ON THE MEAN SQUARE OF THE PRODUCT OF $\zeta(s)$ AND A DIRICHLET POLYNOMIAL

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1. Introduction and statement of results

Let $s=\sigma+it$ be a complex variable, $\zeta(s)$ the Riemann zeta-function, and

$$A(s)=\sum_{m\le M}a(m)m^{-s}$$

be a Dirichlet polynomial, where $M\ge 1$ and $a(m)$'s are complex coefficients satisfying

$$a(m)=O(m^{\varepsilon}) \tag{1.1}$$

for any $\varepsilon>0$. (In what follows, ε is always a small positive number, not necessarily the same at each occurrence.) The mean value

$$I(T,A)=\int_0^T |\zeta(\tfrac12+it)A(\tfrac12+it)|^2dt \qquad (T\ge 2)$$

has been studied by several mathematicians. Iwaniec [4] obtained an upper bound of $I(T,A)$, and then, Balasubramanian, Conrey and Heath-Brown [1] established the asymptotic formula

$$I(T,A)=\sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]}\left(\log\frac{(k,\ell)^2T}{2\pi k\ell}+2\gamma-1\right)T+E(T,A) \tag{1.2}$$

under the assumption $\log M\ll\log T$ (the symbol $f\ll g$ means $f=O(g)$), where (k,ℓ) is the greatest common divisor of k and ℓ, $[k,\ell]=k\ell/(k,\ell)$ is the least common multiple of k and ℓ, $\overline{a(\ell)}$ is the complex conjugate of $a(\ell)$, γ is Euler's constant, and $E(T,A)$ is the error term satisfying

$$E(T,A)\ll M^2T^{\varepsilon}+T(\log T)^{-B} \qquad (\text{for } \log M\ll\log T) \tag{1.3}$$

for any $B>0$, where the implied constant depends on B and ε. They also gave several sharper estimates of $E(T,A)$ under further assumptions, and mentioned an application to the distribution of zeros of $\zeta(s)$. Their condition $\log M\ll\log T$ for (1.3) is actually not necessary (see the remark at the end of this section).

Motohashi [8] stated a different type of estimate, that is

$$E(T,A)\ll M^{4/3}T^{1/3+\varepsilon}, \tag{1.4}$$

with a brief sketch of the proof. His proof, different from that of [1], is a variant of Atkinson's method. Actually his argument is valid only for the integral from $-T$ to T, hence his claim should be understood as

$$\tilde{E}(T,A) \ll M^{4/3}T^{1/3+\varepsilon} \qquad (\text{for } M \ll T^{1/2}(\log T)^{-3/4}), \tag{1.5}$$

where $\tilde{E}(T,A)$ is defined by

$$\begin{aligned}&\frac{1}{2}\int_{-T}^{T}|\zeta(\tfrac{1}{2}+it)A(\tfrac{1}{2}+it)|^2 dt\\ &= \sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]}\left(\log\frac{(k,\ell)^2T}{2\pi k\ell}+2\gamma-1\right)T+\tilde{E}(T,A).\end{aligned} \tag{1.6}$$

The left-hand side of (1.6) coincides with $I(T,A)$ if $a(m)$'s are real, but it is not true in general. Note that the condition $M \ll T^{1/2}(\log T)^{-3/4}$ is necessary, though Motohashi did not state it, because in the proof he used a parameter G which satisfies $G \le T(\log T)^{-1}$, and at the last stage of the proof he chose $G = M^{4/3}T^{1/3}$.

The aim of the present paper is to develop another approach to this problem. Let A_0 be a sufficiently large positive number, $L = A_0(\log T)^{1/2}$, and μ, ρ be non-negative numbers satisfying $\rho < 1$, $\mu+\rho > 0$. We assume

$$L \le M^{\mu}T^{\rho} \le \frac{T}{A_0 L}. \tag{1.7}$$

We shall prove

Theorem 1. *For any $M \ge 1$ and $T \ge 2$ satisfying (1.7), we have*

$$E(T,A) \ll M^{2-\mu/2}T^{1/2-\rho/2+\varepsilon} + M^{\mu}T^{\rho+\varepsilon}. \tag{1.8}$$

In particular, replacing ε on the right-hand side of (1.8) by $\varepsilon/2$, and taking $\mu = 0$, $\rho = 1-\varepsilon$, we obtain

Corollary 1. *For any $M \ge 1$ and $T \ge 2$, we have*

$$E(T,A) \ll M^2T^{\varepsilon} + T^{1-\varepsilon/2}. \tag{1.9}$$

This gives a slight improvement of the estimate (1.3) of Balasubramanian, Conrey and Heath-Brown [1]. On the other hand, if $M \ll T^{1/2}(\log T)^{-3/8}$, we can choose $\mu = 4/3$, $\rho = 1/3$ to obtain the following corollary, which recovers Motohashi's claim.

Corollary 2. *Under the condition $M \ll T^{1/2}(\log T)^{-3/8}$, we have*

$$E(T,A) \ll M^{4/3}T^{1/3+\varepsilon}. \tag{1.10}$$

Our proof of the above thoerem is an analogue of the argument developed in Katsurada and Matsumoto [5]. Therefore it is also a variant of Atkinson's method, and can be regarded as a generalization of the argument described in Section 2.7 of Ivić [3]. By the same method we can treat the case $1/2 < \sigma < 1$. In this case, the integral

$$I_\sigma(T,A) = \int_0^T |\zeta(\sigma+it)A(\sigma+it)|^2 dt$$

satisfies the asymptotic formula of the form

$$\begin{aligned} I_\sigma(T,A) &= \sum_{k\le M}\sum_{\ell\le M} \frac{a(k)\overline{a(\ell)}}{[k,\ell]^{2\sigma}}\zeta(2\sigma)T \\ &+ \sum_{k\le M}\sum_{\ell\le M} \frac{a(k)\overline{a(\ell)}}{k\ell}(k,\ell)^{2-2\sigma}(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} \\ &+ E_\sigma(T,A), \end{aligned} \tag{1.11}$$

where $E_\sigma(T,A)$ is the error term. We can estimate this term as follows:

Theorem 2. *If $\rho < (4\sigma-1)^{-1}$, then for any $M \geq 1$ and $T \geq 2$ satisfying (1.7), we have*

$$E_\sigma(T,A) \ll M^{2-(1-\rho)^{-1}\mu f(\sigma,\rho)}T^{f(\sigma,\rho)+\varepsilon} + M^\mu T^{\rho+\varepsilon} \tag{1.12}$$

for $1/2 < \sigma < 1$, where $f(\sigma,\rho) = \rho(\frac{1}{2}-2\sigma)+\frac{1}{2}$.

Remark 1. The assumption $\rho < (4\sigma-1)^{-1}$ implies $f(\sigma,\rho) > 0$.

The estimates corresponding to Corollaries 1 and 2 can be stated as

$$E_\sigma(T,A) \ll M^2T^\varepsilon + T^{1/(4\sigma-1)-\varepsilon} \tag{1.13}$$

(under the choice $\mu = 0$, $\rho = 1/(4\sigma-1)-\varepsilon$) and

$$E_\sigma(T,A) \ll M^{8\sigma/(1+4\sigma)}T^{1/(1+4\sigma)+\varepsilon} \qquad (\text{for } M \ll T^{1/2}L^{-1/2-1/8\sigma}) \tag{1.14}$$

(under the choice $\mu = 8\sigma/(1+4\sigma)$, $\rho = 1/(1+4\sigma)$) for $1/2 < \sigma < 1$. The asymptotic formula (1.11) with the estimate (1.14) is clearly a generalization of Theorem 1 of the author [7].

Remark 2. Here we show that Theorems 1 and 2 are trivial if

$$M \gg T^b \tag{1.15}$$

with a sufficiently large positive constant b. In fact, since

$$\zeta(\sigma+it)A(\sigma+it) \ll M^{1-\sigma+\varepsilon}(|t|+1)^{(1-\sigma)/3} \qquad (1/2 \le \sigma < 1), \tag{1.16}$$

we have $I(T,A) \ll M^{1+\varepsilon}T^{4/3}$ and $I_\sigma(T,A) \ll M^{2-2\sigma+\varepsilon}T^{2(1-\sigma)/3+1}$. In case $\sigma = 1/2$, by using (3.9) below, we see that the first term on the right-hand side of (1.2) is $O(M^\varepsilon T\log T)$. Hence trivially

$$E(T,A) \ll M^{1+\varepsilon}T^{4/3} + M^\varepsilon T\log T \ll M^{1+4/3b+\varepsilon},$$

which is clearly superseded by the right-hand side of (1.8) if b is sufficiently large. Similarly, in case $1/2 < \sigma < 1$, since the first and the second terms on the right-hand side of (1.11) are $O(T + M^\varepsilon T^{2-2\sigma})$, we have

$$E_\sigma(T,A) \ll M^{2-2\sigma+b^{-1}(2(1-\sigma)/3+1)+\varepsilon}. \tag{1.17}$$

Noting $(1-\rho)^{-1}f(\sigma,\rho) < 1$, we see that (1.17) implies (1.12) for sufficiently large b.

