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ON $\log L$ AND L'/L FOR L-FUNCTIONS AND THE ASSOCIATED "M-FUNCTIONS": CONNECTIONS IN OPTIMAL CASES

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ABSTRACT. Let $\mathcal{L}(s, \chi)$ be either $\log L(s, \chi)$ or $L'/L(s, \chi)$, associated with an (abelian) L-function $L(s, \chi)$ of a global field K. For any quasi-character $\psi\colon \mathbb{C} \to \mathbb{C}^{\times}$ of the additive group of complex numbers, consider the average "$\mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}}$" of $\psi(\mathcal{L}(s, \chi))$ over all Dirichlet characters χ on K with a given prime conductor $\mathfrak{f}$. This paper contains (i) study of the limit as $N(\mathfrak{f}) \to \infty$ of this average, (ii) basic studies of the analytic function $\tilde{M}_s(z_1, z_2)$ in 3 complex variables arising from (i) (here, $(z_1, z_2) \in \mathbb{C}^2$ is the natural parameter for ψ), and (iii) application to value-distribution theory for $\{\mathcal{L}(s, \chi)\}_\chi$. Our base field K is either a function field over a finite field, or a special type of number field: the rational number field $\mathbb{Q}$ or an imaginary quadratic field. But in the number field case, the Generalized Riemann Hypothesis is assumed in (i) and (iii).

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INTRODUCTION

0.1. Our main result on the connection, between the family $\{\log L(s, \chi)\}_\chi$ or $\{L'/L(s, \chi)\}_\chi$ and the corresponding "M-function" $\tilde{M}_s(z_1, z_2)$, is either over a function field over finite field or conditional over a special type of number field. We wish to draw attention to the significance of the study of this complex analytic function in 3 variables, by showing what the most natural direct connection looks like under the optimal circumstances. The study of the analytic function $\tilde{M}_s(z_1, z_2)$ itself (§4) is unconditional over any global field.

This paper is partly a continuation of [10], and is a "companion article" to [11]. But let us start afresh with a direct description of the present conditional results over $\mathbb{Q}$. The L-function $L(s, \chi)$ associated with any non-principal Dirichlet character χ is, under GRH (the Generalized Riemann Hypothesis), holomorphic

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and non-vanishing on $\sigma = \mathrm{Re}(s) > 1/2$; hence its logarithm on this domain can be defined in the natural manner. Thus, the functions

$$\begin{aligned} \mathcal{L}(s,\,\chi) &:= L'(s,\,\chi)/L(s,\,\chi) \quad &&\text{(henceforth called Case 1)} \\ &= \log L(s,\,\chi) &&\text{(henceforth called Case 2)} \end{aligned} \tag{0.1.1}$$

of s are holomorphic on $\sigma > 1/2$. But we shall fix any point s in this domain, consider each $\mathcal{L}(s,\,\chi)$ rather as a function of χ, and for any mild test function Φ on $\mathbb{C}$, study the mean value of $\Phi(\mathcal{L}(s,\,\chi))$ when χ runs over some natural family of characters. For this study, the basic role is played by *quasi-characters* $\Phi = \psi\colon \mathbb{C} \to \mathbb{C}^\times$ of the additive group $\mathbb{C}$. They are parametrized by two complex numbers $z_1,\, z_2$, as

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1\bar{w} + z_2 w)\right) \quad (i = \sqrt{-1}). \tag{0.1.2}$$

They are not only basic in the space of functions on $\mathbb{C}$, but also quite suitable for this study, because each of $\psi_{z_1,z_2}(\mathcal{L}(s,\,\chi))$ has an Euler *product* expansion on $\sigma > 1$ reflecting the Euler *sum* decomposition of $\mathcal{L}(s,\,\chi)$ (which comes from the Euler product decomposition of $L(s,\,\chi)$). In Case 2, we have, by definition,

$$\psi_{z_1,z_2}(\mathcal{L}(s,\,\chi)) = \overline{L(s,\,\chi)}^{\frac{iz_1}{2}} L(s,\,\chi)^{\frac{iz_2}{2}} \quad \text{(Case 2)}. \tag{0.1.3}$$

For each prime divisor[1] $\mathfrak{f}$, let $\mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}}$ denote the (additive) average over all primitive Dirichlet characters χ with conductor $\mathfrak{f}$. One of our main results (Theorem 3 in Section 1.3) is the equality

$$\lim_{\substack{\mathfrak{f}\text{ prime}\\ N(\mathfrak{f})\to\infty}} \mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}}\psi_{z_1,z_2}(\mathcal{L}(s,\,\chi)) = \tilde{M}_\sigma(z_1,\,z_2) \quad (\sigma = \mathrm{Re}(s) > 1/2, \text{ under GRH}), \tag{0.1.4}$$

where the function on the right-hand side is defined as follows. Put

$$\mathcal{Z}(s) = \zeta'(s)/\zeta(s) \quad \text{(Case 1)}, \tag{0.1.5}$$

$$= \log\zeta(s) \quad \text{(Case 2)}, \tag{0.1.6}$$

$\zeta(s)$ being the Riemann zeta function. Consider the Dirichlet series

$$\exp\left(\frac{iz}{2}\mathcal{Z}(s)\right) = \sum_{n=1}^{\infty} \lambda_z(n) n^{-s} \quad (z \in \mathbb{C},\ \sigma > 1). \tag{0.1.7}$$

In each of Cases 1, 2, $\lambda_z(n)$ $(n = 1,\, 2,\, \dots)$ are polynomials in z and multiplicative in n. Now, as in [8] (Case 1) and [18] (Case 2, $s = 1$), consider the following Dirichlet series

$$\tilde{M}_s(z_1,\, z_2) = \sum_{n=1}^{\infty} \lambda_{z_1}(n)\lambda_{z_2}(n) n^{-2s} \quad (\sigma > 1/2). \tag{0.1.8}$$

This series converges absolutely and uniformly in the wider sense and defines an analytic function of 3 complex variables $s,\, z_1,\, z_2$ on $\sigma > 1/2$. For this, GRH is unnecessary. The above function $\tilde{M}_\sigma(z_1,\, z_2)$ is its restriction to $s = \sigma \in \mathbb{R}$.

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[1]Here we use the notation system for the general case of K employed in the main text; thus, here, $\mathfrak{f}$ corresponds to a prime number f, and $N(\mathfrak{f}) = f$.

As for the equality (0.1.4), the difficult case to prove is of course when $\sigma \leqslant 1$. When $s = 1$, Lamzouri [18] (Theorem 9.2 based on "Theorem B" by Granville–Soundararajan) gives an unconditional version of (0.1.4) for Case 2. The earlier works of Elliott [2], [3], [4], and [21], [5] are partially related, but the main difference is that each of them treats a *one-dimensional* distribution. Our work is closer in spirit to the original Bohr–Jessen or Jessen–Wintner theory. For more of the history, cf. [11].

When $z_2 = \bar{z}_1$, ψ_{z_1,z_2} is a character, i.e., $|\psi_{z_1,z_2}(w)| = 1$ $(w \in \mathbb{C})$. In the character case for Case 2, we have proved, in the companion article [11] by a different method, that a certain weaker version of (0.1.4) holds *unconditionally*.

A main point in the present article is: either under GRH or in the case of any function fields (cf. Section 0.2 below), (0.1.4) *itself* holds for any $z_1, z_2 \in \mathbb{C}$. This quasi-character version seems remarkable, because, firstly, the value of $\psi_{z_1,z_2}(\mathcal{L}(s, \chi))$ can then be exponentially large (as $z_1, z_2 \to \infty$ in some directions). Secondly, the equality (0.1.4) asserts that each special value of $\tilde{M}_\sigma(z_1, z_2)$ $(z_1, z_2 \in \mathbb{C})$ has a definite meaning as the corresponding limit average. This strongly motivates a further study of this complex analytic function started for Case 1 in [8]. For this, see Section 0.4 and Section 4.

0.2. As for the base field K, in addition to $K = \mathbb{Q}$, we shall also include imaginary quadratic fields (also under GRH), and any algebraic function field of one variable over a finite field with an assigned "infinite" prime divisor $\mathfrak{p}_\infty$. In short, K is any global field with just one infinite prime. In the function field case, we shall normalize χ by the condition $\chi(\mathfrak{p}_\infty) = 1$ to kill infinitely many trivial twists which do not change the conductor. The Euler factor corresponding to $\mathfrak{p}_\infty$ should be dropped from each of $L(s, \chi)$, $\mathcal{L}(s, \chi)$ and $\tilde{M}_s(z_1, z_2)$. This applies also to the convolution $\mathfrak{p}_\infty$-factor in the function $M_\sigma(z)$ which appears later.

0.3. We shall also study a more general equality of the form

$$\lim_{\substack{\mathfrak{f}\ \text{prime} \\ N(\mathfrak{f})\to\infty}} \mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}} \Phi(\mathcal{L}(s, \chi)) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw| \quad (\sigma = \mathrm{Re(s)} > 1/2), \tag{0.3.1}$$

where $|dw| = du\,dv/2\pi$ for $w = u + vi$ $(u, v \in \mathbb{R})$, Φ is a test function on $\mathbb{C}$, and $M_\sigma(w)$ is the would-be density function for the distribution of values of $\{\mathcal{L}(s, \chi)\}_\chi$ constructed in [8](Case 1) [11](Case 2). The connection between $M_\sigma(w)$ and $\tilde{M}_s(z_1, z_2)$ is

$$\tilde{M}_\sigma(z_1, z_2) = \int_{\mathbb{C}} M_\sigma(w)\psi_{z_1,z_2}(w)|dw| \quad (\sigma > 1/2, \;\; z_1, z_2 \in \mathbb{C}). \tag{0.3.2}$$

Hence (0.1.4) is a special case $\Phi = \psi_{z_1,z_2}$ of (0.3.1), and $\tilde{M}_\sigma(z) := \tilde{M}_\sigma(z, \bar{z})$ is the Fourier dual of $M_\sigma(w)$.

In [10], the following weaker version

$$\lim_{m\to\infty} \mathrm{Avg}_{N(\mathfrak{f})\leqslant m}\big(\mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}} \Phi(\mathcal{L}(s, \chi))\big) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw| \quad (\sigma > 1/2) \tag{0.3.3}$$

is established in Case 1 for the function field case for any test function Φ which is continuous and of at most a polynomial growth (which improves Theorem 7 of [8]).

In [11], it is proved, among other things, that in Case 2, a certain modification of (0.3.3) holds for $K = \mathbb{Q}$ *unconditionally* for any Φ which is continuous and bounded.

In the present article, we shall establish Theorem 4 (Sections 1.4 and 5), which asserts that (0.3.1) itself holds for each of Cases 1, 2, if the ground field is as in Section 0.2 and Φ is any continuous function of at most *exponential* growth. This improves [10] and complements [11]. While in [8] we relied on a quantitative uniform distribution of the points $(\chi(\mathfrak{p}))_{N(\mathfrak{p})\leqslant y}$ $(N(\mathfrak{f}_\chi) \leqslant m)$ on a torus, here, our method relies only on the equality (0.1.4). We found this simpler and stronger, though it hides the uniform distribution property behind.

A brief survey of main results of [10], [11] and of this article is given in [25].

0.4. The above "mean value theorem" (0.1.4) motivates us to study the analytic function $\tilde{M}_s(z_1, z_2)$ itself over *any global field* (*not* assuming GRH). This study was started in [8], [9] for Case 1. The basic properties proved in [8](Case 1) and in this paper (Section 4) include the following.

Firstly, $\tilde{M}_s(z_1, z_2)$ has an Euler product expansion on $\sigma > 1/2$. Moreover, as can be expected, each Euler factor may be interpreted as the limit average of the corresponding Euler factor of $\psi_{z_1,z_2}(\mathcal{L}(s, \chi))$ (see (4.1.3)). Unlike the Euler product expansion of each of $\psi_{z_1,z_2}(\mathcal{L}(s, \chi))$, for which at most a conditional convergence can be expected on $1/2 < \sigma \leqslant 1$, the Euler product expansion for its limit average $\tilde{M}_s(z_1, z_2)$ is *absolutely* convergent on $\sigma > 1/2$ (Theorem $\tilde{M}$ in Section 4.1). Incidentally, in Case 2, the Euler factor $\tilde{M}_{s,\mathfrak{p}}(z_1, z_2)$ is equal to the Gauss hypergeometric function $F(iz_1/2, iz_2/2, 1; N(\mathfrak{p})^{-2s})$ (cf. Section 4.2).

Secondly, this function $\tilde{M}_s(z_1, z_2)$ also admits an everywhere convergent *power series* expansion in z_1, z_2 with Dirichlet series coefficients (Theorem $\tilde{M}$). The coefficient of $z_1^a z_2^b$ is essentially the limit average of $P^{(a,b)}(\mathcal{L}(s, \chi))$, where $P^{(a,b)}(w) = \bar{w}^a w^b$ $(a, b \geqslant 0)$.

As an application we can prove, under GRH, that for any fixed $\sigma > 1/2$, $y > 0$, and for sufficiently large $N(\mathfrak{f})$, the following pair of inequalities holds;

$$\mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}} \exp(2y\mathrm{Re}(L'(s, \chi)/L(s, \chi))) < \mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}} \exp(-2y\mathrm{Re}(L'(s, \chi)/L(s, \chi))), \tag{0.4.1}$$

$$\mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}}|L(s, \chi)|^{2y} > \mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}}|L(s, \chi)|^{-2y}. \tag{0.4.2}$$

Incidentally, when $y = 1$ and $K = \mathbb{Q}$, the limit $N(\mathfrak{f}) \to \infty$ of (0.4.2) becomes[1]

$$\zeta(2\sigma) = \sum_n n^{-2\sigma} > \sum_{n \text{ square free}} n^{-2\sigma}. \tag{0.4.3}$$

These studies draw our attention to the following delicate *sign-problem*. The difference of two sides (l.h.s.)$-$(r.h.s.) in (0.4.1) or (0.4.2) clearly tends to 0 as $y \to 0$, but if one divides this difference by $4y$ and consider its limit-value at $y = 0$ for each $\mathfrak{f}$ (say, with large $N(\mathfrak{f})$), then what is its sign? Experimentally, this turns out to be *the opposite* of what one might expect from the above two inequalities, and offers quite a delicate problem related to some basic arithmetic invariants of

[1]This special limit formula for the left-hand side for $y = 1$ holds unconditionally; [16, Theorem 1].

the ray class field $K_{\mathfrak{f}}$ over K with conductor $\mathfrak{f}$. For example, let $s = 1$. Then the relevant invariants are $\log \kappa_{\mathfrak{f}}$ (Case 2) and $\gamma_{\mathfrak{f}}$ (Case 1), where $\kappa_{\mathfrak{f}}$ (resp. $\kappa_{\mathfrak{f}} \cdot \gamma_{\mathfrak{f}}$) are the residue (resp. the constant term) of the Laurent expansion at $s = 1$ of the Dedekind zeta function of $K_{\mathfrak{f}}$. This will be touched in Section 4.2.

For the next subjects of study related to the function $\tilde{M}_s(z_1, z_2)$ (the zeros, the "Plancherel volume", the analytic continuation), cf. Section 4.3 and the subsequent article [26].

The main results (Theorems 1–4) are summarized in Section 1, except for Theorem $\tilde{M}$ related to the function $\tilde{M}_s(z_1, z_2)$ which is in Section 4. The remaining sections are for their proofs which, we believe, are reasonably self-contained. The treatment of the FF case is not simplified as being "similar". The careful readers will see the reasons especially in Section 2.

1. The Main Results

We shall summarize the main results except (the lengthy) Theorem $\tilde{M}$, which is given in Section 4.1 together with the proof.

1.1. Uniformly admissible family of arithmetic functions. Let K be a global field, i.e., either an algebraic number field of finite degree (NF) or an algebraic function field of one variable over a finite field $\mathbb{F}_q$ (FF), given together with a finite set P_∞ of prime divisors of K. We assume that P_∞ contains all the archimedean primes (NF case) and is *non-empty* also in the FF case. An *integral divisor* will mean any integral ideal (NF case) (resp. effective divisor (FF case)) of K which is coprime with P_∞.

For an integral divisor $\mathfrak{f}$, let $I_{\mathfrak{f}}$ be the group of divisors of K coprime with both $\mathfrak{f}$ and P_∞, and $G_{\mathfrak{f}}$ be the "ray class group" defined by

$$G_{\mathfrak{f}} = I_{\mathfrak{f}}/\{(\alpha)'; \alpha \equiv 1 \pmod{\mathfrak{f}},\ \alpha_v > 0 \text{ (all real archimedean primes } v)\},$$

where for each $\alpha \in K^\times$, $(\alpha)'$ denotes the "prime-to-P_∞" component of the principal divisor generated by α, and α_v, the v-component. Note that $G_{\mathfrak{f}}$ is a *finite* abelian group, including the FF case because P_∞ is non-empty. In fact, if $K_{\mathfrak{f}}$ denotes the maximal abelian extension of K with the conductor dividing $\mathfrak{f}$ in which all prime divisors in P_∞ except[1] the real archimedean primes split completely, then by classfield theory, $G_{\mathfrak{f}}$ is canonically isomorphic with the Galois group $\mathrm{Gal}(K_{\mathfrak{f}}/K)$. Define

$$i_{\mathfrak{f}} \colon I_{\mathfrak{f}} \to G_{\mathfrak{f}}\text{: the projection,}$$

$$\hat{G}_{\mathfrak{f}}\text{: the character group of } G_{\mathfrak{f}}, \text{ with the unit element } \chi_0.$$

For each $\chi \in \hat{G}_{\mathfrak{f}}$ and an integral divisor D, we define $\chi(D) = \chi(i_{\mathfrak{f}}(D))$ if $(D, \mathfrak{f}) = 1$, and $\chi(D) = 0$ otherwise.

An *arithmetic function* will mean a $\mathbb{C}$-valued function $D \mapsto \lambda(D)$ on integral divisors. It will be called *admissible* if it satisfies the following three conditions (A1)–(A3):

[1] This "exception" corresponds to the second system of conditions $\alpha_v > 0$ above. Our main results remain valid if these are dropped.

(A1) $\lambda(D) \ll_{\epsilon'} N(D)^{\epsilon'}$ for any $\epsilon' > 0$.

