EVALUATION FORMULAS OF CAUCHY-MELLIN TYPE FOR CERTAIN SERIES INVOLVING HYPERBOLIC FUNCTIONS

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ABSTRACT. We give evaluation formulas for certain Dirichlet series involving hyperbolic factors at some integer points in terms of π and the lemniscate constant, which have the same flavour as the classical formulas due to Cauchy, Mellin and Ramanujan. We then prove analogous formulas for double series involving hyperbolic functions. These formulas are shown via the functional equation for Barnes multiple zeta-functions, proved in a previous paper of the authors.

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Z}^*$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural numbers, nonnegative integers, rational integers, non-zero integers, rational numbers, real numbers and complex numbers, respectively. Let $i=\sqrt{-1}$.

For $\tau\in\mathbb{C}$ with $\Im\tau>0$, we define

$$S(s;\tau)=\sum_{m=1}^{\infty}\frac{(-1)^m}{\sinh(m\pi i/\tau)m^s}\qquad(s\in\mathbb{C}). \tag{1.1}$$

It is to be noted that $S(s;\tau)$ is holomorphic for all $s\in\mathbb{C}$.

This series was first studied by Cauchy [5], who discovered the following fascinating formulas:

$$S(4k-1;i)=\sum_{m=1}^{\infty}\frac{(-1)^m}{\sinh(m\pi)m^{4k-1}}=\frac{(2\pi)^{4k-1}}{2}\sum_{j=0}^{2k}(-1)^{j+1}\frac{B_{2j}(1/2)}{(2j)!}\frac{B_{4k-2j}(1/2)}{(4k-2j)!}, \tag{1.2}$$

$$S(-1;i)=\sum_{m=1}^{\infty}\frac{(-1)^m m}{\sinh(m\pi)}=-\frac{1}{4\pi}, \tag{1.3}$$

$$S(-4k-1;i)=\sum_{m=1}^{\infty}\frac{(-1)^m m^{4k+1}}{\sinh(m\pi)}=0 \tag{1.4}$$

for $k\in\mathbb{N}$, where $B_n(x)$ is the n-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

In particular, (1.4) implies that $s=-4k-1$ ($k\in\mathbb{N}$) may be regarded as "trivial zeros" of $S(s;i)$.

More generally, Mellin [13] proved

$$\begin{aligned}\alpha^{-N}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{\sinh(m\alpha)m^{2N+1}}-(-\beta)^{-N}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{\sinh(m\beta)m^{2N+1}}\\ =2^{2N+1}\pi\sum_{j=0}^{N+1}(-1)^j\frac{B_{2j}(1/2)}{(2j)!}\frac{B_{2N+2-2j}(1/2)}{(2N+2-2j)!}\alpha^{N+1-j}\beta^j,\end{aligned} \tag{1.5}$$

where N is any integer, α and β are positive numbers such that $\alpha\beta=\pi^2$. Later these were recovered by several mathematicians (for the details, see Berndt [3]).

It is well-known that Ramanujan discovered

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{m}{e^{2\pi m}-1}=\frac{1}{24}-\frac{1}{8\pi},\\
&\sum_{m=1}^{\infty}\frac{m^3}{e^{2\pi m}-1}=\frac{1}{80}\left(\frac{\varpi}{\pi}\right)^4-\frac{1}{240},\\
&\sum_{m=1}^{\infty}\frac{m^5}{e^{2\pi n}-1}=\frac{1}{504},
\end{aligned}\tag{1.6}$$

and so on (see Berndt [4, Chapter 14]), where

$$\varpi=2\int_0^1\frac{dx}{\sqrt{1-x^4}}=\frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}=2.622057\cdots\tag{1.7}$$

is the lemniscate constant. The left-hand sides of formulas (1.6) can also be regarded as series involving hyperbolic functions. From the results mentioned above we may expect that various series involving hyperbolic functions are sometimes evaluated in terms of π and ϖ.

In this paper, we first evaluate $S(-4k+1;i)$ $(k\in\mathbb{N})$ in terms of π and ϖ (see Theorem 3.2 and Example 3.3), by using the functional equation of Barnes double zeta-functions given in our previous paper [10]. Also we evaluate $S(-6k+1;\rho)$ $(k\in\mathbb{N})$, where $\rho=e^{2\pi i/3}$ (see Theorem 3.5 and Example 3.6). These are analogues of (1.6).

The reason why we only consider the cases $\tau=i,\rho$ is mentioned at the end of Section 3 (Remark 3.7).

Our theory on the evaluation of $S(s;\tau)$ is closely related with the theory of some double series (such as Barnes zeta-functions and Eisenstein series), as we will see in Section 3. Therefore it is a natural problem to consider certain double analogues of $S(s;\tau)$ itself. For example, certain double series of Eisenstein type involving hyperbolic functions such as

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^n}{\sinh(m\pi)(m+ni)^k},\quad \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^*}\frac{1}{\sinh(m\pi)^2(m+ni)^k}$$

are studied in [16, 17] and [11]. Another direction of generalization can be found in [10], where we study the series whose each term includes two (or more) hyperbolic factors in the denominator, and prove, for example,

$$\sum_{m=1}^{\infty}\frac{(-1)^m}{\sinh(m\pi i/\rho)\sinh(m\pi i/\rho^2)m^4}=\sum_{m=1}^{\infty}\frac{1}{\sinh(m\pi i/\rho)^2m^4}=-\frac{1}{5670}\pi^4.$$

In this paper we consider another type of double analogue of $S(s;i)$ defined by

$$\mathcal{S}_2(s;i)=\sum_{m\in\mathbb{Z}^*}\sum_{\substack{n\in\mathbb{Z}^*\\ m+n>0}}\frac{(-1)^{m+n}}{\sinh(m\pi)\sinh(n\pi)(m+n)^s}\quad (s\in\mathbb{C}).\tag{1.8}$$

We evaluate $\mathcal{S}_2(-4k;i)$ (see Theorem 4.1) and $\mathcal{S}_2(4k;i)$ (see Theorem 4.7) for $k\in\mathbb{N}$. A key fact for the proof is the existence of trivial zeros (1.4) of $S(s;i)$ (see Remark 4.9).

