SYSTEM RELIABILITY BOUNDS ANALYSIS USING LINEAR PROGRAMMING AND ITS APPLICATION

(線形計画法を用いたシステム信頼性の上下限解析法とその適用に関する研究)

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2014

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Chapter 1

INTRODUCTION

1.1 Research Background

Engineering systems such as buildings, bridges, piping systems, gas treatment systems and power distribution systems usually have a large number of components with complex relationship among the components. The failure of an engineering system could have catastrophic effects involving loss of life and property. For example, in the 2008 Wenchuan Earthquake, the collapses and damages of a large number of buildings and bridges resulted the huge loss to the lives and properties[1,2)].

There are a lot of uncertainties such as the uncertainty in material properties, statistical data, mathematical analysis models, etc. in the engineering system. With the effect of those uncertainties, the performance of engineering systems would be not deterministic. Then, the failure of an engineering system would be probabilistic. In the last decades, engineers have recognized the importance of the analysis of the failure probability of engineering systems, and the related research has become one of the most important research area in the engineering systems.

In general, a system consists of a number of interrelated and interdependent components. Also, a system with a set of components can be considered as a component in a large system. There could be many different types of systems such as a cell, a bridge, a galaxy. Engineering systems considered in this research have two important states, i.e., functional state and failure state, and the probability that a system is in its functional state (design purpose) during a specified period of time is the reliability of the systems. The failure probability of a system is the complement of the reliability of the system.

Many analysis methods have been developed to estimate system reliability; they can generally be categorized into: either analytical methods or simulation based methods[3–16]. The analytical methods such as Dunnet-sobel class correlation matrix method[17] and product of conditional marginals[18] is often based on some assumptions such as an ideal mathematical models of the system. Even though the analytical methods present elegant approaches of handling system reliability problems, each of them has its own assumptions which limited its application.

Monte Carlo (MC) simulations provide an estimate of the failure probability by simulating a large number of samples of the random variables related to the failure of the system[19–21]. It is a simple method which is applicable to a wide range of systems including realistic and precise representations of engineering systems with a large number of components. However, for practical structural systems with high-reliability, a large number of simulations have to be performed, and thus, MC simulation can be computationally expensive and inefficient.

Because estimation of the failure probability of a system usually is difficult or time consuming especially when there exists a dependency among the component states and when the number of components is large. Researchers are trying to find the upper and lower bounds on the exact failure probability of the system, such as simple bounds and Ditlevsen bounds[4]. Among those, there are some methods that use linear programming (LP) for efficiency. Hailperin first explored LP to estimate the best possible inequalities for the probability of a logical function (Boolean function) of events[22]. Because the number of design variables in the LP problem increases rapidly with the number of events, Hailperin's method is only applicable to a small size of logical function of events. The accuracy of some theoretical bounds has been examined by use of the LP estimating the best possible inequalities by Kounias and Marin[23]. Song and Der Kiureghian proposed the linear programming (LP) bounds method for computing the bounds on the failure probability of general systems based on the information of the joint failure probabilities of k components (when $k = 1$, these joint probabilities become the individual component state probabilities, and the different k represents the different level of information)[24]. The LP bounds method has a number of advantages such as providing the narrowest possible result of bounds on the system failure probability for any level of information and having a wide applicability for many systems[25–27].

There exists, however, a critical drawback in the LP bounds method. The size of the LP problem, which is usually related to the number of design variables and the number of constraints, increases exponentially with the number of components. For a system with n components, the number of design variables in the LP bounds method is $N_d = 2^n$. When $n = 18$, $N_d = 262144$ and the problem can barely be solved with common LP programs on a personal computer. When $n = 100$, this number becomes $N_d \approx 1.27 \times 10^{30}$, which is enormously large. The number of constraints, which depends on the number of design variables and the level of joint failure probabilities, also becomes enormously large in the application of the LP bounds method to a large system. This size issue—both the number of design variables and constraints—would be a hindrance if one wants to use the LP bounds method to estimate the bounds on the failure probabilities of a large system.

In order to overcome the size issue of the LP bounds method, Der Kiureghian and Song propose a multi-scale system reliability analysis, whereby the system is decomposed into subsystems and a hierarchy of analysis is performed by considering each subsystem or set of subsystems "separately"[28]. The decomposition facilitates solution of the system reliability by the LP bounds method, whereby the large LP problems for the entire system is replaced by several LP problems of much smaller size. This facility, however, comes at a cost; the system bounds computed for the decomposed system can be wider than the bounds computed for the intact system with the same level of probability information. Also, the best way to decompose the entire system is difficult to find.

Although the above system reliability analysis methods have been developed to estimate the reliability of the system, they are still limited in dealing with a system with small number of components. Therefore, there are pressing needs for developing efficient and accurate system reliability methods.

1.2 Research Objective and Content

As described in the above section, researchers have paid attention to the system reliability analysis, and a lot of methods evaluating the reliability of the system has been proposed in the last decades. All of these methods faced the challenges related to computation burdens in the system reliability analysis, and it also motivated this Ph.D. research, and the goal of this research is to propose new system reliability analysis methods that can estimate the reliability of a large system efficiently and accurately.

Among the methods of the system reliability analysis, the LP bounds method shows some important advantages such as only depending on the information of the joint failure probabilities of a small set of components and providing the narrowest possible result of bounds on the system failure probability for any level of formation of the joint failure probabilities. However, the main drawback of the LP bounds method, i.e., the size issue of the LP problem, limits its application. Based on the LP bounds method, this research will first propose a new method to overcome the size issue of the LP problem for a pure series system and a pure parallel system, and then extend the new method to the system of the combination of a series and parallel subsystems.

The organization of this thesis consists of the following chapters and the purpose of each chapter is depicted.

Chapter 2:

Chapter 2 gives an outline of the past research on system reliability. First, the details of the method of the reliability analysis such as the first order reliability method (FOSM) and the MC simulation are reviewed. Second, the details of the LP bounds method including its advantages and disadvantages has been reviewed. Finally, as an extension of the LP bounds method, the details of the multi-scale system reliability analysis are reviewed.

Chapter 3:

As an efficient reliability tool for a system with a large number of components, the relaxed linear programming (RLP) bounds method is introduced in Chapter 3. First, the universal generating function (UGF) that would be used in the RLP bounds method has been described. Second, the details of the RLP bounds method are explained.

Third, the RLP bounds method has been improved by its variations such as RLP2 and RLP3. Finally, applications of the RLP bounds method to pure series systems as well as pure parallel systems are demonstrated.

Chapter 4:

Chapter 4 will introduce the extended RLP bounds method based on failure modes as an extension of the RLP bounds method. The new approach is developed for a general system consisting of both series and parallel subsystems. Applications of this approach to a general system are demonstrated.

Chapter 5:

Summaries of major findings of this study and the future research topics are provided in this Chapter.

Chapter 2

PAST RESEARCH ON SYSTEM RELIABILITY

2.1 Overview

Reliability of the system has shown its importance in the analysis, design, and planning of the systems especially for the engineering systems, and many researchers have achieved a lot of achievement in the system reliability analysis in the last decades.

In order to acquire the systematic knowledge of reliability analysis, this chapter provide a wide and fundamental introduction, from the basic definition such as component state to estimation method of system reliability such as the linear programming (LP) bounds method. The importance of this chapter is to review the LP bounds method and multi-scale system reliability analysis, their advantages and disadvantages are also summarized.

In section 2.2, some basic concepts of reliability analysis is reviewed.

In section 2.3, the linear programming (LP) bounds method is reviewed.

In section 2.4, the multi-scale system reliability analysis is reviewed.

In section 2.5, the reliability analysis of a system subjected to common source of hazard is reviewed.

In section 2.6, the summary of this chapter is proposed.

2.2 Reliability Analysis Method

2.2.1 Component State

In engineering, it is generally assumed that a component can be in one of two possible states, i.e., functional or failure, and the vector of a component states can be expressed as

$$\boldsymbol{E} = (F, \overline{F}) \tag{2.1}$$

where F denotes the failure state of a component and $\overline{F}$ denotes its complement (functional state).

2.2.2 System Reliability State

In system reliability, it is generally assumed that each component of a system, as well as the entire system, can be in one of two possible states, i.e., functional or failure. The vectors of the component states can be expressed as

$$\boldsymbol{E_i} = (F_i, \overline{F_i}), \qquad i = 1, 2, \ldots, n \tag{2.2}$$

where F_i denotes the failure state of component i and $\overline{F_i}$ denotes its complement (functional state).

The failure state of a system, F_{system}, will be expressed as

$$F_{\text{system}} = f(\boldsymbol{E_1}, \boldsymbol{E_2}, \ldots, \boldsymbol{E_n}) \tag{2.3}$$

where $f(\bullet)$ is the function of failure states of the components or their complements.

A series system fails if any of its component fails. It is typified by a chain, and also called a "weakest link" system. An example of a simple series system is shown in Figure 2.1. The load performed at the system is denoted as S. Failure of any component, such as the failure of the component with resistance R_1, will cause the failure of the system. Thus, the $f(\bullet)$ function have only union operations in a series system, i.e.

$$\begin{aligned} F_{\text{series system}} &= F_1 \cup F_2 \cup F_3 \cup \cdots \cup F_n \\ &= \bigcup_i F_i \end{aligned} \tag{2.4}$$

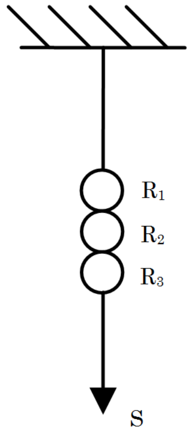

Figure 2.1: An Example of Series Systems

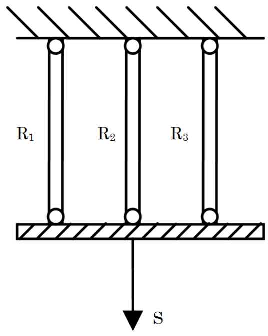

Figure 2.2: An Example of Parallel Systems

A parallel system is a system that fails only when all of the components fail. An example of a simple parallel system is shown in Figure 2.2, in which the load performed at the system is denoted as S. The failure of all components corresponding to the resistance forces R_1, R_2, and R_3 will cause the failure of the system. Thus, the $f(\bullet)$ function have only intersection operations in a parallel system, i.e.

$$\begin{aligned} F_{\text{parallel system}} &= F_1 \cap F_2 \cap F_3 \cap \cdots \cap F_n \\ &= \bigcap_i F_i \end{aligned} \tag{2.5}$$

In general, a general system is a system consists of series and parallel subsystems. Therefore, the function $f(\bullet)$ includes both union and intersection operations. There are two basic formations of the general systems, i.e., a system represented by a series of parallel subsystems and by a parallel of series subsystems. The system represented by a series of parallel system can be expressed in terms of "weakest link", i.e.

$$F_{\text{system}} = \bigcup_{k} \bigcap_{i \in C_k} F_i \tag{2.6}$$

where C_k is the set of component indices of which corresponding components constitute the kth "weakest link".

The system represented by a parallel of series subsystems can be expressed in terms of "link sets", i.e.

$$F_{\text{system}} = \bigcap_{l} \bigcup_{i \in L_l} F_i \tag{2.7}$$

where L_l is the set of component indices of which corresponding components constitute the lth "link set". The complementary system state $\overline{F}_{\text{system}}$ consist of the combination of all unions of the complementary component states $\overline{F_i}$, $i \in L_l$. Using De Morgan's rule one can obtain the formulation as the form in Equation (2.7).

2.2.3 Limit State Function

The limit state function an important function relative to the failure of a system, also is a function connecting the variables represented the system state and the corresponding component states that are relative to the failure of the system.

In engineering system, a system is in a "safe state" if it can fulfill the design requirements. It is in a "failure state" if it fails to fulfill the design requirements. If the system is on the point of failure to fulfill the design requirements, it is at the "limit state". In general, the state of the system can be either the "safe state" or the "failure state".

In order to distinguish the state of a system, the following limit state function is often considered

$$g(\boldsymbol{X}) = g(X_1, X_2, \ldots, X_n) \begin{cases} > 0 & \text{safe state} \\ = 0 & \text{limit state} \\ < 0 & \text{failure state} \end{cases} \tag{2.8}$$

where $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ are the basic variables of the system such as resistances and loads. The surface $g(\boldsymbol{X}) = 0$ is called the limit state surface.

2.2.4 Conventional Reliability Analysis Methods

As some of the conventional reliability analysis methods, the first order reliability method (FORM), product of conditional marginals (PCM), Dunnet-Sobel class correlation matrix method, Monte Carlo (MC) simulation, and bounds on system reliability are outlined in the following.

1) First Order Reliability Method

The first order reliability method (FORM) is one of the most important estimation method for the reliability of engineering systems, and researchers of the engineering systems have paid great attention to the FORM in the last decades. The FORM has a wide application in the practical engineering system.

The "first order" in FORM means that the limit state function $g(\boldsymbol{X})$ is linearly approximated by use of the first order term in its Taylor expansion. The procedure of FOSM can be expressed as

a) Consider a limit state function as

$$g(\boldsymbol{X}) = g(X_1, X_2, \ldots, X_n) \tag{2.9}$$

where X_i are uncorrelated random variables.

b) The random variable X_i (X space) is transformed into standard normal random variable U_i (U space) as follows

$$U_i = \Phi^{-1}(F_X(x_i)) \tag{2.10}$$

where $F_X(\bullet)$ is the marginal cumulative distribution function of X, and $\Phi(\bullet)$ is the cumulative distribution function of the standard normal random variable.

Then, the limit state function in X space is transformed to U space

$$G(\boldsymbol{U}) = G(U_1, U_2, \ldots, U_n) \tag{2.11}$$

c) Reliability index in U space

From the simple geometric meaning in U space, one can find that the smallest distance (β) from the limit state surface $G(\boldsymbol{U}) = 0$, $\boldsymbol{u^*} = ({u_1}^*, {u_2}^*, \ldots, {u_n}^*)$ is the point on the limit state surface closest to the origin, and called design point. β can be expressed as

$$\beta = \frac{-\sum_{i=1}^{n} \left.\frac{\partial G(\boldsymbol{u})}{\partial u_i}\right|_{u^*} \cdot {u_i}^*}{\sqrt{\sum_{i=1}^{n} \left(\left.\frac{\partial G(\boldsymbol{u})}{\partial u_i}\right|_{u^*} \right)^2}} \tag{2.12}$$

Let

$$\alpha_i = \frac{-\left.\frac{\partial G(\boldsymbol{u})}{\partial u_i}\right|_{u^*}}{\sqrt{\sum_{i=1}^{n} \left(\left.\frac{\partial G(\boldsymbol{u})}{\partial u_i}\right|_{u^*} \right)^2}} \tag{2.13}$$

Then,

$${u_i}^* = \alpha_i \cdot \beta \tag{2.14}$$

β can be obtained by solving the simultaneous nonlinear Equations (2.11), (2.12), and (2.13).

d) The failure probability can be expressed as

$$P_f = \Phi(-\beta) \tag{2.15}$$

β is called reliability index.

The details of the FORM can be found in the reference[3, 10, 29].

2) Product of Conditional Marginals

Pandey M.D. proposed the product of conditional marginals (PCM), which is an approach to estimate the multinormal distribution function effectively. The PCM method obtained the advantages of calculation by estimating the joint normal distribution of two variables based on the conditional fractile and conditional correlation coefficient, and could be reasonably accurate up the 20 dimensional multinormal integrals.

In PCM method, instead of conducting the multinormal integral, the integral is estimated by a product of conditional probability as follows[18]:

$$\begin{aligned} \Phi_m(c, R) &= P\left[(X_m \le c_m)| \bigcap_{k=1}^{m-1} (X_k \le c_k)\right] \times P\left[(X_{m-1} \le c_{m-1})| \bigcap_{k=1}^{m-2} (X_k \le c_k)\right] \\ &\quad \times \cdots \times P(X_1 < c_1) \\ &\approx \prod_{k=1}^{m} \Phi(c_{k|(k-1)}) \end{aligned} \tag{2.16}$$

where X_i, $i = 1, 2, \ldots, m$ are normal random variables. $c_{k|(k-1)}$ denotes a conditional normal fractile, and its conditional normal variable can be obtained as[30)]

$$\Phi(c_{k|(k-1)}) \approx P\left[(X_k \le c_k)| \bigcap_{i=1}^{k-1}(X_i \le c_i)\right] \tag{2.17}$$

The general form of the conditional correlation between X_k and X_j can be expressed as[31)]

$$r_{(k+1)(j+1)|k} = \frac{r_{(k+1)(j+1)|(k-1)} - r_{k(k+1)|(k-1)} r_{k(j+1)|(k-1)} B_{k|(k-1)}}{\sqrt{[1 - r^2_{k(k+1)|(k-1)} B_{k|(k-1)}][1 - r^2_{k(j+1)|(k-1)} B_{k|(k-1)}]}} \tag{2.18}$$

Based on the PCM method, Yuan and Pandey[32)] proposed an improved PCM (IPCM) version to improve its accuracy. The details of PCM and IPCM can be found in the reference[18, 32)].

3) Dunnet-Sobel Class Correlation Matrix Method

Dunnett and Sobel proposed a Dunnet-Sobel (DS) class correlation matrix, which is specified as[33)]

$$\begin{aligned} \rho_{ij} &= r_i r_j \\ i &\neq j, \qquad \rho_{ii} = 1. \end{aligned} \tag{2.19}$$

Suppose a system consisting of k components with reliability index β_i, $i = 1, 2, \ldots, k$ and correlation matrix $\mathbf{R}\equiv[\rho_{ij}]$. Based on the DS class correlation matrix, Dunnett and Sobel proposed an approach to estimate the joint probabilities of k variates by the one-dimensional integral as follows[17)]

$$\begin{aligned} P_{12\ldots k} &= P\left(\bigcap_{i=1}^{k} F_i\right) \\ &= \Phi_k\left(\beta_1, \ldots, \beta_k; \mathbf{R}\right) \\ &= \int_{-\infty}^{\infty}\left[\phi(t)\prod_{i=1}^{k}\Phi\left(\frac{\beta_i - r_i t}{\sqrt{1 - r_i^2}}\right)\right] dt \end{aligned} \tag{2.20}$$

where $\Phi_k(u_1, \ldots, u_k; \mathbf{R})$ is the k-variate standard normal cumulative distribution function with correlation matrix $\mathbf{R}\equiv[\rho_{ij}]$, and $\phi(\bullet)$ denotes the one-dimensional standard normal probability density function. The details can be found in reference[17, 33)].

4) Monte Carlo Simulation

Monte Carlo (MC) simulation is an important tool in the pricing of derivative securities in financial, project management, the risk management, and other forecasting fields. The essence of the MC simulation is a mathematical algorithm that involves of repeated random or pseudorandom numbers (sampling) for the solution of the forecasting model. In the system reliability analysis, the MC simulation has widely application to the estimation of the failure probability of the system, and it provides an unique estimate of the failure probability of the system.

In engineering system, the failure probability of the system, P_f, can be expressed as

$$\begin{aligned} P_f &= P(g(\boldsymbol{X}) \leq 0) \\ &= \int \cdots \int_{g(X)\leq 0} f_X(x)dx \end{aligned} \tag{2.21}$$

where $g(\boldsymbol{X}) = g(X_1, X_2, \ldots, X_n)$ is the limit state function of the system, and $f_X(x)$ is the joint probability density function of the random vectors $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. Obviously, the direct numerical integration of Equation (2.21) is usually impossible when n is large.

In the application of MC simulation, a lot of repeated experiments with randomly generated samples $\hat{x}_i$ of random vectors $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ are performed many times. Then the number of the samples with value of the limit state function $g(\hat{x})$ less than 0, i.e., in the failure state, are counted. Suppose N trials have been performed, the failure probability of the system can be approximately expressed as

$$P_f \approx \frac{n(g(\hat{x}_i) < 0)}{N} \tag{2.22}$$

where $n(g(\hat{x}_i) < 0)$ is the number of trails with $g(\hat{x}_i) < 0$. It is obvious that the larger number of experiments, N, will be required when the higher accuracy of the system failure probability is desired.

The efficiency of the MC simulation can be improved by considering the variance reduction techniques such as the importance sampling, the details can be found in the reference[10, 19, 34)]. However, for the practical high reliability structural systems, MC simulation can be computationally expensive and inefficient.

5) Bounds on System Reliability

As described in Chapter 1, the computation of the probability of the system could be an extremely difficult task, particularly when there exists a dependency among the component states and when the number of components is large. Usually, the probabilities of the intersections of all combinations of components states is necessary. For example, for a series system with two-state components, Equation (2.4) can be written as

$$\begin{aligned} P(F_{\text{series system}}) &= P(F_1) + P(F_2) - P(F_1 \cap F_2) + P(F_3) - P(F_1 \cap F_3) \\ &\quad - P(F_2 \cap F_3) + P(F_1 \cap F_2 \cap F_3) + \ldots \qquad (2.23) \\ &= \sum_i P(F_i) - \sum_{i<j} P(F_i \cap F_j) + \sum_{i<j<k} P(F_i \cap F_j \cap F_k) - \ldots \end{aligned}$$

Similarly, the expressions corresponding to Equations (2.6) and (2.7) can be obtained.

Because the computation of the system failure probability is difficult, an alternative approach is to develop upper and lower bounds on the system failure probability. Many researchers are interesting in developing bounds on the system failure probability that employ the individual component failure probabilities, $P_i = P(F_i)$, and the joint probabilities of a small number of component states, i.e., the joint failure probabilities of two components, $P_{ij} = P(F_i) \cap P(F_j)$, $i < j$, the joint failure probabilities of three components, $P_{ijk} = P(F_i) \cap P(F_j) \cap P(F_k)$, $i < j < k$, etc.

Based on the individual component failure probabilities, Boole developed the probability bounds on a series system as[35)]

$$\max P_i \leq P(\overset{n}{\underset{i=1}{\cup}} F_i) \leq \min(1, \sum_{i=1}^{n} P_i) \qquad (2.24)$$

These bounds are called as Boole bounds, and it is the narrowest possible bounds if the given information is limit to the individual component failure probabilities[36)]. Unfortunately, the bounds developed by Boole are usually too wide to be useless for the realistic application. Based on the information of individual component failure probabilities and the joint failure probabilities of two components, Kounias[37)], Hunter[38)], and Ditlevsen[39)] proposed the following widely used bounds for series systems:

$$P_1 + \sum_{i=2}^{n} \max(0, P_i - \sum_{j=1}^{i-1} P_{ij}) \leq P(\overset{n}{\underset{i=1}{\cup}} F_i) \leq P_1 + \sum_{i=2}^{n} (P_i - \max_{j<i} P_{ij}) \qquad (2.25)$$

The accuracy of the bounds by Equation (2.25) depends on the ordering of the component states, and there are $n!$ possible ordering alternatives. The order maximizing the lower bound could be different from the order minimizing the upper bound[40]. Since it is practically impossible to find an ordering rule that can make sure to obtain the narrowest bounds of Equation (2.25), one have to consider all the possible $n!$ ordering alternatives in order to obtain the narrowest bounds of Equation (2.25). Also, the bounds obtained from Equation (2.25) can not guarantee the narrowest possible bounds. These bounds are called as "KHD bounds".

Based on the concept of KHD bounds, Hohenbichler and Rackwitz[41] and Zhang[42] proposed the formula of lower bounds and upper bounds using the joint failure probabilities of three components, the joint failure probabilities of four components, etc, for series system. These bounds are called as "Zhang bounds". Zhang's bounds also have the order-dependency problem similar to KHD bounds. The formulas of lower bounds and upper bounds of Zhang's bounds using the information up to the joint failure probabilities of three components can be expressed in Equations (2.26) and (2.27), respectively.

$$P(\bigcup_{i=1}^{n} F_i) \leq P_1 + P_2 - P_{12} + \sum_{i=3}^{n}[P_i - \max_{k\in(2,3,\ldots,i-1),j<k}(P_{ik} + P_{ij} - P_{ijk})] \tag{2.26}$$

$$P(\bigcup_{i=1}^{n} F_i) \geq P_1 + P_2 + P_{12} + \sum_{i=3}^{n}\max(0, P_i - \sum_{j=i}^{i-1} P_{ij} + \max_{k\in(1,2,\ldots,i-1)} \sum_{j=1,j\neq k}^{i-1} P_{ijk}) \tag{2.27}$$

For the parallel systems, based on the individual component failure probabilities, Boole proposed the narrowest possible bounds as[35,36]

$$\max(0, \sum_{i=1}^{n} P_i - (n-1)) \leq P(\bigcap_{i=1}^{n} F_i) \leq \min P_i \tag{2.28}$$

However, the bounds obtained from Equation (2.28) are usually too wide and useless for the realistic application. There does not exist the theoretical bounds for parallel systems using the individual component failure probabilities, the joint failure probabilities of two components, and the joint failure probabilities of three components. However, the complement of a parallel system can be converted to a series system by use of De Morgan's rule, and then the bounds can be obtained by using Equations (2.25), (2.26), and (2.27).

For a general system, there does not exist the theoretical bounds formulas. One can decompose a general system into a series system of parallel sub-systems or a parallel system of series subsystems, and then use the combination of the Equations (2.25), (2.26), (2.27), and (2.28) to obtain the relaxed bounds. However, these bounds usually are too wide and unacceptable.

2.2.5 Multi-state System Reliability

Because the methods that proposed in this research is applicable to multi-state system, the related concepts will introduce in this section.

1) Multi-state Component

In engineering system, a component could have more than two levels of performance, i.e., two state (perfect functioning and complete failure), during a specified period of time. A component having only two state can be called as a two-state component, and a component having more than two state can be called a multi-state component, which can be expressed as

$$\boldsymbol{E}_{=}(E_1, E_2, \ldots, E_l), \qquad j = 1, 2, \ldots, l \tag{2.29}$$

where l is the number of states of the component. The event that the component being in the jth state can be expressed as E_j.

Similar to the reliability of the two-state component, the probability that a multi-state component being in its functional states (design purpose) during a specified period of time is the reliability of the component. The failure probability of the multi-state component is the complement of its reliability.

2) Multi-state System

If the components of the system, as well as the entire system, only could be in one of two possible states, i.e., functional or failure, this system is called the binary system. If any of the components of the system or the entire system could have more than two states, this system is called the multi-state system. Consider a system with n multi-state components. The vectors of the states of the multi-state components can be expressed as

$$\boldsymbol{E_i} = (E_{i1}, E_{i2}, \ldots, E_{il_i}), \qquad i = 1, 2, \ldots, n \qquad j = 1, 2, \ldots, l_i \tag{2.30}$$

where E_{ij} denotes the event that the component i being in the jth state.