2. The weighted local integral

Now we begin the proof of Theorems 1 and 2. Let Δ be a parameter satisfying

$$L \le \Delta \le \frac{T}{A_0 L}. \tag{2.1}$$

Moreover, in view of Remark 2 in Section 1, we may assume

$$M \le T^b. \tag{2.2}$$

Let u, v be complex variables, and at first assume $\Re u > 1$, $\Re v > 1$. Consider

$$I(u,v;\Delta,A) = \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\zeta(u+iy)\zeta(v-iy)A(u+iy)\bar{A}(v-iy)e^{-(y/\Delta)^2}dy, \tag{2.3}$$

where $\bar{A}(s) = \sum_{m\le M}\overline{a(m)}m^{-s}$. Substituting the Dirichlet series expressions, we have

$$\begin{aligned} I(u,v;\Delta,A) = \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}&\sum_{r=1}^{\infty}r^{-u-iy}\sum_{s=1}^{\infty}s^{-v+iy}\sum_{k\le M}a(k)k^{-u-iy}\\ &\times\sum_{\ell\le M}\overline{a(\ell)}\ell^{-v+iy}e^{-(y/\Delta)^2}dy\\ = \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}&\sum_{m=1}^{\infty}\left(\sum_{k|m}a(k)\right)m^{-u-iy}\sum_{n=1}^{\infty}\left(\sum_{\ell|n}\overline{a(\ell)}\right)n^{-v+iy}e^{-(y/\Delta)^2}dy. \end{aligned}$$

The part corresponding to $m = n$ is equal to

$$\begin{aligned}
&\frac{1}{\Delta\sqrt{\pi}}\sum_{m=1}^{\infty}\left(\sum_{k|m}a(k)\right)\left(\sum_{\ell|m}\overline{a(\ell)}\right)m^{-u-v}\int_{-\infty}^{\infty}e^{-(y/\Delta)^2}dy\\
&=\sum_{m=1}^{\infty}\left(\sum_{k|m}a(k)\right)\left(\sum_{\ell|m}\overline{a(\ell)}\right)m^{-u-v}\\
&=\sum_{k\le M}\sum_{\ell\le M}a(k)\overline{a(\ell)}\sum_{m\equiv 0(\mathrm{mod}\ [k,\ell])}m^{-u-v}\\
&=\zeta(u+v)\sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]^{u+v}}.
\end{aligned}$$

Hence

$$I(u,v;\Delta,A)=\zeta(u+v)\sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]^{u+v}}+I_1(u,v;\Delta,A)+\overline{I_1(\bar{v},\bar{u};\Delta,A)}, \tag{2.4}$$

where

$$I_1(u,v;\Delta,A)=\frac{1}{\Delta\sqrt{\pi}}\sum\sum_{m<n}\left(\sum_{k|m}a(k)\right)\left(\sum_{\ell|n}\overline{a(\ell)}\right)\int_{-\infty}^{\infty}m^{-u-iy}n^{-v+iy}e^{-(y/\Delta)^2}dy.$$

Using the formula

$$\int_{-\infty}^{\infty}\exp(At-Bt^2)dt=\left(\frac{\pi}{B}\right)^{1/2}\exp\left(\frac{A^2}{4B}\right)\qquad(\Re B>0)$$

(see (A.38) of [2]), and then putting $m = km_1$ and $n = km_1 + n_1$, we have

$$\begin{aligned}
I_1(u,v;\Delta,A)&=\sum\sum_{m<n}\left(\sum_{k|m}a(k)\right)\left(\sum_{\ell|n}\overline{a(\ell)}\right)m^{-u}n^{-v}\exp\left(-\frac{1}{4}\Delta^2\log^2\left(\frac{n}{m}\right)\right)\\
&=\sum_{k\le M}a(k)\sum_{m_1=1}^{\infty}\sum_{n_1=1}^{\infty}\left(\sum_{\ell|(km_1+n_1)}\overline{a(\ell)}\right)(km_1)^{-u}(km_1+n_1)^{-v}\\
&\qquad\times\exp\left(-\frac{1}{4}\Delta^2\log^2\left(1+\frac{n_1}{km_1}\right)\right)\\
&=\sum_{k\le M}a(k)k^{-u}\sum_{\ell\le M}\overline{a(\ell)}\sum_{m_1=1}^{\infty}\sum_{n_1=1}^{\infty}m_1^{-u}(km_1+n_1)^{-v}\\
&\qquad\times\exp\left(-\frac{1}{4}\Delta^2\log^2\left(1+\frac{n_1}{km_1}\right)\right)\ell^{-1}\sum_{f=1}^{\ell}\exp\left(2\pi i\frac{(km_1+n_1)f}{\ell}\right),
\end{aligned} \tag{2.5}$$

because the innermost sum is

$$=\begin{cases}\ell & \text{if } \ell|(km_1+n_1),\\ 0 & \text{otherwise.}\end{cases}$$

Let

$$M(s,v;\Delta)=\frac{\Gamma(s)}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\Gamma(v+iy-s)}{\Gamma(v+iy)}e^{-(y/\Delta)^2}dy \tag{2.6}$$

for $\Re v>\Re s>0$. In Section 5.2 of [3], Ivić proved the following properties of $M(s,v;\Delta)$.

Lemma 1. (i) *$M(s,v;\Delta)$ can be continued to the whole space $\mathbb{C}^2$, entire in v, and meromorphic in s. The poles with respect to s are only on $s=0,-1,-2,\ldots$.*

(ii) *For $\Re s>0$ and any v, we have*

$$M(s,v;\Delta)=\int_0^{\infty}x^{s-1}(1+x)^{-v}\exp\left(-\tfrac{1}{4}\Delta^2\log^2(1+x)\right)dx. \tag{2.7}$$

(iii) *For any fixed $c>0$, we have $M(s,v;\Delta)\ll(1+|s|)^{-c}$ as $|\Im s|\to\infty$, uniformly for bounded v and bounded $\Re s$.*

(iv) *If $\Re v>\alpha>0$ and $x>0$, then*

$$\frac{1}{2\pi i}\int_{(\alpha)}M(s,v;\Delta)x^{-s}ds=(1+x)^{-v}\exp\left(-\tfrac{1}{4}\Delta^2\log^2(1+x)\right), \tag{2.8}$$

where the path of integration is the vertical line $\Re s=\alpha$.

From (2.5) and Lemma 1(iv) we have

$$\begin{aligned}I_1(u,v;\Delta,A)=&\sum_{k\le M}a(k)k^{-u}\sum_{\ell\le M}\overline{a(\ell)}\ell^{-1}\sum_{f=1}^{\ell}\sum_{m_1=1}^{\infty}\exp\left(2\pi i\frac{km_1f}{\ell}\right)m_1^{-u}\\&\times\sum_{n_1=1}^{\infty}\exp\left(2\pi i\frac{n_1f}{\ell}\right)(km_1)^{-v}\frac{1}{2\pi i}\int_{(\alpha)}M(s,v;\Delta)\left(\frac{n_1}{km_1}\right)^{-s}ds,\end{aligned}$$

where $1<\alpha<\Re v$. Summation and integration can be interchanged because of absolute convergence, hence

$$\begin{aligned}I_1(u,v;\Delta,A)=&\frac{1}{2\pi i}\int_{(\alpha)}\sum_{k\le M}a(k)k^{-u-v+s}\sum_{\ell\le M}\overline{a(\ell)}\ell^{-1}\\&\times\sum_{f=1}^{\ell}\varphi\left(u+v-s,\frac{kf}{\ell}\right)\varphi\left(s,\frac{f}{\ell}\right)M(s,v;\Delta)ds\end{aligned} \tag{2.9}$$

for $\Re u>1$, $\Re v>\alpha$, where

$$\varphi(s,x)=\sum_{n=1}^{\infty}\exp(2\pi inx)n^{-s} \tag{2.10}$$

is the Lerch zeta-function with the real parameter x. If $x\in\mathbb{Z}$, then $\varphi(s,x)=\zeta(s)$, while if $x\notin\mathbb{Z}$, then $\varphi(s,x)$ is entire in s. Moreover, if $0<x<1$, $\varphi(s,x)$ satisfies the functional equation

$$\varphi(s,x) = (2\pi)^{s-1}\Gamma(1-s)\left\{e\left(\frac{1-s}{4}-x\right)\zeta(1-s,x)\right.$$
$$\left.+\,e\left(-\frac{1-s}{4}-x\right)\zeta(1-s,1-x)\right\}, \tag{2.11}$$

where $\zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}$ is the Hurwitz zeta-function (see Chapter 2 of Laurinčikas and Garunkštis [6]), and $e(x) = \exp(2\pi ix)$.