(A2) For any integral divisor $\mathfrak{f}$ and $\chi \in \hat{G}_{\mathfrak{f}} \backslash \{\chi_0\}$, consider the Dirichlet series

$$g_\lambda(s,\, \chi,\, \mathfrak{f}) = \sum_{(D,\mathfrak{f})=1} \chi(D)\lambda(D)N(D)^{-s}, \tag{1.1.1}$$

where the summation is over all integral divisors D coprime with $\mathfrak{f}$. By (A1), this series converges absolutely and defines a holomorphic function on $\mathrm{Re}(s) > 1$. The condition (A2) imposes that this extends to a holomorphic function on $\mathrm{Re}(s) > 1/2$.

In the FF case over $\mathbb{F}_q$, this means that $g_\lambda = g_\lambda(s,\, \chi,\, \mathfrak{f})$ is a holomorphic function of $u = q^{-s}$ on the open disk $|u| < q^{-1/2}$ and hence that g_λ, as a function of s, is vertically periodic and bounded on $\mathrm{Re}(s) \geqslant 1/2 + \epsilon$ for each $\epsilon > 0$.

(A3) In the FF case, this simply imposes that

$$g_\lambda(s,\, \chi,\, \mathfrak{f}) \ll_{\epsilon,\epsilon'} N(\mathfrak{f})^{\epsilon'} \quad \text{holds on } \mathrm{Re}(s) \geqslant 1/2 + \epsilon \tag{1.1.2}$$

for any $\epsilon,\, \epsilon' > 0$. In the NF cases, the condition is necessarily more complicated;

$$\mathrm{Max}(0,\, \log|g_\lambda(s,\, \chi,\, \mathfrak{f})|) \ll_\epsilon \ell(t)\ell(\mathfrak{f})^{1-2\epsilon} + \ell(t)^2 \quad \text{on } \mathrm{Re}(s) \geqslant 1/2 + \epsilon \tag{1.1.3}$$

for any $0 < \epsilon < 1/2$, where $t = \mathrm{Im}(s)$ and

$$\ell(\mathfrak{f}) = \log(N(\mathfrak{f}) + 2) \quad \text{if } \mathfrak{f} \text{ is an integral divisor,} \tag{1.1.4}$$

$$\ell(t) = \log(|t| + 2) \qquad \text{if } t \in \mathbb{R}. \tag{1.1.5}$$

(The left-hand side of (1.1.3) is, by definition, equal to zero whenever $|g_\lambda(s,\, \chi,\, \mathfrak{f})| \leqslant 1$, including the case $g_\lambda(s,\, \chi,\, \mathfrak{f}) = 0$.)

Remark 1.1.6. A few words about the conditions (1.1.2), (1.1.3). In the FF case, we note first that (1.1.2) is equivalent to the condition that the left-hand side of (1.1.3) is $o_\epsilon(\ell(\mathfrak{f}))$ on $\mathrm{Re}(s) \geqslant 1/2 + \epsilon$. Secondly, the FF counterpart of $\ell(t)$ is the constant 1; hence that of the right-hand side of (1.1.3) is $\ell(\mathfrak{f})^{1-2\epsilon}$, which gives a stronger condition than (1.1.2). We shall prove (Section 3) that the latter stronger condition is satisfied in the present case of interest, i.e., when $\lambda = \lambda_z$. But to prove Theorem 1 below, the former condition is sufficient. In the NF case, the term $\ell(t)\ell(\mathfrak{f})^{1-2\epsilon}$ in (1.1.3) seems to present something fairly essential, but the next term need not be so close to $\ell(t)^2$ (see Remark 2.2.26).

The holomorphic functions $g_\lambda(s,\, \chi,\, \mathfrak{f})$ will be called the *g-functions associated with* λ. If Λ is a family of admissible arithmetic functions such that the implicit constants in (A1) and (A3) can be chosen to be independent of $\lambda \in \Lambda$, then Λ will be called a *uniformly admissible* family of arithmetic functions. Important examples of such families will be given in Theorem 2 (Section 1.2). The notion of uniformly admissible family of arithmetic function is invented because it seems to give a natural setting for the following mean value theorem. Unfortunately, at least at present, we need to assume further in this theorem that $|P_\infty| = 1$, namely, either (a) K is the rational number field or an imaginary quadratic field and P_∞ consists only of the unique archimedean prime, or (b) K is a function field over a finite field and P_∞ consists of just one prime divisor (to be called $\mathfrak{p}_\infty$). The point is that in such a case the group of P_∞-units of K is finite, so that the order of $G_\mathfrak{f}$ is comparable with $N(\mathfrak{f})$.[1]

By $\mathrm{Avg}_{\chi\in X} G(\chi)$, for a finite set X of characters χ and a $\mathbb{C}$-valued function $G(\chi)$ of χ, we shall mean the usual average $|X|^{-1}\sum_{\chi\in X} G(\chi)$.

Theorem 1. *Let Λ be any uniformly admissible family of arithmetic functions, and let λ, λ' run over Λ. Fix any ϵ such that $0<\epsilon<1/2$, and let $s=\sigma+ti$ run over the domain $\sigma\geqslant 1/2+\epsilon$. In the NF case, fix also $T>0$ and impose additionally that $|t|\leqslant T$. Assume $|P_\infty|=1$. Then*:

(i) *Let $\mathfrak{f}$ run over integral divisors. Then*

$$\mathrm{Avg}_{\chi\in\hat{G}_f\setminus\{\chi_0\}}(\overline{g_\lambda(s,\,\chi,\,\mathfrak{f})}\,g_{\lambda'}(s,\,\chi,\,\mathfrak{f}))-\sum_{(D,\mathfrak{f})=1}\overline{\lambda(D)}\,\lambda'(D)N(D)^{-2\sigma}\ll N(\mathfrak{f})^{-\epsilon/2}. \tag{1.1.7}$$

In particular, the quantity on the left-hand side tends to 0 *uniformly as $N(\mathfrak{f})\to\infty$.*

(ii) *Let $\mathfrak{f}$ run only over prime divisors. Then*

$$\lim_{N(\mathfrak{f})\to\infty}\mathrm{Avg}_{\chi\in\hat{G}_f\setminus\{\chi_0\}}(\overline{g_\lambda(s,\,\chi,\,\mathfrak{f})}\,g_{\lambda'}(s,\,\chi,\,\mathfrak{f}))=\sum_{D}\overline{\lambda(D)}\,\lambda'(D)N(D)^{-2\sigma}, \tag{1.1.8}$$

and the convergence is uniform w.r.t. λ, λ' and s. Moreover, the average over $\chi\in\hat{G}_\mathfrak{f}\setminus\{\chi_0\}$ in (1.1.8) *may be replaced by that over all χ with the conductor $\mathfrak{f}_\chi=\mathfrak{f}$.*

This is a natural "χ-analogue" of Carlson's mean value theorem on the limit average of Dirichlet series over a vertical axis inside the critical strip [1], [22], [19], [20]. The proof will be given in Section 2.

Remark 1.1.9. For given $\lambda_1,\,\dots,\,\lambda_k\in\Lambda$, define their $*$-product by

$$(\lambda_1*\cdots*\lambda_k)(D)=\sum_{D=D_1\dots D_k}\lambda_1(D_1)\dots\lambda_k(D_k). \tag{1.1.10}$$

Then this is also admissible, being associated with the product

$$g_{\lambda_1}\cdots g_{\lambda_k}. \tag{1.1.11}$$

This is because if $S(D)$ denotes the number of distinct factors of D, then for any $\epsilon'>0$ we have $S(D)\ll_{\epsilon'}N(D)^{\epsilon'}$, as is well-known in the NF case and can be proved similarly in the FF case (cf. [10, Appendix] for a unified proof). Moreover, if we fix k, then $\Lambda_k:=\{\lambda_1*\cdots*\lambda_k;\,\lambda_1,\,\dots,\,\lambda_k\in\Lambda\}$ is again a uniformly admissible family of arithmetic functions. Thus, Theorem 1 remains valid if g_λ, $g_{\lambda'}$ are replaced by their k-th powers, and $\lambda(D)$, $\lambda'(D)$, by their k-th $*$-powers.

[1] Whether Theorem 1 holds more generally if $N(\mathfrak{f})$ is simply replaced by $|G_\mathfrak{f}|$, is an obvious question. But this involves essentially *different* types of difficulties.

1.2. The families $\{\lambda_z\}_{|z|\leqslant R}$ associated with L-functions. We shall consider the L-function associated with each $\chi \in \hat{G}_{\mathfrak{f}}$, $\chi \neq \chi_0$, *without the P_∞-component.* Namely, define

$$L(s,\,\chi,\,\mathfrak{f}) = \prod_{\mathfrak{p}\notin P_\infty} (1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1}, \tag{1.2.1}$$

which converges absolutely on $\mathrm{Re}(s) > 1$ and extends to a holomorphic function on $\mathrm{Re}(s) > 1/2$. In the FF case, and in the NF case under GRH, it has no zeros on this domain. In these cases, $\log L(s,\,\chi,\,\mathfrak{f})$ on this domain is defined as the unique holomorphic branch that vanishes at $s = +\infty$. Write:

$$\mathcal{L}(s,\,\chi,\,\mathfrak{f}) = \begin{cases} \dfrac{L'}{L}(s,\,\chi,\,\mathfrak{f}) & \text{(Case 1)}, \\ \log L(s,\,\chi,\,\mathfrak{f}) & \text{(Case 2)}. \end{cases} \tag{1.2.2}$$

We shall show in the next Theorem that for any given $R > 0$, the family of functions

$$\exp\left(\frac{iz}{2}\mathcal{L}(s,\,\chi,\,\mathfrak{f})\right) \tag{1.2.3}$$

parametrized by $\{z;\,|z| \leqslant R\}$ forms (in each of Cases 1, 2) a family of g-functions $g_{\lambda_z}(s,\,\chi,\,\mathfrak{f})$ associated with a uniformly admissible family $\{\lambda_z\}_{|z|\leqslant R}$ of arithmetic functions. To explain this, first define the polynomials $G_r(x)$, $H_r(x)$ of x ($r = 0,\,1,\,2,\,\dots$) as

$$\exp(xt/(1-t)) = \sum_{r=0}^{\infty} G_r(x)t^r, \tag{1.2.4}$$

$$\exp\left(-x\log(1-t)\right) = (1-t)^{-x} = \sum_{r=0}^{\infty} H_r(x)t^r, \tag{1.2.5}$$

by generating functions ($|t| < 1$). Explicitly, $G_0(x) = H_0(x) = 1$, and for $r \geqslant 1$,

$$G_r(x) = \sum_{k=1}^{r} \frac{1}{k!}\binom{r-1}{k-1}x^k, \tag{1.2.6}$$

$$H_r(x) = \sum_{k=1}^{r} \frac{1}{k!}\delta_k(r)x^k = \frac{1}{r!}x(x+1)\dots(x+r-1), \tag{1.2.7}$$

where

$$\delta_k(r) = \sum_{\substack{r=r_1+\cdots+r_k \\ r_1,\dots,r_k\geqslant 1}} \frac{1}{r_1\dots r_k}. \tag{1.2.8}$$

Theorem 2. *For each $z \in \mathbb{C}$ and each of Cases* 1, 2, *define the arithmetic function λ_z as follows.*

$$\lambda_z(D) = \prod_{\mathfrak{p}|D} \lambda_z(\mathfrak{p}^{r_\mathfrak{p}}) \quad (\textit{for } D = \prod_{\mathfrak{p}} \mathfrak{p}^{r_\mathfrak{p}}), \tag{1.2.9}$$

$$\lambda_z(\mathfrak{p}^r) = \begin{cases} G_r\Big(-\dfrac{iz}{2}\log N(\mathfrak{p})\Big) & (\text{Case } 1), \\ H_r\Big(\dfrac{iz}{2}\Big) & (\text{Case } 2), \end{cases} \tag{1.2.10}$$

where $i = \sqrt{-1}$. (In particular, $\lambda_z(D) = 1$ for $D = (1)$.) Then, for any K and P_∞,

(i) *the family $\{\lambda_z\}_{|z|\leqslant R}$ satisfies* (A1) *uniformly, i.e., with $\ll$ depending only on ϵ', R.*

(ii) *Moreover, if we assume GRH in the NF case, then $\{\lambda_z\}_{|z|\leqslant R}$ also satisfies* (A2), (A3), *and is a uniformly admissible family of arithmetic functions. The associated g-function is given by*[1]

$$g_{\lambda_z}(s, \chi, \mathfrak{f}) = \exp\left(\frac{iz}{2}\mathcal{L}(s, \chi, \mathfrak{f})\right). \tag{1.2.11}$$

The proof will be given in Section 3.

1.3. Direct consequences of Theorems 1, 2. Now consider the Dirichlet series

$$\tilde{M}_s(z_1, z_2) = \sum_D \lambda_{z_1}(D)\lambda_{z_2}(D)N(D)^{-2s} \quad (\mathrm{Re}(s) > 1/2;\ z_1, z_2 \in \mathbb{C}), \tag{1.3.1}$$

where the summation is over all integral divisors D of K. This converges absolutely and uniformly on $\mathrm{Re}(s) \geqslant 1/2 + \epsilon$, $|z_1|, |z_2| \leqslant R$ for any fixed ϵ, $R > 0$, because $\lambda_{z_1}(D), \lambda_{z_2}(D) \ll_{\epsilon'} N(D)^{\epsilon'}$ (uniformly on $|z_1|, |z_2| \leqslant R$) by Theorem 2(i); hence this is a holomorphic function of three complex variables s, z_1, z_2 on the domain $\mathrm{Re}(s) > 1/2$. Note that $\tilde{M}_s(z_1, z_2)$ is symmetric in z_1, z_2. For $z_1, z_2 \in \mathbb{C}$, let ψ_{z_1,z_2} denote the quasi-character of the additive group $\mathbb{C}$ defined by

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1\bar{w} + z_2 w)\right). \tag{1.3.2}$$

Theorem 3. *Assume $|P_\infty| = 1$, and in the NF case assume also GRH. Then*

$$\lim_{\substack{\mathfrak{f}\ \text{prime} \\ N(\mathfrak{f})\to\infty}} \left(\mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}}\psi_{z_1,z_2}(\mathcal{L}(s, \chi, \mathfrak{f}))\right) = \tilde{M}_\sigma(z_1, z_2) \tag{1.3.3}$$

holds. Moreover, the convergence is uniform on $|z_1|, |z_2| \leqslant R$, $s = \sigma + ti$ with $\sigma \geqslant 1/2 + \epsilon$, and in the NF case, $|t| \leqslant T$.

This is a direct consequence of Theorem 1(ii) and Theorem 2. Indeed, we have $\overline{\lambda_z(D)} = \lambda_{-\bar{z}}(D)$ and

$$\psi_{z_1,z_2}(\mathcal{L}(s, \chi, \mathfrak{f})) = \overline{g_{\lambda_{-\bar{z}_1}}(s, \chi, \mathfrak{f})}\, g_{\lambda_{z_2}}(s, \chi, \mathfrak{f}). \tag{1.3.4}$$

[1]The formula (1.2.11) holds also for $\chi = \chi_0$ when $\sigma > 1$; hence the definition of $\lambda_z(n)$ in the Introduction matches with the present definition; see Section 3.2.

We put $\tilde{M}_\sigma(z) := \tilde{M}_\sigma(z, \bar{z}) = \tilde{M}_\sigma(\bar{z}, z)$.

Remark 1.3.5. When $\sigma > 1$, (1.3.3) holds without GRH, because we may use $\lambda_z(D)N(D)^{-1/2}$ instead of $\lambda_z(D)$.

In Case 2, where $\exp(\mathcal{L}(s, \chi, \mathfrak{f})) = L(s, \chi, \mathfrak{f})$, Theorem 3 gives

$$\lim_{\substack{\mathfrak{f}\text{ prime} \\ N(\mathfrak{f})\to\infty}} \big(\mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}} \overline{L(s, \chi, \mathfrak{f})}^{\frac{iz_1}{2}} L(s, \chi, \mathfrak{f})^{\frac{iz_2}{2}}\big) = \tilde{M}_\sigma(z_1, z_2); \tag{1.3.6}$$

hence in particular:

Corollary 1.3.7 (Case 2)**.** *The assumptions being as in Theorem* 3,

$$\lim_{\substack{\mathfrak{f}\text{ prime} \\ N(\mathfrak{f})\to\infty}} \mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}} |L(s, \chi, \mathfrak{f})^{\frac{i\bar{z}}{2}}|^2 = \tilde{M}_\sigma(-\bar{z}, z); \tag{1.3.8}$$

$$\lim_{\substack{\mathfrak{f}\text{ prime} \\ N(\mathfrak{f})\to\infty}} \mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}} \big(L(s, \chi, \mathfrak{f})^{iz/2} / \overline{L(s, \chi, \mathfrak{f})^{iz/2}}\big) = \tilde{M}_\sigma(\bar{z}) = \tilde{M}_\sigma(z); \tag{1.3.9}$$

in particular, for any $y \in \mathbb{R}$,

$$\lim_{\substack{\mathfrak{f}\text{ prime} \\ N(\mathfrak{f})\to\infty}} \mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}} \big(L(s, \chi, \mathfrak{f}) / \overline{L(s, \chi, \mathfrak{f})}\big)^y = \tilde{M}_\sigma(2iy). \tag{1.3.10}$$

As is shown in [9], $\tilde{M}_\sigma(z)$ has, at least in Case 1, infinitely many purely imaginary zeros, and at most finitely many other zeros.

1.4. Application to the value-distribution.

As before, let

$$\mathcal{L}(s, \chi, \mathfrak{f}) = \begin{cases} L'/L(s, \chi, \mathfrak{f}) & \text{(Case 1)}, \\ \log L(s, \chi, \mathfrak{f}) & \text{(Case 2)}, \end{cases}$$

and let $M_\sigma(w) = *_{\mathfrak{p}\notin P_\infty} M_{\sigma,\mathfrak{p}}(w)$ ($*$: the convolution product) be the associated M-function without the P_∞-component constructed in [8] (for Case 1), [11] (for Case 2, where it is denoted by $\mathcal{M}_\sigma(w)$); see also Section 5.6 of the present paper. At least in Case 2, it has a long history since Bohr and Jessen; for this, cf. the Introduction of [11].