A particular state of the system can be denoted as E_{system}, then, E_{system} can be written as

$$E_{\text{system}} = f(\boldsymbol{E_1}, \boldsymbol{E_2}, \ldots, \boldsymbol{E_n}) \tag{2.31}$$

where the $f(\bullet)$ function consists of the combination of the events of each component, i.e., the unions of the events of the components and/or the intersections of the events of the components.

Similar to the binary system reliability, the probability that a multi-state system being in its functional states (design purpose) during a specified period of time is the reliability of the component. The failure probability of the multi-state system is the complement of its reliability.

Multi-state system reliability models allow both the system and its components to assume more than two levels of performance, and more realistic and more precise representations of engineering systems can be obtained by use of multi-state reliability models. Multi-state system reliability models are much more complex and present major difficulties in system definition and performance evaluation. MC simulations is one of the conventional reliability analysis method of the multi-state system.

2.3 Linear Programming Bounds Method

Linear Programming (LP) is the analysis of problems in which a linear function of a number of variables is to be minimized or maximized when those variables are subject to a number of restraints in the form of linear equalities and inequalities[43–45)]. The feasible region of the solution of the LP is a convex polyhedron, which is a set defined as the intersection of finitely many half spaces, each of which is defined by a linear inequality. Its objective function is real-valued affine function defined on this polyhedron. An LP algorithm finds a point in the polyhedron where this function has the smallest (or largest) value if such a point exists[43–45)].

Song and Der Kiureghian proposed the linear programming (LP) bounds method for computing the bounds on the failure probability of general systems based on the information of the joint failure probabilities of k components. The design variables of

LP in the linear programming (LP) bounds method is the probabilities of the mutually exclusive and collectively exhaustive (MECE) events of the system.

2.3.1 Mutually Exclusive and Collectively Exhaustive Events

If not any of events intersect another event in a sample space, i.e., the intersection of two events is empty, those events are mutually exclusive. If the union of those events cover all the events in a sample space, those events are collectively exhaustive. It is possible that the events in a sample space are both mutually exclusive and collectively exhaustive (MECE).

Considering a system with n two-state components, Hailperin[22)] divided the sample space of component states into 2^n MECE events. Each MECE event consists of a distinct intersection of the failure events F_i and their complements $\overline{F_i}$ (functional events), $i = 1, 2, \ldots, n$. They are called the basic MECE events and denoted by e_r, $r = 1, 2, \ldots, 2^n$. For example, for a system with $n = 3$ two-state components, there are $2^3 = 8$ basic MECE events (see Figure 2.3)

$$\begin{aligned} e_1 &= F_1 \cap F_2 \cap F_3, & e_2 &= \overline{F_1} \cap F_2 \cap F_3, \\ e_3 &= F_1 \cap \overline{F_2} \cap F_3, & e_4 &= F_1 \cap F_2 \cap \overline{F_3}, \\ e_5 &= \overline{F_1} \cap \overline{F_2} \cap F_3, & e_6 &= \overline{F_1} \cap F_2 \cap \overline{F_3}, \\ e_7 &= F_1 \cap \overline{F_2} \cap \overline{F_3}, & e_8 &= \overline{F_1} \cap \overline{F_2} \cap \overline{F_3}. \end{aligned} \tag{2.32}$$

2.3.2 Linear Programming Bounds Method

After Dantzig developed the simplex method in 1947[46)], the LP has became a practical tool. Based on an LP model of the plastic analysis, Nafday et al.[47)] proposed an approach for identifying the mechanism failure modes of building frames. Corotis and Nafday[48)] proposed a combined approach of the MC simulation and the LP to estimate the system reliability of the complex structural system. The approach has the advantage when the traditional MC simulation is inefficient for the practical high reliability structural systems. Although the LP has been introduced for the estimation of system reliability of the structural systems, it has not been used to estimate the

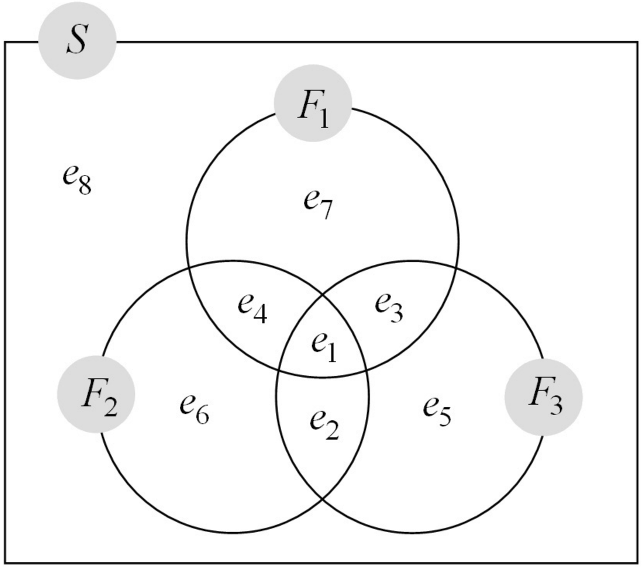

Figure 2.3: Basic MECE event e_r for a system with 3 two-state components.

system failure probability directly. Developing a linear programming (LP) bounds method, Song and Der Kiureghian first show that the LP is useful for estimating bounds on the system failure probability based on the MECE events[24, 49)].

Because that the basic MECE events are mutually exclusive, the probability of any union of events can be obtained by the sum of the corresponding probabilities. Particularly, the probability of any failure event F_i can be obtained by the sum of the probabilities of the basic MECE events that constitute the event F_i. Similarly, any joint failure probability can be obtained by the sum of the basic MECE events that constitute the intersection events. For example, for a system with three two-state components as shown in Figure 2.3, the component failure probability is expressed as

$$\begin{aligned} P(F_1) &= P_1 = p_{m_1} + p_{m_3} + p_{m_4} + p_{m_7} \\ P(F_2) &= P_2 = p_{m_1} + p_{m_2} + p_{m_4} + p_{m_6} \\ P(F_3) &= P_3 = p_{m_1} + p_{m_2} + p_{m_3} + p_{m_5} \end{aligned} \tag{2.33}$$

The joint failure probabilities of two components can be expressed as

$$\begin{aligned} P(F_1 \cap F_2) &= P_{12} = p_{m_1} + p_{m_4} \\ P(F_1 \cap F_3) &= P_{13} = p_{m_1} + p_{m_3} \\ P(F_2 \cap F_3) &= P_{23} = p_{m_1} + p_{m_2} \end{aligned} \tag{2.34}$$

The joint failure probability of three components can be expressed as

$$P(F_1 \cap F_2 \cap F_3) = P_{123} = p_{m_1} \tag{2.35}$$

More generally, we write

$$\begin{aligned} P(F_i) = P_i &= \sum_{m_r : e_r \subseteq F_i} p_{m_r} \\ P(F_i \cap F_j) = P_{ij} &= \sum_{m_r : e_r \subseteq F_i \cap F_j} p_{m_r} \\ P(F_i \cap F_j \cap F_l) = P_{ijl} &= \sum_{m_r : e_r \subseteq F_i \cap F_j \cap F_l} p_{m_r}, \qquad \text{etc.} \end{aligned} \tag{2.36}$$

It should be noted that only the joint failure probability of up to k components such as $P(F_i)$ and $P(F_i \cap F_j), i, j = 1, 2, \ldots, n$, and $i \neq j$, is known, but any probability of the basic MECE event e_r, $p_{m_r} = P(e_r)$, is not known in advance.

Based on the basic axioms of probability, the above probabilities $p_m = \{p_{m_1}, p_{m_2}, \ldots, p_{m_{2^n}}\}$ have the following linear constraints:

$$\sum_{m_r=1}^{2^n} p_{m_r} = 1 \tag{2.37}$$

$$p_{m_r} \geq 0; \qquad r = 1, 2, \ldots, 2^n \tag{2.38}$$

The lower bound and the upper bound of the system failure probability can be obtained as the minimum and the maximum of the objective function of the LP, respectively. We can formulate the LP for this analysis as follows:

$$\begin{aligned} &\text{minimize} \quad (\text{maximize}) && \mathbf{c}^T \mathbf{p}_{\boldsymbol{m}} \\ &\text{subject} \quad \text{to} && \mathbf{A_1} \mathbf{p}_{\boldsymbol{m}} = \mathbf{b_1} \\ & && \mathbf{A_2} \mathbf{p}_{\boldsymbol{m}} \geq \mathbf{b_2} \end{aligned} \tag{2.39}$$

where $\mathbf{p}_{\boldsymbol{m}} = \{p_{m_1}, p_{m_2}, \ldots, p_{m_{2^n}}\}$ is the vector of design variables and represents the probabilities of the basic MECE events; $\mathbf{c}$ is a vector that relates the system failure event to the component failure events; $\mathbf{c}^T \mathbf{p}_{\boldsymbol{m}}$ is the linear objective function; and $\mathbf{A_1}$ and $\mathbf{A_2}$ and $\mathbf{b_1}$ and $\mathbf{b_2}$ are the coefficient matrices and vectors respectively, which represent the information given in terms of joint failure probabilities of k components. $\mathbf{A_1}$ and $\mathbf{b_1}$ are obtained from Equation (2.36). $\mathbf{A_2}$ and $\mathbf{b_2}$ are also obtained from

Equation (2.36), but when one has information of the form $P(F_i) \geq x$ or $P(F_i) \leq x$ rather than $P(F_i) = x$. Also, there are additional linear constraints based on the axioms of probability (Equations (2.37) and (2.38))[24)].

For the above three-component system, if one knows $P(F_1) = 0.01$, $P(F_2) = 0.02$, and $P(F_3) = 0.03$, and the objective function is $P(F_1 \cap F_2 \cap F_3) = p_{m_1}$, then $\mathbf{A_1}$ and $\mathbf{b_1}$ based on Equation (2.36) and $\mathbf{c}^T$ are expressed as

$$\mathbf{A_1} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.40}$$

$$\mathbf{b_1} = \begin{bmatrix} 0.01 \\ 0.02 \\ 0.03 \end{bmatrix} \tag{2.41}$$

$$\mathbf{c}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.42}$$

2.3.3 Size of Linear Programming Problem

The size of LP problem would increase rapidly with the increase of the number of design variables and constraints. For a system with n two-state components, the number of design variables (N_d) can be expressed as

$$N_d = 2^n \tag{2.43}$$

There are also one equality and 2^n inequality constraints resulting from the probability axioms (Equation (2.37) and Equation (2.38)), respectively, and there are $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$ equality or inequality constraints resulting from Equation (2.36) when the complete set of joint failure probabilities of each combination of k components, i.e., the joint failure probabilities of all combinations up to each combination of k component, is available. Thus, the total number of constraints of the LP bounds method, N_c, can be expressed as

$$N_c = 2^n + 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} \tag{2.44}$$

Note that we do not need to know the all set of joint failure probabilities of k components at a particular level. Any partial set of joint failure probabilities of k components can be used.

Kiureghian and Song[28)] mentioned that the LP bounds method developed for two-state systems is equally applicable to multi-state systems. For example, consider a system with n components, and the ith component having m_i states, $i = 1, 2, \ldots, n$. The number of basic MECE events of the system (the number of design variables) can be expressed as

$$N_{d_s} = \prod_{i=1}^{n} m_i \tag{2.45}$$

From the Equations (2.43) and (2.44), one can find that the number of design variables, N_d, grows exponentially with the number of components, and the number of constraints, N_c, would also be enormously large when a system is large. For a system with n two-state components and the complete set of joint failure probabilities up to k components, $k = 3$, then $N_d = 262144$ and $N_c = N_d + 987 + 1$ when $n = 18$. When $n = 100$, $N_d \approx 1.27 \times 10^{30}$ and $N_c = N_d + 166750 + 1$. Also, for the same number of components, the number of design variables, N_{d_s}, as shown in Equation (2.45) will rapidly grow with the increasing number of component states when the LP bounds method is extended to multi-state systems.

2.3.4 Advantages and Disadvantages of the Linear Programming Bounds Method

The advantages and disadvantages of the LP bounds method are summarized as follows:

1) Advantages

The LP bounds method has a number of important advantages over other existing methods (e.g., Boole bounds[35)] or Zhang bounds[41,42)]). They include:

a) any "level," i.e., the number (k) of components considered in the joint probabilities of the states, of information can be used, in the form of both in equalities and inequalities;

b) the statistical dependency among component states is easily accounted for in terms of their joint probabilities;

c) since the LP approach can obtain the optimal solution, the method guarantees the narrowest possible bounds for the given information of the individual component probabilities and the joint probabilities of component states;

d) the method is applicable to a general system, including a system that is neither pure series nor pure parallel and a system for which no theoretical formula exists[25–28, 50].

There are two advantages of the LP bounds method over the MC simulation. One is that the MC simulation can be impractical when the failure probability is very small, whereas the LP bounds method is unaffected by the magnitude of the failure probability. Also, the MC simulation is not applicable when the information on the statistical characteristics of one component or on the correlation between one component and another is missing; however, the LP bounds method is still applicable under such circumstances.

2) Disadvantages

Obviously, the main shortcoming of the LP bounds method is that the size of LP problem grows rapidly with the number of components as shown in Section 2.3.3. Even though there are many advanced algorithms for solving the large size of LP problems, e.g., the column generation method by Jaumard et al.[51], the size issue of LP problem of the LP bounds method would still be a hindrance if one want to use the LP bounds method to estimate the bounds on the failure probabilities of a large system. Suppose one has an LP solver that can barely handle $2^{18} = 262144$ design variables and all the constraints of up to the complete set of joint failure probabilities of each combination of up to 3 components, then in theory the limitation of the number of two-state components in the LP bounds method is 18. Clearly, the size issue of LP problem limited the extension of the LP bounds method for the multi-state systems to a system with a very small number of components.

2.4 Multi-scale System Reliability Analysis

In order to extend the applicability of the LP bounds method, a multi-scale approach has been proposed by Der Kiureghian and Song to deal with a system with a larger number of components. The system is decomposed into subsystems and a hierarchy

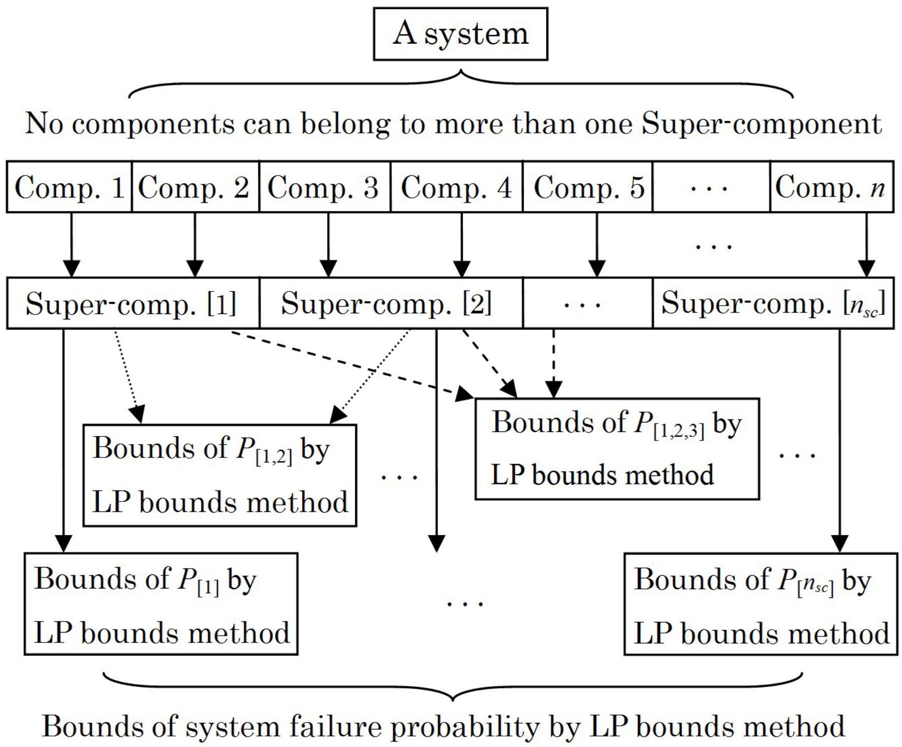

Figure 2.4: Diagram of the multi-scale system reliability analysis.

of analysis is performed by considering each subsystem or set of subsystems "separately"[28)].

2.4.1 Outline of Multi-scale System Reliability Analysis

In multi-scale system reliability analysis, a subset of (a group of) components of the system is considered as a single "super-component"[52)]. If a system with n components has only 1 "super-component" consisting of k components, then this system can be considered as a system with 1 "super-component" and the $n-k$ remaining components. The size of LP problem can be reduced by a factor of 2^{k-1} in this system. Similar to a system with 1 "super-component", the above approach can be proceeded again when a system has more than one "super-component", then the size of LP problem for the entire system can be reduced.

The diagram of multi-scale system reliability analysis is shown in Figure 2.4, where a single "super-component" denoted by the bracket number $[i]$, $i = 1, 2, \ldots, n_{sc}$, and n_{sc} is the number of "super-components". Each "super-component" consists of a group of components, and of course each "super-component" is a system itself. In order to obtain the bounds on the entire system, it is necessary to compute bounds on each "super-component" and the joint failure probabilities of a "super-component" with other components or "super-components". The bounds on failure probability of "super-component" $[i]$ (the bounds on the failure probabilities of subsystem [i]), $P_{[i]}$, can be obtained by the LP bounds method. The bounds on the joint failure probability of "super-components" such as $P_{[i,j]}$ (the bounds on the joint failure probabilities of "super-components" $[i]$ and $[j]$), and the bounds on the joint failure probabilities of "super-components" and components such as $P_{[i],j}$ (the bounds on the joint failure probabilities of "super-component" $[i]$ and component j) can also be obtained by the LP bounds method. Since each "super-component" is treated as a component at the system level, these bounds on the failure probabilities of the "super-components" and the joint failure probabilities of "super-components" with "super-components" or components can be treated as the constraints in solving the LP problem for the entire system. Then the bounds on the entire system can be obtained by the LP bounds method.

Similar to the idea of the "super-components", a subset of the "super-components" can be considered as a "super-super-components". Also, a subset of the "super-super-components" can be considered as a "super-super-super-components" again. Then the size of LP problem for the entire system can be reduced to a manageable size. In essence, instead of solving a single large size of LP problem, the multi-scale system reliability analysis turns to solve a number of smaller size of LP problem by using the LP bounds method.

2.4.2 Guidelines for Effective Selection of "Super-components"

In the multi-scale system reliability analysis, the decomposition of the entire system into subsystems would cause the loss of the information on the components for the entire system level such as the relationship among the "super-components". Such a loss of information leads to the relaxation of the feasible domain that could lead to the wider bounds of the sytem reliability.

There are two factors contribute to the relaxation of the feasible domain by using multi-scale system reliability analysis: the decomposition approach of the entire system (the selection of the "super-components") and the accuracy of the bounds on the failure probability of each "super-component". The different decomposition approaches will lead to the different "super-compnents" in which the number of components and correlation coefficients among compoents are different. Then, different decomposition approaches would cause the different relaxtion of the fesible domain and different accuracy of the failure probability of the entire system. The bounds on the failure probability of each "super-component" that estimated by the LP bounds methods such as bounds on the failure probability of the "super-component" [1] as shown in Figure 2.4, will be used as the information of the failure probabilities of the "super-components". Then, different accuracy of the bounds on the failure probability of each "super-component" could cause the different relaxtion of the fesible domain and a direct effect to the accuracy of the failure probability of the entire system.

Obviously, the selection of the "super-components" is the first step and have important effect on the the accuracy of system failure probabilities of the entire system, thus a set of guidelines for the effective selection of the "super-components" have been listed by Song and Der kiureghian as follows[28)]:

1) Ideally, no components can belong to more than one "super-component".
2) It is desirable that each "super-component" has as few external states as possible, preferably only two.
3) If the selected "super-components" have identical probabilistic characteristics, there is a significant advantage.
4) If the failure probabilities of the selected "super-components" are statistically independent of each other, the significant advantage can be obtained.
5) If the selected "super-components" are disjoint, i.e., the joint failure probabilities of the "super-components" are zero, the advantage can be gained.
6) The most natural candidates for the "super-components" are series and parallel subsystems with the physical representation of the system.
7) Since there could be a lot of decomposition schemes, the decomposition that yields the narrowest bounds is the best scheme.

2.4.3 Advantages and Disadvantages of Multi-scale System Reliability Analysis

The multi-scale system reliability analysis decomposes the entire system into smaller size of the subsystems denoted as the "super-components", then the large size of LP problem on the system failure probabilities can be obtained by solving a number of smaller size of LP problem.

1) Advantages

The multi-scale system reliability analysis provides an approach that allows the determination of the bounds on the reliability of a system with a large number of components by using the LP bounds method. Like the LP bounds method, the wide applicability for many systems is one of the advantages of the multi-scale system reliability analysis. Also like the LP bounds method, the multi-scale system reliability analysis have the same advantages over the MC simulation as shown in Section 2.3.4. The advantages and disadvantages of the LP bounds method are summarized as follows:

2) Disadvantages

There are three obvious disadvantages for the multi-scale system reliability analysis. First, ideally each component of the system should belong to at most one "super-component" for the multi-scale system reliability analysis. Since there are many cases in which numerous components are common in different failure modes, it could be difficult or impossible to select the "super-components" under such circumstances. Second, the size of any "super-component" will still be limited by the restriction on the LP bounds method, which means the limitation of the number of components in each "super-component" is the same with the limitation of the number of components of the LP bounds method. Finally, even though guidelines for the effective selection of the subsystems have been enumerated, the selection might still be difficult or impossible for a general system, rendering the multi-scale approach difficult to apply.

2.5 Reliability Analysis of A System Subjected to Common Source of Hazard

Consider a system subjected to a common source of hazard such as earthquake, and assume that there are several critical failure modes. The correlation among failure modes of the system could be strong because of the common source of hazard. When the extended RLP bounds method based on failure modes is applied, a system with strong correlation among failure modes usually requires values of k_f and $k_{(i)}$ larger than those for a system with weaker correlations in order to achieve comparable accuracy. Such strong correlation among failure modes can be avoided by considering conditional failure probability of the system[53)].

Suppose that X is a random variable causing the statistical dependence among failure modes, which we call a common source random variable (CSRV). Let $f_X(x)$ be the probability density function (PDF) of X. Then, using the theorem of total probability and applying Gaussian integration, we may estimate the system failure probability ($P(F_s)$) by

$$P(F_s) = \int_0^\infty P(F_s|X = x) f_X(x) dx \tag{2.46}$$

where $P(F_s|X = x)$ is the conditional system failure probability given that $X = x$.

The integration of Equation (2.46) can be conducted by using Gaussian integration

$$\begin{aligned} P(F_s) &\approx \int_{x_{\min}}^{x_{\max}} P(F_s|X = x) f_X(x) dx \\ &\approx \sum_{i=1}^{N} w_i P(F_s|X = x) f_X(x_i) \end{aligned} \tag{2.47}$$

where $[x_{\min}, x_{\max}]$ is the range of x used for Gaussian integration. N is the number of points in the Gaussian integration, the x_i ($i = 1, 2, \ldots, N$) are abscissas, and w_i is the corresponding weight factor for x_i.

The values of $x_{\min}$ and $x_{\max}$ can be determined based on the expected order of the accuracy of $P(F_s)$. Since $P(F_s|X = x)$ is a monotonically increasing function, the center of gravity of the product $P(F_s|X = x)f_X(x)$ shifts to the right of the mean of X. In Equation (2.47), because the area under this product to the left of $x_{\min}$

and to the right of $x_{\max}$ is neglected, the range should be wide enough so that the neglected area does not cause a significant error in the estimate. One might consider $P(X \leq x_{\min}) = 10^{-\beta+2}$ and $P(X \geq x_{\max}) = 10^{-\beta-2}$ where the expected order of accuracy is $10^{-\beta}$. For example, if $P(F_s) \approx 10^{-4}$, $x_{\min}$ and $x_{\max}$ was determined so that $P(X \leq x_{\min}) \approx 10^{-2}$ and $P(X \geq x_{\max}) \approx 10^{-6}$.

The number N used in the Gaussian integration is considered acceptable if the differences between the lower bounds and the upper bounds of system failure probability by two Gaussian integration with $N-1$ and N are negligible.

2.6 Summary

In this chapter, some basic concepts of reliability analysis and conventional reliability analysis method are reviewed. The details of the LP bounds method and the multi-scale system reliability analysis have been introduced. The advantages and disadvantages of the LP bounds method have been summarized, respectively. Even though the multi-scale system reliability analysis have extended the applicability of the LP bounds method, the main drawback of the LP bounds method, i.e., the size of LP problem grows rapidly with the number of components, has not been solved. Also, the multi-scale system reliability analysis has its own disadvantages as shown in Section 2.4.3. The challenge of solving the size of LP problem has been remained, which lead to the new method of this research that could solve this size problem of LP.

Chapter 3

RELAXED LINEAR PROGRAMMING BOUNDS METHOD

3.1 Overview

This chapter will introduce a new method named the relaxed linear programming (RLP) bounds method in order to estimate efficiently and accurately the bounds of the reliability of the pure series systems as well as pure parallel systems. The chapter is composed of the following sections:

In section 3.2, the concept of universal generating function (UGF) and the LP bounds method using UGF is introduced.

In section 3.3, the relaxed linear programming (RLP) bounds method and the details of the formulation of constraints of the RLP bounds method is introduced.

In section 3.4, the variation of the RLP bound method (reduction of constraints) are proposed. Numerical examples have been used to demonstrate its application.

In section 3.5, the variation of the RLP bounds method (incomplete information) are introduced. Numerical examples have been used to demonstrate its application.

In section 3.6, the advantages and disadvantages of the RLP bounds method is summarized.

3.2 Universal Generating Function and Reliability Analysis

3.2.1 Introduction to Universal Generating Function

The universal generating function (UGF) is an important tool in discrete mathematics fields, by which various problems can be solved in terms of an uniform program[54)]. Because the failure state and safe state of the system are two discrete states in discrete mathematics fields taking advantage of UGF is considered in this research. A brief introduction of the UGF provided in the following; the further details of the UGF can be found in APPENDIX.

The basic ideas of using the UGF technique in the engineering systems analysis were proposed by Ushakov[55,56)]. In the last decades, Lisniaski and Levitin developed and completed the application of the UGF technique for evaluating and optimizing reliability indices of systems[15,57–59)]. The UGF technique allows one to formulate the entire system states based on the states of its components by using algebraic procedures[60)].

