_0 \to P_0$ such that $t = d_0 u$. Since $d_0(s - u d_1) = 0$, there exists $v : P_1 \to P_1$ such that $s = u d_1 + d_1 v$.

$$
\begin{array}{ccccccc}
& P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & X & \longrightarrow 0 \\
{\scriptstyle v}\,\swarrow & {\scriptstyle s}\downarrow & {\scriptstyle u}\,\swarrow & {\scriptstyle}\downarrow & & {\scriptstyle t}\downarrow & \\
P_1 & \xrightarrow[d_1]{} & P_0 & \xrightarrow[d_0]{} & X & \longrightarrow 0 &
\end{array}
$$

Since d_1 is in the radical, so is s. Thus the assertion is shown. □

The following analog of Wakamatsu's lemma [AR2] will be useful.

Lemma 2.6. *Let $\eta : 0 \to Y \to T' \xrightarrow{f} X$ be an exact sequence in $\mathsf{mod}\,\Lambda$, where T is τ-rigid, and $f : T' \to X$ is a right $(\mathsf{add}\,T)$-approximation. Then we have $Y \in {}^{\perp}(\tau T)$.*

Proof. Replacing X by $\mathrm{Im}\,f$, we can assume that f is surjective. We apply $\mathrm{Hom}_\Lambda(-, \tau T)$ to η to get the exact sequence

$$0 = \mathrm{Hom}_\Lambda(T', \tau T) \to \mathrm{Hom}_\Lambda(Y, \tau T) \to \mathrm{Ext}^1_\Lambda(X, \tau T) \xrightarrow{\mathrm{Ext}^1(f, \tau T)} \mathrm{Ext}^1_\Lambda(T', \tau T),$$

where we have $\mathrm{Hom}_\Lambda(T', \tau T) = 0$ because T is τ-rigid. Since $f : T' \to X$ is a right $(\mathsf{add}\,T)$-approximation, the induced map $(T, f) : \mathrm{Hom}_\Lambda(T, T') \to \mathrm{Hom}_\Lambda(T, X)$ is surjective. Then also the induced map $\underline{\mathrm{Hom}}_\Lambda(T, T') \to \underline{\mathrm{Hom}}_\Lambda(T, X)$ of the maps modulo projectives is

surjective, so by the AR duality the map $\mathrm{Ext}^1(f, \tau T) : \mathrm{Ext}^1_\Lambda(X, \tau T) \to \mathrm{Ext}^1_\Lambda(T', \tau T)$ is injective. It follows that $\mathrm{Hom}_\Lambda(Y, \tau T) = 0$. $\square$

2.2. **τ-rigid modules and torsion classes.** The following correspondence is basic in our paper, where we denote by f-torsΛ the set of functorially finite torsion classes in $\mathsf{mod}\,\Lambda$.

Theorem 2.7. *There is a bijection*

$$\mathrm{s}\tau\text{-tilt}\,\Lambda \longleftrightarrow \text{f-tors}\,\Lambda$$

given by $\mathrm{s}\tau\text{-tilt}\,\Lambda \ni T \mapsto \mathsf{Fac}\,T \in \text{f-tors}\,\Lambda$ *and* $\text{f-tors}\,\Lambda \ni \mathcal{T} \mapsto P(\mathcal{T}) \in \mathrm{s}\tau\text{-tilt}\,\Lambda$.

Proof. Let first $\mathcal{T}$ be a functorially finite torsion class in $\mathsf{mod}\,\Lambda$. Then we know that $T = P(\mathcal{T})$ is τ-rigid by Proposition 1.2(c). Let $e \in \Lambda$ be a maximal idempotent such that $\mathcal{T} \subseteq \mathsf{mod}(\Lambda/\langle e \rangle)$. Then we have $|\Lambda/\langle e \rangle| = |\Lambda/\operatorname{ann}\mathcal{T}|$, and $|\Lambda/\operatorname{ann}\mathcal{T}| = |T|$ by Proposition 1.1(e). Hence $(T, e\Lambda)$ is a support τ-tilting pair for Λ. Moreover we have $\mathcal{T} = \mathsf{Fac}\,P(\mathcal{T})$ by Proposition 1.1(g).

Assume conversely that T is a support τ-tilting Λ-module. Then T is a τ-tilting $(\Lambda/\langle e \rangle)$-module for an idempotent e of Λ. Thus $\mathsf{Fac}\,T$ is a functorially finite torsion class in $\mathsf{mod}(\Lambda/\langle e \rangle)$ such that $T \in \mathsf{add}\,P(\mathsf{Fac}\,T)$ by Proposition 1.2(b). Since $|T| = |\Lambda/\langle e \rangle|$, we have $\mathsf{add}\,T = \mathsf{add}\,P(\mathsf{Fac}\,T)$ by Proposition 1.3. Thus $T \simeq P(\mathsf{Fac}\,T)$. $\square$

We denote by τ-tilt$\,\Lambda$ (respectively, tilt$\,\Lambda$) the set of isomorphism classes of basic τ-tilting Λ-modules (respectively, tilting Λ-modules). On the other hand, we denote by sf-torsΛ (respectively, ff-torsΛ) the set of sincere (respectively, faithful) functorially finite torsion classes in $\mathsf{mod}\,\Lambda$.

Corollary 2.8. *The bijection in Theorem 2.7 induces bijections*

$$\tau\text{-tilt}\,\Lambda \longleftrightarrow \text{sf-tors}\,\Lambda \quad \text{and} \quad \text{tilt}\,\Lambda \longleftrightarrow \text{ff-tors}\,\Lambda.$$

Proof. Let T be a support τ-tilting Λ-module. By Proposition 2.2, it follows that T is a τ-tilting Λ-module (respectively, tilting Λ-module) if and only if T is sincere (respectively, faithful) if and only if $\mathsf{Fac}\,T$ is sincere (respectively, faithful). $\square$

We are interested in the torsion classes where our original module U is a direct summand of $T = P(\mathcal{T})$, since we would like to complete U to a (support) τ-tilting module. The conditions for this to be the case are the following.

Proposition 2.9. *Let $\mathcal{T}$ be a functorially finite torsion class and U a τ-rigid Λ-module. Then $U \in \mathsf{add}\,P(\mathcal{T})$ if and only if $\mathsf{Fac}\,U \subseteq \mathcal{T} \subseteq {}^\perp(\tau U)$.*

Proof. We have $\mathcal{T} = \mathsf{Fac}\,P(\mathcal{T})$ by Proposition 1.1(g).

Assume $\mathsf{Fac}\,U \subseteq \mathcal{T} \subseteq {}^\perp(\tau U)$. Then U is in $\mathcal{T}$. We want to show that U is Ext-projective in $\mathcal{T}$, that is, $\mathrm{Ext}^1_\Lambda(U, \mathcal{T}) = 0$, or equivalently $\mathrm{Hom}_\Lambda(P(\mathcal{T}), \tau U) = 0$, by Proposition 1.2(a). This follows since $P(\mathcal{T}) \in \mathcal{T} \subseteq {}^\perp(\tau U)$. Hence U is a direct summand of $P(\mathcal{T})$.

Conversely, assume $U \in \mathsf{add}\,P(\mathcal{T})$. Then we must have $U \in \mathcal{T}$, and hence $\mathsf{Fac}\,U \subseteq \mathcal{T}$. Since U is Ext-projective in $\mathcal{T}$, we have $\mathrm{Ext}^1_\Lambda(U, \mathcal{T}) = 0$. Since $\mathcal{T} = \mathsf{Fac}\,\mathcal{T}$, we have $\mathrm{Hom}_\Lambda(\mathcal{T}, \tau U) = 0$ by Proposition 1.2(a). Hence we have $\mathcal{T} \subseteq {}^\perp(\tau U)$. $\square$

We now prove the analog, for τ-tilting modules, of the Bongartz completion of classical tilting modules.

Theorem 2.10. *Let U be a τ-rigid Λ-module. Then $\mathcal{T} := {}^\perp(\tau U)$ is a sincere functorially finite torsion class and $T := P(\mathcal{T})$ is a τ-tilting Λ-module satisfying $U \in \mathsf{add}\,T$ and ${}^\perp(\tau T) = \mathsf{Fac}\,T$.*

We call $P(^{\perp}(\tau U))$ the *Bongartz completion* of U.

Proof. The first part follows from the following observation.

Lemma 2.11. *For any τ-rigid Λ-module U, we have a sincere functorially finite torsion class $^{\perp}(\tau U)$.*

Proof. When U is τ-rigid, then $\mathsf{Sub}\tau U$ is a torsionfree class by the dual of Proposition 1.2(b). Then $(^{\perp}(\tau U), \mathsf{Sub}\tau U)$ is a torsion pair, and $\mathsf{Sub}\tau U$ and $^{\perp}(\tau U)$ are functorially finite by Proposition 1.1.

Assume that $^{\perp}(\tau U)$ is not sincere. Then we have $^{\perp}(\tau U) \subseteq \mathsf{mod}(\Lambda/\langle e \rangle)$ for some primitive idempotent e in Λ. The corresponding simple Λ-module S is not a composition factor of any module in $^{\perp}(\tau U)$; in particular $\mathrm{Hom}(^{\perp}(\tau U), D(\Lambda e)) = 0$. Then $D(\Lambda e)$ is in $\mathsf{Sub}\tau U$. But this is a contradiction since τU, and hence also any module in $\mathsf{Sub}\tau U$, has no nonzero injective direct summands. $\qquad\square$

By Corollary 2.8, it follows that T is a τ-tilting Λ-module such that $^{\perp}(\tau U) = \mathsf{Fac}T$. By Proposition 2.9, we have $U \in \mathsf{add}T$. Clearly $^{\perp}(\tau U) \supseteq {}^{\perp}(\tau T)$ since U is in $\mathsf{add}T$. Hence we get $\mathsf{Fac}T = {}^{\perp}(\tau U) \supseteq {}^{\perp}(\tau T) \supseteq \mathsf{Fac}T$, and consequently $^{\perp}(\tau T) = \mathsf{Fac}T$. $\qquad\square$

We have the following characterizations of a τ-rigid module being τ-tilting.

Theorem 2.12. *The following are equivalent for a τ-rigid Λ-module T.*
  (a) *T is τ-tilting.*
  (b) *T is maximal τ-rigid, i.e. if $T \oplus X$ is τ-rigid for some Λ-module X, then $X \in \mathsf{add}T$.*
  (c) *$^{\perp}(\tau T) = \mathsf{Fac}T$.*
  (d) *If $\mathrm{Hom}_{\Lambda}(T, \tau X) = 0$ and $\mathrm{Hom}_{\Lambda}(X, \tau T) = 0$, then $X \in \mathsf{add}T$.*

Proof. (a)$\Rightarrow$(b): Immediate from Proposition 1.3.

(b)$\Rightarrow$(c): Let U be the Bongartz completion of T. Since T is maximal τ-rigid, we have $T \simeq U$, and hence $^{\perp}(\tau T) = {}^{\perp}(\tau U) = \mathsf{Fac}U = \mathsf{Fac}T$, using Theorem 2.10.

(c)$\Rightarrow$(a): Let T be τ-rigid with $^{\perp}(\tau T) = \mathsf{Fac}T$. Let U be the Bongartz completion of T. Then we have

$$\mathsf{Fac}T = {}^{\perp}(\tau T) \supseteq {}^{\perp}(\tau U) \supseteq \mathsf{Fac}U \supseteq \mathsf{Fac}T,$$

and hence all inclusions are equalities. Since $\mathsf{Fac}U = \mathsf{Fac}T$, there exists an exact sequence

$$0 \longrightarrow Y \longrightarrow T' \overset{f}{\longrightarrow} U \longrightarrow 0 \qquad\qquad (1)$$

where $f : T' \to U$ is a right $(\mathsf{add}T)$-approximation. By the Wakamatsu-type Lemma 2.6 we have $\mathrm{Hom}_{\Lambda}(Y, \tau T) = 0$, and hence $\mathrm{Hom}_{\Lambda}(Y, \tau U) = 0$ since $^{\perp}(\tau T) = {}^{\perp}(\tau U)$. By the AR duality we have $\mathrm{Ext}^1_{\Lambda}(U, Y) \simeq D\overline{\mathrm{Hom}}_{\Lambda}(Y, \tau U) = 0$, and hence the sequence (1) splits. Then it follows that U is in $\mathsf{add}T$. Thus T is a τ-tilting Λ-module.

(a)+(c)$\Rightarrow$(d): Assume that (a) and (c) hold, and $\mathrm{Hom}_{\Lambda}(T, \tau X) = 0$ and $\mathrm{Hom}_{\Lambda}(X, \tau T) = 0$. Then $\mathrm{Ext}^1_{\Lambda}(X, \mathsf{Fac}T) = 0$ by Proposition 1.2(a) and X is in $^{\perp}\tau T = \mathsf{Fac}T$. Thus X is in $\mathsf{add}P(\mathsf{Fac}T) = \mathsf{add}T$ by Theorem 2.7.

(d)$\Rightarrow$(b): This is clear. $\qquad\square$

We note the following generalization.

Corollary 2.13. *The following are equivalent for a τ-rigid pair (T, P) for Λ.*
  (a) *(T, P) is a support τ-tilting pair for Λ.*
  (b) *If $(T \oplus X, P)$ is τ-rigid for some Λ-module X, then $X \in \mathsf{add}T$.*

(c) $^{\perp}(\tau T) \cap P^{\perp} = \mathsf{Fac}\, T$.
(d) *If* $\mathrm{Hom}_\Lambda(T, \tau X) = 0$, $\mathrm{Hom}_\Lambda(X, \tau T) = 0$ *and* $\mathrm{Hom}_\Lambda(P, X) = 0$, *then* $X \in \mathsf{add}\, T$.

Proof. In view of Lemma 2.1(b), the assertion follows immediately from Theorem 2.12 by replacing Λ by $\Lambda/\langle e \rangle$ for an idempotent e of Λ satisfying $\mathsf{add}\, P = \mathsf{add}\, e\Lambda$. $\qquad \square$

In the rest of this subsection, we discuss the left-right symmetry of τ-rigid modules. It is somehow surprising that there exists a bijection between support τ-tilting Λ-modules and support τ-tilting Λ^{op}-modules. We decompose M in $\mathsf{mod}\,\Lambda$ as $M = M_{\mathrm{pr}} \oplus M_{\mathrm{np}}$ where M_{pr} is a maximal projective direct summand of M. For a τ-rigid pair (M, P) for Λ, let

$$(M, P)^\dagger := (\mathrm{Tr}\, M_{\mathrm{np}} \oplus P^*, M_{\mathrm{pr}}^*) = (\mathrm{Tr}\, M \oplus P^*, M_{\mathrm{pr}}^*).$$

We denote by τ-rigid Λ the set of isomorphism classes of basic τ-rigid pairs of Λ.

Theorem 2.14. $(-)^\dagger$ *gives bijections*

$$\tau\text{-rigid}\, \Lambda \longleftrightarrow \tau\text{-rigid}\, \Lambda^{\mathrm{op}} \quad and \quad \mathrm{s}\tau\text{-tilt}\, \Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}\, \Lambda^{\mathrm{op}}$$

such that $(-)^{\dagger\dagger} = \mathrm{id}$.

For a support τ-tilting Λ-module M, we simply write $M^\dagger := \mathrm{Tr}\, M_{\mathrm{np}} \oplus P^*$ where (M, P) is a support τ-tilting pair for Λ.

Proof. We only have to show that $(M, P)^\dagger$ is a τ-rigid pair for Λ^{op} since the correspondence $(M, P) \mapsto (M, P)^\dagger$ is clearly an involution. We have

$$0 = \mathrm{Hom}_\Lambda(M_{\mathrm{np}}, \tau M) = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, DM_{\mathrm{np}}) = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \tau\, \mathrm{Tr}\, M). \qquad (2)$$

Moreover we have

$$0 = \mathrm{Hom}_\Lambda(M_{\mathrm{pr}}, \tau M) = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, DM_{\mathrm{pr}}) = D\, \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(M_{\mathrm{pr}}^*, \mathrm{Tr}\, M). \qquad (3)$$

On the other hand, we have

$$0 = \mathrm{Hom}_\Lambda(P, M) = \mathrm{Hom}_\Lambda(P, M_{\mathrm{pr}}) \oplus \mathrm{Hom}_\Lambda(P, M_{\mathrm{np}}). \qquad (4)$$

Thus we have

$$0 = D(P^* \otimes_\Lambda M_{\mathrm{np}}) = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^*, DM_{\mathrm{np}}) = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^*, \tau\, \mathrm{Tr}\, M).$$

This together with (2) shows that $\mathrm{Tr}\, M \oplus P^*$ is a τ-rigid Λ^{op}-module. We have $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(M_{\mathrm{pr}}^*, P^*) = 0$ by (4). This together with (3) shows that $(M, P)^\dagger$ is a τ-rigid pair for Λ^{op}. $\qquad \square$

Now we discuss dual notions of τ-rigid and τ-tilting modules even though we do not use them in this paper.

- We call M in $\mathsf{mod}\,\Lambda$ τ^--*rigid* if $\mathrm{Hom}_\Lambda(\tau^- M, M) = 0$.
- We call M in $\mathsf{mod}\,\Lambda$ τ^--*tilting* if M is τ^--rigid and $|M| = |\Lambda|$.
- We call M in $\mathsf{mod}\,\Lambda$ *support* τ^--*tilting* if M is a τ^--tilting $(\Lambda/\langle e \rangle)$-module for some idempotent e of Λ.

Clearly M is τ^--rigid (respectively, τ^--tilting, support τ^--tilting) Λ-module if and only if DM is τ-rigid (respectively, τ-tilting, support τ-tilting) Λ^{op}-module.

We denote by $\mathrm{cotilt}\,\Lambda$ (respectively, τ^--tilt$\,\Lambda$, $\mathrm{s}\tau^-$-tilt$\,\Lambda$) the set of isomorphism classes of basic cotilting (respectively, τ^--tilting, support τ^--tilting) Λ-modules. On the other hand, we denote by f-torf$\,\Lambda$ the set of functorially finite torsionfree classes in $\mathsf{mod}\,\Lambda$, and by sf-torf$\,\Lambda$ (respectively, ff-torf$\,\Lambda$) the set of sincere (respectively, faithful) functorially finite torsionfree classes in $\mathsf{mod}\,\Lambda$. We have the following results immediately from Theorem 2.7 and Corollary 2.8.

Theorem 2.15. *We have bijections*

$$\mathrm{s}\tau^-\text{-tilt}\,\Lambda \longleftrightarrow \mathrm{f\text{-}torf}\,\Lambda, \quad \tau^-\text{-tilt}\,\Lambda \longleftrightarrow \mathrm{sf\text{-}torf}\,\Lambda \quad and \quad \mathrm{cotilt}\,\Lambda \longleftrightarrow \mathrm{ff\text{-}torf}\,\Lambda$$

given by $\mathrm{s}\tau^-\text{-tilt}\,\Lambda \ni T \mapsto \mathsf{Sub}\,T \in \mathrm{f\text{-}torf}\,\Lambda$ *and* $\mathrm{f\text{-}torf}\,\Lambda \ni \mathcal{F} \mapsto I(\mathcal{F}) \in \mathrm{s}\tau^-\text{-tilt}\,\Lambda$.

On the other hand, we have a bijection

$$\mathrm{s}\tau\text{-tilt}\,\Lambda \longleftrightarrow \mathrm{s}\tau^-\text{-tilt}\,\Lambda$$

given by $(M, P) \mapsto D((M, P)^\dagger) = (\tau M \oplus \nu P, \nu M_{\mathrm{pr}})$. Thus we have bijections

$$\mathrm{f\text{-}tors}\,\Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda \longleftrightarrow \mathrm{s}\tau^-\text{-tilt}\,\Lambda \longleftrightarrow \mathrm{f\text{-}torf}\,\Lambda$$

by Theorems 2.7 and 2.15. We end this subsection with the following observation.

Proposition 2.16. (a) *The above bijections send* $\mathcal{T} \in \mathrm{f\text{-}tors}\,\Lambda$ *to* $\mathcal{T}^\perp \in \mathrm{f\text{-}torf}\,\Lambda$.
(b) *For any support τ-tilting pair (M, P) for Λ, the torsion pairs $(\mathsf{Fac}\,M, M^\perp)$ and $(^\perp(\tau M \oplus \nu P), \mathsf{Sub}(\tau M \oplus \nu P))$ in $\mathrm{mod}\,\Lambda$ coincide.*

Proof. (b) We only have to show $\mathsf{Fac}\,M = {}^\perp(\tau M \oplus \nu P)$. It follows from Proposition 1.2(b) and its dual that $(\mathsf{Fac}\,M, M^\perp)$ and $(^\perp(\tau M \oplus \nu P), \mathsf{Sub}(\tau M \oplus \nu P))$ are torsion pairs in $\mathrm{mod}\,\Lambda$. They coincide since $\mathsf{Fac}\,M = {}^\perp(\tau M) \cap P^\perp = {}^\perp(\tau M \oplus \nu P)$ holds by Corollary 2.13(c).

(a) Let $\mathcal{T} \in \mathrm{f\text{-}tors}\,\Lambda$ and (M, P) be the corresponding support τ-tilting pair for Λ. Since $\mathcal{T}^\perp = M^\perp$ and $D(M^\dagger) = \tau M \oplus \nu P$, the assertion follows from (b). $\square$

2.3. **Mutation of support τ-tilting modules.** In this section we prove our main result on complements for almost complete support τ-tilting pairs. Let us start with the following result.

Proposition 2.17. *Let T be a basic τ-rigid module which is not τ-tilting. Then there are at least two basic support τ-tilting modules which have T as a direct summand.*

Proof. By Theorem 2.12, $\mathcal{T}_1 = \mathsf{Fac}\,T$ is properly contained in $\mathcal{T}_2 = {}^\perp(\tau T)$. By Theorem 2.7 and Lemma 2.11, we have two different support τ-tilting modules $P(\mathcal{T}_1)$ and $P(\mathcal{T}_2)$ up to isomorphism. By Proposition 2.9, they are extensions of T. $\square$

Our aim is to prove the following result.

Theorem 2.18. *Let Λ be a finite dimensional K-algebra. Then any basic almost complete support τ-tilting pair (U, Q) for Λ is a direct summand of exactly two basic support τ-tilting pairs (T, P) and (T', P') for Λ. Moreover we have $\{\mathsf{Fac}\,T, \mathsf{Fac}\,T'\} = \{\mathsf{Fac}\,U, {}^\perp(\tau U) \cap Q^\perp\}$.*

Before proving Theorem 2.18, we introduce a notion of mutation.

Definition 2.19. Two basic support τ-tilting pairs (T, P) and (T', P') for Λ are said to be *mutations* of each other if there exists a basic almost complete support τ-tilting pair (U, Q) which is a direct summand of (T, P) and (T', P'). In this case we write $(T', P') = \mu_X(T, P)$ or simply $T' = \mu_X(T)$ if X is an indecomposable Λ-module satisfying either $T = U \oplus X$ or $P = Q \oplus X$.

We can also describe mutation as follows: Let (T, P) be a basic support τ-tilting pair for Λ, and X an indecomposable direct summand of either T or P.
(a) If X is a direct summand of T, precisely one of the following holds.
  - There exists an indecomposable Λ-module Y such that $X \not\cong Y$ and $\mu_X(T, P) := (T/X \oplus Y, P)$ is a basic support τ-tilting pair for Λ.

- There exists an indecomposable projective Λ-module Y such that $\mu_X(T, P) :=$ $(T/X, P \oplus Y)$ is a basic support τ-tilting pair for Λ.

(b) If X is a direct summand of P, there exists an indecomposable Λ-module Y such that $\mu_X(T, P) := (T \oplus Y, P/X)$ is a basic support τ-tilting pair for Λ.

Moreover, such a module Y in each case is unique up to isomorphism.

In the rest of this subsection, we give a proof of Theorem 2.18. The following is the first step.

Lemma 2.20. *Let (T, P) be a τ-rigid pair for Λ. If U is a τ-rigid Λ-module satisfying $^{\perp}(\tau T) \cap P^{\perp} \subseteq {}^{\perp}(\tau U)$, then there is an exact sequence $U \xrightarrow{f} T' \to C \to 0$ satisfying the following conditions.*

- *f is a minimal left $(\mathsf{Fac}\,T)$-approximation.*
- *T' is in $\mathsf{add}\,T$, C is in $\mathsf{add}\,P(\mathsf{Fac}\,T)$ and $\mathsf{add}\,T' \cap \mathsf{add}\,C = 0$.*

Proof. Consider the exact sequence $U \xrightarrow{f} T' \xrightarrow{g} C \to 0$, where f is a minimal left $(\mathsf{add}\,T)$-approximation. Then $g \in \mathrm{rad}(T', C)$.

(i) f is a minimal left $(\mathsf{Fac}\,T)$-approximation: Take any $X \in \mathsf{Fac}\,T$ and $s : U \to X$. By the Wakamatsu-type Lemma 2.6, there exists an exact sequence

$$0 \to Y \to T'' \xrightarrow{h} X \to 0$$

where h is a right $(\mathsf{add}\,T)$-approximation and $Y \in {}^{\perp}(\tau T)$. Moreover we have $Y \in P^{\perp}$ since $T'' \in P^{\perp}$. By the assumption that $^{\perp}(\tau T) \cap P^{\perp} \subseteq {}^{\perp}(\tau U)$, we have $\mathrm{Hom}_{\Lambda}(Y, \tau U) = 0$, hence $\mathrm{Ext}_{\Lambda}^1(U, Y) = 0$. Then we have an exact sequence

$$\mathrm{Hom}_{\Lambda}(U, T'') \to \mathrm{Hom}_{\Lambda}(U, X) \to \mathrm{Ext}_{\Lambda}^1(U, Y) = 0.$$

Thus there is some $t : U \to T''$ such that $s = ht$.

$$
\begin{array}{ccc}
 & & U \xrightarrow{\ f\ } T' \\
 & {\scriptstyle t}\nearrow\!\!\!\diagup & \downarrow{\scriptstyle s} \\
0 \longrightarrow Y \longrightarrow T'' \xrightarrow{\ h\ } X \longrightarrow 0
\end{array}
$$

Since $T'' \in \mathsf{add}\,T$ and f is a left $(\mathsf{add}\,T)$-approximation, there is some $u : T' \to T''$ such that $t = uf$. Hence we have $hu : T' \to X$ such that $(hu)f = ht = s$, and the claim follows.

(ii) $C \in \mathsf{add}\,P(\mathsf{Fac}\,T)$: We have an exact sequence $0 \to \mathrm{Im}\,f \xrightarrow{i} T' \to C \to 0$, which gives rise to an exact sequence

$$\mathrm{Hom}_{\Lambda}(T', \mathsf{Fac}\,T) \xrightarrow{(i, \mathsf{Fac}\,T)} \mathrm{Hom}_{\Lambda}(\mathrm{Im}\,f, \mathsf{Fac}\,T) \to \mathrm{Ext}_{\Lambda}^1(C, \mathsf{Fac}\,T) \to \mathrm{Ext}_{\Lambda}^1(T', \mathsf{Fac}\,T).$$

We know from (i) that $(f, \mathsf{Fac}\,T) : \mathrm{Hom}_{\Lambda}(T', \mathsf{Fac}\,T) \to \mathrm{Hom}_{\Lambda}(U, \mathsf{Fac}\,T)$ is surjective, and hence $(i, \mathsf{Fac}\,T)$ is surjective. Further, $\mathrm{Ext}_{\Lambda}^1(T', \mathsf{Fac}\,T) = 0$ by Proposition 1.2 since T' is in $\mathsf{add}\,T$ and T is τ-rigid. Then it follows that $\mathrm{Ext}_{\Lambda}^1(C, \mathsf{Fac}\,T) = 0$. Since $C \in \mathsf{Fac}\,T$, this means that C is Ext-projective in $\mathsf{Fac}\,T$.

(iii) $\mathsf{add}\,T' \cap \mathsf{add}\,C = 0$: To show this, it is clearly sufficient to show $\mathrm{Hom}_{\Lambda}(T', C) \subseteq \mathrm{rad}(T', C)$.

Let $s : T' \to C$ be an arbitrary map. We have an exact sequence $\mathrm{Hom}_{\Lambda}(U, T') \to \mathrm{Hom}_{\Lambda}(U, C) \to \mathrm{Ext}_{\Lambda}^1(U, \mathrm{Im}\,f)$. Since $\mathrm{Ext}_{\Lambda}^1(U, \mathrm{Im}\,f) = 0$ because $\mathrm{Im}\,f$ is in $\mathsf{Fac}\,U$, and U is τ-tilting, there is a map $t : U \to T'$ such that $sf = gt$. Since f is a left $(\mathsf{add}\,T)$-approximation, and T' is in $\mathsf{add}\,T$, there is a map $u : T' \to T'$ such that $t = uf$. Then

$(s - gu)f = sf - gt = 0$, hence there is some $v : C \to C$ such that $s - gu = vg$, and hence $s = gu + vg$.

Since $g \in \mathrm{rad}(T', C)$, it follows that $s \in \mathrm{rad}(T', C)$. Hence $\mathrm{Hom}_\Lambda(T', C) \subseteq \mathrm{rad}(T', C)$, and consequently $\mathsf{add}\, T' \cap \mathsf{add}\, C = 0$. $\square$

The following information on the previous lemma is useful.

Lemma 2.21. *In Lemma 2.20, assume $C = 0$. Then $f : U \to T'$ induces an isomorphism $U/U\langle e \rangle \simeq T'$ for a maximal idempotent e of Λ satisfying $Te = 0$. In particular, if T is sincere, then $U \simeq T'$.*

Proof. By our assumption, we have an exact sequence

$$0 \longrightarrow \mathrm{Ker}\, f \longrightarrow U \overset{f}{\longrightarrow} T' \longrightarrow 0. \tag{5}$$

Applying $\mathrm{Hom}_\Lambda(-, \mathsf{Fac}\, T)$, we have an exact sequence

$$\mathrm{Hom}_\Lambda(T', \mathsf{Fac}\, T) \overset{(f, \mathsf{Fac}\, T)}{\longrightarrow} \mathrm{Hom}_\Lambda(U, \mathsf{Fac}\, T) \to \mathrm{Hom}_\Lambda(\mathrm{Ker}\, f, \mathsf{Fac}\, T) \to \mathrm{Ext}^1_\Lambda(T', \mathsf{Fac}\, T).$$

We have $\mathrm{Ext}^1_\Lambda(T', \mathsf{Fac}\, T) = 0$ because T' is in $\mathsf{add}\, T$ and T is τ-tilting. Since $(f, \mathsf{Fac}\, T)$ is surjective, it follows that $\mathrm{Hom}_\Lambda(\mathrm{Ker}\, f, \mathsf{Fac}\, T) = 0$ and so $\mathrm{Ker}\, f \in {}^\perp(\mathsf{Fac}\, T)$. On the other hand, since T is a sincere $(\Lambda/\langle e \rangle)$-module, $\mathsf{mod}(\Lambda/\langle e \rangle)$ is the smallest torsionfree class of $\mathsf{mod}\,\Lambda$ containing $\mathsf{Fac}\, T$. Thus we have a torsion pair $({}^\perp(\mathsf{Fac}\, T), \mathsf{mod}(\Lambda/\langle e \rangle))$, and the canonical sequence for X associated with this torsion pair is given by

$$0 \longrightarrow X\langle e \rangle \longrightarrow X \longrightarrow X/X\langle e \rangle \longrightarrow 0.$$

Since $\mathrm{Ker}\, f \in {}^\perp(\mathsf{Fac}\, T)$ and $T' \in \mathsf{Fac}\, T \subseteq \mathsf{mod}(\Lambda/\langle e \rangle)$, the canonical sequence of U is given by (5). Thus we have $U/U\langle e \rangle \simeq T'$. $\square$

In the next result we prove a useful restriction on X when $T = X \oplus U$ is τ-tilting and X is indecomposable.

Proposition 2.22. *Let $T = X \oplus U$ be a basic τ-tilting Λ-module, with X indecomposable. Then exactly one of ${}^\perp(\tau U) \subseteq {}^\perp(\tau X)$ and $X \in \mathsf{Fac}\, U$ holds.*

Proof. First we assume that ${}^\perp(\tau U) \subseteq {}^\perp(\tau X)$ and $X \in \mathsf{Fac}\, U$ both hold. Then we have

$$\mathsf{Fac}\, U = \mathsf{Fac}\, T = {}^\perp(\tau T) = {}^\perp(\tau U),$$

which implies that U is τ-tilting by Theorem 2.12, a contradiction.

Let $Y \oplus U$ be the Bongartz completion of U. Then we have ${}^\perp\tau(Y \oplus U) = {}^\perp(\tau U) \supseteq {}^\perp\tau T$. Using the triple $(T, 0, Y \oplus U)$ instead of (T, P, U) in Lemma 2.20, there is an exact sequence

$$Y \oplus U \overset{\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\longrightarrow} T' \oplus U \longrightarrow T'' \longrightarrow 0,$$

where $f : Y \to T'$ and $\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right) : Y \oplus U \to T' \oplus U$ are minimal left $(\mathsf{Fac}\, T)$-approximations, T' and T'' are in $\mathsf{add}\, T$ and $\mathsf{add}(T' \oplus U) \cap \mathsf{add}\, T'' = 0$. Then we have $T'' \in \mathsf{add}\, X$.