Theorem 4. *Let* $\sigma := \mathrm{Re}(s) > 1/2$, *assume* $|P_\infty| = 1$, *and in the NF case assume also GRH. Then*

$$\lim_{\substack{\mathfrak{f}\text{ prime} \\ N(\mathfrak{f})\to\infty}} \big(\mathrm{Avg}_{\mathfrak{f}_\chi=\mathfrak{f}} \Phi(\mathcal{L}(s, \chi, \mathfrak{f}))\big) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw| \tag{1.4.1}$$

holds for any continuous function Φ *on* $\mathbb{C}$ *of at most exponential growth, i.e., when* $\Phi(w) \ll e^{a|w|}$ *holds with some* $a > 0$. *The equality* (1.4.1) *holds also when* Φ *is the characteristic function of either a compact subset of* $\mathbb{C}$ *or the complement of such a subset. When moreover* $\sigma > 1$, *then* (1.4.1) *holds unconditionally for any continuous function* Φ *on* $\mathbb{C}$.

The proof will be given in Section 5. This Theorem for Case 1 for the FF case strengthens Theorem B of [10] (and Theorem 7 of [8]) in various sense. The condition on the test function Φ is now considerably loosened, and here, the assertion is on the limit of the average over $\mathfrak{f}$, which is stronger than the previous assertions on the limit, as $m \to \infty$, of a weighted average over $N(\mathfrak{f}) \leqslant m$. It should also be added, however, that Theorem A of [10], and the direct method for proving Theorem B of [10] as its application, may still deserve attention for independent interest.

2. Proof of Theorem 1

There is a traditional method for studying similar types of mean values using approximate functional equations ("reflection principle"; cf. [13, Section 4.4]). But here, no functional equations are assumed. So, although we have a common basis, we must proceed differently. The main key will be the decomposition $g = g_+ - g_-$ (Proposition 2.2.1) of the function $g = g_\lambda(s,\, \chi,\, \mathfrak{f})$ defined by (1.1.1). Each of $g_+,\ g_-$ depends also on an auxiliary parameter $X \geqslant 1$. Roughly speaking, the "$-$" part will be estimated for each individual χ by using (A3); the larger the parameter X is, the stronger this estimate will be. The "$+$" part is an absolutely convergent Dirichlet series over the integral divisors D with $(D,\, \mathfrak{f}) = 1$, and the required average over χ can be estimated through the orthogonality relation for characters and by a careful treatment of the partial sum over $N(D) \ll N(\mathfrak{f})$ based on Proposition 2.1.1 below; the smaller the X is, the stronger this estimate. A careful choice of X depending on $N(\mathfrak{f})$ will lead us to the proof.

Throughout this section, we assume that $|P_\infty| = 1$; i.e., either (a) K is rational or imaginary quadratic and P_∞ consists only of the unique archimedean prime (NF case), or (b) $P_\infty = \{\mathfrak{p}_\infty\}$ for a given prime divisor $\mathfrak{p}_\infty$ (FF case).

2.1. Preliminaries. Notations being as in Section 1.1, for each $x \geqslant 1$ and an integral divisor $\mathfrak{f}$, let $n(c,\, \mathfrak{f}; x)$ for each $c \in G_\mathfrak{f}$ denote the number of integral divisors D of K with $N(D) \leqslant x$ satisfying $(D,\, \mathfrak{f}) = 1$ and $i_\mathfrak{f}(D) = c$. The point in the following Proposition 2.1.1 is that $\mathfrak{f}$ is *not fixed*.

Proposition 2.1.1. (i) *For any* $\mathfrak{f}$ *and* x,

$$\operatorname*{Max}_{c \in G_\mathfrak{f}} n(c,\, \mathfrak{f}; x) \ll 1 + N(\mathfrak{f})^{-1} x.$$

(ii) *There exists* $A = A_K > 0$ *such that for any* $\mathfrak{f}$ *and any* $x < A \cdot N(\mathfrak{f})$,

$$\operatorname*{Max}_{c \in G_\mathfrak{f}} n(c,\, \mathfrak{f}; x) \leqslant 1. \tag{2.1.2}$$

Proof. First, let K be a function field over $\mathbb{F}_q$. Then, since principal divisors have norm equal to 1, two integral divisors belonging to the same class c must have the equal norm (recall that integral divisors are coprime with $\mathfrak{p}_\infty$). Now, Proposition 3.3.16 of [10] asserts that the number of integral divisors D with the given norm $N(D) = q^m$ satisfying $(D,\, \mathfrak{f}) = 1$ and $i_\mathfrak{f}(D) = c$ cannot exceed $\mathrm{Max}(1,\, q^{m+1}/N(\mathfrak{f}))$. Therefore,

$$n(c,\, \mathfrak{f}; x) \leqslant \mathrm{Max}(1,\, qx/N(\mathfrak{f})) \ll 1 + N(\mathfrak{f})^{-1} x. \tag{2.1.3}$$

Moreover, if $q^m < N(\mathfrak{f})$ (so that $q^{m+1} \leqslant N(\mathfrak{f})$), there is at most one such D. Hence (ii) holds with $A_K = 1$.

When $K = \mathbb{Q}$ and $f \in \mathbb{N}$, $n(c, (f); x) \leqslant x/f + 1$; whence (i). Moreover, $n(c, (f); x) \leqslant 1$ when $x < f$; hence (ii) holds with $A_{\mathbb{Q}} = 1$.

Now let K be imaginary quadratic, with class number h. To prove (i), let $\mathfrak{A}_i$ $(1 \leqslant i \leqslant h)$ be a set of representatives of the ideal classes in K, and for each i $(1 \leqslant i \leqslant h)$, choose a fundamental domain (a parallelogram) Ω_i for the lattice $\mathfrak{A}_i$ embedded in $\mathbb{C}$. Then as a fundamental domain $\Omega_{\mathfrak{f}}$ for any divisor $\mathfrak{f} \neq (0)$, we may choose some complex scalar multiple of one of the Ω_i. For each i, the number of distinct translations of Ω_i (by an element of the lattice $\mathfrak{A}_i$) that meet the disk $\{|\xi|^2 \leqslant x\}$ is $\ll 1 + x$; hence the number of distinct translations of $\Omega_{\mathfrak{f}}$ (by an element of $\mathfrak{f}$) that meet $\{|\xi|^2 \leqslant x\}$ is $\ll 1 + x/N(\mathfrak{f})$. Now (i) follows easily from the finiteness of the unit group of K. As for (ii), suppose that D, D' are distinct integral divisors belonging to c such that $N(D') \leqslant N(D)$. This means that $D' = (\alpha)D$ with some $\alpha \equiv 1 \pmod{\mathfrak{f}}$, $\alpha \neq 1$, $N(\alpha) \leqslant 1$. The last inequality gives $N(\alpha - 1) \leqslant 2(N(\alpha) + 1) \leqslant 4$. On the other hand, $(\alpha - 1)D$ must be integral, $\neq (0)$, and divisible by $\mathfrak{f}$; hence $N(\alpha - 1)N(D) \geqslant N(\mathfrak{f})$. Therefore, $4 \geqslant N(\alpha - 1) \geqslant N(D)^{-1}N(\mathfrak{f})$; i.e., $N(D) \geqslant N(\mathfrak{f})/4$. Therefore, (ii) holds with $A_K = 1/4$. □

We shall also need the formula for the cardinality of the group $G_{\mathfrak{f}}$;

$$|G_{\mathfrak{f}}| = \delta_K h_K w_K^{-1} N(\mathfrak{f}) \prod_{\mathfrak{p}|\mathfrak{f}} (1 - N(\mathfrak{p})^{-1}), \tag{2.1.4}$$

where δ_K equals 1 in the NF case and equals $\deg \mathfrak{p}_\infty$ in the FF case, h_K is the class number of K, and w_K is the number of residue classes mod $\mathfrak{f}$ represented by some root of unity, except that it is 1 when $K = \mathbb{Q}$. Note that $w_K = q - 1$ in the FF case over $\mathbb{F}_q$. Since

$$\prod_{N(\mathfrak{p}) \leqslant y} (1 - N(\mathfrak{p})^{-1}) \gg (\log y)^{-1} \tag{2.1.5}$$

(cf. [10, Section 3.7] for a proof for the FF case), the above formula gives

$$N(\mathfrak{f})/\log N(\mathfrak{f}) \ll |G_{\mathfrak{f}}| \ll N(\mathfrak{f}). \tag{2.1.6}$$

2.2. The integral expression. The basic notations are as follows.

- ϵ: a fixed positive number $< 1/2$. The symbol $\ll$ will depend on ϵ, but this dependence will be suppressed from the notations.
- $s \in \mathbb{C}$ will always satisfy $\sigma := \mathrm{Re}(s) \geqslant 1/2 + \epsilon$;
- $\mathfrak{f}$: any integral divisor;
- X: a real parameter $\geqslant 1$. Later, we shall choose $X = N(\mathfrak{f})^\beta$, with $\beta = 1 + \epsilon/2$;
- Λ: a given uniformly admissible family of arithmetic functions;
- λ, $\lambda' \in \Lambda$; write $g = g_\lambda$, $g' = g_{\lambda'}$ (cf. Section 1.1).

Proposition 2.2.1. (i) *On the space* $\mathrm{Re}(s) \geqslant 1/2+\epsilon$, *one can express* $g = g(s, \chi, \mathfrak{f})$ $(\chi \in \hat{G}_{\mathfrak{f}} \backslash \{\chi_0\})$ *as the difference*

$$g = g_+ - g_- \tag{2.2.2}$$

of two holomorphic functions

$$g_+ = g_+(s,\,\chi,\,\mathfrak{f};X) = \frac{1}{2\pi i}\int_{\mathrm{Re}(w)=c}\Gamma(w)g(s+w,\,\chi,\,\mathfrak{f})X^w\,dw, \tag{2.2.3}$$

and

$$g_- = g_-(s,\,\chi,\,\mathfrak{f};X) = \frac{1}{2\pi i}\int_{\mathrm{Re}(w)=\epsilon'-\epsilon}\Gamma(w)g(s+w,\,\chi,\,\mathfrak{f})X^w\,dw, \tag{2.2.4}$$

where c and ϵ' are any positive real numbers satisfying $c > \mathrm{Max}(0,\,1-\sigma)$ and $0<\epsilon'<\epsilon$. Each of g_+ and g_- depends on the parameter X but not on c or ϵ'.

(ii) *g_+ has a Dirichlet series expansion*

$$g_+ = \sum_{(D,\mathfrak{f})=1}\chi(D)\lambda(D)\exp(-N(D)/X)N(D)^{-s}, \tag{2.2.5}$$

which is absolutely convergent for any $\chi\in\hat{G}_{\mathfrak{f}}$ and any $s\in\mathbb{C}$.

Proof. First, note that

$$g(s,\,\chi,\,\mathfrak{f}) = \frac{1}{2\pi i}\int_B\Gamma(w)g(s+w,\,\chi,\,\mathfrak{f})X^w\,dw \tag{2.2.6}$$

holds, where B is the positively oriented rectangle bordering

$$\epsilon'-\epsilon\leqslant\mathrm{Re}(w)\leqslant c,\qquad |\mathrm{Im}(w)|\leqslant T \tag{2.2.7}$$

($T>0$). This is clear, because the integrand is holomorphic in w on (2.2.7) except for a simple pole at $w=0$ with the residue $g(s,\,\chi,\,\mathfrak{f})$ (in fact, since $-1<\epsilon'-\epsilon<0$, the only pole of $\Gamma(w)$ on (2.2.7) is a simple pole at $w=0$ (with the residue 1), and since $\mathrm{Re}(s+w)\geqslant 1/2+\epsilon' > 1/2$, $g(s+w,\,\chi,\,\mathfrak{f})$ is holomorphic on (2.2.7), by (A2)).

To prove (i), let us estimate the integrand on $\epsilon'-\epsilon\leqslant\mathrm{Re}(w)\leqslant c$; $|\mathrm{Im}(w)|\geqslant T$. First, $|X^w|\leqslant X^c$ (because $X\geqslant 1$); secondly, in the FF case, $g(s+w,\,\chi,\,\mathfrak{f})$ is holomorphic and vertically periodic; hence bounded, and in the NF case, for each fixed s, $\mathfrak{f}$, χ,

$$|g(s+w,\,\chi,\,\mathfrak{f})|\ll\exp(C\log^2(|\mathrm{Im}(w)|+2)) \tag{2.2.8}$$

with some $C=C_{s,\chi,\mathfrak{f}}>0$, by (A3). Thirdly,

$$|\Gamma(w)|\ll|\mathrm{Im}(w)|^{c-1/2}\exp\Bigl(-\frac{\pi}{2}|\mathrm{Im}(w)|\Bigr) \tag{2.2.9}$$

for $|\mathrm{Im}(w)|\geqslant 1$ and $\mathrm{Re}(w)\leqslant c$. Now (i) follows directly from these by letting $T\to\infty$ in (2.2.6).

(ii) Since $\sigma+c>1$, the Dirichlet series expansion

$$g(s+w,\,\chi,\,\mathfrak{f}) = \sum_{(D,\mathfrak{f})=1}\chi(D)\lambda(D)N(D)^{-s-w} \tag{2.2.10}$$

is absolutely convergent on $\mathrm{Re}(w) = c$, and the convergence is uniform with respect to $\mathrm{Im}(w)$. Therefore,

$$\begin{aligned} g_+ &= g_+(s,\, \chi,\, \mathfrak{f}; X) \\ &= \frac{1}{2\pi i} \int_{\mathrm{Re}(w)=c} \Gamma(w) \biggl(\sum_{(D,\mathfrak{f})=1} \chi(D)\lambda(D)N(D)^{-s-w} \biggr) X^w \, dw \\ &= \sum_{(D,\mathfrak{f})=1} \chi(D)\lambda(D)N(D)^{-s} \biggl(\frac{1}{2\pi i} \int_{\mathrm{Re}(w)=c} \Gamma(w)N(D)^{-w}X^w \, dw \biggr). \end{aligned} \tag{2.2.11}$$

But since

$$\frac{1}{2\pi i} \int_{\mathrm{Re}(u)=c} \Gamma(u)a^{-u} du = e^{-a} \quad (a,\, c > 0), \tag{2.2.12}$$

we obtain the desired Dirichlet series expansion (2.2.5). Because of the exponential factor, this converges absolutely for any $s \in \mathbb{C}$ and any $\chi \in \hat{G}_{\mathfrak{f}}$. This can be seen easily by noting that $\lambda(D) \ll N(D)$, and that the number of D with $N(D) = n$ is certainly $\ll n$. □

We define $g_+(s,\, \chi,\, \mathfrak{f}; X)$ for any $\chi \in \hat{G}_{\mathfrak{f}}$ *including* $\chi = \chi_0$, by (2.2.5).

Proposition 2.2.13. *Let* $\sigma = \mathrm{Re}(s) \geqslant 1/2 + \epsilon$. *Then*

(i) *For any* $\epsilon' > 0$ *and* $\chi \in \hat{G}_{\mathfrak{f}}$,

$$|g_+(s,\, \chi,\, \mathfrak{f}; X)| \ll_{\epsilon'} X^{1/2+\epsilon'-\epsilon}. \tag{2.2.14}$$

(ii) *For any* ϵ' $(0 < \epsilon' < \epsilon)$, $T > 0$, *and for* $|\mathrm{Im}(s)| \leqslant T$, $\chi \in \hat{G}_{\mathfrak{f}} \backslash \{\chi_0\}$,

$$|g_-(s,\, \chi,\, \mathfrak{f}; X)| \ll_{\epsilon',T} (N(\mathfrak{f})X)^{\epsilon'} X^{-\epsilon}. \tag{2.2.15}$$

Proof. (i) Since

$$g_+(s,\, \chi,\, \mathfrak{f}; X) = \sum_{(D,\mathfrak{f})=1} \chi(D)\lambda(D) \exp(-N(D)/X)N(D)^{-s}, \tag{2.2.16}$$

we have, by (A1),

$$\begin{aligned} |g_+(s,\, \chi,\, \mathfrak{f}; X)| &\ll \sum_{(D,\mathfrak{f})=1} N(D)^{\epsilon'} \exp(-N(D)/X)N(D)^{-\sigma} \\ &\leqslant \sum_{n=1}^{\infty} a(n)n^{\epsilon'-(1/2+\epsilon)}e^{-n/X}, \end{aligned} \tag{2.2.17}$$

where $a(n)$ denotes the number of D with $N(D) = n$. But since $\sum_{n \leqslant x} a(n) \ll x$ for any $x \geqslant 1$, we obtain, by partial summation,

$$|g_+(s,\, \chi,\, \mathfrak{f}; X)| \ll \int_1^{\infty} t|f'(t)|dt, \tag{2.2.18}$$

where $f(t) = t^{-a}e^{-t/X}$, with $a = (1/2+\epsilon)-\epsilon'$. But $f'(t)/f(t) = -(X^{-1}+at^{-1})$; hence $t|f'(t)| \ll_{\epsilon'} (X^{-1}t+1)f(t)$; hence

$$|g_+(s,\,\chi,\,\mathfrak{f};X)| \ll_{\epsilon'} \int_1^\infty (X^{-1}t+1)t^{-a}e^{-t/X}\,dt = X^{1-a}\int_{1/X}^\infty (u+1)u^{-a}e^{-u}\,du$$
$$\ll X^{1-a}(\Gamma(2-a)+\Gamma(1-a)) \ll X^{1-a} = X^{1/2+\epsilon'-\epsilon}.$$

This settles (i).

(ii) By definition,

$$g_-(s,\,\chi,\,\mathfrak{f};X) = \frac{1}{2\pi i}\int_{\mathrm{Re}(w)=\epsilon'-\epsilon} \Gamma(w)g(s+w,\,\chi,\,\mathfrak{f})X^w\,dw. \tag{2.2.19}$$

Since $\mathrm{Re}(w)=\epsilon'-\epsilon$, we have $|X^w| = X^{\epsilon'-\epsilon}$, and

$$\Gamma(w) \ll \exp\Bigl(-\frac{\pi}{2}|\mathrm{Im}(w)|\Bigr). \tag{2.2.20}$$

In the FF case, since $\mathrm{Re}(s+w) \geqslant 1/2+\epsilon'$, we have, by (A3),

$$|g(s+w,\,\chi,\,\mathfrak{f})| \ll_{\varepsilon',\epsilon''} N(\mathfrak{f})^{\epsilon''}; \tag{2.2.21}$$

for any $\epsilon',\,\epsilon''>0$; in particular, for $\epsilon''=\epsilon'$; whence (2.2.15).