Consider a discrete random variable X with a sampling space $\boldsymbol{x}$ and the corresponding probability mass function (pmf), p_j, which can be expressed as

$$\begin{aligned}
\boldsymbol{x} &= (x_1, x_2, \ldots, x_m) \\
p_j &= P(X = x_j); \qquad j = 1, 2, \ldots, m \\
\boldsymbol{p} &= (p_1, p_2, \ldots, p_m)
\end{aligned} \tag{3.1}$$

Similar to a moment generating function of X(details of moment generating function can be found in APPENDIX), another function related to X that determines its pmf can be expressed as

$$\begin{aligned}
u_X(z) &= E(z^X) \\
&= p_1 z^{x_1} + p_2 z^{x_2} + \cdots + p_n z^{x_{m_i}} \\
&= \sum_{j=1}^{m} p_j z^{x_j}
\end{aligned} \tag{3.2}$$

in which $E(z^X)$ denotes the expectation of z^X.

This function is usually called the z-transform of X. The more properties and details about the generating function and z-transform is referred to the books by Grimmentt and Strizaker[61)] and Ross[62)].

Consider n independent discrete random variables $X_1, X_2, \ldots, X_n$ with respective sample space $\boldsymbol{x_i} = (x_{i,1},\, x_{i,2},\, \ldots,\, x_{i,m})$, and the corresponding pmf $\boldsymbol{p_i} = (p_{i,1},\, p_{i,2},\, \ldots,\, p_{i,m})$. The z-transform of each random variable X_i can be expressed as

$$u_{X_i}(z) = \sum_{j=1}^{m_i} p_{i,j} z^{x_{i,j}} \tag{3.3}$$

and the z-transform of the sum of X_i is the product of the individual z-transform of these variables as follows:

$$\begin{aligned} u_{\sum\limits_{i=1}^{n} X_i}(z) &= E\left(z^{\left(\sum\limits_{i=1}^{n} X_i\right)}\right) \\ &= E\left(\prod_{i=1}^{n} z^{X_i}\right) \\ &= \prod_{i=1}^{n} u_{X_i}(z) \\ &= \sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\cdots\sum_{j_n=1}^{m_n}\left[\left(\prod_{i=1}^{n} p_{i,j_i}\right) z^{(x_{1,j_1}+x_{2,j_2}+\cdots+x_{n,j_n})}\right] \end{aligned} \tag{3.4}$$

Note that when random variables X and Y are independent of each other, $E(XY) = E(X) \cdot E(Y)$.

Levitin[60)] defined the UGF of the function by replacing $(x_{1,j_1} + x_{2,j_2} + \cdots + x_{n,j_n})$ in Equation (3.4) with $f(x_{1,j_1}, x_{2,j_2}, \ldots, x_{n,j_n})$, which denotes an arbitrary function of $X_1, X_2, \ldots, X_n$, as

$$U(z) = \sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\cdots\sum_{j_n=1}^{m_n}\left[\left(\prod_{i=1}^{n} p_{i,j_i}\right) z^{f(x_{1,j_1},x_{2,j_2},\ldots,x_{n,j_n})}\right] \tag{3.5}$$

Then, the UGF of a random variable can simply be defined by Equation (3.3).

For example, consider two random variables X_1 and X_2 with a sample space $x_1 = (1, 2)$ and pmf $p_1 = (0.3, 0.7)$, and $x_2 = (1, 2, 4)$ and $p_2 = (0.2, 0.3, 0.5)$. In order to obtain the pmf of the function $Y = {X_1}^{X_2}$, all of the possible combinations of the values of X_1 and X_2 should be considered. The UGF corresponding to this pmf takes the form:

$$\begin{aligned}
U_Y(z) &= u_{X_1}(z) \underset{f}{\otimes} u_{X_2}(z) \\
&= (0.3z^1 + 0.7z^2) \underset{f}{\otimes} (0.2z^1 + 0.3z^2 + 0.5z^4) \\
&= 0.06z^{f(1,1)} + 0.14z^{f(2,1)} + 0.09z^{f(1,2)} + 0.21z^{f(2,2)} \\
&\quad +0.15z^{f(1,4)} + 0.35z^{f(2,4)} \\
&= 0.06z^{(1^1)} + 0.14z^{(2^1)} + 0.09z^{(1^2)} + 0.21z^{(2^2)} + 0.15z^{(1^4)} + 0.35z^{(2^4)} \\
&= 0.06z^1 + 0.14z^2 + 0.09z^1 + 0.21z^4 + 0.15z^1 + 0.35z^{16} \qquad (3.6)
\end{aligned}$$

where $f(X_1, X_2) = {X_1}^{X_2}$ and $\underset{f}{\otimes}$ is an operation based on the function f.

By collecting the like terms in Equation (3.6), one can obtain

$$U_Y(z) = 0.30z^1 + 0.14z^2 + 0.21z^4 + 0.35z^{16} \qquad (3.7)$$

The sample space and pmf of the variable Y is y=(1, 2, 4, 16) and p_y=(0.30, 0.14, 0.21, 0.35), respectively.

Note that all of the combinations of the values of variables $X_1, X_2, \ldots, X_n$ are MECE and the total number of possible combinations is

$$N = \prod_{i=1}^{n} m_i \qquad (3.8)$$

3.2.2 LP Bounds Method using UGF

This section will propose an approach to introduce the UGF into the LP bounds method. The number of design variables in the LP of this approach is still the same with that of the LP bounds method; however, it will be extend further in the next section.

Consider a system consisting of n statistically dependent components and suppose that the ith component has m_i possible states, $i = 1, 2, \ldots, n$, then the UGF of the component can be defined as

$$u_i(z) = \sum_{j=1}^{m_i} p_{i,j} z^{x_{i,j}}; \qquad i = 1, \ldots, n \qquad (3.9)$$

where the exponent $x_{i,j}$ of $z^{x_{i,j}}$ encodes the state that the ith component is in state j, $p_{i,j}$ denotes the probability corresponding to that state[63].

For example, an UGF of a component with two states, F and $\overline{F}$, has two terms:

$$u(z) = p_1 z^0 + p_2 z^x \tag{3.10}$$

where the 0 of z^0 encodes the state that the component fails, while the x of z^x encodes the state that this component survives; p_1 and p_2, are the failure probability of this component and its complement, respectively.

By extending Equation (3.5), the UGF of the system with statistically dependent components can be expressed as

$$U(z) = \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \cdots \sum_{j_n=1}^{m_n} \left(p_{j_1,j_2,\ldots,j_n} z^{f(x_{1,j_1}, x_{2,j_2}, \ldots, x_{n,j_n})} \right) \tag{3.11}$$

where the exponents $f(x_{1,j_1}, x_{2,j_2}, \ldots, x_{n,j_n})$ of $z^{f(x_{1,j_1}, x_{2,j_2}, \ldots, x_{n,j_n})}$ encode the system states consisting of its components states (each system state corresponds to one basic MECE events), $p_{j_1,j_2,\ldots,j_n}$ denote the probabilities corresponding to its system states.

Because all of the combinations of the values of $X_1, X_2, \ldots, X_n$ are MECE, the number of possible combinations of the values of statistically dependent $X_1, X_2, \ldots, X_n$ is also expressed by Equation (3.8).

In the system with 3 two-state components as shown in Figure 3.1, the UGF of the system can be expressed as

$$\begin{aligned} U(z) \quad = \quad & p_{1,1,1} z^0 + p_{2,1,1} z^{f(x_1)} + p_{1,2,1} z^{f(x_2)} + p_{1,1,2} z^{f(x_3)} \\ & + p_{2,2,1} z^{f(x_1,x_2)} + p_{2,1,2} z^{f(x_1,x_3)} + p_{1,2,2} z^{f(x_2,x_3)} \\ & + p_{2,2,2} z^{f(x_1,x_2,x_3)} \end{aligned} \tag{3.12}$$

where the 0 of z^0 encodes the state that no component survives, and $p_{1,1,1}$ is the probability corresponding to the state encoded by the 0 of z^0; the $f(x_1)$ of $z^{f(x_1)}$, $f(x_2)$ of $z^{f(x_2)}$, and $f(x_3)$ of $z^{f(x_3)}$, respectively, encodes the state that only component 1, only component 2, and only component 3 survives, and $p_{2,1,1}$, $p_{1,2,1}$, and $p_{1,1,2}$ is the probability corresponding to the state encoded by the $f(x_1)$ of $z^{f(x_1)}$, $f(x_2)$ of $z^{f(x_2)}$, and $f(x_3)$ of $z^{f(x_3)}$, respectively; the $f(x_1, x_2)$ of $z^{f(x_1,x_2)}$ encodes the state that only both components 1 and 2 survive, and $p_{2,2,1}$ is the probability corresponding to the state encoded by the $f(x_1, x_2)$ of $z^{f(x_1,x_2)}$; the $f(x_1, x_2, x_3)$ of $z^{f(x_1,x_2,x_3)}$ encodes the

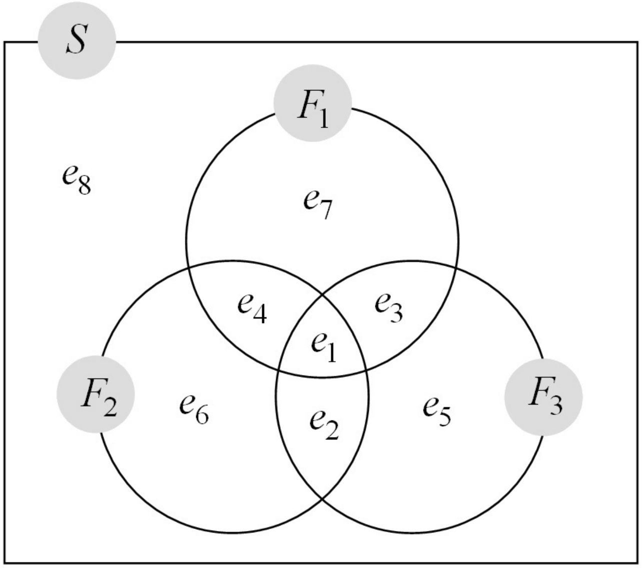

Figure 3.1: Basic MECE event e_r for a system with 3 two-state components.

state that all components survive, and $p_{2,2,2}$ is the probability corresponding to the state encoded by the $f(x_1, x_2, x_3)$ of $z^{f(x_1,x_2,x_3)}$, and so on.

The UGF of the system expressed by Equation (3.12) has $2^3 = 8$ terms, each of which corresponds to one of the basic MECE events of the system and its corresponding probability as described in Section 2.3.2. Taking the definition of the probability of basic MECE events described in the Section 2.3.2 into account, the failure probabilities of the basic MECE events of this system can be described as

$$\begin{aligned} p_{m_1} &= p(e_1), & p_{m_2} &= p(e_2), \\ p_{m_3} &= p(e_3), & p_{m_4} &= p(e_4), \\ p_{m_5} &= p(e_5), & p_{m_6} &= p(e_6), \\ p_{m_7} &= p(e_7), & p_{m_8} &= p(e_8). \end{aligned} \tag{3.13}$$

Let $e(z^{f(\bullet)})$ denote the event that the system is in the state encoded by $f(\bullet)$ of $z^{f(\bullet)}$. Comparing $f(\bullet)$ with Figure 2.3, one can find the following relationships.

$$\begin{aligned} e(z^0) &= e_1, & e(z^{f(x_1)}) &= e_2, \\ e(z^{f(x_2)}) &= e_3, & e(z^{f(x_3)}) &= e_4, \\ e(z^{f(x_1,x_2)}) &= e_5, & e(z^{f(x_1,x_3)}) &= e_6, \\ e(z^{f(x_2,x_3)}) &= e_7, & e(z^{f(x_1,x_2,x_3)}) &= e_8. \end{aligned} \tag{3.14}$$

Then, the following probability relationships can be obtained.

$$\begin{aligned} p_{1,1,1} &= p_{m_1}, & p_{2,1,1} &= p_{m_2}, \\ p_{1,2,1} &= p_{m_3}, & p_{1,1,2} &= p_{m_4}, \\ p_{2,2,1} &= p_{m_5}, & p_{2,1,2} &= p_{m_6}, \\ p_{1,2,2} &= p_{m_7}, & p_{2,2,2} &= p_{m_8}. \end{aligned} \tag{3.15}$$

Similarly, one can find that the number of terms of the UGF for a system with n two-state components is 2^n, each of which corresponds to one of the basic MECE events of the system and its corresponding probability. Obviously, the probabilities p_{m_r}'s serve as the design variable in LP similar to Equation (2.39). Thus, we still have the same hindrance in the application of the UGF to a large system as the LP bounds method. Yet, the UGF allows one to reduce considerably the computational burden by using simple algebraic procedures as described in the following section.

3.3 Relaxed Linear Programming Bounds Method

3.3.1 Introduction to Relaxed Linear Programming Bounds Method

1) UGF for Relaxed LP Bounds Method

Consider a system consisting of n components, and suppose that the component i has two possible states, for $i = 1, 2, \ldots, n$. A conceptually simplified UGF of the system can be expressed as

$$\begin{aligned} U(z) &= p_1 z^0 + p_2 z^{x_1} + p_3 z^{x_2} + \cdots + p_{n+1} z^{x_n} \\ &\quad + p_{n+2} z^{2x_1} + p_{n+3} z^{2x_2} + \cdots + p_{2n+1} z^{2x_n} \\ &\quad + p_{2n+2} z^{3x_1} + p_{2n+3} z^{3x_2} + \cdots + p_{3n+1} z^{3x_n} + \ldots \\ &\quad + p_{(n-2)n+2} z^{(n-1)x_1} + p_{(n-2)n+3} z^{(n-1)x_2} + \cdots + p_{(n-1)n+1} z^{(n-1)x_n} \\ &\quad + p_{n^2-n+2} z^{nx} \end{aligned} \tag{3.16}$$

where the 0 of z^0 encodes the subset of the state in which no component survives, and p_1 is the probability corresponding to the state encoded by the 0 of z^0; the x_1

of z^{x_1} encodes the subset of the state in which only component 1 survives, and p_2 is the probability corresponding to the state encoded by the x_1 of z^{x_1}; the $2x_1$ of z^{2x_1} encodes the portion of the subsets of the states in which only two components including component 1 survive, and p_{n+2} is the probability corresponding to the state encoded by the $2x_1$ of z^{2x_1}; the union of all states encoded by the $2x_i$ of z^{2x_i}, $i = 1, 2, \ldots, n$, is the subset of the states in which only two components survive, the sum of associated probabilities is the probability corresponding to the states that only two components survive; the nx of z^{nx} encodes the subset of the state in which all components survive, and p_{n^2-n+2} is the probability corresponding to the state encoded by the nx of z^{nx}, and so on[64,65].

Note that these events, like the one encoded by z^{2x_i}, are not the basic MECE events. However, the union of all events encoded by z^{jx_i} for $i = 1, 2, \ldots, n$, (E_j), is mutually exclusive with z^0, z^{nx}, and the union of all events encoded by z^{kx_i} (E_k), $(j, k$=1,2,$\ldots, n-1,\ k \neq j)$. Note that E_j is the event that exactly j component survives. Then, clearly z^0, z^{nx}, and E_j $(j = 1, 2, \ldots, n-1)$ are collectively exhaustive.

2) LP Problem

Similar to the LP bounds method, the linear objective function can be obtained by the corresponding matrix related to the system failure based on Equation (3.16). $\boldsymbol{p} = \{p_1, p_2, \ldots, p_{n^2-n+2}\}$ in Equation (3.16) can serve as the design variables of LP in Equation (2.39). The component failure probabilities and the joint failure probabilities of components are also the equality constraints or inequalities constraints but not the same with that of the LP bounds method. The details is expressed as follows:

a) Objective Function

From Equation (3.16), one can easily find the objective function of the LP for a series system, $\mathbf{c}^T\mathbf{p}_{\boldsymbol{r}}$, and the vector $\mathbf{c}$ that relates the system failure event can be expressed as

$$\mathbf{c}^T = [1 \quad 1 \quad \ldots \quad 1 \quad 0] \tag{3.17}$$

Also, one can easily find the objective function of the LP for a parallel system, $\mathbf{c}^T\mathbf{p}_{\boldsymbol{r}}$, and the vector $\mathbf{c}$ that relates the system failure event can be expressed as

$$\mathbf{c}^T = [1 \quad 0 \quad 0 \quad \ldots \quad 0] \tag{3.18}$$

b) Design Variables

Similar to Equation (2.39), the probabilities p_r's in the UGF of the system serve as the design variables in the LP, and the number of design variables can be expressed as

$$N_d = n^2 - n + 2 \tag{3.19}$$

Also, the given information are component failure probabilities, the joint failure probabilities of components, and the probabilities of design variables is unknown in advance.

c) Constraints

Based on the basic axioms of probability, the probabilities p_r's in Equation (3.16) have the following linear constraints:

$$\sum_{r=1}^{n^2-n+2} p_r = 1 \tag{3.20}$$

$$p_r \geq 0; \qquad r = 1, \ldots, n^2 - n + 2 \tag{3.21}$$

Similar to the LP bounds method, the constraints of the above approach is usually based on the component failure probabilities and the joint failure probabilities of two components or three components; however, the formulation of constraints in this approach is much different from that of the LP bounds method. The constraints of this approach consisting of equalities and inequalities are based on the relaxed bounds on the component failure probabilities and relaxed bounds on the joint failure probabilities of components, which could cause a relaxed bounds on the failure probability of the system. We called the above approach for bounds on the system reliability based on Equation (3.16) as the Relaxed Linear Programming (RLP) bounds method, and the constraints of the RLP bounds method will be introduced in the following Subsection.

3.3.2 Formulation of Constraints

The given information, i.e., the component failure probabilities and the joint failure probabilities of two components or three components, will take the expression of the relaxed bounds in the RLP bounds method. The details of these relaxed bounds will be introduced step by step in this Section.

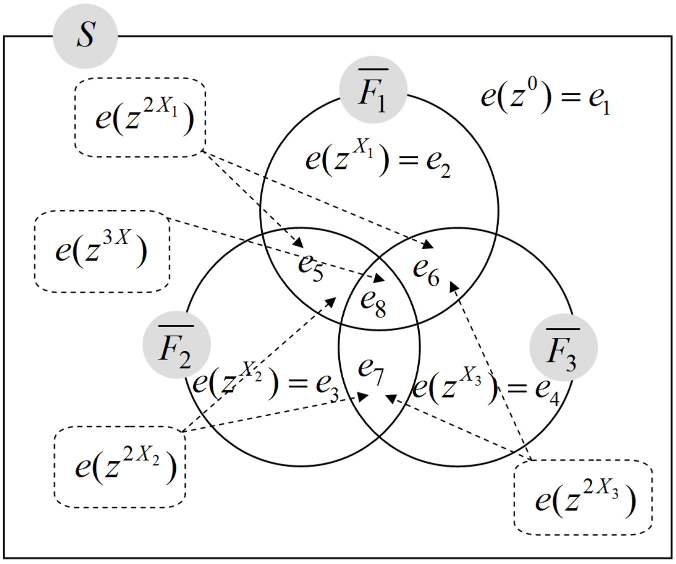

Figure 3.2: MECE event e_r for a three-event sample space.

Relaxed Bounds on Component Failure Probability

Using a system with 3 two-state components as shown in Figure 3.1, the concept of the relaxed bounds on the component failure probabilities is introduced first. The UGF of the 3 two-state components system can be expressed as

$$\begin{aligned} U(z) &= p_1 z^0 + p_2 z^{x_1} + p_3 z^{x_2} + p_4 z^{x_3} \\ &\quad + p_5 z^{2x_1} + p_6 z^{2x_2} + p_7 z^{2x_3} + p_8 z^{3x} \end{aligned} \tag{3.22}$$

The MECE events corresponding to the above system states are shown in Figure 3.2.

Comparing Figure 3.1 with Figure 3.2, the following relationships among the events encoded by z^0, z^{3x}, z^{jx_i} $(i = 1,2,3,\ j = 1,2)$, and the basic MECE events can be expressed as

$$\begin{aligned} &e(z^0) = e_1, && e(z^{x_1}) = e_2, \\ &e(z^{x_2}) = e_3, && e(z^{x_3}) = e_4, \\ &e(z^{3x}) = e_8, && e(z^{2x_1}) \subset (e_5 \cup e_6)\,, \\ &e(z^{2x_2}) \subset (e_5 \cup e_7)\,, && e(z^{2x_3}) \subset (e_6 \cup e_7)\,. \end{aligned} \tag{3.23}$$

The corresponding probability relationships can be expressed as

$$\begin{aligned}
&p_1 = p_{m_1}, && p_2 = p_{m_2}, \\
&p_3 = p_{m_3}, && p_4 = p_{m_4}, \\
&p_8 = p_{m_8}, && p_5 < p_{m_5} + p_{m_6}, \\
&p_6 < p_{m_5} + p_{m_7}, && p_7 < p_{m_6} + p_{m_7}.
\end{aligned} \tag{3.24}$$

The relationships among the union of the events encoded by z^{jx_i} $(i = 1, 2, 3,\ j = 1, 2)$, and the basic MECE events can be find as

$$\begin{aligned}
(e(z^{x_1}) \cup e(z^{x_2}) \cup e(z^{x_3})) &= (e_2 \cup e_3 \cup e_4)\,, \\
\left(e(z^{2x_1}) \cup e(z^{2x_2}) \cup e(z^{2x_3})\right) &= (e_5 \cup e_6 \cup e_7)\,.
\end{aligned} \tag{3.25}$$

The corresponding probability relationships can be expressed as

$$\begin{aligned}
p_2 + p_3 + p_4 &= p_{m_2} + p_{m_3} + p_{m_4}, \\
p_5 + p_6 + p_7 &= p_{m_5} + p_{m_6} + p_{m_7}.
\end{aligned} \tag{3.26}$$

The constraints of this system can be derived as follows:

1) The bounds of component failure probabilities can be expressed as

$$\begin{aligned}
P(F_1) = P_1 &\begin{cases} > p_1 + p_3 + p_4 \\ < p_1 + p_3 + p_4 + p_6 + p_7 \end{cases} \\
P(F_2) = P_2 &\begin{cases} > p_1 + p_2 + p_4 \\ < p_1 + p_2 + p_4 + p_5 + p_7 \end{cases} \\
P(F_3) = P_3 &\begin{cases} > p_1 + p_2 + p_3 \\ < p_1 + p_2 + p_3 + p_5 + p_6 \end{cases}
\end{aligned} \tag{3.27}$$

$p_1 + p_3 + p_4$ corresponds to the probabilities of states that no component survives and only one component except component 1 survives. $p_5 + p_6 + p_7$ corresponds to the probabilities of states that only two components survive. One can easily find that P_1 is greater than $p_1 + p_3 + p_4$, and smaller than $p_1 + p_3 + p_4 + p_5 + p_6 + p_7$. Furthermore, because p_5 is the probability corresponding to the state encoded by the $2x_1$ of z^{2x_1}, i.e., the part of the states that only two components including component 1 survive, p_5 can be excluded from the inequality. The other inequalities can be derived similarly.

2) The joint failure probabilities of two components can be expressed as

$$\begin{aligned} P(F_1 \cap F_2) &= P_{12} \\ &= p_1 + p_4 \\ P(F_1 \cap F_3) &= P_{13} \\ &= p_1 + p_3 \\ P(F_2 \cap F_3) &= P_{23} \\ &= p_1 + p_2 \end{aligned} \tag{3.28}$$

$p_1 + p_4$ corresponds to the probabilities of states that no component survives and only component 3 survives. One can easily find that P_{12} equals to $p_1 + p_4$ in this system. The other inequalities can be derived similarly.

3) Obviously the joint failure probability of three components is the failure probability of all components in this system, and it can be expressed as

$$\begin{aligned} P(F_1 \cap F_2 \cap F_3) &= P_{123} \\ &= p_1 \end{aligned} \tag{3.29}$$

4) From Equation (2.33), one can find that the sum of $P(F_1)$, $P(F_2)$, and $P(F_3)$ can be expressed as

$$\begin{aligned} & P(F_1) + P(F_2) + P(F_3) \\ = \ & P_1 + P_2 + P_3 \\ = \ & 3p_{m_1} + 2(p_{m_2} + p_{m_3} + p_{m_4}) + p_{m_5} + p_{m_6} + p_{m_7} \\ = \ & \binom{3}{1} p_{m_1} + \binom{2}{1} (p_{m_2} + p_{m_3} + p_{m_4}) + \binom{1}{1} (p_{m_5} + p_{m_6} + p_{m_7}) \end{aligned} \tag{3.30}$$

Comparing the relationships shown in Equations (3.24), (3.26), and (3.30), the sum of $P(F_1)$, $P(F_2)$, and $P(F_3)$ can also be expressed as

$$\begin{aligned} & P(F_1) + P(F_2) + P(F_3) \\ = \ & \binom{3}{1} p_1 + \binom{2}{1} (p_2 + p_3 + p_4) + \binom{1}{1} (p_5 + p_6 + p_7) \end{aligned} \tag{3.31}$$

Note that when one has information such as $P(F_i) \geq x$ or $P(F_i) \leq x$ rather than $P(F_i) = x$, the equalities (Equations (3.30) and (3.31)) will also change to the inequalities.

From this simple example, we have shown the basic concept of the relaxed bounds on the component failure probabilities (Equations (3.27) and (3.31)) in the RLP bounds method. From Equations (3.28) and (3.29), one can also find that the joint failure probabilities of two components or three components in the RLP bounds method are still equalities in a system with three two-state components. However, when the number of components of the system is greater than 3, the joint failure probabilities of two components will take the expression of the bounds in the RLP bounds method. Since the bounds on system reliability based on the component failure probabilities is often unacceptably wide, we will introduce the relaxed bounds on the joint failure probabilities of two components in the following example.