Assume first $T'' \neq 0$. Then $T'' \simeq X^\ell$ for some $\ell \geq 1$, so we have $T' \in \mathsf{add} U$. Since we have a surjective map $T' \to T''$, we have $X \in \mathsf{Fac} T' \subseteq \mathsf{Fac} U$.

Assume now that $T'' = 0$. Applying Lemma 2.21, we have that $\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right) : Y \oplus U \to T' \oplus U$ is an isomorphism since T is sincere. Thus $Y \in \mathsf{add} T$, and we must have $Y \simeq X$. Thus $^\perp(\tau X) = {}^\perp(\tau Y) \supseteq {}^\perp(\tau U)$.   □

Now we are ready to prove Theorem 2.18.

(i) First we assume that $Q = 0$ (i.e. U is an almost complete τ-tilting module).

In view of Proposition 2.17 it only remains to show that there are at most two extensions of U to a support τ-tilting module. Using the bijection in Theorem 2.7, we only have to show that for any support τ-tilting module $X \oplus U$, the torsion class $\mathsf{Fac}(X \oplus U)$ is either $\mathsf{Fac} U$ or $^\perp(\tau U)$. If $X = 0$ (i.e. U is a support τ-tilting module), then this is clear. If $X \neq 0$, then $X \oplus U$ is a τ-tilting Λ-module. Moreover by Proposition 2.22 either $X \in \mathsf{Fac} U$ or $^\perp(\tau U) \subseteq {}^\perp(\tau X)$ holds. If $X \in \mathsf{Fac} U$, then we have $\mathsf{Fac}(X \oplus U) = \mathsf{Fac} U$. If $^\perp(\tau U) \subseteq {}^\perp(\tau X)$, then we have $\mathsf{Fac}(X \oplus U) = {}^\perp(\tau(X \oplus U)) = {}^\perp(\tau U)$. Thus the assertion follows.

(ii) Let (U, Q) be a basic almost complete support τ-tilting pair for Λ and e be an idempotent of Λ such that $\mathsf{add} Q = \mathsf{add} e\Lambda$. Then U is an almost complete τ-tilting $(\Lambda/\langle e \rangle)$-module by Proposition 2.3(a). It follows from (i) that U is a direct summand of exactly two basic support τ-tilting $(\Lambda/\langle e \rangle)$-modules. Thus the assertion follows since basic support τ-tilting $(\Lambda/\langle e \rangle)$-modules which have U as a direct summand correspond bijectively to basic support τ-tilting pairs for Λ which have (U, Q) as a direct summand.   □

The following special case of Lemma 2.20 is useful.

Proposition 2.23. *Let T be a support τ-tilting Λ-module. Assume that one of the following conditions is satisfied.*

(i) *U is a τ-rigid Λ-module such that $\mathsf{Fac} T \subseteq {}^\perp(\tau U)$.*
(ii) *U is a support τ-tilting Λ-module such that $U \geq T$.*

Then there exists an exact sequence $U \xrightarrow{f} T^0 \to T^1 \to 0$ such that f is a minimal left $(\mathsf{Fac} T)$-approximation of U and T^0 and T^1 are in $\mathsf{add} T$ and satisfy $\mathsf{add} T^0 \cap \mathsf{add} T^1 = 0$.

Proof. Let (T, P) be a support τ-tilting pair for Λ. Then $^\perp(\tau T) \cap P^\perp = \mathsf{Fac} T$ holds by Corollary 2.13(c). Thus $^\perp(\tau T) \cap P^\perp \subseteq {}^\perp(\tau U)$ holds for both cases. Hence the assertion is immediate from Lemma 2.20 since C is in $\mathsf{add} P(\mathsf{Fac} T) = \mathsf{add} T$ by Theorem 2.7.   □

The following well-known result [HU1] can be shown as an application of our results.

Corollary 2.24. *Let Λ be a finite dimensional K-algebra and U a basic almost complete tilting Λ-module. Then U is faithful if and only if U is a direct summand of precisely two basic tilting Λ-modules.*

Proof. It follows from Theorem 2.18 that U is a direct summand of exactly two basic support τ-tilting Λ-modules T and T' such that $\mathsf{Fac} T = \mathsf{Fac} U$. If U is faithful, then T and T' are tilting Λ-modules by Proposition 2.2(b). Thus the 'only if' part follows. If U is not faithful, then T is not a tilting Λ-module since it is not faithful because $\mathsf{Fac} T = \mathsf{Fac} U$. Thus the 'if' part follows.   □

2.4. **Partial order, exchange sequences and Hasse quiver.** In this section we investigate two quivers. One is defined by partial order, and the other one by mutation. We show that they coincide.

Since we have a bijection $T \mapsto \mathsf{Fac}\, T$ between sτ-tilt Λ and f-tors Λ, then inclusion in f-tors Λ gives rise to a partial order on sτ-tilt Λ, and we have an associated Hasse quiver. Note that sτ-tilt Λ has a unique maximal element Λ and a unique minimal element 0.

The following description of when $T \geq U$ holds will be useful.

Lemma 2.25. *Let (T, P) and (U, Q) be support τ-tilting pairs for Λ. Then the following conditions are equivalent.*

    (a) $T \geq U$.
    (b) $\mathrm{Hom}_\Lambda(U, \tau T) = 0$ *and* $\mathsf{add}\, P \subseteq \mathsf{add}\, Q$.
    (c) $\mathrm{Hom}_\Lambda(U_{\mathrm{np}}, \tau T_{\mathrm{np}}) = 0$, $\mathsf{add}\, T_{\mathrm{pr}} \supseteq \mathsf{add}\, U_{\mathrm{pr}}$ *and* $\mathsf{add}\, P \subseteq \mathsf{add}\, Q$.

Proof. (a)$\Rightarrow$(c) Since $\mathsf{Fac}\, T \supseteq \mathsf{Fac}\, U$, we have $\mathsf{add}\, T_{\mathrm{pr}} \supseteq \mathsf{add}\, U_{\mathrm{pr}}$ and $\mathrm{Hom}_\Lambda(U, \tau T) = 0$. Moreover $\mathsf{add}\, P \subseteq \mathsf{add}\, Q$ holds by Proposition 2.2(a).

(b)$\Rightarrow$(a) We have $\mathsf{Fac}\, T = {}^\perp(\tau T) \cap P^\perp$ by Corollary 2.13(c). Since $\mathsf{add}\, P \subseteq \mathsf{add}\, Q$, we have $U \in Q^\perp \subseteq P^\perp$. Since $\mathrm{Hom}_\Lambda(U, \tau T) = 0$, we have $U \in {}^\perp(\tau T) \cap P^\perp = \mathsf{Fac}\, T$, which implies $\mathsf{Fac}\, T \supseteq \mathsf{Fac}\, U$.

(c)$\Rightarrow$(b) This is clear. $\qquad\square$

Also we shall need the following.

Proposition 2.26. *Let $T, U, V \in$ sτ-tilt Λ such that $T \geq U \geq V$. Then $\mathsf{add}\, T \cap \mathsf{add}\, V \subseteq \mathsf{add}\, U$.*

Proof. Clearly we have $P(\mathsf{Fac}\, T) \cap \mathsf{Fac}\, U \subseteq P(\mathsf{Fac}\, U) = \mathsf{add}\, U$. Thus we have $\mathsf{add}\, T \cap \mathsf{add}\, V \subseteq P(\mathsf{Fac}\, T) \cap \mathsf{Fac}\, U \subseteq \mathsf{add}\, U$. $\qquad\square$

The following observation is immediate.

Proposition 2.27.   (a) *For any idempotent e of Λ, the inclusion* sτ-tilt $(\Lambda/\langle e \rangle) \to$ sτ-tilt Λ *preserves the partial order.*
    (b) *The bijection* $(-)^\dagger :$ sτ-tilt $\Lambda \to$ sτ-tilt Λ^{op} *in Theorem 2.14 reverses the partial order.*

Proof. (a) This is clear.

(b) Let (T, P) and (U, Q) be support τ-tilting pairs of Λ. By Lemma 2.25, $T \geq U$ if and only if $\mathrm{Hom}_\Lambda(U_{\mathrm{np}}, \tau T_{\mathrm{np}}) = 0$, $\mathsf{add}\, T_{\mathrm{pr}} \supseteq \mathsf{add}\, U_{\mathrm{pr}}$ and $\mathsf{add}\, P \subseteq \mathsf{add}\, Q$. This is equivalent to $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, T_{\mathrm{np}}, \tau\, \mathrm{Tr}\, U_{\mathrm{np}}) = 0$, $\mathsf{add}\, T_{\mathrm{pr}}^* \supseteq \mathsf{add}\, U_{\mathrm{pr}}^*$ and $\mathsf{add}\, P^* \subseteq \mathsf{add}\, Q^*$. By Lemma 2.25 again, this is equivalent to $(\mathrm{Tr}\, T_{\mathrm{np}} \oplus P^*, T_{\mathrm{pr}}^*) \leq (\mathrm{Tr}\, U_{\mathrm{np}} \oplus Q^*, U_{\mathrm{pr}}^*)$. $\qquad\square$

In the rest of this section, we study a relationship between partial order and mutation.

Definition-Proposition 2.28. *Let $T = X \oplus U$ and T' be support τ-tilting Λ-modules such that $T' = \mu_X(T)$ for some indecomposable Λ-module X. Then either $T > T'$ or $T < T'$ holds by Theorem 2.18. We say that T' is a left mutation (respectively, right mutation) of T and we write $T' = \mu_X^-(T)$ (respectively, $T' = \mu_X^+(T)$) if the following equivalent conditions are satisfied.*

    (a) $T > T'$ *(respectively, $T < T'$).*
    (b) $X \notin \mathsf{Fac}\, U$ *(respectively, $X \in \mathsf{Fac}\, U$).*
    (c) ${}^\perp(\tau X) \supseteq {}^\perp(\tau U)$ *(respectively, ${}^\perp(\tau X) \not\supseteq {}^\perp(\tau U)$).*

If T is a τ-tilting Λ-module, then the following condition is also equivalent to the above conditions.

    (d) *T is a Bongartz completion of U (respectively, T is a non-Bongartz completion of U).*

Proof. This follows immediately from Theorem 2.18 and Proposition 2.22. □

Definition 2.29. We define the *support τ-tilting quiver* $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ of Λ as follows:
- The set of vertices is $\mathrm{s}\tau\text{-tilt}\,\Lambda$.
- We draw an arrow from T to U if U is a left mutation of T.

Next we show that one can calculate left mutation of support τ-tilting Λ-modules by exchange sequences which are constructed from left approximations.

Theorem 2.30. *Let $T = X \oplus U$ be a basic τ-tilting module which is the Bongartz completion of U, where X is indecomposable. Let $X \xrightarrow{f} U' \xrightarrow{g} Y \to 0$ be an exact sequence, where f is a minimal left $(\mathsf{add}\,U)$-approximation. Then we have the following.*

   (a) *If U is not sincere, then $Y = 0$. In this case $U = \mu_X^-(T)$ holds and this is a basic support τ-tilting Λ-module which is not τ-tilting.*

   (b) *If U is sincere, then Y is a direct sum of copies of an indecomposable Λ-module Y_1 and is not in $\mathsf{add}\,T$. In this case $Y_1 \oplus U = \mu_X^-(T)$ holds and this is a basic τ-tilting Λ-module.*

Proof. We first make some preliminary observations. We have $^\perp(\tau U) \subseteq {}^\perp(\tau X)$ because T is a Bongartz completion of U. By Lemma 2.20, we have an exact sequence
$$X \xrightarrow{f} U' \xrightarrow{g} Y \to 0$$
such that U' is in $\mathsf{add}\,U$, Y is in $\mathsf{add}\,P(\mathsf{Fac}\,U)$, $\mathsf{add}\,U' \cap \mathsf{add}\,Y = 0$ and f is a left $(\mathsf{Fac}\,U)$-approximation. We have $\mathrm{Ext}_\Lambda^1(Y, \mathsf{Fac}\,U) = 0$ since $Y \in \mathsf{add}\,P(\mathsf{Fac}\,U)$, and hence $\mathrm{Hom}_\Lambda(U, \tau Y) = 0$ by Proposition 1.2. We have an injective map $\mathrm{Hom}_\Lambda(Y, \tau(Y \oplus U)) \to \mathrm{Hom}_\Lambda(U', \tau(Y \oplus U))$. Since U is τ-rigid, we have that $\mathrm{Hom}_\Lambda(U', \tau(Y \oplus U)) = 0$, and consequently $\mathrm{Hom}_\Lambda(Y, \tau(Y \oplus U)) = 0$. It follows that $Y \oplus U$ is τ-rigid.

We show that $g : U' \to Y$ is a right $(\mathsf{add}\,T)$-approximation. To see this, consider the exact sequence
$$\mathrm{Hom}_\Lambda(T, U') \to \mathrm{Hom}_\Lambda(T, Y) \to \mathrm{Ext}_\Lambda^1(T, \mathrm{Im}\,f).$$
Since $\mathrm{Im}\,f \in \mathsf{Fac}\,T$, we have $\mathrm{Ext}_\Lambda^1(T, \mathrm{Im}\,f) = 0$, which proves the claim.

We have that Y does not have any indecomposable direct summand from $\mathsf{add}\,T$. For if T' in $\mathsf{add}\,T$ is an indecomposable direct summand of Y, then the natural inclusion $T' \to Y$ factors through $g : U' \to Y$. This contradicts the fact that $f : X \to U'$ is left minimal.

Now we are ready to prove the claims (a) and (b).

(a) Assume first that U is not sincere. Let e be a primitive idempotent with $Ue = 0$. Then U is a τ-rigid $(\Lambda/\langle e \rangle)$-module. Since $|U| = |\Lambda| - 1 = |\Lambda/\langle e \rangle|$, we have that U is a τ-tilting $(\Lambda/\langle e \rangle)$-module, and hence a support τ-tilting Λ-module which is not τ-tilting.

(b) Next assume that U is sincere. Since we have already shown that $Y \oplus U$ is τ-rigid and $Y \notin \mathsf{add}\,T$, it is enough to show $Y \neq 0$. Otherwise we have $X \simeq U'$ by Lemma 2.21 since U is sincere. This is not possible since U' is in $\mathsf{add}\,U$, but X is not. Hence it follows that $Y \neq 0$. □

We do not know the answer to the following.

Question 2.31. *Is Y always indecomposable in Theorem 2.30(b)?*

Note that right mutation can not be calculated as directly as left mutation.

Remark 2.32. Let T and T' be support τ-tilting Λ-modules such that $T' = \mu_X(T)$ for some indecomposable Λ-module X.

(a) If $T' = \mu_X^-(T)$, then we can calculate T' by applying Theorem 2.30.
(b) If $T' = \mu_X^+(T)$, then we can calculate T' using the following three steps: First calculate $T^\dagger$. Then calculate $T'^\dagger$ by applying Theorem 2.30 to $T^\dagger$. Finally calculate T' by applying $(-)^\dagger$ to $T'^\dagger$.

Our next main result is the following.

Theorem 2.33. *For $T, U \in s\tau$-tilt Λ, the following conditions are equivalent.*

   (a) *U is a left mutation of T.*
   (b) *T is a right mutation of U.*
   (c) *$T > U$ and there is no $V \in s\tau$-tilt Λ such that $T > V > U$.*

Before proving Theorem 2.33, we give the following result as a direct consequence.

Corollary 2.34. *The support τ-tilting quiver $Q(s\tau$-tilt $\Lambda)$ is the Hasse quiver of the partially ordered set $s\tau$-tilt Λ.*

The following analog of [AI, Proposition 2.36] is a main step to prove Theorem 2.33.

Theorem 2.35. *Let U and T be basic support τ-tilting Λ-modules such that $U > T$. Then:*

   (a) *There exists a right mutation V of T such that $U \geq V$.*
   (b) *There exists a left mutation V' of U such that $V' \geq T$.*

Before proving Theorem 2.35, we finish the proof of Theorem 2.33 by using Theorem 2.35.

(a)$\Leftrightarrow$(b) Immediate from the definitions.

(a)$\Rightarrow$(c) Assume that $V \in s\tau$-tilt Λ satisfies $T > V \geq U$. Then we have $\mathsf{add}\,T \cap \mathsf{add}\,U \subseteq \mathsf{add}\,V$ by Proposition 2.26. Thus T and V have an almost complete support τ-tilting pair for Λ as a common direct summand. Hence we have $V \simeq U$ by Theorem 2.18.

(c)$\Rightarrow$(a) By Theorem 2.35, there exists a left mutation V of T such that $T > V \geq U$. Then $V \simeq U$ by our assumption. Thus U is a left mutation of T. $\qquad\square$

To prove Theorem 2.35, we shall need the following results.

Lemma 2.36. *Let U and T be basic support τ-tilting Λ-modules such that $U > T$. Let $U \xrightarrow{f} T^0 \to T^1 \to 0$ be an exact sequence as given in Proposition 2.23. If X is an indecomposable direct summand of T which does not belong to $\mathsf{add}\,T^0$, then we have $U \geq \mu_X(T) > T$.*

Proof. First we show $\mu_X(T) > T$. Since X is in $\mathsf{Fac}\,T \subseteq \mathsf{Fac}\,U$, there exists a surjective map $a : U^\ell \to X$ for some $\ell > 0$. Since $f^\ell : U^\ell \to (T^0)^\ell$ is a left $(\mathsf{add}\,T)$-approximation, a factors through f^ℓ and we have $X \in \mathsf{Fac}\,T^0$. It follows from $X \notin \mathsf{add}\,T^0$ that $X \in \mathsf{Fac}\,T^0 \subseteq \mathsf{Fac}\,\mu_X(T)$. Thus $\mathsf{Fac}\,T \subseteq \mathsf{Fac}\,\mu_X(T)$ and we have $\mu_X(T) > T$.

Next we show $U \geq \mu_X(T)$. Let $(U, e\Lambda)$ and $(T, e'\Lambda)$ be support τ-tilting pairs for Λ. By Proposition 2.27(b), we know that $U^\dagger = \mathrm{Tr}\,U \oplus \Lambda e$ and $T^\dagger = \mathrm{Tr}\,T \oplus \Lambda e'$ are support τ-tilting Λ^{op}-modules such that $U^\dagger < T^\dagger$. In particular, any minimal right $(\mathsf{add}\,T^\dagger)$-approximation

$$\mathrm{Tr}\,T_0 \oplus P \to U^\dagger \tag{6}$$

of $U^\dagger$ with $T_0 \in \mathsf{add}\,T_{\mathrm{np}}$ and $P \in \mathsf{add}\,\Lambda e'$ is surjective. The following observation shows $T_0 \in \mathsf{add}\,T^0$.

Lemma 2.37. *Let X and Y be in $\mathsf{mod}\,\Lambda$ and P in $\mathsf{proj}\,\Lambda^{\mathrm{op}}$. Let $f : Y \to X^0$ be a left $(\mathsf{add}\,X)$-approximation of Y and $g : \operatorname{Tr} X_0 \oplus P_0 \to \operatorname{Tr} Y$ be a minimal right $(\mathsf{add}\,\operatorname{Tr} X \oplus P)$-approximation of $\operatorname{Tr} Y$ with $X_0 \in \mathsf{add}\,X_{\mathrm{np}}$ and $P_0 \in \mathsf{add}\,P$. If g is surjective, then X_0 is a direct summand of X^0.*

Proof. Assume that g is surjective and consider the exact sequence

$$0 \longrightarrow K \overset{h}{\longrightarrow} \operatorname{Tr} X_0 \oplus P_0 \overset{g}{\longrightarrow} \operatorname{Tr} Y \longrightarrow 0.$$

Then h is in $\operatorname{rad}(K, \operatorname{Tr} X_0 \oplus P_0)$ since g is right minimal. It is easy to see that in the stable category $\underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$, a pseudokernel of g is given by h, which is in the radical of $\underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$. In particular, g is a minimal right $(\mathsf{add}\,\operatorname{Tr} X)$-approximation in $\underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$. Since $\operatorname{Tr} : \underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Lambda^{\mathrm{op}}$ is a duality, we have that $\operatorname{Tr} g : \operatorname{Tr}\operatorname{Tr} Y \to \operatorname{Tr}(\operatorname{Tr} X_0 \oplus P_0) = X_0$ is a minimal left $(\mathsf{add}\,X)$-approximation of $\operatorname{Tr}\operatorname{Tr} Y$ in $\underline{\mathsf{mod}}\,\Lambda$. On the other hand, $f : Y \to X^0$ is clearly a left $(\mathsf{add}\,X)$-approximation of Y in $\underline{\mathsf{mod}}\,\Lambda$. Since $\operatorname{Tr}\operatorname{Tr} Y$ is a direct summand of Y, we have that X_0 is a direct summand of X^0 in $\underline{\mathsf{mod}}\,\Lambda$. Thus the assertion follows. $\square$

We now finish the proof of Lemma 2.36.

Since $T_0 \in \mathsf{add}\,T^0$ and $X \notin \mathsf{add}\,T^0$, we have $X \notin \mathsf{add}\,T_0$ and hence $U^\dagger \in \mathsf{Fac}(\operatorname{Tr}(T/X) \oplus \Lambda e')$ by (6). Hence we have $U^\dagger \le \mu_X(T)^\dagger$, which implies $U \ge \mu_X(T)$ by Proposition 2.27(b). $\square$

Now we are ready to prove Theorem 2.35.

We only prove (a) since (b) follows from (a) and Proposition 2.27(b).

(i) Let $(U, e\Lambda)$ and $(T, e'\Lambda)$ be support τ-tilting pairs for Λ. Let

$$U \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0$$

be an exact sequence given by Proposition 2.23. If $T \notin \mathsf{add}\,T^0$, then any indecomposable direct summand X of T which is not in $\mathsf{add}\,T^0$ satisfies $U \ge \mu_X(T) > T$ by Lemma 2.36. Thus we assume $T \in \mathsf{add}\,T^0$ in the rest of proof. Since $\mathsf{add}\,T^0 \cap \mathsf{add}\,T^1 = 0$, we have $T^1 = 0$ which implies $T^0 = U/U\langle e'\rangle$ by Lemma 2.21.

(ii) By Proposition 2.27(b), we know that $U^\dagger = \operatorname{Tr} U \oplus \Lambda e$ and $T^\dagger = \operatorname{Tr} T \oplus \Lambda e'$ are support τ-tilting Λ^{op}-modules such that $U^\dagger < T^\dagger$. Let

$$T_0^\dagger \overset{f}{\longrightarrow} U^\dagger \longrightarrow 0$$

be a minimal right $(\mathsf{add}\,T^\dagger)$-approximation of $U^\dagger$. If $\Lambda e' \notin \mathsf{add}\,T_0^\dagger$, then any indecomposable direct summand Q of $\Lambda e'$ which is not in $\mathsf{add}\,T_0^\dagger$ satisfies $U^\dagger \in \mathsf{Fac}(T^\dagger/Q)$. Thus we have $U^\dagger \le \mu_Q(T^\dagger)$ and $U \ge \mu_{Q^*}(T) > T$ by Proposition 2.27. We assume $\Lambda e' \in \mathsf{add}\,T_0^\dagger$ in the rest of proof.

(iii) We show that there exists an exact sequence

$$P_1 \overset{a}{\longrightarrow} \operatorname{Tr} T^0 \oplus P_0 \longrightarrow \operatorname{Tr} U \longrightarrow 0 \tag{7}$$

in $\mathsf{mod}\,\Lambda^{\mathrm{op}}$ such that $P_0 \in \mathsf{proj}\,\Lambda^{\mathrm{op}}$, $P_1 \in \mathsf{add}\,\Lambda e'$, $a \in \operatorname{rad}(P_1, \operatorname{Tr} T^0 \oplus P_0)$ and the map

$$(a, U^\dagger) : \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} T^0 \oplus P_0, U^\dagger) \longrightarrow \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P_1, U^\dagger) \tag{8}$$

is surjective.

Let $Q_1 \xrightarrow{d} Q_0 \to U \to 0$ be a minimal projective presentation of U. Let $d' : Q_1' \to Q_0$ be a right $(\mathsf{add}\, e'\Lambda)$-approximation of Q_0. Since $T^0 = U/U\langle e' \rangle$ by (i), we have a projective presentation $Q_1' \oplus Q_1 \xrightarrow{\binom{d'}{d}} Q_0 \to T^0 \to 0$ of T^0. Thus we have an exact sequence

$$Q_0^* \xrightarrow{(d'^*\ d^*)} Q_1'^* \oplus Q_1^* \xrightarrow{\binom{c'}{c}} \operatorname{Tr} T^0 \oplus Q \longrightarrow 0$$

for some projective Λ^{op}-module Q. We have a commutative diagram

$$
\begin{array}{ccccccc}
Q_0^* & \xrightarrow{\ d^*\ } & Q_1^* & \longrightarrow & \operatorname{Tr} U & \longrightarrow & 0 \\
\downarrow{\scriptstyle d'^*} & & \downarrow{\scriptstyle -c} & & \| & & \\
Q_1'^* & \xrightarrow{\ c'\ } & \operatorname{Tr} T^0 \oplus Q & \longrightarrow & \operatorname{Tr} U & \longrightarrow & 0
\end{array}
$$

of exact sequences. Now we decompose the morphism c' as

$$c' = \begin{pmatrix} a & 0 \\ 0 & 1_{Q''} \end{pmatrix} : Q_1'^* = P_1 \oplus Q'' \longrightarrow \operatorname{Tr} T^0 \oplus Q = \operatorname{Tr} T^0 \oplus P_0 \oplus Q'',$$

where a is in the radical. Then we naturally have an exact sequence (7), and clearly we have $P_0 \in \mathsf{proj}\,\Lambda^{\mathrm{op}}$ and $P_1 \in \mathsf{add}\,\Lambda e'$ by our construction. It remains to show that (8) is surjective. We only have to show that the map

$$(c', U^{\dagger}) : \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} T^0 \oplus Q, U^{\dagger}) \longrightarrow \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(Q_1'^*, U^{\dagger})$$

is surjective. Take any map $s : Q_1'^* \to U^{\dagger}$. By Proposition 2.4(c), there exists $t : Q_1^* \to U^{\dagger}$ such that $sd'^* = td^*$. Thus there exists $u : \operatorname{Tr} T^0 \oplus Q \to U^{\dagger}$ such that $s = uc'$ and $t = -uc$, which shows the assertion.

(iv) First we assume P_1 in (iii) is non-zero. Since $\Lambda e' \in \mathsf{add}\,T_0^{\dagger}$ by (ii) and $P_1 \in \mathsf{add}\,\Lambda e'$, we have $P_1 \in \mathsf{add}\,T_0^{\dagger}$. Thus there exists a morphism $s : P_1 \to T_0^{\dagger}$ which is not in the radical. Since (8) is surjective, there exists $t : \operatorname{Tr} T^0 \oplus P_0 \to U^{\dagger}$ such that $ta = fs$. Since f is a surjective right $(\mathsf{add}\,T^{\dagger})$-approximation and P_0 is projective, there exists $u : \operatorname{Tr} T^0 \oplus P_0 \to T_0^{\dagger}$ such that $t = fu$.

$$
\begin{array}{ccccccc}
P_1 & \xrightarrow{\ a\ } & \operatorname{Tr} T^0 \oplus P_0 & \longrightarrow & \operatorname{Tr} U & \longrightarrow & 0 \\
\downarrow{\scriptstyle s} & \nearrow{\scriptstyle u} & \downarrow{\scriptstyle t} & & & & \\
T_0^{\dagger} & \xrightarrow{\ f\ } & U^{\dagger} & \longrightarrow & 0 & &
\end{array}
$$

Since $f(s - ua) = 0$ and f is right minimal, we have that $s - ua$ is in the radical. Since a is in the radical, so is s, a contradiction.

Consequently, we have $P_1 = 0$. Thus $\operatorname{Tr} T^0 \oplus P_0 \simeq \operatorname{Tr} U$ and $\operatorname{Tr} T^0 \simeq \operatorname{Tr} U$. Since $T \in \mathsf{add}\,T^0$ by our assumption, we have $\mathsf{add}\,T_{\mathrm{np}} = \mathsf{add}\,U_{\mathrm{np}}$. Since $U > T$, we have $T_{\mathrm{pr}} \in \mathsf{add}\,U_{\mathrm{pr}}$. Thus $U \simeq T \oplus P$ for some projective Λ-module P.

(v) It remains to consider the case $U \simeq T \oplus P$ for some projective Λ-module P.

Since $U > T$, we have $\mathsf{add}\,e\Lambda \subsetneq \mathsf{add}\,e'\Lambda$. Take any indecomposable summand $e''\Lambda$ of $(e' - e)\Lambda$ and let $V := \mu_{e''\Lambda}(T, e'\Lambda)$, which has a form $(T \oplus X, (e' - e'')\Lambda)$ with X indecomposable. Clearly $V > T$ holds. Since $\tau U \in \mathsf{add}\,\tau(T \oplus X)$ by our assumption and $e\Lambda \in \mathsf{add}\,(e' - e'')\Lambda$ by our choice of e'', we have

$$\mathsf{Fac}\,U = {}^{\perp}(\tau U) \cap (e\Lambda)^{\perp} \supseteq {}^{\perp}(\tau(T \oplus X)) \cap ((e' - e'')\Lambda)^{\perp} = \mathsf{Fac}\,V$$

by Corollary 2.13(c). Thus $U \geq V$ holds. $\square$

We end this section with the following application, which is an analog of [HU2, Corollary 2.2].

Corollary 2.38. *If* $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ *has a finite connected component* C*, then* $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda) = C$*.*

Proof. Fix T in C. Applying Theorem 2.35(a) to $\Lambda \geq T$, we have a sequence $T = T_0 < T_1 < T_2 < \cdots$ of right mutations of support τ-tilting modules such that $\Lambda \geq T_i$ for any i. Since C is finite, this sequence must be finite. Thus $\Lambda = T_i$ for some i, and Λ belongs to C. Now we fix any $U \in \mathrm{s}\tau\text{-tilt}\,\Lambda$. Applying Theorem 2.35(b) to $\Lambda \geq U$, we have a sequence $\Lambda = V_0 > V_1 > V_2 > \cdots$ of left mutations of support τ-tilting modules such that $V_i \geq U$ for any i. Since C is finite, this sequence must be finite. Thus $U = V_j$ for some j, and U belongs to C. $\square$

3. CONNECTION WITH SILTING THEORY

Throughout this section, let Λ be a finite dimensional K-algebra. Any almost complete silting complex has infinitely many complements. But if we restrict to two-term silting complexes, we get another class of objects extending the (classical) tilting modules and satisfying the two complement property (Corollary 3.8). Moreover we will show that there is a bijection between support τ-tilting Λ-modules and two-term silting complexes for Λ, which is of independent interest (Theorem 3.2). The two-term silting complexes are defined as follows.

Definition 3.1. We call a complex $P = (P^i, d^i)$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ *two-term* if $P^i = 0$ for all $i \neq 0, -1$. Clearly $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ is two-term if and only if $\Lambda \geq P \geq \Lambda[1]$.

We denote by 2-silt Λ (respectively, 2-presilt Λ) the set of isomorphism classes of basic two-term silting (respectively, presilting) complexes for Λ.

Clearly any two-term complex is isomorphic to a two-term complex $P = (P^i, d^i)$ satisfying $d^{-1} \in \mathrm{rad}(P^{-1}, P^0)$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Moreover, for any two-term complexes P and Q, we have $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, Q[i]) = 0$ for any $i \neq -1, 0, 1$.

The aim of this section is to prove the following result.

Theorem 3.2. *Let* Λ *be a finite dimensional* K*-algebra. Then there exists a bijection*

$$2\text{-silt}\,\Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda$$

given by $2\text{-silt}\,\Lambda \ni P \mapsto H^0(P) \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ *and* $\mathrm{s}\tau\text{-tilt}\,\Lambda \ni (M, P) \mapsto (P_1 \oplus P \xrightarrow{(f\ 0)} P_0) \in 2\text{-silt}\,\Lambda$ *where* $f : P_1 \to P_0$ *is a minimal projective presentation of* M*.*

The following result is quite useful.

Proposition 3.3. *Let* P *be a two-term presilting complex for* Λ*.*
 (a) *P is a direct summand of a two-term silting complex for* Λ*.*
 (b) *P is a silting complex for* Λ *if and only if* $|P| = |\Lambda|$*.*

Proof. (a) This is shown in [Ai, Proposition 2.16].
 (b) The 'only if' part follows from Proposition 1.6(a). We will show the 'if' part. Let P be a two-term presilting complex for Λ with $|P| = |\Lambda|$. By (a), there exists a complex X such that $P \oplus X$ is silting. Then we have $|P \oplus X| = |\Lambda| = |P|$ by Proposition 1.6(a), so X is in $\mathsf{add}\,P$. Thus P is silting. $\square$

The following lemma is important.