2. Preliminary results on Eisenstein series

We begin with recalling several known results on Eisenstein series. Let

$$G_{2j}(\tau)=\sum_{m\in\mathbb{Z}}\sum_{\substack{n\in\mathbb{Z}\\ (m,n)\neq(0,0)}}\frac{1}{(m+n\tau)^{2j}}\tag{2.1}$$

be the Eisenstein series, where $j \in \mathbb{N}$ and $\tau \in \mathbb{C}$ with $\Im \tau > 0$. Note that even if $j = 1$, we define $G_2(\tau)$ by (2.1) which converges not absolutely but conditionally. Denote the lattice $\mathbb{Z} + \mathbb{Z}\tau$ by $L(\tau)$ and define the Weierstrass $\wp$-function by

$$\wp(z; L(\tau)) = \frac{1}{z^2} + \sum_{\substack{\lambda \in L(\tau) \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right). \tag{2.2}$$

Then we see that

$$\wp(z; L(\tau)) = \frac{1}{z^2} + \sum_{j=1}^{\infty} (2j+1) G_{2j+2}(\tau) z^{2j} \tag{2.3}$$

(see, for example, [9, Chapter 1, §6]). When $\tau = i$, we have $G_2(i) = -\pi$, and

$$G_{4k}(i) = \mathcal{E}_{4k} \frac{(2\varpi)^{4k}}{(4k)!}, \quad G_{4k+2}(i) = 0 \quad (k \in \mathbb{N}), \tag{2.4}$$

where $\mathcal{E}_{4k} \in \mathbb{Q}$. The numbers $\mathcal{E}_{4k}$ ($k \in \mathbb{N}$) are called Hurwitz numbers, because (2.4) is due to Hurwitz [7]. For example, we see that

$$G_4(i) = \frac{1}{15}\varpi^4,\ G_8(i) = \frac{1}{525}\varpi^8,\ G_{12}(i) = \frac{2}{53625}\varpi^{12}, \ldots \tag{2.5}$$

(see [12]). Analogously, Katayama [8, (6.8)] showed that

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{(2m+1+(2n+1)i)^{4k}} = \mathcal{E}_{4k}^{(1,1)} \frac{(2\varpi)^{4k}}{(4k)!} \quad (k \in \mathbb{N}) \tag{2.6}$$

with $\mathcal{E}_{4k}^{(1,1)} \in \mathbb{Q}$ ($k \in \mathbb{N}$), which Katayama called 2-division Hurwitz numbers. The values of $\mathcal{E}_{4k}^{(1,1)}$ are, for example,

$$\mathcal{E}_4^{(1,1)} = -\frac{1}{2^5},\ \mathcal{E}_8^{(1,1)} = \frac{3^2}{2^9},\ \mathcal{E}_{12}^{(1,1)} = -\frac{3^4 \cdot 7}{2^{13}}, \ \cdots.$$

In the case $\tau = \rho = e^{2\pi i/3}$, it holds that $G_{6k}(\rho) \in \mathbb{Q} \cdot \widetilde{\varpi}^{6k}$ ($k \in \mathbb{N}$), where

$$\widetilde{\varpi} = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.4286506\cdots$$

(see [14, 18]). For example, we have

$$G_6(\rho) = \frac{\widetilde{\varpi}^6}{35},\ G_{12}(\rho) = \frac{\widetilde{\varpi}^{12}}{7007},\ G_{18}(\rho) = \frac{\widetilde{\varpi}^{18}}{1440257}, \ldots \tag{2.7}$$

3. Barnes zeta-functions and $S(s;\tau)$

Now we recall the Barnes multiple zeta-function [1, 2] defined by

$$\zeta_n(s, a; \omega_1, \ldots, \omega_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{1}{(a + \omega_1 m_1 + \cdots + \omega_n m_n)^s} \tag{3.1}$$

for $a, \omega_1, \ldots, \omega_n \in H(\theta)$ for some $\theta \in \mathbb{R}$, where

$$H(\theta) = \{z = re^{i(\theta+\phi)} \in \mathbb{C} \mid r > 0, -\pi/2 < \phi < \pi/2\}$$

is the open half plane whose boundary line is vertical with $e^{i\theta}$. Then $\zeta_n(s, a; \omega_1, \ldots, \omega_n)$ converges absolutely and uniformly on any compact subset in $\Re s > n$, and is continued meromorphically to the whole complex plane.

Recently we showed the following functional equation.

Theorem 3.1 ([10], Theorem 2.1)**.** *For $y \in [0,1)$,*

$$\zeta_n(s,a(y);\omega_1,\ldots,\omega_n) = -\frac{2\pi i}{\Gamma(s)(e^{2\pi is}-1)} \times \sum_{k=1}^{n}\sum_{m\in\mathbb{Z}^*}\omega_k^{-1}\Big(\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{e^{(2m\pi i\omega_j/\omega_k)y}}{e^{2m\pi i\omega_j/\omega_k}-1}\Big)(2m\pi i\omega_k^{-1})^{s-1}e^{2m\pi iy},$$

where

$$a(y) = \omega_1(1-y)+\cdots+\omega_n(1-y) \in H(\theta),$$

and the argument of $2m\pi i\omega_k^{-1}$ is to be taken as$(\pi/2)-\arg\omega_k$. Note that the right-hand side converges absolutely uniformly on the whole space $\mathbb{C}$ if $0<y<1$, and on the region $\Re s<0$ if $y=0$.