Let $\beta > \max\{2,\alpha\}$, and assume $\Re(u+v) < \beta$. We shift the path of integration on the right-hand side of (2.9) to $\Re s = \beta$. Lemma 1(iii) implies that this shifting is possible. The function $\varphi(u+v-s, kf/\ell)$ has a pole at $s = u+v-1$ only when $kf/\ell \in \mathbb{Z}$. Hence we have

$$I_1(u,v;\Delta,A) = P(u,v;\Delta,A)$$
$$+\sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{k\ell}\sum_{f=1}^{\ell}\xi(f;k,\ell)\varphi\left(u+v-1,\frac{f}{\ell}\right)M(u+v-1,v;\Delta), \tag{2.12}$$

where

$$P(u,v;\Delta,A) = \frac{1}{2\pi i}\int_{(\beta)}\sum_{k\le M}a(k)k^{-u-v+s}\sum_{\ell\le M}\overline{a(\ell)}\ell^{-1}$$
$$\times\sum_{f=1}^{\ell}\varphi\left(u+v-s,\frac{kf}{\ell}\right)\varphi\left(s,\frac{f}{\ell}\right)M(s,v;\Delta)ds \tag{2.13}$$

and

$$\xi(f;k,\ell) = \begin{cases} 1 & \text{if } \ell | kf, \\ 0 & \text{otherwise.}\end{cases}$$

Since $M(s,v;\Delta)$ is entire in v (by Lemma 1(i)), the expression (2.13) is valid for any u, v satisfying $\Re(u+v) < \beta+1$. Hence (2.12) gives the meromorphic continuation of $I_1(u,v;\Delta,A)$ to the region $\Re(u+v) < \beta+1$. Therefore now we can put $u = \sigma+it$, $v = \sigma-it$ in (2.12), where $1/2 < \sigma < 1$, $t \ge 2$, and $T \ll t \ll T$. Substituting the resulting expression and its complex conjugate into (2.4), we obtain

$$I(\sigma+it,\sigma-it;\Delta,A) = \zeta(2\sigma)\sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]^{2\sigma}}$$
$$+\,P_\sigma(t;\Delta,A) + \overline{P_\sigma(t;\Delta,A)}$$
$$+\sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{k\ell}\sum_{f=1}^{\ell}\xi(f;k,\ell)\varphi\left(2\sigma-1,\frac{f}{\ell}\right)M(2\sigma-1,\sigma-it;\Delta)$$
$$+\sum_{k\le M}\sum_{\ell\le M}\frac{\overline{a(k)}a(\ell)}{k\ell}\sum_{f=1}^{\ell}\xi(f;k,\ell)\overline{\varphi\left(2\sigma-1,\frac{f}{\ell}\right)}\,\overline{M(2\sigma-1,\sigma-it;\Delta)}, \tag{2.14}$$

where $P_\sigma(t;\Delta,A)=P(\sigma+it,\sigma-it;\Delta,A)$. Changing the letters k and ℓ in the last member of the right-hand side, we obtain

$$\begin{aligned}I(\sigma+it,\sigma-it;\Delta,A)&=P_\sigma(t;\Delta,A)+\overline{P_\sigma(t;\Delta,A)}\\&\quad+\sum_{k\leq M}\sum_{\ell\leq M}\frac{a(k)\overline{a(\ell)}}{k\ell}H_\sigma(t;k,\ell)\end{aligned}\tag{2.15}$$

for $1/2<\sigma<1$, where

$$\begin{aligned}&H_\sigma(t;k,\ell)=\frac{k\ell}{[k,\ell]^{2\sigma}}\zeta(2\sigma)\\&+\sum_{f=1}^{\ell}\xi(f;k,\ell)\varphi\left(2\sigma-1,\frac{f}{\ell}\right)M(2\sigma-1,\sigma-it;\Delta)\\&+\sum_{f=1}^{k}\xi(f;\ell,k)\varphi\left(2\sigma-1,-\frac{f}{k}\right)M(2\sigma-1,\sigma+it;\Delta).\end{aligned}\tag{2.16}$$

3. The case on the critical line

In this section we show an expression, analogous to (2.15), for $\sigma=1/2$. Let $\sigma=\frac{1}{2}+\delta$, where δ is a small positive number. Then from (2.6), noting $\Gamma(2\delta)=(2\delta)^{-1}-\gamma+O(\delta)$, we can easily see that

$$M(2\sigma-1,\sigma+it;\Delta)=\frac{1}{2\delta}-B(t;\Delta)-\gamma+O(\delta),$$

where

$$B(t;\Delta)=\frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}\left(\tfrac{1}{2}+i(t+y)\right)e^{-(y/\Delta)^2}dy$$

((3.6) and (3.7) of [5]). Also,

$$\zeta(2\sigma)[k,\ell]^{-2\sigma}=\frac{1}{[k,\ell]}\left(\frac{1}{2\delta}-\log[k,\ell]+\gamma+O(\delta)\right),$$

$$\varphi(2\sigma-1,x)=\varphi(0,x)+2\delta\varphi'(0,x)+O(\delta^2).$$

Hence we have

$$
\begin{aligned}
H_\sigma(t;k,\ell) &= \frac{k\ell}{[k,\ell]}\left(\frac{1}{2\delta} - \log[k,\ell] + \gamma + O(\delta)\right) \\
&+ \frac{1}{2\delta}\sum_{f=1}^{\ell}\xi(f;k,\ell)\varphi\left(0,\frac{f}{\ell}\right) + \frac{1}{2\delta}\sum_{f=1}^{k}\xi(f;\ell,k)\varphi\left(0,-\frac{f}{k}\right) \\
&+ \sum_{f=1}^{\ell}\xi(f;k,\ell)\left\{\varphi'\left(0,\frac{f}{\ell}\right) - (B(-t,\Delta)+\gamma)\varphi\left(0,\frac{f}{\ell}\right)\right\} \\
&+ \sum_{f=1}^{k}\xi(f;\ell,k)\left\{\varphi'\left(0,-\frac{f}{k}\right) - (B(t,\Delta)+\gamma)\varphi\left(0,-\frac{f}{k}\right)\right\} \\
&+ O(\delta).
\end{aligned} \tag{3.1}
$$

We note that

$$\frac{k\ell}{[k,\ell]} + \sum_{f=1}^{\ell}\xi(f;k,\ell)\varphi\left(0,\frac{f}{\ell}\right) + \sum_{f=1}^{k}\xi(f;\ell,k)\varphi\left(0,-\frac{f}{k}\right) = 0. \tag{3.2}$$

In fact, if $(k,\ell)=d$, then we can write $k=d\kappa$, $\ell=d\lambda$, $(\kappa,\lambda)=1$. Then $\xi(f;k,\ell)=1$ if and only if $\ell|kf$, that is $\lambda|f$. Hence, putting $f=j\lambda$, we have

$$\sum_{f=1}^{\ell}\xi(f;k,\ell)\varphi\left(0,\frac{f}{\ell}\right) = \sum_{j=1}^{d}\varphi\left(0,\frac{j}{d}\right). \tag{3.3}$$

When $\Re s>1$, we have

$$\sum_{j=1}^{d}\varphi\left(s,\frac{j}{d}\right) = d^{1-s}\zeta(s), \tag{3.4}$$

because the left-hand side is

$$= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{d}\exp(2\pi i n j/d)\right)n^{-s} = d\sum_{n\equiv 0 (\mathrm{mod}\ d)} n^{-s}.$$

The relation (3.4) is valid, by analytic continuation, at $s=0$. Hence from (3.3) we have

$$\sum_{f=1}^{\ell}\xi(f;k,\ell)\varphi\left(0,\frac{f}{\ell}\right) = d\zeta(0) = -\frac{d}{2} = -\frac{k\ell}{2[k,\ell]}. \tag{3.5}$$

The value of $\sum_{f=1}^{k}\xi(f;\ell,k)\varphi(0,-f/k)$ is the same, hence (3.2) follows.