Relaxed Bounds on Joint Failure Probability of Two Components

Second, let us consider a system with four two-state components. Then, the UGF of the system can be expressed as

$$\begin{aligned} U(z) \;=\; & p_1 z^0 + p_2 z^{x_1} + p_3 z^{x_2} + p_4 z^{x_3} + p_5 z^{x_4} \\ & + p_6 z^{2x_1} + p_7 z^{2x_2} + p_8 z^{2x_3} + p_9 z^{2x_4} \\ & + p_{10} z^{3x_1} + p_{11} z^{3x_2} + p_{12} z^{3x_3} + p_{13} z^{3x_4} + p_{14} z^{4x} \end{aligned} \tag{3.32}$$

This system consists of $2^4 = 16$ basic MECE events, they can be expressed as

$$\begin{aligned} e_1 &= F_1 \cap F_2 \cap F_3 \cap F_4, & e_2 &= \overline{F_1} \cap F_2 \cap F_3 \cap F_4, \\ e_3 &= F_1 \cap \overline{F_2} \cap F_3 \cap F_4, & e_4 &= F_1 \cap F_2 \cap \overline{F_3} \cap F_4, \\ e_5 &= F_1 \cap F_2 \cap F_3 \cap \overline{F_4}, & e_6 &= \overline{F_1} \cap \overline{F_2} \cap F_3 \cap F_4, \\ e_7 &= \overline{F_1} \cap F_2 \cap \overline{F_3} \cap F_4, & e_8 &= \overline{F_1} \cap F_2 \cap F_3 \cap \overline{F_4}, \\ e_9 &= F_1 \cap \overline{F_2} \cap \overline{F_3} \cap F_4, & e_{10} &= F_1 \cap \overline{F_2} \cap F_3 \cap \overline{F_4}, \\ e_{11} &= F_1 \cap F_2 \cap \overline{F_3} \cap \overline{F_4}, & e_{12} &= \overline{F_1} \cap \overline{F_2} \cap \overline{F_3} \cap F_4, \\ e_{13} &= \overline{F_1} \cap \overline{F_2} \cap F_3 \cap \overline{F_4}, & e_{14} &= \overline{F_1} \cap F_2 \cap \overline{F_3} \cap \overline{F_4}, \end{aligned}$$

$$e_{15} = F_1 \cap \overline{F_2} \cap \overline{F_3} \cap \overline{F_4}, \qquad e_{16} = \overline{F_1} \cap \overline{F_2} \cap \overline{F_3} \cap \overline{F_4}.$$

From the the definition of the events encoded by z^0, z^{4x}, z^{jx_i} ($i = 1, 2, 3, 4,\ j = 1, 2, 3$), the following relationships among the events encoded by z^0, z^{4x}, z^{jx_i} ($i = 1, 2, 3, 4,\ j = 1, 2, 3$), and the basic MECE events can be expressed as

$$\begin{aligned}
&e(z^0) = e_1, && e(z^{x_1}) = e_2,\\
&e(z^{x_2}) = e_3, && e(z^{x_3}) = e_4,\\
&e(z^{x_4}) = e_5, && e(z^{4x}) = e_{16},\\
&e(z^{2x_1}) \subset (e_6 \cup e_7 \cup e_8)\,, && e(z^{2x_2}) \subset (e_6 \cup e_9 \cup e_{10})\,,\\
&e(z^{2x_3}) \subset (e_7 \cup e_9 \cup e_{11})\,, && e(z^{2x_4}) \subset (e_8 \cup e_{10} \cup e_{11})\,,\\
&e(z^{3x_1}) \subset (e_{12} \cup e_{13} \cup e_{14})\,, && e(z^{3x_2}) \subset (e_{12} \cup e_{13} \cup e_{15})\,,\\
&e(z^{3x_3}) \subset (e_{12} \cup e_{14} \cup e_{15})\,, && e(z^{3x_4}) \subset (e_{13} \cup e_{14} \cup e_{15})\,.
\end{aligned} \tag{3.33}$$

The corresponding probability relationships can be expressed as

$$\begin{aligned}
&p_1 = p_{m_1}, && p_2 = p_{m_2},\\
&p_3 = p_{m_3}, && p_4 = p_{m_4},\\
&p_5 = p_{m_5}, && p_6 < p_{m_6} + p_{m_7} + p_{m_8},\\
&p_7 < p_{m_6} + p_{m_9} + p_{m_{10}}, && p_8 < p_{m_7} + p_{m_9} + p_{m_{11}},\\
&p_9 < p_{m_8} + p_{m_{10}} + p_{m_{11}}, && p_{10} < p_{m_{12}} + p_{m_{13}} + p_{m_{14}},\\
&p_{11} < p_{m_{12}} + p_{m_{13}} + p_{m_{15}}, && p_{12} < p_{m_{12}} + p_{m_{14}} + p_{m_{15}},\\
&p_{13} < p_{m_{13}} + p_{m_{14}} + p_{m_{15}}, && p_{14} = p_{m_{16}}.
\end{aligned} \tag{3.34}$$

The relationships among the union of the events encoded by z^{jx_i} ($i = 1, 2, 3, 4$, $j = 1, 2, 3$), and the basic MECE events can be find as

$$\begin{aligned}
&(e(z^{x_1}) \cup e(z^{x_2}) \cup e(z^{x_3}) \cup e(z^{x_4})) = (e_2 \cup e_3 \cup e_4 \cup e_5)\,,\\
&\big(e(z^{2x_1}) \cup e(z^{2x_2}) \cup e(z^{2x_3}) \cup e(z^{2x_4})\big) = (e_6 \cup e_7 \cup e_8 \cup e_9 \cup e_{10} \cup e_{11})\,,\\
&\big(e(z^{3x_1}) \cup e(z^{3x_2}) \cup e(z^{3x_3}) \cup e(z^{3x_4})\big) = (e_{12} \cup e_{13} \cup e_{14} \cup e_{15})\,.
\end{aligned} \tag{3.35}$$

The corresponding probability relationships can be expressed as

$$\begin{aligned} p_2 + p_3 + p_4 + p_5 &= p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5}, \\ p_6 + p_7 + p_8 + p_9 &= p_{m_6} + p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}}, \\ p_{10} + p_{11} + p_{12} + p_{13} &= p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}}, \end{aligned} \tag{3.36}$$

The constraints of this system can be derived as follows:

1) The bounds of component failure probabilities can be expressed as

$$\begin{aligned} P(F_1) = P_1 &\begin{cases} > p_1 + p_3 + p_4 + p_5 \\ < p_1 + p_3 + p_4 + p_5 + p_7 + p_8 + p_9 + p_{11} + p_{12} + p_{13} \end{cases} \\ P(F_2) = P_2 &\begin{cases} > p_1 + p_2 + p_4 + p_5 \\ < p_1 + p_2 + p_4 + p_5 + p_6 + p_8 + p_9 + p_{10} + p_{12} + p_{13} \end{cases} \\ P(F_3) = P_3 &\begin{cases} > p_1 + p_2 + p_3 + p_5 \\ < p_1 + p_2 + p_3 + p_5 + p_6 + p_7 + p_9 + p_{10} + p_{11} + p_{13} \end{cases} \\ P(F_4) = P_4 &\begin{cases} > p_1 + p_2 + p_3 + p_4 \\ < p_1 + p_2 + p_3 + p_4 + p_6 + p_7 + p_8 + p_{10} + p_{11} + p_{12} \end{cases} \end{aligned} \tag{3.37}$$

Similar to the system with 3 two-state components, $p_1 + p_3 + p_4 + p_5$ corresponds to the probabilities of states that no component survives and only one component except component 1 survives. $p_6 + p_7 + p_8 + p_9$ corresponds to the probabilities of states that only two components survive. $p_{10} + p_{11} + p_{12} + p_{13}$ corresponds to the probabilities of states that only three components survive. One can easily find that P_1 is greater than $p_1 + p_3 + p_4 + p_5$, and smaller than $p_1 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 + p_{10} + p_{11} + p_{12} + p_{13}$. Furthermore, because p_6 and p_{10} is the probability corresponding to the state encoded by the $2x_1$ of z^{2x_1} and the $3x_1$ of z^{3x_1}, i.e., the part of the states that only two components including component 1 survive and the part of the states that only three components including component 1 survive, respectively, p_6 and p_{10} can be excluded from the inequality. The other inequalities can be derived similarly.

2) The bounds of joint failure probabilities of two components can be expressed as

$$
\begin{aligned}
P(F_1 \cap F_2) = P_{12} &\begin{cases} > p_1 + p_4 + p_5 \\ < p_1 + p_4 + p_5 + p_8 + p_9 \end{cases} \\
P(F_1 \cap F_3) = P_{13} &\begin{cases} > p_1 + p_3 + p_5 \\ < p_1 + p_3 + p_5 + p_7 + p_9 \end{cases} \\
P(F_1 \cap F_4) = P_{14} &\begin{cases} > p_1 + p_3 + p_4 \\ < p_1 + p_3 + p_4 + p_7 + p_8 \end{cases} \\
P(F_2 \cap F_3) = P_{23} &\begin{cases} > p_1 + p_2 + p_5 \\ < p_1 + p_2 + p_5 + p_6 + p_9 \end{cases} \\
P(F_2 \cap F_4) = P_{24} &\begin{cases} > p_1 + p_2 + p_4 \\ < p_1 + p_2 + p_4 + p_6 + p_8 \end{cases} \\
P(F_3 \cap F_4) = P_{34} &\begin{cases} > p_1 + p_2 + p_3 \\ < p_1 + p_2 + p_3 + p_6 + p_7 \end{cases}
\end{aligned}
\tag{3.38}
$$

$p_1+p_4+p_5$ corresponds to the probabilities of states that no component survives and only components 3 or 4 survives. Since $p_6+p_7+p_8+p_9$ corresponds to the probabilities of states that only two components survive, one can easily find that P_{12} is greater than $p_1+p_4+p_5$, and smaller than $p_1+p_3+p_4+p_5+p_6+p_7+p_8+p_9$. Furthermore, because p_6 and p_7 is the probability corresponding to the state encoded by the $2x_1$ of z^{2x_1} and by the $2x_2$ of z^{2x_2}, i.e., the part of the states that only two components including component 1 survive and the part of the states that only two components including component 2 survive, respectively, p_6 and p_7 can be excluded from the inequality. The other inequalities can be derived similarly.

3) The joint failure probabilities of three components can be expressed as

$$
\begin{aligned}
P(F_1 \cap F_2 \cap F_3) &= P_{123} \\
&= p_1 + p_5 \\
P(F_1 \cap F_2 \cap F_4) &= P_{124} \\
&= p_1 + p_4
\end{aligned}
$$

$$\begin{aligned} P(F_1 \cap F_3 \cap F_4) &= P_{134} \\ &= p_1 + p_3 \\ P(F_2 \cap F_3 \cap F_4) &= P_{234} \\ &= p_1 + p_2 \end{aligned} \tag{3.39}$$

$p_1 + p_5$ corresponds to the probabilities of states that no component survives and only components 4 survives. One can easily find that P_{123} equals to $p_1 + p_5$ in this system. The other inequalities can be derived similarly.

4) Obviously the joint failure probability of three components is the failure probability of all components in this system, and it can be expressed as

$$\begin{aligned} P(F_1 \cap F_2 \cap F_3 \cap F_4) &= P_{1234} \\ &= p_1 \end{aligned} \tag{3.40}$$

5) From Equation (2.33), one can find that the sum of $P(F_1)$, $P(F_2)$, $P(F_4)$, and $P(F_4)$ can be expressed as

$$\begin{aligned} & P(F_1) + P(F_2) + P(F_3) + P(F_4) \\ = \; & P_1 + P_2 + P_3 + P_4 \\ = \; & 4p_{m_1} + 3(p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5}) \\ & +2(p_{m_6} + p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}}) \\ & +(p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}}) \\ = \; & \binom{4}{1} p_{m_1} + \binom{3}{1}(p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5}) \\ & + \binom{2}{1}(p_{m_6} + p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}}) \\ & + \binom{1}{1}(p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}}) \end{aligned} \tag{3.41}$$

Comparing the relationships shown in Equations (3.36), (3.46), and (3.41), the sum of $P(F_1)$, $P(F_2)$, $P(F_4)$, and $P(F_4)$ can also be expressed as

$$
\begin{aligned}
& P(F_1) + P(F_2) + P(F_3) + P(F_4) \\
= & \binom{4}{1} p_1 + \binom{3}{1}(p_2 + p_3 + p_4 + p_5) \\
& + \binom{2}{1}(p_6 + p_7 + p_8 + p_9) \\
& + \binom{1}{1}(p_{10} + p_{11} + p_{12} + p_{13})
\end{aligned} \tag{3.42}
$$

6) Similarly, from Equation (2.34), one can find that the sum of $P(F_{12})$, $P(F_{13})$, $P(F_{14})$, $P(F_{23})$, $P(F_{24})$, and $P(F_{34})$ can be expressed as

$$
\begin{aligned}
& P(F_{12}) + P(F_{13}) + P(F_{14}) + P(F_{23}) + P(F_{24}) + P(F_{34}) \\
= & P_{12} + P_{13} + P_{14} + P_{23} + P_{24} + P_{34} \\
= & 6p_{m_1} + 3(p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5}) \\
& + p_{m_6} + p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}} \\
= & \binom{4}{2} p_{m_1} + \binom{3}{2}(p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5}) \\
& + \binom{2}{2}(p_{m_6} + p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}})
\end{aligned} \tag{3.43}
$$

Comparing the relationships shown in Equations (3.36), (3.46), and (3.43), the sum of $P(F_{12})$, $P(F_{13})$, $P(F_{14})$, $P(F_{23})$, $P(F_{24})$, and $P(F_{34})$ can also be expressed as

$$
\begin{aligned}
& P(F_{12}) + P(F_{13}) + P(F_{14}) + P(F_{23}) + P(F_{24}) + P(F_{34}) \\
= & \binom{4}{2} p_1 + \binom{3}{2}(p_2 + p_3 + p_4 + p_5) + \binom{2}{2}(p_6 + p_7 + p_8 + p_9)
\end{aligned} \tag{3.44}
$$

Similar to bounds on component failure probability, one can easily find the formulation of the bounds on the joint failure probabilities of two components in the RLP bounds method Equations (3.38) and (3.43). Also, from Equations (3.39) and (3.40), one can find that the joint failure probabilities of three components or four components in the RLP bounds method are still equalities in a system with four two-state components. Generally, the acceptable bounds on system reliability is based on the given information of the joint failure probabilities of two components or three components, we will introduce the relaxed bounds on the joint failure probabilities of three components in the following example.

Relaxed Bounds on Joint Failure Probability of Three Components

Third, let us consider a system with five two-state components. Then, the UGF of the system can be expressed as

$$\begin{aligned} U(z) &= p_1 z^0 + p_2 z^{x_1} + p_3 z^{x_2} + p_4 z^{x_3} + p_5 z^{x_4} + p_6 z^{x_5} \\ &\quad + p_7 z^{2x_1} + p_8 z^{2x_2} + p_9 z^{2x_3} + p_{10} z^{2x_4} + p_{11} z^{2x_5} \\ &\quad + p_{12} z^{3x_1} + p_{13} z^{3x_2} + p_{14} z^{3x_3} + p_{15} z^{3x_4} + p_{16} z^{3x_5} \\ &\quad + p_{17} z^{4x_1} + p_{18} z^{4x_2} + p_{19} z^{4x_3} + p_{20} z^{4x_4} + p_{21} z^{4x_5} + p_{22} z^{5x} \end{aligned} \tag{3.45}$$

This system consists of $2^5 = 32$ basic MECE events, they can be expressed as

$$\begin{aligned}
e_1 &= F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5, & e_2 &= \overline{F_1} \cap F_2 \cap F_3 \cap F_4 \cap F_5, \\
e_3 &= F_1 \cap \overline{F_2} \cap F_3 \cap F_4 \cap F_5, & e_4 &= F_1 \cap F_2 \cap \overline{F_3} \cap F_4 \cap F_5, \\
e_5 &= F_1 \cap F_2 \cap F_3 \cap \overline{F_4} \cap F_5, & e_6 &= F_1 \cap F_2 \cap F_3 \cap F_4 \cap \overline{F_5}, \\
e_7 &= \overline{F_1} \cap \overline{F_2} \cap F_3 \cap F_4 \cap F_5, & e_8 &= \overline{F_1} \cap F_2 \cap \overline{F_3} \cap F_4 \cap F_5, \\
e_9 &= \overline{F_1} \cap F_2 \cap F_3 \cap \overline{F_4} \cap F_5 & e_{10} &= \overline{F_1} \cap F_2 \cap F_3 \cap F_4 \cap \overline{F_5}, \\
e_{11} &= F_1 \cap \overline{F_2} \cap \overline{F_3} \cap F_4 \cap F_5, & e_{12} &= F_1 \cap \overline{F_2} \cap F_3 \cap \overline{F_4} \cap F_5, \\
e_{13} &= F_1 \cap \overline{F_2} \cap F_3 \cap F_4 \cap \overline{F_5}, & e_{14} &= F_1 \cap F_2 \cap \overline{F_3} \cap \overline{F_4} \cap F_5, \\
e_{15} &= F_1 \cap F_2 \cap \overline{F_3} \cap F_4 \cap \overline{F_5}, & e_{16} &= F_1 \cap F_2 \cap F_3 \cap \overline{F_4} \cap \overline{F_5}, \\
e_{17} &= \overline{F_1} \cap \overline{F_2} \cap \overline{F_3} \cap F_4 \cap F_5, & e_{18} &= \overline{F_1} \cap \overline{F_2} \cap F_3 \cap \overline{F_4} \cap F_5, \\
e_{19} &= \overline{F_1} \cap \overline{F_2} \cap F_3 \cap F_4 \cap \overline{F_5}, & e_{20} &= \overline{F_1} \cap F_2 \cap \overline{F_3} \cap \overline{F_4} \cap F_5, \\
e_{21} &= \overline{F_1} \cap F_2 \cap \overline{F_3} \cap F_4 \cap \overline{F_5}, & e_{22} &= \overline{F_1} \cap F_2 \cap F_3 \cap \overline{F_4} \cap \overline{F_5}, \\
e_{23} &= F_1 \cap \overline{F_2} \cap \overline{F_3} \cap \overline{F_4} \cap F_5, & e_{24} &= F_1 \cap \overline{F_2} \cap \overline{F_3} \cap F_4 \cap \overline{F_5}, \\
e_{25} &= F_1 \cap \overline{F_2} \cap F_3 \cap \overline{F_4} \cap \overline{F_5}, & e_{26} &= F_1 \cap F_2 \cap \overline{F_3} \cap \overline{F_4} \cap \overline{F_5}, \\
e_{27} &= \overline{F_1} \cap \overline{F_2} \cap \overline{F_3} \cap \overline{F_4} \cap F_5, & e_{28} &= \overline{F_1} \cap \overline{F_2} \cap \overline{F_3} \cap F_4 \cap \overline{F_5}, \\
e_{29} &= \overline{F_1} \cap \overline{F_2} \cap F_3 \cap \overline{F_4} \cap \overline{F_5}, & e_{30} &= \overline{F_1} \cap F_2 \cap \overline{F_3} \cap \overline{F_4} \cap \overline{F_5}, \\
e_{31} &= F_1 \cap \overline{F_2} \cap \overline{F_3} \cap \overline{F_4} \cap \overline{F_5}, & e_{32} &= \overline{F_1} \cap \overline{F_2} \cap \overline{F_3} \cap \overline{F_4} \cap \overline{F_5}.
\end{aligned}$$

From the the definition of the events encoded by z^0, z^{5x}, z^{jx_i} ($i = 1, 2, 3, 4, 5$, $j = 1, 2, 3, 4$), the following relationships among the events encoded by z^0, z^{5x}, z^{jx_i}

($i = 1, 2, 3, 4, 5,\ j = 1, 2, 3, 4$), and the basic MECE events can be expressed as

$$
\begin{aligned}
&e(z^0) = e_1, && e(z^{x_1}) = e_2,\\
&e(z^{x_2}) = e_3, && e(z^{x_3}) = e_4,\\
&e(z^{x_4}) = e_5, && e(z^{x_5}) = e_6,\\
&e(z^{5x}) = e_{32}, &&\\
&e(z^{2x_1}) \subset (e_7 \cup e_8 \cup e_9 \cup e_{10}), && e(z^{2x_2}) \subset (e_7 \cup e_{11} \cup e_{12} \cup e_{13}),\\
&e(z^{2x_3}) \subset (e_8 \cup e_{11} \cup e_{14} \cup e_{15}), && e(z^{2x_4}) \subset (e_9 \cup e_{12} \cup e_{14} \cup e_{16}),\\
&e(z^{2x_5}) \subset (e_{10} \cup e_{13} \cup e_{15} \cup e_{16}), &&\\
&e(z^{3x_1}) \subset (e_{17} \cup e_{18} \cup e_{19} \cup e_{20} \cup e_{21} \cup e_{22}), &&\\
&e(z^{3x_2}) \subset (e_{17} \cup e_{18} \cup e_{19} \cup e_{23} \cup e_{24} \cup e_{25}), &&\\
&e(z^{3x_3}) \subset (e_{17} \cup e_{20} \cup e_{21} \cup e_{23} \cup e_{24} \cup e_{26}), &&\\
&e(z^{3x_4}) \subset (e_{18} \cup e_{20} \cup e_{22} \cup e_{23} \cup e_{25} \cup e_{26}), &&\\
&e(z^{3x_5}) \subset (e_{19} \cup e_{21} \cup e_{22} \cup e_{24} \cup e_{25} \cup e_{26}), &&\\
&e(z^{4x_1}) \subset (e_{27} \cup e_{28} \cup e_{29} \cup e_{30}), && e(z^{4x_2}) \subset (e_{27} \cup e_{28} \cup e_{29} \cup e_{31}),\\
&e(z^{4x_3}) \subset (e_{27} \cup e_{28} \cup e_{30} \cup e_{31}), && e(z^{4x_4}) \subset (e_{27} \cup e_{29} \cup e_{30} \cup e_{31}),\\
&e(z^{4x_5}) \subset (e_{28} \cup e_{29} \cup e_{30} \cup e_{31}). &&
\end{aligned}
\tag{3.46}
$$

The corresponding probability relationships can be expressed as

$$
\begin{aligned}
&p_1 = p_{m_1}, && p_2 = p_{m_2},\\
&p_3 = p_{m_3}, && p_4 = p_{m_4},\\
&p_5 = p_{m_5}, && p_6 = p_{m_6},\\
&p_7 < p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}}, && p_8 < p_{m_7} + p_{m_{11}} + p_{m_{12}} + p_{m_{13}},\\
&p_9 < p_{m_8} + p_{m_{11}} + p_{m_{14}} + p_{m_{15}}, && p_{10} < p_{m_9} + p_{m_{12}} + p_{m_{14}} + p_{m_{16}},\\
&p_{11} < p_{m_{10}} + p_{m_{13}} + p_{m_{15}} + p_{m_{16}}, &&\\
&p_{12} < p_{m_{17}} + p_{m_{18}} + p_{m_{19}} + p_{m_{20}} + p_{m_{21}} + p_{m_{22}}, &&\\
&p_{13} < p_{m_{17}} + p_{m_{18}} + p_{m_{19}} + p_{m_{23}} + p_{m_{24}} + p_{m_{25}}, &&\\
&p_{14} < p_{m_{17}} + p_{m_{20}} + p_{m_{21}} + p_{m_{23}} + p_{m_{24}} + p_{m_{26}}, &&
\end{aligned}
\tag{3.47}
$$

$p_{15} < p_{m_{18}} + p_{m_{20}} + p_{m_{22}} + p_{m_{23}} + p_{m_{25}} + p_{m_{26}},$

$p_{16} < p_{m_{19}} + p_{m_{21}} + p_{m_{22}} + p_{m_{24}} + p_{m_{25}} + p_{m_{26}},$

$p_{17} < p_{m_{27}} + p_{m_{28}} + p_{m_{29}} + p_{m_{30}},$ $\quad p_{18} < p_{m_{27}} + p_{m_{28}} + p_{m_{29}} + p_{m_{31}},$

$p_{19} < p_{m_{27}} + p_{m_{28}} + p_{m_{29}} + p_{m_{31}},$ $\quad p_{20} < p_{m_{27}} + p_{m_{29}} + p_{m_{30}} + p_{m_{31}},$

$p_{21} < p_{m_{27}} + p_{m_{29}} + p_{m_{30}} + p_{m_{31}},$ $\quad p_{22} = p_{m_{32}}.$

The relationships among the union of the events encoded by z^{jx_i} ($i = 1,2,3,4,5$, $j = 1,2,3,4$), and the basic MECE events can be expressed as

$$\begin{aligned}
&(e(z^{x_1}) \cup e(z^{x_2}) \cup e(z^{x_3}) \cup e(z^{x_4}) \cup e(z^{x_5})) \\
&= (e_2 \cup e_3 \cup e_4 \cup e_5 \cup e_6)\,, && (3.48) \\
&\big(e(z^{2x_1}) \cup e(z^{2x_2}) \cup e(z^{2x_3}) \cup e(z^{2x_4}) \cup e(z^{2x_5})\big) \\
&= (e_7 \cup e_8 \cup e_9 \cup e_{10} \cup e_{11} \cup e_{12} \cup e_{13} \cup e_{14} \cup e_{15} \cup e_{16})\,, && (3.49) \\
&\big(e(z^{3x_1}) \cup e(z^{3x_2}) \cup e(z^{3x_3}) \cup e(z^{3x_4}) \cup e(z^{3x_5})\big) \\
&= (e_{17} \cup e_{18} \cup e_{19} \cup e_{20} \cup e_{21} \cup e_{22} \cup e_{23} \cup e_{24} \cup e_{25} \cup e_{26})\,, && (3.50) \\
&\big(e(z^{4x_1}) \cup e(z^{4x_2}) \cup e(z^{4x_3}) \cup e(z^{4x_4}) \cup e(z^{4x_5})\big) \\
&= (e_{27} \cup e_{28} \cup e_{29} \cup e_{30} \cup e_{31})\,. && (3.51)
\end{aligned}$$

The corresponding probability relationships can be expressed as

$$\begin{aligned}
&p_2 + p_3 + p_4 + p_5 + p_6 \\
&= p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5} + p_{m_6}, && (3.52) \\
&p_7 + p_8 + p_9 + p_{10} + p_{11} \\
&= p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}} + p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}} + p_{m_{16}}, && (3.53) \\
&p_{12} + p_{13} + p_{14} + p_{15} + p_{16} \\
&= p_{m_{17}} + p_{m_{18}} + p_{m_{19}} + p_{m_{20}} + p_{m_{21}} + p_{m_{22}} + p_{m_{23}} + p_{m_{24}} + p_{m_{25}} + p_{m_{26}}, && (3.54) \\
&p_{17} + p_{18} + p_{19} + p_{20} + p_{21} \\
&= p_{m_{27}} + p_{m_{28}} + p_{m_{29}} + p_{m_{30}} + p_{m_{31}}. && (3.55)
\end{aligned}$$

The constraints of this system can be derived as follows:

1) The bounds of component failure probabilities can be expressed as

$$
\begin{aligned}
P(F_1) = P_1 &\begin{cases} > p_1 + p_3 + p_4 + p_5 + p_6 \\ < p_1 + p_3 + p_4 + p_5 + p_6 \\ \quad + p_8 + p_9 + p_{10} + p_{11} + p_{13} + p_{14} + p_{15} + p_{16} + p_{18} + p_{19} + p_{20} + p_{21} \end{cases} \\
P(F_2) = P_2 &\begin{cases} > p_1 + p_2 + p_4 + p_5 + p_6 \\ < p_1 + p_2 + p_4 + p_5 + p_6 \\ \quad + p_7 + p_9 + p_{10} + p_{11} + p_{12} + p_{14} + p_{15} + p_{16} + p_{17} + p_{19} + p_{20} + p_{21} \end{cases} \\
P(F_3) = P_3 &\begin{cases} > p_1 + p_2 + p_3 + p_5 + p_6 \\ < p_1 + p_2 + p_3 + p_5 + p_6 \\ \quad + p_7 + p_8 + p_{10} + p_{11} + p_{12} + p_{13} + p_{15} + p_{16} + p_{17} + p_{18} + p_{20} + p_{21} \end{cases} \\
P(F_4) = P_4 &\begin{cases} > p_1 + p_2 + p_3 + p_4 + p_6 \\ < p_1 + p_2 + p_3 + p_4 + p_6 \\ \quad + p_7 + p_8 + p_9 + p_{11} + p_{12} + p_{13} + p_{14} + p_{16} + p_{17} + p_{18} + p_{19} + p_{21} \end{cases} \\
P(F_5) = P_5 &\begin{cases} > p_1 + p_2 + p_3 + p_4 + p_5 \\ < p_1 + p_2 + p_3 + p_4 + p_5 \\ \quad + p_7 + p_8 + p_9 + p_{10} + p_{12} + p_{13} + p_{14} + p_{15} + p_{17} + p_{18} + p_{19} + p_{20} \end{cases}
\end{aligned}
\tag{3.56}
$$

Similar to the system with 3 two-state components, $p_1 + p_3 + p_4 + p_5 + p_6$ corresponds to the probabilities of states that no component survives and only one component except component 1 survives. $p_7 + p_8 + p_9 + p_{10} + p_{11}$ corresponds to the probabilities of states that only two components survive. $p_{12} + p_{13} + p_{14} + p_{15} + p_{16}$ corresponds to the probabilities of states that only three components survive. $p_{17} + p_{18} + p_{19} + p_{20} + p_{21}$ corresponds to the probabilities of states that only four components survive. One can easily find that P_1 is greater than $p_1 + p_3 + p_4 + p_5 + p_6$, and smaller than $p_1 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 + p_{10} + p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{16} + p_{17} + p_{18} + p_{19} + p_{20} + p_{21}$. Furthermore, because p_7, p_{12}, and p_{17} is the probability corresponding to the state encoded by the $2x_1$ of z^{2x_1}, by the $3x_1$ of z^{3x_1}, and by the $4x_1$ of z^{4x_1}, i.e., the part of the states that only two components including component 1 survive, the part of the states that only three components including component 1 survive, and the part of the states that only four components including component 1 survive, respectively, p_7, p_{12}, and p_{17} can be excluded from the inequality. The other inequalities can be derived similarly.