This theorem connects the Barnes multiple zeta-function with Dirichlet series involving hyperbolic functions. In particular when $(n,y,\omega_1,\omega_2)=(2,1/2,1,i)$, we can obtain

$$\zeta_2(s,a(1/2);1,i) = \frac{(2\pi)^s}{2\Gamma(s)\left(e^{\pi is}+1\right)\left(e^{\pi is/2}+1\right)}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m m^{s-1}}{\sinh(m\pi)} \tag{3.2}$$

for any $s\in\mathbb{C}$. Note that, by calculating the values at nonpositive integers on both sides of (3.2), we can obtain Cauchy's formula (1.2) (see [10, Corollary 6.2 and Corollary 6.3]).

On the other hand, when s is a positive integer, (3.2) gives the following consequence. We can easily see that, for $k\in\mathbb{N}$,

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\frac{1}{(2m+1+(2n+1)i)^{4k}} &= \frac{4}{2^{4k}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+1/2+(n+1/2)i)^{4k}}\\ &= \frac{4}{2^{4k}}\zeta_2(4k,a(1/2);1,i)\end{aligned} \tag{3.3}$$

(divide the left-hand side into four parts according to the signs of $2m+1$ and $2n+1$, and use the fact $(p-qi)^4=(q+pi)^4$). Hence, by combining (2.6), (3.2) and (3.3), we obtain the following theorem.

Theorem 3.2. *For $k\in\mathbb{N}$,*

$$S(-4k+1;i) = \sum_{m=1}^{\infty}\frac{(-1)^m m^{4k-1}}{\sinh(m\pi)} = \frac{2^{4k-2}}{k}\mathcal{E}_{4k}^{(1,1)}\left(\frac{\varpi}{\pi}\right)^{4k}. \tag{3.4}$$

Example 3.3.

$$\begin{aligned}
S(-3;i) &= \sum_{m=1}^{\infty}\frac{(-1)^m m^{3}}{\sinh(m\pi)} = -\frac{1}{8}\left(\frac{\varpi}{\pi}\right)^{4},\\
S(-7;i) &= \sum_{m=1}^{\infty}\frac{(-1)^m m^{7}}{\sinh(m\pi)} = \frac{9}{16}\left(\frac{\varpi}{\pi}\right)^{8},\\
S(-11;i) &= \sum_{m=1}^{\infty}\frac{(-1)^m m^{11}}{\sinh(m\pi)} = -\frac{189}{8}\left(\frac{\varpi}{\pi}\right)^{12},\\
S(-15;i) &= \sum_{m=1}^{\infty}\frac{(-1)^m m^{15}}{\sinh(m\pi)} = \frac{130977}{32}\left(\frac{\varpi}{\pi}\right)^{16},\\
S(-19;i) &= \sum_{m=1}^{\infty}\frac{(-1)^m m^{19}}{\sinh(m\pi)} = -\frac{16110171}{8}\left(\frac{\varpi}{\pi}\right)^{20}.
\end{aligned}$$

Next we consider the case $\tau = \rho = e^{2\pi i/3}$. We first show

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \frac{1}{(2m+1+(2n+1)\rho)^{6k}} = \frac{1}{3}\left(1-\frac{1}{2^{6k}}\right) G_{6k}(\rho) \qquad (k\in\mathbb{N}). \tag{3.5}$$

To show this, we divide the definition (2.1) of $G_{6k}(\rho)$ as $A_{00}+A_{01}+A_{10}+A_{11}$, where A_{ij} denotes the partial sum running over m and n with $m\equiv i$ and $n\equiv j$ (mod 2). Since $\rho^3=1$ and $\rho^2=-\rho-1$, we see that

$$\begin{aligned}(2m+(2n+1)\rho)^{6k} &= (-2m\rho-(2n+1)\rho^2)^{6k}\\ &= (-2m\rho+(2n+1)(\rho+1))^{6k} = (2n+1+(2n-2m+1)\rho)^{6k},\end{aligned}$$

from which we find $A_{01}=A_{11}$. Similarly we see that $A_{10}=A_{01}$. On the other hand it is obvious that $A_{00}=2^{-6k}G_{6k}(\rho)$. Since A_{11} is equal to the left-hand side of (3.5), collecting the above results we obtain the conclusion.

From Theorem 3.1 we can deduce the following.

Lemma 3.4. *For $\Re s>2$, we have*

$$\begin{aligned}&(-\rho)^s\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+1/2+(n+1/2)\rho)^s}+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+1/2+(n+1/2)(-\rho^{-1}))^s}\\ &\quad=\frac{(2\pi)^s e^{-\pi i s/2}}{2\Gamma(s)\,(e^{\pi i s}+1)}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m m^{s-1}}{\sinh(m\pi i/\rho)},\end{aligned}\tag{3.6}$$

where $(-\rho)^s=(e^{-\pi i}\rho)^s$.