Therefore, letting $\delta\to 0$ in (3.1), we have

$$\begin{aligned}
H_\sigma(t;k,\ell)|_{\sigma\to 1/2+0} &= \frac{k\ell}{[k,\ell]}(-\log[k,\ell]+\gamma)\\
&+\sum_{f=1}^{\ell}\xi(f;k,\ell)\left\{\varphi'\left(0,\frac{f}{\ell}\right)-(B(-t,\Delta)+\gamma)\varphi\left(0,\frac{f}{\ell}\right)\right\}\\
&+\sum_{f=1}^{k}\xi(f;\ell,k)\left\{\varphi'\left(0,-\frac{f}{k}\right)-(B(t,\Delta)+\gamma)\varphi\left(0,-\frac{f}{k}\right)\right\}. \qquad (3.6)
\end{aligned}$$

Differentiating the both sides of (3.4) and putting $s=0$, we have

$$\sum_{j=1}^{d}\varphi'\left(0,\frac{j}{d}\right) = -\zeta(0)d\log d+\zeta'(0)d = \frac{1}{2}d\log d-\frac{1}{2}\log(2\pi)d. \qquad (3.7)$$

Using (3.5) and (3.7), we see that the right-hand side of (3.6) is

$$\begin{aligned}
&\frac{k\ell}{[k,\ell]}(-\log[k,\ell]+\gamma)+d\log d-\log(2\pi)d+\frac{d}{2}\left(B(t,\Delta)+B(-t,\Delta)+2\gamma\right)\\
&\qquad = \frac{k\ell}{[k,\ell]}\left(\log\frac{(k,\ell)^2}{2\pi k\ell}+2\gamma+\frac{1}{2}B(t,\Delta)+\frac{1}{2}B(-t,\Delta)\right).
\end{aligned}$$

Moreover, using Stirling's formula we have

$$B(\pm t,\Delta)=\pm\frac{1}{2}\pi i+\log t+O\left(\frac{\Delta}{t}\right)$$

((3.9) of [5]). Hence

$$H_\sigma(t;k,\ell)|_{\sigma\to 1/2+0} = \frac{k\ell}{[k,\ell]}\left(\log\frac{(k,\ell)^2}{2\pi k\ell}+2\gamma+\log t+O\left(\frac{\Delta}{t}\right)\right). \qquad (3.8)$$

We let $\sigma\to 1/2+0$ in (2.15), with using (3.8). The contribution of the error term on the right-hand side of (3.8) to (2.14) is $O(\Delta t^{-1+\varepsilon})$. To prove this estimate, it suffices to show

$$\sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]}\ll M^\varepsilon, \qquad (3.9)$$

because $M^\varepsilon\ll t^\varepsilon$ in view of (2.2). The left-hand side of (3.9) is

$$\begin{aligned}
&\ll \sum_{k\le M}\sum_{\ell\le M}(k\ell)^{-1+\varepsilon}d \ll \sum_{d\le M}d\sum_{\kappa\le M/d}\sum_{\lambda\le M/d}(\kappa\lambda d^2)^{-1+\varepsilon}\\
&\ll \sum_{d\le M}d^{-1+\varepsilon}\left(\frac{M}{d}\right)^\varepsilon \ll M^\varepsilon,
\end{aligned}$$

hence (3.9) follows. Therefore now we obtain

$$I\left(\tfrac{1}{2}+it, \tfrac{1}{2}-it; \Delta, A\right) = P_{1/2}(t;\Delta,A) + \overline{P_{1/2}(t;\Delta,A)} + \sum_{k\le M}\sum_{\ell\le M} \frac{a(k)\overline{a(\ell)}}{[k,\ell]}\left(\log\frac{(k,\ell)^2 t}{2\pi k\ell} + 2\gamma\right) + O\left(\Delta t^{-1+\varepsilon}\right). \tag{3.10}$$

4. The weighted mean square

Now we go back to the integral (2.3), and continue it meromorphically by a different method. At first assume $\Re u > 1$ and $\Re v > 1$. Let $K > \max\{1, \Re v - 1\}$, and shift the path of integration on the right-hand side of (2.3) to $\Im y = -K$. The residue at $y = i(1-v)$ appears. The resulting expression

$$I(u,v;\Delta,A) = \frac{2\sqrt{\pi}}{\Delta}\zeta(u+v-1)A(u+v-1)\bar{A}(1)e^{(v-1)^2/\Delta^2} + \frac{1}{\Delta\sqrt{\pi}}\int_{\Im y=-K}\zeta(u+iy)\zeta(v-iy)A(u+iy)\bar{A}(v-iy)e^{-(y/\Delta)^2}dy \tag{4.1}$$

can be continued meromorphically to the region $\Re u > -K+1$, $\Re v < K+1$, which includes $\mathcal{D} = \{(u,v) \mid 0 < \Re u < 1, 0 < \Re v < 1\}$.

Now assume $(u,v)\in\mathcal{D}$, and shift back the path of integration on the right-hand side of (4.1) to $\Im y = 0$. This time the residue at $y = i(u-1)$ appears, and

$$\begin{aligned} I(u,v;\Delta,A) = {}& \frac{2\sqrt{\pi}}{\Delta}\zeta(u+v-1)\Big\{A(u+v-1)\bar{A}(1)e^{(v-1)^2/\Delta^2} \\ & + A(1)\bar{A}(u+v-1)e^{(u-1)^2/\Delta^2}\Big\} \\ & + \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\zeta(u+iy)\zeta(v-iy)A(u+iy)\bar{A}(v-iy)e^{-(y/\Delta)^2}dy \end{aligned} \tag{4.2}$$

for $(u,v)\in\mathcal{D}$. In particular, we can now put $u=\sigma+it$, $v=\sigma-it$, where $1/2\le\sigma<1$, $t\ge 2$ and $T\ll t\ll T$, in (4.2). Then

$$\begin{aligned} I(\sigma+it,\sigma-it;\Delta,A) = {}& J_\sigma(t;\Delta,A) \\ & + \frac{2\sqrt{\pi}}{\Delta}\zeta(2\sigma-1)\{A(2\sigma-1)\bar{A}(1)\exp((\sigma-1-it)^2/\Delta^2) \\ & + A(1)\bar{A}(2\sigma-1)\exp((\sigma-1+it)^2/\Delta^2)\}, \end{aligned} \tag{4.3}$$

where

$$J_\sigma(t;\Delta,A) = \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}|\zeta(\sigma+i(t+y))A(\sigma+i(t+y))|^2 e^{-(y/\Delta)^2}dy.$$

Since $\Delta\le T/A_0L$ by (2.1), the second member on the right-hand side of (4.3) is

$$\ll M^{1+\varepsilon}\Delta^{-1}e^{-C_1L^2} \ll T^{-C_2},$$

where C_1, C_2 are positive constants. Since we assume (2.2), if A_0 is sufficiently large, then C_1, C_2 are also large. Therefore

$$I(\sigma+it,\sigma-it;\Delta,A)=J_\sigma(t;\Delta,A)+O(T^{-C_2}) \tag{4.4}$$

with a large C_2 for $1/2\leq\sigma<1$.