Lemma 3.4. *Let* $M, N \in \mathsf{mod}\Lambda$. *Let* $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ *and* $Q_1 \xrightarrow{q_1} Q_0 \xrightarrow{q_0} N \to 0$ *be minimal projective presentations of* M *and* N *respectively. Let* $P = (P_1 \xrightarrow{p_1} P_0)$ *and* $Q = (Q_1 \xrightarrow{q_1} Q_0)$ *be two-term complexes for* Λ. *Then the following conditions are equivalent:*

(a) $\mathrm{Hom}_\Lambda(N, \tau M) = 0$.
(b) $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, Q[1]) = 0$.

In particular, M *is a* τ-*rigid* Λ-*module if and only if* P *is a presilting complex for* Λ.

Proof. The condition (a) is equivalent to the fact that $(p_1, N) : \mathrm{Hom}_\Lambda(P_0, N) \to \mathrm{Hom}_\Lambda(P_1, N)$ is surjective by Proposition 2.4(b).

(a)$\Rightarrow$(b) Any morphism $f \in \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, Q[1])$ is given by some $f \in \mathrm{Hom}_\Lambda(P_1, Q_0)$. Since (p_1, N) is surjective, there exists $g : P_0 \to N$ such that $q_0 f = g p_1$. Moreover, since P_0 is projective, there exists $h_0 : P_0 \to Q_0$ such that $q_0 h_0 = g$. Since $q_0(f - h_0 p_1) = 0$, we have $h_1 : P_1 \to Q_1$ with $f = q_1 h_1 + h_0 p_1$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & M & \longrightarrow & 0 \\
& {\scriptstyle h_1} \swarrow & {\scriptstyle f}\downarrow & {\scriptstyle h_0}\swarrow & \downarrow {\scriptstyle g} & & & & \\
0 & \longrightarrow & Q_1 & \xrightarrow[q_1]{} & Q_0 & \xrightarrow[q_0]{} & N & \longrightarrow & 0.
\end{array}
$$

Hence we have $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, Q[1]) = 0$.

(b)$\Rightarrow$(a) Take any $f \in \mathrm{Hom}_\Lambda(P_1, N)$. Since P_1 is projective, there exists $g : P_1 \to Q_0$ such that $q_0 g = f$.

$$
\begin{array}{ccc}
P_1 & \xrightarrow{p_1} & P_0 \\
{\scriptstyle g}\downarrow & \searrow {\scriptstyle f} & \\
Q_1 \xrightarrow[q_1]{} Q_0 & \xrightarrow[q_0]{} & N \longrightarrow 0.
\end{array}
$$

Then g gives a morphism $P \to Q[1]$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$. Since $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, Q[1]) = 0$, there exist $h_0 : P_0 \to Q_0$ and $h_1 : P_1 \to Q_1$ such that $g = q_1 h_1 + h_0 p_1$. Hence we have $f = q_0(q_1 h_1 + h_0 p_1) = q_0 h_0 p_1$. Therefore (p_1, N) is surjective. $\square$

We also need the following observation.

Lemma 3.5. *Let* $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ *be a minimal projective presentation of* M *in* $\mathsf{mod}\Lambda$ *and* $P := (P_1 \xrightarrow{p_1} P_0)$ *be a two-term complex for* Λ. *Then for any* Q *in* $\mathsf{proj}\Lambda$, *the following conditions are equivalent.*

(a) $\mathrm{Hom}_\Lambda(Q, M) = 0$.
(b) $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(Q, P) = 0$.

Proof. The proof is left to the reader since it is straightforward. $\square$

The following result shows that silting complexes for Λ give support τ-tilting modules.

Proposition 3.6. *Let* $P = (P_1 \xrightarrow{d} P_0)$ *be a two-term complex for* Λ *and* $M := \mathrm{Cok}\, d$.

(a) *If* P *is a silting complex for* Λ *and* d *is right minimal, then* M *is a* τ-*tilting* Λ-*module.*
(b) *If* P *is a silting complex for* Λ, *then* M *is a support* τ-*tilting* Λ-*module.*

Proof. (b) We write $d = (d'\ 0) : P_1 = P_1' \oplus P_1'' \to P_0$, where d' is right minimal. Then the sequence $P_1' \xrightarrow{d'} P_0 \to M \to 0$ is a minimal projective presentation of M. We show that (M, P_1'') is a support τ-tilting pair for Λ. Since P is silting, M is a τ-rigid Λ-module by Lemma 3.4. On the other hand, since P is silting, we have $\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(P_1'', P) = 0$. By Lemma 3.5, we have $\mathrm{Hom}_\Lambda(P_1'', M) = 0$. Thus (M, P_1'') is a τ-rigid pair for Λ. Since d' is a minimal projective presentation of M, we have $|M| = |P_1' \xrightarrow{d'} P_0|$. Thus we have

$$|M| + |P_1''| = |P_1' \xrightarrow{d'} P_0| + |P_1''| = |P|,$$

which is equal to $|\Lambda|$ by Proposition 1.6(a). Hence (M, P_1'') is a support τ-tilting pair for Λ.

(a) This is the case $P_1'' = 0$ in (b). $\qquad\square$

The following result shows that support τ-tilting Λ-modules give silting complexes for Λ.

Proposition 3.7. *Let $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ be a minimal projective presentation of M in $\mathsf{mod}\,\Lambda$.*

  (a) *If M is a τ-tilting Λ-module, then $(P_1 \xrightarrow{d_1} P_0)$ is a silting complex for Λ.*
  (b) *If (M, Q) is a support τ-tilting pair for Λ, then $P_1 \oplus Q \xrightarrow{(d_1\ 0)} P_0$ is a silting complex for Λ.*

Proof. (b) We know that $(P_1 \xrightarrow{d_1} P_0)$ is a presilting complex for Λ by Lemma 3.4. Let $P := (P_1 \oplus Q \xrightarrow{(d_1\ 0)} P_0)$. By Lemmas 3.4 and 3.5, we have that P is a presilting complex for Λ. Since d_1 is a minimal projective presentation, we have $|P_1 \xrightarrow{d_1} P_0| = |M|$. Moreover, since (M, Q) is a support τ-tilting pair for Λ, we have $|M| + |Q| = |\Lambda|$. Thus we have

$$|P| = |P_1 \xrightarrow{d_1} P_0| + |Q| = |M| + |Q| = |\Lambda|.$$

Hence P is a silting complex for Λ by Proposition 3.3(b).

(a) This is the case $Q = 0$ in (b). $\qquad\square$

Now Theorem 3.2 follows from Propositions 3.6 and 3.7. $\qquad\square$

We give some applications of Theorem 3.2.

Corollary 3.8. *Let Λ be a finite dimensional K-algebra.*

  (a) *Any basic two-term presilting complex P for Λ with $|P| = |\Lambda| - 1$ is a direct summand of exactly two basic two-term silting complexes for Λ.*
  (b) *Let $P, Q \in 2\text{-silt}\,\Lambda$. Then P and Q have all but one indecomposable direct summand in common if and only if P is a left or right mutation of Q.*

Proof. (a) This follows from Theorems 2.18 and 3.2.
(b) This is immediate from (a). $\qquad\square$

Now we define $\mathrm{Q}(2\text{-silt}\,\Lambda)$ as the full subquiver of $\mathrm{Q}(\mathrm{silt}\,\Lambda)$ with vertices corresponding to two-term silting complexes for Λ.

Corollary 3.9. *The bijection in Theorem 3.2 is an isomorphism of the partially ordered sets. In particular, it induces an isomorphism between the two-term silting quiver $\mathrm{Q}(2\text{-silt}\,\Lambda)$ and the support τ-tilting quiver $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$.*

Proof. Let $(M, e\Lambda)$ and $(N, f\Lambda)$ be support τ-tilting pairs for Λ. Let $P := (P_1 \to P_0)$ and $Q := (Q_1 \to Q_0)$ be minimal projective presentations of M and N respectively. We only have to show that $M \geq N$ if and only if $\mathrm{Hom}_{\mathsf{K^b(proj}\Lambda)}(P \oplus e\Lambda[1], (Q \oplus f\Lambda[1])[1]) = 0$.

We know that $M \geq N$ if and only if $\mathrm{Hom}_\Lambda(N, \tau M) = 0$ and $e\Lambda \in \mathsf{add}\, f\Lambda$ by Lemma 2.25. Moreover $\mathrm{Hom}_\Lambda(N, \tau M) = 0$ if and only if $\mathrm{Hom}_{\mathsf{K^b(proj}\Lambda)}(P, Q[1]) = 0$ by by Lemma 3.4. On the other hand $e\Lambda \in \mathsf{add}\, f\Lambda$ if and only if $\mathrm{Hom}_\Lambda(e\Lambda, N) = 0$ since N is a sincere $(\Lambda/\langle f \rangle)$-module. Thus $e\Lambda \in \mathsf{add}\, f\Lambda$ is equivalent to $\mathrm{Hom}_{\mathsf{K^b(proj}\Lambda)}(e\Lambda, Q) = 0$ by Lemma 3.5. Consequently $M \geq N$ if and only if $\mathrm{Hom}_{\mathsf{K^b(proj}\Lambda)}(P \oplus e\Lambda[1], Q[1]) = 0$, and this is equivalent to $\mathrm{Hom}_{\mathsf{K^b(proj}\Lambda)}(P \oplus e\Lambda[1], (Q \oplus f\Lambda[1])[1]) = 0$ since $\mathrm{Hom}_{\mathsf{K^b(proj}\Lambda)}(P \oplus e\Lambda[1], f\Lambda[2]) = 0$ is automatic. Thus the assertion follows. $\square$

Immediately we have the following application.

Corollary 3.10. *If* $\mathrm{Q}(2\text{-silt}\,\Lambda)$ *has a finite connected component* C*, then* $\mathrm{Q}(2\text{-silt}\,\Lambda) = C$.

Proof. This is immediate from Corollaries 2.38 and 3.9. $\square$

Note also that Theorem 3.2 and Corollary 3.9 give an alternative proof of Theorem 2.35 since the corresponding property for two-term silting complexes holds by [AI, Proposition 2.36].

4. CONNECTION WITH CLUSTER-TILTING THEORY

Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt 2-Calabi-Yau (2-CY for short) triangulated category (for example, the cluster category $\mathcal{C}_Q$ associated with a finite acyclic quiver Q [BMRRT]). We shall assume that our category $\mathcal{C}$ has a cluster-tilting object T. Associated with T, we have by definition the 2-CY-tilted algebra $\Lambda = \mathrm{End}_{\mathcal{C}}(T)$, whose module category is closely connected with the 2-CY-category $\mathcal{C}$. In particular, there is an equivalence of categories [BMR1, KR]:

$$\overline{(-)} := \mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C}/[T[1]] \to \mathsf{mod}\,\Lambda. \tag{9}$$

In this section we investigate this relationship more closely by giving a bijection between cluster-tilting objects in $\mathcal{C}$ and support τ-tilting Λ-modules (Theorem 4.1). This was the starting point for the theory of τ-rigid and τ-tilting modules. As an application, we give a proof of some known results for cluster-tilting objects (Corollary 4.5). Also we give a direct connection between cluster-tilting objects in $\mathcal{C}$ and two-term silting complexes for Λ (Theorem 4.7). There is an induced isomorphism between the associated graphs (Corollary 4.8).

4.1. Support τ-tilting modules and cluster-tilting objects. In this subsection we show that there is a close relationship between the cluster-tilting objects in $\mathcal{C}$ and support τ-tilting Λ-modules. We use this to apply our main Theorem 2.18 to get a new proof of the fact that almost complete cluster-tilting objects have exactly two complements, and of the fact that all maximal rigid objects are cluster-tilting, as first proved in [IY] and [ZZ], respectively.

We denote by $\mathsf{iso}\,\mathcal{C}$ the set of isomorphism classes of objects in a category $\mathcal{C}$. From our equivalence (9), we have a bijection

$$\widetilde{(-)} : \mathsf{iso}\,\mathcal{C} \longleftrightarrow \mathsf{iso}(\mathsf{mod}\,\Lambda) \times \mathsf{iso}(\mathsf{proj}\,\Lambda)$$

given by $X = X' \oplus X'' \mapsto \widetilde{X} := (\overline{X'}, \overline{X''[-1]})$, where X'' is a maximal direct summand of X which belongs to $\mathsf{add}\,T[1]$. We denote by $\mathsf{rigid}\,\mathcal{C}$ (respectively, $\mathsf{m\text{-}rigid}\,\mathcal{C}$) the set

of isomorphism classes of basic rigid (respectively, maximal rigid) objects in $\mathcal{C}$, and by
c-tilt$_T\mathcal{C}$ the set of isomorphism classes of basic cluster-tilting objects in $\mathcal{C}$ which do not
have non-zero direct summands in $\mathsf{add}\,T[1]$.

Our main result in this section is the following.

Theorem 4.1. *The bijection $\widetilde{(-)}$ induces bijections*

$$\mathrm{rigid}\,\mathcal{C} \longleftrightarrow \tau\text{-rigid}\,\Lambda, \quad \text{c-tilt}\,\mathcal{C} \longleftrightarrow \text{s}\tau\text{-tilt}\,\Lambda \quad and \quad \text{c-tilt}_T\mathcal{C} \longleftrightarrow \tau\text{-tilt}\,\Lambda.$$

Moreover we have $\text{c-tilt}\,\mathcal{C} = \text{m-rigid}\,\mathcal{C} = \{U \in \mathrm{rigid}\,\mathcal{C} \mid |U| = |T|\}$.

We start with the following easy observation (see [KR]).

Lemma 4.2. *The functor $\overline{(-)}$ induces an equivalence of categories between $\mathsf{add}\,T$ (respectively, $\mathsf{add}\,T[2]$) and $\mathsf{proj}\,\Lambda$ (respectively, $\mathsf{inj}\,\Lambda$). Moreover we have an isomorphism $\overline{(-)} \circ [2] \simeq \nu \circ \overline{(-)} : \mathsf{add}\,T \to \mathsf{inj}\,\Lambda$ of functors.*

Now we express $\mathrm{Ext}^1_{\mathcal{C}}(X,Y)$ in terms of the images $\overline{X}$ and $\overline{Y}$ in our fixed 2-CY tilted
algebra Λ. We let

$$\langle X, Y \rangle_\Lambda = \langle X, Y \rangle := \dim_K \mathrm{Hom}_\Lambda(X, Y).$$

Proposition 4.3. *Let X and Y be objects in $\mathcal{C}$. Assume that there are no nonzero
indecomposable direct summands of $T[1]$ for X and Y.*

(a) *We have $\overline{X[1]} \simeq \tau\overline{X}$ and $\overline{Y[1]} \simeq \tau\overline{Y}$ as Λ-modules.*
(b) *We have an exact sequence*

$$0 \to D\,\mathrm{Hom}_\Lambda(\overline{Y}, \tau\overline{X}) \to \mathrm{Ext}^1_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_\Lambda(\overline{X}, \tau\overline{Y}) \to 0.$$

(c) $\dim \mathrm{Ext}^1_{\mathcal{C}}(X,Y) = \langle \overline{X}, \tau\overline{Y} \rangle_\Lambda + \langle \overline{Y}, \tau\overline{X} \rangle_\Lambda$.

Proof. (a) This can be shown as in the proof of [BMR1, Proposition 3.2]. Here we give a
direct proof. Take a triangle

$$T_1 \xrightarrow{\ g\ } T_0 \xrightarrow{\ f\ } X \longrightarrow T_1[1] \tag{10}$$

with a minimal right ($\mathsf{add}\,T$)-approximation f and $T_0, T_1 \in \mathsf{add}\,T$. Applying $\overline{(\ )}$ to (10),
we have an exact sequence

$$\overline{T_1} \xrightarrow{\ \overline{g}\ } \overline{T_0} \xrightarrow{\ \overline{f}\ } \overline{X} \longrightarrow 0. \tag{11}$$

This gives a minimal projective presentation of $\overline{X}$ since X has no nonzero indecomposable
direct summands of $T[1]$. Applying the Nakayama functor to (11) and $\mathrm{Hom}_{\mathcal{C}}(T, -)$ to (10)
and comparing them by Lemma 4.2, we have the following commutative diagram of exact
sequences:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \tau\overline{X} & \longrightarrow & \nu\overline{T_1} & \xrightarrow{\nu\overline{g}} & \nu\overline{T_0} \\
 & & & & \downarrow{\wr} & & \downarrow{\wr} \\
0 = \overline{T_0[1]} & \longrightarrow & \overline{X[1]} & \longrightarrow & \overline{T_1[2]} & \xrightarrow{\overline{g[2]}} & \overline{T_0[2]}.
\end{array}
$$

Thus we have $\tau\overline{X} \simeq \overline{X[1]}$.

(b) We have an exact sequence

$$0 \to [T[1]](X, Y[1]) \to \mathrm{Hom}_{\mathcal{C}}(X, Y[1]) \to \mathrm{Hom}_{\mathcal{C}/[T[1]]}(X, Y[1]) \to 0,$$

where $[T[1]]$ is the ideal of $\mathcal{C}$ consisting of morphisms which factor through $\mathsf{add}\, T[1]$. We have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}/[T[1]]}(X, Y[1]) \simeq \mathrm{Hom}_\Lambda(\overline{X}, \overline{Y[1]}) \overset{\text{(a)}}{\simeq} \mathrm{Hom}_\Lambda(\overline{X}, \tau\overline{Y}). \tag{12}$$

On the other hand, the first of following functorial isomorphism was given in [Pa, 3.3].

$$[T[1]](X, Y[1]) \simeq D\,\mathrm{Hom}_{\mathcal{C}/[T[1]]}(Y, X[1]) \overset{(12)}{\simeq} D\,\mathrm{Hom}_\Lambda(\overline{Y}, \tau\overline{X}).$$

Thus the assertion follows.

(c) This is immediate from (b). $\qquad\square$

We now consider the general case, where we allow indecomposable direct summands from $T[1]$ in X or Y.

Proposition 4.4. *Let $X = X' \oplus X''$ and $Y = Y' \oplus Y''$ be objects in $\mathcal{C}$ such that X'' and Y'' are the maximal direct summands of X and Y respectively, which belong to $\mathsf{add}\, T[1]$. Then*

$$\dim \mathrm{Ext}^1_{\mathcal{C}}(X, Y) = \langle \overline{X'}, \tau\overline{Y'} \rangle_\Lambda + \langle \overline{Y'}, \tau\overline{X'} \rangle_\Lambda + \langle \overline{X''[-1]}, \overline{Y'} \rangle_\Lambda + \langle \overline{Y''[-1]}, \overline{X'} \rangle_\Lambda.$$

Proof. Since $\mathrm{Ext}^1_{\mathcal{C}}(X'', Y'') = 0$, we have

$$\dim \mathrm{Ext}^1_{\mathcal{C}}(X, Y) = \dim \mathrm{Ext}^1_{\mathcal{C}}(X', Y') + \dim \mathrm{Ext}^1_{\mathcal{C}}(X'', Y') + \dim \mathrm{Ext}^1_{\mathcal{C}}(X', Y'').$$

By Proposition 4.3, the first term equals $\langle \overline{X'}, \tau\overline{Y'} \rangle_\Lambda + \langle \overline{Y'}, \tau\overline{X'} \rangle_\Lambda$. Clearly the second term equals $\langle \overline{X''[-1]}, \overline{Y'} \rangle_\Lambda$, and the third term equals $\langle \overline{Y''[-1]}, \overline{X'} \rangle_\Lambda$. $\qquad\square$

Now we are ready to prove Theorem 4.1.

By Proposition 4.4, we have that X is rigid if and only if $\widetilde{X}$ is a τ-rigid pair for Λ. Thus we have bijections $\mathrm{rigid}\,\mathcal{C} \leftrightarrow \tau\text{-rigid}\,\Lambda$, which induces a bijection $\mathrm{m\text{-}rigid}\,\mathcal{C} \leftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda$ by Corollary 2.13(a)$\Leftrightarrow$(b).

On the other hand we show that a bijection $\mathrm{c\text{-}tilt}\,\mathcal{C} \leftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda$ is induced. Since $\mathrm{c\text{-}tilt}\,\mathcal{C} \subseteq \mathrm{m\text{-}rigid}\,\mathcal{C}$, we only have to show that any $X \in \mathrm{rigid}\,\mathcal{C}$ satisfying that $\widetilde{X}$ is a support τ-tilting pair for Λ is a cluster-tilting object in $\mathcal{C}$. Assume that $Y \in \mathcal{C}$ satisfies $\mathrm{Ext}^1_{\mathcal{C}}(X, Y) = 0$. By Proposition 4.4, we have $\mathrm{Hom}_\Lambda(\overline{X'}, \tau\overline{Y'}) = 0$, $\mathrm{Hom}_\Lambda(\overline{Y'}, \tau\overline{X'}) = 0$, $\mathrm{Hom}_\Lambda(\overline{X''[-1]}, \overline{Y'}) = 0$ and $\mathrm{Hom}_\Lambda(\overline{Y''[-1]}, \overline{X'}) = 0$. By the first 3 equalities, we have $\overline{Y'} \in \mathsf{add}\,\overline{X'}$ by Corollary 2.13(a)$\Leftrightarrow$(d). By the last equality we have $\overline{Y''[-1]} \in \mathsf{add}\,\overline{X''[-1]}$. Thus $Y \in \mathsf{add}\, X$ holds, which shows that X is a cluster-tilting object in $\mathcal{C}$.

The remaining statements follow immediately. $\qquad\square$

Now we recover the following results in [IY] and [ZZ].

Corollary 4.5. *Let $\mathcal{C}$ be a 2-CY triangulated category with a cluster-tilting object T.*

(a) *[IY] Any basic almost complete cluster-tilting object is a direct summand of exactly two basic cluster-tilting objects. In particular, T is a mutation of V if and only if T and V have all but one indecomposable direct summand in common.*

(b) *[ZZ] An object X in $\mathcal{C}$ is cluster-tilting if and only if it is maximal rigid if and only if it is rigid and $|X| = |T|$.*

Proof. (a) This is immediate from the bijections given in Theorem 4.1 and the corresponding result for support τ-tilting pairs given in Theorem 2.18.

(b) This is the last equality in Theorem 4.1. $\qquad\square$

Connections between cluster-tilting objects in $\mathcal{C}$ and tilting Λ-modules have been investigated in [Smi, FL]. It was shown that a tilting Λ-module always comes from a cluster-tilting object in $\mathcal{C}$, but the image of a cluster-tilting object is not always a tilting Λ-module. This is explained by Theorem 4.1 asserting that the Λ-modules corresponding to the cluster-tilting objects of $\mathcal{C}$ are the support τ-tilting Λ-modules, which are not necessarily tilting Λ-modules.

4.2. **Two-term silting complexes and cluster-tilting objects.** Throughout this subsection, let $\mathcal{C}$ be a 2-CY category with a cluster-tilting object T. Fix a cluster-tilting object $T \in \mathcal{C}$. Let $\Lambda := \mathrm{End}_{\mathcal{C}}(T)$ and let $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ be the homotopy category of bounded complexes of finitely generated projective Λ-modules. In this section, we shall show that there is a bijection between cluster-tilting objects in $\mathcal{C}$ and two-term silting complexes for Λ and that the mutations are compatible with each other.

The following result will be useful, where we denote by $\mathsf{K}^2(\mathsf{proj}\,\Lambda)$ the full subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ consisting of two-term complexes for Λ.

Proposition 4.6. *There exists a bijection*

$$\mathrm{iso}\,\mathcal{C} \longleftrightarrow \mathrm{iso}(\mathsf{K}^2(\mathsf{proj}\,\Lambda))$$

which preserves the number of non-isomorphic indecomposable direct summands.

Proof. For any object $U \in \mathcal{C}$, there exists a triangle

$$T_1 \xrightarrow{\ g\ } T_0 \xrightarrow{\ f\ } U \longrightarrow T_1[1]$$

where $T_1, T_0 \in \mathsf{add}\,T$ and f is a minimal right ($\mathsf{add}\,T$)-approximation. By Lemma 4.2, we have a two-term complex $\overline{T_1} \xrightarrow{\ \overline{g}\ } \overline{T_0}$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.

Conversely, let $P_1 \xrightarrow{\ d\ } P_0$ be a two-term complex for Λ. By Lemma 4.2, there exists a morphism $g : T_1 \to T_0$ in $\mathsf{add}\,T$ such that $\overline{g} = d$. Taking the cone of g, we have an object U in $\mathcal{C}$. Then we can easily check that the correspondence gives a bijection and preserves the number of non-isomorphic indecomposable direct summands. $\qquad\square$

Using this, we get the desired correspondence.

Theorem 4.7. *The bijection in Proposition 4.6 induces bijections*

$$\mathrm{rigid}\,\mathcal{C} \longleftrightarrow 2\text{-}\mathrm{presilt}\,\Lambda \quad and \quad \mathrm{c\text{-}tilt}\,\mathcal{C} \longleftrightarrow 2\text{-}\mathrm{silt}\,\Lambda.$$

Proof. (i) For any rigid object $U \in \mathcal{C}$, we have a triangle

$$T_1 \xrightarrow{\ g\ } T_0 \xrightarrow{\ f\ } U \xrightarrow{\ h\ } T_1[1]$$

where $T_1, T_0 \in \mathsf{add}\,T$ and f is a minimal right ($\mathsf{add}\,T$)-approximation. Let $a : T_1 \to T_0$ be an arbitrary morphism in $\mathcal{C}$. Since U is rigid, we have $fah[-1] = 0$. Thus we have a commutative diagram

$$
\begin{array}{ccccccc}
U[-1] & \xrightarrow{h[-1]} & T_1 & \xrightarrow{\ g\ } & T_0 & \xrightarrow{\ f\ } & U \\
\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow \\
T_1 & \xrightarrow{\ g\ } & T_0 & \xrightarrow{\ f\ } & U & \xrightarrow{\ h\ } & T_1[1]
\end{array}
$$

of triangles in $\mathcal{C}$. Since $hb = 0$, there exists $k_0 : T_0 \to T_0$ such that $b = fk_0$. Since $f(a - k_0g) = 0$, there exists $k_1 : T_1 \to T_1$ such that $gk_1 = a - k_0g$. Therefore we have

$$\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}((\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0}), (\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0})[1]) = 0.$$

Thus $\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0}$ is a presilting complex for Λ.

(ii) Let $P := (P_1 \xrightarrow{d} P_0)$ be a two-term presilting complex for Λ. There exists a unique $g : T_1 \to T_0$ in $\mathsf{add}\,T$ such that $\overline{g} = d$. We consider a triangle

$$T_1 \xrightarrow{g} T_0 \xrightarrow{f} U \xrightarrow{h} T_1[1]$$

in $\mathcal{C}$. We take a morphism $a : U \to U[1]$ in $\mathcal{C}$. Then we have the commutative diagram

$$
\begin{array}{ccccc}
T_1 & \xrightarrow{g} & T_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle h[1]af} & & \downarrow \\
0 & \longrightarrow & T_1[2] & \xrightarrow{g[2]} & T_0[2].
\end{array}
$$

Applying $\overline{(-)}$, we have a commutative diagram

$$
\begin{array}{ccccc}
P_1 & \xrightarrow{d} & P_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \overline{h[1]af}} & & \downarrow \\
0 & \longrightarrow & \nu P_1 & \xrightarrow{\nu d} & \nu P_0.
\end{array}
$$

Thus we have a morphism $P \to \nu P[-1]$ in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. Since P is a presilting complex for Λ, we have

$$\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(P, \nu P[-1]) \simeq \mathrm{D}\,\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(P[-1], P) = 0.$$

Therefore $\overline{h[1]af} = 0$, and the morphism $h[1]af$ factors through $\mathsf{add}\,T[1]$. Hence we have $h[1]af = 0$. Thus we have a commutative diagram

$$
\begin{array}{ccccccc}
T_1 & \xrightarrow{g} & T_0 & \xrightarrow{f} & U & \xrightarrow{h} & T_1[1] \\
& & \downarrow{\scriptstyle a_0} & & \downarrow{\scriptstyle a} & & \\
T_1[1] & \xrightarrow{g[1]} & T_0[1] & \xrightarrow{f[1]} & U[1] & \xrightarrow{h[1]} & T_1[2].
\end{array}
$$

Since $T_0 \in \mathsf{add}\,T$, we have $a_0 = 0$. Thus $af = 0$, so there exists $\varphi : T_1[1] \to U[1]$ such that $a = \varphi h$. Since $T_1 \in \mathsf{add}\,T$, we have $h[1]\varphi = 0$. Thus there exists $b : T_1[1] \to T_0[1]$ such that $\varphi = f[1]b$. Consequently, we have commutative diagrams

$$
\begin{array}{ccccc}
0 & \longrightarrow & T_1 & \xrightarrow{g} & T_0 \\
\downarrow & & \downarrow{\scriptstyle b[-1]} & & \downarrow \\
T_1 & \xrightarrow{g} & T_0 & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccc}
0 & \longrightarrow & P_1 & \xrightarrow{d} & P_0 \\
\downarrow & & \downarrow{\scriptstyle \overline{b[-1]}} & & \downarrow \\
P_1 & \xrightarrow{d} & P_0 & \longrightarrow & 0
\end{array}
$$

Since P is a presilting complex for Λ, there exist $s : T_0[1] \to T_0[1]$ and $t : T_1[1] \to T_1[1]$ such that $b = sg[1] + g[1]t$. Therefore we have

$$a = \varphi h = f[1]bh = f[1]sg[1]h + f[1]g[1]th = 0.$$

Hence $\mathrm{Hom}_{\mathcal{C}}(U, U[1]) = 0$, that is, U is rigid, and the claim follows. $\qquad\square$

Corollary 4.8. *The bijections in Theorems 3.2 and 4.7 induce isomorphisms of the following graphs.*

   (a) *The underlying graph of the support τ-tilting quiver $\mathrm{Q}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ of Λ.*
   (b) *The underlying graph of the two-term silting quiver $\mathrm{Q}(2\text{-silt}\,\Lambda)$ of Λ.*
   (c) *The cluster-tilting graph $\mathrm{G}(\text{c-tilt}\,\mathcal{C})$ of $\mathcal{C}$.*

Proof. (a) and (b) are the same by Corollary 3.9.

We show that (b) and (c) are the same. Let U and V be cluster-tilting objects in $\mathcal{C}$. Let P and Q be the two-term silting complexes for Λ corresponding respectively to U and V by Theorem 4.7. By Corollary 4.5(a) the following conditions are equivalent:

   (a) There exists an edge between U and V in the exchange graph.
   (b) U and V have all but one indecomposable direct summand in common.

Clearly (b) is equivalent to the following condition:

   (c) P and Q have all but one indecomposable direct summand in common.

Now (c) is equivalent to the following condition by Corollary 3.8(b).

   (d) There exists an edge between P and Q in the underlying graph of the silting quiver.

Therefore the exchange graph of $\mathcal{C}$ and the underlying graph of the silting full subquiver consisting of two-term complexes for Λ coincide. $\square$

We end this section with the following application.

Corollary 4.9. *If $\mathrm{G}(\text{c-tilt}\,\mathcal{C})$ has a finite connected component C, then $\mathrm{G}(\text{c-tilt}\,\mathcal{C}) = C$.*

Proof. This is immediate from Corollaries 2.38 and 4.8. $\square$

5. Numerical invariants

In this section, we introduce g-vectors following [AR1] and [DeKe]. We show that g-vectors of indecomposable direct summands of support τ-tilting modules form a basis of the Grothendieck group (Theorem 5.1). Moreover we observe that non-isomorphic τ-rigid pairs have different g-vectors (Theorem 5.5). In [DWZ] the authors defined what they called E-invariants of finite dimensional decorated representations of Jacobian algebras, and used this to solve several conjectures from [FZ]. In the case of finite dimensional Jacobian algebras they showed that the E-invariants were given by formulas which we were led to in section 4.1, by considering $\dim_K \mathrm{Ext}^1_{\mathcal{C}}(T,T)$ for a cluster-tilting object T in $\mathcal{C}$. We here consider E-invariants for any finite dimensional algebra, using the same formula, and show that they can be expressed in terms of homomorphism spaces, dimension vectors and g-vectors. We give some further results on the case of 2-CY tilted algebras, including a comparison for neighbouring 2-CY tilted algebras (Theorem 5.7).

5.1. g-vectors and E-invariants for finite dimensional algebras. Recall from [DeKe] that the g-vectors are defined as follows: Let $K_0(\mathsf{proj}\,\Lambda)$ be the Grothendieck group of the additive category $\mathsf{proj}\,\Lambda$. Then the isomorphism classes $P(1), \ldots, P(n)$ of indecomposable projective Λ-modules form a basis of $K_0(\mathsf{proj}\,\Lambda)$. Consider M in $\mathsf{mod}\,\Lambda$ and let

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be its minimal projective presentation in $\mathsf{mod}\,\Lambda$. Then we write

$$P_0 - P_1 = \sum_{i=1}^{n} g_i^M P(i),$$

where by definition $g^M = (g_1^M, \ldots, g_n^M)$ is the *g-vector* of M. The element $P_0 - P_1$ is also called an *index* of M, which was investigated in [AR1], in connection with studying modules determined by their composition factors, and in [DeKe].