In the NF case, the situation is more complicated. Put $\mathrm{Im}(w)=u$, so that $\mathrm{Im}(s+w) = t+u$. Then by (1.1.3) (since $\mathrm{Re}(s+w)\geqslant 1/2+\epsilon'$) there exists $C=C_{\epsilon'}>0$ such that

$$|g(s+w,\,\chi,\,\mathfrak{f})| \leqslant \exp\bigl\{C(\ell(t+u)\ell(\mathfrak{f})^{1-2\epsilon'}+\ell(t+u)^2)\bigr\}. \tag{2.2.22}$$

But since $|t+u|+2 \leqslant (|t|+2)(|u|+1)$, we may replace $\ell(t+u)$ by $\ell(t)+\log(|u|+1)$; hence also $\ell(t+u)^2$ by $2(\ell(t)^2+\log^2(|u|+1))$. Therefore, there exists $C'=C'_{\epsilon',T}>0$ such that when $|t|\leqslant T$,

$$|g(s+w,\,\chi,\,\mathfrak{f})| \leqslant \exp(C'\ell(\mathfrak{f})^{1-2\epsilon'})\exp\bigl\{C'(\ell(\mathfrak{f})^{1-2\epsilon'}\log(|u|+1)+\log^2(|u|+1))\bigr\}, \tag{2.2.23}$$

which, together with (2.2.19)(2.2.20)(and $e^{-\pi u/2}\leqslant e^{-u}$), gives

$$|g_-(s,\,\chi,\,\mathfrak{f};X)| \ll X^{\epsilon'-\epsilon}\exp(C'\ell(\mathfrak{f})^{1-2\epsilon'})\int_0^\infty e^{-u}(u+1)^{C'\ell(\mathfrak{f})^{1-2\epsilon'}}e^{C'\log^2(u+1)}\,du. \tag{2.2.24}$$

By using the Schwarz inequality

$$\left(\int_0^\infty f_1(u)f_2(u)\,du\right)^2 \leqslant \left(\int_0^\infty f_1(u)^2\,du\right)\left(\int_0^\infty f_2(u)^2\,du\right) \tag{2.2.25}$$

for $f_1(u) = e^{-u/2}(u+1)^{C'\ell(\mathfrak{f})^{1-2\epsilon'}}$, $f_2(u) = e^{-u/2}e^{C'\log^2(u+1)}$, and by noting that the integral of $f_2(u)^2du$ for this case is $\ll_{\epsilon',T} 1$, we obtain

$$|g_-(s,\,\chi,\,\mathfrak{f};X)| \ll_{\epsilon',T} X^{\epsilon'-\epsilon}\exp(C'\ell(\mathfrak{f})^{1-2\epsilon'})\left(\int_0^\infty e^{-u}(u+1)^{2C'\ell(\mathfrak{f})^{1-2\epsilon'}}\,du\right)^{1/2}.$$

By putting $u+1=v$ and comparing the integral with the Γ-integral, we obtain

$$\begin{aligned}|g_-(s,\chi,\mathfrak{f};X)| &\ll_{\epsilon',T} X^{\epsilon'-\epsilon}\exp(C'\ell(\mathfrak{f})^{1-2\epsilon'})\Gamma(2C'\ell(\mathfrak{f})^{1-2\epsilon'}+1)^{1/2}\\ &\ll X^{\epsilon'-\epsilon}\exp(C'\ell(\mathfrak{f})^{1-2\epsilon'})\exp(C'\ell(\mathfrak{f})^{1-2\epsilon'}\log(2C'\ell(\mathfrak{f})^{1-2\epsilon'}))\\ &\ll X^{\epsilon'-\epsilon}\exp(C''\ell(\mathfrak{f})^{1-2\epsilon'}\log\ell(\mathfrak{f})),\end{aligned}$$

with some $C''=C''_{\epsilon',T}>0$. But $C''\ell(\mathfrak{f})^{-2\epsilon'}\log(\ell(\mathfrak{f}))<\epsilon'$ holds for $N(\mathfrak{f})$ sufficiently large (depending on ϵ', T). Hence this is

$$\ll_{\epsilon',T} X^{\epsilon'-\epsilon}\exp(\epsilon'\ell(\mathfrak{f}))=X^{\epsilon'-\epsilon}(N(\mathfrak{f})+2)^{\epsilon'}\ll X^{\epsilon'-\epsilon}N(\mathfrak{f})^{\epsilon'}.$$

This settles the proof of (ii) also in the NF case. □

Remark 2.2.26. As can be seen from the proofs of Proposition 2.2.1(i) and Proposition 2.2.13(ii) (which are the only places where the assumption (A3) is used), the second term $\ell(t)^2$ on the right-hand side of (1.1.3) may be replaced by any non-negative valued continuous even function $f(t)$ on $\mathbb{R}$ satisfying $f(t+t')\ll f(t)+f(t')$ and $\lim_{t\to\infty} f(t)/t=0$.

2.3. Study of $\mathrm{Avg}_{\chi\in\hat{G}_\mathfrak{f}}\big(\overline{g_+(\chi)}g'_+(\chi)\big)$. This average will give the main term of $\mathrm{Avg}_{\chi\in\hat{G}_\mathfrak{f}\setminus\{\chi_0\}}\big(\overline{g(\chi)}g'(\chi)\big)$, and this estimation depends only on the property (A1) of the admissible family. Here, and in what follows in this subsection, we shall suppress from the notations the dependence on s, $\mathfrak{f}$, X. Thus,

$$g_+(\chi)=\sum_{(D,\mathfrak{f})=1}\chi(D)\lambda(D)\exp(-N(D)/X)N(D)^{-s}, \tag{2.3.1}$$

$$g'_+(\chi)=\sum_{(D,\mathfrak{f})=1}\chi(D)\lambda'(D)\exp(-N(D)/X)N(D)^{-s} \tag{2.3.2}$$

($\chi\in\hat{G}_\mathfrak{f}$). The orthogonality relation for characters gives directly

$$S:=\mathrm{Avg}_{\chi\in\hat{G}_\mathfrak{f}}(\overline{g_+(\chi)}g'_+(\chi))=\sum_{c\in G_\mathfrak{f}}\overline{T(c)}T'(c), \tag{2.3.3}$$

where

$$\begin{aligned}T(c)&=\sum_{i_\mathfrak{f}(D)=c}\lambda(D)\exp(-N(D)/X)N(D)^{-s},\\ T'(c)&=\sum_{i_\mathfrak{f}(D)=c}\lambda'(D)\exp(-N(D)/X)N(D)^{-s}.\end{aligned} \tag{2.3.4}$$

Now we shall make full use of Proposition 2.1.1. Let $A=A_K>0$ be as in Proposition 2.1.1(ii), and decompose as $T(c)=T_1(c)+T_2(c)$, where $T_1(c)$ (resp. $T_2(c)$) denotes the partial sum over $N(D)<A\cdot N(\mathfrak{f})$ (resp. $N(D)\geqslant A\cdot N(\mathfrak{f})$). Define $T'_i(c)$ ($i=1,2$) similarly. By definition, the sum for $T_1(c)$ has at most *one* term. Call $c\in G_\mathfrak{f}$ *small* when there exists an integral divisor D such that $i_\mathfrak{f}(D)=c$ and $N(D)<A\cdot N(\mathfrak{f})$. In this case, call D_c the unique such D. Thus,

$$T_1(c)=\begin{cases}\lambda(D_c)\exp(-N(D_c)/X)N(D_c)^{-s}, & c\text{ small},\\ 0 & \text{otherwise.}\end{cases}$$

Since $c \mapsto D_c$ gives a bijection between small classes in $G_{\mathfrak{f}}$ and integral divisors D satisfying $(D, \mathfrak{f}) = 1$ and $N(D) < A \cdot N(\mathfrak{f})$, we obtain

$$S_1 := \sum_{c \in G_{\mathfrak{f}}} \overline{T_1(c)}\, T_1'(c) = \sum_{\substack{(D,\mathfrak{f})=1 \\ N(D)<AN(\mathfrak{f})}} \overline{\lambda(D)}\, \lambda'(D) \exp(-2N(D)/X) N(D)^{-2\sigma}. \tag{2.3.5}$$

Note that

$$S_1 \ll \sum_D N(D)^{\epsilon-2\sigma} \ll \sum_D N(D)^{-1-\epsilon} \ll 1. \tag{2.3.6}$$

As for

$$T_2(c) = \sum_{\substack{i_{\mathfrak{f}}(D)=c \\ N(D)\geqslant AN(\mathfrak{f})}} \lambda(D) \exp(-N(D)/X) N(D)^{-s}, \tag{2.3.7}$$

we shall prove

$$T_2(c) \ll_{\epsilon'} N(\mathfrak{f})^{-1} X^{1/2+\epsilon'-\epsilon} \tag{2.3.8}$$

for any $\epsilon' > 0$. Since $\lambda(D) \ll N(D)^{\epsilon'}$, we have

$$T_2(c) \ll \sum_{n \geqslant AN(\mathfrak{f})} a_c(n) n^{\epsilon'-\sigma} e^{-n/X}, \tag{2.3.9}$$

where $a_c(n)$ denotes the number of D satisfying $N(D) = n$ and $i_{\mathfrak{f}}(D) = c$. But since $\sum_{n \leqslant x} a_c(n) \ll N(\mathfrak{f})^{-1} x$ for $x \geqslant A \cdot N(\mathfrak{f})$ by Proposition 2.1.1(i), we obtain (2.3.8) exactly by the same argument as in the proof of Proposition 2.2.13(i). Therefore, by (2.1.6), $E_2 := \sum_{c \in G_{\mathfrak{f}}} |T_2(c)|^2$, $E_2' := \sum_{c \in G_{\mathfrak{f}}} |T_2'(c)|^2$ satisfy

$$E_2,\, E_2' \ll |G_{\mathfrak{f}}| N(\mathfrak{f})^{-2} X^{1-2(\epsilon-\epsilon')} \ll N(\mathfrak{f})^{-1} X^{1-2(\epsilon-\epsilon')}. \tag{2.3.10}$$

Therefore, by (2.3.6) for $(T_1 = T_1')$, (2.3.10), and by the Schwarz inequality, we obtain

$$\begin{aligned} S - S_1 &= \sum_{c \in G_{\mathfrak{f}}} \left(\overline{(T_1(c) + T_2(c))} (T_1'(c) + T_2'(c)) - \overline{T_1(c)} T_1'(c) \right) \\ &\ll (N(\mathfrak{f})^{-1} X^{\alpha})^{1/2} + N(\mathfrak{f})^{-1} X^{\alpha}, \end{aligned} \tag{2.3.11}$$

where $\alpha = 1 - 2(\epsilon - \epsilon') > 0$. We shall choose

$$X = N(\mathfrak{f})^{\beta}, \quad \text{with } 0 < \beta < \alpha^{-1}, \tag{2.3.12}$$

so that $N(\mathfrak{f})^{-1} X^{\alpha}$ is a negative power of $N(\mathfrak{f})$; hence

$$S - S_1 \ll (N(\mathfrak{f})^{-1} X^{\alpha})^{1/2} = N(\mathfrak{f})^{(-1+\alpha\beta)/2}. \tag{2.3.13}$$

We shall now treat the difference between S_1 and

$$S_0 = \sum_{(D,\mathfrak{f})=1} \overline{\lambda(D)} \lambda'(D) N(D)^{-2\sigma} \tag{2.3.14}$$

(the second term in (1.1.7) in Theorem 1). By the definitions of S_1, S_0, we have $S_0 - S_1 = E + E'$, with

$$\begin{aligned} E &= \sum_{\substack{(D,\mathfrak{f})=1 \\ N(D)\geqslant AN(\mathfrak{f})}} \overline{\lambda(D)}\,\lambda'(D)N(D)^{-2\sigma}, \\ E' &= \sum_{\substack{(D,\mathfrak{f})=1 \\ N(D)<AN(\mathfrak{f})}} \overline{\lambda(D)}\,\lambda'(D)\,(1-\exp(-2N(D)/X))\,N(D)^{-2\sigma}. \end{aligned} \tag{2.3.15}$$

As for E,

$$E \ll \sum_{N(D)\geqslant AN(\mathfrak{f})} N(D)^{\epsilon-2\sigma} \leqslant \sum_{N(D)\geqslant AN(\mathfrak{f})} N(D)^{-1-\epsilon}. \tag{2.3.16}$$

But since the number of D with norm $\leqslant x$ is $\ll x$, this gives

$$E \ll \int_{AN(\mathfrak{f})}^{\infty} t|d(t^{-1-\epsilon})/dt|\,dt \ll N(\mathfrak{f})^{-\epsilon}. \tag{2.3.17}$$

As for E', since $0 < 1-\exp(-a) < a$ holds for any $a > 0$,

$$\begin{aligned} E' &\ll \sum_{N(D)<AN(\mathfrak{f})} N(D)^{\epsilon-2\sigma}(1-\exp(-2N(D)/X)) \\ &< 2AN(\mathfrak{f})X^{-1}\sum_{D} N(D)^{-1-\epsilon} \ll N(\mathfrak{f})X^{-1}; \end{aligned} \tag{2.3.18}$$

hence for the above choice of X we have $E' \ll N(\mathfrak{f})^{1-\beta}$; hence

$$S_0 - S_1 \ll N(\mathfrak{f})^{-\epsilon} + N(\mathfrak{f})^{1-\beta}. \tag{2.3.19}$$

Therefore, combining this with (2.3.13) we obtain (for the above choice of X)

$$S - S_0 \ll N(\mathfrak{f})^{(-1+\alpha\beta)/2} + N(\mathfrak{f})^{-\epsilon} + N(\mathfrak{f})^{1-\beta}. \tag{2.3.20}$$

Now the question is how to choose $\beta > 0$ so that all the exponents of $N(\mathfrak{f})$ on the right-hand side of (2.3.20) are negative and the minimal of their absolute values is large enough. One of such choices is where $\epsilon' = \epsilon/4$, $\beta = 1+\epsilon/2$, in which case $\alpha = 1-(3/2)\epsilon$, and the three exponents are

$$(-\epsilon-(3/4)\epsilon^2)/2, \quad -\epsilon, \quad -\epsilon/2;$$

hence

$$S - S_0 \ll N(\mathfrak{f})^{-\epsilon/2}. \tag{2.3.21}$$

(We shall see in Section 2.5 that this choice of β is appropriate also for the estimation of the counterpart related to $g_-(\chi)$.)

2.4. Differences between modified averages. We now compare the averages of $\overline{g_+(\chi)}\,g'_+(\chi)$ over the whole group $\chi \in \hat{G}_{\mathfrak{f}}$, with that over the complement of χ_0, and when $\mathfrak{f}$ is a prime divisor, also with that over $\{\chi; \mathfrak{f}_\chi = \mathfrak{f}\}$ (Note that when the class number is greater than one, there can be non-principal characters with the conductor (1).) It is easy to see that these differences are

$$\ll \frac{1}{|G_{\mathfrak{f}}|}\left(\operatorname*{Max}_{\chi\in\hat{G}_{\mathfrak{f}}}|g_+(\chi)|\operatorname*{Max}_{\chi\in\hat{G}_{\mathfrak{f}}}|g'_+(\chi)|\right). \tag{2.4.1}$$

Hence by Proposition 2.2.13(i) and by (2.1.6), this is

$$\ll |G_{\mathfrak{f}}|^{-1} X^{\alpha} \ll (\log N(\mathfrak{f})) N(\mathfrak{f})^{-1+\alpha\beta} \ll N(\mathfrak{f})^{(-1+\alpha\beta)/2} \ll N(\mathfrak{f})^{-\epsilon/2}.$$

Therefore, by combining this with the main estimation (2.3.21) of the previous subsection, we obtain

$$\mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} (\overline{g_+(\chi)}\, g'_+(\chi)) - S_0 \ll N(\mathfrak{f})^{-\epsilon/2}, \tag{2.4.2}$$

together with the additional statement that when $\mathfrak{f}$ is a prime divisor, the average may be replaced by that over $\{\chi; \mathfrak{f}_\chi = \mathfrak{f}\}$.

2.5. Final stage of the proof. It remains to estimate the difference

$$\mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} (\overline{g(\chi)}\, g'(\chi)) - \mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} (\overline{g_+(\chi)} g'_+(\chi)). \tag{2.5.1}$$

Recall that $g = g_+ - g_-$, $g' = g'_+ - g'_-$. But

$$\mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} |g_+(\chi)|^2, \quad \mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} |g'_+(\chi)|^2 \ll 1, \tag{2.5.2}$$

because of $S \ll 1$ (which follows from (2.3.6), (2.3.13)) and of the above estimations in Section 2.4. On the other hand, by Proposition 2.2.13(ii) (since $X^{-2\epsilon} \leqslant X^{-\epsilon}$),

$$\mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} |g_-(\chi)|^2, \quad \mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} |g'_-(\chi)|^2 \ll (N(\mathfrak{f})X)^{2\epsilon''} X^{-\epsilon} \tag{2.5.3}$$

for any $\epsilon'' > 0$. Hence if we choose ϵ'' so small that $2\epsilon''(1+\beta) \leqslant \epsilon(\beta - 1)$, which is possible since $\beta > 1$, we obtain

$$\mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} |g_-(\chi)|^2, \quad \mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} |g'_-(\chi)|^2 \ll N(\mathfrak{f})^{-\epsilon}. \tag{2.5.4}$$

Therefore, by the Schwarz inequality, (2.5.1) is $\ll N(\mathfrak{f})^{-\epsilon/2}$. Therefore, together with (2.4.2) we obtain

$$\mathrm{Avg}_{\chi \in \hat{G}_{\mathfrak{f}} \setminus \{\chi_0\}} (\overline{g(\chi)} g'(\chi)) - S_0 \ll N(\mathfrak{f})^{-\epsilon/2}. \tag{2.5.5}$$

This settles the proof of the first statement of Theorem 1.