Now, combining (4.4) with the results proved in Sections 2 and 3, we obtain the following expressions of $J_\sigma(t;\Delta,A)$, which are fundamental in our analysis. First, in the case $\sigma=1/2$, from (3.10) and (4.4) we immediately obtain the following

Lemma 2. *For $T\ll t\ll T$, we have*

$$\begin{aligned}J_{1/2}(t;\Delta,A)=&\sum_{k\leq M}\sum_{\ell\leq M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]}\left(\log\frac{(k,\ell)^2t}{2\pi k\ell}+2\gamma\right)\\&+P_{1/2}(t;\Delta,A)+\overline{P_{1/2}(t;\Delta,A)}+O\left(\Delta T^{-1+\varepsilon}\right).\end{aligned} \tag{4.5}$$

Next consider the case $1/2<\sigma<1$. Since

$$\sum_{f=1}^{\ell}\xi(f;k,\ell)\varphi\left(2\sigma-1,\frac{f}{\ell}\right)=\sum_{j=1}^{d}\varphi\left(2\sigma-1,\frac{j}{d}\right)=d^{2-2\sigma}\zeta(2\sigma-1)$$

by (3.4), we can rewrite (2.16) as

$$\begin{aligned}H_\sigma(t;k,\ell)=&\frac{k\ell}{[k,\ell]^{2\sigma}}\zeta(2\sigma)\\&+d^{2-2\sigma}\zeta(2\sigma-1)\{M(2\sigma-1,\sigma-it;\Delta)+M(2\sigma-1,\sigma+it;\Delta)\}.\end{aligned} \tag{4.6}$$

It has been shown in the proof of Lemma 1 of [5] that

$$M(2\sigma-1,\sigma-it;\Delta)+M(2\sigma-1,\sigma+it;\Delta)=2\Gamma(2\sigma-1)\left\{t^{1-2\sigma}\sin(\pi\sigma)+O\left(\frac{\Delta}{t^{2\sigma}}\right)\right\},$$

and the functional equation of $\zeta(s)$ implies

$$2\Gamma(2\sigma-1)\zeta(2\sigma-1)\sin(\pi\sigma)=(2\pi)^{2\sigma-1}\zeta(2-2\sigma).$$

Hnece from (4.6) we have

$$H_\sigma(t;k,\ell)=\frac{k\ell}{[k,\ell]^{2\sigma}}\zeta(2\sigma)+d^{2-2\sigma}(2\pi)^{2\sigma-1}\zeta(2-2\sigma)t^{1-2\sigma}+O\left(d^{2-2\sigma}\frac{\Delta}{t^{2\sigma}}\right). \tag{4.7}$$

Substitute this into the right-hand side of (2.15), and combine with (4.4). By using (3.9) we see that the contribution of the error term on the right-hand side of (4.7) is $O(\Delta t^{-2\sigma+\varepsilon})$. Hence we obtain

Lemma 3. *For $1/2 < \sigma < 1$ and $T \ll t \ll T$, we have*

$$\begin{aligned}
J_\sigma(t;\Delta,A) &= \sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]^{2\sigma}}\zeta(2\sigma) \\
&+ \sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{k\ell}(k,\ell)^{2-2\sigma}(2\pi)^{2\sigma-1}\zeta(2-2\sigma)t^{1-2\sigma} \\
&+ P_\sigma(t;\Delta,A) + \overline{P_\sigma(t;\Delta,A)} + O\left(\Delta T^{-2\sigma+\varepsilon}\right). \qquad (4.8)
\end{aligned}$$

We can connect $J_\sigma(t;\Delta,A)$ with the mean square of $\zeta(\sigma+it)A(\sigma+it)$ by the following inequalities:

$$\begin{aligned}
\int_{T-L\Delta}^{2T+L\Delta} J_\sigma(t;\Delta,A)dt \ge \int_T^{2T} &|\zeta(\sigma+it)A(\sigma+it)|^2 dt \\
&+ O\left(M^{2(1-\sigma)+\varepsilon}T^{2(1-\sigma)/3+1}e^{-L^2}\right) \qquad (4.9)
\end{aligned}$$

and

$$\begin{aligned}
\int_{T+L\Delta}^{2T-L\Delta} J_\sigma(t;\Delta,A)dt \le \int_T^{2T} &|\zeta(\sigma+it)A(\sigma+it)|^2 dt \\
&+ O\left(M^{2(1-\sigma)+\varepsilon}T^{2(1-\sigma)/3+1}e^{-L^2}\right) \qquad (4.10)
\end{aligned}$$

for $1/2 \le \sigma < 1$. These can be proved analogously to Lemma 3 of [5], so we omit the details of the proof. We only note that, instead of (4.6) of [5], we use the bound (1.16) to obtain the above error estimates.

On the other hand, from Lemma 2 we have

$$\begin{aligned}
&\int_{T\mp L\Delta}^{2T\pm L\Delta} J_{1/2}(t;\Delta,A)dt \\
&= \sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]}\left(\log\frac{(k,\ell)^2 t}{2\pi k\ell} + 2\gamma - 1\right)t\,\Bigg|_{t=T\mp L\Delta}^{2T\pm L\Delta} \\
&+ \int_{T\mp L\Delta}^{2T\pm L\Delta}\left(P_{1/2}(t;\Delta,A) + \overline{P_{1/2}(t;\Delta,A)}\right)dt + O\left(\Delta T^\varepsilon\right) \\
&= \sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]}\left\{\left(\log\frac{(k,\ell)^2 t}{2\pi k\ell} + 2\gamma - 1\right)t\,\Bigg|_{t=T}^{2T} + O(L\Delta\log T)\right\} \\
&+ \int_{T\mp L\Delta}^{2T\pm L\Delta}\left(P_{1/2}(t;\Delta,A) + \overline{P_{1/2}(t;\Delta,A)}\right)dt + O\left(\Delta T^\varepsilon\right).
\end{aligned}$$

Comparing this with (1.2), we obtain the case $\sigma=1/2$ of

$$\begin{aligned} E_\sigma(2T,A)-E_\sigma(T,A)&=\int_T^{2T}|\zeta(\sigma+it)A(\sigma+it)|^2dt\\ &-\int_{T\mp L\Delta}^{2T\pm L\Delta}J_\sigma(t;\Delta,A)dt+\int_{T\mp L\Delta}^{2T\pm L\Delta}\left(P_\sigma(t;\Delta,A)+\overline{P_\sigma(t;\Delta,A)}\right)dt\\ &+O(L\Delta(T^\varepsilon)^\omega), \end{aligned} \tag{4.11}$$

where $E_{1/2}(T,A)=E(T,A)$ and $\omega=1$ or 0 according as $\sigma=1/2$ or $1/2<\sigma<1$. The case $1/2<\sigma<1$ can be shown similarly from Lemma 3 and (1.11). It is an analogue of (4.8) and (4.9) of [5]. Combining (4.11) with (4.9) and (4.10), again similarly to [5], we obtain

Lemma 4. *For $1/2\le\sigma<1$, we have*

$$\begin{aligned} |E_\sigma(2T,A)-E_\sigma(T,A)|&\le\left|\int_{T-L\Delta}^{2T+L\Delta}\left(P_\sigma(t;\Delta,A)+\overline{P_\sigma(t;\Delta,A)}\right)dt\right|\\ &+\left|\int_{T+L\Delta}^{2T-L\Delta}\left(P_\sigma(t;\Delta,A)+\overline{P_\sigma(t;\Delta,A)}\right)dt\right|+O(L\Delta(T^\varepsilon)^\omega). \end{aligned} \tag{4.12}$$

Therefore, now our problem is reduced to the evaluation of the integral

$$\int_{T'}^{T''}\left(P_\sigma(t;\Delta,A)+\overline{P_\sigma(t;\Delta,A)}\right)dt, \tag{4.13}$$

where $T'=T\mp L\Delta$ and $T''=2T\pm L\Delta$. Since A_0 is sufficiently large (see (2.1)), we see that $T\ll T'\ll T$, $T\ll T''\ll T$.

5. An infinite series expression of $P_\sigma(t;\Delta,A)$

Let $1/2\le\sigma<1$. From (2.13) we have

$$\begin{aligned} P_\sigma(t;\Delta,A)&=\frac{1}{2\pi i}\int_{(\beta)}\sum_{k\le M}a(k)k^{-2\sigma+s}\sum_{\ell\le M}\overline{a(\ell)}\ell^{-1}\\ &\times\sum_{f=1}^{\ell}\varphi\left(2\sigma-s,\frac{kf}{\ell}\right)\varphi\left(s,\frac{f}{\ell}\right)M(s,\sigma-it;\Delta)ds. \end{aligned} \tag{5.1}$$