Another useful vector associated with M is the dimension vector $c^M = (c_1^M, \ldots, c_n^M)$. Denote by $S(i)$ the simple top of $P(i)$. Then c_i^M is by definition the multiplicity of the simple module $S(i)$ as composition factor of M. This vector has played an important role in cluster theory for the acyclic case, since the denominators of cluster variables are determined by dimension vectors of indecomposable rigid modules over path algebras [BMRT, CK]. Now this result is not true in general [BMR2].

We have the following result on g-vectors of support τ-tilting modules.

Theorem 5.1. *Let (M, P) be a support τ-tilting pair for Λ with $M = \bigoplus_{i=1}^{\ell} M_i$ and $P = \bigoplus_{i=\ell+1}^{n} P_i$ with M_i and P_i indecomposable. Then $g^{M_1}, \cdots, g^{M_\ell}, g^{P_{\ell+1}}, \cdots, g^{P_n}$ form a basis of the Grothendieck group $K_0(\mathsf{proj}\Lambda)$.*

Proof. By Theorem 3.2, we have a corresponding silting complex $Q = \bigoplus_{i=1}^{n} Q_i$ for Λ with indecomposable Q_i, where the vectors $g^{M_1}, \cdots, g^{M_\ell}, g^{P_{\ell+1}}, \cdots, g^{P_n}$ are exactly the classes of $Q_1, \cdots, Q_n$ in the Grothendieck group $K_0(\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\Lambda)) = K_0(\mathsf{proj}\Lambda)$. By Proposition 1.6(b), we have the assertion. $\square$

This gives a result below due to Dehy-Keller. Recall that for a cluster-tilting object $T \in \mathcal{C}$ and an object $X \in \mathcal{C}$, there exists a triangle

$$T'' \to T' \to X \to T''[1]$$

in $\mathcal{C}$ with $T', T'' \in \mathsf{add}T$. We call $\mathrm{ind}_T(X) := T' - T'' \in K_0(\mathsf{add}T)$ the *index* of X.

Corollary 5.2. [DeKe, Theorem 2.4] *Let $\mathcal{C}$ be a 2-CY triangulated category, and T and $U = \bigoplus_{i=1}^{n} U_i$ be basic cluster-tilting objects with U_i indecomposable. Then the indices $\mathrm{ind}_T(U_1), \cdots, \mathrm{ind}_T(U_n)$ form a basis of the Grothendieck group $K_0(\mathsf{add}T)$ of the additive category $\mathsf{add}T$.*

Proof. We can assume that $U_i \notin \mathsf{add}T[1]$ for $1 \le i \le \ell$, and $U_i \in \mathsf{add}T[1]$ for $\ell+1 \le i \le n$. Then $(\bigoplus_{i=1}^{\ell} \overline{U_i}, \bigoplus_{i=\ell+1}^{n} \overline{U_i[-1]})$ is a support τ-tilting pair for Λ by Theorem 4.1. The equivalence $\mathrm{Hom}_{\mathcal{C}}(T, -) : \mathsf{add}T \to \mathsf{proj}\Lambda$ gives an isomorphism $K_0(\mathsf{add}T) \simeq K_0(\mathsf{proj}\Lambda)$. This sends $\mathrm{ind}_T(U_i)$ to $g^{\overline{U_i}}$ for $1 \le i \le \ell$, and to $-g^{\overline{U_i[-1]}}$ for $\ell+1 \le i \le n$. Thus the assertion follows from Theorem 5.1. $\square$

Now we consider a pair $M = (X, P)$ of a Λ-module X and a projective Λ-module P. We regard a Λ-module X as a pair $(X, 0)$. For such pairs $M = (X, P)$ and $N = (Y, Q)$, let

$$\begin{aligned}
g^M &:= g^X - g^P, \\
E'_\Lambda(M, N) &:= \langle X, \tau Y \rangle + \langle P, Y \rangle, \\
E_\Lambda(M, N) &:= E'_\Lambda(M, N) + E'_\Lambda(N, M), \\
E_\Lambda(M) &:= E_\Lambda(M, M).
\end{aligned}$$

We call g^M the *g-vector* of M, and $E_\Lambda(M, N)$ the *E-invariant* of M and N. Clearly a pair $(M, 0)$ is τ-rigid if and only if $E_\Lambda(M) = 0$.

There is the following relationship between E-invariants and g-vectors, where we denote by $a \cdot b$ the standard inner product $\sum_{i=1}^{n} a_i b_i$ for vectors $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$.

Proposition 5.3. *Let* Λ *be a finite dimensional algebra, and let* X *and* Y *be in* $\mathsf{mod}\,\Lambda$. *Then we have the following.*

$$
\begin{aligned}
E'_\Lambda(X,Y) &= \langle Y, X \rangle - g^Y \cdot c^X, \\
E_\Lambda(X,Y) &= \langle Y, X \rangle + \langle X, Y \rangle - g^Y \cdot c^X - g^X \cdot c^Y, \\
E_\Lambda(X) &= 2(\langle X, X \rangle - g^X \cdot c^X).
\end{aligned}
$$

Proof. We only have to show the first equality. Since $P_0 - P_1 = \sum_{i=1}^{n} g_i^Y P(i)$, then $\langle P_0, X \rangle - \langle P_1, X \rangle = g^Y \cdot c^X$. By Proposition (2.4)(a), we have

$$
E'_\Lambda(X,Y) = \langle X, \tau Y \rangle = \langle Y, X \rangle + \langle P_1, X \rangle - \langle P_0, X \rangle = \langle Y, X \rangle - g^Y \cdot c^X.
$$

$\square$

The following more general description of E-invariants is also clear.

Proposition 5.4. *For any pair* $M = (X, P)$ *and* $N = (Y, Q)$, *we have*

$$
E_\Lambda(M, N) = \langle Y, X \rangle + \langle X, Y \rangle - g^M \cdot c^Y - g^N \cdot c^X.
$$

We end this subsection with the following analog of [DeKe, Theorem 2.3], which was also observed by Plamondon.

Theorem 5.5. *The map* $M \mapsto g^M$ *gives an injection from the set of isomorphism classes of* τ-*rigid pairs for* Λ *to* $K_0(\mathsf{proj}\,\Lambda)$.

Proof. The proof is based on Propositions 2.4(c) and 2.5, and is the same as that of [DeKe, Theorem 2.3]. $\square$

5.2. *E-invariants for 2-CY tilted algebras.* In the rest of this section, let $\mathcal{C}$ be a 2-CY triangulated K-category and let T be a cluster-tilting object in $\mathcal{C}$. Let $\Lambda := \mathrm{End}_\mathcal{C}(T)$. For any object $X \in \mathcal{C}$, we take a decomposition $X = X' \oplus X''$ where X'' is a maximal direct summand of X which belongs to $\mathsf{add}\,T[1]$ and define a pair by

$$
\widetilde{X}_\Lambda := (\overline{X'}, \overline{X''[-1]}),
$$

where $\overline{(-)}$ is an equivalence $\mathrm{Hom}_\mathcal{C}(T, -) : \mathcal{C}/[T[1]] \to \mathsf{mod}\,\Lambda$ given in (9).

We have the following interpretation of E-invariants.

Proposition 5.6. *We have* $E_\Lambda(\widetilde{X}_\Lambda, \widetilde{Y}_\Lambda) = \dim_K \mathrm{Ext}^1_\mathcal{C}(X, Y)$ *for any* $X, Y \in \mathcal{C}$.

Proof. This is immediate from Proposition 4.4 and our definition of E-invariants. $\square$

Now let T' be a cluster-tilting mutation of T. Then we refer to the 2-CY-tilted algebras $\Lambda = \mathrm{End}_\mathcal{C}(T)$ and $\Lambda' = \mathrm{End}_\mathcal{C}(T')$ as *neighbouring* 2-CY-tilted algebras. We define a pair $\widetilde{X}_{\Lambda'}$ for Λ' in a similar way to $\widetilde{X}_\Lambda$ by using the equivalence $\mathrm{Hom}_\mathcal{C}(T', -) : \mathcal{C}/[T'[1]] \to \mathsf{mod}\,\Lambda'$.

By our approach to the E-invariant, the following is now a direct consequence.

Theorem 5.7. *With the above notation, let* M *and* N *be objects in* $\mathcal{C}$. *Then* $E_\Lambda(\widetilde{M}_\Lambda, \widetilde{N}_\Lambda) = E_{\Lambda'}(\widetilde{M}_{\Lambda'}, \widetilde{N}_{\Lambda'})$.

Proof. This is clear from Proposition 5.6 since both sides are equal to $\dim_K \mathrm{Ext}^1_\mathcal{C}(M, N)$. $\square$

In particular, $\widetilde{M}_\Lambda$ is τ-rigid if and only if $\widetilde{M}_{\Lambda'}$ is τ-rigid.

This result is analogous to the corresponding result for (neighbouring) Jacobian algebras proved in [DWZ], in a larger generality. It is however not clear whether the two concepts of neighbouring algebras coincide for finite dimensional neighbouring Jacobian algebras. See [BIRS] for more information.

6. Examples

In this section we illustrate some of our work with easy examples.

Example 6.1. Let Λ be a local finite dimensional K-algebra. Then we have sτ-tilt$\Lambda = \{\Lambda, 0\}$ since the condition $\mathrm{Hom}_\Lambda(M, \tau M) = 0$ implies either $M = 0$ or $\tau M = 0$ (i.e. M is projective). We have Q(sτ-tiltΛ) = ($\Lambda \longrightarrow 0$), Q(f-torsΛ) = ($\mathrm{mod}\Lambda \longrightarrow 0$) and Q(2-siltΛ) = ($\Lambda \longrightarrow \Lambda[1]$).

Example 6.2. Let Λ be a finite dimensional K-algebra given by the quiver $1 \underset{b}{\overset{a}{\rightleftarrows}} 2$ with relations $ab = ba = 0$. Then Q(sτ-tiltΛ), Q(f-torsΛ) and Q(2-siltΛ) are the following:

Example 6.3. Let Λ be a finite dimensional K-algebra given by the quiver $\begin{smallmatrix} & 2 & \\ a\nearrow & & \searrow b \\ 1 & \xleftarrow{c} & 3 \end{smallmatrix}$
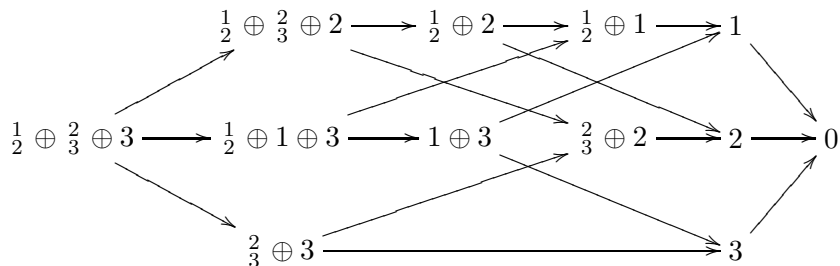
with relations $ab = bc = ca = 0$. Then Λ is a cluster-tilted algebra of type A_3, and there are 14 elements in c-tilt$\mathcal{C}$ for the cluster category $\mathcal{C}$ of type A_3. By our bijections, we know that there are 14 elements in each set sτ-tiltΛ, f-torsΛ and 2-siltΛ.

$$\begin{array}{ccc}
\frac{1}{2} \oplus \frac{2}{3} \oplus 2 & \longrightarrow & \frac{2}{3} \oplus 2 \\
& \searrow & \\
& \frac{1}{2} \oplus 2 \longrightarrow 2 \\
\frac{1}{2} \oplus \frac{2}{3} \oplus \frac{3}{1} \longrightarrow \frac{1}{2} \oplus 1 \oplus \frac{3}{1} \longrightarrow \frac{1}{2} \oplus 1 \\
& \frac{3}{1} \oplus 1 \longrightarrow 1 \longrightarrow 0 \\
3 \oplus \frac{2}{3} \oplus \frac{3}{1} \longrightarrow \frac{3}{1} \oplus 3 \\
& \frac{2}{3} \oplus 3 \longrightarrow 3
\end{array}$$
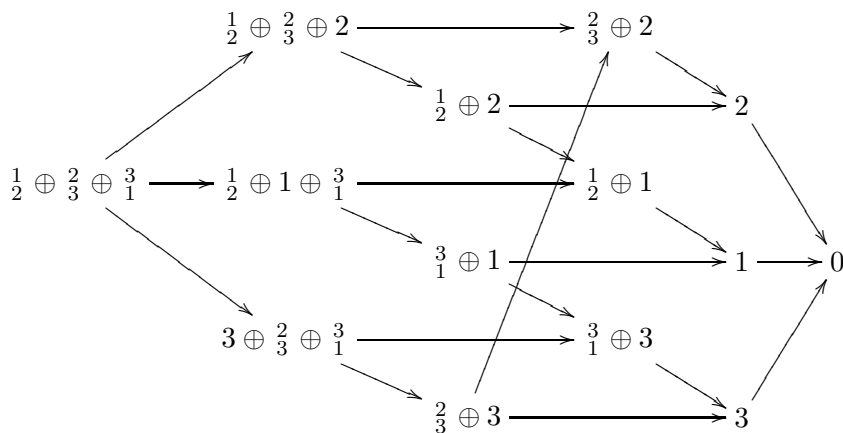
Example 6.4. Let $\Lambda = KQ/\langle \alpha\beta \rangle$, where Q is the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Then $T = S_1 \oplus P_1 \oplus P_3$ is a τ-tilting module which is not a tilting module. Here S_i denotes the simple Λ-module associated with the vertex i, and P_i denotes the corresponding indecomposable projective Λ-module.

In this case there are 12 basic support τ-tilting Λ-modules, and $\mathrm{Q}(\mathrm{s}\tau\text{-}\mathrm{tilt}\,\Lambda)$ is the following.

$$\begin{array}{ccccc}
\frac{1}{2} \oplus \frac{2}{3} \oplus 2 & \longrightarrow & \frac{1}{2} \oplus 2 & \longrightarrow \frac{1}{2} \oplus 1 \longrightarrow 1 \\
\frac{1}{2} \oplus \frac{2}{3} \oplus 3 \longrightarrow \frac{1}{2} \oplus 1 \oplus 3 \longrightarrow 1 \oplus 3 & & \frac{2}{3} \oplus 2 \longrightarrow 2 \longrightarrow 0 \\
& \frac{2}{3} \oplus 3 & \longrightarrow & 3
\end{array}$$

Part 2. The classification of τ-tilting modules over Nakayama algebras

This part is based on the paper [Ad1].

7. Preliminaries

Let Λ be a basic finite dimensional K-algebra. In this section, we collect basic results which are necessary in this part.

By the following lemma, we may regard (Λ/I)-modules as Λ-modules.

Lemma 7.1. [ASS, A.6.1] *Let I be a two-sided ideal of Λ. Then the natural surjection $\Lambda \to \Lambda/I$ induces a fully faithful functor $\mathsf{mod}(\Lambda/I) \to \mathsf{mod}\,\Lambda$.*

The following elementary lemma is often used.

Lemma 7.2. *Let $M \in \mathsf{mod}\,\Lambda$ be indecomposable, $P \in \mathsf{mod}\,\Lambda$ indecomposable projective, and $E \in \mathsf{mod}\,\Lambda$ indecomposable injective.*
  (1) *[ARS, II.1] The following hold.*
      (a) $\mathrm{Hom}_\Lambda(P, M) \neq 0$ *if and only if M has $\mathrm{top}\,P$ as a composition factor.*
      (b) $\mathrm{Hom}_\Lambda(M, E) \neq 0$ *if and only if M has $\mathrm{soc}\,E$ as a composition factor.*
  (2) *[ASS, IV.3.5] The following hold.*

(a) *Assume that $M \not\simeq P$. The natural injection* $\operatorname{rad} P \to P$ *induces an isomorphism* $\operatorname{Hom}_\Lambda(M, P) \simeq \operatorname{Hom}_\Lambda(M, \operatorname{rad} P)$.

(b) *Assume that $M \not\simeq E$. The natural surjection* $E \to E/\operatorname{soc} E$ *induces an isomorphism* $\operatorname{Hom}_\Lambda(E, M) \simeq \operatorname{Hom}_\Lambda(E/\operatorname{soc} E, M)$.

We recall the definition and basic properties of τ-tilting modules. For more details, we refer to [AIR]. We denote by $|M|$ the number of pairwise nonisomorphic indecomposable summands of a Λ-module M.

Definition 7.3. (1) We call M in $\operatorname{mod}\Lambda$ *τ-rigid* if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$.

(2) We call M in $\operatorname{mod}\Lambda$ *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$.

It is known that

(i) If M is τ-rigid, then $|M| \leq |\Lambda|$.

(ii) Each τ-tilting Λ-module M is *sincere* (*i.e.*, every simple Λ-module appears as a composition factor in M).

For τ-tilting modules, we have an analog of Bongartz Lemma for tilting modules.

Proposition 7.4. [AIR, Theorem 2.10 and 2.12] *Let M be a τ-rigid Λ-module. Then there exists $N \in \operatorname{mod}\Lambda$ such that $M \oplus N$ is a τ-tilting Λ-module. Moreover, M is a τ-tilting Λ-module if and only if it is a maximal τ-rigid Λ-module (i.e if $M \oplus L$ is τ-rigid for some Λ-module L, then $L \in \operatorname{add} M$).*

Definition 7.5. We call M in $\operatorname{mod}\Lambda$ *support τ-tilting* if there exists an idempotent $e_M \in \Lambda$ such that M is a τ-tilting $(\Lambda/\langle e_M \rangle)$-module. Note that e_M is uniquely determined since M is a sincere $(\Lambda/\langle e_M \rangle)$-module by (ii) above. If moreover $e_M \neq 0$, M is called a *proper support τ-tilting* Λ-module.

Throughout this part, we denote by $\operatorname{tilt}\Lambda$ (respectively, τ-$\operatorname{tilt}\Lambda$, $\operatorname{s}\tau$-$\operatorname{tilt}\Lambda$, $\operatorname{ps}\tau$-$\operatorname{tilt}\Lambda$) the set of isomorphism classes of basic tilting (respectively, τ-tilting, support τ-tilting, proper support τ-tilting) Λ-modules. The following obsevations are clear.

Proposition 7.6. (1) $\operatorname{s}\tau$-$\operatorname{tilt}\Lambda = \tau$-$\operatorname{tilt}\Lambda \coprod \operatorname{ps}\tau$-$\operatorname{tilt}\Lambda$.

(2) $\operatorname{ps}\tau$-$\operatorname{tilt}\Lambda = \coprod_{e \in \mathcal{E}_\Lambda \setminus \{0\}} \tau$-$\operatorname{tilt}(\Lambda/\langle e \rangle)$.

(3) *If Λ is hereditary, then τ-$\operatorname{tilt}\Lambda = \operatorname{tilt}\Lambda$.*

The following lemma is useful.

Lemma 7.7. [AIR, Lemma 2.1] *Let I be a two-sided ideal of Λ, and $M, N \in \operatorname{mod}(\Lambda/I)$. Then the following hold.*

(1) *If $\operatorname{Hom}_\Lambda(N, \tau_\Lambda M) = 0$, then $\operatorname{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.*

(2) *The converse of (1) holds if $I = \langle e \rangle$ for an idempotent $e \in \Lambda$.*

We give a criterion for τ-rigid modules to be support τ-tilting modules. We denote by $\operatorname{s}(M)$ the number of nonisomorphic simple modules appearing in a composition series of $M \in \operatorname{mod}\Lambda$.

Proposition 7.8. *Let M be a τ-rigid Λ-module. Then the following are equivalent:*

(1) *M is a support τ-tilting Λ-module.*

(2) *$|M| = \operatorname{s}(M)$.*

Proof. Let $e \in \Lambda$ be a maximal idempotent such that $\operatorname{Hom}_\Lambda(e\Lambda, M) = 0$. Namely, M does not have top $e\Lambda$ as a composition factor by Lemma 7.2(1). Then we have $\operatorname{s}(M) = |\Lambda| - |e\Lambda|$. Thus (1)$\Rightarrow$(2) holds clearly. On the other hand, (2)$\Rightarrow$(1) holds since M is a τ-rigid $(\Lambda/\langle e \rangle)$-module by Lemma 7.7(1). $\qquad\square$

We call M in $\mathsf{mod}\,\Lambda$ *almost complete support τ-tilting* if there exists an idempotent $e \in \Lambda$ such that M is a τ-rigid $(\Lambda/\langle e \rangle)$-module and $|M| = |\Lambda| - |e\Lambda| - 1$. In this case, we write (M, e) instead of M.

Proposition 7.9. [AIR, Theorem 2.18] *Any basic almost complete support τ-tilting Λ-module (M, e) is a direct summand of exactly two basic support τ-tilting Λ-modules L and N such that $e\Lambda \in \mathsf{add}\,e_L\Lambda \cap \mathsf{add}\,e_N\Lambda$.*

For a Λ-module M, we denote by $\mathsf{Fac}(M)$ the full subcategory of $\mathsf{mod}\,\Lambda$ consisting of factor modules of direct sums of copies of M.

Definition-Proposition 7.10. [AIR, Theorem 2.7] *For any $M, N \in \mathrm{s}\tau\text{-tilt}\,\Lambda$, we write $M \geq N$ if $\mathsf{Fac}(M) \supseteq \mathsf{Fac}(N)$. Then $\geq$ gives a partial order on $\mathrm{s}\tau\text{-tilt}\,\Lambda$.*

We introduce the Hasse quiver of $\mathrm{s}\tau\text{-tilt}\,\Lambda$.

Definition 7.11. We define the *Hasse quiver* $\mathrm{H}(\Lambda)$ of $\mathrm{s}\tau\text{-tilt}\,\Lambda$ as follows:
- The set of vertices is $\mathrm{s}\tau\text{-tilt}\,\Lambda$.
- We draw an arrow from M to N if $M > N$ and there exists no $L \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ such that $M > L > N$.

Remark 7.12. By [AIR, Corollary 2.34], the Hasse quiver $\mathrm{H}(\Lambda)$ is a $|\Lambda|$-regular graph.

8. Classification of τ-tilting modules over Nakayama algebras

In this section, for Nakayama algebras, we study a connection between (1) τ-tilting modules, (2) proper support τ-tilting modules, (3) triangulations of a regular polygon with a puncture, and (4) certain integer sequences. At Table 1 in Subsection 8.4, we give an example of these correspondences.

Recall the definition and basic properties of Nakayama algebras. A module M is said to be *uniserial* if it has a unique composition series. A finite dimensional algebra is said to be *Nakayama* if every indecomposable projective module and every indecomposable injective module are uniserial. The following quivers will play a central role in this part.
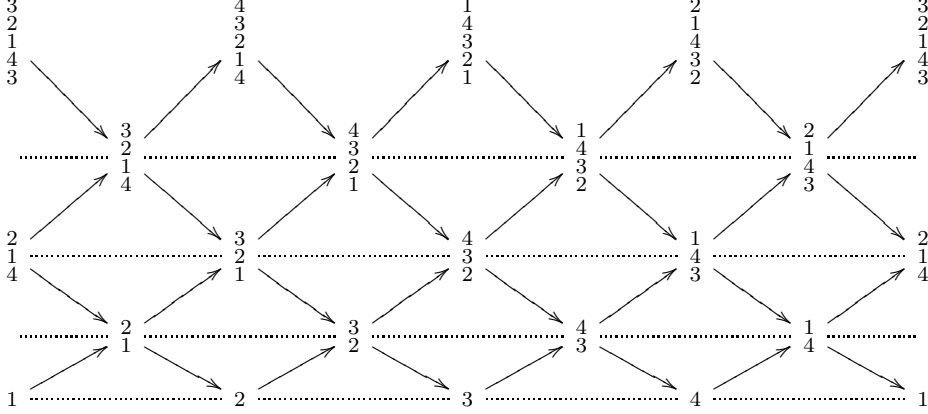

Proposition 8.1. [ASS, V.3.2] *A basic connected algebra is Nakayama if and only if its quiver is either $\vec{A}_n$ or $\vec{\Delta}_n$.*

Throughout this section, we assume that Λ is a basic connected Nakayama algebra with n simple modules. We give a concrete description of indecomposable modules over Nakayama algebras. We denote by $\ell(M)$ the Loewy length of $M \in \mathsf{mod}\,\Lambda$.

Proposition 8.2. [ASS, V.3.5, V.4.1 and V.4.2] *For any indecomposable Λ-module M, there exists $i \in [1, n]$ and $t \in [1, \ell(P(i))]$ such that $M \simeq P(i)/\mathrm{rad}^t P(i)$ and $t = \ell(M)$. Moreover, if M is not projective, then we have $\tau M \simeq \mathrm{rad}\,P(i)/\mathrm{rad}^{t+1} P(i)$ and $\ell(\tau M) = \ell(M)$.*

We let $\Lambda_n^r := K\vec{\Delta}_n/J^r$, where J is the arrow ideal of $K\vec{\Delta}_n$. The Auslander-Reiten quiver of Λ_n^r can be drawn easily [ASS, V.4.1]. For example, the AR-quiver of Λ_4^5 is given by the following, where we identify the extreme left of the quiver with the extreme right of the quiver, and the broken arrows are the action of the AR-translation τ:

$$
\begin{array}{ccccccccc}
\begin{smallmatrix}3\\2\\1\\4\\3\end{smallmatrix} & & \begin{smallmatrix}4\\3\\2\\1\\4\end{smallmatrix} & & \begin{smallmatrix}1\\4\\3\\2\\1\end{smallmatrix} & & \begin{smallmatrix}2\\1\\4\\3\\2\end{smallmatrix} & & \begin{smallmatrix}3\\2\\1\\4\\3\end{smallmatrix} \\
& \begin{smallmatrix}3\\2\\1\\4\end{smallmatrix} & & \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} & & \begin{smallmatrix}1\\4\\3\\2\end{smallmatrix} & & \begin{smallmatrix}2\\1\\4\\3\end{smallmatrix} & \\
\begin{smallmatrix}2\\1\\4\end{smallmatrix} & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} & & \begin{smallmatrix}4\\3\\2\end{smallmatrix} & & \begin{smallmatrix}1\\4\\3\end{smallmatrix} & & \begin{smallmatrix}2\\1\\4\end{smallmatrix} \\
& \begin{smallmatrix}2\\1\end{smallmatrix} & & \begin{smallmatrix}3\\2\end{smallmatrix} & & \begin{smallmatrix}4\\3\end{smallmatrix} & & \begin{smallmatrix}1\\4\end{smallmatrix} & \\
1 & & 2 & & 3 & & 4 & & 1
\end{array}
$$

By Proposition 8.2, each indecomposable Λ-module M is uniquely determined, up to isomorphism, by its simple top $S(j)$ and the Loewy length $l := \ell(M)$. In this case, M has a unique composition series with the associated composition factors $S((j)_n), S((j-1)_n), \cdots, S((j-l+1)_n)$. Thus we can easily understand the existence of homomorphisms between indecomposable Λ-modules.

Lemma 8.3. *Let $M = P(j)/\operatorname{rad}^l P(j)$ and $N = P(i)/\operatorname{rad}^k P(i)$ for $i, j, k, l \in [1, n]$. The following conditions are equivalent:*

(1) $\operatorname{Hom}_\Lambda(M, N) \neq 0$.
(2) $j \in [i - k + 1, i]_n$ *and* $(i - k + 1)_n \in [j - l + 1, j]_n$.

Moreover, if $l \geq k$, then the following condition is also equivalent:

(3) $\operatorname{Hom}_\Lambda(P(j), N) \neq 0$.

Proof. $(1) \Rightarrow (2)$: If $\operatorname{Hom}_\Lambda(M, N) \neq 0$, then M has $S((i-k+1)_n) = \operatorname{soc} N$ as a composition factor and N has $S(j) = \operatorname{top} M$ as a composition factor by Lemma 7.2(1). Hence the assertion follows.

$(2) \Rightarrow (1)$: By our assumption, there exists an indecomposable Λ-module L such that $\operatorname{top} L = S(j)$, $\operatorname{soc} L = S((i - k + 1)_n)$ and $\ell(L) \leq n$. Then L is a factor module of M and a submodule of N. Thus we have $\operatorname{Hom}_\Lambda(M, N) \neq 0$.

$(1) \Rightarrow (3)$: This is clear since M is a factor module of $P(j)$.

$(3) \Rightarrow (2)$: Since N has $S(j) = \operatorname{top} M$ as a composition factor by Lemma 7.2(1), $l \geq k$ imply that M has $S((i - k + 1)_n) = \operatorname{soc} N$ as a composition factor. Hence the assertion follows. $\qquad \square$

We give a criterion for indecomposable modules to be τ-rigid.

Proposition 8.4. *Let M be an indecomposable nonprojective Λ-module. Then M is τ-rigid if and only if $\ell(M) < n$.*

Proof. By Proposition 8.2, we can assume that $M = P(j)/\operatorname{rad}^l P(j)$ and $\tau M = P(j-1)/\operatorname{rad}^l P(j-1)$. Then we have

$$\operatorname{Hom}_\Lambda(M, \tau M) \neq 0 \overset{8.3}{\Leftrightarrow} \begin{cases} j \in [j-l, j-1]_n \\ (j-l)_n \in [j-l+1, j]_n \end{cases}$$
$$\Leftrightarrow \ell(M) \geq n.$$

$\square$

In view of [AIR, Theorem 3.2], this was independently obtained by Antipov-Zvonareva [AZ, Proposition 5.3].

8.1. **τ-tilting modules and triangulations.** In this subsection, we give a connection between τ-tilting Λ-modules and triangulations of an n-regular polygon with a puncture. Recall the definition and basic results of triangulations. Let $\mathcal{G}_n$ be an n-regular polygon with a puncture. We label the points of $\mathcal{G}_n$ counterclockwise around the boundary by $1, 2, \cdots, n$.

Definition 8.5. Let $i, j \in [1, n]$.

(1) An *inner arc* $\langle i, j \rangle$ in $\mathcal{G}_n$ is a path from the point i to the point j homotopic to the boundary path through $i, i+1, \cdots, i+t = j \pmod n$, where t is an integer satisfying $2 \leq t \leq n$. Then we call i an *initial* point, j a *terminal* point, and $\ell(\langle i, j \rangle) := t$ the *length* of the inner arc. By definition, $\ell(\langle i, j \rangle) \leq n$ holds for any inner arc in $\mathcal{G}_n$.

(2) A *projective arc* $\langle \bullet, j \rangle$ in $\mathcal{G}_n$ is a path from the puncture to the point j. Then we call j a *terminal* point.

(3) An *admissible arc* is an inner arc or a projective arc. Namely,

$$\operatorname{Arc}(n) := \{\text{admissible arcs in } \mathcal{G}_n\} = \{\langle i, j \rangle \mid i, j \in [1, n]\} \coprod \{\langle \bullet, j \rangle \mid j \in [1, n]\}.$$

Note that, if $i \neq j$, $\langle i, j \rangle$ and $\langle j, i \rangle$ are different arcs as the picture in Figure 1 shows.
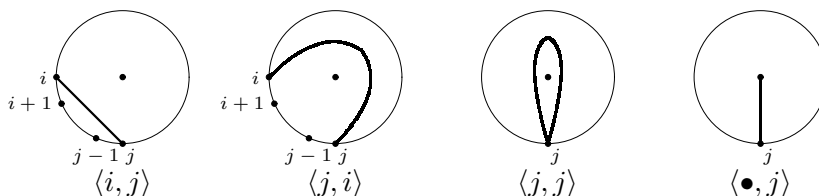

FIGURE 1. Admissible arcs in a polygon with a puncture

Definition 8.6. (1) Two admissible arcs in $\mathcal{G}_n$ are called *compatible* if they do not intersect in $\mathcal{G}_n$ (except their initial and terminal points).

(2) A *triangulation* of $\mathcal{G}_n$ is a maximal set of distinct pairwise compatible admissible arcs. We denote by $\mathcal{T}(n)$ the set of triangulations of $\mathcal{G}_n$.

(3) For integers $l_1, l_2, \cdots, l_n \geq 1$, we denote by $\mathcal{T}(n; l_1, l_2, \cdots, l_n)$ the subset of $\mathcal{T}(n)$ consisting of triangulations such that the length of every inner arc with the terminal point j is at most l_j for any $j \in [1, n]$.

For example, the set of all projective arcs gives a triangulation of $\mathcal{G}_n$.