When $\mathfrak{f}$ is a prime divisor, the above average may be replaced by that over $\{\chi;\, \mathfrak{f}_\chi = \mathfrak{f}\}$. Moreover, the sum S_0, which is over $(D,\, \mathfrak{f}) = 1$, and the corresponding sum over all (integral divisors) D differ only by a quantity $\ll N(\mathfrak{f})^{\epsilon' - 2\sigma} \ll N(\mathfrak{f})^{-1}$. Thus, the proof of the second statement of Theorem 1 is also settled.

3. Proof of Theorem 2

The substantial part in the proof of Theorem 2 (Section 1.2) is that of (ii), especially that of (A3) in the NF-case, to which all but the first two subsections will be devoted. What we need is a sufficiently strong *absolute* GRH-bound of $|\mathcal{L}(s,\, \chi,\, \mathfrak{f})|$ on $\sigma \geqslant 1/2 + \epsilon$ with an explicit description of the dependence on $N(\mathfrak{f})$ and $|\mathrm{Im}(s)|$ (Corollary-Est in Section 3.3). We shall first reduce this to a "universal" estimate on $\mathrm{Re}(s) \geqslant 1/2 + \epsilon$ for $|L'/L(s,\, \chi,\, \mathfrak{f})|$ (Theorem-Est in Section 3.3), which will then be proved by using one of the "explicit formulas" (Theorem-Exp in Section 3.5).

Throughout Section 3, K and P_∞ are general, i.e., as at the beginning of Section 1.1.

3.1. Estimations of $\lambda_z(D)$. Recall the definition of $\lambda_z(D)$ given in Theorem 2. Let z run only over $|z| \leqslant R$. We shall prove that

$$\lambda_z(D) \ll_{R,\epsilon'} N(D)^{\epsilon'} \tag{3.1.1}$$

holds for any $\epsilon' > 0$. First, since $H_r(x)$, $G_r(x)$ (Section 1.2) are polynomials with positive coefficients and since $\delta_k(r) \leqslant \binom{r-1}{k-1}$, we have

$$|H_r(iz/2)| \leqslant H_r(|z|/2) \leqslant G_r(|z|/2) \leqslant G_r(|z| \log N(\mathfrak{p})). \tag{3.1.2}$$

But by [8] Sublemma 3.10.1 and $\binom{r-1}{k-1} \leqslant \binom{r}{k}$, we have $G_r(x) \leqslant \exp(2\sqrt{rx})$ $(x \geqslant 0)$; hence

$$|\lambda_z(\mathfrak{p}^r)| \leqslant G_r(|z| \log N(\mathfrak{p})) \leqslant \exp(2\sqrt{r|z|\log N(\mathfrak{p})})$$

holds in both Cases 1,2. Now since we may assume $\lambda_z(D) \neq 0$ in proving (3.1.1), we may take the logarithm of $|\lambda_z(D)|$ for estimation. Denoting by $\mathrm{Supp}(D)$ the set of prime factors of D, we obtain

$$\begin{aligned} \log|\lambda_z(D)| &\leqslant 2\sqrt{|z|} \sum_{\mathfrak{p}|D} \sqrt{r_{\mathfrak{p}} \log N(\mathfrak{p})} \leqslant 2\sqrt{|z|}\sqrt{|\mathrm{Supp}(D)|}\sqrt{\sum_{\mathfrak{p}|D} r_{\mathfrak{p}} \log N(\mathfrak{p})} \\ &\leqslant 2\sqrt{R}\sqrt{|\mathrm{Supp}(D)| \log N(D)}. \end{aligned} \tag{3.1.3}$$

(The second inequality is by the Schwarz inequality.) On the other hand, we have

$$|\mathrm{Supp}(D)| \ll \frac{\log N(D)}{\log\log N(D) + 2} \tag{3.1.4}$$

[8, Sublemma 3.10.5]. Therefore, (3.1.3) gives

$$\log|\lambda_z(D)| \ll_R \frac{\log N(D)}{\sqrt{\log\log N(D) + 2}}. \tag{3.1.5}$$

Therefore, for any $\epsilon' > 0$, $\log|\lambda_z(D)| \leqslant \epsilon' \log N(D)$, if $N(D) \gg_{\epsilon',R} 1$. This proves (3.1.1) and hence the statement (i) of Theorem 2.

3.2. The function $g_{\lambda_z}(s, \chi, \mathfrak{f})$. To prove (ii), first we note that

$$\exp\left(\frac{iz}{2}\mathcal{L}(s, \chi, \mathfrak{f})\right) = \sum_{(D,\mathfrak{f})=1} \chi(D)\lambda_z(D)N(D)^{-s} \tag{3.2.1}$$

holds on $\mathrm{Re}(s) > 1$ for any $\chi \in \hat{G}_{\mathfrak{f}}$. Indeed, on this domain, each side of (3.2.1) has an absolutely convergent Euler product decomposition, and the equality between their $\mathfrak{p}$-components is given by the equality (1.2.4) with $x = -\frac{iz}{2}\log N(\mathfrak{p})$, $t = \chi(\mathfrak{p})N(\mathfrak{p})^{-s}$ in Case 1, and by (1.2.5) with $x = \frac{iz}{2}$, $t = \chi(\mathfrak{p})N(\mathfrak{p})^{-s}$ in Case 2. And when $\chi \neq \chi_0$, the left-hand side of (3.2.1) is holomorphic on $\mathrm{Re}(s) > 1/2$ under GRH. This settles the proof of (A2) for $\lambda = \lambda_z$ and of the equality

$$g_{\lambda_z}(s, \chi, \mathfrak{f}) = \exp\left(\frac{iz}{2}\mathcal{L}(s, \chi, \mathfrak{f})\right). \tag{3.2.2}$$

Now we are going to prove (A3) in several steps.

3.3. Reduction of A3 to Theorem-Est. The property (A3) will be proved as a Corollary of the following estimation theorem.

Theorem-Est. *Let $\chi \in \hat{G}_{\mathfrak{f}} \backslash \{\chi_0\}$, and $s = \sigma + ti$, with $\sigma \geqslant 1/2 + \epsilon$ $(\epsilon > 0)$. Then*

$$\left|\frac{L'}{L}(s,\,\chi,\,\mathfrak{f})\right| \ll_\epsilon \frac{\ell(\mathfrak{f})^{2-2\sigma}-1}{1-\sigma} \qquad \text{(FF)}$$

$$\ll_\epsilon \frac{\ell(\mathfrak{f})^{2-2\sigma}-1}{1-\sigma}\left(\ell(t)+\frac{\ell(t)^2}{\ell(\mathfrak{f})}\right) \qquad \text{(NF; under GRH).}$$

When $\sigma = 1$, $(\ell(\mathfrak{f})^{2-2\sigma}-1)/(1-\sigma)$ should be replaced by its limit at $\sigma = 1$; namely by $2\log\ell(\mathfrak{f})$.

In [14, Theorem 5.17], a similar GRH-bound for $|L'/L(f,\,s)|$ is given for a wide class of L-functions called $L(f,\,s)$.[1] Unfortunately, this class does not contain our L-functions for the FF-case (the Euler products have essentially different features), nor does it contain L-functions for the NF-case for imprimitive characters χ. Reduction of Theorem-Est (or even the Key Lemma in Section 3.4) to [14, Section 5.7] would require several minor but tedious arguments. So, we shall keep our original line of proof, despite the fact that it is also of a standard nature.

Corollary-Est. *Let $\mathcal{L}(s,\,\chi,\,\mathfrak{f})$ be either $L'/L(s,\,\chi,\,\mathfrak{f})$* (Case 1) *or $\log L(s,\,\chi,\,\mathfrak{f})$* (Case 2), *and for any $0 < \epsilon < 1/2$, let $\sigma \geqslant 1/2 + \epsilon$. Then*

$$|\mathcal{L}(s,\,\chi,\,\mathfrak{f})| \ll_\epsilon \ell(\mathfrak{f})^{1-2\epsilon} \qquad \text{(FF)}$$

$$\ll_\epsilon \ell(\mathfrak{f})^{1-2\epsilon}\left(\ell(t)+\frac{\ell(t)^2}{\ell(\mathfrak{f})}\right) \qquad \text{(NF; under GRH).}$$

That λ_z $(|z| \leqslant R)$ satisfies (A3) uniformly follows directly from (3.2.2) and this corollary.

Reduction of Corollary-Est to Theorem-Est. (Case 1) For each $y > 1$,

$$\frac{y^{1-\sigma}-1}{1-\sigma} = \int_1^y u^{-\sigma}du \tag{3.3.1}$$

is monotone decreasing with σ. Therefore,

$$\frac{\ell(\mathfrak{f})^{2-2\sigma}-1}{1-\sigma} \leqslant \frac{\ell(\mathfrak{f})^{1-2\epsilon}-1}{1-(1/2+\epsilon)}. \tag{3.3.2}$$

But since the right-hand side of (3.3.2) is $\ll_\epsilon \ell(\mathfrak{f})^{1-2\epsilon}$, the corollary for Case 1 follows immediately from Theorem-Est.

(Case 2) Put $\sigma_0 := \mathrm{Max}(\sigma,\,2)$. Then

$$\log L(s,\,\chi,\,\mathfrak{f}) = \int_{\sigma_0+ti}^{\sigma+ti} \frac{L'}{L}(s,\,\chi,\,\mathfrak{f})\,ds + \log L(\sigma_0+ti,\,\chi,\,\mathfrak{f}). \tag{3.3.3}$$

Since $|\log L(\sigma_0+ti,\,\chi,\,\mathfrak{f})| \leqslant |\log\zeta_K(2)| \ll_K 1$ (by the Dirichlet series expansion for $\log L(s,\,\chi,\,\mathfrak{f})$), where $\zeta_K(s)$ denotes the Dedekind zeta function of K without the P_∞-components, and since $|\sigma-\sigma_0| < 2-1/2 \ll 1$, the Corollary for Case 2 follows immediately from that for Case 1 by estimation of the integrand. □

[1]The authors thank the referee for this information.

Thus, Theorem 2 is reduced to Theorem-Est.

3.4. Reduction of Theorem-Est to a Key Lemma. As usual, for any integral divisor D of K, let $\Lambda(D) = \log N(\mathfrak{p})$ when $D = \mathfrak{p}^r$ for some prime divisor $\mathfrak{p}$ and $r \geqslant 1$, and $\Lambda(D) = 0$ otherwise. For $y > 1$ and $\chi \in \hat{G}_{\mathfrak{f}}$, put

$$\psi(s,\,\chi,\,\mathfrak{f};<y) = \sum_{N(D)<y} \chi(D)\Lambda(D)N(D)^{-s}, \tag{3.4.1}$$

$$\psi(s,\,\chi,\,\mathfrak{f};y) = \psi(s,\,\chi,\,\mathfrak{f};<y) + \frac{1}{2}\sum_{N(D)=y} \chi(D)\Lambda(D)N(D)^{-s}. \tag{3.4.2}$$

When $\mathfrak{f} = \mathfrak{f}_\chi$ (the conductor of χ), we shall suppress $\mathfrak{f}$ from these notations and write as $\psi(s,\,\chi;<y)$, $\psi(s,\,\chi;y)$. We shall assume hereafter that y is *separated from* 1, i.e., $1 - y^{-1} \gg 1$. Then we have

$$\frac{y^{1-\sigma}-1}{1-\sigma} = \int_1^y u^{-\sigma}du \geqslant (y-1)y^{-\sigma} \gg y^{1-\sigma}, \tag{3.4.3}$$

and also an elementary unconditional estimate (cf. [8](6.4.9)):

$$|\psi(s,\,\chi,\,\mathfrak{f};y)| \leqslant \sum_{N(\mathfrak{p})\leqslant y} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^\sigma - 1} \ll \int_1^y u^{-\sigma}du + y^{1-\sigma} \ll \frac{y^{1-\sigma}-1}{1-\sigma}. \tag{3.4.4}$$

We shall reduce the proof of Theorem-Est to the following

Key Lemma. *For $s = \sigma + ti$ with $\sigma \geqslant 1/2 + \epsilon$ and for $y > 1$ separated from 1,*

$$\left|\frac{L'}{L}(s,\,\chi) + \psi(s,\,\chi;y)\right| \ll_\epsilon y^{1/2-\sigma}\ell(\mathfrak{f}) \qquad \text{(FF)},$$

$$\ll_\epsilon y^{1/2-\sigma}(\ell(\mathfrak{f})\ell(t) + \ell(t)^2) + y^{1-\sigma} \qquad \text{(NF; under GRH)}.$$

(The term $y^{1-\sigma}$ can be replaced by a term which tends to 0 as $y \to \infty$ whenever $\sigma > 1/2$; but this is more complicated and not necessary for the present purpose.)

Now this reduction can be done by using the following intermediate objects and the decomposition, for a suitable choice of y.

$$\left|\frac{L'}{L}(s,\,\chi,\,\mathfrak{f})\right| \leqslant \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \tag{3.4.5}$$

where

$$\mathrm{I} = \left|\frac{L'}{L}(s,\,\chi,\,\mathfrak{f}) - \frac{L'}{L}(s,\,\chi)\right|, \quad \mathrm{II} = \left|\frac{L'}{L}(s,\,\chi) + \psi(s,\,\chi;y)\right|,$$
$$\mathrm{III} = |\psi(s,\,\chi,\,\mathfrak{f};y) - \psi(s,\,\chi;y)|, \quad \mathrm{IV} = |\psi(s,\,\chi,\,\mathfrak{f};y)|.$$

Firstly, (3.4.4) gives

$$\mathrm{IV} \ll \frac{y^{1-\sigma}-1}{1-\sigma}. \tag{3.4.6}$$

Secondly, I and III are also minor terms obviously bounded by

$$\sum_{\mathfrak{p}|\mathfrak{f}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^\sigma - 1} = \mathrm{I}_1 + \mathrm{I}_2, \tag{3.4.7}$$

where I_1 (resp. I_2) are the partial sums over $N(\mathfrak{p}) \leqslant \ell(\mathfrak{f})^2$ (resp. $N(\mathfrak{p}) > \ell(\mathfrak{f})^2$). By (3.4.4) we have

$$\mathrm{I}_1 \ll \frac{\ell(\mathfrak{f})^{2-2\sigma} - 1}{1-\sigma}. \tag{3.4.8}$$

As for I_2, since $(\log y)/(y^\sigma - 1)$ is monotone decreasing for $y > 1$, and since $\sum_{\mathfrak{p}|\mathfrak{f}} 1 \ll \ell(\mathfrak{f})/\log \ell(\mathfrak{f})$ by [8, Sublemma 3.10.5] we have

$$\mathrm{I}_2 \leqslant \frac{2\log \ell(\mathfrak{f})}{\ell(\mathfrak{f})^{2\sigma} - 1} \sum_{\mathfrak{p}|\mathfrak{f}} 1 \ll \ell(\mathfrak{f})^{1-2\sigma} \ll \ell(\mathfrak{f})^{2-2\sigma} \ll \frac{\ell(\mathfrak{f})^{2-2\sigma} - 1}{1-\sigma}. \tag{3.4.9}$$

(The last $\ll$ is by (3.4.3).) Therefore,

$$\mathrm{I}, \mathrm{III} \ll \frac{\ell(\mathfrak{f})^{2-2\sigma} - 1}{1-\sigma}. \tag{3.4.10}$$

Now put $y = \ell(\mathfrak{f})^2$ ($\geqslant (\log 3)^2 > 1$). Then (3.4.6) and (3.4.10) give

$$\mathrm{I} + \mathrm{III} + \mathrm{IV} \ll \frac{\ell(\mathfrak{f})^{2-2\sigma} - 1}{1-\sigma}; \tag{3.4.11}$$

while the Key lemma for this y gives

$$\begin{aligned} \mathrm{II} &\ll_\epsilon \ell(\mathfrak{f})^{2-2\sigma} && \text{(FF)} \\ &\ll_\epsilon \ell(\mathfrak{f})^{2-2\sigma}(\ell(t) + \ell(t)^2/\ell(\mathfrak{f})) + \ell(\mathfrak{f})^{2-2\sigma} && \text{(NF; under GRH);} \end{aligned} \tag{3.4.12}$$

hence by combining (3.4.11), (3.4.12) and (3.4.3), we obtain Theorem-Est. Thus, Theorem-Est is reduced to the Key Lemma.

The Key Lemma in the FF case is proved in [8](6.8.4). To prove this in the NF case, we shall make use of the following "explicit formula".

3.5. An explicit formula. Let K be any number field, let P_∞ consist only of the archimedean primes of K, and let χ be a *primitive* Dirichlet character on K, so that $L(s, \chi, \mathfrak{f})$ is the usual L-function $L(s, \chi)$. Put $\delta_\chi = 1$ (resp. 0) for $\chi = \chi_0$ (resp. $\chi \neq \chi_0$).

Theorem-Exp. *Let $\sigma = \mathrm{Re}(s) > 1/2$ and $y > 1$. Then*:

$$\frac{L'}{L}(s, \chi) + \psi(s, \chi; y) = -\delta_\chi\Big(\frac{y^{-s}}{s} + \frac{y^{1-s}}{s-1}\Big) + \sum_\rho{}' \frac{y^{\rho - s}}{s-\rho} + \ell(s, \mathrm{sign}(\chi); y), \tag{3.5.1}$$

where ρ runs over all non-trivial zeros of $L(s, \chi)$, $\sum_\rho' = \lim_{T\to\infty} \sum_{|\rho| \leqslant T}$, and

$$\begin{aligned} \ell(s, \mathrm{sign}(\chi); y) &= \sum_{\mu:\ \text{trivial zeros}} \frac{y^{\mu - s}}{s-\mu} \\ &= (a + r_2) \sum_{i \geqslant 0,\ \text{even}} \frac{y^{-i-s}}{s+i} + (a' + r_2) \sum_{i\geqslant 1,\ \text{odd}} \frac{y^{-i-s}}{s+i}. \end{aligned} \tag{3.5.2}$$

Here, a (resp. a') denotes the number of real places of K at which χ is unramified (resp. ramified), and r_2 is the number of complex places of K.

For the proof see Section 3.7.