Proof. Using Theorem 3.1 with $(n,y,\omega_1,\omega_2)=(2,1/2,1,\rho)$ and $(n,y,\omega_1,\omega_2)=(2,1/2,1,-\rho^{-1})$, where $-\rho^{-1}=e^{\pi i/3}$, we find that the left-hand side of (3.6) is equal to

$$-\frac{2\pi i}{\Gamma(s)(e^{2\pi i s}-1)}B,$$

where

$$\begin{aligned}B&=(-\rho)^s\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m e^{m\pi i\rho}}{e^{2m\pi i\rho}-1}(2m\pi i)^{s-1}+\frac{1}{\rho}(-\rho)^s\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m e^{m\pi i/\rho}}{e^{2m\pi i/\rho}-1}(2m\pi i/\rho)^{s-1}\\ &\quad+\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m e^{m\pi i/\rho}}{1-e^{2m\pi i/\rho}}(2m\pi i)^{s-1}-\rho\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m e^{m\pi i\rho}}{1-e^{2m\pi i\rho}}(2m\pi\rho/i)^{s-1}.\end{aligned}$$

The first and the fourth sums of B cancel with each other, and the remaining part (the second and the third sums) gives

$$B=\frac{-1+e^{-\pi i s}}{2}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m(2m\pi i)^{s-1}}{\sinh(m\pi i/\rho)},$$

which implies the lemma. □

Putting $s=6k$ ($k\in\mathbb{N}$) in equation (3.6) and noting

$$\begin{aligned}&2^{6k}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\frac{1}{(2m+1+(2n+1)\rho)^{6k}}\\ &\quad=2\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+1/2+(n+1/2)\rho)^{6k}}+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+1/2+(n+1/2)(-\rho^{-1}))^{6k}}\right\}\end{aligned}$$

(which can be seen by dividing the left-hand side into four parts according to the signs of $2m+1$ and $2n+1$, and using the fact $(p-q\rho)^6=(q+p(-\rho^{-1}))^6$), we have

$$\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m m^{6k-1}}{\sinh(m\pi i/\rho)}=\frac{2(-1)^k(6k-1)!}{\pi^{6k}}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\frac{1}{(2m+1+(2n+1)\rho)^{6k}}\tag{3.7}$$

for $k \in \mathbb{N}$. Hence, combining (3.5) and (3.7), we obtain the following.

Theorem 3.5. *For* $k \in \mathbb{N}$,

$$S(-6k+1;\rho) \in \mathbb{Q} \cdot \left(\frac{\widetilde{\varpi}}{\pi}\right)^{6k}. \tag{3.8}$$

Example 3.6. Substituting (2.7) into (3.5), and using (3.7), we obtain the following:

$$\begin{aligned} S(-5;\rho) &= \sum_{m=1}^{\infty} \frac{(-1)^m m^5}{\sinh(m\pi i/\rho)} = -\frac{9}{8}\left(\frac{\widetilde{\varpi}}{\pi}\right)^{6}, \\ S(-11;\rho) &= \sum_{m=1}^{\infty} \frac{(-1)^m m^{11}}{\sinh(m\pi i/\rho)} = \frac{30375}{16}\left(\frac{\widetilde{\varpi}}{\pi}\right)^{12}, \\ S(-17;\rho) &= \sum_{m=1}^{\infty} \frac{(-1)^m m^{17}}{\sinh(m\pi i/\rho)} = -\frac{658560375}{8}\left(\frac{\widetilde{\varpi}}{\pi}\right)^{18}. \end{aligned}$$

Remark 3.7. The principle of the proofs of theorems in this section is to express the special values of $S(s;\tau)$ in terms of Eisenstein series, and use known facts on Eisenstein series. Those expressions are proved by using specific properties of numbers i and ρ. In general, $S(s;\tau)$ can be written in terms of Barnes multiple zeta-functions by Theorem 3.1, but Eisenstein series is usually a linear combination of (two or more) Barnes zeta-functions, so the above argument cannot be applied to other values of τ. This is the reason why we only consider the cases $\tau = i,\rho$ in this section.

4. Some relations among $\mathcal{S}_2(s;i)$ and $S(s;i)$

In this section, we will give some relation formulas among $\mathcal{S}_2(s;i)$ and $S(s;i)$. First we prove the following theorem concerning the values at negative integers.

Theorem 4.1. *For* $p \in \mathbb{N}$,

$$\mathcal{S}_2(-4p;i) = -\frac{4p}{\pi} S(-4p+1;i) = -\frac{2^{4p}}{\pi}\mathcal{E}_{4p}^{(1,1)}\left(\frac{\varpi}{\pi}\right)^{4p}. \tag{4.1}$$

In order to prove this theorem, we prepare some notation and lemmas. Let

$$h(t) = e^t - e^{-t} + \frac{e^{it}-e^{-it}}{i} = 4\sum_{j=0}^{\infty} \frac{t^{4j+1}}{(4j+1)!}, \tag{4.2}$$

and

$$J(t) = \sum_{n=1}^{\infty} \frac{(-1)^n h(nt)}{\sinh(n\pi)} + \frac{t}{\pi} \quad (|t|<\pi). \tag{4.3}$$

The right-hand side of (4.3) is absolutely convergent when $|t|<\pi$, so is holomorphic. Substituting (4.2) into (4.3), and using (1.3) and (1.4), we have

$$\begin{aligned} J(t) &= 4\sum_{j=0}^{\infty} S(-4j-1;i)\frac{t^{4j+1}}{(4j+1)!} + \frac{t}{\pi} \\ &= 4S(-1;i)t + \frac{t}{\pi} = 0 \quad (|t|<\pi). \end{aligned} \tag{4.4}$$

Let

$$F(t) = \sum_{m\in\mathbb{Z}^*} \frac{(-1)^m e^{mt}}{\sinh(m\pi)} \cdot J(t). \tag{4.5}$$

Then $F(t)$ is absolutely convergent when $|t| < \pi$. By (4.4), we have $F(t) \equiv 0$ for $|t| < \pi$. Substituting (4.2) and (4.3) into (4.5), we have

$$\begin{aligned} F(t) = &\sum_{m\in\mathbb{Z}^*}\sum_{n=1}^{\infty} \frac{(-1)^{m+n}\left\{e^{(m+n)t} - e^{(m-n)t} + i^{-1}e^{(m+ni)t} - i^{-1}e^{(m-ni)t}\right\}}{\sinh(m\pi)\sinh(n\pi)} \\ &+ \frac{t}{\pi}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m e^{mt}}{\sinh(m\pi)}. \end{aligned} \tag{4.6}$$

