

A dissertation for the degree of Doctor of Philosophy

**Characteristics of flow field and wall mass transfer rates downstream of a circular squared-edged orifice plate in a round pipe**

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# Abstract

This dissertation concentrates on the experimental study of characteristics of flow field and wall mass transfer behind an orifice in a round pipe. The main objective of this thesis is to understand the pipe-wall-thinning phenomenon behind an orifice in a round pipe from the viewpoint of hydrodynamics.

The experimental work can be divided by two parts, in the first part, Particle Image Velocimetry (PIV) was used to measure the flow field systematically. The definitions of the core-, recirculation-, and shear-layer regions were given according to our systematic measurements for three different orifice-to-pipe-diameter ratios (defined as  $\beta$  ratio), in which  $\beta = 0.41, 0.50$  and  $0.62$ . In the experiments, the flow is fully developed pipe flow well upstream of the orifice plate, and the Reynolds number is in the range of  $25000 \sim 55000$ . The flow field is found to be nearly independent of Reynolds number, whereas the orifice  $\beta$  ratio has very important effects on the downstream flow field. More specifically, the effects of  $\beta$  ratio on the mean flow field, Reynolds stress, shear layer growth rate and energy budget terms are discussed. We find out that the profiles of the mean velocity, Reynolds stress or even two-point correlation in the recirculation region are independent of  $\beta$  ratio when the orifice height and the mean streamwise velocity are selected as the typical velocity and length scales. However, in the shear-layer region, only profile of mean streamwise velocity has self-similar property and shows  $\beta$  ratio independence. The profiles of Reynolds stresses and turbulent kinetic energy budget terms are only independent of  $\beta$  ratio very close to the orifice plate. As the flow progresses, the core-region collapses and the pipe wall exerts more and more effects on the shear-layer region, therefore the profiles of Reynolds stresses and turbulent kinetic energy budget terms show strong dependence on the  $\beta$  ratio.

We also presents the uncertainty analysis on the instantaneous velocity and statistical quantities based on two state-of-art methods, namely, the image-matching method and Taylor-series expansion method. It is found out that both uncertainties of the instantaneous velocities and statistical quantities are strongly dependent on the location of the flow field. Therefore, it is impossible to use a uniform uncertainty value for the whole flow field. The experimental results and uncertainty values of statistical quantities makes the experimental data a useful reference for the validation of numerical codes in orifice flows.

In the second experimental part, the flow field and wall mass transfer rate downstream an orifice were measured simultaneously probably for the first time by

means of PIV and Limiting Diffusion Current Technique (LDCT), respectively. The hydrodynamic conditions in the simultaneous measurement were exactly the same as those in the PIV measurement of the first part, and the arrangement of the electrodes were exactly the same as that in the mass transfer measurement in our group. The Schmidt number of the solution is approximately 2100, and the flow field and wall mass transfer results were validated by our previous data and good agreements were found. There is a response time lag between the flow field and wall mass transfer rate. Even at high Schmidt numbers where the concentration boundary layer is very thin, the large-scale flow structures are found to be important to the wall mass transfer rate. There is a compression and expansion eigenmode in the recirculation region, which accounts for about 10% of total turbulent kinetic energy in the recirculation region and is independent of  $\beta$  ratio. The stagnation plane of this compression and expansion eigenmode in the recirculation region is found out to agree very well with the position of the maximum mean wall mass transfer rate.

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