2) The bounds of joint failure probabilities of two components can be expressed as

$$
\begin{aligned}
P(F_1 \cap F_2) = P_{12} &\begin{cases} > p_1 + p_4 + p_5 + p_6 \\ < p_1 + p_4 + p_5 + p_6 + p_9 + p_{10} + p_{11} + p_{14} + p_{15} + p_{16} \end{cases} \\
P(F_1 \cap F_3) = P_{13} &\begin{cases} > p_1 + p_3 + p_5 + p_6 \\ < p_1 + p_3 + p_5 + p_6 + p_8 + p_{10} + p_{11} + p_{13} + p_{15} + p_{16} \end{cases} \\
P(F_1 \cap F_4) = P_{14} &\begin{cases} > p_1 + p_3 + p_4 + p_6 \\ < p_1 + p_3 + p_4 + p_6 + p_8 + p_9 + p_{11} + p_{13} + p_{14} + p_{16} \end{cases} \\
P(F_1 \cap F_5) = P_{15} &\begin{cases} > p_1 + p_3 + p_4 + p_5 \\ < p_1 + p_3 + p_4 + p_5 + p_8 + p_9 + p_{10} + p_{13} + p_{14} + p_{15} \end{cases} \\
P(F_2 \cap F_3) = P_{23} &\begin{cases} > p_1 + p_2 + p_5 + p_6 \\ < p_1 + p_2 + p_5 + p_6 + p_7 + p_{10} + p_{11} + p_{12} + p_{15} + p_{16} \end{cases} \\
P(F_2 \cap F_4) = P_{24} &\begin{cases} > p_1 + p_2 + p_4 + p_6 \\ < p_1 + p_2 + p_4 + p_6 + p_7 + p_9 + p_{11} + p_{12} + p_{14} + p_{16} \end{cases} \\
P(F_2 \cap F_5) = P_{25} &\begin{cases} > p_1 + p_2 + p_4 + p_5 \\ < p_1 + p_2 + p_4 + p_5 + p_7 + p_9 + p_{10} + p_{12} + p_{14} + p_{15} \end{cases} \\
P(F_3 \cap F_4) = P_{34} &\begin{cases} > p_1 + p_2 + p_3 + p_6 \\ < p_1 + p_2 + p_3 + p_6 + p_7 + p_8 + p_{11} + p_{12} + p_{13} + p_{16} \end{cases} \\
P(F_3 \cap F_5) = P_{35} &\begin{cases} > p_1 + p_2 + p_3 + p_5 \\ < p_1 + p_2 + p_3 + p_5 + p_7 + p_8 + p_{10} + p_{12} + p_{13} + p_{15} \end{cases} \\
P(F_4 \cap F_5) = P_{45} &\begin{cases} > p_1 + p_2 + p_3 + p_4 \\ < p_1 + p_2 + p_3 + p_4 + p_7 + p_8 + p_9 + p_{12} + p_{13} + p_{14} \end{cases}
\end{aligned}
\tag{3.57}
$$

$p_1 + p_4 + p_5 + p_6$ corresponds to the probabilities of states that no component survives and only components 3, 4, or 5 survives. Since $p_7 + p_8 + p_9 + p_{10} + p_{11}$ and $p_{12} + p_{13} + p_{14} + p_{15} + p_{16}$ correspond to the probabilities of states that only two components survive and the probabilities of states that only three components survive, respectively, one can easily find that P_{12} is greater than $p_1 + p_4 + p_5 + p_6$, and smaller than $p_1 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 + p_{10} + p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{16}$. Furthermore, because p_7 and p_8 is the probability corresponding to the state encoded by the $2x_1$ of z^{2x_1} and by the $2x_2$ of z^{2x_2}, i.e., the part of the states that only two components including component 1 survive and the part of the states that only two components including component 2 survive, respectively, p_7 and p_8 can be excluded from the inequality. Similarly, p_{12} and p_{13} can also be excluded from the inequality. The other inequalities can be derived similarly.

3) The bounds of joint failure probabilities of three components can be expressed as

$$
\begin{aligned}
P(F_1 \cap F_2 \cap F_3) = P_{123} &\begin{cases} > p_1 + p_5 + p_6 \\ < p_1 + p_5 + p_6 + p_{10} + p_{11} \end{cases} \\
P(F_1 \cap F_2 \cap F_4) = P_{124} &\begin{cases} > p_1 + p_4 + p_6 \\ < p_1 + p_4 + p_6 + p_9 + p_{11} \end{cases} \\
P(F_1 \cap F_2 \cap F_5) = P_{125} &\begin{cases} > p_1 + p_4 + p_5 \\ < p_1 + p_4 + p_5 + p_9 + p_{10} \end{cases} \\
P(F_1 \cap F_3 \cap F_4) = P_{134} &\begin{cases} > p_1 + p_3 + p_6 \\ < p_1 + p_3 + p_6 + p_8 + p_{11} \end{cases} \\
P(F_1 \cap F_3 \cap F_5) = P_{135} &\begin{cases} > p_1 + p_3 + p_5 \\ < p_1 + p_3 + p_5 + p_8 + p_{10} \end{cases} \\
P(F_1 \cap F_4 \cap F_5) = P_{145} &\begin{cases} > p_1 + p_3 + p_4 \\ < p_1 + p_3 + p_4 + p_8 + p_9 \end{cases} \\
P(F_2 \cap F_3 \cap F_4) = P_{234} &\begin{cases} > p_1 + p_2 + p_6 \\ < p_1 + p_2 + p_6 + p_7 + p_{11} \end{cases} \\
P(F_2 \cap F_3 \cap F_5) = P_{235} &\begin{cases} > p_1 + p_2 + p_5 \\ < p_1 + p_2 + p_5 + p_7 + p_{10} \end{cases}
\end{aligned}
\tag{3.58}
$$

$$P(F_2 \cap F_4 \cap F_5) = P_{245} \begin{cases} > p_1 + p_2 + p_4 \\ < p_1 + p_2 + p_4 + p_7 + p_9 \end{cases}$$

$$P(F_3 \cap F_4 \cap F_5) = P_{345} \begin{cases} > p_1 + p_2 + p_3 \\ < p_1 + p_2 + p_3 + p_7 + p_8 \end{cases}$$

$p_1 + p_5 + p_6$ corresponds to the probabilities of states that no component survives and only components 4 or 5 survives. Since $p_7 + p_8 + p_9 + p_{10} + p_{11}$ corresponds to the probabilities of states that only two components survive, one can easily find that P_{123} is greater than $p_1 + p_5 + p_6$, and smaller than $p_1 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 + p_{10} + p_{11}$. Furthermore, because p_7, p_8 and p_9 is the probability corresponding to the state encoded by the $2x_1$ of z^{2x_1}, by the $2x_2$ of z^{2x_2}, and by the $2x_3$ of z^{2x_3}, i.e., the part of the states that only two components including component 1 survive, the part of the states that only two components including component 2 survive, and the part of the states that only two components including component 3 survive, respectively, p_7, p_8, and p_9 can be excluded from the inequality. The other inequalities can be derived similarly.

4) The joint failure probabilities of four components can be expressed as

$$\begin{aligned} P(F_1 \cap F_2 \cap F_3 \cap F_4) &= P_{1234} \\ &= p_1 + p_6 \\ P(F_1 \cap F_2 \cap F_3 \cap F_5) &= P_{1235} \\ &= p_1 + p_5 \\ P(F_1 \cap F_2 \cap F_4 \cap F_5) &= P_{1245} \\ &= p_1 + p_4 \\ P(F_1 \cap F_3 \cap F_4 \cap F_5) &= P_{1345} \\ &= p_1 + p_3 \\ P(F_2 \cap F_3 \cap F_4 \cap F_5) &= P_{2345} \\ &= p_1 + p_2 \end{aligned} \tag{3.59}$$

$p_1 + p_6$ corresponds to the probabilities of states that no component survives and only components 5 survives. One can easily find that P_{1234} equals to $p_1 + p_6$ in this system. The other inequalities can be derived similarly.

5) Obviously the joint failure probability of five components is the failure probability of all components in this system, and it can be expressed as

$$\begin{aligned} P(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5) &= P_{12345} \\ &= p_1 \end{aligned} \tag{3.60}$$

6) From Equation (2.33), one can find that the sum of $P(F_1)$, $P(F_2)$, $P(F_3)$, $P(F_4)$, and $P(F_5)$ can be expressed as

$$\begin{aligned} & P(F_1) + P(F_2) + P(F_3) + P(F_4) + P(F_5) \\ = \; & P_1 + P_2 + P_3 + P_4 + P_5 \\ = \; & 5p_{m_1} + 4(p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5} + p_{m_6}) \\ & +3(p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}} + p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}} + p_{m_{16}}) \\ & +2(p_{m_{17}} + p_{m_{18}} + p_{m_{19}} + p_{m_{20}} + p_{m_{21}} + p_{m_{22}} + p_{m_{23}} + p_{m_{24}} + p_{m_{25}} + p_{m_{26}}) \\ & +p_{m_{27}} + p_{m_{28}} + p_{m_{29}} + p_{m_{30}} + p_{m_{31}} \\ = \; & \binom{5}{1} p_{m_1} + \binom{4}{1}(p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5} + p_{m_6}) \\ & + \binom{3}{1}(p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}} + p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}} + p_{m_{16}}) \\ & + \binom{2}{1}(p_{m_{17}} + p_{m_{18}} + p_{m_{19}} + p_{m_{20}} + p_{m_{21}} + p_{m_{22}} + p_{m_{23}} + p_{m_{24}} + p_{m_{25}} + p_{m_{26}}) \\ & + \binom{1}{1}(p_{m_{27}} + p_{m_{28}} + p_{m_{29}} + p_{m_{30}} + p_{m_{31}}) \end{aligned} \tag{3.61}$$

Comparing the relationships shown in Equations (3.47), (3.52), (3.53),(3.54),(3.55), and (3.61), the sum of $P(F_1)$, $P(F_2)$, $P(F_3)$, $P(F_4)$, and $P(F_5)$ can also be expressed as

$$\begin{aligned} & P(F_1) + P(F_2) + P(F_3) + P(F_4) + P(F_5) \\ = \; & \binom{5}{1} p_1 + \binom{4}{1}(p_2 + p_3 + p_4 + p_5 + p_6) \\ & + \binom{3}{1}(p_7 + p_8 + p_9 + p_{10} + p_{11}) \\ & + \binom{2}{1}(p_{12} + p_{13} + p_{14} + p_{15} + p_{16}) \\ & + \binom{1}{1}(p_{17} + p_{18} + p_{19} + p_{20} + p_{21}) \end{aligned} \tag{3.62}$$

7) Similarly, from Equation (2.34), one can find that the sum of $P(F_{12})$, $P(F_{13})$, $P(F_{14})$, $P(F_{15})$, $P(F_{23})$, $P(F_{24})$, $P(F_{25})$, $P(F_{34})$, $P(F_{35})$, and $P(F_{45})$ can be expressed as

$$\begin{aligned}
& P(F_{12}) + P(F_{13}) + P(F_{14}) + P(F_{15}) + P(F_{23}) \\
& + P(F_{24}) + P(F_{25}) + P(F_{34}) + P(F_{35}) + P(F_{45}) \\
= \;& P_{12} + P_{13} + P_{14} + P_{15} + P_{23} + P_{24} + P_{25} + P_{34} + P_{35} + P_{45} \\
= \;& 10p_{m_1} + 6(p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5} + p_{m_6}) \\
& + 3(p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}} + p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}} + p_{m_{16}}) \\
& + p_{m_{17}} + p_{m_{18}} + p_{m_{19}} + p_{m_{20}} + p_{m_{21}} + p_{m_{22}} + p_{m_{23}} + p_{m_{24}} + p_{m_{25}} + p_{m_{26}} \\
= \;& \binom{5}{2} p_{m_1} + \binom{4}{2} (p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5} + p_{m_6}) \qquad (3.63) \\
& + \binom{3}{2} (p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}} + p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}} + p_{m_{16}}) \\
& + \binom{2}{2} (p_{m_{17}} + p_{m_{18}} + p_{m_{19}} + p_{m_{20}} + p_{m_{21}} + p_{m_{22}} + p_{m_{23}} + p_{m_{24}} + p_{m_{25}} + p_{m_{26}})
\end{aligned}$$

Comparing the relationships shown in Equations (3.47), (3.52), (3.53),(3.54),(3.55), and (3.63), the sum of $P(F_{12})$, $P(F_{13})$, $P(F_{14})$, $P(F_{15})$, $P(F_{23})$, $P(F_{24})$, $P(F_{25})$, $P(F_{34})$, $P(F_{35})$, and $P(F_{45})$ can also be expressed as

$$\begin{aligned}
& P(F_{12}) + P(F_{13}) + P(F_{14}) + P(F_{15}) + P(F_{23}) \\
& + P(F_{24}) + P(F_{25}) + P(F_{34}) + P(F_{35}) + P(F_{45}) \\
= \;& \binom{5}{2} p_1 + \binom{4}{2} (p_2 + p_3 + p_4 + p_5 + p_6) + \binom{3}{2} (p_7 + p_8 + p_9 + p_{10} + p_{11}) \\
& + \binom{2}{2} (p_{12} + p_{13} + p_{14} + p_{15} + p_{16}) \qquad (3.64)
\end{aligned}$$

8) Similar to the bounds on the joint failure probabilities of two components, one can easily find the formulation of the bounds on the joint failure probabilities of

three components can be expressed as

$$
\begin{aligned}
& P(F_{123}) + P(F_{124}) + P(F_{125}) + P(F_{134}) + P(F_{135}) \\
& +P(F_{145}) + P(F_{234}) + P(F_{235}) + P(F_{245}) + P(F_{345}) \\
= \;& P_{123} + P_{124} + P_{125} + P_{134} + P_{135} + P_{145} + P_{234} + P_{235} + P_{245} + P_{345} \\
= \;& 10p_{m_1} + 4(p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5} + p_{m_6}) \\
& +(p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}} + p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}} + p_{m_{16}}) \\
= \;& \binom{5}{3} p_{m_1} + \binom{4}{3} (p_{m_2} + p_{m_3} + p_{m_4} + p_{m_5} + p_{m_6}) \qquad (3.65) \\
& + \binom{3}{3} (p_{m_7} + p_{m_8} + p_{m_9} + p_{m_{10}} + p_{m_{11}} + p_{m_{12}} + p_{m_{13}} + p_{m_{14}} + p_{m_{15}} + p_{m_{16}})
\end{aligned}
$$

Comparing the relationships shown in Equations (3.47), (3.52), (3.53),(3.54),(3.55), and (3.65), the sum of $P(F_{123})$, $P(F_{124})$, $P(F_{125})$, $P(F_{134})$, $P(F_{135})$, $P(F_{145})$, $P(F_{234})$, $P(F_{235})$, $P(F_{245})$, and $P(F_{345})$ can also be expressed as

$$
\begin{aligned}
& P(F_{123}) + P(F_{124}) + P(F_{125}) + P(F_{134}) + P(F_{135}) \\
& +P(F_{145}) + P(F_{234}) + P(F_{235}) + P(F_{245}) + P(F_{345}) \qquad (3.66) \\
= \;& \binom{5}{3} p_1 + \binom{4}{3} (p_2 + p_3 + p_4 + p_5 + p_6) + \binom{3}{3} (p_7 + p_8 + p_9 + p_{10} + p_{11})
\end{aligned}
$$

Similar to the bounds on the joint failure probabilities of two components, one can easily obtain the formulation of bounds on the joint failure probabilities of three components in the RLP bounds method (Equations (3.59) and (3.66)). Also, from Equations (3.59) and (3.60), one can find that the joint failure probabilities of four components and five components in the RLP bounds method are equalities. Based on the above examples and formulation of bounds on the component failure probabilities, bounds on the joint failure probabilities of two components, and the bounds on the joint failure probabilities of three components the general formulation of constraints can be generalized as show in the following Section.

General Formulation of Constraints

For a system with n two-state components, one can generalize the formulation of bounds on the component failure probabilities, $P(F_i)$, $i = 1, 2, \ldots, n$, as

$$P(F_i) = P_i \begin{cases} > p_1 + \sum_{j=2}^{n+1} p_j - p_{i+1} \\ < p_1 + \sum_{j=2}^{n+1} p_j - p_{i+1} \\ \quad + \sum_{j=n+2}^{2n+1} p_j - p_{n+i+1} + \ldots \\ \quad + \sum_{j=(n-2)n+2}^{(n-1)n+1} p_j - p_{(n-2)n+i+1} \end{cases} \tag{3.67}$$

The sum of the component failure probabilities, $P(F_i)$, $i = 1, 2, \ldots, n$, can be expressed as

$$\begin{aligned} & \sum_{i=1}^{n} P(F_i) \\ = \; & \binom{n}{1} p_1 + \binom{n-1}{1} (p_2 + p_3 + \cdots + p_{n+1}) \\ & + \binom{n-2}{1} (p_{n+2} + p_{n+3} + \cdots + p_{2n+1}) \\ & + \binom{n-3}{1} (p_{2n+2} + p_{2n+3} + \cdots + p_{3n+1}) + \ldots \\ & + \binom{1}{1} (p_{(n-2)n+2} + p_{(n-2)n+3} + \cdots + p_{(n-1)n+1}) \end{aligned} \tag{3.68}$$

The formulation of bounds on the joint failure probabilities of two components, $P(F_i \cap F_j)$, $i = 1, 2, \ldots, n-1$, $j = 2, 3, \ldots, n$, $i < j$, can be expressed as

$$P(F_i \cap F_j) = P_{ij} \begin{cases} > p_1 + \sum_{k=2}^{n+1} p_k - p_{i+1} - p_{j+1} \\ < p_1 + \sum_{k=2}^{n+1} p_k - p_{i+1} - p_{j+1} \\ \quad + \sum_{k=n+2}^{2n+1} p_k - p_{n+i+1} - p_{n+j+1} + \ldots \\ \quad + \sum_{k=(n-3)n+2}^{(n-2)n+1} p_k - p_{(n-3)n+i+1} - p_{(n-3)n+j+1} \end{cases} \tag{3.69}$$

The sum of the all combination of the joint failure probabilities of two components, $P(F_i \cap F_j)$, $i = 1, 2, \ldots, n-1$, $j = 2, 3, \ldots, n$, $i < j$, can be expressed as

$$\begin{aligned}
& \sum_{i=1}^{n-1} \sum_{j=2}^{n} P(F_i \cap F_j) \\
= \quad & \binom{n}{2} p_1 + \binom{n-1}{2} (p_2 + p_3 + \cdots + p_{n+1}) \\
& + \binom{n-2}{2} (p_{n+2} + p_{n+3} + \cdots + p_{2n+1}) \\
& + \binom{n-3}{2} (p_{2n+2} + p_{2n+3} + \cdots + p_{3n+1}) + \ldots \\
& + \binom{2}{2} (p_{(n-3)n+2} + p_{(n-3)n+3} + \cdots + p_{(n-2)n+1})
\end{aligned} \tag{3.70}$$

The formulation of bounds on the joint failure probabilities of three components, $P(F_i \cap F_j \cap F_k)$, $i = 1, 2, \ldots, n-2$, $j = 2, 3, \ldots, n-1$, $k = 3, 4, \ldots, n$, $i < j < k$, can be expressed as

$$P(F_i \cap F_j \cap F_k) = P_{ijk} \begin{cases} > p_1 + \sum_{l=2}^{n+1} p_l - p_{i+1} - p_{j+1} - p_{k+1} \\ < p_1 + \sum_{l=2}^{n+1} p_l - p_{i+1} - p_{j+1} - p_{k+1} \\ \quad + \sum_{l=n+2}^{2n+1} p_l - p_{n+i+1} - p_{n+j+1} - p_{n+k+1} + \ldots \\ \quad + \sum_{l=(n-4)n+2}^{(n-3)n+1} p_l - p_{(n-3)n+i+1} - p_{(n-3)n+j+1} - p_{(n-3)n+k+1} \end{cases} \tag{3.71}$$

The sum of the all combination of the joint failure probabilities of three components, $P(F_i \cap F_j \cap F_k)$, $i = 1, 2, \ldots, n-2$, $j = 2, 3, \ldots, n-1$, $k = 3, 4, \ldots, n$, $i < j < k$, can be expressed as

$$\begin{aligned}
&\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n} P(F_i \cap F_j \cap F_k) \\
= &\binom{n}{3}p_1 + \binom{n-1}{3}(p_2 + p_3 + \cdots + p_{n+1}) \\
&+\binom{n-2}{3}(p_{n+2} + p_{n+3} + \cdots + p_{2n+1}) \\
&+\binom{n-3}{3}(p_{2n+2} + p_{2n+3} + \cdots + p_{3n+1}) + \ldots \\
&+\binom{3}{3}(p_{(n-4)n+2} + p_{(n-4)n+3} + \cdots + p_{(n-3)n+1})
\end{aligned} \tag{3.72}$$

The formulation of bounds on the joint failure probabilities of k components and its sum can be derived similarly.

From Equation (3.28), one can find that the joint failure probabilities of two components is not the bounds when $n = 2 + 1 = 3$, and it can be expressed as

$$\begin{aligned}
&P(F_i \cap F_j) \\
= \; &p_1 + \sum_{k=2}^{n+1} p_k - p_{i+1} - p_{j+1}
\end{aligned} \tag{3.73}$$

Similarly, from Equation (3.39), one can find that the joint failure probabilities of three components is not the bounds when $n = 3 + 1 = 4$, and it can be expressed as

$$\begin{aligned}
&P(F_i \cap F_j \cap F_k) \\
= \; &p_1 + \sum_{l=2}^{n+1} p_l - p_{i+1} - p_{j+1} - p_{k+1}
\end{aligned} \tag{3.74}$$

In conclusion, the joint failure probabilities of k components can be expressed as equalities in then RLP bounds method when $n = k + 1$.

In general, when the complete set of joint failure probabilities of each k components is available, the number of inequalities similar to Equations (3.67), (3.69), and (3.71) is

$$2\left(\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}\right) \tag{3.75}$$

Also similar to Equations (3.68), (3.70), and (3.72), one can find k equalities when the complete set of joint failure probabilities of each k component are available. Because the number of the constraints based on the axioms of probability is n^2-n+3 (Equations

(3.20) and (3.21)), the total number of constraints of the RLP bounds method is given by

$$N_c = (n^2 - n + 3) + 2\left(\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}\right) + k \tag{3.76}$$

3.4 Variation of RLP: Reduction of Constraints

3.4.1 Basic RLP

In the application of the RLP bounds method, one would simply consider the constraints of all equalities and inequalities (such as Equations (3.67) and (3.68)) and those resulting from the probability axioms (Equations (3.20) and (3.21)) in the RLP bounds method; we denoted this method as RLP1. For RLP1, the number of constraints is shown in Equation (3.76).

3.4.2 RLP2

When some of the joint failure probabilities of k components are identical, the inequality constraints based on these joint failure probabilities have the same effect in the calculation of the RLP bounds method. For such a case, one can sum up each side of these constraints, respectively, and thereby reduce the number of constraints without losing any accuracy. When some of the joint failure probabilities of k components are not identical but close to each other in value, one can still sum up each side of the inequality constraints with only a slight loss of accuracy. Thus, we sum up the inequality constraints related to the components whose joint failure probabilities are close to each other, and denoted this method as RLP2.

3.4.3 RLP3

If some of the inequality constraints in the RLP bounds method are redundant, they would have the same effect in the calculation. One can only remain one of the redundant inequality constraints for the RLP bounds method without losing any accuracy. If all of the constraints are redundant for the joint failure probabilities of any identical k components, for example, each component have the same failure probability (k=1), all of inequality constraints such as Equation (3.67) can be represented by one equality constraint such as Equation (3.68).

On the course of this research, we also found that the equality constraints such as Equation (3.68) and the probability axioms seem to be much important than the constraints of inequalities, and to be the "stronger" constraints in the LP algorithem. Thus, one could delete all of the inequalities, and consider only remain the constraints of equalities such as Equation (3.68) and those resulting from the probability axioms (Equations (3.20) and (3.21)) as constraints of the RLP bounds method. We called this method as RLP3, and the number of constraints can be expressed as

$$N_c = n^2 - n + k + 3 \tag{3.77}$$

Note the number of design variables of RLP1, RLP2, and RLP3 is the same each other as expressed by Equation (3.19). The rule for decreasing the number of constraints in RLP2 will further discussed in Section 3.4.6. The accuracy of RLP1, RLP2, and RLP3 are investigated later using numerical examples.