3.6. Reduction of the Key Lemma (NF case) to Theorem-Exp. To avoid inessential complications we shall restrict our attention to the case where P_∞ consists only of the archimedean primes. (The difference arising in the general case can be estimated easily as in the estimations of I, III in Section 3.4.) Recall Theorem-Exp for $\chi \neq \chi_0$, which reads as

$$\frac{L'}{L}(s, \chi) + \psi(s, \chi; y) = \sum_\rho{}' \frac{y^{\rho-s}}{s-\rho} + \ell(s, \operatorname{sign}(\chi); y). \tag{3.6.1}$$

As before, let $\sigma_0 = \operatorname{Max}(\sigma, 2)$. Then (3.6.1) for s and σ_0 give

$$\frac{L'}{L}(s, \chi) + \psi(s, \chi; y) = P + Q + R, \tag{3.6.2}$$

where

$$P = y^{\sigma_0 - s}\left(\frac{L'}{L}(\sigma_0, \chi) + \psi(\sigma_0, \chi; y)\right), \tag{3.6.3}$$

$$Q = \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{\sigma_0 - \rho}\right) y^{\rho - s}, \tag{3.6.4}$$

$$R = \ell(s, \operatorname{sign}(\chi); y) - y^{\sigma_0 - s}\ell(\sigma_0, \operatorname{sign}(\chi); y). \tag{3.6.5}$$

The sum over ρ in Q is, unlike that in the above explicit formula itself, absolutely convergent.

(*Estimation of* P) Since $\sigma_0 \geqslant 2 > 1$, $L'/L(\sigma_0, \chi)$ has an absolutely convergent Dirichlet series expansion

$$\frac{L'}{L}(\sigma_0, \chi) = -\sum_D \Lambda(D)\chi(D)N(D)^{-\sigma_0}; \tag{3.6.6}$$

hence

$$\left|\frac{L'}{L}(\sigma_0, \chi) + \psi(\sigma_0, \chi; y)\right| \leqslant \sum_{N(D) \geqslant y} \Lambda(D)N(D)^{-\sigma_0} \ll \frac{y^{1-\sigma_0}}{\sigma_0 - 1} \ll y^{1-\sigma_0}.$$

(By partial summation, using $\sum_{N(D)\leqslant x} \Lambda(D) \ll x$.) Hence

$$P \ll y^{1-\sigma}. \tag{3.6.7}$$

(*Estimation of* R) It is easy to see that $\ell(s, \operatorname{sign}(\chi); y) \ll y^{-\sigma}$; hence

$$R \ll y^{-\sigma}. \tag{3.6.8}$$

(*Estimation of* Q*, under GRH*) By definition, and by GRH,

$$|Q| \leqslant y^{1/2-\sigma} \sum_\rho \frac{|\sigma_0 - s|}{|(s-\rho)(\sigma_0 - \rho)|}. \tag{3.6.9}$$

Write $\rho = 1/2 + i\gamma$. The only property on the distribution of γ on the real axis that we are going to use is the standard estimation, cf., e.g., [17];

$$n_\chi(x) := \#\{\rho;\ |\gamma - x| \leqslant 1\} \ll \log d_\chi + [K:\mathbb{Q}]\log(|x|+2) \tag{3.6.10}$$

for any $x \in \mathbb{R}$, where $d_\chi = |d_K| N(\mathfrak{f}_\chi)$ (d_K: the discriminant of K). Thus, in our notations,

$$n_\chi(x) \ll \ell(\mathfrak{f}) + \ell(x). \tag{3.6.11}$$

Now since

$$\begin{aligned} |\sigma_0 - s| &\leqslant |t| + |\sigma_0 - \sigma| < |t| + 3/2 < |t| + 2, \\ \sqrt{2} \cdot |s - \rho| &\geqslant (\sigma - 1/2) + |t - \gamma| \geqslant \epsilon + |t - \gamma| \gg_\epsilon 2 + |t - \gamma|, \\ \sqrt{2} \cdot |\sigma_0 - \rho| &\geqslant (\sigma_0 - 1/2) + |\gamma| \geqslant 3/2 + |\gamma| \gg 2 + |\gamma|, \end{aligned}$$

we have, by (3.6.9) and (3.6.11),

$$|Q| \ll y^{1/2-\sigma}(|t| + 2) \sum_\rho \frac{1}{(|\gamma| + 2)(|t - \gamma| + 2)} \tag{3.6.12}$$

$$\ll y^{1/2-\sigma}(|t| + 2)(\ell(\mathfrak{f}) B_1(t) + B_2(t)), \tag{3.6.13}$$

where

$$B_1(t) = \int_{-\infty}^{\infty} \frac{1}{(|x| + 2)(|t - x| + 2)} dx, \quad B_2(t) = \int_{-\infty}^{\infty} \frac{\log(|x| + 2)}{(|x| + 2)(|t - x| + 2)} dx. \tag{3.6.14}$$

It is easy to see that

$$B_1(t) \ll \frac{\log(|t| + 2)}{|t| + 2}, \quad B_2(t) \ll \frac{\log^2(|t| + 2)}{|t| + 2}. \tag{3.6.15}$$

Therefore,

$$|Q| \ll y^{1/2-\sigma}(\ell(\mathfrak{f})\ell(t) + \ell(t)^2). \tag{3.6.16}$$

Therefore, (3.6.7)(3.6.8)(3.6.16) combined gives

$$\left| \frac{L'}{L}(s, \chi) + \psi(s, \chi; y) \right| \ll_\epsilon y^{1-\sigma} + y^{-\sigma} + y^{1/2-\sigma}(\ell(\mathfrak{f})\ell(t) + \ell(t)^2). \tag{3.6.17}$$

But since $y^{-\sigma} \ll y^{1-\sigma}$, this settles the proof of the Key Lemma assuming Theorem-Exp.

3.7. Proof of Theorem-Exp. First, Weil's explicit formula ((11) in [24]), applied to the function $F(x)$ on $\mathbb{R}$ defined by $F(x) = e^{(1/2-s)x}$ ($0 < x < \log y$), $F(x) = 0$ ($x < 0$ or $x > \log y$), $F(x) = (F(x+0) + F(x-0))/2$ (everywhere), gives directly:

$$\psi(s, \chi; y) + \delta_\chi \left(\frac{y^{-s} - 1}{s} + \frac{y^{1-s} - 1}{s - 1} \right) = \sum_\rho{}' \frac{y^{\rho - s} - 1}{s - \rho} + \frac{1}{2}(\log d_\chi - N \log \pi)$$
$$+ \frac{a + r_2}{2} G\Big(\frac{s}{2}\Big) + \frac{a' + r_2}{2} G\Big(\frac{s+1}{2}\Big) + \ell(s, \mathrm{sign}(\chi); y), \tag{3.7.1}$$

where $d_\chi = |d_K| N(\mathfrak{f}_\chi)$, $N = [K : \mathbb{Q}]$ and $G(s) = \Gamma'(s)/\Gamma(s)$. (Note: $\pm\frac{N}{2}\log 2$ appears in two different terms in the Weil formula, but they cancel each other.)

On the other hand, the partial fractional decomposition of $L'/L(s,\chi)$ gives

$$\frac{L'}{L}(s,\chi)+\delta_\chi\left(\frac{1}{s}+\frac{1}{s-1}\right)=\sum_\rho{}' \frac{1}{s-\rho}-\frac{1}{2}(\log d_\chi - N\log\pi) \\ -\frac{a+r_2}{2}G\Big(\frac{s}{2}\Big)-\frac{a'+r_2}{2}G\Big(\frac{s+1}{2}\Big). \quad (3.7.2)$$

Here, the key formula is in [17] (the formula (5.9)), but in addition, we need the (conditional) convergence of the sums $\sum'_\rho(1-\rho)^{-1}$ and $\sum'_\rho \rho^{-1}$ (cf. [12, Section 2]) and the formula in Theorem 2 of *loc. cit.*, which asserts that the limit of the left-hand side of (3.7.2) as $s\to 1$ is equal to

$$\sum_\rho{}' \frac{1}{1-\rho}-\frac{1}{2}\log d_\chi+\frac{a+r_2}{2}(\gamma+\log 4\pi)+\frac{a'+r_2}{2}(\gamma+\log\pi), \quad (3.7.3)$$

where γ denotes the usual Euler constant. By summing up (3.7.1), (3.7.2), we obtain the desired explicit formula.

This completes the proof of Theorem 2.

4. The Analytic Function $\tilde{M}_s(z_1, z_2)$

4.1. The basic theorem. Let K and P_∞ be as at the beginning of Section 1.1. In the following theorem, we shall exhibit some basic properties of the complex analytic function

$$\tilde{M}_s(z_1,z_2)=\sum_{D:\text{ integral}}\lambda_{z_1}(D)\lambda_{z_2}(D)N(D)^{-2s} \quad (4.1.1)$$

of s, z_1, z_2 ($\mathrm{Re}(s)>1/2$) defined in Section 1.3. This includes reviews of some results given in [8]Section 3.7 (Case 1) and [11] (Case 2). The GRH is *not* assumed. First, note that $\lambda_z(D)$ (as well as $\chi(D)$) is multiplicative in D and hence $\tilde{M}_s(z_1,z_2)$ has an Euler product expansion, at least formally.

Theorem $\tilde{M}$. (i) *Let $\mathfrak{p}$ be any prime divisor of K not contained in P_∞, and* $\mathrm{Re}(s)>0$. *Define a continuous function $g_{s,\mathfrak{p}}(t)$ on $\mathbb{C}^1=\{t\in\mathbb{C}; |t|=1\}$ by*

$$\begin{aligned} g_{s,\mathfrak{p}}(t) &= \frac{-(\log N(\mathfrak{p}))N(\mathfrak{p})^{-s}t}{1-N(\mathfrak{p})^{-s}t} \quad &\text{(Case 1)}\\ &= -\log(1-N(\mathfrak{p})^{-s}t) \quad &\text{(Case 2)}\end{aligned} \quad (4.1.2)$$

(*the principal branch of the logarithm*), *and for $z_1, z_2\in\mathbb{C}$, put*

$$\tilde{M}_{s,\mathfrak{p}}(z_1,z_2)=\int_{\mathbb{C}^1}\exp\left(\frac{i}{2}(z_1 g_{s,\mathfrak{p}}(t^{-1})+z_2 g_{s,\mathfrak{p}}(t))\right)d^\times t, \quad (4.1.3)$$

where $d^{\times}t$ *denotes the normalized Haar measure on* $\mathbb{C}^1$. *Then with the notations of Section* 1.2,

$$\begin{aligned}\tilde{M}_{s,\mathfrak{p}}(z_1,\, z_2) &= \sum_{r=0}^{\infty} \lambda_{z_1}(\mathfrak{p}^r)\lambda_{z_2}(\mathfrak{p}^r)N(\mathfrak{p})^{-2rs} \\ &= 1 + \sum_{a,b\geqslant 1} (\pm i/2)^{a+b} \mu_{s,\mathfrak{p}}^{(a,b)} \frac{z_1^a z_2^b}{a!\, b!},\end{aligned} \tag{4.1.4}$$

where the sign is minus (*resp. plus*) *for Case* 1 (*resp. Case* 2), *and*

$$\begin{aligned}\mu_{s,\mathfrak{p}}^{(a,b)} &= (\log N(\mathfrak{p}))^{a+b} \sum_{r\geqslant \mathrm{Max}(a,b)} \binom{r-1}{a-1}\binom{r-1}{b-1} N(\mathfrak{p})^{-2rs} \quad \text{(Case 1)} \\ &= \sum_{r\geqslant \mathrm{Max}(a,b)} \delta_a(r)\delta_b(r)N(\mathfrak{p})^{-2rs} \quad \text{(Case 2)}.\end{aligned} \tag{4.1.5}$$

(ii) $\tilde{M}_s(z_1,\, z_2)$ *has an absolutely convergent Euler product expansion on* $\mathrm{Re}(s) > 1/2$;

$$\tilde{M}_s(z_1,\, z_2) = \prod_{\mathfrak{p}\notin P_\infty} \tilde{M}_{s,\mathfrak{p}}(z_1,\, z_2). \tag{4.1.6}$$

This convergence is uniform on $\mathrm{Re}(s) \geqslant 1/2 + \epsilon$, $|z_1|, |z_2| \leqslant R$, *for each fixed* ϵ, $R > 0$.

(iii) $\tilde{M}_s(z_1,\, z_2)$ *for each* s *with* $\mathrm{Re}(s) > 1/2$ *has an everywhere absolutely convergent power series expansion*

$$\tilde{M}_s(z_1,\, z_2) = 1 + \sum_{a,b\geqslant 1} (\pm i/2)^{a+b} \mu_s^{(a,b)} \frac{z_1^a z_2^b}{a!\, b!}, \tag{4.1.7}$$

with the same choice of the sign as above. Here, $\mu_s^{(a,b)}$ *denotes the following Dirichlet series, which is absolutely convergent on* $\mathrm{Re}(s) > 1/2$;

$$\mu_s^{(a,b)} = \sum_{D:\ \text{integral}} \Lambda_a(D)\Lambda_b(D)N(D)^{-2s}, \tag{4.1.8}$$

where each $\Lambda_k(D)$ *is a non-negative real number determined from the polynomial coefficients of* $\lambda_z(D)$ *by the formula*

$$\lambda_z(D) = \sum_{k=0}^{\infty} \frac{\Lambda_k(D)}{k!} (\pm iz/2)^k \tag{4.1.9}$$

(*the same choice of the sign*).

(iv) *As before* (*cf. Sections* 0.1, 0.3), *put*

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1\overline{w} + z_2 w)\right) \tag{4.1.10}$$

$(z_1, z_2, w \in \mathbb{C})$, and for $\sigma > 1/2$, let $M_\sigma(w)$ denote the "M-function" defined and studied in [8] (Case 1), [11] (Case 2). *(In the latter, it is denoted as $\mathcal{M}_\sigma(w)$.) Then*

$$\tilde{M}_\sigma(z_1, z_2) = \int_{\mathbb{C}} M_\sigma(w)\psi_{z_1,z_2}(w)|dw|. \tag{4.1.11}$$

In particular, $\tilde{M}_\sigma(z, \overline{z})$ is the Fourier dual $\tilde{M}_\sigma(z)$ of $M_\sigma(w)$.

Proof. (i) Recall basic definitions in Section 1.2. By (1.2.4) for $x \to (-iz/2)\log N(\mathfrak{p})$, $t \to N(\mathfrak{p})^{-s}t$, and (1.2.5) for $x \to iz/2$, $t \to N(\mathfrak{p})^{-s}t$, we obtain

$$\exp\left(\frac{iz}{2}g_{s,\mathfrak{p}}(t)\right) = \sum_{r=0}^{\infty} \lambda_z(\mathfrak{p}^r)(N(\mathfrak{p})^{-s}t)^r. \tag{4.1.12}$$

But since (by (4.1.3)) $\tilde{M}_{s,\mathfrak{p}}(z_1, z_2)$ is nothing but the constant term in the Fourier expansion of

$$\exp\left(\frac{i}{2}(z_1 g_{s,\mathfrak{p}}(t^{-1}) + z_2 g_{s,\mathfrak{p}}(t))\right) \tag{4.1.13}$$

in t^n $(n \in \mathbb{Z})$, (4.1.4) follows directly.

(ii) Fix any ϵ, $R > 0$, and let s, z_1, z_2 run over $\mathrm{Re}(s) \geqslant 1/2 + \epsilon$, $|z_1|$, $|z_2| \leqslant R$. First, it is easy to see that the product over $N(\mathfrak{p}) \leqslant y$ $(\mathfrak{p} \notin P_\infty)$ of $\tilde{M}_{s,\mathfrak{p}}(z_1, z_2)$ converges to $\tilde{M}_s(z_1, z_2)$ uniformly as $y \to \infty$. (In fact, compare the Dirichlet series expansion (4.1.1) for $\tilde{M}_s(z_1, z_2)$ and that for this product. The latter corresponds to the partial sum of the former over all such D that satisfy the property "$\mathfrak{p}|D$ implies $N(\mathfrak{p}) \leqslant y$". So, the difference is *a* partial sum of *the* partial sum over $N(D) > y$. Since (4.1.1) converges absolutely and uniformly, this difference tends to 0 uniformly.) Our assertion, the absolute convergence of the infinite product, states also that the sum

$$\sum_{\mathfrak{p} \notin P_\infty} |\tilde{M}_{s,\mathfrak{p}}(z_1, z_2) - 1| \tag{4.1.14}$$

converges uniformly. But by (4.1.4) and Theorem 2 (i),

$$\begin{aligned} |\tilde{M}_{s,\mathfrak{p}}(z_1, z_2) - 1| &\leqslant \sum_{r=1}^{\infty} |\lambda_{z_1}(\mathfrak{p}^r)\lambda_{z_2}(\mathfrak{p}^r)| N(\mathfrak{p})^{-2r\sigma} \\ &\ll_{\epsilon',R} \sum_{r=1}^{\infty} N(\mathfrak{p})^{(2\epsilon'-2\sigma)r} \leqslant \sum_{r=1}^{\infty} N(\mathfrak{p})^{(-1-\epsilon)r} < 2N(\mathfrak{p})^{-1-\epsilon} \end{aligned} \tag{4.1.15}$$

(take $\epsilon' = \epsilon/2$); hence this is clear.

(iii) First, a few preliminary remarks on $\Lambda_k(D)$ defined by (4.1.9). When $k = 0$, we have $\Lambda_0(D) = 0$ (resp. 1) for $D \neq (1)$ (resp. $D = (1)$). When $k = 1$, $\Lambda_1(D) = 0$ unless $D = \mathfrak{p}^r$ with some $\mathfrak{p} \notin P_\infty$ and $r \geqslant 1$, and in this case,

$$\Lambda_1(D) = \begin{cases} \log N(\mathfrak{p}) & \text{(Case 1)}, \\ 1/r & \text{(Case 2)}. \end{cases} \tag{4.1.16}$$

For $k \geqslant 1$, Λ_k can also be expressed as the k-th iteration of Λ_1;

$$\Lambda_k(D) = \sum_{D=D_1\ldots D_k} \Lambda_1(D_1)\ldots\Lambda_1(D_k). \tag{4.1.17}$$

In Case 1, this is shown in [8, Section 3.8]. In Case 2, the proof runs as follows. Let $D = \prod_\nu \mathfrak{p}_\nu^{r_\nu}$ be the prime factorization of D, and t_ν be independent variables. Then, being multiplicative, $\lambda_z(D)$ is equal to the coefficient of $\prod_\nu t_\nu^{r_\nu}$ in

$$\begin{aligned}\prod_\nu\Big(\sum_{r=0}^{\infty}\lambda_z(\mathfrak{p}_\nu^r)t_\nu^r\Big) &= \prod_\nu\Big(\sum_{r=0}^{\infty}H_r(iz/2)t_\nu^r\Big)\\ &= \exp\Big(-(iz/2)\sum_\nu\log(1-t_\nu)\Big)\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}(iz/2)^k\Big(-\sum_\nu\log(1-t_\nu)\Big)^k. \end{aligned}\tag{4.1.18}$$

But the coefficient of $\prod_\nu t_\nu^{r_\nu}$ in $(-\sum_\nu \log(1-t_\nu))^k$ is nothing but the right-hand side of (4.1.17). Therefore, by (4.1.9) we obtain the desired equality (4.1.17).