In the numerator of the first part on the right-hand side of (4.6), replacing $-n$ by n in the second and fourth terms in braces and using $\sinh(-x) = -\sinh(x)$, we can rewrite (4.6) as

$$F(t) = \sum_{m\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*} \frac{(-1)^{m+n}\left\{e^{(m+n)t} + i^{-1}e^{(m+ni)t}\right\}}{\sinh(m\pi)\sinh(n\pi)} + \frac{t}{\pi}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m e^{mt}}{\sinh(m\pi)}. \tag{4.7}$$

We further let

$$\begin{aligned} \widetilde{F}(t) = &\sum_{m\in\mathbb{Z}^*}\sum_{\substack{n\in\mathbb{Z}^*\\ m+n\neq 0}} \frac{(-1)^{m+n}e^{(m+n)t}}{\sinh(m\pi)\sinh(n\pi)} \\ &+ \frac{1}{i}\sum_{m\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*} \frac{(-1)^{m+n}e^{(m+ni)t}}{\sinh(m\pi)\sinh(n\pi)} + \frac{t}{\pi}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m e^{mt}}{\sinh(m\pi)}. \end{aligned} \tag{4.8}$$

Then $\widetilde{F}(t)$ is absolutely convergent for $|t| < \pi$, hence we can write

$$\widetilde{F}(t) = \sum_{j=0}^{\infty} \widetilde{D}_j \frac{t^j}{j!}.$$

Lemma 4.2.

$$\widetilde{D}_0 = \sum_{m\in\mathbb{Z}^*}\frac{1}{\sinh^2(m\pi)}, \qquad \widetilde{D}_j = 0 \ \ (j \geq 1).$$

Proof. Comparing (4.7) and (4.8), we have

$$F(t) = \widetilde{F}(t) - \sum_{m\in\mathbb{Z}^*}\frac{1}{\sinh^2(m\pi)}.$$

Since $F(t) \equiv 0$ for $|t| < \pi$, we complete the proof. □

We let

$$\begin{aligned} \mathcal{T}_2(k;i) = &\sum_{m\in\mathbb{Z}^*}\sum_{\substack{n\in\mathbb{Z}^*\\ m+n\neq 0}} \frac{(-1)^{m+n}}{\sinh(m\pi)\sinh(n\pi)(m+n)^k} \\ &+ \frac{1}{i}\sum_{m\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*} \frac{(-1)^{m+n}}{\sinh(m\pi)\sinh(n\pi)(m+ni)^k} \qquad (k\in\mathbb{Z}). \end{aligned} \tag{4.9}$$

Lemma 4.3. *For* $j \in \mathbb{N}_0$,

$$\widetilde{D}_j = \mathcal{T}_2(-j;i) + \frac{j}{\pi}S(1-j;i).$$

Proof. Considering the Maclaurin expansion of e^t in (4.8) and using (4.9), we have

$$\begin{aligned} \widetilde{F}(t) &= \sum_{j=0}^{\infty}\mathcal{T}_2(-j;i)\frac{t^j}{j!} + \frac{1}{\pi}\sum_{j=0}^{\infty}S(-j;i)\frac{t^{j+1}}{j!} \\ &= \sum_{j=0}^{\infty}\left\{\mathcal{T}_2(-j;i) + \frac{j}{\pi}S(1-j;i)\right\}\frac{t^j}{j!}. \end{aligned}$$

Thus we complete the proof. □

Proof of Theorem 4.1. When $k=-4p$ $(p\in\mathbb{N})$, the second sum on the right-hand side of (4.9) is

$$I:=\sum_{m\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*}\frac{(-1)^{m+n}(m+ni)^{4p}}{\sinh(m\pi)\sinh(n\pi)}.$$

But we see that $I=0$, because, replacing m by $-m$ in I, we have

$$\begin{aligned}I&=-\sum_{m\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*}\frac{(-1)^{m+n}(-m+ni)^{4p}}{\sinh(m\pi)\sinh(n\pi)}\\&=-i^{4p}\sum_{m\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*}\frac{(-1)^{m+n}(mi+n)^{4p}}{\sinh(m\pi)\sinh(n\pi)}=-I.\end{aligned}$$

Therefore

$$\mathcal{T}_2(-4p;i)=\mathcal{S}_2(-4p;i). \tag{4.10}$$

Combining this with $\mathcal{T}_2(-j;i)+\frac{j}{\pi}S(1-j;i)=0$ (for $j\in\mathbb{N}$), which follows from Lemmas 4.2 and 4.3, we obtain the first equation of Theorem 4.1. The second equation of Theorem 4.1 follows from Theorem 3.2. $\square$

Next we evaluate $\mathcal{S}_2(4p;i)$ $(p\in\mathbb{N})$. For this aim, we prepare the following three lemmas. Note that the former two lemmas are quoted from the previous papers of the third-named author.

Lemma 4.4 ([16] Theorem 3.1)**.** *For $k\in\mathbb{N}_0$, let*

$$\mathcal{G}_{2k+1}(i)={\sum_{n\in\mathbb{Z}}}'\sum_{m\in\mathbb{Z}^*}\frac{(-1)^n}{\sinh(m\pi)(m+ni)^{2k+1}}, \tag{4.11}$$

where $\sum'$ means that, when $k=0$, we first sum in the region $|n|\le N$ and then take the limit $N\to\infty$. Then

$$\begin{aligned}\mathcal{G}_{2k+1}(i)=&\frac{2(-1)^{k+1}}{\pi}\sum_{j=0}^{k}\left(2^{1-2k+2j}-1\right)\zeta(2k-2j)\\&\times\left\{(-1)^jG_{2j+2}(i)+2\zeta(2j+2)\right\},\end{aligned} \tag{4.12}$$

where $\zeta(s)$ is the Riemann zeta-function.