We rewrite the factor $\varphi(2\sigma-s,kf/\ell)$ by using the functional equation, that is (2.11) if $kf/\ell\notin\mathbb{Z}$, and

$$\zeta(2\sigma-s)=2^{2\sigma-s}\pi^{-1+2\sigma-s}\cos\left(\frac{\pi}{2}(1-2\sigma+s)\right)\Gamma(1-2\sigma+s)\zeta(1-2\sigma+s)$$

if $kf/\ell \in \mathbb{Z}$. We have

$$\begin{aligned}
P_\sigma(t;\Delta,A) &= \frac{1}{2\pi i}\int_{(\beta)} \sum_{k\le M} a(k)k^{-2\sigma+s} \sum_{\ell\le M} \overline{a(\ell)}\ell^{-1}(2\pi)^{-1+2\sigma-s}\Gamma(1-2\sigma+s) \\
&\times \Bigg\{ 2\sum_{j=1}^{d} \cos\left(\frac{\pi}{2}(1-2\sigma+s)\right)\zeta(1-2\sigma+s)\varphi\left(s,\frac{j}{d}\right) \\
&+ \sum_f{}^{*} \left(e\left(\frac{1-2\sigma+s}{4} - \frac{kf}{\ell}\right)\zeta\left(1-2\sigma+s, \frac{\{kf\}_\ell}{\ell}\right)\right. \\
&\left. + \ e\left(-\frac{1-2\sigma+s}{4} - \frac{kf}{\ell}\right)\zeta\left(1-2\sigma+s, \frac{\{-kf\}_\ell}{\ell}\right)\right)\varphi\left(s,\frac{f}{\ell}\right)\Bigg\} \\
&\times M(s,\sigma-it;\Delta)ds, \qquad (5.2)
\end{aligned}$$

where the summation $\sum^*$ runs over all f satisfying $1 \le f \le \ell$ and $f \not\equiv 0 (\mathrm{mod}\ \lambda)$, and $\{x\}_\ell$ means the integer determined uniquely by $\{x\}_\ell \equiv x (\mathrm{mod}\ \ell)$, $0 < \{x\}_\ell < \ell$.

From the assumption $\beta > 2$ it follows that $\Re(1-2\sigma+s) > 1$, hence the zeta factors on the right-hand side can be written down as Dirichlet series. Let

$$\sigma_a(n;x) = \sum_{m|n} e^{2\pi i m x} m^a$$

and

$$\sigma_a(n;x,\ell,b) = \sum_{\substack{m|n \\ n/m\equiv b (\mathrm{mod}\ \ell)}} e^{2\pi i m x} m^a,$$

where a, x are real numbers and n, ℓ, b are positive integers. We see that

$$\begin{aligned}
&\zeta(1-2\sigma+s, b/\ell)\varphi(s,x) \\
&= \ell^{1-2\sigma+s} \sum_{\substack{1\le k<\infty \\ k\equiv b (\mathrm{mod}\ \ell)}} k^{-1+2\sigma-s} \sum_{m=1}^{\infty} e^{2\pi i m x} m^{1-2\sigma} m^{-1+2\sigma-s} \\
&= \ell^{1-2\sigma+s} \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n;x,\ell,b) n^{-1+2\sigma-s}, \qquad (5.3)
\end{aligned}$$

and especially

$$\zeta(1-2\sigma+s)\varphi(s,x) = \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n;x) n^{-1+2\sigma-s}. \qquad (5.4)$$

Applying (5.3), (5.4), and the formula

$$\cos\left(\frac{\pi}{2}(1-2\sigma+s)\right) = \frac{1}{2}\left(e\left(\frac{1-2\sigma+s}{4}\right) + e\left(-\frac{1-2\sigma+s}{4}\right)\right)$$

to (5.2), and changing the order of integration and summation (which can be verified by absolute convergence), we obtain

$$\begin{aligned} P_\sigma(t;\Delta,A) &= \sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{k\ell} \\ &\times\left\{\sum_{j=1}^{d}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}\left(n;\frac{j}{d}\right)\left(Q_\sigma^+(t;n,k)+Q_\sigma^-(t;n,k)\right)\right. \\ &+\sum_f{}^{*} e\left(-\frac{kf}{\ell}\right)\sum_{n=1}^{\infty}\sigma_{1-2\sigma}\left(n;\frac{f}{\ell},\ell,\{kf\}_\ell\right)Q_\sigma^-(t;n,k\ell) \\ &\left.+\sum_f{}^{*} e\left(-\frac{kf}{\ell}\right)\sum_{n=1}^{\infty}\sigma_{1-2\sigma}\left(n;\frac{f}{\ell},\ell,\{-kf\}_\ell\right)Q_\sigma^+(t;n,k\ell)\right\}, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} Q_\sigma^\pm(t;n,q) = \frac{1}{2\pi i}\int_{(\beta)} e\left(\mp\frac{1-2\sigma+s}{4}\right)\Gamma(1-2\sigma+s) \\ \times\left(\frac{q}{2\pi n}\right)^{1-2\sigma+s} M(s,\sigma-it;\Delta)ds. \end{aligned} \tag{5.6}$$

The above (5.5) is the analogue of (5.1) of [5]. (The condition $\sigma<(\beta+1)/2$ stated there is to be read as $\sigma<\beta/2$.)

Define

$$S_\sigma(n;k,\ell)=\sum_{\mu|n}{}^{\#}\mu^{2\sigma-1}e(\mu/\lambda),$$

where the summation $\sum^{\#}$ runs over all positive integer μ satisfying $\mu|n$, $\mu^{-1}n\equiv k(\mathrm{mod}\ d)$ and $\mu\not\equiv 0(\mathrm{mod}\ \lambda)$. Then from (5.5) we can show the following

Lemma 5. *For $1/2\le\sigma<1$, we have*

$$\begin{aligned} P_\sigma(t;\Delta,A) &= \sum_{k\le M}\sum_{\ell\le M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]} \\ &\times\left\{d^{1-2\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)(Q_\sigma^+(t;dn,k)+Q_\sigma^-(t;dn,k))\right. \\ &+\sum_{n=1}^{\infty}n^{1-2\sigma}e\left(\frac{n\bar\kappa}{d\lambda}\right)\overline{S_\sigma(n;k,\ell)}Q_\sigma^-(t;dn,k\ell) \\ &\left.+\sum_{n=1}^{\infty}n^{1-2\sigma}e\left(-\frac{n\bar\kappa}{d\lambda}\right)S_\sigma(n;k,\ell)Q_\sigma^+(t;dn,k\ell)\right\}, \end{aligned} \tag{5.7}$$

where $\sigma_{1-2\sigma}(n)=\sum_{m|n}m^{1-2\sigma}$ and $\bar\kappa$ is determined by $\kappa\bar\kappa\equiv 1(\mathrm{mod}\ \lambda)$.

Proof. We first note that

$$\sum_{j=1}^{d}\sigma_{1-2\sigma}\left(n;\frac{j}{d}\right)=\sum_{m|n}m^{1-2\sigma}\sum_{j=1}^{d}e^{2\pi imj/d}=d\sum_{\substack{m|n\\ m\equiv 0(\mathrm{mod}\ d)}}m^{1-2\sigma}$$

vanishes unless $d|n$. If $d|n$, putting $n=d\nu$ and $m=d\mu$, we find that the above is

$$=d\sum_{\mu|\nu}(d\mu)^{1-2\sigma}=d^{2-2\sigma}\sigma_{1-2\sigma}(\nu).$$

Hence

$$\begin{aligned}&\sum_{j=1}^{d}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}\left(n;\frac{j}{d}\right)\left(Q_\sigma^+(t;n,k)+Q_\sigma^-(t;n,k)\right)\\&=\sum_{\nu=1}^{\infty}d^{2-2\sigma}\sigma_{1-2\sigma}(\nu)(Q_\sigma^+(t;d\nu,k)+Q_\sigma^-(t;d\nu,k)).\end{aligned}\tag{5.8}$$

Next consider

$$\begin{aligned}&{\sum_f}^{*}e\left(-\frac{kf}{\ell}\right)\sigma_{1-2\sigma}\left(n;\frac{f}{\ell},\ell,\{kf\}_\ell\right)\\&={\sum_f}^{*}e\left(-\frac{kf}{\ell}\right)\sum_{\substack{m|n\\ m\equiv kf(\mathrm{mod}\ \ell)}}e\left(\frac{nf}{m\ell}\right)\left(\frac{n}{m}\right)^{1-2\sigma}.\end{aligned}\tag{5.9}$$