Now define $\mu_s^{(a,b)}$ by (4.1.8). Note that $\mu_s^{(0,0)} = 1$, $\mu_s^{(a,0)} = \mu_s^{(0,b)} = 0$ for $ab \neq 0$. Since the Dirichlet coefficients in (4.1.8) for $a,\, b \geqslant 1$ are non-negative for all D and positive for, say, $D = \mathfrak{p}^{a+b}$, we have

$$\mu_\sigma^{(a,b)} > 0 \quad (a,\, b \geqslant 1). \tag{4.1.19}$$

Now let us prove (iii). In Case 1, this is proved in [8, Theorem 5 in Section 3.7]. The proof for Case 2 is almost parallel, but let us sketch this proof. Fix s with $\mathrm{Re}(s) > 1/2$, and put $\zeta_j = iz_j/2$ $(j = 1,\, 2)$. Since the Dirichlet series (4.1.1) converges uniformly on $|\zeta_1|,\, |\zeta_2| \leqslant 1$, we obtain, first differentiating termwise, then putting $\zeta_1 = \zeta_2 = 0$, and then using (4.1.9),

$$\begin{aligned}\left(\frac{\partial^{a+b}\tilde{M}_s(z_1,\, z_2)}{\partial\zeta_1^a\partial\zeta_2^b}\right)_{(0,0)} &= \sum_D \left(\frac{\partial^a\lambda_{z_1}(D)}{\partial\zeta_1^a}\right)_0\left(\frac{\partial^b\lambda_{z_2}(D)}{\partial\zeta_2^b}\right)_0 N(D)^{-2s}\\ &= \sum_D \Lambda_a(D)\Lambda_b(D)N(D)^{-2s} = \mu_s^{(a,b)}.\end{aligned}\tag{4.1.20}$$

Since $\tilde{M}_s(z_1,\, z_2)$ is an entire function, the power series (4.1.7) must converge (absolutely) everywhere.

(iv) The integral on the right-hand side of (4.1.11) is the limit of that over $|w| \leqslant R$, and the convergence as $R \to \infty$ is uniform on $|z_1|,\, |z_2| \leqslant r$ for any $r > 0$, because $M_\sigma(w) = O(e^{-\lambda|w|^2})$ holds for any $\lambda > 0$, as will be proved in Section 5 (Lemma B). Therefore, each side of (4.1.11) is a holomorphic function of $z_1,\, z_2$. Since they are equal when $z_2 = \overline{z}_1$, as is proved in [8] (Case 1), [11] (Case 2), they must be equal for any $z_1,\, z_2 \in \mathbb{C}$.

This completes the proof of Theorem $\tilde{M}$. □

4.2. Related comments. (I) From (4.1.7) and (4.1.11), we obtain

$$\mu_\sigma^{(a,b)} = (\pm 1)^{a+b} \int_{\mathbb{C}} M_\sigma(w) P^{(a,b)}(w)|dw| \tag{4.2.1}$$

by partial derivation. Here, $\pm 1 = -1$ (resp. 1) for Case 1 (resp. Case 2) throughout this subsection, and $P^{(a,b)}(w) = \bar{w}^a w^b$ $(a,\, b \geqslant 0)$. Thus, by Theorem 4 (to be proved later) applied to $\Phi = P^{(a,b)}$, we also obtain

$$\mu_\sigma^{(a,b)} = (\pm 1)^{a+b} \lim_{\substack{\mathfrak{f}\ \mathrm{prime} \\ N(\mathfrak{f})\to\infty}} \mathrm{Avg}_{\mathfrak{f}_\chi = \mathfrak{f}} P^{(a,b)}(\mathcal{L}(s,\, \chi)), \tag{4.2.2}$$

under the same assumption as in Theorem 4. In Case 1, this equality for $s = 1$ is proved unconditionally over $K = \mathbb{Q}$ [12], and over function fields, for any s with $\sigma > 1/2$ (Theorem 7(iii) of [8], under an additional inessential assumption $\deg \mathfrak{p}_\infty = 1$).

(II) When $iz_1/2 = iz_2/2 = y \in \mathbb{R}$, so that $\psi_{z_1,z_2}(w) = \exp(2y\mathrm{Re}(w))$, (4.1.7) gives

$$\big(\tilde{M}_\sigma(2y/i,\, 2y/i) - \tilde{M}_\sigma(-2y/i,\, -2y/i)\big)/4y = \frac{(\pm 1)}{2} \sum_{k\geqslant 3,\ \mathrm{odd}} \Bigg(\sum_{\substack{k=a+b \\ a,b\geqslant 1}} \frac{\mu_\sigma^{(a,b)}}{a!\, b!} \Bigg) y^{k-1}. \tag{4.2.3}$$

By (4.1.19), the right-hand side is non-zero when $y \neq 0$, and has the same sign as (± 1). Under the assumption of Theorem 3, this is the limit of

$$h_s(\mathfrak{f},\, y) = \mathrm{Avg}_{\mathfrak{f}_\chi = \mathfrak{f}} \big(e^{2y\,\mathrm{Re}\,\mathcal{L}(s,\chi,\mathfrak{f})} - e^{-2y\,\mathrm{Re}\,\mathcal{L}(s,\chi,\mathfrak{f})}\big)/4y \tag{4.2.4}$$

as $N(\mathfrak{f}) \to \infty$; hence for any *fixed* s and $y \neq 0$, the inequalities

$$h_s(\mathfrak{f},\, y) < 0 \quad \text{(Case 1)}, \tag{4.2.5}$$

$$> 0 \quad \text{(Case 2)}, \tag{4.2.6}$$

hold when $N(\mathfrak{f})$ is *sufficiently large*. On the other hand, we have

$$h_s(\mathfrak{f},\, 0) = \mathrm{Avg}_{\mathfrak{f}_\chi = \mathfrak{f}}\, \mathrm{Re}(\mathcal{L}(s,\, \chi,\, \mathfrak{f})), \tag{4.2.7}$$

and since the expression (4.2.3) is 0 for $y = 0$, this must tend to 0 as $N(\mathfrak{f}) \to \infty$.

The sign of $h_s(\mathfrak{f},\, 0)$ for each *individual* $\mathfrak{f}$ offers a more delicate problem. For example, let $K = \mathbb{Q}$, $P_\infty = (\infty)$ and $s = 1$. Then for any odd prime f, $(f-2)h_1((f),\, 0)$ is equal to

$$\sum_{f_\chi = f} \mathrm{Re}(L'(1,\, \chi)/L(1,\, \chi)) = \gamma_f - \gamma \quad \text{(Case 1)}, \tag{4.2.8}$$

$$\sum_{f_\chi = f} \log|L(1,\, \chi)| = \log \kappa_f \quad \text{(Case 2)}. \tag{4.2.9}$$

Here, κ_f denotes the residue of the Dedekind zeta function $\zeta_{K_f}(s)$ of the cyclotomic field $K_f = \mathbb{Q}(\mu_f)$ at $s = 1$, γ_f denotes the quotient of the constant term of its Laurent expansion at $s = 1$ divided by κ_f (the "Euler–Kronecker constant (invariant)" in the sense of [6]), and $\gamma = \gamma_1$, the usual Euler constant. The first named author thinks it very likely that, in contrast to the above inequalities (4.2.5), (4.2.6),

$$h_1((f),\,0) > 0 \quad \text{(Case 1)}, \tag{4.2.10}$$
$$< 0 \quad \text{(Case 2)} \tag{4.2.11}$$

both hold[2]. Among these, the first inequality is essentially a part of the conjectures on the behaviour of γ_f raised in [7]. The second, which is equivalent to $\kappa_f < 1$, maybe new even as a conjecture. A more basic question is whether $\zeta_{K_f}(\sigma)/\zeta(\sigma) = 1 - 2^{-\sigma} - \cdots$ is everywhere monotone *increasing* on $\sigma > 1 - \epsilon$, as some numerical evidences suggest. In fact, if this is true, then both (4.2.10), (4.2.11) follow immediately.

(III) In Case 2, the second formula for $H_r(x)$ in (1.2.7) shows that the local factor $\tilde{M}_{s,\mathfrak{p}}(z_1,\, z_2)$ is nothing but the Gauss hypergeometric function

$$F(a,\, b,\, c;\, t) = 1 + \frac{a \cdot b}{1 \cdot c}\, t + \frac{a(a+1) \cdot b(b+1)}{1 \cdot 2 \cdot c(c+1)}\, t^2 + \cdots ,$$

for

$$a = iz_1/2, \qquad b = iz_2/2, \qquad c = 1; \qquad t = N(\mathfrak{p})^{-2s}.$$

In particular, when $a = b = y = \pm 1$, $F(a,\, b,\, c;\, t) = (1 - N(\mathfrak{p})^{-2s})^{-1}$ or $1 + N(\mathfrak{p})^{-2s}$, for $y = 1$ or -1, respectively; hence

$$\tilde{M}_s(-2i,\, -2i) = \sum_{D \text{ integral}} N(D)^{-2s}; \quad \tilde{M}_s(2i,\, 2i) = \sum_{\substack{D \text{ integral} \\ \text{square free}}} N(D)^{-2s}. \tag{4.2.12}$$

4.3. Further prospects. Let us briefly mention here some further results on this function by the first named author, just to indicate that this direction of research does not come at least so soon to a dead-end. They are on (i) the zeros of $\tilde{M}_s(z_1,\, z_2)$ cf. [9], (ii) the "Plancherel volume" cf. [9], [26], and (iii) the analytic continuation to the left of $\sigma > 1/2$ cf. [26].

As for the zeros, they are "merely" the collection of the zeros of local Euler factors, but still, a non-trivial basic object of study. For example, when $s = \sigma > 1/2$ is fixed and $z := z_1 = \bar{z}_2$, one can prove that there are infinitely many (discrete set of) zeros z on the imaginary axis $\mathrm{Re}(z) = 0$. As for the "exceptional zeros" off this axis, at least in Case 1 they are finite in number, none if σ is sufficiently large, but they actually do exist for some small σ (say, for $K = \mathbb{Q}$, $\sigma = 1/2 + 1/20$)); cf. [9].

As for (ii), the "Plancherel volume" refers to the integral invariant

$$\nu_\sigma := \int_{\mathbb{C}} |\tilde{M}_\sigma(z)|^2 |dz| = \int_{\mathbb{C}} M_\sigma(w)^2 |dw|. \tag{4.3.1}$$

The product $\mu_\sigma \nu_\sigma$ with the variance

$$\mu_\sigma := \mu_\sigma^{(1,1)} = \int_{\mathbb{C}} M_\sigma(w)|w|^2 |dw| \tag{4.3.2}$$

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[2] These may look rather contradictory to the above inequalities (4.2.5), (4.2.6), but imagine the last moment of *sunset* for Case 1, and that of *sunrise* for Case 2. Namely, the graph of $h_1((f),\, y)$ for each f crosses the horizontal axis near $y = 0$ on both sides and, as $f \to \infty$, the graph tends to that of $-Cy^2 + \cdots$ (Case 1), $Cy^2 + \cdots$ (Case 2), where $C = \mu_1^{(1,2)}/2 > 0$.

is another interesting and probably basic object of study (*loc. cit.*). For example, the determination of $\lim_{\sigma\to 1/2}(\mu_\sigma\nu_\sigma)$ requires further arithmetic study, and this turns out to be equal to 1 (as in the case of the Gaussian distribution with the center 0).

As for (iii), one can prove that $\tilde{M}_s(z_1, z_2)$ extends to an analytic function on a wider domain

$$\{\sigma > 0\} \setminus \{\rho/2n\}_{n\in\mathrm{N},\zeta_K(\rho)=0 \text{ or } \infty}, \tag{4.3.3}$$

(univalent in Case 1 and multivalent in Case 2), and this property helps study the behavior of our "M-functions" at (say) $s = 1/2$. There is nothing mysterious in this, as the following first step continuation indicates. *The function*

$$\tilde{M}_s(z_1, z_2)\exp(\phi(2s)z_1z_2/4) \tag{4.3.4}$$

extends to a holomorphic function of (s, z_1, z_2) *on* $\sigma > 1/4$. *Here,*

$$\phi(s) = d^2/ds^2(\log\zeta_K(s)) \quad \text{(Case 1; \textit{not} the first derivative)}$$
$$\phi(s) = \log\zeta_K(s) \quad \text{(Case 2)},$$

$\zeta_K(s)$ *being the Dedekind zeta function of* K *without the* P_∞*-components.*

5. Proof of Theorem 4

The most basic ingredient for the proof of Theorem 4 is Theorem 3, but we shall also need two other fairly basic results, Lemma A (Section 5.1) and Lemma B (Section 5.2). In Section 5.3, we shall give a proof of Theorem 4 assuming these two lemmas. Then, in later subsections, we shall prove these lemmas.

5.1. Changing test functions. Let $\mathbb{R}^d = \{x = (x_1, \dots, x_d); x_i \in \mathbb{R}\ (1 \leqslant i \leqslant d)\}$ be the d-dimensional Euclidean space ($d \geqslant 1$), and $|dx| = (dx_1 \cdots dx_d)/(2\pi)^{d/2}$ be the self-dual Haar measure with respect to the self-dual pairing $e^{i\langle x,x'\rangle}$ of $\mathbb{R}^d$, where $\langle x, x'\rangle = \sum_{i=1}^d x_i x_i'$. Write, as usual, $|x| = \langle x, x\rangle^{1/2}$. In what follows, "function" will always mean a $\mathbb{C}$-valued function on $\mathbb{R}^d$.

For any function f belonging to L^1, its Fourier transform $f^\wedge$ and the inverse Fourier transform $f^\vee$ are defined by

$$f^\wedge(x) = \int f(y)e^{i\langle x,y\rangle}|dy|, \quad f^\vee(x) = \int f(y)e^{-i\langle x,y\rangle}|dy|. \tag{5.1.1}$$

Let $\Lambda = \Lambda(\mathbb{R}^d)$ denote the space of all $f \in L^1 \cap L^\infty$ such that $f^\wedge$ also belongs to $L^1 \cap L^\infty$ and that $(f^\wedge)^\vee = f$ holds. (By definition, L^∞ consists of all continuous functions which vanish at infinity; $f \in L^1 \cap L^\infty$ implies $f \in L^p$ for all $1 \leqslant p \leqslant \infty$.) Let us recall here the following basic facts (cf., e.g., [23]). If $f \in L^1$, then $f \in \Lambda$ holds if and only if f is continuous and $f^\wedge$ belongs to L^1. Moreover, for any $f, g \in \Lambda$, we have

$$\int \overline{f^\wedge(x)}\, g^\wedge(x)|dx| = \int \overline{f(x)}\, g(x)|dx|. \tag{5.1.2}$$

Call $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ the Schwartz space, i.e., the set of all C^∞-functions f such that for any partial derivative D of any order and for any $k \geqslant 0$, $|x|^k D(f)$ tends to 0 as

$|x| \to \infty$. Then $\mathcal{S}$ is contained in Λ, and is stable under the Fourier transform. In particular, any compactly supported C^∞-function belongs to Λ.

By a *good density function* on $\mathbb{R}^d$, we shall mean any non-negative real valued continuous function $M(x)$ on $\mathbb{R}^d$ that belongs to Λ and satisfies

$$\int M(x)|dx| = 1. \tag{5.1.3}$$

Note that $M^\wedge$ necessarily satisfies

$$|M^\wedge(x)| \leqslant 1, \quad \overline{M^\wedge(x)} = M^\wedge(-x). \tag{5.1.4}$$

Consider any *finite* measure space $X^* = (X, \omega)$ with the total measure $\omega(X) = 1$. In other words, a finite set X is equipped with a weight function $\omega(\chi) \geqslant 0$ $(\chi \in X)$ satisfying $\sum_\chi \omega(\chi) = 1$. For any $\mathbb{C}$-valued function ϕ on X, we define the weighted average

$$\mathrm{Avg}_{X^*}\phi = \int \phi\omega = \sum_{\chi \in X} \omega(\chi)\phi(\chi). \tag{5.1.5}$$

Consider, now, any pair $X^{**} = (X^*, \ell)$ of a finite measure space $X^* = (X, \omega)$ and a mapping $\ell\colon X \to \mathbb{R}^d$. We shall need some terminology related to "approximation" of a given measure space $(\mathbb{R}^d, M(x)|dx|)$ by the ℓ-images of such finite measure spaces X^*. Namely, for a sequence $\{X_n^{**}\}$ of $X_n^{**} = (X_n^*, \ell_n)$, and a test function Φ on $\mathbb{R}^d$, consider the following condition on Φ;

$$\lim_{n\to\infty} \mathrm{Avg}_{X_n^*}(\Phi \circ \ell_n) = \int M(x)\Phi(x)|dx|. \tag{5.1.6}$$

(So to speak, "approximation at the level Φ".) What we shall need is to deduce, from the validity of (5.1.6) for some special classes of functions Φ, that for more general Φ.