Lemma 4.5 ([15] Lemma 8)**.** *Suppose $\{P_k\}_{k\ge0}$ and $\{Q_k\}_{k\ge0}$ are sequences which satisfy the relations*

$$\sum_{\mu=0}^{k}P_{k-\mu}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}=Q_k$$

for any $k\in\mathbb{N}_0$. Then the relation

$$P_k=-2\sum_{\nu=0}^{k}\left(2^{1-2k+2\nu}-1\right)\zeta(2k-2\nu)Q_\nu$$

holds for any $k\in\mathbb{N}_0$.

Lemma 4.6. *For $p\in\mathbb{N}_0$,*

$$\frac{1}{\pi}\mathcal{G}_{2p+1}(i)=\sum_{j=0}^{p}\left\{\mathcal{T}_2(2p-2j;i)+\frac{2j-2p}{\pi}S(2p+1-2j;i)\right\}\frac{(i\pi)^{2j}}{(2j+1)!}. \tag{4.13}$$

Proof. Let $\theta\in(-\pi,\pi)$, and let

$$\begin{aligned}C(\theta)=&i\sum_{n\in\mathbb{Z}^*}\sum_{\substack{m\in\mathbb{Z}^*\\m+n\neq0}}\frac{(-1)^{m+n}\sin((m+n)\theta)}{\sinh(m\pi)\sinh(n\pi)(m+n)^{2p+1}}\\&-i{\sum_{n\in\mathbb{Z}^*}}'\sum_{m\in\mathbb{Z}^*}\frac{(-1)^{m+n}\sinh((m+ni)i\theta)}{\sinh(m\pi)\sinh(n\pi)(m+ni)^{2p+1}}\end{aligned} \tag{4.14}$$

$$-\frac{(2p+1)i}{\pi}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m\sin(m\theta)}{\sinh(m\pi)m^{2p+2}}+\frac{i\theta}{\pi}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^m\cos(m\theta)}{\sinh(m\pi)m^{2p+1}}.$$

Since $|\theta|<\pi$, all of the above sums are convergent absolutely. Hence, using the Maclaurin expansions of $\sin x$, $\cos x$ and $\sinh x$ and applying Lemma 4.3, we have

$$\begin{aligned}C(\theta)&=\sum_{j=0}^{\infty}\left\{\mathfrak{T}_2(2p-2j;i)+\frac{2j-2p}{\pi}S(2p+1-2j;i)\right\}\frac{(i\theta)^{2j+1}}{(2j+1)!}\\&=\sum_{j=0}^{p}\left\{\mathfrak{T}_2(2p-2j;i)+\frac{2j-2p}{\pi}S(2p+1-2j;i)\right\}\frac{(i\theta)^{2j+1}}{(2j+1)!}\\&\quad+\sum_{j=p+1}^{\infty}\widetilde{D}_{2j-2p}\frac{(i\theta)^{2j+1}}{(2j+1)!}.\end{aligned}\tag{4.15}$$

By Lemma 4.2, we have $\widetilde{D}_{j-p}=0$ for $j\geq p+1$. Therefore the right-hand side of (4.15) are continuous for any $\theta\in\mathbb{R}$. On the other hand, the first, the third, and the fourth sums on the right-hand side of (4.14) are clearly convergent uniformly in $\theta\in\mathbb{R}$, hence continuous for any $\theta\in\mathbb{R}$. Next we claim that the second sum on the right-hand side of (4.14) is continuous for $\theta\in(-\pi,\pi]$. (Note here that our original proof of this claim was erroneous. The following proof of the claim is due to the referee.)

To show this claim, it is enough to consider the case $p=0$. Then

$$\sinh((m+ni)i\theta)=\frac{1}{2}(e^{im\theta-n\theta}-e^{-im\theta+n\theta}),$$

and the contribution of $e^{im\theta-n\theta}$ $(n>0)$ and of $e^{-im\theta+n\theta}$ $(n<0)$ to the second sum are obviously absolutely convergent for any $\theta\in\mathbb{R}$. The contribution of the remaining part is (with replacing n by $-n$ when $n<0$)

$$\begin{aligned}&\lim_{N\to\infty}\left\{-\sum_{n=1}^{N}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^{m+n}e^{-im\theta+n\theta}}{2\sinh(m\pi)\sinh(n\pi)(m+ni)}-\sum_{n=1}^{N}\sum_{m\in\mathbb{Z}^*}\frac{(-1)^{m+n}e^{im\theta+n\theta}}{2\sinh(m\pi)\sinh(n\pi)(m-ni)}\right\}\\&=-\frac{1}{2}\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}e^{-im\theta+n\theta}}{\sinh(m\pi)\sinh(n\pi)(m+ni)}+\sum_{n=1}^{N}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}e^{im\theta+n\theta}}{\sinh(-m\pi)\sinh(n\pi)(-m+ni)}\right.\\&\qquad\left.+\sum_{n=1}^{N}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}e^{im\theta+n\theta}}{\sinh(m\pi)\sinh(n\pi)(m-ni)}+\sum_{n=1}^{N}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}e^{-im\theta+n\theta}}{\sinh(-m\pi)\sinh(n\pi)(-m-ni)}\right\}\\&=-\lim_{N\to\infty}\left\{\sum_{m=1}^{\infty}\frac{(-1)^me^{-im\theta}}{\sinh(m\pi)}\sum_{n=1}^{N}\frac{(-1)^ne^{n\theta}}{\sinh(n\pi)(m+ni)}+\sum_{m=1}^{\infty}\frac{(-1)^me^{im\theta}}{\sinh(m\pi)}\sum_{n=1}^{N}\frac{(-1)^ne^{n\theta}}{\sinh(n\pi)(m-ni)}\right\}\\&=\lim_{N\to\infty}\left\{-\sum_{m=1}^{\infty}\frac{(-1)^m2m\cos(m\theta)}{\sinh(m\pi)}\sum_{n=1}^{N}\frac{(-1)^ne^{n\theta}}{\sinh(n\pi)(m^2+n^2)}\right.\\&\qquad\left.+\sum_{m=1}^{\infty}\frac{(-1)^m2\sin(m\theta)}{\sinh(m\pi)}\sum_{n=1}^{N}\frac{(-1)^ne^{n\theta}}{\sinh(n\pi)}\frac{n}{(m^2+n^2)}\right\}.\end{aligned}$$