For a subset X of $\operatorname{Arc}(n)$, we denote by $X_{i,j}$ the subset of X consisting of all inner arcs contained in the fan whose boundary is $\langle \bullet, i \rangle$, $\langle \bullet, j \rangle$ and the edge connecting from i to j in the counterclockwise direction.
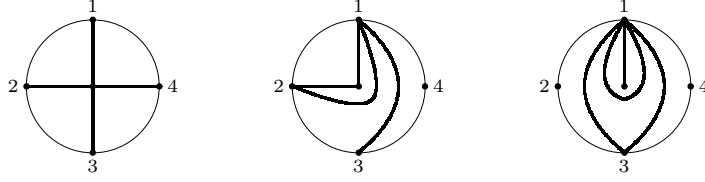
FIGURE 2. Triangulations of $\mathcal{G}_4$

Remark 8.7. Let $X \in \mathcal{T}(n)$. Assume that X contains an inner arc $\langle i, j \rangle$. Then we regard $X_{i,j}$ as a triangulation of the $((j-i)_n + 1)$-regular polygon (with no puncture) by identifying the inner arc $\langle i, j \rangle$ with a side of the polygon. Thus the cardinality of $X_{i,j}$ is equal to $(j-i)_n - 1$.
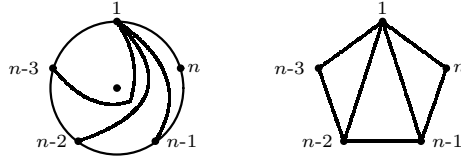
FIGURE 3. Examples of $X_{n-3,1}$ and triangulation of a 5-regular polygon

Triangulations of $\mathcal{G}_n$ have the following properties. Let $X \in \mathcal{T}(n)$. Assume that $1 \leq j_1 < j_2 < \cdots < j_r \leq n$ are all integers satisfying $\langle \bullet, j_i \rangle \in X$, and let $j_i^+ := j_{i+1}$ for any $i \in [1, r]$, where $j_{r+1} := j_1 + n$. Note that, if $j_i^+ - j_i > 0$, then X must contain the inner arc $\langle j_i, j_i^+ \rangle$.

Proposition 8.8. *Each triangulation of $\mathcal{G}_n$ consists of exactly n admissible arcs and contains at least one projective arc.*

Proof. Let $X \in \mathcal{T}(n)$. For a longest inner arc in X with the initial point i and the terminal point j, there exist projective arcs $\langle \bullet, i \rangle, \langle \bullet, j \rangle \in X$ by the maximality of triangulations. Thus any triangulation contains a projective arc. Moreover, by Remark 8.7, the cardinality of X is

$$|X| = \sum_{i=1}^{r} \{1 + (j_i^+ - j_i - 1)\} = j_{r+1} - j_1 = n.$$

$\square$

We give a correspondence between indecomposable τ-rigid modules and admissible arcs. We denote by iτ-rigidΛ the set of isomorphism classes of indecomposable τ-rigid Λ-modules. By Proposition 8.4, every indecomposable nonprojective τ-rigid Λ-module M is uniquely determined by its simple top $S(j)$ and its simple socle $S(k)$. Such an indecomposable τ-rigid module is denoted by $M_{k-2,j}$. Moreover, let $M_{\bullet,j} := P(j)$.

Proposition 8.9. *Let Λ be a Nakayama algebra and $\ell_j := \ell(P(j))$. The following hold.*
  (1) *There is a bijection*

$$\{\langle \bullet, i \rangle \mid i \in [1, n]\} \coprod \{\langle i, j \rangle \mid i, j \in [1, n], \ \ell(\langle i, j \rangle) \leq \ell_j\} \longrightarrow i\tau\text{-rigid}\Lambda$$

  given by $\langle i, j \rangle \mapsto M_{i,j}$ for $i \in [1, n] \coprod \{\bullet\}$ and $j \in [1, n]$.
  (2) *For any $i, k \in [1, n] \coprod \{\bullet\}$ and $j, l \in [1, n]$, $\langle i, j \rangle$ and $\langle k, l \rangle$ are compatible if and only if $M_{i,j} \oplus M_{k,l}$ is τ-rigid.*

Proof. (1) By Proposition 8.4, every indecomposable Λ-module M is either a projective Λ-module or a Λ-module with $\ell(M) < \min\{\ell(P), n\}$, where P is a projective cover of M. Thus there are one-to-one correspondences

$$\{\langle \bullet, j \rangle \mid j \in [1, n]\} \longleftrightarrow \{P(j) \mid j \in [1, n]\}$$

$$\{\langle i, j \rangle \mid i, j \in [1, n], \quad \ell(\langle i, j \rangle) \le \ell_j\} \longleftrightarrow \{M_{i,j} \mid i, j \in [1, n], \quad \ell(M_{i,j}) < \min\{\ell_j, n\}\}.$$

(2) If $i = k \in \{\bullet\}$, then the assertion is clear. We may assume that $i \in [1, n] \coprod \{\bullet\}$ and $j, k, l \in [1, n]$. Since we have $\tau M_{k,l} = M_{k-1,l-1}$ by Proposition 8.2, thus, by Lemma 8.3, $\mathrm{Hom}_\Lambda(M_{i,j}, \tau M_{k,l}) \ne 0$ if and only if $j \in [k+1, l-1]_n$ and $(k+1)_n \in [j - \ell(M_{i,j}) + 1, j]_n$. This means that $\langle i, j \rangle$ and $\langle k, l \rangle$ are compatible. $\square$

As a conclusion, we obtain the following theorem. A similar observation was given independently in [AZ, Proposition 5.4].

Theorem 8.10. *Let Λ be a Nakayama algebra with n simple modules and $\ell_j := \ell(P(j))$ for any $j \in [1, n]$. Then the map in Proposition 8.9 induces a bijection*

$$\tau\text{-tilt}\,\Lambda \longrightarrow \mathcal{T}(n; \ell_1, \ell_2, \cdots, \ell_n).$$

Proof. Each τ-tilting Λ-module is maximal τ-rigid with exactly n indecomposable direct summands by Proposition 7.4. On the other hand, each triangulation of $\mathcal{G}_n$ is a maximal set of pairwise compatible admissible arcs with the cardinality n by Proposition 8.8. Thus the assertion follows from Proposition 8.9(2). $\square$

As an application, we have the following corollary.

Corollary 8.11. *Each τ-tilting Λ-module has a nonzero projective Λ-module as a direct summand.*

Proof. It follows from Theorem 8.10, since any triangulation of $\mathcal{G}_n$ contains a projective arc by Proposition 8.8. $\square$

8.2. **Triangulations and integer sequences.** In this subsection, we give a simple description of τ-tilt Λ and $\mathcal{T}(n)$ in terms of certain integer sequences. Let

$$\mathcal{Z}(n) := \left\{ (a_1, a_2, \cdots, a_n) \in \mathbb{Z}_{\ge 0}^n \mid \sum_{i=1}^n a_i = n \right\}.$$

Let X be a subset of the set $\mathrm{Arc}(n)$ of all admissible arcs in $\mathcal{G}_n$. In view of Proposition 8.9, we define $\mathrm{top}(X) = (a_1, a_2, \cdots, a_n)$, where a_p is the number of admissible arcs in X with a terminal point p. If $X \in \mathcal{T}(n)$, then $\mathrm{top}(X) \in \mathcal{Z}(n)$ since any triangulation of $\mathcal{G}_n$ contains exactly n admissible arcs by Proposition 8.8.

We give the following observation.

Theorem 8.12. *There is a bijection*

$$\mathcal{T}(n) \longrightarrow \mathcal{Z}(n)$$

given by $X \mapsto \mathrm{top}(X)$.

To prove Theorem 8.12, we construct the inverse map $\mathcal{Z}(n) \to \mathcal{T}(n)$. Let $a = (a_1, \cdots, a_n) \in \mathcal{Z}(n)$. We associate to a a sequence $a' = (a'_1, \cdots, a'_n)$ given by

$$a'_i = \sum_{p=1}^i (a_p - 1)$$

and let

$$||a'|| = \max\{a'_1, \cdots, a'_n\}.$$

We define the subset X_a of $\mathrm{Arc}(n)$ as follows: Assume that $1 \leq j_1 < j_2 < \cdots < j_r \leq n$ are all integers such that $a'_{j_i} = ||a'||$ and let $j_i^+ := j_{i+1}$ for any $i \in [1, r]$, where $j_{r+1} := j_1 + n$ and $a_p := a_{(p)_n}$ if p is an integer. For any $l \in [1, n]$, we define $(a_l - \delta_l)$ inner arcs with the terminal point l as follows, where,

$$\delta_\ell := \begin{cases} 1 & ((\ell)_n \in \{j_1, j_2, \cdots, j_r\}) \\ 0 & (else). \end{cases}$$

for any integer ℓ. Note that, if $j_i^+ - j_i = 1$, then $a_{j_i^+} - \delta_{j_i^+} = 0$ holds. Then there does not exist inner arcs with the terminal point j_i^+. Thus we assume that $j_i^+ - j_i > 1$. Moreover, we may assume that $l \in [j_i + 2, j_i^+]$ for some $i \in [1, r]$. For each $s \in [1, a_l - \delta_l]$, there exists unique $k := k_s^{(l)_n} \in [j_i + 1, l - 1]$ satisfying the following conditions:

(a) $a_k = 0$,
(b) $(a_{k+1} - 1) + \cdots + (a_{l-1} - 1) + (s - 1) = 0$,
(c) $(a_{k+1} - 1) + \cdots + (a_{m-1} - 1) + (a_m - 1) \leq 0$ for any integer $k < m < l - 1$.

We define a set X_a of admissible arcs by

$$X_a := \{\langle \bullet, j_i \rangle \mid i \in [1, r]\} \coprod \{\langle (k_s^l - 1)_n, l \rangle \mid l \in [1, n], \ s \in [1, a_l - \delta_l]\}.$$

We explain the construction of X_a by the following example.

Example 8.13. Let $n = 8$ and $a := (0, 4, 1, 0, 1, 0, 2, 0) \in \mathcal{Z}(8)$. Now we associate a to Figure A, where the coordinates of black points are $(i, a'_i + i)$. The number $||a'||$ can be detected by drawing parallel diagonal lines. In this example, $||a'|| = a'_2 = a'_3$ hold because the points $(2, a'_2 + 2)$ and $(3, a'_3 + 3)$ are on the highest diagonal line.

Moreover, we can also detect k_s^l. For example, if $l = 2$ and $s = 3$, then $k_3^2 = 4$.

The corresponding X_a can be obtained in the following way. First we slightly move diagonal lines in Figure A as Figure B. Next, regarding all diagonal lines (except the highest diagonal line) as inner arcs, we get X_a in Figure C.

FIGURE A.
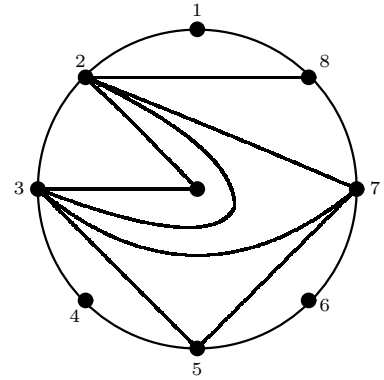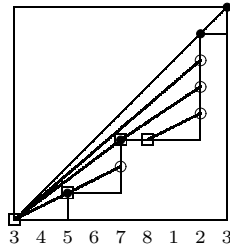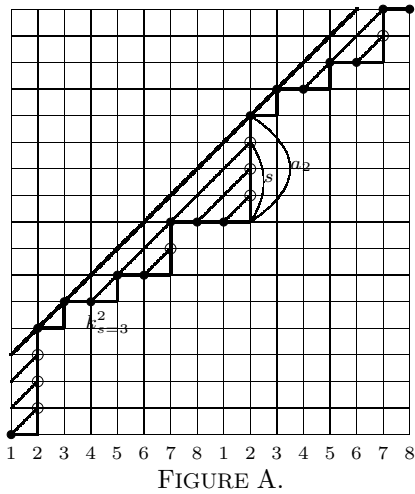
FIGURE B.

FIGURE C.

We note that X_a in Figure C is a triangulation of $\mathcal{G}_8$. This is always the case as the following result shows.

Proposition 8.14. *We have $X_a \in \mathcal{T}(n)$ and $\mathrm{top}(X_a) = a$.*

Proof. Since there are exactly a_l admissible arcs in X_a with a terminal point l by definition, $\mathrm{top}(X_a) = a$ holds. We claim that $X_a \in \mathcal{T}(n)$. Since each projective arc in X_a is compatible with all admissible arcs in X_a, it is enough to show that two inner arcs $\langle (k_s^l - 1)_n, l \rangle, \langle (k_u^h - 1)_n, h \rangle \in X_a$ are compatible. If they are not compatible, we may assume that $1 \le j < k_u^h < k_s^l < h < l \le j^+ \le n$, where j satisfies $a_j' = \|a'\|$. Then, by (b) and (c) in the conditions of k_s^l and k_u^h, we have

$$(a_{k_u^h+1} - 1) + \cdots + (a_{k_s^l-1} - 1) + (a_{k_s^l} - 1) + (a_{k_s^l+1} - 1) + \cdots + (a_h - 1) = a_h - u(\ge 0).$$

Moreover, by (a) and (b), the left-hand side of the equation above is less than zero because we have

$$(a_{k_u^h+1} - 1) + \cdots + (a_{k_s^l-1} - 1) \le 0$$
$$(a_{k_s^l} - 1) = -1$$
$$(a_{k_s^l+1} - 1) + \cdots + (a_h - 1) \le 0.$$

Hence the claim follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now, we are ready to prove Theorem 8.12.

Proof of Theorem 8.12. By Proposition 8.14, we have $\mathrm{top}(X_a) = a$ for any $a \in \mathcal{Z}(n)$. We have only to show that $X_{\mathrm{top}(X)} = X$ for any $X \in \mathcal{T}(n)$. Let $a := \mathrm{top}(X) = (a_1, a_2, \cdots, a_n)$.

(i) First, we show that X and X_a contain exactly the same projective arcs. Namely, we claim that $\langle \bullet, j \rangle \in X$ if and only if $a_j' = \|a'\|$. Indeed, By Proposition 8.8, we can fix a projective arc $\langle \bullet, j \rangle \in X$. The number $a_{j+1} + a_{j+2} + \cdots + a_{j^+}$ is exactly one plus the number of inner arcs in X_{j,j^+} and the number $a_{j+1} + a_{j+2} + \cdots + a_i$ is at most the number of inner arcs in $X_{j,i}$ for any $i \in [j+1, j^+ - 1]$. Thus we have

$$\sum_{p=1}^{j^+} a_p - \sum_{p=1}^{j} a_p = a_{j+1} + a_{j+2} + \cdots + a_{j^+} = j^+ - j$$

$$\sum_{p=1}^{i} a_p - \sum_{p=1}^{j} a_p = a_{j+1} + a_{j+2} + \cdots + a_i < i - j.$$

This implies $a_j' = a_{j^+}'$ and $a_j' > a_i'$. Repeating the same argument, we have $a_j' = a_k'$ if $\langle \bullet, k \rangle \in X$ and $a_j' > a_k'$ otherwise. Hence, the claim follows.

(ii) Next we show that, for any $j, l \in [1, n]$, the inner arc $\langle j, l \rangle \in X$ satisfies (a), (b), and (c) in the conditions of inner arcs in $X_{\mathrm{top}(X)}$. We take an integer $l - n + 1 \le k \le l - 1$ with $j = (k-1)_n$. Since $a_i - \delta_i$ gives the number of inner arcs with a terminal point i, clearly $a_k = 0$ holds. Since $X_{j,l}$ is an triangulation of $(l - k + 2)$-gon by Remark 8.7, we have

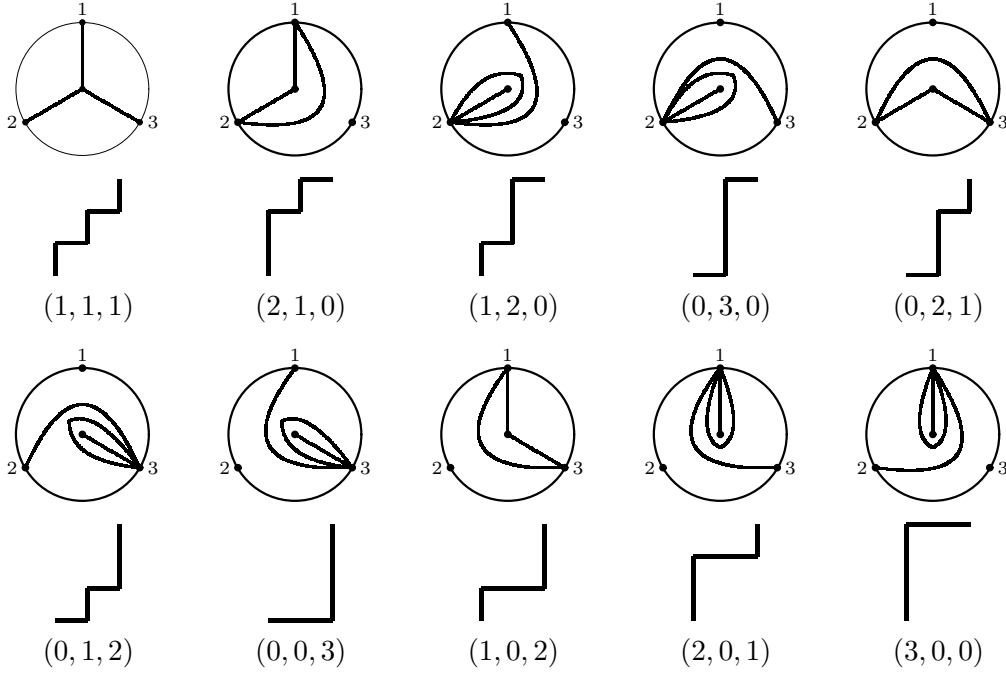$$a_{k+1} + a_{k+2} + \cdots + a_{l'-1} + s = 1 + |X_{j,l}| = l - k$$

for some $1 \le s \le a_l - \delta_l$. Hence (b) holds. Moreover, for any $k < m < l$, since the number $\sum_{p=k+1}^{m} a_p$ is at most one plus the cardinality of $X_{j,(m)_n}$, we have

$$a_{k+1} + a_{k+2} + \cdots + a_{m-1} + a_m \le 1 + |X_{j,(m)_n}| = m - k.$$

Therefore (c) holds. Thus we have $X \subseteq X_a$, and hence $X = X_a$ because each triangulation consists exactly n admissible arcs by Proposition 8.8. □

We give an example of Theorem 8.12.

Example 8.15. Let $n = 3$.

$(1,1,1)$ $\qquad$ $(2,1,0)$ $\qquad$ $(1,2,0)$ $\qquad$ $(0,3,0)$ $\qquad$ $(0,2,1)$

$(0,1,2)$ $\qquad$ $(0,0,3)$ $\qquad$ $(1,0,2)$ $\qquad$ $(2,0,1)$ $\qquad$ $(3,0,0)$

The next result gives a generalization of Theorem 8.12, For $l_1, l_2, \cdots, l_n \geq 1$, we denote by $\mathcal{Z}(n; l_1, \cdots, l_n)$ the subset of $\mathcal{Z}(n)$ consisting of the integer sequences $a = (a_1, \cdots, a_n)$ such that $\ell_j(a) \leq l_j$ for any $j \in [1, n]$, where $\ell_j(a)$ is given by

$$\ell_j(a) := \begin{cases} 0 & \text{if } a_j = 0. \\ (j - k^j_{a_j - \delta_j} + 1)_n & \text{if } a_j > 0. \end{cases}$$

Note that $\ell_j(a)$ is at most n and equals to the maximal length of inner arcs in X_a with a terminal point j.

Theorem 8.16. *Let* $l_1, \cdots, l_n \geq 1$. *There are mutually inverse bijections*

$$\mathcal{T}(n; l_1, \cdots, l_n) \longleftrightarrow \mathcal{Z}(n; l_1, \cdots, l_n)$$

given by $X \mapsto \mathrm{top}(X)$ *and* $a \mapsto X_a$.

Proof. It is clear from Theorem 8.10 and the definition of X_a. □

8.3. τ-**tilting modules and proper support** τ-**tilting modules.** In this subsection, we study a connection between τ-tilting Λ-modules and proper support τ-tilting Λ-modules.

Let $\{e_1, e_2, \cdots, e_n\}$ be a complete set of primitive orthogonal idempotents of Λ and $\mathcal{E}_\Lambda := \{\sum_{j \in J} e_j \mid J \subset [1, n]\}$. We define the bijection

$$\phi : \mathcal{E}_\Lambda \longrightarrow \mathcal{E}_\Lambda$$

given by $\phi(\sum_{i \in I} e_i) = \sum_{i \in I} e_{i-1}$, where $e_0 := e_n$.

We denote by $\mathsf{mod}_{\mathrm{np}}\Lambda$ the full subcategory of $\mathsf{mod}\Lambda$ consisting of Λ-modules which does not have nonzero projective direct summands, and let $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda := \mathrm{ps}\tau\text{-tilt}\Lambda \cap \mathsf{mod}_{\mathrm{np}}\Lambda$. We decompose $M \in \mathsf{mod}\Lambda$ as $M = M_{\mathrm{np}} \oplus M_{\mathrm{pr}}$, where $M_{\mathrm{np}} \in \mathsf{mod}_{\mathrm{np}}\Lambda$ and M_{pr} is a maximal projective direct summand of M.

We state our main theorem in this subsection, where e_M is the idempotent in Definition 7.5.

Theorem 8.17. *Let Λ be a Nakayama algebra. Then the following hold.*

(1) *There are mutually inverse bijections*

$$\tau\text{-tilt}\Lambda \longleftrightarrow \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$$

given by $\tau\text{-tilt}\Lambda \ni M \mapsto M_{\mathrm{np}}$ and $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda \ni M \mapsto M \oplus \phi(e_M)\Lambda$.

(2) *If $\ell(P(i)) \geq n$ for any $i \in [1, n]$, then $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda = \mathrm{ps}\tau\text{-tilt}\Lambda$. In particular, we have a bijection*

$$\tau\text{-tilt}\Lambda \longleftrightarrow \mathrm{ps}\tau\text{-tilt}\Lambda.$$

In the rest of this subsection, we will give a proof of Theorem 8.17. First, we show that the map $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda \ni M \mapsto M \oplus \phi(e_M)\Lambda \in \tau\text{-tilt}\Lambda$ is well-defined.

Proposition 8.18. *If $M \in \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$, then we have $M \oplus \phi(e_M)\Lambda \in \tau\text{-tilt}\Lambda$.*

Proof. Since M is annihilated by e_M, M does not have top $e_M\Lambda$ as a composition factor. By Proposition 8.2, τM does not have $\mathrm{top}(\phi(e_M)\Lambda)$ as a composition factor. Hence $M \oplus \phi(e_M)\Lambda$ is a τ-rigid Λ-module. Moreover, by $M \in \mathsf{mod}_{\mathrm{np}}\Lambda$, we have

$$|M \oplus \phi(e_M)\Lambda| = |M| + |\phi(e_M)\Lambda| = |M| + |e_M\Lambda| = |\Lambda|.$$

Thus $M \oplus \phi(e_M)\Lambda$ is a τ-tilting Λ-module. $\qquad\square$

Conversely, for given a τ-tilting Λ-module, we give a construction of a certain proper support τ-tilting Λ-module. By Corollary 8.11, every τ-tilting Λ-module has a nonzero projective direct summand.

Proposition 8.19. *If $M \in \tau\text{-tilt}\Lambda$, then we have $M_{\mathrm{np}} \in \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$ and $M = M_{\mathrm{np}} \oplus \phi(e_{M_{\mathrm{np}}})\Lambda$.*

Proof. Let $M = M_{\mathrm{np}} \oplus M_{\mathrm{pr}} \in \tau\text{-tilt}\Lambda$. We may assume that $M_{\mathrm{pr}} = e\Lambda$, where $e \in \Lambda$ is a nonzero idempotent. Since M is τ-rigid, τM_{np} does not have top $e\Lambda$ as a composition factor by Lemma 7.2(1). Thus M_{np} does not have $\mathrm{top}(\phi^{-1}(e)\Lambda)$ as a composition factor by Proposition 8.2. Hence M_{np} is a τ-rigid $(\Lambda/\langle\phi^{-1}(e)\rangle)$-module. Moreover, we have

$$|M_{\mathrm{np}}| = |M| - |e\Lambda| = |\Lambda| - |\phi^{-1}(e)\Lambda|.$$

Thus M_{np} is a τ-tilting $(\Lambda/\langle\phi^{-1}(e)\rangle)$-module. Hence, $M_{\mathrm{np}} \in \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$ holds, and moreover we have $\phi^{-1}(e) = e_{M_{\mathrm{np}}}$. Thus $M_{\mathrm{pr}} = e\Lambda = \phi(e_{M_{\mathrm{np}}})\Lambda$. $\qquad\square$

Now we are ready to prove Theorem 8.17.

Proof of Theorem 8.17. (1) It follows from Proposition 8.18 and 8.19.

(2) This is clear because $\ell(P(i)) \geq n$ implies that $P(i)$ is sincere. $\qquad\square$

We give another proof of Corollary 8.11 without using combinatorics.

Proof of Corollary 8.11. Let $M \in \tau$-tilt Λ and L be an indecomposable direct summand of M with the maximal Loewy length. We write $M = L \oplus N$. Then $(N, 0)$ is an almost complete support τ-tilting Λ-module. Assume that M does not have a nonzero projective Λ-module as a direct summand. Since M is τ-rigid, $\mathrm{Hom}_\Lambda(L, \tau N)$ vanishes. Since the Loewy length of L is at least the Loewy length of any indecomposable direct summand of N by the choice of L, we have $\mathrm{Hom}_\Lambda(P_L, \tau N) = 0$ by Lemma 8.3, where P_L is a projective cover of L. Thus $P_L \oplus N$ is a τ-rigid Λ-module. Moreover, since P_L is indecomposable, we have

$$|P_L \oplus N| = |P_L| + |N| = |P_L| + |M| - |L| = |M| = |\Lambda|.$$

Hence, $P_L \oplus N$ is a τ-tilting Λ-module. Moreover, N is a support τ-tilting Λ-module by Proposition 8.19. This means that almost complete support τ-tilting Λ-module $(N, 0)$ is a direct summand of three support τ-tilting Λ-modules $N, L \oplus N$ and $P_L \oplus N$. However, this contradicts Proposition 7.9. □

As an immediate consequence of Theorem 8.17, we have the following statement, where τ-tilt$_{\mathrm{np}}(\Lambda/\langle e \rangle) := \tau$-tilt$(\Lambda/\langle e \rangle) \cap \mathsf{mod}_{\mathrm{np}}\Lambda$.

Corollary 8.20. *Let Λ be a Nakayama algebra.*

(1) *We have*
$$\mathsf{s}\tau\text{-tilt}\,\Lambda = \coprod_{e \in \mathcal{E}_\Lambda \setminus \{0\}} \left(\tau\text{-tilt}\,(\Lambda/\langle e \rangle) \coprod \{M \oplus \phi(e)\Lambda \mid M \in \tau\text{-tilt}_{\mathrm{np}}(\Lambda/\langle e \rangle)\} \right).$$

(2) *If $\ell(P(i)) \geq n$ for any $i \in [1, n]$, we have*
$$\mathsf{s}\tau\text{-tilt}\,\Lambda = \coprod_{e \in \mathcal{E}_\Lambda \setminus \{0\}} \{M,\ M \oplus \phi(e)\Lambda \mid M \in \mathrm{tilt}\,(\Lambda/\langle e \rangle)\}.$$

Proof. By Theorem 8.17(1), we have
$$\tau\text{-tilt}\,\Lambda = \coprod_{e \in \mathcal{E}_\Lambda \setminus \{0\}} \{M \oplus \phi(e)\Lambda \mid M \in \tau\text{-tilt}_{\mathrm{np}}(\Lambda/\langle e \rangle)\}. \tag{13}$$

(1) The assertion follows from (13), Proposition 7.6(1) and (2).
(2) By Theorem 8.17(2), we can omit "np" in (13), and hence we have
$$\mathsf{s}\tau\text{-tilt}\,\Lambda = \coprod_{e \in \mathcal{E}_\Lambda \setminus \{0\}} \{M,\ M \oplus \phi(e)\Lambda \mid M \in \tau\text{-tilt}\,(\Lambda/\langle e \rangle)\}. \tag{14}$$

By Proposition 7.6(3), we can replace τ-tilt$(\Lambda/\langle e \rangle)$ in (14) by tilt$(\Lambda/\langle e \rangle)$. Thus the assertion follows. □

Finally, we give an example.

Example 8.21. Let $\Lambda := \Lambda_3^3 = K\vec{\Delta}_3/J^3$ (see Proposition 8.1). To obtain τ-tilting Λ-modules, let us calculate $\Lambda/\langle e \rangle$ for any idempotent $e \in \mathcal{E}_\Lambda$. We have $\Lambda/\langle e_i \rangle \simeq K\vec{A}_2$, $\Lambda/\langle e_i + e_{i+1} \rangle \simeq K\vec{A}_1$, and $\Lambda/\langle e_1 + e_2 + e_3 \rangle = \{0\}$ for $i \in [1, 3]$. Thus we have

$$\tau\text{-tilt}\,(\Lambda/\langle e_i \rangle) = \mathrm{tilt}\,(K\vec{A}_2) = \{\,{}^{i+2}_{i+1} \oplus {}_{i+2},\ {}^{i+2}_{i+1} \oplus {}_{i+1}\,\}$$
$$\tau\text{-tilt}\,(\Lambda/\langle e_i + e_{i+1} \rangle) = \mathrm{tilt}\,(K\vec{A}_1) = \{\,{}_{i+2}\,\}$$
$$\tau\text{-tilt}\,(\Lambda/\langle e_1 + e_2 + e_3 \rangle) = \{0\}.$$

Applying Theorem 8.17, we have

$$\text{s}\tau\text{-tilt}\,\Lambda = \big\{\,\{0\}\,,\,\substack{1}\,,\,\substack{2}\,,\,\substack{3}\,,\,\tfrac{1}{3}\oplus\substack{1}\,,\,\tfrac{1}{3}\oplus\substack{3}\,,\,\tfrac{2}{1}\oplus\substack{2}\,,\,\tfrac{2}{1}\oplus\substack{1}\,,\,\tfrac{3}{2}\oplus\substack{3}\,,\,\tfrac{3}{2}\oplus\substack{2}\,\big\}$$

$$\coprod\big\{\substack{1\\3\\2}\oplus\substack{2\\1\\3}\oplus\substack{3\\2\\1}\,,\,\substack{1}\oplus\substack{1\\3\\2}\oplus\substack{2\\1\\3}\,,\,\substack{2}\oplus\substack{2\\1\\3}\oplus\substack{3\\2\\1}\,,\,\substack{3}\oplus\substack{3\\2\\1}\oplus\substack{1\\3\\2}\,,$$

$$\substack{1\\3}\oplus\substack{1}\oplus\substack{1\\3\\2}\,,\,\substack{1\\3}\oplus\substack{3}\oplus\substack{3\\2}\,,\,\substack{2\\1}\oplus\substack{2}\oplus\substack{2\\1\\3}\,,\,\substack{2\\1}\oplus\substack{1}\oplus\substack{1\\3}\,,\,\substack{3\\2}\oplus\substack{3}\oplus\substack{3\\2\\1}\,,\,\substack{3\\2}\oplus\substack{2}\oplus\substack{2\\1}\big\}.$$

Moreover, the Hasse quiver $H(\Lambda)$ is the following:
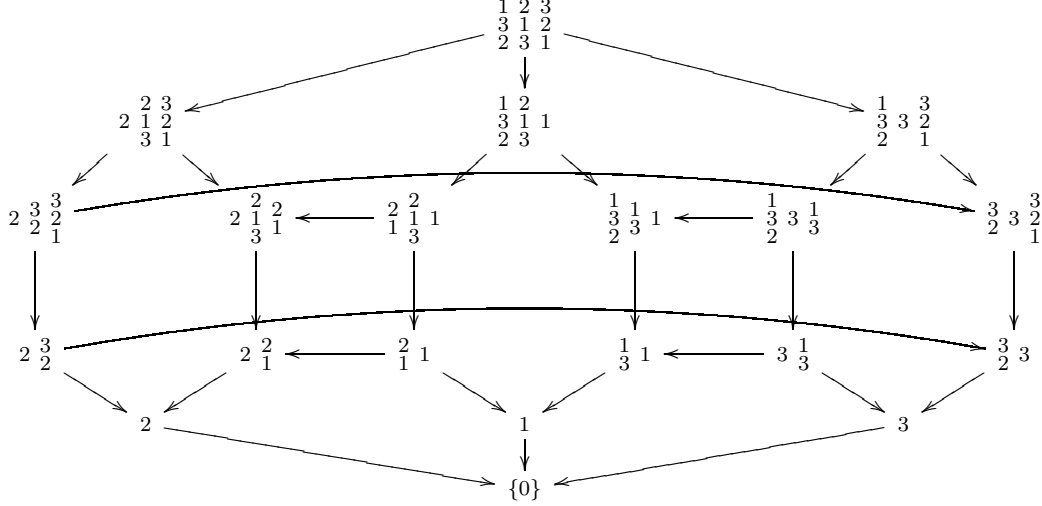
8.4. **Summary and applications.** Summarizing Theorem 8.10, 8.16 and 8.17, we have the following result.