The condition $m\equiv kf(\mathrm{mod}\ \ell)$ implies $d|m$, hence the above double sum vanishes unless $d|n$. If $d|n$, we write $n=d\nu$, $m=d\mu$ to obtain that the right-hand side of (5.9) is

$$\begin{aligned}&={\sum_f}^{*}e\left(-\frac{kf}{\ell}\right)\sum_{\substack{\mu|\nu\\ \mu\equiv\kappa f(\mathrm{mod}\ \lambda)}}e\left(\frac{\nu f}{\mu\ell}\right)\left(\frac{\nu}{\mu}\right)^{1-2\sigma}\\&=\nu^{1-2\sigma}\sum_{\mu|\nu}\mu^{2\sigma-1}{\sum_f}^{**}e\left(\left(\frac{\nu}{\mu}-k\right)\frac{f}{\ell}\right),\end{aligned}\tag{5.10}$$

where $\sum^{**}$ runs over all f such that $1\le f\le\ell$, $f\not\equiv 0(\mathrm{mod}\ \lambda)$, and $f\equiv\mu\bar{\kappa}(\mathrm{mod}\ \lambda)$. From these conditions it follows that $\mu\not\equiv 0(\mathrm{mod}\ \lambda)$. We see that

$$\begin{aligned}{\sum_f}^{**}e\left(\left(\frac{\nu}{\mu}-k\right)\frac{f}{\ell}\right)&=\sum_{j=0}^{d-1}e\left(\left(\frac{\nu}{\mu}-k\right)\frac{\{\mu\bar{\kappa}\}_\lambda+j\lambda}{\ell}\right)\\&=e\left(\left(\frac{\nu}{\mu}-k\right)\frac{\{\mu\bar{\kappa}\}_\lambda}{\ell}\right)\times\begin{cases}d & \text{if } d|(\mu^{-1}\nu-k),\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Hence the right-hand side of (5.10) is

$$
\begin{aligned}
&= d\nu^{1-2\sigma}\sideset{}{^\#}\sum_{\mu|\nu}\mu^{2\sigma-1}e\left(\left(\frac{\nu}{\mu}-k\right)\frac{\{\mu\bar{\kappa}\}_\lambda}{d\lambda}\right)\\
&= d\nu^{1-2\sigma}\sideset{}{^\#}\sum_{\mu|\nu}\mu^{2\sigma-1}e\left(\left(\frac{\nu}{\mu}-k\right)\frac{\mu\bar{\kappa}}{d\lambda}\right)\\
&= d\nu^{1-2\sigma}e\left(\frac{\nu\bar{\kappa}}{d\lambda}\right)\overline{S_\sigma(\nu;k,\ell)}.
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
&\sideset{}{^*}\sum_{f} e\left(-\frac{kf}{\ell}\right)\sum_{n=1}^\infty \sigma_{1-2\sigma}\left(n;\frac{f}{\ell},\ell,\{kf\}_\ell\right)Q_\sigma^-(t;n,k\ell)\\
&\qquad = d\sum_{\nu=1}^\infty \nu^{1-2\sigma}e\left(\frac{\nu\bar{\kappa}}{d\lambda}\right)\overline{S_\sigma(\nu;k,\ell)}Q_\sigma^-(t;d\nu,k\ell),
\end{aligned}
$$

and similarly we can show that the last double sum in the curly parenthesis on the right-hand side of (5.5) is equal to

$$
d\sum_{\nu=1}^\infty \nu^{1-2\sigma}e\left(-\frac{\nu\bar{\kappa}}{d\lambda}\right)S_\sigma(\nu;k,\ell)Q_\sigma^+(t;d\nu,k\ell).
$$

Substituting these formulas and (5.8) into (5.5), and using $d/k\ell=[k,\ell]^{-1}$, we arrive at the assertion of Lemma 5.

6. Completion of the proof

Now we evaluate the integral (4.13) by using Lemma 5. Since

$$
\left|n^{1-2\sigma}e\left(-\frac{n\bar{\kappa}}{d\lambda}\right)S_\sigma(n;k,\ell)\right|\le n^{1-2\sigma}\sum_{\mu|n}\mu^{2\sigma-1}=\sigma_{1-2\sigma}(n), \tag{6.1}
$$

from Lemma 5 we have

$$
\begin{aligned}
&\int_{T'}^{T''}P_\sigma(t;\Delta,A)dt \ll \sum_{k\le M}\sum_{\ell\le M}(k\ell)^{-1+\varepsilon}d\\
&\times\left\{d^{1-2\sigma}\sum_{n=1}^\infty\sigma_{1-2\sigma}(n)\left(\left|\int_{T'}^{T''}Q_\sigma^+(t;dn,k)dt\right|+\left|\int_{T'}^{T''}Q_\sigma^-(t;dn,k)dt\right|\right)\right.\\
&\left.+\sum_{n=1}^\infty\sigma_{1-2\sigma}(n)\left(\left|\int_{T'}^{T''}Q_\sigma^+(t;dn,k\ell)dt\right|+\left|\int_{T'}^{T''}Q_\sigma^-(t;dn,k\ell)dt\right|\right)\right\}.
\end{aligned}\tag{6.2}
$$

The quantity $Q_\sigma^\pm(t;n,q)$, defined by (5.6), is exactly the same as $Q_\sigma^\pm(t;n,q)$ introduced in Section 5 of [5] and studied in Sections 6, 7 and 8 of [5]. In Section

6 of [5], by using Lemma 1(ii), it is shown that

$$\int_{T'}^{T''} Q_\sigma^+(t;n,q)dt = \frac{1}{i}\int_0^\infty h_\sigma^+(y;n,q)dy \tag{6.3}$$

and

$$\int_{T'}^{T''} Q_\sigma^-(t;n,q)dt = \frac{1}{i}\int_0^\infty \overline{h_\sigma^-(y;n,q)}dy, \tag{6.4}$$

where

$$\begin{aligned} h_\sigma^\pm(y;n,q) &= \frac{\exp\left(-\frac{1}{4}\Delta^2\log^2(1+y^{-1})\right)}{y^\sigma(1+y)^\sigma\log(1+y^{-1})} \\ &\times\left\{e\left(\frac{\pm T''}{2\pi}\log\left(1+y^{-1}\right)\right)-e\left(\frac{\pm T'}{2\pi}\log\left(1+y^{-1}\right)\right)\right\}e(-ny/q) \end{aligned} \tag{6.5}$$

(see (6.3) and (8.2) of [5]). Let $N \gg qTL^2\Delta^{-2}$. Then Lemma 7 of [5] implies

$$\sum_{n>N}\sigma_{1-2\sigma}(n)\left|\int_{T'}^{T''} Q_\sigma^\pm(t;n,q)dt\right| \ll q^{1+\sigma+\varepsilon}e^{-AT}+(qT)^{-C_3}, \tag{6.6}$$

where A is a positive constant and C_3 is a large positive constant. The remaining part $n \le N$ has been discussed in Section 8 of [5]. From Lemma 8 of [5] (and its proof) we have

$$\sum_{n\le N}\sigma_{1-2\sigma}(n)\left|\int_0^{L^{-1}\Delta} h_\sigma^\pm(y;n,q)dy\right| \ll e^{-AL^2}NT(\log N\log T)^\omega, \tag{6.7}$$

where A is a positive constant and ω is as in Section 4. Also we have

$$\begin{aligned} &\int_{L^{-1}\Delta}^\infty h_\sigma^\pm(y;n,q)dy \\ &\ll (L^{-1}\Delta)^{1-2\sigma}\left\{\frac{q}{n}\left(1+\left(\frac{nT}{q}\right)^{1/4}\right)+\Delta^{3/2}L^{1/2}T^{-1/2}\right\} \end{aligned} \tag{6.8}$$

if $n \le qT_1L^2\Delta^{-2}$ ((8.17) of [5]).

By k^* we denote k or $k\ell$. From (5.6) it follows that $Q_\sigma^\pm(t;dn,k^*) = Q_\sigma^\pm(t;n,d^{-1}k^*)$. Hence

$$\begin{aligned} &\sum_{n=1}^\infty\sigma_{1-2\sigma}(n)\left|\int_{T'}^{T''} Q_\sigma^\pm(t;dn,k^*)dt\right| \le \sum_{n>N^*}\sigma_{1-2\sigma}(n)\left|\int_{T'}^{T''} Q_\sigma^\pm(t;n,d^{-1}k^*)dt\right| \\ &+\sum_{n\le N^*}\sigma_{1-2\sigma}(n)\left|\int_0^{L^{-1}\Delta} h_\sigma^\pm(y;n,d^{-1}k^*)dy\right| + \sum_{n\le N^*}\sigma_{1-2\sigma}(n)\left|\int_{L^{-1}\Delta}^\infty h_\sigma^\pm(y;n,d^{-1}k^*)dy\right| \\ &= \Sigma_1+\Sigma_2+\Sigma_3, \end{aligned} \tag{6.9}$$

say, where $N^* = d^{-1}k^*T_1L^2\Delta^{-2}$.