We denote by J_1 and J_2 the first and the second sums on the right-hand side, respectively. Then we see that J_1 is convergent absolutely and uniformly in $\theta\in(-\pi,\pi]$. Next we rewrite the inner sum of J_2 as

$$\begin{aligned}\sum_{n=1}^{N}\frac{(-1)^ne^{n\theta}}{\sinh(n\pi)}\frac{n}{m^2+n^2}&=\sum_{n=1}^{N}\frac{(-1)^ne^{n\theta}}{\sinh(n\pi)}\left(\frac{n}{m^2+n^2}-\frac{1}{n}\right)+\sum_{n=1}^{N}\frac{e^{n\theta}}{\sinh(n\pi)}\frac{(-1)^n}{n}\\&=-\sum_{n=1}^{N}\frac{(-1)^ne^{n\theta}}{\sinh(n\pi)}\frac{m^2}{n(m^2+n^2)}+\sum_{n=1}^{N}\frac{e^{n\theta}}{\sinh(n\pi)}\frac{(-1)^n}{n}.\end{aligned}$$

Hence we can rewrite J_2 to

$$-\sum_{m=1}^{\infty}\frac{(-1)^m 2m^2\sin(m\theta)}{\sinh(m\pi)}\sum_{n=1}^{N}\frac{(-1)^n e^{n\theta}}{\sinh(n\pi)}\frac{1}{n(m^2+n^2)}$$
$$+\left(\sum_{m=1}^{\infty}\frac{(-1)^m 2\sin(m\theta)}{\sinh(m\pi)}\right)\left(\sum_{n=1}^{N}\frac{e^{n\theta}}{\sinh(n\pi)}\frac{(-1)^n}{n}\right).$$

Let N tend to infinity. Then the first double sum and the sum in the former parentheses of the second part are convergent absolutely and uniformly in $\theta\in(-\pi,\pi]$. The sum in the latter parentheses of the second part is not convergent absolutely for $\theta=\pi$ but convergent conditionally for $\theta=\pi$ and convergent absolutely for $\theta\in\mathbb{C}$ with $|\theta|<\pi$. Therefore it is continuous for $\theta\in(-\pi,\pi]$ by Abel's theorem. Hence the claim follows.

Therefore the both expressions (4.14) and (4.15) of $C(\theta)$ are continuous when $\theta\to\pi-0$. Therefore, letting $\theta\to\pi-0$ and using $\sin((m+n)\pi)=0$ and

$$\sinh((m+ni)i\pi)=\sinh(mi\pi-n\pi)=-(-1)^m\sinh(n\pi),$$

we see that

$$i\sum_{m\in\mathbb{Z}^*}\sum_{n\in\mathbb{Z}^*}\frac{(-1)^n}{\sinh(m\pi)(m+ni)^{2p+1}}+i\sum_{m\in\mathbb{Z}^*}\frac{1}{\sinh(m\pi)m^{2p+1}}$$
$$=\sum_{j=0}^{p}\left\{\mathcal{T}_2(2p-2j;i)+\frac{2j-2p}{\pi}S(2p+1-2j;i)\right\}\frac{(i\pi)^{2j+1}}{(2j+1)!}.$$

Note that the left-hand side is equal to $i\mathcal{G}_{2p+1}(i)$. Therefore, dividing the both sides by $i\pi$, we obtain (4.13). □

Theorem 4.7. *For $p\in\mathbb{N}_0$,*

$$\mathcal{S}_2(4p;i)=\frac{4p}{\pi}S(4p+1;i)+2\left(1-\frac{\pi}{3}\right)S(4p-1;i)-\frac{4}{\pi}\sum_{j=1}^{p}\zeta(4j+2)S(4p-4j-1;i). \tag{4.16}$$

Remark 4.8. By (1.2) and (1.3), we see that $S(4k-1;i)\in\mathbb{Q}\cdot\pi^{4k-1}$ $(k\in\mathbb{N}_0)$. Hence (4.16) gives that

$$\pi\mathcal{S}_2(4p;i)-4pS(4p+1;i)\in\mathbb{Q}[\pi]\quad(p\in\mathbb{N}_0).$$

However it is not known whether $S(4p+1;i)$ can be written as a closed form in terms of π, ϖ and so on. In fact, from (1.5) in the case $\alpha=\beta=\pi$, we have no information about $S(4p+1;i)$ unlike the situation of $S(4p-1;i)$.

By (4.12) and the property of $G_{2j}(i)$ (see Section 2), we see that $\pi\mathcal{G}_{2p+1}(i)\in\mathbb{Q}\left[\pi,\varpi^4\right]$ $(p\in\mathbb{N}_0)$. Therefore, putting $k=2p+1$ in (4.17) below, we have $\pi^2\mathcal{T}_2(4p+2;i)\in\mathbb{Q}\left[\pi,\varpi^4\right]$ $(p\in\mathbb{N}_0)$, while we cannot evaluate $\mathcal{S}_2(4p+2;i)$ individually.