Theorem 8.22. *Let Λ be a Nakayama algebra with n simple modules and $\ell_i := \ell(P(i))$ for any $i \in [1,n]$. Then there are bijections between*

    (1) $\tau\text{-tilt}\,\Lambda$,
    (2) $\text{ps}\tau\text{-tilt}_{\text{np}}\Lambda$,
    (3) $\mathcal{T}(n;\ell_1,\ell_2,\cdots,\ell_n)$,
    (4) $\mathcal{Z}(n;\ell_1,\ell_2,\cdots,\ell_n)$.

The following corollary is an immediate consequence of Theorem 8.22. We identify $\text{top}\,M = S(1)^{a_1} \oplus S(2)^{a_2} \oplus \cdots \oplus S(n)^{a_n}$ with a sequence $(a_1, a_2, \cdots, a_n)$.

Corollary 8.23. *Assume that $\ell(P(i)) \geq n$ for any $i \in [1,n]$. Then the map $M \mapsto \text{top}\,M$ gives a bijection*

$$\tau\text{-tilt}\,\Lambda \longrightarrow \mathcal{Z}(n).$$

In particular, the cardinality of $\text{s}\tau\text{-tilt}\,\Lambda$ is

$$|\text{s}\tau\text{-tilt}\,\Lambda| = \binom{2n}{n}.$$

Proof. For any $M \in \tau\text{-tilt}\,\Lambda$, we have

$$\text{top}\,M = \text{top}\,X,$$

where X is the triangulation corresponding to M by Theorem 8.12. Thus the first assertion follows from Theorem 8.22. Next, we have

$$|\text{s}\tau\text{-tilt}\,\Lambda| = |\tau\text{-tilt}\,\Lambda| + |\text{ps}\tau\text{-tilt}\,\Lambda| \overset{8.17(2)}{=} 2|\tau\text{-tilt}\,\Lambda| \overset{8.22}{=} 2|\mathcal{Z}(n)|.$$

$\mathcal{T}(4)$	$\mathcal{Z}(4)$	τ-tilt Λ_4^4	psτ-tilt Λ_4^4	τ-tilt Λ_4^3	τ-tilt Λ_4^2
	$(1,1,1,1)$	$\begin{smallmatrix}1&2&3&4\\4&1&2&3\\3&4&1&2\\2&3&4&1\end{smallmatrix}$	$\{0\}$	$\begin{smallmatrix}1&2&3&4\\4&1&2&3\\3&4&1&2\end{smallmatrix}$	$\begin{smallmatrix}1&2&3&4\\4&1&2&3\end{smallmatrix}$
	$(1,0,2,1)$	$\begin{smallmatrix}1&3&4\\4&2&3\\3&1&2\\2&4&1\end{smallmatrix}\,3$	3	$\begin{smallmatrix}1&3&4\\4&2&3\\3&1&2\end{smallmatrix}\,3$	$\begin{smallmatrix}1&3&4\\4&2&3\end{smallmatrix}\,3$
	$(2,0,2,0)$	$\begin{smallmatrix}1&3\\4&2\\3&1\\2&4\end{smallmatrix}\,1\,3$	$1\,3$	$\begin{smallmatrix}1&3\\4&2\\3&1\end{smallmatrix}\,1\,3$	$\begin{smallmatrix}1&3\\4&2\end{smallmatrix}\,1\,3$
	$(1,0,1,2)$	$\begin{smallmatrix}1&4\\4&3\\3&2\\2&1\end{smallmatrix}\begin{smallmatrix}4\\3\end{smallmatrix}\,3$	$\begin{smallmatrix}4\\3\end{smallmatrix}\,3$	$\begin{smallmatrix}1&4\\4&3\\3&2\end{smallmatrix}\begin{smallmatrix}4\\3\end{smallmatrix}\,3$	
	$(1,0,0,3)$	$\begin{smallmatrix}1&4\\4&3\\3&2\\2&1\end{smallmatrix}\begin{smallmatrix}4\\3\end{smallmatrix}\,4$	$\begin{smallmatrix}4\\3\end{smallmatrix}\,4$	$\begin{smallmatrix}1&4\\4&3\\3&2\end{smallmatrix}\begin{smallmatrix}4\\3\end{smallmatrix}\,4$	
	$(2,0,1,1)$	$\begin{smallmatrix}1&1\\4&4\\3&3\\2\end{smallmatrix}\begin{smallmatrix}4\\3\end{smallmatrix}\,3$	$\begin{smallmatrix}1\\4\\3\end{smallmatrix}\begin{smallmatrix}4\\3\end{smallmatrix}\,3$		
	$(3,0,1,0)$	$\begin{smallmatrix}1&1\\4&4\\3&3\\2\end{smallmatrix}\,1\,3$	$\begin{smallmatrix}1\\4\\3\end{smallmatrix}\,1\,3$		
	$(4,0,0,0)$	$\begin{smallmatrix}1&1\\4&4\\3&3\\2\end{smallmatrix}\begin{smallmatrix}1\\4\end{smallmatrix}\,1$	$\begin{smallmatrix}1\\4\\3\end{smallmatrix}\begin{smallmatrix}1\\4\end{smallmatrix}\,1$		
	$(3,0,0,1)$	$\begin{smallmatrix}1&1\\4&4\\3&3\\2\end{smallmatrix}\begin{smallmatrix}1\\4\end{smallmatrix}\,4$	$\begin{smallmatrix}1\\4\\3\end{smallmatrix}\begin{smallmatrix}1\\4\end{smallmatrix}\,4$		
	$(2,0,0,2)$	$\begin{smallmatrix}1&1\\4&4\\3&3\\2\end{smallmatrix}\begin{smallmatrix}4\\3\end{smallmatrix}\,4$	$\begin{smallmatrix}1\\4\\3\end{smallmatrix}\begin{smallmatrix}4\\3\end{smallmatrix}\,4$		

TABLE 1. Example of Theorem 8.22

Now, it is elementary that $2|\mathcal{Z}(n)| = 2\binom{2n-1}{n-1} = \binom{2n}{n}$ holds. $\qquad\square$

As an application of Theorem 8.22, we give a proof of the following well-known result (*e.g.* [BK]).

Corollary 8.24. *Let* $\Lambda := K\vec{A}_n$ *be a path algebra. Then there are bijections between*

(1) $\mathrm{tilt}\,\Lambda$,
(2) $\mathcal{S}(n) := \{X \in \mathcal{T}(n) \mid \langle \bullet, n \rangle \in X\}$,
(3) $\mathcal{Y}(n) := \{a \in \mathcal{Z}(n) \mid ||a'|| = 0\}$.

In particular, we have

$$|\mathrm{tilt}\,\Lambda| = C_n,$$

where C_n *is the* n-*th Catalan number* $\frac{1}{n+1}\binom{2n}{n}$.

Note that $\mathcal{S}(n)$ can identify the set of triangulations of $(n+2)$-regular polygon (with no puncture). Therefore the cardinality of $\mathcal{S}(n)$ is equal to the n-th Catalan number C_n.

Proof. Since Λ is hereditary, we have τ-$\mathrm{tilt}\,\Lambda = \mathrm{tilt}\,\Lambda$ by Proposition 7.6(3). Moreover, there are bijections between $\mathrm{tilt}\,\Lambda$, $\mathcal{T}(n; 1, 2, \cdots, n)$, and $\mathcal{Z}(n; 1, 2, \cdots, n)$ by Theorem 8.22.

First, we show that

$$\mathcal{S}(n) = \mathcal{T}(n; 1, 2, \cdots, n). \tag{15}$$

Indeed, assume that $X \in \mathcal{S}(n)$. Since X contains the projective arc $\langle \bullet, n \rangle$, we have $\ell(\langle i, j \rangle) \leq j$ for each inner arc $\langle i, j \rangle \in X$. Thus, we have $X \in \mathcal{T}(n; 1, 2, \cdots, n)$. Conversely, assume that $X \in \mathcal{T}(n; 1, 2, \cdots, n)$. Clearly, the projective arc $\langle \bullet, n \rangle$ is compatible with all admissible arc in X. Thus, we have $\langle \bullet, n \rangle \in X$, and hence $X \in \mathcal{S}(n)$.

Next, we show that

$$\mathcal{Y}(n) = \mathcal{Z}(n; 1, 2, \cdots, n).$$

Indeed, if $a \in \mathcal{Z}(n; 1, 2, \cdots, n)$, then X_a contains the projective arc $\langle \bullet, n \rangle$ by (15). Thus we have $||a'|| = a'_n(= 0)$, and hence $a \in \mathcal{Y}(n)$. Conversely, if $a \in \mathcal{Y}(n)$, then $a'_n = 0 = ||a'||$. Thus we have $1 \leq k^i_{a_i - \delta_i} < i \leq n$ for any i, and hence $\ell_i(a) \leq i$. Therefore $a \in \mathcal{Z}(n; 1, 2, \cdots, n)$. $\qquad\square$

8.5. **Miscellaneous results on Nakayama algebras of type** A. In this subsection, we give another classification of τ-tilting modules over Nakayama algebras of type A. The calculation of proper support τ-tilting modules over Nakayama algebras can be reduced to that of τ-tilting modules over smaller Nakayama algebras of type A. Moreover, as an application, we give the number of τ-tilting modules as a recurrence relation.

Throughout this subsection, we assume that Λ is a Nakayama algebra of type $\vec{A}_n^{\mathrm{op}}$. Here we use the quiver $\vec{A}_n^{\mathrm{op}}$ instead of $\vec{A}_n$ because the opposite index is more convenient for our setting. Namely, its quiver is isomorphic to $\vec{A}_n^{\mathrm{op}}$, that is,

$$\vec{A}_n^{\mathrm{op}} : \quad n \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_{n-2}} \cdots \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1$$

Thus τM has the following property for any $M \in \mathsf{mod}\,\Lambda$.

Lemma 8.25. *If* M *is a* τ-*rigid* Λ-*module, then* $M \oplus P(1)$ *is also a* τ-*rigid* Λ-*module.*

Proof. Since the vertex 1 in $\vec{A}_n^{\mathrm{op}}$ is a source, τM does not have $S(1)$ as a composition factor by Proposition 8.2. Hence $\mathrm{Hom}_\Lambda(P(1), \tau M) = 0$ by Lemma 7.2. Thus $M \oplus P(1)$ is τ-rigid. $\qquad\square$

By Lemma 8.25, we have the following result for τ-tilting modules.

Proposition 8.26. *Each τ-tilting Λ-module has $P(1)$ as a direct summand.*

Proof. Let M be a basic τ-tilting Λ-module. Then $M \oplus P(1)$ is a τ-rigid Λ-module by Lemma 8.25. Thus we have $P(1) \in \mathsf{add}\, M$ by Proposition 7.4. $\square$

Our aim in this subsection is to show the following theorem.

Theorem 8.27. *Let Λ be a Nakayama algebra of type $\vec{A}_n^{\mathrm{op}}$. Then there are mutually inverse bijections*

$$\tau\text{-tilt}\,\Lambda \longleftrightarrow \coprod_{i=1}^{\ell(P(1))} \tau\text{-tilt}\,(\Lambda/\langle e_i \rangle)$$

given by τ-tilt $\Lambda \ni M \mapsto M/P(1)$ and $N \mapsto N \oplus P(1) \in \tau\text{-tilt}\,\Lambda$.

Proof. Let $N \in \tau\text{-tilt}\,(\Lambda/\langle e_i \rangle)$ for $i \in [1, \ell(P(1))]$. Then $N \oplus P(1)$ is a τ-tilting Λ-module because it is τ-rigid by Lemma 8.25 and $|N \oplus P(1)| = |\Lambda|$ by $P(1) \notin \mathsf{mod}\,(\Lambda/\langle e_i \rangle)$.

Conversely, let M be a basic τ-tilting Λ-module. Then, by Proposition 8.26, we decompose M as $M = P(1) \oplus N_1 \oplus N_2$, where N_1 is a maximal direct summand of M consisting of $(\Lambda/\langle e_1 \rangle)$-modules and N_2 is a direct summand of M with top $N_2 \in \mathsf{add}\,S(1)$. If $N_2 = 0$, then N_1 is a τ-tilting $(\Lambda/\langle e_1 \rangle)$-module clearly. Assume that $N_2 \neq 0$. Then we have $N_2 \in \mathsf{mod}\,(\Lambda/\langle e_{j+1} \rangle)$ for $j := \ell(N_2)$. Note that $1 \leq j < \ell(P(1))$. Now, we claim that $N_1 \in \mathsf{mod}\,(\Lambda/\langle e_{j+1} \rangle)$. Indeed, if it does not hold, then there exists an indecomposable Λ-module $Y \in \mathsf{add}\,N_1$ which has $S(j+1)$ as a composition factor. Let $X := P(1)/\mathrm{rad}^j P(1) \in \mathsf{add}\,N_2$. Since $\mathrm{soc}(\tau X) = S(j+1)$ and $\mathrm{top}(\tau X) = S(2)$ hold by Proposition 8.2, we have $\mathrm{Hom}_\Lambda(Y, \tau X) \neq 0$ by Lemma 8.3(2)$\Rightarrow$(1). However, this contradicts that M is τ-rigid. Thus $N_1 \oplus N_2$ is a τ-tilting $(\Lambda/\langle e_{j+1} \rangle)$-module. $\square$

As an application of Theorem 8.27, we have a recurrence relation for the cardinality of τ-tilt Λ.

Corollary 8.28.   (1) *Let Λ be a Nakayama algebra of type A_n^{op}. Then we have*

$$|\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^{\ell(P(1))} C_{i-1} \cdot |\tau\text{-tilt}\,(\Lambda/\langle e_{\leq i} \rangle)|$$

where $e_{\leq i} := e_1 + e_2 + \cdots + e_i$.
   (2) *Let $\Lambda = \Gamma_n^r := K\vec{A}_n^{\mathrm{op}}/\mathrm{rad}^r K\vec{A}_n^{\mathrm{op}}$. Then we have*

$$|\tau\text{-tilt}\,\Gamma_n^r| = \sum_{i=1}^{r} C_{i-1} \cdot |\tau\text{-tilt}\,\Gamma_{n-i}^r|.$$

Proof. (1) By Theorem 8.27, we have

$$|\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^{\ell(P(1))} |\tau\text{-tilt}\,(\Lambda/\langle e_i \rangle)|.$$

Since the quiver of Λ is a tree, we have $\Lambda/\langle e_i \rangle \simeq (\Lambda/\langle e_{\geq i} \rangle) \times (\Lambda/\langle e_{\leq i} \rangle)$, where $e_{\geq i} := e_i + e_{i+1} + \cdots + e_n$ and $e_{\leq i} := e_1 + e_2 + \cdots + e_i$ for $i \in [1, n]$. Thus there is a bijection

$$\tau\text{-tilt}\,(\Lambda/\langle e_{\geq i} \rangle) \times \tau\text{-tilt}\,(\Lambda/\langle e_{\leq i} \rangle) \longrightarrow \tau\text{-tilt}\,(\Lambda/\langle e_i \rangle)$$

given by $(N, L) \mapsto N \oplus L$. Hence we have

$$|\tau\text{-tilt}\,(\Lambda/\langle e_i \rangle)| = |\tau\text{-tilt}\,(\Lambda/\langle e_{\geq i} \rangle)| \cdot |\tau\text{-tilt}\,(\Lambda/\langle e_{\leq i} \rangle)|.$$

If $i \leq \ell(P(1))$, then $\Lambda/\langle e_{\geq i} \rangle$ is isomorphic to $K\vec{A}_{i-1}^{\text{op}}$. Thus, by Corollary 8.24, we have

$$|\tau\text{-tilt}\,(\Lambda/\langle e_{\geq i} \rangle)| = |\text{tilt}\,(K\vec{A}_{i-1})| = C_{i-1}.$$

Therefore the assertion follows.

(2) If $\Lambda = \Gamma_n^r$, then we have $\Lambda/\langle e_{\leq i} \rangle \simeq \Gamma_{n-i}^r$. Hence it is clear from (1). $\qquad\square$

Example 8.29. We calculate $\tau\text{-tilt}\,\Gamma_n^2$ inductively. First, we already know that

$$\text{tilt}\,(K\vec{A}_1) = \{\, {}_1 \,\}, \quad \text{tilt}\,(K\vec{A}_2) = \{\, {}_2 \oplus {}_{\frac{1}{2}}, \, {}_1 \oplus {}_{\frac{1}{2}} \,\}.$$

By Theorem 8.27,

$$\tau\text{-tilt}\,\Gamma_3^2 \longleftrightarrow \tau\text{-tilt}\,(\Gamma_3^2/\langle e_2 \rangle) \coprod \tau\text{-tilt}\,(\Gamma_3^2/\langle e_1 \rangle) = \{\, {}_1 \oplus {}_3 \,\} \coprod \{\, {}_3 \oplus {}_{\frac{2}{3}}, \, {}_2 \oplus {}_{\frac{2}{3}} \,\}.$$

Thus we have

$$\tau\text{-tilt}\,\Gamma_3^2 = \{\, {}_3 \oplus {}_{\frac{2}{3}} \oplus {}_{\frac{1}{2}}, \, {}_2 \oplus {}_{\frac{2}{3}} \oplus {}_{\frac{1}{2}}, \, {}_1 \oplus {}_3 \oplus {}_{\frac{1}{2}} \,\}.$$

Similarly, we have

$$\tau\text{-tilt}\,\Gamma_4^2 = \{\, {}_4 \oplus {}_{\frac{3}{4}} \oplus {}_1 \oplus {}_{\frac{1}{2}}, \, {}_3 \oplus {}_{\frac{3}{4}} \oplus {}_1 \oplus {}_{\frac{1}{2}} \,\}$$

$$\coprod \{\, {}_4 \oplus {}_{\frac{3}{4}} \oplus {}_{\frac{2}{3}} \oplus {}_{\frac{1}{2}}, \, {}_3 \oplus {}_{\frac{3}{4}} \oplus {}_{\frac{2}{3}} \oplus {}_{\frac{1}{2}}, \, {}_2 \oplus {}_4 \oplus {}_{\frac{2}{3}} \oplus {}_{\frac{1}{2}} \,\}.$$

As a similar result of Corollary 8.28, Jasso [Ja2] showed a recurrence relation

$$|\text{s}\tau\text{-tilt}\,\Gamma_n^2| = 2|\text{s}\tau\text{-tilt}\,\Gamma_{n-1}^2| + |\text{s}\tau\text{-tilt}\,\Gamma_{n-2}^2|.$$

We give examples of the number of support τ-tilting modules over some Nakayama algebras.

TABLE 2. $|\tau\text{-tilt}\,\Gamma_n^r|$

r \ n	1	2	3	4	5
1	1	1	1	1	1
2	1	2	3	5	8
3	1	2	5	9	18
4	1	2	5	14	28
5	1	2	5	14	42

TABLE 3. $|\text{s}\tau\text{-tilt}\,\Gamma_n^r|$

r \ n	1	2	3	4	5
1	2	4	8	16	32
2	2	5	12	29	70
3	2	5	14	37	98
4	2	5	14	42	118
5	2	5	14	42	132

TABLE 4. $|\tau\text{-tilt}\,\Lambda_n^r|$

r \ n	1	2	3	4	5
1	1	1	1	1	1
2	1	3	4	7	11
3	1	3	10	15	31
4	1	3	10	35	56
5	1	3	10	35	126

TABLE 5. $|\text{s}\tau\text{-tilt}\,\Lambda_n^r|$

r \ n	1	2	3	4	5
1	2	4	8	16	32
2	2	6	14	34	82
3	2	6	20	50	132
4	2	6	20	70	182
5	2	6	20	70	252

9. τ-TILTING MODULES AND DROZD-KIRICHENKO REJECTION

Our aim in this section is to study a connection of support τ-tilting modules between two algebras related by Drozd-Kirichenko rejection. As an application, we give an algorithm to construct the Hasse quivers of support τ-tilting modules over Nakayama algebras.

Rejection Lemma of Drozd-Kirichenko is the following statement, which give explicit relationship between representation theory of Λ and that of the factor algebra Λ/S of Λ. We denote by $\mathrm{ind}\,\Lambda$ the set of isomorphism classes of indecomposable Λ-modules.

Proposition 9.1. [DrKi] *Let Λ be a basic finite dimensional K-algebra and Q an indecomposable projective-injective summand of Λ as a Λ-module. Then the following hold.*

(1) $S := \mathrm{soc}\,Q$ *is a two-sided ideal of* Λ.
(2) $\mathrm{ind}(\Lambda/S) = \mathrm{ind}(\Lambda) \setminus \{Q\}$.
(3) $\tau_\Lambda U \simeq \tau_{\Lambda/S} U$ *for any* $U \in \mathsf{mod}(\Lambda/S)$ *with* $U \not\simeq Q/S$, *and* $\tau_\Lambda(Q/S) \simeq \mathrm{rad}\,Q$.
(4) *Any almost split sequence in* $\mathsf{mod}(\Lambda/S)$ *is an almost split sequence in* $\mathsf{mod}\,\Lambda$.
(5) *All almost split sequences in* $\mathsf{mod}\,\Lambda$ *are either almost split sequences in* $\mathsf{mod}(\Lambda/S)$ *or*

$$0 \longrightarrow \mathrm{rad}\,Q \longrightarrow Q \oplus \mathrm{rad}\,Q/S \longrightarrow Q/S \longrightarrow 0.$$

(6) Q/S *is an indecomposable projective* (Λ/S)-*module and* $\mathrm{rad}\,Q$ *is an indecomposable injective* (Λ/S)-*module.*

9.1. Main results.

Let Λ be an arbitrary basic finite dimensional K-algebra (we do not assume that Λ is Nakayama). We always assume that Λ has an indecomposable projective-injective summand Q as a Λ-module. Moreover, let $S := \mathrm{soc}\,Q$ and $\overline{\Lambda} := \Lambda/S$. We consider the functor

$$\overline{(-)} := - \otimes_\Lambda \overline{\Lambda} : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\overline{\Lambda}.$$

Then $\overline{Q} = Q/S$. Note that, for every indecomposable Λ-module $M \not\simeq Q$, we have an isomorphism $\overline{M} \simeq M$ as $\overline{\Lambda}$-module.

In this subsection, we show that the poset $\mathrm{s}\tau\text{-tilt}\,\Lambda$ can be constructed from the poset $\mathrm{s}\tau\text{-tilt}\,\overline{\Lambda}$. The following construction is crucial.

Definition 9.2. Let $\Omega = (\Omega, \geq)$ be a poset and N a subposet of Ω. We define a new poset $\Omega^N = (\Omega^N, \geq_N)$ as follows, where $N^+ := \{n^+ \mid n \in N\}$ is a copy of N and $\omega_1, \omega_2 \in \Omega$, $w \in \Omega \setminus N$, and $n_1, n_2 \in N$ are arbitrary elements:

$$\Omega^N := \Omega \coprod N^+,$$
$$\omega_1 \geq_N \omega_2 :\Leftrightarrow \omega_1 \geq \omega_2$$
$$w \geq_N n_1^+ :\Leftrightarrow w \geq n_1$$
$$n_2^+ \geq_N \omega_2 :\Leftrightarrow n_2 \geq \omega_2$$
$$n_1^+ \geq_N n_2^+ :\Leftrightarrow n_1 \geq n_2.$$

In particular, $n_1 \geq_N n_2^+$ never hold. It is easily checked that $(\Omega^N, \geq_N)$ forms a poset.

Now we observe that the Hasse quiver $\mathrm{H}(\Omega^N)$ of the poset Ω^N is constructed from the Hasse quiver $\mathrm{H}(\Omega)$ of the poset Ω by the combinatorial operation given as follows. For a quiver H, we denote by H_0 the set of vertices of H and by H_1 the set of arrows of H.

Definition-Proposition 9.3. *In the setting of Definition 9.2, let* $\mathrm{H}(\Omega) := (\mathrm{H}_0, \mathrm{H}_1)$ *and* $\mathrm{H}(\Omega^N) := (\mathrm{H}'_0, \mathrm{H}'_1)$ *be the Hasse quivers of* Ω *and* Ω^N, *respectively. We define a new quiver* $\mathrm{H}(\Omega)^N := (\mathrm{H}''_0, \mathrm{H}''_1)$ *as follows, where* ω_1, ω_2 *are arbitrary elements in* $\Omega \setminus N$ *and* n_1, n_2 *are arbitrary elements in* N:

$$\mathrm{H}''_0 = \mathrm{H}_0 \coprod N^+$$

$$\mathrm{H}''_1 = \{\omega_1 \to \omega_2 \mid \omega_1 \to \omega_2 \text{ in } \mathrm{H}_1\} \coprod \{n_2 \to \omega_2 \mid n_2 \to \omega_2 \text{ in } \mathrm{H}_1\}$$

$$\coprod \{n_1 \to n_2, \ n_1^+ \to n_2^+ \mid n_1 \to n_2 \text{ in } \mathrm{H}_1\}$$

$$\coprod \{\omega_1 \to n_1^+ \mid \omega_1 \to n_1 \text{ in } \mathrm{H}_1\} \coprod \{n_1^+ \to n_1 \mid n_1 \in \mathrm{H}_0\}.$$

Then we have

$$\mathrm{H}(\Omega)^N = \mathrm{H}(\Omega^N).$$

$$\mathrm{H}(\Omega) \qquad\qquad\qquad \mathrm{H}(\Omega)^N$$
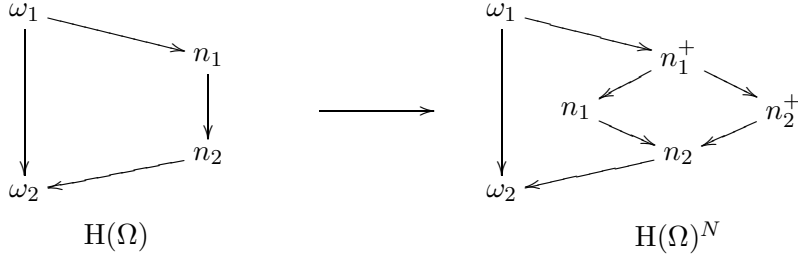
Proof. By the definition of Ω^N, $\mathrm{H}'_0 = \mathrm{H}_0 \coprod N^+ = \mathrm{H}''_0$ clearly holds.

To show the statement for H'_1, we give the following observation.

Lemma 9.4. *Let* $\omega_1, \omega_2 \in \Omega \setminus N$ *and* $n_1, n_2 \in N$ *be any elements.*

(1) *The following hold.*
  (a) *There is an arrow* $\omega_1 \to \omega_2$ *in* H_1 *if and only if so is in* H'_1.
  (b) *There is an arrow* $n_2 \to \omega_2$ *in* H_1 *if and only if so is in* H'_1.
  (c) *There is an arrow* $n_1 \to n_2$ *in* H_1 *if and only if so is in* H'_1 *if and only if there is an arrow* $n_1^+ \to n_2^+$ *in* H'_1.
  (d) *There is an arrow* $\omega_1 \to n_1$ *in* H_1 *if and only if there is an arrow* $\omega_1 \to n_1^+$ *in* H'_1.
  (e) *For any* $n_1 \in N$, *there is an arrow* $n_1^+ \to n_1$ *in* H'_1.
(2) *There are no arrows* $n_1 \to n_2^+$, $n_1^+ \to \omega_2$ *and* $\omega_1 \to n_1$ *in* H'_1.

Proof. (1) We only prove (a); the proofs of (b), (c) and (d) are similar.

(a) It follows from that the following conditions are equivalent by definition.

  (i) There is an arrow $\omega_1 \to \omega_2$ in H_1.
  (ii) $\omega_1 > \omega_2$ in Ω and there does not exist $x \in \Omega$ such that $\omega_1 > x > \omega_2$.
  (iii) $\omega_1 >_N \omega_2$ in Ω^N and there does not exist $y \in \Omega^N$ such that $\omega_1 >_N y >_N \omega_2$.
  (iv) There is an arrow $\omega_1 \to \omega_2$ in H'_1.

(e) By definition, we have $n_1^+ >_N n_1$. Assume that there exists $y \in \Omega^N$ such that $n_1^+ \geq_N y \geq_N n_1$. Then $y = n_1^+$ or n_1 holds clearly. Hence the assertion follows.

(2) It is clear from the definition of Ω^N. $\square$

By Lemma 9.4, the assertion follows. $\square$

The following theorem is our main result. Let

$$\mathcal{N} := \{N \in \mathrm{s}\tau\text{-tilt}\,\overline{\Lambda} \mid \overline{Q} \in \mathsf{add}\,N \text{ and } \mathrm{Hom}_\Lambda(N, Q) = 0\}.$$

Applying Definition 9.2, we have a poset $(\mathrm{s}\tau\text{-tilt}\,\overline{\Lambda})^{\mathcal{N}}$. For any Λ-module M, we denote by $\alpha(M)$ a basic Λ-module satisfying $\mathsf{add}(\alpha(M)) = \mathsf{add}\,\overline{M}$.

Theorem 9.5. *Let Λ be a basic finite dimensional algebra and Q an indecomposable projective-injective summand of Λ as a Λ-module. Then $M \mapsto \alpha(M)$ gives an isomorphism of posets*

$$\mathrm{s}\tau\text{-tilt}\,\Lambda \longrightarrow (\mathrm{s}\tau\text{-tilt}\,\overline{\Lambda})^{\mathcal{N}}.$$

In particular, we have an isomorphism of Hasse quivers

$$\mathrm{H}(\Lambda) \simeq \mathrm{H}((\mathrm{s}\tau\text{-tilt}\,\overline{\Lambda})^{\mathcal{N}}).$$

The proof of Theorem 9.5 will be given at the end of this subsection. We illustrate Theorem 9.5 with the following example.

Example 9.6. Let Λ be the preprojective algebra of Dynkin type A_3. Then we have $\Lambda = {\small\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \oplus {\small\begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}} \oplus {\small\begin{smallmatrix}3\\2\\1\end{smallmatrix}}$. Let $Q := {\small\begin{smallmatrix}1\\2\\3\end{smallmatrix}}$. Then we have $\overline{\Lambda} = {\small\begin{smallmatrix}1\\2\end{smallmatrix}} \oplus {\small\begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}} \oplus {\small\begin{smallmatrix}3\\2\\1\end{smallmatrix}}$. The Hasse quiver $\mathrm{H}(\overline{\Lambda})$ is given by
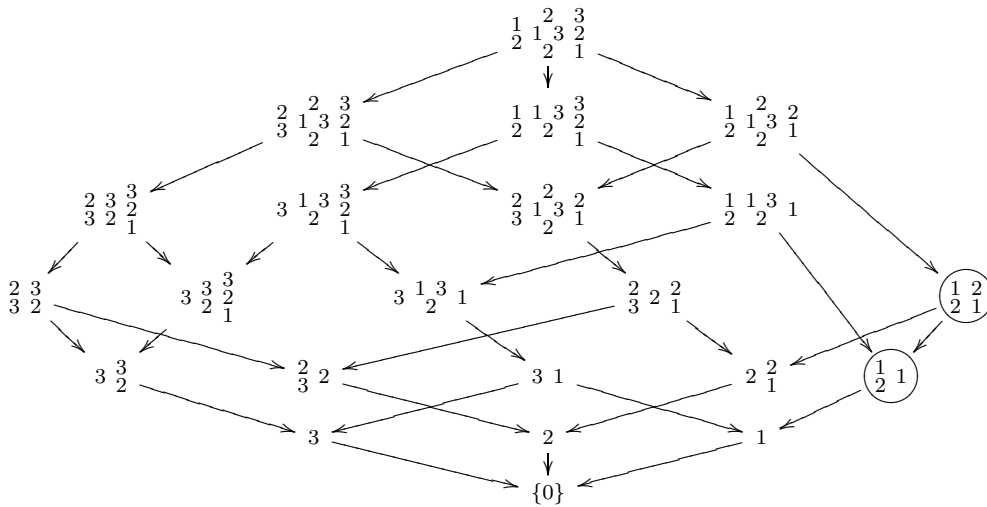
where the elements in $\mathcal{N}$ are marked by circles.