3.4.4 Size of Linear Programming Problem

For a system with n two-state components, the number of design variables (N_d) for all RLP1, RLP2, and RLP3 are identical as shown in Equation (3.19). However, the number of constraints (N_c) for RLP1, RLP2, and RLP3 are different, and then, the relative size of LP problem are different. The size of LP problem of RLP1 is determined by Equations (3.19) and (3.76), and that of RLP3 is determined by Equations (3.19) and (3.77).

Suppose one has the same LP solver described in Section 2.3.3 that can solve an LP problem with $N_d = 262144$ and $N_c = N_d + 987 + 1$ in the LP bounds method when $n = 18$. The theoretical limitation of the number of components in RLP1 and RLP3 that can be solved by the LP solver are shown in Table 3.1, in which k means that the complete set of the joint failure probabilities up to k components in the RLP bounds method is available.

Note that the estimation of the joint failure probabilities of k components requires additional computational effort. Most of the constraints in the LP bounds method,

Table 3.1: Theoretical limitation of the number of components.

k	RLP1	RLP3
1	512	512
2	362	512
3	91	512

i.e., the part determined by $2^n + 1$ in Equation (2.44), is the one only based on the basic axioms of probability (Equations (2.37) and (2.38)). Because the number of components (n) is usually small in the LP bounds method, the computational effort for the remaining part of the constraints ($\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$), i.e., the joint failure probabilities of k component, is also small.

On the contrary, most of the constraints in the RLP bounds method, i.e., the part determined by Equation (3.75), require additional computational effort. When the number of components (n) is large, a lot of CPU times is required for the calculation of the joint failure probabilities of k components. Thus, when the estimation of the joint failure probabilities of k components require additional computational effort, the RLP bounds method usually takes most of CPU times to estimate the joint failure probabilities of k components not for the LP.

3.4.5 Numerical Examples 1

Example 1: Truss with Seven Components as a Series System

Consider a truss as shown in Figure 3.3, which was used by Song and Der Kiureghian[24)] as a series system example. One can easily find that this truss is a statically determinate structure, and the failure of any component will cause the failure of the truss. Therefore, this truss can be considered as a series system. The load performed at the truss is denoted as S. For the sake of simplicity, we neglect the buckling failure mode. Let X_i, $i = 1, 2, \ldots, 7$, denote either tensile strength for a component in tension or compressive strength for a component in compression. From the distribution of internal forces shown in Figure 3.3, one can find that the failure states of the individual components are $F_i = \{X_i \leq S/(2\sqrt{3})\}$ for $i = 1$ and 2 and $F_i = \{X_i \leq S/\sqrt{3}\}$ for

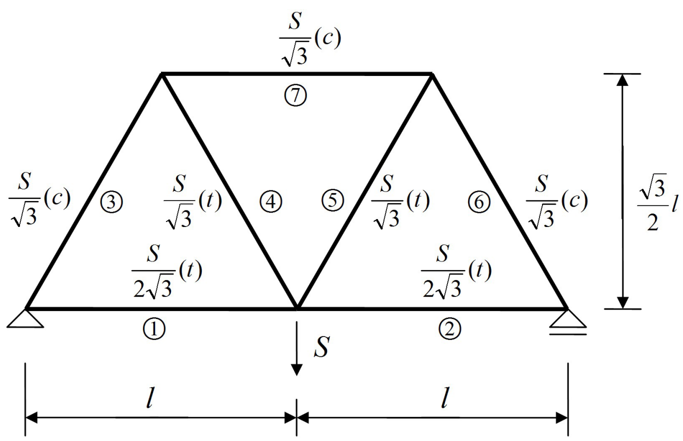

Figure 3.3: A statically determinate truss as a series system.

(t: intension, c: compression)

$i = 3, 4, \ldots, 7$. Suppose the intensity of the load is deterministic,i.e., $S = 100$, and the component strengths, i.e., the random variables X_i, $i = 1, 2, \ldots, 7$, are jointly normally distributed. Let X_1 and X_2 have the means of 100 and the standard deviations of 20, and X_3–X_7 have the means of 200 and the standard deviations of 40. Based on the above conditions, the component failure probabilities are equal, and it can be expressed as

$$P_i = P(F_i) = \Phi\left(\frac{100/2\sqrt{3} - 100}{20}\right) = 1.88 \times 10^{-4}; \qquad i = 1, 2$$

$$P_i = P(F_i) = \Phi\left(\frac{100/\sqrt{3} - 200}{40}\right) = 1.88 \times 10^{-4}; \qquad i = 3, \ldots, 7 \qquad (3.78)$$

where $\Phi(\bullet)$ denotes the standard normal cumulative distribution function. Further, let the X_i's have a Dunnet-Sobel (DS) class correlation matrix (See Section 2.2.4), then the joint failure probabilities can be estimated by Equation (2.20).

1) A system with complete information

First consider the case $r_1 = 0.90$, $r_2 = 0.96$, $r_3 = 0.91$, $r_4 = 0.95$, $r_5 = 0.92$, $r_6 = 0.94$, and $r_7 = 0.93$. The joint probabilities of two components P_{ij} computed using Equation (2.20) are $P_{12} = 5.73 \times 10^{-5}$, $P_{13} = 4.35 \times 10^{-5}$, $P_{14} = 5.42 \times 10^{-5}$, $P_{15} = 4.59 \times 10^{-5}$, $P_{16} = 5.13 \times 10^{-5}$, $P_{17} = 4.85 \times 10^{-5}$, $P_{23} = 6.08 \times 10^{-5}$, $P_{24} = 7.79 \times 10^{-5}$, $P_{25} = 6.47 \times 10^{-5}$, $P_{26} = 7.42 \times 10^{-5}$, $P_{27} = 6.87 \times 10^{-5}$, $P_{34} = 5.75 \times 10^{-5}$, $P_{35} = 4.86 \times 10^{-5}$, $P_{36} = 5.43 \times 10^{-5}$, $P_{37} = 5.14 \times 10^{-5}$, $P_{45} = 6.10 \times 10^{-5}$, $P_{46} = 6.88 \times 10^{-5}$, $P_{47} = 6.48 \times 10^{-5}$, $P_{56} = 5.76 \times 10^{-5}$, $P_{57} = 5.44 \times 10^{-5}$, and $P_{67} = 6.11 \times 10^{-5}$.

The joint probabilities of three components P_{ijl} computed using Equation (2.20) are $P_{123} = 2.81 \times 10^{-5}$, $P_{124} = 3.58 \times 10^{-5}$, $P_{125} = 2.99 \times 10^{-5}$, $P_{126} = 3.37 \times 10^{-5}$, $P_{127} = 3.17 \times 10^{-5}$, $P_{134} = 2.66 \times 10^{-5}$, $P_{135} = 2.25 \times 10^{-5}$, $P_{136} = 2.52 \times 10^{-5}$, $P_{137} = 2.38 \times 10^{-5}$, $P_{145} = 2.82 \times 10^{-5}$, $P_{146} = 3.18 \times 10^{-5}$, $P_{147} = 2.99 \times 10^{-5}$, $P_{156} = 2.67 \times 10^{-5}$, $P_{157} = 2.52 \times 10^{-5}$, $P_{167} = 2.83 \times 10^{-5}$, $P_{234} = 3.80 \times 10^{-5}$, $P_{235} = 3.17 \times 10^{-5}$, $P_{236} = 3.58 \times 10^{-5}$, $P_{237} = 3.37 \times 10^{-5}$, $P_{245} = 4.04 \times 10^{-5}$, $P_{246} = 4.58 \times 10^{-5}$, $P_{247} = 4.30 \times 10^{-5}$, $P_{256} = 3.80 \times 10^{-5}$, $P_{257} = 3.58 \times 10^{-5}$, $P_{267} = 4.04 \times 10^{-5}$, $P_{345} = 2.99 \times 10^{-5}$, $P_{346} = 3.37 \times 10^{-5}$, $P_{347} = 3.18 \times 10^{-5}$, $P_{356} = 2.83 \times 10^{-5}$, $P_{357} = 2.67 \times 10^{-5}$, $P_{367} = 3.00 \times 10^{-5}$, $P_{456} = 3.58 \times 10^{-5}$, $P_{457} = 3.37 \times 10^{-5}$, $P_{467} = 3.81 \times 10^{-5}$, and $P_{567} = 3.18 \times 10^{-5}$.

Song and Der Kiureghian[24)] used the above probability information as the exact results which is a little different from the results estimated by Equation (2.20). For the purpose of comparison, the above information is also used to estimate the bounds by the LP bounds method, RLP1, and RLP3 (RLP2 is not considered in this example), and the results are shown in Table 3.2, in which LP denotes the LP bounds method. Note that the narrower bounds can be obtained by considering a higher level of joint failure probability (i.e., large k).

The LP bounds method involves $2^7 = 128$ design variables whereas RLP1 and RLP3 involve $7^2 - 7 + 2 = 44$ design variables, and 1 equality and $2^7 = 128$ inequality constrains in the LP bounds method and 1 equality and $7^2 - 7 + 2 = 44$ inequality constrains in the RLP bounds method result from the basic axioms of probability (Equations (2.37) and (2.38), Equations (3.20) and (3.21)), respectively. The number of equality and inequality constrains for each method resulted from Equations (2.44) and (3.76) is shown in Table 3.3.

In Table 3.2 the bounds by the LP bounds method, RLP1, and RLP3 are found to be identical when $k = 1$. When $k = 2$ and $k = 3$, the bounds of the LP bounds method are found to be a little narrower than the bounds of RLP1 and RLP3, and the bounds of RLP1 are also found to be a little narrower than the bounds of RLP3, but the difference is not notable. Note that the narrower bounds can be obtained by considering a higher level of joint failure probability.

2) A series equireliable and equicorrelated system

Table 3.2: Bounds on the series system failure probability with equal component failure probabilities.

Bounds ($\times 10^{-3}$)	LP	RLP1	RLP3
$k = 1$	0.188–1.316	0.188–1.316	0.188–1.316
$k = 2$	0.477–0.912	0.469–0.934	0.469–0.966
$k = 3$	0.631–0.796	0.613–0.796	0.576–0.796

Table 3.3: The number of constraints.

	LP		RLP1		RLP3	
	Equalities	Inequalities	Equalities	Inequalities	Equalities	Inequalities
$k = 1$	8	128	2	58	2	44
$k = 2$	29	128	3	100	3	44
$k = 3$	64	128	4	170	4	44

As another example, consider the case where $r_i = \sqrt{\rho}$ in the DS correlation model, such that $\rho_{ij} = \rho$ for $i \neq j$ and $\rho_{ii} = 1$. This is the case of a series equireliable and equicorrelated system. The bounds of system failure probability estimated by the LP bounds method, RLP1, and RLP3 are shown in Table 3.4 for $k = 2$ and in Table 3.5 for $k = 3$. It is assumed that $\rho = 0.2, 0.4, 0.6, 0.8$, or 0.9. These bounds are found to be identical for all different ρ and k; in fact, the difference of the bounds by the LP bounds method, RLP1, and RLP3 is very small.

3) A system with inequality information

Finally, consider the case where the information of probabilities includes not only equalities but also inequalities. Suppose that for the equicorrelated system with $\rho = 0.9$, the inequalities $P_{ijl} \leq 5 \times 10^{-5}$, instead of equalities $P_{ijl} = 4.47 \times 10^{-5}$, $(1 \leq i < j < l \leq 7)$, are available. The bounds for LP bounds method, RLP1, and RLP3 for this case can be obtained by simply replacing the equality constraints with the corresponding inequality constraints. The result is shown in Table 3.6. One can easily find that these bounds are wider than the bounds ($k = 3$ and $\rho = 0.9$) in Table 3.5. The bounds of the LP bounds method, RLP1, and RLP3 are also found to be identical

Table 3.4: Bounds on the series system failure probability with equal component failure probabilities and correlations ($k = 2$).

		Bounds ($\times 10^{-3}$)		
ρ	P_{ij} $(1 \leq i < j < l \leq 7)$	LP	RLP1	RLP3
0.2	4.11×10^{-7}	1.307–1.314	1.307–1.314	1.307–1.314
0.4	2.56×10^{-6}	1.262–1.301	1.262–1.301	1.262–1.301
0.6	1.10×10^{-5}	1.085–1.250	1.085–1.250	1.085–1.250
0.8	3.87×10^{-5}	0.606–1.084	0.606–1.084	0.606–1.084
0.9	7.20×10^{-5}	0.406–0.884	0.406–0.884	0.406–0.884

Table 3.5: Bounds on the series system failure probability with equal component failure probabilities and correlations ($k = 3$).

		Bounds ($\times 10^{-3}$)		
ρ	P_{ij} $(1 \leq i < j < l \leq 7)$	LP	RLP1	RLP3
0.2	3.86×10^{-7}	1.307–1.308	1.307–1.308	1.307–1.308
0.4	1.68×10^{-6}	1.265–1.268	1.265–1.268	1.265–1.268
0.6	2.26×10^{-5}	1.119–1.163	1.119–1.163	1.119–1.163
0.8	1.72×10^{-5}	0.761–0.928	0.761–0.928	0.761–0.928
0.9	4.47×10^{-5}	0.516–0.722	0.516–0.722	0.516–0.722

Table 3.6: Bounds on the series system failure probability with equal component failure probabilities and correlations ($k = 3$).

		Bounds ($\times 10^{-3}$)		
ρ	P_{ij} $(1 \leq i < j < l \leq 7)$	LP	RLP1	RLP3
0.9	$\leq 5 \times 10^{-5}$	0.406–0.759	0.406–0.759	0.406–0.759

Table 3.7: Bounds on the series system failure probability with equal correlations.

ρ	k	Bounds ($\times 10^{-2}$) LP	RLP1	RLP2	RLP3
0.1	2	1.275–1.281	1.275–1.285	1.275–1.286	1.275–1.287
0.5	2	1.070–1.174	1.070–1.218	1.070–1.225	1.070–1.235
0.9	2	0.641–0.707	0.621–0.847	0.621–0.878	0.403–0.927

to each other.

Example 2: General Series Systems

1) An equicorrelated series system

Consider an equicorrelated series system with n=8 components and $\rho = 0.1$, 0.5, or 0.9, β's of the component are assumed to be 2.5, 3.0, 3.0, 3.1, 3.1, 3.2, 3.2, and 3.2. It is assumed that the performance function of each component is normally distributed. Then the joint failure probability of a small number of components can be estimated by the product of conditional marginal (PCM) method,[18)32)] which is an easy and fast method with a reasonable accuracy for the joint normal distribution function with small dimensions. Note that the PCM method is just used to demonstrate the efficiency and applicability of the RLP bounds method. The other reliability calculation methods can also be used. The system failure probabilities estimated by the LP bounds method, RLP1, RLP2, and RLP3 are shown in Table 3.7 for $k = 2$ and in Table 3.8 for $k = 3$. Since the failure probability of component with β=2.5 is much different from the others, we only consider the constraints related to this component as the important constraints in RLP2.

As expected, the bounds of the LP bounds method are found to be narrower than the bounds of RLP1, RLP2, and RLP3, and the bounds of RLP2 are also found to be narrower than the bounds of RLP3. The difference between the bounds by RLP1 and those by RLP2 is not notable. The difference between the bounds by LP bounds method and those by RLP1 will slightly increase with an increase of ρ, but the bounds of RLP1 and RLP2 are still close to the bounds of the LP bounds method.

Table 3.8: Bounds on the series system failure probability with equal correlations.

		Bounds ($\times 10^{-2}$)			
ρ	k	LP	RLP1	RLP2	RLP3
0.1	3	1.275–1.276	1.275–1.276	1.275–1.276	1.275–1.276
0.5	3	1.111–1.136	1.101–1.140	1.097–1.140	1.096–1.140
0.9	3	0.661–0.673	0.621–0.760	0.621–0.760	0.522–0.760

2) An equireliable series system

Consider an equireliable system with $n = 8$ components and $\beta = 3.5$. Assume that the correlation matrix $\mathbf{R}\equiv[\rho_{ij}]$ has the following form:

$$\mathbf{R} = \begin{bmatrix} 1 & 0.9 & 0.8 & 0.8 & 0.7 & 0.7 & 0.5 & 0.5 \\ & 1 & 0.9 & 0.8 & 0.8 & 0.7 & 0.7 & 0.5 \\ & & 1 & 0.9 & 0.8 & 0.8 & 0.7 & 0.7 \\ & & & 1 & 0.9 & 0.8 & 0.8 & 0.7 \\ & & & & 1 & 0.9 & 0.8 & 0.8 \\ & & \text{sym.} & & & 1 & 0.9 & 0.8 \\ & & & & & & 1 & 0.9 \\ & & & & & & & 1 \end{bmatrix}$$

The system failure probabilities estimated by the LP bounds method, RLP1, RLP2, and RLP3 are shown in Table 3.9. In RLP2 only the constraints related to the components having the correlation coefficient of 0.5 are considered to be important and taken into account in the LP problem. As expected, the bounds of the LP bounds method are found to be narrower than the bounds of RLP1, RLP2, and RLP3, and the bounds of RLP2 are also found to be narrower than the bounds of RLP3.

3) A system with 8 components

Consider an series system with n=8 components and the correlation matrix $\mathbf{R}\equiv[\rho_{ij}]$

Table 3.9: Bounds on the series system failure probability with equal component failure probabilities.

	Bounds ($\times 10^{-3}$)			
k	LP	RLP1	RLP2	RLP3
2	0.775–1.145	0.771–1.384	0.771–1.408	0.771–1.508
3	1.068–1.089	0.969–1.201	0.960–1.201	0.867–1.201

has the following form

$$\mathbf{R} = \begin{bmatrix} 1 & 0.9 & 0.8 & 0.8 & 0.7 & 0.7 & 0.7 & 0.5 \\ & 1 & 0.9 & 0.8 & 0.8 & 0.7 & 0.7 & 0.7 \\ & & 1 & 0.9 & 0.8 & 0.8 & 0.7 & 0.7 \\ & & & 1 & 0.9 & 0.8 & 0.8 & 0.7 \\ & & & & 1 & 0.9 & 0.8 & 0.8 \\ & & \text{sym.} & & & 1 & 0.9 & 0.8 \\ & & & & & & 1 & 0.9 \\ & & & & & & & 1 \end{bmatrix}$$

β's of the components are assumed to be 2.5, 3.0, 3.0, 3.1, 3.1, 3.2, 3.2, and 3.2. The system failure probabilities estimated by the LP bounds method, RLP1, RLP2, and RLP3 are shown in Table 3.10. In RLP2(1) only the constraints related to the component with β=2.5 are considered as important constraints, whereas in RLP2(2) only the constraints related to the components having a correlation coefficient of 0.5 are considered important. As expected, it is found that the bounds of the LP bounds method are the smallest, and that the bounds of RLP2(1) and RLP2(2) are also narrower than the bounds of RLP3. Because the system considered here is a little complex, the important constraints are not easy to find out. The rules for decreasing the number of constraint will be discussed in Section 3.4.6.

4) An equireliable and equicorrelated system with 15 components

Consider an equireliable and equicorrelated system with n=15 components, $\beta = 3$, and $\rho = 0.1$, 0.5, or 0.9. It is also assumed that the performance function of each component is normally distributed. The joint failure probability of k components can

Table 3.10: Bounds on the series system failure probability with equal component failure probabilities.

	Bounds ($\times 10^{-2}$)				
k	LP	RLP1	RLP2(1)	RLP2(2)	RLP3
2	0.719–0.847	0.621–0.954	0.621–0.965	0.535–1.025	0.535–1.047
3	0.772–0.806	0.717–0.876	0.691–0.876	0.667–0.876	0.650–0.876

Table 3.11: Bounds on the series system failure probability with equal component failure probabilities and correlations.

		Bounds ($\times 10^{-2}$)			
ρ	k	LP	RLP1	RLP3	MC
0.1	2	1.973–2.018	1.973–2.018	1.973–2.018	1.976
0.5	2	1.167–1.910	1.167–1.910	1.167–1.910	1.524
0.9	2	0.271–1.148	0.271–1.148	0.271–1.148	0.554
0.9	3	0.360–0.816	0.360–0.816	0.360–0.816	0.554

be estimated by the PCM method. The failure probabilities for series systems estimated by the LP bounds method, RLP1, RLP3, and the Monte Carlo (MC) method are shown in Table 3.11. MC simulations are conducted with 10^7 samples. As expected, the results of the LP bounds method, RLP1, and RLP3 are identical to each other; the difference of the bounds found by the LP bounds method, RLP1, and RLP3 is usually very smaller. The CPU time required for each analysis is shown in Table 3.12; one can find that the RLP bounds method is much faster than other methods. Note that RLP2 is not considered because the bounds of RLP2 will also identical to each other.

5) An equireliable and equicorrelated system with 100 components

Consider the case of an equireliable and equicorrelated system with n=100 components, $\beta = 3$, and $\rho = 0.1$, 0.3, or 0.5. It is also assumed that the performance function of each component is normally distributed. Then the joint failure probability of two or three components can be estimated by the PCM method. The failure probabilities for the series system estimated by RLP3 and MC method are shown in Table 3.13. MC

Table 3.12: CPU times for the series system failure probability with equal component failure probabilities and correlations.

		CPU times (s)			
ρ	k	LP	RLP1	RLP3	MC
0.1	2	10.617	1.265	0.406	6.748
0.5	2	10.074	1.213	0.430	6.814
0.9	2	10.117	1.173	0.453	6.771
0.9	3	1647.210	4.693	0.529	6.771

Table 3.13: Bounds on the failure probability of a series system with 100 components and equal component correlations ($\beta = 3$).

ρ	k	Bounds of RLP3	MC
0.1	2	0.111–0.135	0.115
0.3	2	0.051–0.133	0.086
0.5	2	0.019–0.127	0.056
0.5	3	0.022–0.094	0.056

simulations are conducted with 10^7 samples. The CPU time required for each analysis is shown in Table 3.14.

The bounds of RLP3 is getting narrower with the increase of k; however,one can see that when $k = 3$ the CPU time of RLP3 is much longer than the others. In fact, the linear programming in RLP3 takes only 0.482 s to calculate the bounds of the system failure probability, whereas the PCM method takes 115.027 s to calculate the joint failure probability of up to k components ($k = 3$). Because the number of combinations would increase rapidly with an increase of k and n, one needs longer and longer CPU times to calculate the joint failure probability of these combinations.

6) A series system with 100 components

Consider a series system with 100 components in which the correlation coefficient among components is considered in a product form, $\rho_{ij} = r_i r_j$ for $i \neq j$, $\rho_{ii} = 1$, and $r_i = \sqrt{(101 - i)/100}$, $i, j = 1, 2, \ldots, 100$. The component reliability indices varied as

Table 3.14: CPU times for a series system failure probability with 100 components and equal component correlations ($\beta = 3$).

		CPU times (s)	
ρ	k	RLP3	MC
0.1	2	1.346	57.239
0.3	2	1.331	57.785
0.5	2	1.290	57.767
0.5	3	115.509	57.767

Table 3.15: Bounds on the failure probability of a series system with 100 components.

k	Bounds of RLP3 ($\times 10^{-3}$)	MC
1	0.084–8.390	
2	6.611–8.354	7.878
3	6.891–8.189	

$\beta_i = 3.0 + 1.5 \times r_i$, $i = 1, 2, \ldots, 100$. The failure probabilities for the series system estimated by RLP3 and MC method are shown in Table 3.15. MC simulations are conducted with 10^7 samples. The CPU time required for each analysis is shown in Table 3.16.

As expected, the bounds of RLP3 is getting narrower with the increase of k, and one can see that when $k = 3$ the CPU time of RLP3 is much longer than the others. During the computation, the linear programming in RLP3 takes only 0.863 s to calculate the bounds of the system failure probability, whereas the PCM method takes 115.759 s to calculate the joint failure probability of up to 3 components ($k = 3$).

Example 3: A Parallel System with 6 Components

Consider a parallel system with six components, which was used by Pandey[18)] as a parallel system example. The correlation coefficient is considered in a product form, $\rho_{ij} = r_i r_j$ for $i \neq j$, $\rho_{ii} = 1$, and $r_i = \sqrt{(13 - 2i)/12}$, $i, j = 1, 2, \ldots, 6$[9)].

1) An equireliable parallel system

Table 3.16: CPU times for a series system failure probability with 100 components.

	CPU times (s)	
k	RLP3	MC
1	0.903	
2	1.680	58.280
3	116.622	

Table 3.17: Bounds on the parallel system failure probability with equal component failure probabilities.

			Bounds		
β	k	Numerical integration	LP	RLP1	RLP3
3	4	0.704×10^{-7}	$(0\text{–}1.035)\times10^{-7}$	$(0\text{–}1.035)\times10^{-7}$	$(0\text{–}3.271)\times10^{-6}$
2	4	0.528×10^{-4}	$(0\text{–}0.733)\times10^{-4}$	$(0\text{–}0.733)\times10^{-4}$	$(0\text{–}4.372)\times10^{-4}$
1	4	0.593×10^{-2}	$(0.100\text{–}0.767)\times10^{-2}$	$(0\text{–}0.767)\times10^{-2}$	$(0\text{–}1.363)\times10^{-2}$
0	3	0.117	$(0.171\text{–}1.778)\times10^{-1}$	$(0.071\text{–}1.778)\times10^{-1}$	$(0\text{–}2.113)\times10^{-1}$

Assume all six components having the identical β. The system failure probabilities that estimated by the LP bounds method, RLP1, RLP3, and numerical integration are reported for β ranging from 0 to 3 as shown in Table 3.17.

2) A unequireliable and unequicorrelated parallel system

Consider a second case in which the component reliability indices varied as $\beta_i = \beta + (7-2i)/5$, but the correlation matrix is the same as in the first case. The system failure probabilities estimated by the LP bounds method, RLP1, and RLP3 are presented in Table 3.18. As expected, the bounds of the LP bounds method are found to be narrower than the bounds of RLP1 and RLP3, and the bounds of RLP1 are also found to be narrower than the bounds of RLP3.

In the numerical examples in this Subsection, only multivariate normal random variables are considered in order to compare the efficiency of the proposed method with the other methods. However, it should be noted that the RLP bounds method can handle any kind of distribution.

Table 3.18: Bounds on the parallel system failure probability.