Applying (6.8), we have

$$\begin{aligned}
\Sigma_3 &\ll \sum_{n\le N^*}\sigma_{1-2\sigma}(n)(L^{-1}\Delta)^{1-2\sigma}\left\{\frac{k^*}{nd}\left(1+\frac{ndT}{k^*}\right)^{1/4}+\Delta^{3/2}L^{1/2}T^{-1/2}\right\}\\
&\ll (L^{-1}\Delta)^{1-2\sigma}\Bigg\{\sum_{n\le N^*}\sigma_{1-2\sigma}(n)\left(\frac{k^*}{nd}+\left(\frac{k^*}{nd}\right)^{3/4}T^{1/4}\right)\\
&\qquad +\Delta^{3/2}L^{1/2}T^{-1/2}\sum_{n\le N^*}\sigma_{1-2\sigma}(n)\Bigg\}\\
&\ll \Delta^{1-2\sigma}\left\{\frac{k^*}{d}+\left(\frac{k^*}{d}\right)^{3/4}(TN^*)^{1/4}+\Delta^{3/2}T^{-1/2}N^*\right\}T^{\varepsilon}\\
&\ll \frac{k^*}{d}\Delta^{1/2-2\sigma}T^{1/2+\varepsilon},
\end{aligned}$$

hence the contribution of Σ_3 to the right-hand side of (6.2) is

$$\ll \sum_{k\le M}\sum_{\ell\le M}(k\ell)^{-1+\varepsilon}k^*\Delta^{1/2-2\sigma}T^{1/2+\varepsilon}\ll M^2\Delta^{1/2-2\sigma}T^{1/2+\varepsilon}.$$

Using (6.6) and (6.7) we can see that the contributions of Σ_1 and Σ_2 are negligible. Hence

$$\int_{T'}^{T''}P_\sigma(t;\Delta,A)dt\ll M^2\Delta^{1/2-2\sigma}T^{1/2+\varepsilon}. \tag{6.10}$$

Combining (6.10) with Lemma 4, we obtain

$$|E_\sigma(2T,A)-E_\sigma(T,A)|\ll M^2\Delta^{1/2-2\sigma}T^{1/2+\varepsilon}+L\Delta(T^\varepsilon)^\omega \tag{6.11}$$

for $1/2\le\sigma<1$.

Recall that we assume (2.2) to obtain (6.11). The inequality $M\le(2^{-j}T)^b$ is valid for $0\le j\le j_1$, where $j_1=[\log(M^{-1/b}T)/\log 2]$. Let

$$\Delta_j=\left(\max\{2^{-cj}M,\ T_j^\varepsilon\}\right)^\mu T_j^\rho \qquad (0\le j\le j_1),$$

where $T_j=2^{-j}T$ and $c>0$. Note that $2^{-cj}M\le T_j^\varepsilon$ for $j>j_2$, where $j_2=[\log(MT^{-\varepsilon})/(c-\varepsilon)\log 2]$. We prove that, if we choose c suitably, then there exists a constant $C_0>0$ such that

$$L_j\le\Delta_j\le\frac{T_j}{A_0L_j} \tag{6.12}$$

for $T_j\ge C_0$, where $L_j=A_0(\log T_j)^{1/2}$. In fact, we see that $\Delta_j\ge T_j^{\mu\varepsilon+\rho}\gg L_j$, which yields the first inequality of (6.12). Since $\rho<1$, it is clear that

$$T_j^{\mu\varepsilon+\rho}\ll\frac{T_j}{L_j}.$$

Next, if $\mu > 0$, we choose c for which $c\mu + \rho = 1$ holds, then

$$(2^{-cj}M)^\mu T_j^\rho = 2^{-(c\mu+\rho)j} M^\mu T^\rho = 2^{-j} M^\mu T^\rho \le \frac{2^{-j}T}{A_0 L} \le \frac{T_j}{A_0 L_j}$$

by (1.7). If $\mu = 0$, then clearly

$$(2^{-cj}M)^\mu T_j^\rho = T_j^\rho \ll \frac{T_j}{L_j}$$

because $\rho < 1$. Hence the second inequality of (6.12) follows.

Let j_3 be the largest integer for which $2^{-j_3}T \ge C_0$ holds, and put $j_0 = \min\{j_1, j_3\}$. Then (6.12) implies that (2.1) and (2.2) are valid for T_j and Δ_j $(0 \le j \le j_0)$ instead of T and Δ, respectively. Hence we may replace T and Δ in (6.11) by T_j and Δ_j respectively to obtain

$$\begin{aligned} &|E_\sigma(T_{j-1}, A) - E_\sigma(T_j, A)| \\ &\ll M^{2+\mu(1/2-2\sigma)} (2^{-j})^{(c\mu+\rho)(1/2-2\sigma)+1/2+\varepsilon} T^{f(\sigma,\rho)+\varepsilon} \\ &\quad + L_j M^\mu (2^{-j})^{c\mu+\rho+\varepsilon\omega} T^{\rho+\varepsilon\omega} \end{aligned} \tag{6.13}$$

if $\mu > 0$ and $1 \le j \le \min\{j_0, j_2\}$, and

$$\ll M^2 (2^{-j})^{f(\sigma,\rho)+\varepsilon} T^{f(\sigma,\rho)+\varepsilon} + L_j (2^{-j})^{\rho+\varepsilon} T^{\rho+\varepsilon} \tag{6.14}$$

if $\mu > 0$ and $j_2 < j \le j_0$, or if $\mu = 0$.

We sum up these inequalities for $j = 1, 2, \ldots, j_0$. In case $\mu > 0$, noting $c\mu + \rho = 1$, $f(\sigma, \rho) > 0$ and $M^{-1/c}T^\varepsilon \ll 2^{-j_2} \ll M^{-1/c}T^\varepsilon$, we find that

$$\begin{aligned} &|E_\sigma(T, A) - E_\sigma(T_{j_0}, A)| \\ &\ll M^{2+\mu(1/2-2\sigma)} T^{f(\sigma,\rho)+\varepsilon} \sum_{j=1}^{j_2} (2^{-j})^{1-2\sigma+\varepsilon} + M^\mu T^{\rho+\varepsilon\omega} L \sum_{j=1}^{j_2} (2^{-j})^{1+\varepsilon\omega} \\ &\quad + M^2 T^{f(\sigma,\rho)+\varepsilon} \sum_{j=j_2+1}^{j_0} (2^{-j})^{f(\sigma,\rho)+\varepsilon} + T^{\rho+\varepsilon} \sum_{j=j_2+1}^{j_0} (2^{-j})^{\rho+\varepsilon} \\ &\ll M^{2+\mu(1/2-2\sigma)+c^{-1}(2\sigma-1)} T^{f(\sigma,\rho)+\varepsilon} + M^\mu T^{\rho+\varepsilon} \\ &\quad + M^{2-c^{-1}f(\sigma,\rho)} T^{f(\sigma,\rho)+\varepsilon} + M^{-c^{-1}\rho} T^{\rho+\varepsilon}, \end{aligned}$$

hence

$$|E_\sigma(T, A) - E_\sigma(T_{j_0}, A)| \ll M^{2-(1-\rho)^{-1}\mu f(\sigma,\rho)} T^{f(\sigma,\rho)+\varepsilon} + M^\mu T^{\rho+\varepsilon}, \tag{6.15}$$

because

$$\mu\left(\tfrac{1}{2} - 2\sigma\right) + \frac{2\sigma - 1}{c} = -\frac{f(\sigma,\rho)}{c} = -\frac{\mu f(\sigma,\rho)}{1-\rho}.$$

From (6.14) we can easily see that (6.15) is also valid in case $\mu = 0$.

On the other hand, if $j_0 = j_1$, from Remark 2 in Section 1 we have

$$|E_\sigma(T_{j_0}, A)| \ll M^{2-(1-\rho)^{-1}\mu f(\sigma,\rho)} T_{j_0}^{f(\sigma,\rho)+\varepsilon} + M^\mu T_{j_0}^{\rho+\varepsilon} \\ \ll M^{2-(1-\rho)^{-1}\mu f(\sigma,\rho)} T^{f(\sigma,\rho)+\varepsilon} + M^\mu T^{\rho+\varepsilon}. \qquad (6.16)$$

If $j_0 = j_3$, then $T_{j_0} \ll 1$. Hence (1.15) is valid for $M \gg 1$, and in this case (6.16) again follows from Remark 2. If $T_{j_0} \ll 1$ and $M \ll 1$ then $E_\sigma(T_{j_0}, A)$ is clearly bounded. Hence (6.16) is true in all the cases. Combining (6.15) and (6.16), we obtain the assertions of Theorems 1 and 2.

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