On the other hand, the Hasse quiver $H(\Lambda)$ is given by

$$\begin{smallmatrix} \frac{1}{2} & \frac{2}{1}3 & \frac{3}{2} \\ 3 & 2 & 1 \end{smallmatrix}$$

(Hasse quiver diagram)
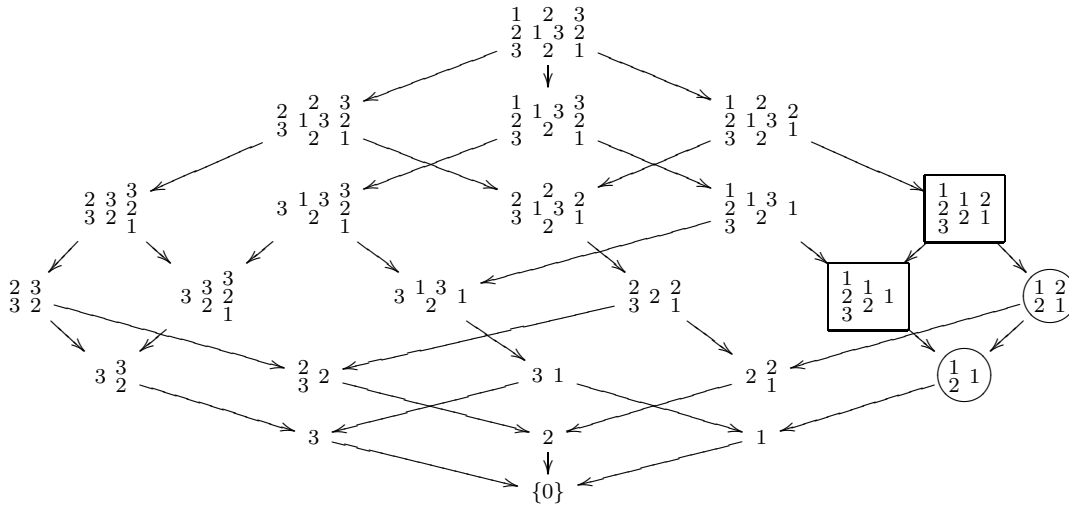
where the elements in $\mathcal{N}^+$ are marked by rectangles. One can easily check that the relationship between $s\tau$-tilt Λ and $s\tau$-tilt $\overline{\Lambda}$ is given by Theorem 9.8.

We refer to [Miz2] for more information on $H(\Lambda)$ for a preprojective algebra Λ of Dynkin type.

Next we give the following corollary of Theorem 9.5.

Corollary 9.7. *In the setting of Theorem 9.5, assume that $\overline{Q}$ has S as a composition factor. Then $\mathcal{N} = \emptyset$ holds, and we have an isomorphism of posets*

$$s\tau\text{-tilt}\,\Lambda \simeq s\tau\text{-tilt}\,\overline{\Lambda}$$

and an isomorphism of Hasse quivers

$$H(\Lambda) \simeq H(\overline{\Lambda}).$$

Proof. Since $\overline{Q}$ has S as a composition factor, we have $\mathrm{Hom}_\Lambda(\overline{Q}, Q) \neq 0$ by Lemma 7.2(1). Thus $\mathcal{N} = \emptyset$ holds. The assertion follows from Theorem 9.5 since $(s\tau\text{-tilt}\,\overline{\Lambda})^\emptyset = s\tau\text{-tilt}\,\overline{\Lambda}$. $\square$

In the rest of this subsection, we give a proof of Theorem 9.5. We will describe the relationship between $s\tau$-tilt Λ and $s\tau$-tilt $\overline{\Lambda}$ more explicitly. We decompose $s\tau$-tilt Λ as $s\tau\text{-tilt}\,\Lambda = \mathcal{M}_1 \coprod \mathcal{M}_2^- \coprod \mathcal{M}_2^+ \coprod \mathcal{M}_3$, where

$$\mathcal{M}_1 = \mathcal{M}_1(Q) := \{M \in s\tau\text{-tilt}\,\Lambda \mid Q, \overline{Q} \notin \mathsf{add}\,M\},$$
$$\mathcal{M}_2^- = \mathcal{M}_2^-(Q) := \{M \in s\tau\text{-tilt}\,\Lambda \mid Q \notin \mathsf{add}\,M \text{ and } \overline{Q} \in \mathsf{add}\,M\},$$
$$\mathcal{M}_2^+ = \mathcal{M}_2^+(Q) := \{M \in s\tau\text{-tilt}\,\Lambda \mid Q \oplus \overline{Q} \in \mathsf{add}\,M\},$$
$$\mathcal{M}_3 = \mathcal{M}_3(Q) := \{M \in s\tau\text{-tilt}\,\Lambda \mid Q \in \mathsf{add}\,M \text{ and } \overline{Q} \notin \mathsf{add}\,M\}.$$

Theorem 9.8. *Let Λ be a basic finite dimensional algebra and Q an indecomposable projective-injective summand of Λ as a Λ-module. Then the following hold.*

(1) *The map $M \mapsto \alpha(M)$ gives bijections*

$$\mathcal{M}_1 \to \mathcal{N}_1, \ \mathcal{M}_2^- \to \mathcal{N}_2, \ \mathcal{M}_2^+ \to \mathcal{N}_2^+, \ \mathcal{M}_3 \to \mathcal{N}_3,$$

where

$$\mathcal{N}_1 = \mathcal{N}_1(Q) := \{N \in \text{s}\tau\text{-tilt}\,\overline{\Lambda} \mid \overline{Q} \notin \text{add}\,N\},$$

$$\mathcal{N}_2 = \mathcal{N}_2(Q) := \mathcal{N} = \{N \in \text{s}\tau\text{-tilt}\,\overline{\Lambda} \mid \overline{Q} \in \text{add}\,N \text{ and } \text{Hom}_\Lambda(N, Q) = 0\},$$

$$\mathcal{N}_3 = \mathcal{N}_3(Q) := \{N \in \text{s}\tau\text{-tilt}\,\overline{\Lambda} \mid \overline{Q} \in \text{add}\,N \text{ and } \text{Hom}_\Lambda(N, Q) \neq 0\}.$$

and $\mathcal{N}_2^+$ is a copy of $\mathcal{N}_2$. In particular, there is a bijection

$$\alpha : \text{s}\tau\text{-tilt}\,\Lambda \to (\text{s}\tau\text{-tilt}\,\overline{\Lambda})^{\mathcal{N}_2}$$

given by the bijection above.

(2) *We have*

$$\text{s}\tau\text{-tilt}\,\Lambda = \{N \mid N \in \mathcal{N}_1 \textstyle\coprod \mathcal{N}_2\} \textstyle\coprod \{Q \oplus N \mid N \in \mathcal{N}_2\} \textstyle\coprod \{Q \oplus (N/\overline{Q}) \mid N \in \mathcal{N}_3\}.$$

Proof. We only have to give a proof of (1) because (2) follows from (1) immediately. We start with an easy lemma.

Lemma 9.9. *Assume $\overline{Q} \neq 0$ and that $U \in \text{mod}\,\overline{\Lambda}$ does not have $\overline{Q}$ as a direct summand. The following are equivalent.*

(a) $\overline{Q} \oplus U$ *is a τ-rigid Λ-module.*
(b) $\overline{Q} \oplus U$ *is a τ-rigid $\overline{\Lambda}$-module with $\text{Hom}_\Lambda(\overline{Q} \oplus U, Q) = 0$.*

Proof. We have isomorphisms

$$\text{Hom}_\Lambda(\overline{Q} \oplus U, \tau_\Lambda(\overline{Q} \oplus U)) \overset{9.1(3)}{\simeq} \text{Hom}_\Lambda(\overline{Q} \oplus U, \text{rad}\,Q) \oplus \text{Hom}_{\overline{\Lambda}}(\overline{Q} \oplus U, \tau_{\overline{\Lambda}}U)$$

$$\overset{7.2(2)}{\simeq} \text{Hom}_\Lambda(\overline{Q} \oplus U, Q) \oplus \text{Hom}_{\overline{\Lambda}}(\overline{Q} \oplus U, \tau_{\overline{\Lambda}}(\overline{Q} \oplus U)).$$

Thus the assertion follows. $\qquad\square$

The following proposition plays an important role.

Proposition 9.10. *Assume $\overline{Q} \neq 0$ and that $U \in \text{mod}\,\overline{\Lambda}$ does not have $\overline{Q}$ as a direct summand.*

(1) *The following are equivalent:*
  (a) U *is a support τ-tilting Λ-module.*
  (b) U *is a support τ-tilting $\overline{\Lambda}$-module.*
(2) *The following are equivalent:*
  (a) $Q \oplus \overline{Q} \oplus U$ *is a support τ-tilting Λ-module.*
  (b) $\overline{Q} \oplus U$ *is a support τ-tilting Λ-module.*
  (c) $\overline{Q} \oplus U$ *is a support τ-tilting $\overline{\Lambda}$-module with $\text{Hom}_\Lambda(\overline{Q} \oplus U, Q) = 0$.*
(3) *The following are equivalent:*
  (a) $Q \oplus U$ *is a support τ-tilting Λ-module.*
  (b) $\overline{Q} \oplus U$ *is a support τ-tilting $\overline{\Lambda}$-module with $\text{Hom}_\Lambda(\overline{Q} \oplus U, Q) \neq 0$.*

Proof. (1) Since we have $\tau_\Lambda U \simeq \tau_{\overline{\Lambda}}U$ by Proposition 9.1(3), the assertion follows.

(2) (a) $\Leftrightarrow$ (b): We claim that $M := Q \oplus \overline{Q} \oplus U$ is a τ-rigid Λ-module if and only if $N := \overline{Q} \oplus U$ is a τ-rigid Λ-module. Indeed, this follows from isomorphisms

$$\text{Hom}_\Lambda(M, \tau_\Lambda M) \simeq \text{Hom}_\Lambda(Q, \tau_\Lambda N) \oplus \text{Hom}_\Lambda(N, \tau_\Lambda N)$$

$$\overset{7.2(2)}{\simeq} \text{Hom}_\Lambda(\overline{Q}, \tau_\Lambda N) \oplus \text{Hom}_\Lambda(N, \tau_\Lambda N),$$

and $\overline{Q} \in \text{add}\,N$.

By Lemma 9.9, both (a) and (b) imply $\mathrm{Hom}_\Lambda(N, Q) = 0$, or equivalently N does not have S as a composition factor by Lemma 7.2(2). Hence $\mathrm{s}(M) = \mathrm{s}(N) + 1$ holds, where $\mathrm{s}(M)$ is the number of nonisomorphic simple modules appearing in a composition series of M.. Thus the assertion follows from Proposition 7.8 and $|M| = |N| + 1$.

(b) $\Leftrightarrow$ (c): This is immediate from Lemma 9.9.

(3) First, we claim that $Q \oplus U$ is a τ-rigid Λ-module if and only if $\overline{Q} \oplus U$ is a τ-rigid $\overline{\Lambda}$-module. Indeed, this follows from isomorphisms

$$\mathrm{Hom}_\Lambda(Q \oplus U, \tau_\Lambda U) \overset{7.2(2)}{\simeq} \mathrm{Hom}_\Lambda(\overline{Q} \oplus U, \tau_\Lambda U) \overset{9.1(3)}{\simeq} \mathrm{Hom}_{\overline{\Lambda}}(\overline{Q} \oplus U, \tau_{\overline{\Lambda}} U).$$

Next, we claim that, if $Q \oplus U \in \mathrm{s}\tau\text{-tilt}\,\Lambda$, then $\mathrm{Hom}_\Lambda(\overline{Q} \oplus U, Q) \neq 0$. Indeed, assume that $\mathrm{Hom}_\Lambda(\overline{Q} \oplus U, Q) = 0$. Since $Q \oplus U$ is τ-rigid, $\overline{Q} \oplus U$ is a τ-rigid $\overline{\Lambda}$-module by Lemma 7.2(2) and Proposition 9.1(3), and hence $\overline{Q} \oplus U$ is a τ-rigid Λ-module by by Lemma 9.9. Thus $Q \oplus \overline{Q} \oplus U$ is a τ-rigid $(\Lambda/\langle e_{Q \oplus U}\rangle)$-module by Lemma 7.7. This contradicts that τ-tilting modules are maximal τ-rigid by Proposition 7.4. Since $\mathrm{Hom}_\Lambda(\overline{Q} \oplus U, Q) \neq 0$ holds by our assumption, $\overline{Q} \oplus U$ has S as a composition factor by Lemma 7.2(2). Thus the assertion follows from Proposition 7.8 since we have $\mathrm{s}(Q \oplus U) = \mathrm{s}(\overline{Q} \oplus U)$. $\square$

Remark 9.11. If Q is a simple projective-injective Λ-module, or equivalently $\overline{Q} = 0$, then we have $\mathrm{s}\tau\text{-tilt}\,\overline{\Lambda} = \mathcal{N}_2$. In this case, Proposition 9.10(2) holds because we have $\overline{\Lambda} \simeq \Lambda/\langle e_i\rangle$, where $\mathrm{soc}(Q) = S(i) = \mathrm{top}(e_i\Lambda)$.

We now finish the proof of Theorem 9.8(1).

By Proposition 9.10, the map $M \mapsto \alpha(M)$ gives bijections

$$\mathcal{M}_1 \to \mathcal{N}_1, \ \mathcal{M}_2^- \to \mathcal{N}_2, \ \mathcal{M}_2^+ \to \mathcal{N}_2^+, \ \mathcal{M}_3 \to \mathcal{N}_3.$$

Hence $\alpha : \mathrm{s}\tau\text{-tilt}\,\Lambda \to (\mathrm{s}\tau\text{-tilt}\,\overline{\Lambda})^{\mathcal{N}}$ is a bijection. $\square$

From now, we give a proof of Theorem 9.5. The following lemma is useful to understand the structure of the poset $\mathrm{s}\tau\text{-tilt}\,\Lambda$.

Lemma 9.12.   (1) *Let $N, N' \in \mathrm{s}\tau\text{-tilt}\,\overline{\Lambda}$. If $N > N'$ holds, then we have*

$$(N, N') \notin (\mathcal{N}_1 \times \mathcal{N}_2) \coprod (\mathcal{N}_1 \times \mathcal{N}_3) \coprod (\mathcal{N}_2 \times \mathcal{N}_3).$$

(2) *Let $M, M' \in \mathrm{s}\tau\text{-tilt}\,\Lambda$. If $M > M'$ holds, then we have*

$$(M, M') \notin (\mathcal{M}_1 \times \mathcal{M}_2^{\pm}) \coprod (\mathcal{M}_1 \times \mathcal{M}_3) \coprod (\mathcal{M}_2^{\pm} \times \mathcal{M}_3) \coprod (\mathcal{M}_2^- \times \mathcal{M}_2^+).$$

Proof. We only prove (1); the proof of (2) is similar. If $N \geq N'$ holds and $P \in \mathsf{add}\,N'$ is projective, then $P \in \mathsf{add}\,N$. Thus the assertion except $(N, N') \notin \mathcal{N}_2 \times \mathcal{N}_3$ follows. Assume that $(N, N') \in \mathcal{N}_2 \times \mathcal{N}_3$. Then we have $N' \in \mathsf{Fac}(N)$. This is a contradiction since N does not have S as a composition factor and N' has S as a composition factor. $\square$

Now, we are ready to prove Theorem 9.5.

Proof of Theorem 9.5. Since the map $\alpha : \mathrm{s}\tau\text{-tilt}\,\Lambda \to (\mathrm{s}\tau\text{-tilt}\,\overline{\Lambda})^{\mathcal{N}}$ is a bijection by Theorem 9.8, we have only to show that, for any $M, L \in \mathrm{s}\tau\text{-tilt}\,\Lambda$, $M \geq L$ holds if and only if $\alpha(M) \geq \alpha(L)$ holds. Indeed, if $M \geq L$, then $L \in \mathsf{Fac}\,M$, and hence $\overline{L} \in \mathsf{Fac}\,\overline{M}$ hold, which implies $\alpha(M) \geq \alpha(L)$. Conversely, assume that $\alpha(M) \geq \alpha(L)$. If both M and L are in either $\mathcal{M}_1$, $\mathcal{M}_2^{\pm}$ or $\mathcal{M}_3$, then $M \geq L$ holds clearly. Otherwise, by Lemma 9.12, we have

$$(M, L) \in (\mathcal{M}_3 \times \mathcal{M}_2^{\pm}) \coprod (\mathcal{M}_3 \times \mathcal{M}_1) \coprod (\mathcal{M}_2^{\pm} \times \mathcal{M}_1) \coprod (\mathcal{M}_2^+, \mathcal{M}_2^-).$$

By the definition of the functor $\overline{(-)}$, we have $\overline{M} \in \mathsf{Fac}\,M$ in $\mathsf{mod}\,\Lambda$. Hence $M \geq \alpha(M) \geq \alpha(L)$ holds in $\mathsf{mod}\,\Lambda$. If we have $L \in \mathcal{M}_1 \coprod \mathcal{M}_2^-$, then $\alpha(L) = \overline{L} = L$ and hence $M \geq L$ hold. On the other hand, if we have $L \in \mathcal{M}_2^+$, then $L = Q \oplus \overline{L} \in \mathsf{Fac}\,M$ holds because $M \in \mathcal{M}_3$ has Q as a direct summand. Hence the assertion follows. $\qquad\square$

9.2. **Applications to Nakayama algebras.** In this subsection, we apply the results in the previous subsection to a Nakayama algebra Λ. We give a combinatorial method to construct the Hasse quiver of support τ-tilting Λ-modules.

It is basic that any Nakayama algebra Λ has an at least one indecomposable projective-injective module Q and its factor algebra $\Lambda/\mathrm{soc}\,Q$ is again Nakayama (see [ASS, V.3.3 and V.3.4]). Thus we can iteratively apply Drozd-Kirichenko rejection to Nakayama algebras. We have the following algorithm for a construction of the Hasse quiver.

Algorithm 9.13. Let Λ be a Nakayama algebra.

(0) Take a (non-unique) sequence of Nakayama algebras

$$\Lambda =: \Lambda_1 \longrightarrow\!\!\!\!\longrightarrow \Lambda_2 \longrightarrow\!\!\!\!\longrightarrow \cdots \longrightarrow\!\!\!\!\longrightarrow \Lambda_{m-1} \longrightarrow \Lambda_m := 0$$

such that $\Lambda_{i+1} := \Lambda_i/\mathrm{soc}\,Q_i$, where Q_i is an indecomposable projective-injective Λ_i-module, and $m > 0$ is an integer.

(1) First, the Hasse quiver $\mathrm{H}(\Lambda_m)$ consists of one vertex and no arrows.

(2) Secondly, the Hasse quiver $\mathrm{H}(\Lambda_{m-1})$ is given by $\mathrm{H}((\mathsf{s}\tau\text{-tilt}\,\Lambda_m)^{\mathcal{N}_2(Q_{m-1})})$ (see Definition-Proposition 9.3).

(3) Thirdly, the Hasse quiver $\mathrm{H}(\Lambda_{m-2})$ is given by $\mathrm{H}((\mathsf{s}\tau\text{-tilt}\,\Lambda_{m-1})^{\mathcal{N}_2(Q_{m-2})})$.

$\vdots$

(l) The Hasse quiver $\mathrm{H}(\Lambda_{m-l+1})$ is given by $\mathrm{H}((\mathsf{s}\tau\text{-tilt}\,\Lambda_{m-l+2})^{\mathcal{N}_2(Q_{m-l+1})})$.

$\vdots$

(m) Finally, the Hasse quiver $\mathrm{H}(\Lambda_1)$ is given by $\mathrm{H}((\mathsf{s}\tau\text{-tilt}\,\Lambda_2)^{\mathcal{N}_2(Q_1)})$.

As a direct consequence of Theorem 9.5, we have the following result.

Theorem 9.14. *Let Λ be a Nakayama algebra with n simple modules. Assume that $\ell(P(i)) \geq n$ for any $i \in [1, n]$. Then we have*

$$\mathrm{H}(\Lambda) \simeq \mathrm{H}(\Lambda_n^n),$$

where Λ_n^n is the self-injective Nakayama algebra with n simple modules and the Loewy length n.
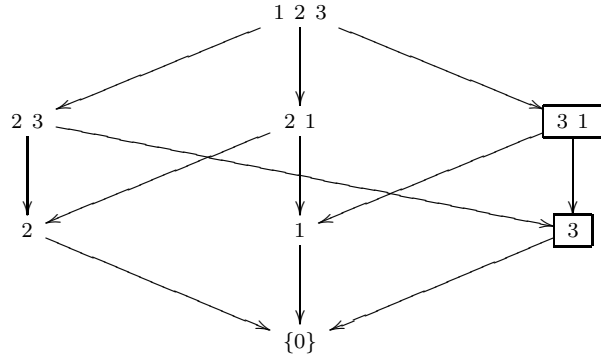
We give an example of calculation of the Hasse quiver using Algorithm 9.13.

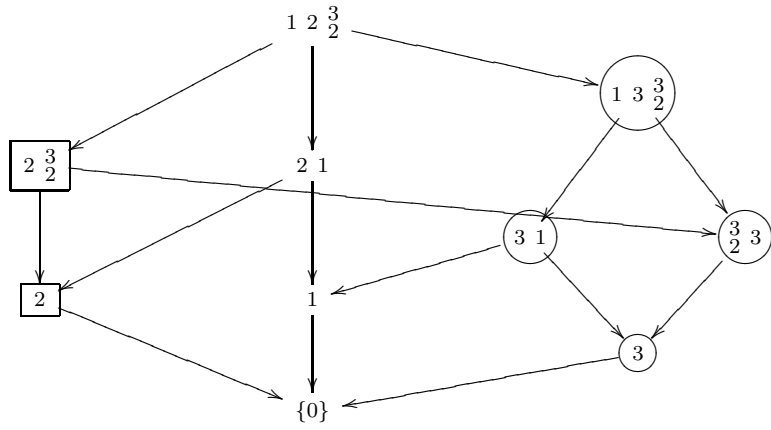**Example 9.15.** Let $\Lambda_1 := \Lambda_3^4 = K\vec{\Delta}_3/J^4$. Then we have sequence

$$\Lambda_1 \longrightarrow\!\!\!\!\longrightarrow \Lambda_2 \longrightarrow\!\!\!\!\longrightarrow \Lambda_3 \longrightarrow\!\!\!\!\longrightarrow \cdots \longrightarrow\!\!\!\!\longrightarrow \Lambda_8 \longrightarrow\!\!\!\!\longrightarrow \Lambda_9 \longrightarrow\!\!\!\!\longrightarrow \Lambda_{10} =: K^3 \ .$$

of Nakayama algebras, where each Λ_i is explicitly given bellow. We will describe $\mathrm{H}(\Lambda_i)$ inductively, where the elements in $\mathcal{N}_2(Q_i)$ are marked by rectangles and those in $\mathcal{M}_2(Q_i)$ are marked by circles. Note that $\mathrm{H}(K^m)$ is isomorphic to the Hasse quiver of the set of all subsets of an m-element set ordered by inclusion. Thus we may begin with a semisimple algebra.
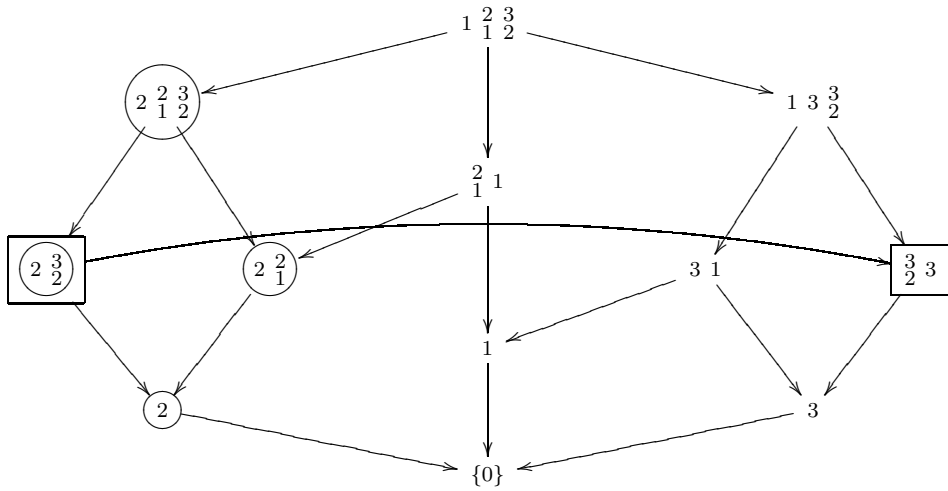
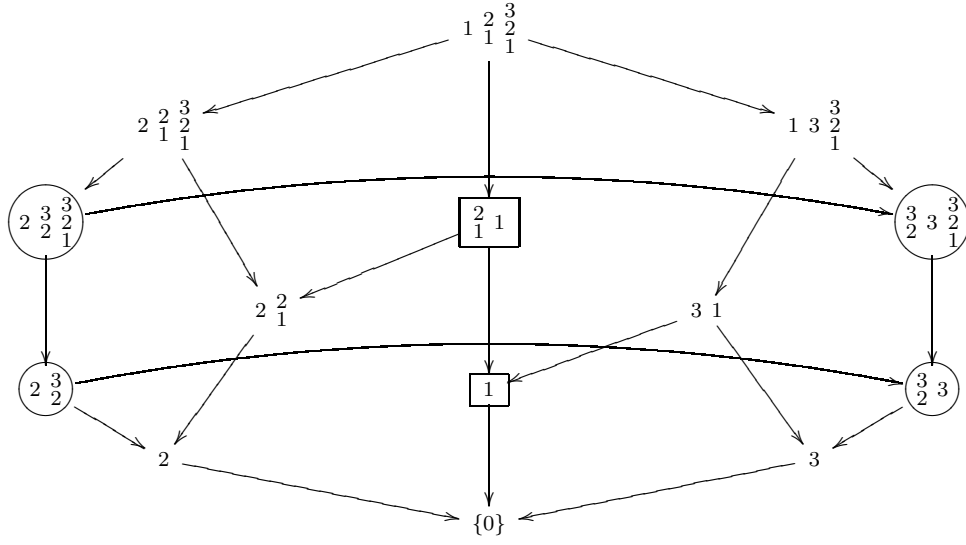(1) $\mathrm{H}(\Lambda_{10})$ is the following, where $\Lambda_{10} = K^3$ and $Q_9 = \genfrac{}{}{0pt}{}{3}{2}$

1 2 3

2 3　　　　　2 1　　　　3 1

2　　　　　　1　　　　　3

{0}

(2) H(Λ_9) is the following, where $\Lambda_9 = K(\, 3 \longrightarrow 2 \qquad 1 \,)$ and $Q_8 = {}^2_1$

1 2 $\frac{3}{2}$

1 3 $\frac{3}{2}$

2 $\frac{3}{2}$　　　　2 1

3 1　　　　$\frac{3}{2}$ 3

2　　　　　1

3

{0}

(3) H(Λ_8) is the following, where $\Lambda_8 = K\vec{A}_3/\langle \alpha_2 \alpha_1 \rangle$ and $Q_7 = \frac{3}{2}_1$

1 $\frac{2}{1}$ $\frac{3}{2}$

2 $\frac{2}{1}$ $\frac{3}{2}$　　　　　　　1 3 $\frac{3}{2}$

$\frac{2}{1}$ 1

2 $\frac{3}{2}$　　　2 $\frac{2}{1}$　　　3 1　　　$\frac{3}{2}$ 3

1

2　　　　　　　　3

{0}

(4) H(Λ_7) is the following, where $\Lambda_7 = K\vec{A}_3$ and $Q_6 = \frac{1}{3}$

(5) H(Λ_6) is the following, where $\Lambda_6 = K\vec{\Delta}_3/\langle \alpha_3\alpha_2,\ \alpha_1\alpha_3 \rangle$ and $Q_5 = \begin{smallmatrix}2\\1\\3\end{smallmatrix}$
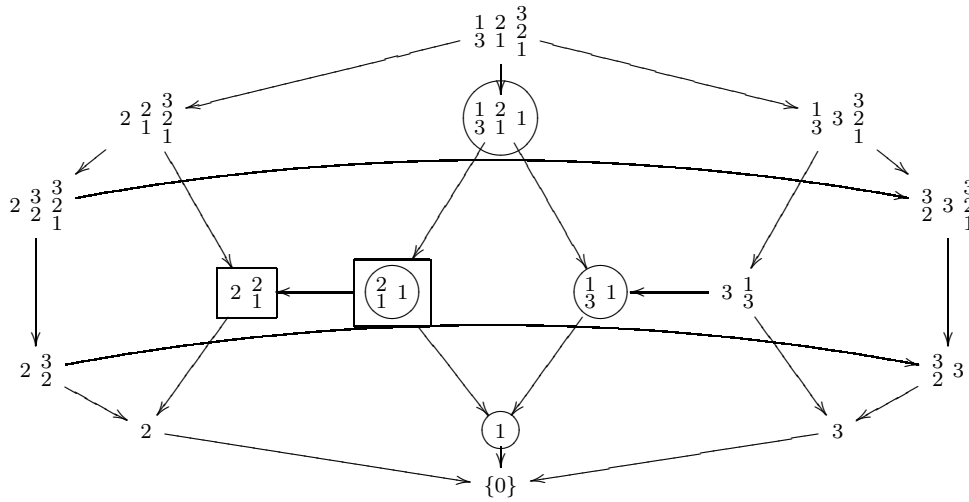
(6) H(Λ_5) is the following, where $\Lambda_5 = K\vec{\Delta}_3/\langle \alpha_3\alpha_2, \alpha_2\alpha_1\alpha_3 \rangle$ and $Q_4 = \begin{smallmatrix}1\\3\\2\end{smallmatrix}$
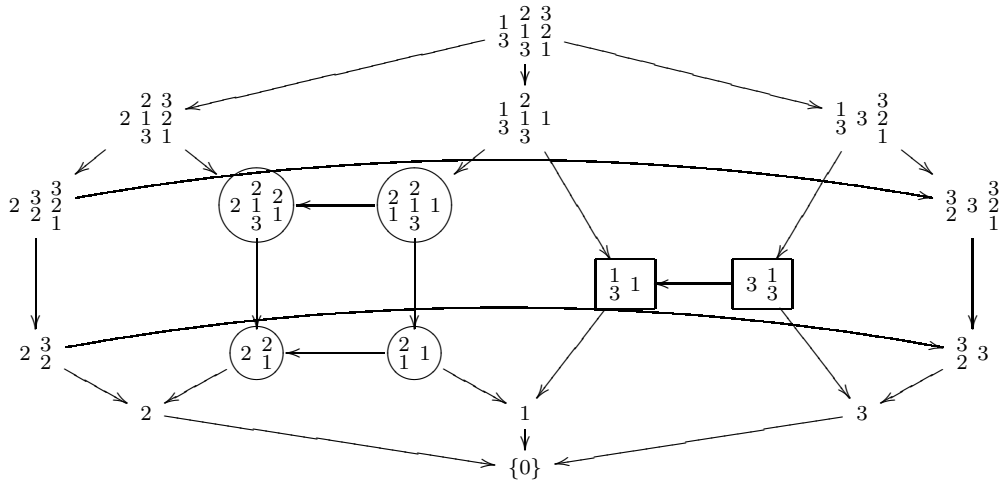
(7) $H(\Lambda_4)$ is the following, where $\Lambda_4 = \Lambda_3^3$ and $Q_3 = \begin{smallmatrix}3\\2\\1\\3\end{smallmatrix}$
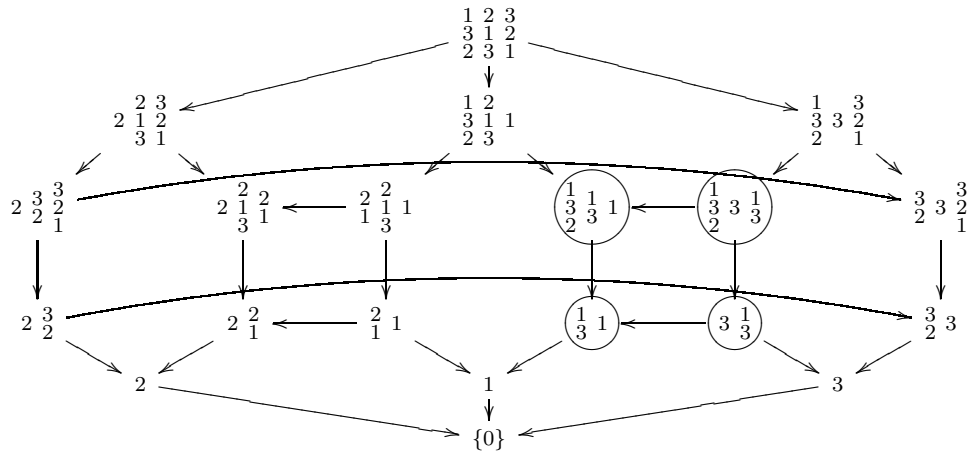
(8) $H(\Lambda_3)$ is the following, where $\Lambda_3 = K\vec{\Delta}_3/\langle \alpha_3\alpha_2\alpha_1, \alpha_1\alpha_3\alpha_2\rangle$ and $Q_2 = \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix}$
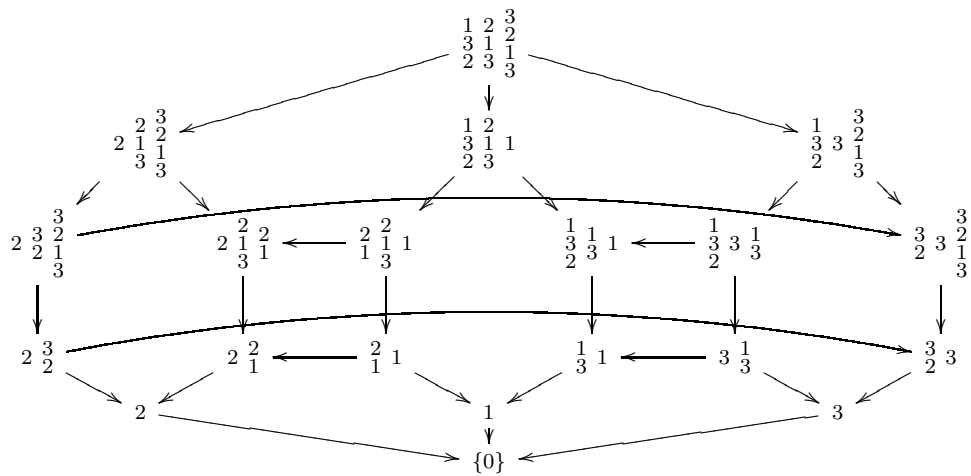
(9) H(Λ_2) is the following, where $\Lambda_2 = K\vec{\Delta}_3/\langle \alpha_3\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_3\alpha_2 \rangle$ and $Q_1 = \begin{smallmatrix} 1 \\ 3 \\ 2 \\ 1 \end{smallmatrix}$
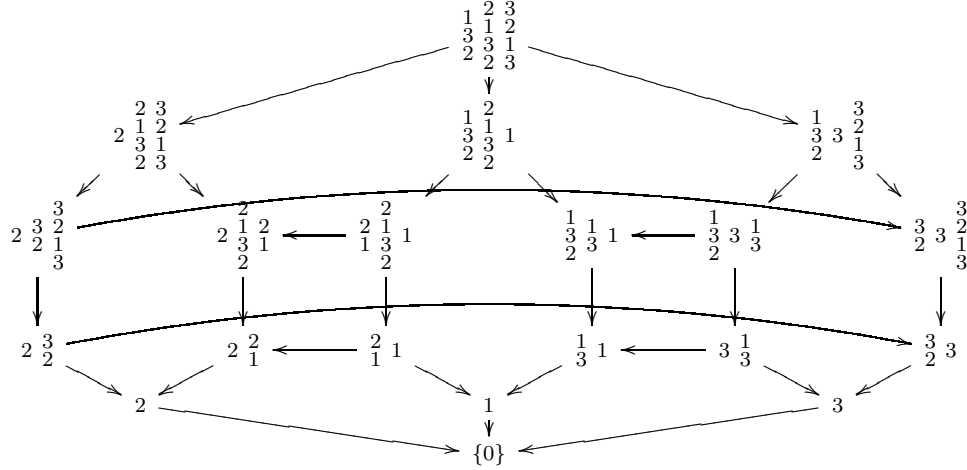
(10) H(Λ_1) is the following, where $\Lambda_1 = \Lambda_3^4$
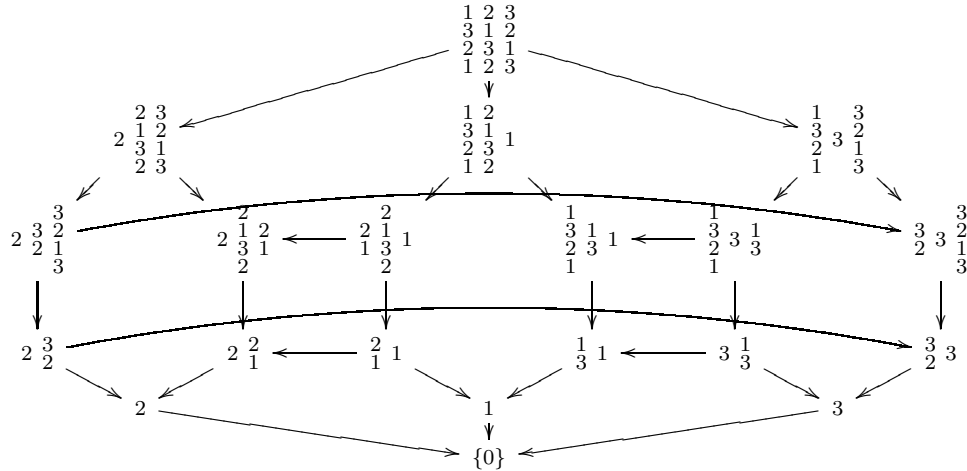
Note that the Hasse quivers Λ_i ($i \leq 4$) are the same shapes. This is a consequence of Theorem 9.14.