β	k	Numerical integration	Bounds LP	RLP1	RLP3
3	4	0.322×10^{-6}	$(0–0.950)\times10^{-6}$	$(0–0.950)\times10^{-6}$	$(0–1.905)\times10^{-6}$
2	4	0.112×10^{-3}	$(0.023–0.189)\times10^{-3}$	$(0–0.231)\times10^{-3}$	$(0–0.337)\times10^{-3}$
1	4	0.699×10^{-2}	$(0.640–0.780)\times10^{-2}$	$(0.106–1.023)\times10^{-2}$	$(0–1.365)\times10^{-2}$
0	3	0.990×10^{-1}	$(0.946–1.043)\times10^{-1}$	$(0.479–1.212)\times10^{-1}$	$(0–1.614)\times10^{-1}$

3.4.6 Strategy for RLP2

From examples presented in Section 3.4.5, one can find that the RLP bounds method is more efficient and can solve problems involving larger systems than the LP bounds method by decreasing the number of design variables and constraints. Also, one can find that RLP2 as described in Section 3.4 would have a wide applicability. However, the rule for decreasing the number of constraints is not quite clear; the number of constraints may still be enormously large for a large system. In this Section, we will improve the guideline for RLP2 by introducing a strategy to decrease the number of constraints.

Based on observations of examples of the RLP bounds method, it is proposed to divide the range of the joint failure probabilities of k components (i.e., from their minimum value to the maximum value) into $t_k \cdot n$, (in which t_k is an arbitrary selected natural number for each k) intervals evenly using a log scale, and to sum up the inequality constraints of the joint failure probabilities of k components in the same interval. The border of the i-th interval, P_{b_i}, is expressed as

$$\begin{aligned} \ln P_{b_i} &= \ln P_{\min} + (\ln P_{\max} - \ln P_{\min}) \cdot \frac{i}{t_k \cdot n} \\ i &= 1, 2, \ldots, t_k \cdot n \end{aligned} \tag{3.79}$$

where $P_{\min}$ and $P_{\max}$ are the minimum of the joint failure probabilities of k components and maximum of them, respectively. Note that the left border of the first interval is $\ln P_{\min}$ not $\ln P_{b_1}$. By using this procedure, the maximum of the total number of constraints can be reduced from that given by Equation (3.76) to

$$n_c = (n^2 - n + 3) + 2 \cdot \sum_{i=1}^{k} t_i \cdot n + k \tag{3.80}$$

Because that the probabilities are scattered, some of the internal would be empty, i.e., no probabilities fails into those intervals, and those intervals can be easily excluded. Note that if each probabilities for any identical k are the same as described in the RLP3 (Section 3.9), the sum of those probabilities can represent each of them, i.e., only one equality remains. Comparing Equation (3.76) and (3.80), one can obtain the controllable number of constraints by controlling the number of t_k.

For example, for a system with 3 two-state components as shown in Equation (3.22), when $k = 1$ and $t_k = 1$, each of component failure probability can be expressed as two inequalities in the RLP bounds method as shown in Equation (3.27). If $P(F_1)$ and $P(F_2)$ are in the same interval, one of the intervals would be empty. Because that $P(F_1)$ and $P(F_2)$ can be replaced by $P(F_1) + P(F_2)$, and the four constraints relative to $P(F_1)$ and $P(F_2)$ can be decreased to two constraints relative to $P(F_1) + P(F_2)$. Thus, the number of inequality constraints in the RLP bounds method for $k = 1$ may be reduced from 6 to 4.

Take another example, consider a system with 20 two-state components and suppose $t_k = k$. Then, the number of constraints of RLP1, and the maximum of the number of constraints of the method based on the proposed strategy can be obtained as shown in Table 3.19. The RLP2 in Table 3.19 denotes the RLP bounds method with the proposed strategy. Because there could exist some empty intervals, the number of constraints of RLP2 is the maximum. Clearly, the efficiency increases rapidly with an increase in the value of n and k.

There could be the cases that two joint failure probabilities are close to each other but not in the same internal. To cope with such cases, we propose to consider another set of internals whose border is given by

$$\begin{aligned} \ln P_{b_i} &= \ln P_{\min} + (\ln P_{\max} - \ln P_{\min}) \cdot \frac{(i - 0.5)}{(t_k \cdot n)} \\ i &= 1, 2, \ldots, t_k \cdot (n-1) \end{aligned} \tag{3.81}$$

The selection of t_k depends on the dispersion degree of the joint failure probabilities

Table 3.19: The number of constraints.

	RLP1		RLP2	
	Equalities	Inequalities	Equalities	Inequalities
$k = 1$	2	422	2	422
$k = 2$	3	802	3	462
$k = 3$	4	2700	4	502

of k components. A large dispersion requires a correspondingly large t_k. The accuracy of the RLP bounds method using this procedure is investigated in the following numerical example.

3.4.7 Numerical Example 2

1) A Truss with 7 component

Consider the same truss as shown in Figure 3.3, to see the effect of the reduction of the number of constraints in the RLP bounds method by considering the the proposed strategy.

In Tables 3.20 and 3.21, LP denotes the LP bounds method, RLP1 denotes the RLP bounds method considering all equalities and inequalities, RLP2 denotes the RLP bounds method with the strategy proposed in Section 3.4.6 when $t_k = 1$, and k is the number of components up to which considered in the joint failure probability. The failure probabilities and the number of equality and inequality constrains for each method are shown in Tables 3.20 and 3.21. Note that the number of constraints of RLP2 shown in Table 3.21 is the realistic number not the maximum shown in Equation (3.80).

Table 3.20: Bounds on the truss system failure probability.

Bounds ($\times 10^{-3}$)	LP	RLP1	RLP2
$k = 1$	0.188–1.316	0.188–1.316	0.188–1.316
$k = 2$	0.477–0.912	0.469–0.934	0.469–0.936
$k = 3$	0.631–0.796	0.613–0.796	0.607–0.796

Table 3.21: The number of constraints of the truss system.

	LP		RLP1		RLP2	
	Equalities	Inequalities	Equalities	Inequalities	Equalities	Inequalities
$k = 1$	8	128	2	58	2	44
$k = 2$	29	128	3	100	3	70
$k = 3$	64	128	4	170	4	96

In Table 3.20, the bounds estimated by the LP bounds method, RLP1, and RLP2, are identical when $k = 1$. When $k = 2$ and $k = 3$, the bounds of RLP1 and RLP2 are found to be somewhat wider than those of LP. The bounds of RLP2 are also slightly wider than those of RLP1; however, the difference between RLP1 and RLP2 is negligible, and the number of constraints in RLP2 is smaller than that of RLP1. The CPU times for all cases are found to be less than 0.5s, and the CPU times using RLP2 is the shortest among them. Note that the CPU times for the LP algorithm are determined by the number of design variables and constraints.

2) A series system with 20 components

Consider a series system with 20 components in which the correlation coefficient among components is considered in a product form, $\rho_{ij} = r_i r_j$ for $i \neq j$, $\rho_{ii} = 1$, and $r_i = \sqrt{(21-i)/20}$, $i, j = 1, 2, \ldots, 20$. The component reliability indices varied as $\beta_i = 3.0 + 1.5 \times r_i$, $i = 1, 2, \ldots, 20$.

The failure probabilities and the CPU times estimated by RLP1 and RLP2 are shown in Tables 3.22 and 3.23, in which RLP1 denotes the RLP bounds method considering all equalities and inequalities, RLP2 denotes the RLP bounds method with the strategy proposed in Section 3.4.6 when $t_k = k$, and k is the number of components up to which considered in the joint failure probability. The number of constraints of RLP1, and the maximum of the number of constraints of RLP2 are shown in Table 3.19. The failure probability estimated by MC simulations conducting with 10^7 samples is 1.312×10^{-3}, and the CPU times is 9.8 s.

In Table 3.22, the bounds of RLP2 are slightly wider than those of RLP1; however, the difference between RLP1 and RLP2 is negligible. The number of constraints in

Table 3.22: Bounds on the failure probability of a series system.

Bounds ($\times 10^{-3}$)	RLP1	RLP2
$k = 1$	0.426–1.366	0.426–1.366
$k = 2$	1.295–1.357	1.295–1.358
$k = 3$	1.308–1.338	1.307–1.338

Table 3.23: CPU times for a series system with 20 components.

	CPU times (s)	
k	RLP1	RLP2
1	0.091	0.090
2	5.610	1.231
3	73.226	3.627

RLP2 is smaller than that of RLP1 as shown in Table 3.19, and as expected the CPU times for RLP2 are shorter than that of RLP1.

3.5 Variation of RLP: Incomplete Information

3.5.1 LP+RLP Approach

Sometimes one can only have the joint failure probability of not all of but only some of the k components, i.e., incomplete set of failure probability, still the RLP bounds method can be applied with less accuracy. In such a case, one take the n_1 components that the complete set of joint failure probabilities of each k components are available as a subsystem, of which can be bounds on reliability by the RLP bounds method. Because this subsystem have $({n_1}^2 - n_1 + 2)$ design variables in the RLP bounds method, it also can be considered as a $({n_1}^2 - n_1 + 2)$-state component.

Combining the subsystem with the $(n - n_1)$ remaining components having the incomplete set of joint failure probability of k components, the system now consists of $(n - n_1)$ two-state components and one $({n_1}^2 - n_1 + 2)$-state component. The bounds on system reliability is estimated using LP bounds method; the number of design variables in the LP bounds method is

$$N_d = ({n_1}^2 - n_1 + 2) \cdot 2^{n-n_1} \tag{3.82}$$

This kind of combination of the LP bounds method and RLP bounds method is denoted here as LP+RLP. Obviously, the size of design variables will increase quickly, which will limit its application.

Since there are three types of RLP bounds methods, i.e., RLP1, RLP2, and RLP3, there are three combinations of the LP bounds method and RLP bounds method (denoted as LP+RLP1, LP+RLP2, and LP+RLP3, respectively). For example, suppose a system consists of seven components while only six components having the complete set of joint failure probabilities of each k components. Then these six components form a subsystem formulated by the RLP bounds method with $6^2 - 6 + 2 = 32$ design variables, this subsystem serves as a 32-state component in the LP bounds method, and the number of design variables is 64 in the LP+RLP approach. The accuracy of the LP+RLP approaches is investigated later using numerical examples.

3.5.2 Numerical Examples for Variation 2

1) A Truss with 7 component

Consider the same truss as shown in Figure 3.3, to investigate the accuracy of the LP+RLP approach based on the incomplete information. Suppose the information of the joint failure probabilities involving component 1 are not available, i.e., P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_{17}, P_{123}, P_{124}, P_{125}, P_{126}, P_{127}, P_{134}, P_{135}, P_{136}, P_{137}, P_{145}, P_{146}, P_{147}, P_{156}, P_{157}, and P_{167} are unknown. For the LP bounds method, this problem can be solved by simply removing 6 constraints (6 joint failure probabilities of two components) for $k = 2$ and 15 constraints (15 joint failure probabilities of three components) for $k = 3$. For the LP+RLP approaches, one can take components 2–7 as a subsystem, which can be formulated by RLP1 and RLP3, respectively, and then take the subsystem as a 32-state component in the LP bounds method. The number of design variables of the LP+RLP approaches is $2 \cdot (6^2 - 6 + 2) = 64$ resulted from Equation (3.82), whereas that of the LP bounds method is $2^7 = 128$ resulted from Equation (2.43). For the correlation model shown in the example, i.e., 1) A system with complete information), the result

Table 3.24: Bounds on the series system failure probability with equal component failure probabilities and incomplete probability information.

Bounds ($\times 10^{-3}$)	LP	LP+RLP1	LP+RLP3
$k=2$	0.443–0.970	0.443–0.982	0.443–1.007
$k=3$	0.585–0.877	0.579–0.879	0.553–0.879

is shown in Table 3.24. These bounds are wider than the corresponding bounds shown in Table 3.2 because of the incomplete information. The bounds of the LP+RLP1 and LP+RLP3 are also found to be a little wider than that of the LP bounds method, and the bounds of the LP+RLP1 are also found to be narrower than that of the LP+RLP3.

2) An equireliable and equicorrelated system with 100 components

Consider the case of an equireliable and equicorrelated system with n=100 components, $\beta = 3$, and $\rho = 0.1$, 0.3, or 0.5. It is also assumed that the performance function of each component is normally distributed. Then the joint failure probability of two or three components can be estimated by the PCM method. Suppose the information of the joint failure probabilities involving one component are not available. For the LP+RLP approach, one can take 99 components which have complete sets of information to form a subsystem formulated by the RLP bounds method with $99^2 - 99 + 2 = 9704$ design variables resulted from Equation (3.19). This subsystem serves as a 9704-state component in the LP bounds method, and the number of design variables is 19408 resulted from Equation (3.82) in the LP. The failure probabilities for the series system estimated by LP+RLP3 are also shown in Table 3.25. The CPU time required for each analysis is shown in Table 3.26.

The bounds of LP+RLP3 shown in Table 3.25 are wider than that of RLP3 shown in Table 3.13 because of the missing information. Similar to the result of RLP3, one can see that when $k = 3$ the CPU time of LP+RLP3 is much longer than the others. In fact, the linear programming in LP+RLP3 takes only 2.064 s to calculate the bounds of the system failure probability, whereas the PCM method takes 109.113 s to calculate the joint failure probability of up to k components ($k = 3$). Because the number of combinations would increase rapidly with an increase of k and n, one needs longer and

Table 3.25: Bounds on the failure probability of a series system with 100 components and equal component correlations ($\beta = 3$).

ρ	k	Bounds of LP+RLP3
0.1	2	0.110–0.143
0.3	2	0.051–0.141
0.5	2	0.019–0.136
0.5	3	0.022–0.103

Table 3.26: CPU times for a series system failure probability with 100 components and equal component correlations ($\beta = 3$).

		CPU times (s)
ρ	k	LP+RLP3
0.1	2	3.197
0.3	2	3.050
0.5	2	2.879
0.5	3	111.177

longer CPU times to calculate the joint failure probability of these combinations.

3.6 Summary of Relaxed Linear Programming Bounds Method

As an efficient reliability tool for a system with a large number of components, the relaxed linear programming (RLP) bounds method is introduced in this chapter. The RLP bounds method can be used to estimate the bounds for a series system as well as a parallel system.

3.6.1 Advantages

The most important advantage of the RLP bounds method over the LP bounds method is that the RLP bounds method can handle a system with much larger number of components than the LP bounds method. For example, suppose authors have the same LP solve described in Section 2.3.4, then in theory the limitation of the number of components for RLP3 is 512, whereas that of the LP bounds method is 18.

Similar to the LP bounds method, the RLP bounds method has a lot of advantages over other existing methods (e.g., Boole bounds[35] or Zhang bounds[41,42]). The main advantages include: (a) any "level," i.e., the number (k) of components considered in the joint probabilities of the states, of information can be used, including equalities and inequalities; (b) the statistical dependency among component states is easily accounted for in terms of their joint probabilities; (c) the method provides the result comparable to the LP bounds method for the given information of individual and joint component states probabilities; (d) the method is applicable to a pure series system as well as a pure parallel system.

Like the LP bounds method, there are also two advantages of the RLP bounds method over the MC simulation. One is that the RLP bounds method is unaffected by the magnitude of the failure probability, whereas the MC simulation is not. Another advantage is that the RLP bounds method is still applicable when the information is not incomplete, whereas MC simulation is not.

3.6.2 Disadvantages

The main drawback of the RLP bounds method is that the method is only applicable to a pure series system and a pure parallel system, not for a general system.

In this chapter, as an efficient reliability tool, the RLP bounds method has been developed. The RLP bounds method is applicable to a pure large series system and a pure large parallel system. The number of design variables can be decreased from 2^n to n^2-n+2 by employing the UGF, and the number of constraints has also been decreased extremely by considering the decreasing strategy as described in Section 3.4.6. The RLP bounds method is efficient than the LP bounds method, and it can provide a result comparable to that of the LP bounds method. The RLP bounds method has also a lot advantages similar to the LP bounds method. However, the RLP bounds method is not applicable to the combination system of series subsystem and parallel subsystem. In order to extend the RLP bounds method a general system, an new approach will be introduced in the next chapter.

Chapter 4

EXTENDED RLP BOUNDS METHOD BASED ON FAILURE MODES

4.1 Overview

Although the RLP bounds method is more efficient and can solve problems involving larger systems than the LP bounds method, the method can still be improved to handle a general system consisting of both series and parallel subsystem[66–69]. This chapter will introduce the extended RLP bounds method based on failure modes to the reliability analysis of the general system. The chapter is composed of the following sections:

In section 4.2, the outline of the extended RLP bounds method based on failure modes is introduced.

In section 4.3, the procedure of the extended RLP bounds method based on failure modes is introduced.

In section 4.4, the limitation of size of the extended RLP bounds method based on failure modes is introduced.

In section 4.5, numerical examples have been used to prove the application of the extended RLP bounds method based on failure modes.

In section 4.6, the advantages and disadvantages of the extended RLP bounds method based on failure modes is summarized.

4.2 Outline of Extended RLP Bounds Method Based on Failure Modes

Since the RLP bounds method is not applicable to a general system that consists of both series subsystems and parallel subsystems, in this section, we propose an approach to extend the applicability of the combination of the RLP bounds method and the LP bounds method to handle a general system by decomposing the entire system into subsystems based on the failure modes.

A system with multi-failure modes can be modeled as a series system if it fails whenever any of its critical failure modes occur. Many methods have been introduced and used to determine the critical failure modes of a system[27,70]. In such a system each critical failure mode or so called cut set can be considered as a "component".

Each critical failure mode is also a system (subsystem) itself. The bounds on its failure probability can be computed by the RLP bounds method if it is a series or parallel system, and the bounds on its joint failure probability can also be computed by the RLP bounds method. These bounds estimated by the RLP bounds method are then used as constraints in solving the LP problem in order to estimate the failure probability of the entire system. Since this approach is based on the failure modes of the system, we called it as extended RLP bounds method based on failure modes.

4.3 Procedure of Extended RLP Bounds Method Based on Failure Modes

The extended RLP bounds method based on the failure modes takes the following procedure (see Figure 4.1).

1) The first step is to define the components which is related to the failure of the system.

2) The critical failure modes for the system are identified. Each critical failure mode can be considered as a "component" in the entire system. It should be noted that unlike the multi-scale system reliability analysis, a component can belong to more

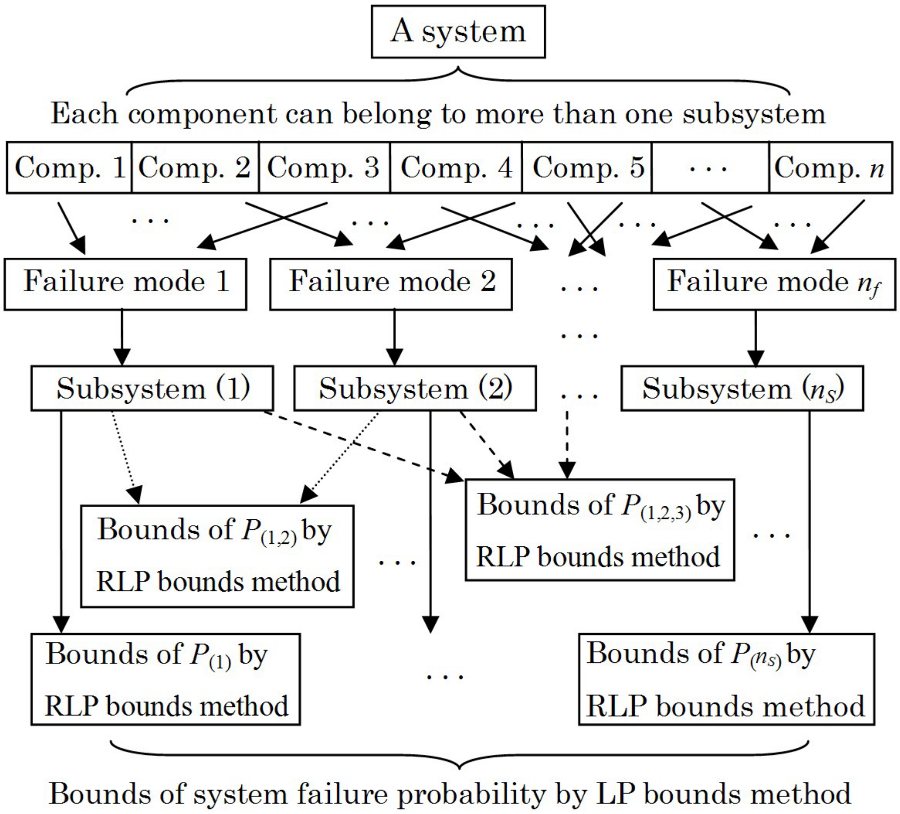

Figure 4.1: Diagram of extended RLP bounds method based on failure modes.

than one critical failure mode. Then, the extended RLP bounds method based on failure modes can do better than the multi-scale system reliability analysis for maintaining the correlation among the components, i.e., provides a more realistic model.

Since the system can be modeled as a series system based on the critical failure modes, the failure probabilities of the system can be expressed as

$$P_f = P(F_1 \cup F_2 \cup F_3 \cup \cdots \cup F_{n_f}) \tag{4.1}$$

where n_f is the number of the critical failure modes, and F_i, $i = 1, 2, \ldots, n_f$, is the event that ith failure mode occurs.

When the number of critical failure modes is large, a group of failure modes is treated as a subsystem. The grouping is made based on the degree of correlation; failure modes having a strong correlation can be combined into a group. The

Equation (4.1) can be rewritten as

$$P_f = P(F_{(1)} \cup F_{(2)} \cup F_{(3)} \cup \cdots \cup F_{(n_S)}) \tag{4.2}$$

where n_S, $n_S \leq n_f$, is the number of the subsystems, and $F_{(j)}$, $j = 1, 2, \ldots, n_S$, is the event that jth subsystem fails.

3) Based on the component failure probabilities and the joint failure probabilities of the components, the bounds on the subsystem failure probabilities and the joint failure probabilities of the subsystems can be estimated by the RLP bounds method.

The failure probabilities of the subsystem i, $P_{(i)}$, can be expressed as

$$\begin{aligned} P_{(i)} &= P(F_{(i)}) \\ i &= 1, 2, ..., n_S \end{aligned} \tag{4.3}$$

Note that (i) in $P_{(i)}$ denotes subsystem i. The bounds on failure probability of the subsystem (i) can be estimated by the RLP bounds method if it is a series or parallel system. In fact, the subsystem transformed from the critical failure mode is usually a parallel system.

The joint failure probabilities of the subsystem i and j, $P_{(i,j)}$, can be expressed as

$$\begin{aligned} P_{(i,j)} &= P(F_{(i)} \cup F_{(j)}) \\ i &= 1, 2, ..., n_S - 1, \quad j = 1, 2, ..., n_S, \quad i \neq j \end{aligned} \tag{4.4}$$

The bounds on the joint failure probability of the subsystem i and j can also be estimated by the RLP bounds method because a new parallel subsystem consists of those two subsystems.

The joint failure probabilities of the subsystem i, j, and k, $P_{(i,j,k)}$, can be expressed as

$$\begin{aligned}
P_{(i,j,k)} &= P(F_{(i)} \cup F_{(j)} \cup F_{(k)}) \\
i &= 1, 2, ..., n_S - 2, \\
j &= 1, 2, ..., n_S - 1, \\
k &= 1, 2, ..., n_S, \\
i &\neq j \neq k
\end{aligned} \tag{4.5}$$

4) Based on the bounds on the subsystem failure probabilities estimated by the RLP bounds method and the joint failure probabilities of the subsystems estimated by the RLP bounds method, the bounds on the failure probabilities of the entire system can be estimated by the LP bounds method.

On the course of this research, we found that the equalities have an important effect to the accuracy of the RLP bounds method. When the information for many of the probabilities is given by inequalities, the bounds for the entire system could be wide using the RLP bounds method. Therefore, the LP bounds method is preferred for determining the bounds for the entire system.

When a system has a very large number of failure modes, sub-subsystems below each subsystem are considered again, i.e., a group of sub-subsystems are grouped together as a single subsystem. Also, if a subsystem is not a pure series system or a pure parallel system, the diagram shown in Figure 4.1 is applied to estimate the bounds of its failure probability.

4.4 Limitation of Size of Extended RLP Bounds Method Based on Failure Modes

Since the bounds of the entire system and subsystems are obtained by using the LP bounds method and the RLP bounds method, the application of the failure mode analysis method is limited by the size of the LP problem of the entire system and of each subsystem.

1) Size of LP Problem for Subsystem i

The number of design variables for subsystem i is

$$N_{d_{(i)}} = {n_{(i)}}^2 - n_{(i)} + 2 \tag{4.6}$$

where $n_{(i)}$ denotes the number of components in subsystem i.

If the complete set of joint failure probabilities of each $k_{(i)}$ components is available, the constraints for subsystem i have the form of the RLP bounds method, and the number of constraints for subsystem i is

$$N_{c_{(i)}} = ({n_{(i)}}^2 - n_{(i)} + 3) + 2\left(\binom{n_{(i)}}{1} + \binom{n_{(i)}}{2} + \cdots + \binom{n_{(i)}}{k_{(i)}}\right) + k_{(i)} \tag{4.7}$$

2) Size of LP Problem for Entire System

The number of design variables for the entire system is

$$N_d = 2^{n_S} \tag{4.8}$$

If the complete set of joint failure probabilities of each k_f subsystems is available, the constraints for the entire system have the form of the LP bounds method, and the number of constraints for the entire system is

$$N_c = (2^{n_S} + 1) + 2\left(\binom{n_S}{1} + \binom{n_S}{2} + \cdots + \binom{n_S}{k_f}\right) \tag{4.9}$$

Note that the failure probabilities of subsystems and the joint failure probabilities of subsystems are usually given as the bounds on the failure probabilities, leading to inequality constraints.

Suppose we have the same LP solver described in Section 2.3.4. Then, in theory the bounds for the system failure probability can be obtained using the extended RLP bounds method based on failure modes when $n_S \leq 18$, $k_f \leq 3$, $n_{(i)} \leq 512$, and $k_{(i)} \leq 3$.

4.5 Numerical Examples

4.5.1 Example 1: Rigid-plastic Structure as a General System

As a general system example, consider an ideally elastic-plastic cantilever beam including an ideally rigid-brittle bar as shown in Figure 4.2[24]. It is assumed that the

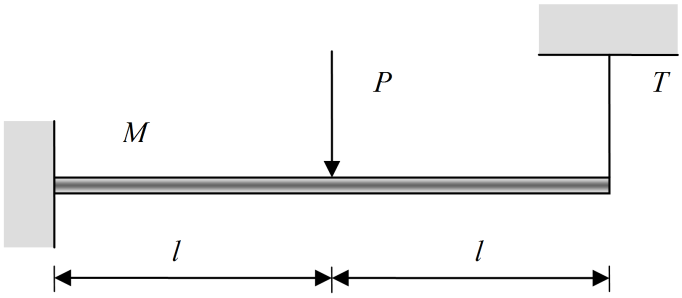

Figure 4.2: Cantilever beam-bar system.