Proof of Theorem 4.7. By Lemma 4.6 we find that the choice

$$P_k=\mathcal{T}_2(2k;i)-\frac{2k}{\pi}S(2k+1;i),\quad Q_k=\frac{1}{\pi}\mathcal{G}_{2k+1}(i)\qquad(k\in\mathbb{N}_0)$$

satisfies the condition of Lemma 4.5. Therefore by Lemma 4.5 we obtain

$$\mathcal{T}_2(2k;i)-\frac{2k}{\pi}S(2k+1;i)=-\frac{2}{\pi}\sum_{\nu=0}^{k}\left(2^{1-2k+2\nu}-1\right)\zeta(2k-2\nu)\mathcal{G}_{2\nu+1}(i). \tag{4.17}$$

Similarly to (4.10), we have $\mathcal{T}_2(4p;i)=\mathcal{S}_2(4p;i)$. Hence, by (4.17) with $k=2p$ and (4.12), we have

$$\mathcal{S}_2(4p;i)-\frac{4p}{\pi}S(4p+1;i)$$

$$
\begin{aligned}
&= -\frac{2}{\pi}\sum_{\nu=0}^{2p}\left(2^{1-4p+2\nu}-1\right)\zeta(4p-2\nu)\mathcal{G}_{2\nu+1}(i)\\
&= \frac{4}{\pi^2}\sum_{\nu=0}^{2p}\left(2^{1-4p+2\nu}-1\right)\zeta(4p-2\nu)(-1)^{\nu}\\
&\qquad\times\sum_{j=0}^{\nu}\left(2^{1-2\nu+2j}-1\right)\zeta(2\nu-2j)\left\{(-1)^{j}G_{2j+2}(i)+2\zeta(2j+2)\right\}\\
&= \frac{4}{\pi^2}\sum_{j=0}^{2p}\sum_{\nu=j}^{2p}\left\{\left(2^{1-4p+2\nu}-1\right)\zeta(4p-2\nu)(-1)^{\nu}\left(2^{1-2\nu+2j}-1\right)\zeta(2\nu-2j)\right\}\\
&\qquad\times\left\{(-1)^{j}G_{2j+2}(i)+2\zeta(2j+2)\right\}.
\end{aligned}
$$

Dividing the right-hand side into two subsums according as $j=2l$ ($0\le l\le p$) and $j=2l+1$ ($0\le l\le p-1$), we find that the right-hand side is

$$
\begin{aligned}
&\frac{4}{\pi^2}\sum_{l=0}^{p}\sum_{\nu=2l}^{2p}\left(2^{1-4p+2\nu}-1\right)\zeta(4p-2\nu)(-1)^{\nu}\left(2^{1-2\nu+4l}-1\right)\zeta(2\nu-4l)\\
&\qquad\times\left(G_{4l+2}(i)+2\zeta(4l+2)\right)\\
&\quad+\frac{4}{\pi^2}\sum_{l=0}^{p-1}\sum_{\nu=2l+1}^{2p}\left(2^{1-4p+2\nu}-1\right)\zeta(4p-2\nu)(-1)^{\nu}\left(2^{3-2\nu+4l}-1\right)\zeta(2\nu-4l-2)\\
&\qquad\quad\times\left(-G_{4l+4}(i)+2\zeta(4l+4)\right).
\end{aligned}
\tag{4.18}
$$

The second member of (4.18) vanishes because, letting

$$
\Lambda=\sum_{\nu=2l+1}^{2p}\left(2^{1-4p+2\nu}-1\right)\zeta(4p-2\nu)(-1)^{\nu}\left(2^{3-2\nu+4l}-1\right)\zeta(2\nu-4l-2)
$$

and putting $\mu=2p-\nu+2l+1$, we find $\Lambda=-\Lambda$, hence $\Lambda=0$. On the other hand, by using

$$
\left(2^{1-2\nu}-1\right)\zeta(2\nu)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m^{2\nu}}=-\frac{(2\pi i)^{2\nu}B_{2\nu}(1/2)}{2(2\nu)!}\qquad(\nu\in\mathbb{N})
$$

and $\zeta(0)=-1/2$ (see [6, Chapter 1]), we see that the first member of (4.18) is

$$
\begin{aligned}
&\frac{4}{\pi^2}\sum_{l=0}^{p}\left(G_{4l+2}(i)+2\zeta(4l+2)\right)\\
&\qquad\times(2\pi i)^{4p-4l}\sum_{\mu=0}^{2p-2l}(-1)^{\mu}\frac{B_{4p-4l-2\mu}(1/2)}{2(4p-4l-2\mu)!}\frac{B_{2\mu}(1/2)}{2(2\mu)!}.
\end{aligned}
$$

By (1.2), this coincides with

$$
-\frac{2}{\pi}\sum_{l=0}^{p}\left(G_{4l+2}(i)+2\zeta(4l+2)\right)S(4p-4l-1;i).
$$

By the property of $G_{2j}(i)$ (see Section 2), we obtain (4.16). Thus we complete the proof of Theorem 4.7. $\square$

Remark 4.9. (i) In both the proofs of Theorem 4.1 and Theorem 4.7, a key role is played by Lemma 4.2, which is based on the fact $J(t)\equiv 0$. The latter fact is shown by using Cauchy's (1.3), (1.4), which are known for $\tau=i$. This is the reason why we only consider the case $\tau=i$ in this section.

(ii) The idea of Lemma 4.2 is to begin with $J(t)\equiv 0$, which is a consequence of the existence of trivial zeros (1.4) of $S(s;i)$, and multiply $J(t)$ by another series to obtain a new identity

$F(t) \equiv 0$. This type of argument has been repeatedly used by the third-named author (see, e.g., [15] [16] [17]).

(iii) On the other hand, the method in [11] is quite different. We will develop this direction of research further in a forthcoming paper.

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