Part 3. Characterizing τ-rigid-finite algebras with radical square zero

This part is based on the paper [Ad2].

10. Preliminaries

In this section, we collect some results which are necessary in this part. Let Λ be a basic finite dimensional K-algebra and $J := J_\Lambda$ a Jacobson radical of Λ.

10.1. τ-rigid modules. We recall basic properties of τ-rigid modules.

Definition 10.1. A Λ-module X is called τ-rigid if $\mathrm{Hom}_\Lambda(X, \tau X) = 0$. We denote by i$\tau$-rigid$\Lambda$ the set of isomorphism classes of indecomposable τ-rigid Λ-modules. An algebra Λ is called τ-rigid-finite if iτ-rigidΛ is a finite set.

By Auslander-Reiten duality $\operatorname{Ext}_\Lambda^1(X,Y) \simeq \operatorname{D}\overline{\operatorname{Hom}}_\Lambda(Y,\tau X)$, every τ-rigid Λ-module X is *rigid* (*i.e.* $\operatorname{Ext}_\Lambda^1(X,X) = 0$), and the converse is true if Λ is hereditary (*e.g.*, the path algebra KQ of an acyclic quiver Q).

For a Λ-module X, we denote by

$$P_1^X \xrightarrow{p} P_0^X \xrightarrow{q} X \to 0$$

a minimal projective presentation. The following proposition plays an important role in this part.

Proposition 10.2. [AIR, Proposition 2.4 and 2.5] *For a Λ-module X, the following hold.*
  (1) *X is τ-rigid if and only if the map $(p,X) : \operatorname{Hom}_\Lambda(P_0^X, X) \to \operatorname{Hom}_\Lambda(P_1^X, X)$ is surjective.*
  (2) *If X is τ-rigid, then we have $\operatorname{add} P_0^X \cap \operatorname{add} P_1^X = 0$.*

For an idempotent $e \in \Lambda$, we consider two K-linear functors

$$L_e(-) := - \otimes_{e\Lambda e} e\Lambda : \operatorname{mod}(e\Lambda e) \to \operatorname{mod}\Lambda, \quad R_e(-) := (-)e : \operatorname{mod}\Lambda \to \operatorname{mod}(e\Lambda e).$$

Then (L_e, R_e) is an adjoint pair. Moreover, the following result gives a connection between τ-rigid $(e\Lambda e)$-modules and τ-rigid Λ-modules.

Lemma 10.3. [ASS, I.6.8] *Let Λ be an algebra and $e \in \Lambda$ an idempotent.*
  (1) *The functor L_e is fully faithful and there exists a functorial isomorphism $R_e L_e \simeq 1_{\operatorname{mod} eAe}$. In particular, L_e and R_e induce mutually quasi-inverse equivalences between categories $\operatorname{mod}(e\Lambda e)$ and $\operatorname{Im} L_e := \{L_e(X) \mid X \in \operatorname{mod}(e\Lambda e)\}$.*
  (2) *A Λ-module X is in the category $\operatorname{Im} L_e$ if and only if $P_0^X \oplus P_1^X \in \operatorname{add} e\Lambda$.*

We have the following result.

Proposition 10.4. *Let Λ be an algebra and $e \in \Lambda$ an idempotent. Assume that a Λ-module X is in $\operatorname{Im} L_e$. Then X is τ-rigid if and only if the $(e\Lambda e)$-module Xe is τ-rigid. In particular, L_e and R_e induce mutually inverse bijections*

$$\operatorname{i}\tau\text{-rigid}(e\Lambda e) \longleftrightarrow \operatorname{i}\tau\text{-rigid}\Lambda \cap \operatorname{Im} L_e.$$

Proof. By Lemma 10.3(2), we have $P_0^X \oplus P_1^X \in \operatorname{add} e\Lambda$. Hence the sequence

$$P_1^X e \xrightarrow{pe} P_0^X e \xrightarrow{qe} Xe \to 0$$

is a projective presentation. By Lemma 10.3(1), the projective presentation is minimal, and moreover we have a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Hom}_A(P_0^X, X) & \xrightarrow{\;(p,X)\;} & \operatorname{Hom}_A(P_1^X, X) \\
\Big\downarrow{\scriptstyle\simeq} & & \Big\downarrow{\scriptstyle\simeq} \\
\operatorname{Hom}_{eAe}(P_0^X e, Xe) & \xrightarrow{(pe,Xe)} & \operatorname{Hom}_{eAe}(P_1^X e, Xe)
\end{array}
$$

where the vertical maps are isomorphisms. By using Proposition 10.2(1), we have that X is a τ-rigid Λ-module if and only if Xe is a τ-rigid $(e\Lambda e)$-module. $\qquad\square$

The following proposition is a well-known result for path algebras.

Proposition 10.5. [ASS, VII.5.1, VIII.2.7 and VIII.2.9] *Let Q be a connected acyclic quiver and $\Lambda := KQ$ the path algebra of Q. Then the following hold.*

(1) Λ *is representation-finite if and only if Q is a Dynkin quiver. In this case, every indecomposable Λ-module is rigid.*

(2) *If Λ is not representation-finite, then there exist infinitely many isomorphism classes of indecomposable rigid Λ-modules. Moreover, there exists an indecomposable Λ-module which is not rigid.*

Immediately, we have the following characterization of τ-rigid-finiteness for path algebras of acyclic quivers.

Theorem 10.6. *Let Q be a connected acyclic quiver and $\Lambda := KQ$ the path algebra of Q. Then the following are equivalent:*

(1) Λ *is representation-finite.*

(2) Λ *is τ-rigid-finite.*

(3) Q *is a Dynkin quiver.*

Proof. It follows from Proposition 10.5 because rigid modules are exactly τ-rigid modules for any hereditary algebra. $\qquad\qquad\square$

10.2. **Algebras with radical square zero.** Throughout this subsection, we assume that Λ is an algebra with radical square zero (*i.e.*, $J^2 = 0$). For algebras with radical square zero, the following triangular matrix algebra plays an important role:

$$\Delta := \Delta(\Lambda) := \begin{bmatrix} \Lambda/J & J \\ 0 & \Lambda/J \end{bmatrix}.$$

Each Δ-module is given by a triplet $(X', X''; \varphi)$, where X', X'' are (Λ/J)-modules and φ is a morphism

$$\varphi : X' \otimes_{\Lambda/J} J \to X''$$

in $\mathsf{mod}(\Lambda/J)$. A morphism $f : (X', X''; \varphi) \to (Y', Y''; \psi)$ in $\mathsf{mod}\,\Delta$ is given by a pair (f', f''), where $f' : X' \to Y'$ and $f'' : X'' \to Y''$ are morphisms in $\mathsf{mod}(\Lambda/J)$ such that $\psi f' = f'' \varphi$ (see [ASS, A.2.7] and [ARS, III.2] for details).

$$\begin{array}{ccc}
X' & \xrightarrow{\ \varphi\ } & X'' \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} \\
Y' & \xrightarrow{\ \psi\ } & Y''
\end{array}$$

We recall some properties of the triangular matrix algebra Δ. Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the vertex set and Q_1 is the arrow set. Then we define a new quiver $Q^{\mathrm{s}} = (Q_0^{\mathrm{s}}, Q_1^{\mathrm{s}})$, called *separated quiver*, as follows: Let $Q_0^+ := \{i^+ \mid i \in Q_0\}$ and $Q_0^- := \{i^- \mid i \in Q_0\}$ be copies of Q_0. Then

$$Q_0^{\mathrm{s}} := Q_0^+ \coprod Q_0^-, \quad Q_1^{\mathrm{s}} := \{i^+ \to j^- \mid i \to j \text{ in } Q_1\}.$$

Note that the separated quiver Q^{s} is not necessarily connected even if Q is connected. For example, the separated quiver Q^{s} of the following quiver Q is not connected:
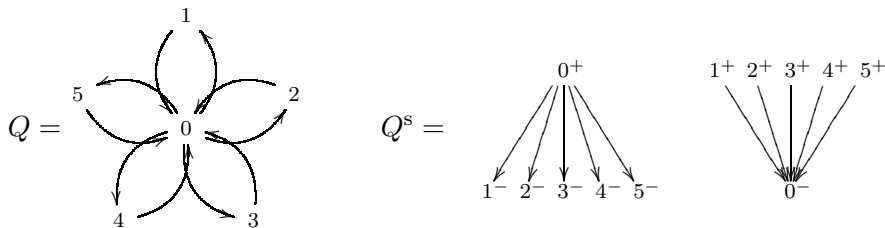
We call a quiver *bipartite* if each vertex is either a sink or a source.

Proposition 10.7. [ARS, III.2.5 and X.2.6] *Let Q be the quiver of Λ. The following hold.*

(1) *The separated quiver Q^{s} is bipartite.*

(2) *The algebra Δ is isomorphic to the path algebra of Q^{s}. In particular, Δ is a hereditary algebra with radical square zero.*

(3) *Each simple Δ-module is one of the form $(S, 0; 0)$ or $(0, S; 0)$, where S is a simple Λ-module.*

(4) *Each indecomposable projective Δ-module is one of the form $(P/PJ, PJ; 1_{PJ})$ or $(0, P/PJ; 0)$, where P is an indecomposable projective Λ-module.*

Now we recall results on representation theory of algebras with radical square zero. We define a functor $F : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Delta$ as follows: For any Λ-module X, we let

$$F(X) := (X/XJ, XJ; \varphi_X),$$

where the map $\varphi_X : X/XJ \otimes_{\Lambda/J} J \to XJ$ is naturally induced by the natural multiplication morphism $X \otimes_\Lambda J \to XJ$ since $J^2 = 0$. For any morphism $g : X \to Y$, we let

$$F(g) := (g', g''),$$

where $g' : X/XJ \to Y/YJ$ is induced by g and $g'' : XJ \to YJ$ is the restriction of XJ.

Proposition 10.8. [ARS, X.2.1 and X.2.2] *The following hold.*

(1) *The functor F is full and induces an equivalence of categories $\underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Delta$.*

(2) *A Λ-module X is indecomposable (respectively, projective) if and only if FX is an indecomposable (respectively, a projective) Δ-module.*

(3) *The following are equivalent:*
   (a) *Λ is representation-finite.*
   (b) *The separated quiver of the quiver for Λ is a disjoint union of Dynkin quivers.*

Remark 10.9. Clearly a stable equivalence preserves representation-finiteness. However a stable equivalence does not preserve τ-rigid-finiteness in general. Indeed, let Λ be an algebra with radical square zero whose quiver consists of one vertex and n loops with $n \geq 2$. Since Λ is local, every indecomposable τ-rigid Λ-module is projective, and in particular Λ is τ-rigid-finite. On the other hand, since the separated quiver is the n-Kronecker quiver

$$\circ \; \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \; \circ \;,$$

Δ is not τ-rigid-finite by Theorem 10.6.

11. Main results

Throughout this section, let Λ be an algebra with radical square zero, and Δ, F as in Subsection 10.2. Let Q be the quiver of Λ and Q^{s} the separated quiver of Q. A full subquiver Q' of Q^{s} is called a *single subquiver* (respectively, a *maximal single subquiver*) if, for any $i \in Q_0$, the vertex set Q'_0 contains at most (respectively, exactly) one of i^+ or i^-. We denote by $\mathcal{S}$ the set of all single subquivers of Q^{s}.

The following theorem is our main result of this part.

Theorem 11.1. *Let Λ be an algebra with radical square zero and Q^{s} the separated quiver of the quiver Q for Λ. Then the following are equivalent:*

(1) *Λ is τ-rigid-finite.*

(2) *Every single subquiver of Q^{s} is a disjoint union of Dynkin quivers.*

(3) *Every maximal single subquiver of Q^{s} is a disjoint union of Dynkin quivers.*

The proof of Theorem 11.1 will be given in the rest of this section. A key result is the following criterion for indecomposable Λ-modules to be τ-rigid in terms of the triangular matrix algebra Δ.

Theorem 11.2. *Let X be an indecomposable Λ-module. The following are equivalent:*

(1) *X is a τ-rigid Λ-module.*
(2) *FX is a τ-rigid Δ-module and $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$.*

We prove Theorem 11.2 by comparing a minimal projective presentation of a Λ-module X with that of the Δ-module FX. We start with an easy lemma.

Lemma 11.3. *Let $f : X \to Y$ be a nonzero morphism between indecomposable modules in $\mathsf{mod}\,\Lambda$. If $\mathrm{Im}\,f$ is contained in YJ, then there exists a unique morphism $\tilde{f} : X/XJ \to YJ$ such that*

$$f = (X \xrightarrow{\pi} X/XJ \xrightarrow{\tilde{f}} YJ \xrightarrow{\iota} Y),$$

where π and ι are natural morphisms.

Proof. This is clear since $f(XJ) \subseteq (YJ)J = 0$ holds by $J^2 = 0$. $\qquad\square$

The following lemma gives a construction of a minimal projective presentation of FX from that of X.

Lemma 11.4. *Let $P_1^X \xrightarrow{p} P_0^X \xrightarrow{q} X \to 0$ be a minimal projective presentation of an indecomposable Λ-module X. Then*

$$0 \longrightarrow (0, P_1^X/P_1^X J; 0) \xrightarrow{(0,\tilde{p})} FP_0^X \xrightarrow{Fq} FX \longrightarrow 0 \qquad (16)$$

is a minimal projective resolution of the Δ-module FX.

Proof. Since $J^2 = 0$ holds, $\ker q$ is semisimple. Hence $\ker q = P_1^X/P_1^X J$ holds. Since $\ker q$ is contained in $P_0^X J$, by Lemma 11.3, we have a decomposition

$$p = (P_1^X \xrightarrow{\pi} P_1^X/P_1^X J \xrightarrow{\tilde{p}} P_0^X J \xrightarrow{\iota} P_0^X),$$

where π and ι are natural morphisms. Thus, we have the following commutative diagram

where $Fq = (q', q'')$. Thus the exact sequence

$$0 \to (0, P_1^X/P_1^X J; 0) \xrightarrow{(0,\tilde{p})} (P_0^X/P_0^X J, P_0^X J; 1_{P_0^X J}) \xrightarrow{(q',q'')} (X/XJ, XJ; \varphi_X) \to 0$$

in $\mathsf{mod}\,\Delta$ is a minimal projective resolution of FX, because Δ is hereditary and $(0,\tilde{p})$ is in the radical of $\mathsf{mod}\,\Lambda$. $\hfill\square$

Now we are ready to prove Theorem 11.2.

Proof of Theorem 11.2. Let $\iota : P_0^X J \to P_0^X$, $\iota' : XJ \to X$ and $\pi : P_1^X \to P_1^X/P_1^X J$ are natural morphisms.

(1)$\Rightarrow$(2): Assume that X is τ-rigid. By Proposition 10.2(2), we have $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$. Now we show that FX is a τ-rigid Δ-module. We have a minimal projective resolution (16) in Lemma 11.4. By Proposition 10.2(1), we have only to show that

$$((0,\tilde{p}), FX) : \mathrm{Hom}_\Delta(FP, FX) \to \mathrm{Hom}_\Delta((0, P_1^X/P_1^X J; 0), FX) \qquad (17)$$

is surjective. Let

$$(0, h) : (0, P_1^X/P_1^X J; 0) \to FX = (X/XJ, XJ; \varphi_X)$$

be a morphism in $\mathsf{mod}\,\Delta$ and $f := \iota' h \pi : P_1^X \to X$ a morphism in $\mathsf{mod}\,\Lambda$. Since X is τ-rigid, there exists a morphism $g : P_0^X \to X$ such that $f = gp$. Let

$$Fg := (g', g'') : (P_0^X/P_0^X J, P_0^X J; 1_{P_0^X J}) \to (X/XJ, XJ; \varphi_X).$$

Then we have

$$\iota' h \pi = f = gp = g \iota \tilde{p} \pi = \iota' g'' \tilde{p} \pi,$$

and hence we have $h = g'' \tilde{p}$. Thus we have $(0, h) = (g', g'')(0, \tilde{p})$. Consequently, the map (17) is surjective.

(2)$\Rightarrow$(1): Assume that FX is τ-rigid and $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$. We have only to show that

$$(p, X) : \mathrm{Hom}_\Lambda(P_0^X, X) \to \mathrm{Hom}_\Lambda(P_1^X, X) \qquad (18)$$

is surjective by Proposition 10.2(1). Let $f : P_1^X \to X$ be a morphism in $\mathsf{mod}\,\Lambda$. Since $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$ holds, $\mathrm{Im}\,f$ is contained in XJ. Thus, by Lemma 11.3, there exists a morphism $\tilde{f} : P_1^X/P_1^X J \to XJ$ in $\mathsf{mod}\,\Lambda$ such that $f = \iota' \tilde{f} \pi$. Now we consider a morphism

$$(0, \tilde{f}) : (0, P_1^X/P_1^X J; 0) \to FX = (X/XJ, XJ; \varphi_X)$$

in $\mathsf{mod}\,\Delta$. Since (16) in Lemma 11.4 gives a minimal projective resolution and FX is τ-rigid, there exists a morphism $(g', g'') : FP_0^X \to FX$ in $\mathsf{mod}\,\Delta$ such that $(0, \tilde{f}) = (g', g'')(0, \tilde{p})$. In particular, we have $\tilde{f} = g'' \tilde{p}$. Since F is full by Proposition 10.8(1), there exists a morphism $g : P_0^X \to X$ such that $Fg = (g', g'')$. Then g'' is a restriction of g by construction of F, and we have

$$gp = g \iota \tilde{p} \pi = \iota' g'' \tilde{p} \pi = \iota' \tilde{f} \pi = f.$$

Consequently, the map (18) is surjective.

$$
\begin{array}{ccc}
P_1^X & \xrightarrow{\ \pi\ } & P_1^X/P_1^X J \\
\end{array}
$$

This finishes the proof. $\hfill\square$

For any indecomposable Λ-module X, we decompose the terms P_0^X and P_1^X in a minimal projective presentation of X as

$$
P_0^X := \bigoplus_{i \in Q_0} (e_i \Lambda)^{n_i}, \ \ P_1^X := \bigoplus_{i \in Q_0} (e_i \Lambda)^{m_i},
$$

where n_i and m_i are multiplicities of the indecomposable projective Λ-module corresponding to $i \in Q_0$. Then, by Lemma 11.4, the terms P_0^{FX} and P_1^{FX} in a minimal projective presentation of the Δ-module FX can be written as

$$
P_0^{FX} = \bigoplus_{i \in Q_0} (e_{i^+} \Delta)^{n_i}, \ \ P_1^{FX} := \bigoplus_{i \in Q_0} (e_{i^-} \Delta)^{m_i}.
$$

We denote by Q^X the full subquiver of Q^{s} with $Q_0^X := \{i^+ \in Q_0^{\mathrm{s}} \mid n_i \neq 0\} \coprod \{i^- \in Q_0^{\mathrm{s}} \mid m_i \neq 0\}$. Then, the condition $\mathsf{add}\, P_0^X \cap \mathsf{add}\, P_1^X = 0$ is satisfied if and only if Q^X is a single subquiver of Q^{s}. In particular, if X is τ-rigid, then Q^X is a single subquiver of Q^s by Proposition 10.2(2).

Now we are ready to prove Theorem 11.1. For any full subquiver Q' of Q^{s}, let

$$
\mathrm{i}\tau\text{-rigid}\,(\Delta, Q') := \mathrm{i}\tau\text{-rigid}\,\Delta \cap \mathrm{Im}\, L_{e_{Q'}},
$$

where $e_{Q'} := \sum_{i \in Q_0'} e_i$ and $L_{e_{Q'}}$ is the functor in Subsection 10.1. We denote by $\mathrm{i}\tau\text{-rigid}_{\mathrm{np}}\Lambda$ the subset of $\mathrm{i}\tau\text{-rigid}\,\Lambda$ consisting of nonprojective modules.

Proof of Theorem 11.1. (1)$\Leftrightarrow$(2): First we claim that the functor $F : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Delta$ induces a bijection

$$
\mathrm{i}\tau\text{-rigid}_{\mathrm{np}}\Lambda \to \bigcup_{Q' \in \mathcal{S}} \mathrm{i}\tau\text{-rigid}_{\mathrm{np}}(\Delta, Q').
$$

Indeed, by Proposition 10.8(2) and Theorem 11.2, X is an indecomposable nonprojective τ-rigid Λ-module if and only if FX is an indecomposable nonprojective τ-rigid Δ-module satisfying $\mathsf{add}\, P_0^X \cap \mathsf{add}\, P_1^X = 0$, or equivalently $Q^X \in \mathcal{S}$. In this case, $FX \in \mathrm{Im}\, L_{e_{Q^X}}$ clearly holds. Hence the claim follows from that the functor F induces a stable equivalence $\underline{\mathsf{mod}}\,\Lambda \to \underline{\mathsf{mod}}\,\Delta$ by Proposition 10.8(1).

Next, by Proposition 10.4, for any full subquiver Q' of Q^{s}, we have bijections

$$
\mathrm{i}\tau\text{-rigid}\,(\Delta, Q') \leftrightarrow \mathrm{i}\tau\text{-rigid}\,(e_{Q'} \Delta e_{Q'}).
$$

Since Q' is bipartite, there is an isomorphism $e_{Q'} \Delta e_{Q'} \simeq K Q'$. Since there are only finitely many single subquiver of Q^{s}, we have that Λ is τ-rigid-finite if and only if $K Q'$ is τ-rigid-finite for every single subquiver Q' of Q^{s}. Hence the assertion follows from Theorem 10.6.

(2)$\Rightarrow$(3): It is clear.

(3)$\Rightarrow$(2): Since every single subquiver of the separated quiver Q^s is contained in some maximal single subquiver, the assertion follows from the fact that every subquiver of Dynkin quivers is a disjoint union of Dynkin quivers. $\square$

12. APPLICATIONS AND EXAMPLES

In this section, we give applications and examples of results in previous section. As an immediate consequence of Theorem 11.2, we have the following two corollaries.

Corollary 12.1. *Let Λ be a representation-finite algebra with radical square zero and X an indecomposable Λ-module. Then X is a τ-rigid Λ-module if and only if $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$.*

Proof. The 'only if' part follows from Proposition 10.2(2). We show the 'if' part. Since Λ is representation-finite, Δ is a finite product of path algebras with Dynkin quivers by Proposition 10.8(3). Hence FX is a τ-rigid Δ-module by Proposition 10.5(1). Thus, if $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$ holds, then X is τ-rigid by Theorem 11.2. $\square$

We give a positive answer to a question posed by Zhang [Zh].

Corollary 12.2. *Let Λ be an algebra with radical square zero. If every indecomposable Λ-module is τ-rigid, then Λ is representation-finite.*

Proof. Assume that Λ is not representation-finite. By Proposition 10.8(3), the separated quiver contains a non-Dynkin quiver as a subquiver. By Proposition 10.5(2), there exists an indecomposable Δ-module M which is not rigid. By Proposition 10.8(1), there exists an indecomposable nonprojective Λ-module X such that $FX \simeq M$. The Λ-module X is not τ-rigid by Theorem 11.2. $\square$

At the end of this part, we apply our main results to the following algebras which associate with Brauer graph algebras. We start with the following observation.

Proposition 12.3. [Ad1, Corollary 3.7] *Let Λ be a ring-indecomposable non-semisimple symmetric algebra and $\overline{\Lambda} := \Lambda/\operatorname{soc}\Lambda$. Then there is a bijection*

$$\mathrm{i}\tau\text{-rigid}\,\Lambda \to \mathrm{i}\tau\text{-rigid}\,\overline{\Lambda}$$

given by $X \mapsto X \otimes_\Lambda \overline{\Lambda}$. In particular, Λ is τ-rigid-finite if and only if $\overline{\Lambda}$ is τ-rigid-finite.

Note that, for every indecomposable projective Λ-module P, the module $P/\operatorname{soc}P$ is a τ-rigid $\overline{\Lambda}$-module but not a τ-rigid Λ-module.

First, we give a classification of indecomposable τ-rigid modules over a multiplicity-free Brauer line algebra. This is a special case of results in [AZ] and [AAC]. We denote by $\operatorname{ind}\Lambda$ the set of isomorphism classes of indecomposable Λ-modules.

Proposition 12.4. *Let Q be the following quiver:*

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

(1) *Let Λ be an algebra with radical square zero whose quiver is Q. Then Λ is a representation-finite algebra with*

$$\mathrm{i}\tau\text{-rigid}\,\Lambda = \operatorname{ind}\Lambda.$$

(2) *Let Γ is a multiplicity-free Brauer line algebra, that is, $\Gamma \simeq KQ/I$, where*

$$I = \langle \alpha_1 \beta_1 \alpha_1, \beta_{n-1} \alpha_{n-1} \beta_{n-1}, \alpha_i \alpha_{i+1}, \beta_{i+1} \beta_i, \beta_i \alpha_i - \alpha_{i+1} \beta_{i+1} \mid i = 1, 2, \cdots, n-2 \rangle.$$

Then we have

$$\text{i}\tau\text{-rigid}\,\Gamma = \text{ind}\,\Gamma \setminus \{e_i \Gamma / \text{soc}(e_i \Gamma) \mid i \in Q_0\}.$$

Proof. (1) Since the underlying graph of the separated quiver Q^{s} is a disjoint union of the following two Dynkin graphs of type A

$$1^+ \rule{1cm}{0.4pt} 2^- \rule{1cm}{0.4pt} \cdots \rule{1cm}{0.4pt} n{-}1^{-\epsilon} \rule{1cm}{0.4pt} n^{\epsilon} \qquad 1^- \rule{1cm}{0.4pt} 2^+ \rule{1cm}{0.4pt} \cdots \rule{1cm}{0.4pt} n{-}1^{\epsilon} \rule{1cm}{0.4pt} n^{-\epsilon}$$

where $\epsilon = -$ if n is even and $\epsilon = +$ if n is odd, Λ is representation-finite by Proposition 10.8. Moreover, for any indecomposable Λ-module X, the Δ-module FX is rigid by Proposition 10.5(1), or equivalently τ-rigid. Moreover, we have $\mathsf{add}\,P_0^X \cap \mathsf{add}\,P_1^X = 0$ by Lemma 11.4. Hence, every indecomposable Λ-module is always τ-rigid by Theorem 11.2.
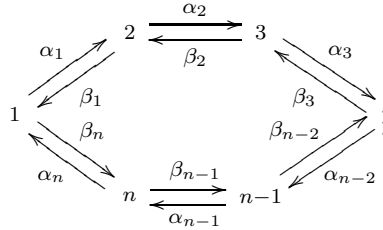
(2) Since Γ is a symmetric algebra, there is a bijection

$$\text{i}\tau\text{-rigid}\,\Gamma \to \text{i}\tau\text{-rigid}\,\overline{\Gamma}$$

by Proposition 12.3. Since $\overline{\Gamma}$ is isomorphic to Λ, the assertion follows from (1). $\qquad\square$

Finally, we give an example of τ-rigid-finite algebras which is not representation-finite.

Proposition 12.5. *Let Q be the following quiver:*

(1) *Let Λ be an algebra with radical square zero whose quiver is Q. Then the following hold.*

  (a) *Λ is not representation-finite.*

  (b) *Λ is τ-rigid-finite if and only if n is odd.*

(2) *Let Γ be a multiplicity-free Brauer cyclic graph algebra, that is, $\Gamma \simeq KQ/I$, where*

$$I = \langle \alpha_n \alpha_1, \beta_1 \beta_n, \beta_n \alpha_n - \alpha_1 \beta_1, \alpha_i \alpha_{i+1}, \beta_{i+1} \beta_i, \beta_i \alpha_i - \alpha_{i+1} \beta_{i+1} \mid i = 1, 2, \cdots, n-1 \rangle.$$

Then Γ is τ-rigid-finite if and only if n is odd.

Proof. (1) The separated quiver Q^{s} is one of the following quivers:
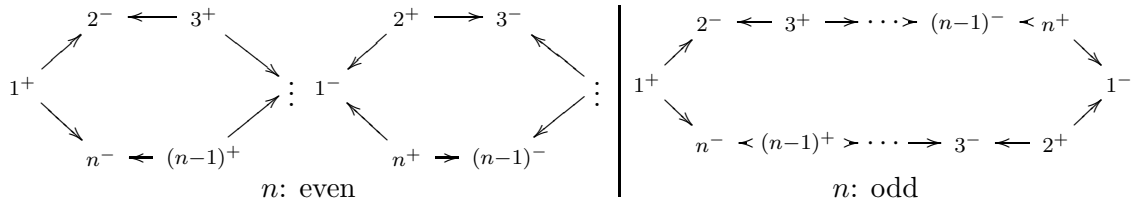
Thus Λ is not representation-finite by Proposition 10.8.

If n is odd, then every maximal single subquiver is a disjoint union of Dynkin quivers. Thus Λ is τ-rigid-finite by Theorem 11.1. On the other hand, if n is even, then two connected components are non-Dynkin maximal single subquivers. Thus Λ is not τ-rigid-finite by Theorem 11.1.

(2) Since Γ is a symmetric algebra, by Proposition 12.3, we have only to claim that $\overline{\Gamma}$ is τ-rigid-finite if and only if n is odd. Indeed, since $\overline{\Gamma}$ is isomorphic to Λ, the claim follows from (1). $\qquad\square$

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