Lemma A *Let $M(x)$ be any good density function on $\mathbb{R}^d$, and $\{X_n^{**}\}_{n\geqslant 1}$ be a sequence of pairs $X_n^{**} = (X_n^*, \ell_n)$ of a finite measure space X_n^* and a mapping $\ell_n\colon X_n \to \mathbb{R}^d$.*

(i) *Suppose that (5.1.6) holds for any additive characters $\Phi = \psi^{(y)}\colon x \to e^{i\langle x,y\rangle}$, and that the convergence $\lim_{n\to\infty}$ is uniform in the wider sense (i.e., uniform when the parameter y runs over any compact subset of $\mathbb{R}^d$). Then (5.1.6) holds for any function Φ belonging to Λ. In particular, it holds for any compactly supported C^∞-function.*

(ii) *Suppose (5.1.6) holds for all compactly supported C^∞-functions Φ on $\mathbb{R}^d$. Then:*

(a) *it holds for any bounded continuous function Φ;*
(b) *it holds for any continuous function Φ satisfying*

$$|\Phi(x)| \leqslant \phi_0(|x|), \tag{5.1.7}$$

if there exists a continuous monotone non-decreasing function $\phi_0(r) > 0$ of $r \geqslant 0$ satisfying $\lim_{r\to\infty} \phi_0(r) = \infty$ and

$$\int M(x)\phi_0(|x|)|dx| < \infty, \tag{5.1.8}$$

$$\mathrm{Avg}_{X_n^*}(\phi_0 \circ |\ell_n|)^2 \ll 1. \tag{5.1.9}$$

(c) *Its holds when Φ is the characteristic function of either a compact subset of $\mathbb{R}^d$ or the complement of such a subset.*

5.2. Rapid decay of $M_\sigma(w)$, *a la* Jessen–Wintner. We shall need the following property of $M_\sigma(w)$, which essentially goes back to Jessen–Wintner [15].

Lemma B *Let K and P_∞ be as in Section* 1.1, *and the function $M_\sigma(w)$ be as in Section* 4.1. *Fix $\sigma > 1/2$. Then in each of Cases* 1, 2, *we have*

$$M_\sigma(w) = O(e^{-\lambda|w|^2}) \quad \text{for any } \lambda > 0. \tag{5.2.1}$$

This proof will be given in Section 5.6.

5.3. Proof of Theorem 4 assuming Lemmas A, B. We shall apply Lemma A to the following situation:

$\mathbb{C}$ for $\mathbb{R}^2$ (note that $\mathrm{Re}(\bar{z}w) = \langle z, w\rangle$ $(z,\ w \in \mathbb{C})$),
$M_\sigma(w)$ $(\sigma > 1/2)$ for $M(x)$.

(That $M_\sigma(w)$ belongs to Λ is proved in [8], [11].)

The set of prime divisors $\mathfrak{f}$ ($\neq \mathfrak{p}_\infty$ in the FF case) of K, instead of $n = 1, 2, \ldots$,
The set $\hat{G}'_{\mathfrak{f}} := \{\chi \in \hat{G}_{\mathfrak{f}};\ \mathfrak{f}_\chi = \mathfrak{f}\}$ for X_n, with $\omega_\chi = 1/|\hat{G}'_{\mathfrak{f}}|$ for all $\chi \in \hat{G}'_{\mathfrak{f}}$;

and finally,

$\mathcal{L}(s, \chi, \mathfrak{f})$ for $\ell_n(\chi)$ (for each s with $\sigma = \mathrm{Re}(s) > 1/2$).

Since $\psi_{z,\bar{z}}$ $(z \in \mathbb{C})$ runs over all additive characters of $\mathbb{C}$, Theorem 3 for the case $z_2 = \bar{z_1}$ asserts that the assumption of Lemma A(i), and hence its conclusion, in particular the first common assumption of Lemma A(ii), are satisfied. Now take any $a > 0$ and put $\phi_0(r) = \exp(ar)$. It remains to show that this satisfies the assumption of (ii)(b). But (5.1.8) is obvious by Lemma B, while (5.1.9) is nothing but the following direct consequence of Theorem 3.

Sublemma. *The assumptions being as in Theorem* 3, *fix any $\epsilon > 0$, $T > 0$, $a > 0$, and let $s = \sigma + ti$ run over $\sigma \geqslant 1/2 + \epsilon$, and in the NF case, only over $|t| \leqslant T$. Then for any prime divisor $\mathfrak{f}$ we have*

$$\mathrm{Avg}_{\mathfrak{f}_\chi = \mathfrak{f}} \exp(a|\mathcal{L}(s, \chi, \mathfrak{f})|) \ll 1. \tag{5.3.1}$$

Proof of the sublemma. Write $\ell_\chi = \mathcal{L}(s, \chi, \mathfrak{f})$, $\mathrm{Avg}_\chi = \mathrm{Avg}_{\mathfrak{f}_\chi = \mathfrak{f}}$. Since $e^{a|\ell_\chi|} \leqslant e^{a|\mathrm{Re}(\ell_\chi)|}e^{a|\mathrm{Im}(\ell_\chi)|}$, the Schwarz inequality reduces the sublemma to

$$\mathrm{Avg}_\chi e^{2a|\mathrm{Re}(\ell_\chi)|}, \quad \mathrm{Avg}_\chi e^{2a|\mathrm{Im}(\ell_\chi)|} \ll 1.$$

But since $e^{2a|\mathrm{Re}(\ell_\chi)|} < |e^{a\ell_\chi}|^2 + |e^{-a\ell_\chi}|^2$ and $e^{2a|\mathrm{Im}(\ell_\chi)|} < |e^{-ai\ell_\chi}|^2 + |e^{ai\ell_\chi}|^2$, and moreover since $\mathrm{Avg}_\chi |e^{z\ell_\chi}|^2 \ll 1$ holds for $z = \pm a, \pm ai$, in view of Theorem 3, the sublemma follows. □

Finally, suppose that $\sigma > 1$. Then Theorem 3 holds unconditionally (Remark 1.3.5), $|\mathcal{L}(s, \chi, \mathfrak{f})|$ is bounded, and $M_\sigma(w)$ is compactly supported [8] (Case 1), [11] (Case 2); hence the validity of (1.4.1) for any continuous function Φ is a trivial consequence of that for any compactly supported continuous function. Thus, Theorem 4 is reduced to Lemmas A and B.

5.4. Proof of Lemma A(i). By assumption and by (5.1.1), we have

$$\lim_{n\to\infty} \mathrm{Avg}_{X_n^*}(\psi^{(y)} \circ \ell_n) = M^\wedge(y) \quad \text{(uniformly on } |y| \leqslant R) \tag{5.4.1}$$

for any $R > 0$, where $\psi^{(y)}(x) = e^{i\langle x,y\rangle}$. Now let Φ be any element of Λ and put

$$\Delta_n(\Phi) = \mathrm{Avg}_{X_n^*}(\Phi \circ \ell_n) - \int M(x)\Phi(x)|dx|. \tag{5.4.2}$$

(Since M, $\Phi \in \Lambda$, the above integral is finite.) Write $X_n^* = (X_n, \omega_n)$. Then since $(\Phi^\wedge)^\vee = \Phi$, $\overline{M(x)} = M(x)$ and $\overline{M^\wedge(y)} = M^\wedge(-y)$, (5.1.2) gives

$$\begin{aligned}
\Delta_n(\Phi) &= \sum_{\chi\in X_n} \omega_n(\chi)\Phi(\ell_n(\chi)) - \int M(x)\Phi(x)|dx| \\
&= \sum_{\chi\in X_n} \omega_n(\chi) \int \Phi^\wedge(y) e^{-i\langle \ell_n(\chi), y\rangle}|dy| - \int M^\wedge(-y)\Phi^\wedge(y)|dy| \\
&= \int \Big(\sum_{\chi\in X_n} \omega_n(\chi) e^{i\langle \ell_n(\chi), -y\rangle} - M^\wedge(-y) \Big) \Phi^\wedge(y)|dy| \\
&= \int \big(\mathrm{Avg}_{X_n^*}(\psi^{(-y)} \circ \ell_n) - M^\wedge(-y)\big)\Phi^\wedge(y)|dy|.
\end{aligned}$$

But since $|\psi^{(-y)}(x)|$, $|M^\wedge(y)| \leqslant 1$, we obtain for any $R > 0$,

$$|\Delta_n(\Phi)| \leqslant \int_{|y|\leqslant R} |\mathrm{Avg}_{X_n^*}(\psi^{(-y)} \circ \ell_n) - M^\wedge(-y)|\,|\Phi^\wedge(y)|\,|dy| + 2\int_{|y|\geqslant R} |\Phi^\wedge(y)|\,|dy|. \tag{5.4.3}$$

Since $\Phi \in \Lambda$ and hence in particular $\Phi^\wedge \in L^1$, the total integral of $|\Phi^\wedge|$ is finite. Call this value I. Now, given any $\epsilon > 0$, choose R so large that the second term on the right-hand side of (5.4.3) is $< \epsilon$. Then choose $\epsilon' > 0$ such that $\epsilon' I < \epsilon$. By (5.4.1),

$$|\mathrm{Avg}_{X_n^*}(\psi^{(-y)} \circ \ell_n) - M^\wedge(-y)| < \epsilon' \tag{5.4.4}$$

holds on $|y| \leqslant R$ for sufficiently large n, which implies $|\Delta_n(\Phi)| < 2\epsilon$ for such large n. This settles the proof of (i).

5.5. Proof of Lemma A(ii). First, the validity of (5.1.6) for any compactly supported C^∞-function implies that for any compactly supported continuous function, because the latter can be approximated by the former.

As for (c), this can be proved directly by the approximation of the characteristic function of a given compact set by continuous compactly supported functions, as is explained in detail in the two dimensional case in [10, Section 4.3].

Now we shall prove (a) and (b). Let Φ be continuous and either (a) bounded or (b) satisfying (5.1.7). Put $\alpha_n = \mathrm{Avg}_{X_n^*}(\Phi \circ \ell_n)$. In each case, α_n $(n = 1, 2, \dots)$ is a bounded sequence (by (5.1.9) in case (b)). Let α be any of its limit points. It is the limit of some subsequence α_{n_ν}. The goal is to prove

$$\alpha = \int M(x)\Phi(x)|dx|. \tag{5.5.1}$$

To prove this, note first that for any $R > 0$ there exists a compactly supported continuous function Φ_R satisfying

$$|\Phi(x) - \Phi_R(x)| \leqslant (1 - \mathrm{ch}_R(x))|\Phi(x)|, \tag{5.5.2}$$

where ch_R denotes the characteristic function of $\{|x| \leqslant R\}$. Indeed, if $E(u)$ is any compactly supported continuous function such that $0 \leqslant E(u) \leqslant 1$ everywhere and $E(u) = 1$ for $|u| \leqslant 1$, then $\Phi_R(x) = \Phi(x)E(x/R)$ has this property. Now choose such Φ_R for each R, and put $\alpha_{n,R} = \mathrm{Avg}_{X_n^*}(\Phi_R \circ \ell_n)$. Since Φ_R is compactly supported and continuous, we have

$$\lim_{n\to\infty} \alpha_{n,R} = \int M(x)\Phi_R(x)|dx|. \tag{5.5.3}$$

Now, (5.5.2) gives

$$\begin{aligned} \alpha_n - \alpha_{n,R} &= \mathrm{Avg}_{X_n^*}((\Phi - \Phi_R) \circ \ell_n) \\ &\ll \beta_{n,R} := \mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R)|\Phi| \circ \ell_n), \end{aligned} \tag{5.5.4}$$

and (5.5.2) together with (5.1.8) in case (b) gives

$$\lim_{R\to\infty} \int M(x)\Phi_R(x)|dx| = \int M(x)\Phi(x)|dx|. \tag{5.5.5}$$

Now suppose that Φ is bounded. Then $\beta_{n,R} \ll \mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R) \circ \ell_n)$, which tends to $\int_{|x|\geqslant R} M(x)|dx|$ as $n \to \infty$ since we already know that (5.1.6) holds for $1 - \mathrm{ch}_R$. Therefore, (5.5.4) for n_ν, with $\nu \to \infty$ gives

$$\alpha - \int M(x)\Phi_R(x)|dx| \ll \int_{|x|\geqslant R} M(x)|dx|. \tag{5.5.6}$$

Therefore, by letting $R \to \infty$ we obtain (5.5.1) when Φ is bounded.

When $|\Phi(x)| \leqslant \phi_0(|x|)$ as in (b), (5.1.7)(5.1.9) and the Schwarz inequality give (note that $1 - \mathrm{ch}_R$ is the same as its square):

$$\begin{aligned} \beta_{n,R}^2 &\leqslant (\mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R) \circ \ell_n))(\mathrm{Avg}_{X_n^*}(\phi_0 \circ |\ell_n|)^2) \\ &\ll \mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R) \circ \ell_n). \end{aligned} \tag{5.5.7}$$

We also have

$$(1 - \mathrm{ch}_R)(x)\phi_0(R)^2 \leqslant \phi_0(|x|)^2,$$

because the left-hand side equals 0 (resp. $\phi_0(R)^2$) for $|x| \leqslant R$ (resp. $|x| > R$), while $\phi_0(r)$ is positive and monotone non-decreasing. Therefore, again by (5.1.9),

$$\mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R) \circ \ell_n) \leqslant \phi_0(R)^{-2}\mathrm{Avg}_{X_n^*}(\phi_0 \circ |\ell_n|)^2 \ll \phi_0(R)^{-2}. \tag{5.5.8}$$

Therefore, by (5.5.7), we obtain $\beta_{n,R} \ll \phi_0(R)^{-1}$. Therefore, (5.5.4) for n_ν with $\nu \to \infty$ gives

$$\alpha - \int M(x)\Phi_R(x)|dx| \ll \phi_0(R)^{-1}; \tag{5.5.9}$$

hence by letting $R \to \infty$, we obtain (5.5.1) also for this case. This completes the proof of Lemma A.

5.6. Proof of Lemma B. This is based on the following (slight) generalizations of Theorems 14 and 16 of [15].

Theorem (*a la* Jessen–Wintner). *Let $F(z) = \sum_{n\geqslant 1} a_n z^n$, with $a_1 \neq 0$, be convergent for $|z| < \rho \leqslant \infty$. Let $0 < r < \rho$ be given, and $\{r_n\}_{n\geqslant 1}$, $\{\lambda_n\}_{n\geqslant 1}$ be two sequences of positive numbers satisfying $r_n \leqslant r$ for all $n \geqslant 1$, $\lambda_n^{-1} \ll 1$ and $\sum_{n\geqslant 1} \lambda_n^2 r_n^2 < \infty$. Then* (i) *the distribution*

$$\sum_{n=1}^{N} \lambda_n F(r_n e^{2\pi i\theta_n}) \quad (0 \leqslant \theta_1, \ldots, \theta_N < 1) \tag{5.6.1}$$

on $\mathbb{C}$ converges absolutely as $N \to \infty$ and the limit distribution possesses a continuous density. In other words, there exists a unique continuous function $D(w)$ on $\mathbb{C}$ which satisfies

$$\lim_{N\to\infty} \int_0^1 \cdots \int_0^1 \Phi\left(\sum_{n=1}^{N} \lambda_n F(r_n e^{2\pi i\theta_n})\right) d\theta_1 \cdots d\theta_N = \int_{\mathbb{C}} D(w)\Phi(w)|dw| \tag{5.6.2}$$

for any continuous function Φ on $\mathbb{C}$ with compact support.

(ii) *For any $\lambda > 0$, we have*

$$D(w) = O(e^{-\lambda|w|^2}) \quad (|w| \to \infty). \tag{5.6.3}$$

The case $\lambda_n = 1$ $(n \geqslant 1)$ of (i) (resp. (ii)) corresponds to (the main results of) Theorem 14 (resp. 16) of [15]. Their proofs apply also to this generalized case involving $\{\lambda_n\}_{n\geqslant 1}$.

Now, take

$$F(z) = \begin{cases} z/(z-1) & \text{(Case 1)}, \\ -\log(1-z) & \text{(Case 2)}, \end{cases}$$

(so that $\rho = 1$). Moreover, take $\{\mathfrak{p} \notin P_\infty\}$ instead of $n \in \mathbb{N}$, take $N(\mathfrak{p})^{-\sigma}$ for r_n, take $\log N(\mathfrak{p})$ (resp. 1) for λ_n in Case 1 (resp. Case 2). Note that $\lambda_n^{-1} \ll 1$, and that $\sum_n \lambda_n^2 r_n^2 < \infty$ holds for $\sigma > 1/2$. So it suffices to show that $D = M_\sigma$.

For each $y > 1$, let $P = P_y = \{\mathfrak{p} \notin P_\infty,\, N(\mathfrak{p}) \leqslant y\}$, and $N = |P|$. Put $T_P = \prod_{\mathfrak{p}\in P} \mathbb{C}^1$ and $g_{\sigma,P}(t_P) = \sum_{\mathfrak{p}\in P} g_{\sigma,\mathfrak{p}}(t_\mathfrak{p})$ $(t_P = (t_\mathfrak{p})_{\mathfrak{p}\in P} \in T_P)$, where $g_{\sigma,\mathfrak{p}}$ is as defined by (4.1.2). Then for any continuous function $\Phi(w)$ on $\mathbb{C}$ with compact support, we have

$$\begin{aligned}\int_0^1 \cdots \int_0^1 \Phi\left(\sum_{n=1}^{N} \lambda_n F(r_n e^{2\pi i\theta_n})\right) d\theta_1 \cdots d\theta_N &= \int_{T_P} \Phi(g_{\sigma,P}(t_P)) d^\times t_P \\ &= \int_{\mathbb{C}} M_{\sigma,P}(w)\Phi(w)|dw| \end{aligned} \tag{5.6.4}$$

($d^\times t_P$: the normalized Haar measure of T_P). The first equality is obvious from the above definitions of λ_n, r_n, $g_{\sigma,P}$, and second equality is by [8, Theorem 1] (Case 1), [11, Proposition 3.1] (Case 2). As $y \to \infty$, $M_{\sigma,P}(w)$ converges uniformly to $M_\sigma(w)$

([8, Th. 2] (Case 1), [11, Proposition 3.5] (Case 2)). Therefore, as $N \to \infty$, (5.6.4) converges to

$$\int D(w)\Phi(w)|dw| = \int M_\sigma(w)\Phi(w)|dw| \tag{5.6.5}$$

for all such $\Phi(w)$, which proves $D(w) = M_\sigma(w)$, as desired. (Cf. also Theorem 19 of [15] for Case 2, with $N(\mathfrak{p})$ in place of p_n).

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