Table 4.1: The mean and standard deviation of normal distribution *M, T, and X.*

	Mean	Standard Deviation
M	1000	300
T	110	20
P	150	30

moment capacity of the beam, M, the tensile strength of the bar, T, and the load applied at the midspan of the beam, P, are uncorrelated normal random variables with statistical characteristics shown in Table 4.1. For the sake of simplicity, we assume in this example that, the length of the beam, l, is deterministic and equals 5. Ignoring shear failure, there are three possible paths, or cut sets, for the failure of this structural system as follows:

1) Cut set 1: The rigid-brittle bar failed first (failure event F_1), then the cantilever beam failed after forming a hinge at the fixed end of the beam (failure event F_2).

2) Cut set 2: A hinge forms at the fixed end of the beam first (failure event F_3), then another hinge forms at the midspan of the beam (failure event F_4).

3) Cut set 3: A hinge forms at the fixed end of the beam first (failure event F_3), then the rigid-brittle bar failed (failure event F_5).

The procedure of the extended RLP bounds method based on the failure modes can be described by the following.

[1] Considering the sequence of the failure events within the above cut sets, there are five different failure events, that can be considered as the failure of five virtual

Table 4.2: Bounds on the failure probability of the cantilever beam-bar system.

Bounds	proposed method		
($\times 10^{-3}$)	k_f=1	k_f=2	LP
$k_m = 1$	0–9.24	0–9.24	0–9.24
$k_m = 2$	6.83–8.01	7.75–7.79	7.75–7.75

components forming a virtual system. Three cut sets in the structural system correspond to three subsystems in the virtual system. The failure events of the subsystem i, $F_{(i)}$, $i = 1, 2, 3$, can be defined as

$$F_{(1)} = F_1 \cap F_2, \quad F_{(2)} = F_3 \cap F_4, \quad F_{(3)} = F_3 \cap F_5$$

Then, the system failure event, F_s, can be expressed as

$$\begin{aligned} F_s &= F_{(1)} \cup F_{(2)} \cup F_{(3)} \\ &= (F_1 \cap F_2) \cup (F_3 \cap F_4) \cup (F_3 \cap F_5) \end{aligned} \tag{4.10}$$

[2] After the structural analysis of the structural system, the failure events of the virtual components can be expressed as follows:

a) The failure event F_1: the rigid-brittle bar failed first,

$$F_1 = \left\{ X_1 = T - \frac{5P}{16} < 0 \right\} \tag{4.11}$$

b) The failure event F_2: the cantilever beam failed as the result of forming a hinge at the fixed end of the beam after the failure event F_1,

$$F_2 = \{ X_2 = M - lP < 0 \} \tag{4.12}$$

c) The failure event F_3: a hinge forms at the fixed end of the beam first,

$$F_3 = \left\{ X_3 = M - \frac{3lP}{8} < 0 \right\} \tag{4.13}$$

d) The failure event F_4: another hinge forms at the midspan of the beam after the failure event F_3,

$$F_4 = \left\{ X_4 = M - \frac{lP}{3} < 0 \right\} \tag{4.14}$$

e) The failure event F_5: the rigid-brittle bar failed after the failure event F_3,

$$F_5 = \{X_5 = M + 2lT - lP < 0\} \tag{4.15}$$

Because M, T, and P are normal random variables, X_i's in Equations (4.11) - (4.15) are jointly normally distributed. The failure probabilities of the virtual component and the joint failure probabilities of the virtual components can be computed from the standard normal cumulative distribution function (CDF) and the joint normal CDF, respectively, using the PCM.

[3] Based on the failure probabilities of the virtual component and the joint failure probabilities of the two virtual components estimating in step [1], the bounds on the failure probabilities of the subsystem and the joint failure probabilities of the subsystems can be estimated by the RLP bounds method. Note that when the information of the joint failure probabilities of the two virtual components is available, the failure probability of each subsystem is an exact value in this structural system.

[4] Based on the information of the failure probabilities of the subsystem and the joint failure probabilities of the subsystems estimated in step [3], the bounds on the failure probability of the entire system can be estimated by the LP bounds method.

The bounds on the failure probability of the system estimated by the extended RLP bounds method based on failure modes and the LP bounds method are shown in Table 4.2. Their k_m denotes the number of components considered to calculate the bounds of the subsystem failure probability. By MC simulation with 10^7 simulations, the system failure probability is estimated as 7.75×10^{-3}.

In Table 4.2, one can find that when $k_m = 1$ the bounds determined by the extended RLP bounds method based on failure modes are rather wide, but identical to those resulting from the LP bounds method. When $k_m = 2$ and $k_f = 1$, the bounds estimated by the extended RLP bounds method based on failure modes are found to be acceptable. The difference between the bounds is negligible, particularly when $k_m = 2$ and $k_f = 2$. The CPU times of the extended RLP bounds method based on failure modes and the

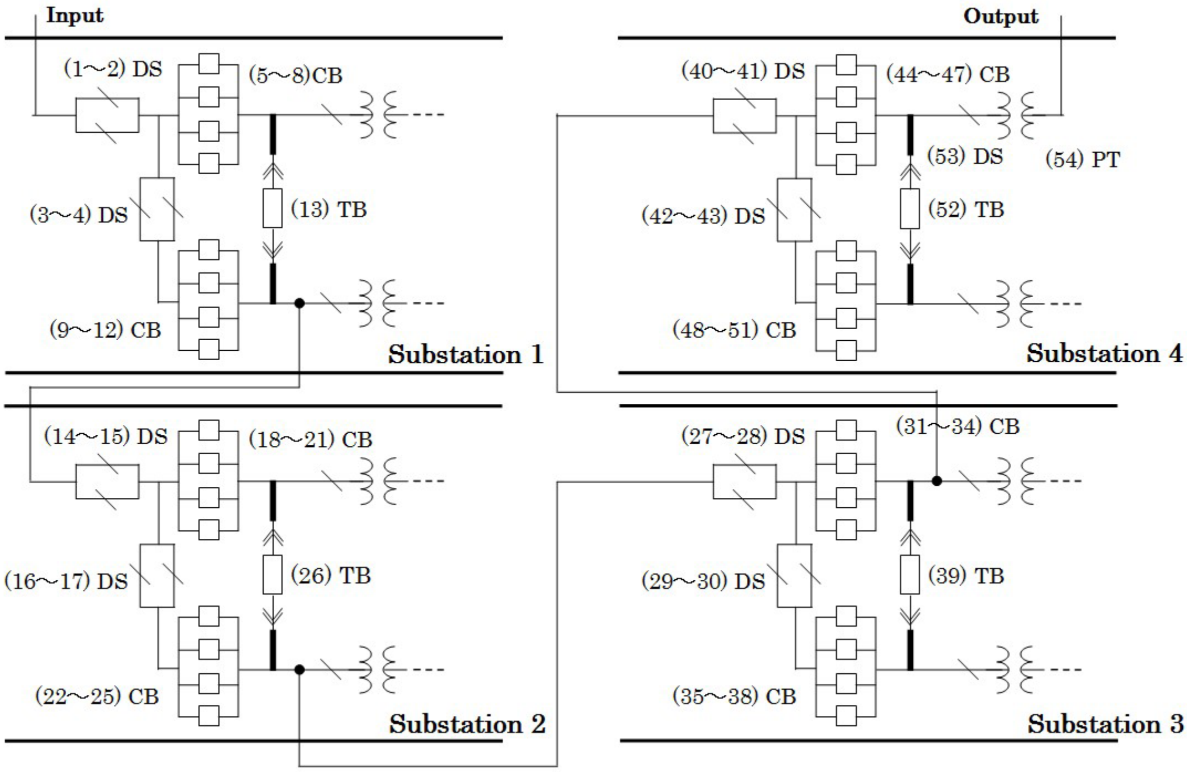

Figure 4.3: Four substation power network.

LP bounds method are both less than 1 second in this example; however, the extended RLP bounds method based on failure modes can handle a much larger system efficiently, while the LP bounds method cannot, as shown in the next section.

4.5.2 Example 2: Seismic reliability of a substation power network

Consider the four substation power network system shown in Figure 4.3, which is designed based on the hypothetical substation example[71, 72]. Equipment items of the system of the substation power network are shown in Table 4.3.

Assume that the substation power network is located in an earthquake-prone region. Let A denote the bedrock peak ground acceleration (PGA) in the region of the substations, and let S_i denote a factor representing the local site response of equipment item i, such that $A \cdot S_i$ is the actual peak acceleration experienced by the i_{th} equipment item. Assume that A is a lognormal random variable with mean $0.15g$ (in units of

Table 4.3: The mean and coefficient of variation of equipment capacities.

Equipment Items	Mean	c.o.v.
Disconnect Switch (DS)	0.7g	0.3
Circuit Breaker (CB)	0.6g	0.3
Tie Breaker (TB)	1.0g	0.3
Power Transformer (PT)	1.5g	0.5

gravity acceleration, g) and a coefficient of variation (c.o.v.) equal to 0.5. Assume also that the S_i, $i = 1, 2, \ldots, n$, are lognormal random variables independent of each other and also independent of A, with means 1.0 and c.o.v. ' s 0.2. Let R_i denote the capacity of the i_{th} equipment item with respect to the base acceleration in units of g, and assume that R_i are lognormally distributed with statistics as shown in Table 4.3. Assume that the tie breakers have an equal correlation coefficient of 0.5, and that the other equipment items have an equal correlation coefficient of 0.3. The capacities of equipment items in different categories are assumed to be statistically independent of each other.

The ability of the substation power network to supply power from the input line of Substation 1 (Input) to the output line of Substation 4 (Output) is assumed to be the performance criterion. For this criterion, there are 54 components in the system of the substation power network as shown in Figure 4.3. For example, ($1 \sim 2$) DS means that there are components 1 and 2, and that they are a Disconnect Switch.

The failure events of the individual equipment items are formulated as $F_i = \ln R_i - \ln A - \ln S_i \leq 0$, $i = 1, 2, \ldots, 54$. Let $V_i = \ln R_i - \ln A - \ln S_i$. Because R_i, A, and S_i are lognormally distributed, V_i is normally distributed.

The system has 22 cut sets which are expressed using component identification numbers shown in Figure 4.3 as follows: $F_{n_1} = \{1, 2\}$, $F_{n_2} = \{3, 4, 5, 6, 7, 8\}$, $F_{n_3} = \{3, 4, 13\}$, $F_{n_4} = \{5, 6, 7, 8, 9, 10, 11, 12\}$, $F_{n_5} = \{9, 10, 11, 12, 13\}$, $F_{n_6} = \{14, 15\}$, $F_{n_7} = \{16, 17, 18, 19, 20, 21\}$, $F_{n_8} = \{16, 17, 26\}$, $F_{n_9} = \{18, 19, 20, 21, 22, 23, 24, 25\}$, $F_{n_{10}} = \{22, 23, 24, 25, 26\}$, $F_{n_{11}} = \{27, 28\}$, $F_{n_{12}} = \{29, 30, 31, 32, 33, 34\}$, $F_{n_{13}} = \{29, 30, 39\}$, $F_{n_{14}} = \{31, 32, 33, 34, 35, 36, 37, 38\}$, $F_{n_{15}} = \{35, 36, 37, 38, 39\}$, $F_{n_{16}} = \{40, 41\}$, $F_{n_{17}} =$

Table 4.4: Bounds calculated by the extended RLP bounds method based on failure modes.

Bounds	Case 1		Case 2
($\times 10^{-3}$)	k_f=1	k_f=2	k_f=1
$k_m = 1$	2.90–37.88	2.90-37.88	2.90-43.71
$k_m = 2$	2.90-9.55	2.92-9.54	2.90-10.13
$k_m = 3$	2.90-7.27	3.53-7.19	2.90-7.38

$\{42, 43, 44, 45, 46, 47\}$, $F_{n_{18}} = \{42, 43, 52\}$, $F_{n_{19}} = \{44, 45, 46, 47, 48, 49, 50, 51\}$, $F_{n_{20}} = \{48, 49, 50, 51, 52\}$, $F_{n_{21}} = \{53\}$, $F_{n_{22}} = \{54\}$. Because the number of components is large and the relationships among the components are complex in this system, the LP bounds method is not applicable and a multi-scale approach is also difficult to apply.

For the extended RLP bounds method based on failure mode, the failure probability can be obtained from Equation (2.47) by replacing X by A. The failure events of the individual equipment items are formulated as $F_i = \{\ln R_i - \ln a - \ln S_i \leq 0 | A = a\}$, $i = 1, 2, \ldots, 54$, where a is the value of A. Then, $V_i = \ln R_i - \ln a - \ln S_i$, and the failure probabilities information is given by the product of the conditional marginal (PCM) method[18,32)]. Since authors want to obtain an accuracy of approximately 10^{-4}, $a_{\min}$ and $a_{\max}$ are determined to be $0.045g$ and $1.2g$, respectively, so that $P(A \leq a_{\min}) \approx 10^{-2}$ and $P(A \geq a_{\max}) \approx 10^{-6}$. The number N in the Gaussian integration is set equal to 11. In this example, because the number of failure modes is larger than 18, the set of failure modes is divided into 7 subsystems as follows: (F_{n_1}, F_{n_6}, $F_{n_{11}}$, $F_{n_{16}}$, $F_{n_{21}}$), (F_{n_2}, F_{n_3}, F_{n_5}), (F_{n_7}, F_{n_8}, $F_{n_{10}}$), ($F_{n_{12}}$, $F_{n_{13}}$, $F_{n_{15}}$), ($F_{n_{17}}$, $F_{n_{18}}$, $F_{n_{20}}$), (F_{n_4}, F_{n_9}, $F_{n_{14}}$, $F_{n_{19}}$), ($F_{n_{22}}$).

The failure probabilities and the CPU times of the extended RLP bounds method based on failure modes are shown in Tables 4.4 and 4.5, respectively, as Case 1. Using MC simulations with 10^7 simulations, the system failure probability is estimated as 4.57×10^{-3} and the corresponding CPU time is 39.7 s.

Furthermore, suppose that the information for the statistics of component 52 is missing; in this case, the probabilities involving this equipment item are not available.

Table 4.5: CPU times (seconds) of the extended RLP bounds method based on failure modes.

Bounds	Case 1		Case 2
($\times 10^{-3}$)	k_f=1	k_f=2	k_f=1
$k_m = 1$	8.9	8.9	7.7
$k_m = 2$	9.0	84.2	9.9
$k_m = 3$	20.8	630.7	20.8

With the extended RLP bounds method based on failure modes, the relative constraints are removed. The failure probabilities and the CPU times of the extended RLP bounds method based on failure modes are also shown in Tables 4.4 and 4.5, respectively, as Case 2. Note that with incomplete probability information, MC simulations cannot be performed.

From Tables 4.4 and 4.5, one finds that the accuracy of the extended RLP bounds method based on failure modes is acceptable when $k_m \geq 2$ and $k_f \geq 1$, and that narrower bounds can be obtained by increasing k_m and k_f.

4.6 Summary of Extended RLP Bounds Method Based on Failure Modes

As an efficient reliability tool for a general system with a large number of components, the extended RLP bounds method based on failure modes is introduced in this chapter.

4.6.1 Advantages

The main advantage of the extended RLP bounds method based on failure modes over the multi-scale approach is that each component of the system can belong to more than one subsystem in the extended RLP bounds method based on failure modes because the subsystems are based on the failure modes. In the multi-scale approach, any component of the system can belong to only one subsystem because it is not possible to define the system failure event (the union of failure modes) when one component belongs to more than one subsystem. There are many cases in which many components are common in different failure modes. In such cases, the multi-scale approach is difficult or impossible to apply, while the extended RLP bounds method based on failure modes is applicable. In addition, an advantage of the RLP bounds method over the LP bounds method is that a subsystem with a much larger number of components can be handled, while the limitation on the number of components in each subsystem remains the same as that of the RLP bounds method.

Like the LP bounds method, there are also two advantages of the extended RLP bounds method based on failure modes over the MC simulation. One is that the extended RLP bounds method based on failure modes is unaffected by the magnitude of the failure probability, whereas the MC simulation is not. Another advantage is that the extended RLP bounds method based on failure modes is still applicable when the information is not incomplete, whereas MC simulation is not.

4.6.2 Disadvantages

The system failure probability is often determined by the failure probability of some important failure modes or components, and the extended RLP bounds method based on failure modes shows a big advantage if only the important failure modes is considered. However, finding failure modes of the large and complex systems is difficult, and it is a very important research topic.

4.6.3 Summary

As an efficient reliability tool for a general system with a large number of components, the extended RLP bounds method based on failure modes is developed in this chapter. It is based on the information of a few probabilities, and can provide the bounds of the failure probability of the large system especially when the other method, such as the multi-scale system reliability analysis, is not applicable. The extended RLP bounds method based on failure modes provides a result comparable to that of the LP bounds method when the LP bounds method is applicable. It is found to show a big advantage if only important failure modes are considered.

Chapter 5

CONCLUSION

5.1 Summary

The present study proposed the system reliability analysis methods using linear programming based on the individual component failure probabilities and the joint failure probabilities of a small set of components (usually two or three). The original point of this study is to propose new system reliability analysis methods that can estimate the system reliability efficiently and accurately. The important findings and engineering contributions of this research are summarized as follows.

Chapter 1

In this chapter, the background and objective of this research was introduced. The estimation of the system reliability is difficult when the number of components of the system is large or the system is complex. Researchers are trying to find out the bounds on the system failure probability based on the information of the individual component failure probabilities and the joint failure probabilities of a small set of components. There are no theoretical bounds for the general systems. The challenges related to computation burdens in the system reliability analysis constitutes the focus of the present research.

Chapter 2

This chapter reviewed the past research concerning the existing formulation of bounds on system reliability, the LP bounds method, and the multi-scale system reliability analysis. The LP bounds method was proposed as a general method to estimate the system failure probability. It can provide the most narrowest possible bounds based on the information of the individual component failure probabilities and the joint failure probabilities of components. However, the drawback of the LP bounds method is that the size of LP problem grows rapidly with the number of components. This is the hindrance in the application of the LP bounds method to a system with large number of components. In theory, the limitation of the number of components in the LP bounds method is 18. In order to solve the above drawback, the multi-scale system reliability analysis has been proposed, The multi-scale system reliability analysis decomposed the entire system into smaller size of the subsystems, then the large size of LP problem can be obtained by solving a number of smaller size of LP problem. However, the selection of subsystem in the multi-scale system reliability analysis could difficult or impossible in some systems. Also, the limitation of the number of components is the same with that of the LP bounds method in each subsystem. Therefore, the more efficient reliability methods are required.

Chapter 3

In this chapter, as an efficient reliability tool for a system with a large number of components, the RLP bounds method has been introduced. The RLP bounds method is still based on the information that the individual components failure probabilities and the joint failure probabilities of components, and the information both including equalities and inequalities can be used. Like the LP bounds method, there are also two advantages of the RLP bounds method over the MC simulation. One is that the RLP bounds method is unaffected by the magnitude of the failure probability, whereas the MC simulation is not. Another advantage is that the RLP bounds method is still applicable when the information is not incomplete, whereas MC simulation is not. The most important contribution of the RLP bounds method is that the size of LP problem can be well solved, e.g., the number of design variables can be decreased from 2^n to

$n^2 - n + 2$ by employing the UGF, and the number of constraints can also be reduced by using the decreasing strategy. The RLP bounds method can provide the result comparable to that of the LP bounds method. The main drawback of the RLP bounds method is only applicable to a pure series system as well as a pure parallel system.

Chapter 4

In order to extend the applicability of the RLP bounds method, the extended RLP bounds method based on failure modes has been proposed in this chapter. It is applicable to a general system insisting of series subsystems and parallel subsystems. The main advantage over the multi-scale approach is that each component of the system can belong to more than one subsystem in the proposed approach because the subsystems are based on the failure modes. In the multi-scale approach, because any component of the system can belong to only one subsystem, it is not possible to define the system failure event (the union of failure modes) when one component belongs to more than one subsystem. In some cases, the multi-scale system reliability analysis approach is difficult or impossible to apply, while the extended RLP bounds method based on failure modes is applicable.

5.2 Expectation

System Reliability Analysis with Multi-state Components

1) Reliability Analysis with Brittle Components

Because a component is usually considered as a ductile component in engineering system, the brittle state can be considered as a special state of multi-state component. After one component fails, the load transfers to or redistributes among the remaining components, thus changing and, in general, increasing the demand on the remaining components, this behavior and process of the whole system is very complex.

The ideal structure from the safety point of view is a statically indeterminate structure with nothing but ductile components. The designer will in general strive to design his structure in that way. Yet even then the possibility that one of the components will display brittle behavior must be taken into account. The brittle behaviors lie in imperfections of construction, in damage that occurs during the service life of the structure, or in fatigue phenomena. Among a structure system, one of its subsystems may collapse when it breaks abruptly, moreover, the whole system will be disordered and it can collapse consequently. However, the estimation of the reliability of system with brittle components is very difficult.

Because the UGF technique can formulate the entire system states based on the states of its components by using algebraic procedures as describe in this research, authors have some clues to estimate the reliability of the system with brittle components[63)]. Consequently, finding an appreciate approach by using the UGF technique and linear programming to estimate the reliability of the system with brittle components would be an interesting research topic in the future.

2) Reliability Analysis with multi-state Components

In real world problems, a realistic model of the engineering systems usually needs to consider a lot of system states, and the idea of multi-state systems was first introduced by Hirsch, Meisner and Boll[73)]. Equation (2.31) can be considered as the formulation of the state of the multi-state system. Increasingly high requirements for accurate reliability evaluation of multi-state system make the calculation of system reliability is extremely difficult or impossible by use of the classical binary reliability theory. Therefore, the theory and methods of multi-state system reliability is highly needed. In the last decades, more and more people joined the research of the multi-state system reliability[74–79]. The early advances in multi-state system reliability theory and methods were summarized by Lisnianski and Levitin[15].

In reality, any realistic system would be the multi-state system, then the evaluation of the reliability of multi-state systems would be very important. However, the multi-state system reliability analysis would be very complex. Since the UGF technique have been used to define the states of the multi-state system and to estimate the reliability[60], the computational burden is the crucial factor when one solves reliability analysis. Since the methods proposed in this research have proved that the bounds on the system reliability can be estimate efficiently and accurately by using the UGF and the LP, it would be important research area to extend the application of methods by use of the UGF and the LP.

APPENDIX A

UNIVERSAL GENERATING FUNCTION AND ITS PROPERTIES

A.1 Overview

In this appendix, the concept of universal generating function (UGF) is reviewed and its properties is specifically described.

In section A.2, the concepts of the generating function, the moment generating function, and the universal generating function are described.

In section A.3, the important properties of the universal generating function are depicted.

A.2 Concept of Universal Generating Function

A.2.1 Concept of Generating Function

Generating functions[54)] are one of the most surprising and useful inventions in discrete mathematics, and it become a bridge between discrete mathematics and continuous analysis by transforming problems about sequences into problems about functions. This mathematical machinery can be apply to problems about sequences, for example, all sorts of counting problems can be solved by use of generating functions.

The concept of generating functions can be considered as a powerful tool and technique for solving discrete problems. The general idea of generating function is as follows. In counting problems, an infinite sequence a_n of real numbers can be expressed as

$$a_0, \quad a_1, \quad a_2, \quad \dots \tag{A.1}$$

where a_n represents different values for different n. A formal power series $G(x)$ can be defined as

$$\begin{aligned} G(x) &= a_0 + a_1 x + a_2 x^2 + \dots \\ &= \sum_{n}^{\infty} a_n x^n \end{aligned} \tag{A.2}$$

The above $G(x)$ is the generating function for the infinite sequence a_0, a_1, a_2, ... This $G(x)$ is also called ordinary generating function[60)].

Let us take a simple example. Consider a sequence as

$$1, \quad 1, \quad 1,, \quad 1, \quad 1, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0, \quad \dots \tag{A.3}$$

The corresponding generating function is

$$\begin{aligned} G(x) &= 1 + 1x + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 0x^6 + 0x^7 + \dots \\ &= 1 + x + x^2 + x^3 + x^4 + x^5 \end{aligned} \tag{A.4}$$

For the sum of the above geometric series, we can write

$$\begin{aligned} G(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 \\ &= \frac{1 - x^6}{1 - x} \end{aligned} \tag{A.5}$$

A.2.2 Concept of Moment Generating Function

In probability theory and statistics, the moment generating function of a discrete random variable is an alternative specification of its probability distribution. The moment generating function[60)] associated with a discrete random variable X is a function $m(t)$ defined by

$$\begin{aligned} m(t) &= E(e^{tX}) \\ &= \sum_{i=0}^{k} e^{tx_i} p_i \end{aligned} \tag{A.6}$$

where the vector $x = (x_0, x_1, \ldots, x_k)$ consisting of the possible values of X and the vector $p = (p_0, p_1, \ldots, p_k)$ consisting of the corresponding probabilities $p_i = P\{X = x_i\}$ represent the probabilistic distribution of variable X.

The expected values $E(X)$, $E(X^2)$, $E(X^3)$, $\ldots$, and $E(X^r)$ are called moments. Sometimes the moments are difficult to find. However, all of the moments of discrete random variable X can be obtained by successively differentiating $m(t)$. For this reason, the function $m(t)$ is called the moment generating function. For example, the first derivative of $m(t)$ can be expressed as

$$\begin{aligned} m'(t) &= \frac{d}{dt}(\sum_{i=0}^{k} e^{tx_i} p_i) \\ &= \sum_{i=0}^{k} x_i e^{tx_i} p_i \end{aligned} \tag{A.7}$$

Then, the first moment can be obtained as

$$\begin{aligned} m'(0) &= \sum_{i=0}^{k} x_i p_i \\ &= E(X) \end{aligned} \tag{A.8}$$

The second derivative of $m(t)$ can be expressed as

$$\begin{aligned} m''(t) &= \frac{d}{dt}(m'(t)) \\ &= \frac{d}{dt}(\sum_{i=0}^{k} x_i e^{tx_i} p_i) \\ &= \sum_{i=0}^{k} {x_i}^2 e^{tx_i} p_i \end{aligned} \tag{A.9}$$

Then, the second moment can be obtained as

$$\begin{aligned} m''(0) &= \sum_{i=0}^{k} {x_i}^2 p_i \\ &= E(X^2) \end{aligned} \tag{A.10}$$

The nth derivative of $m(t)$ is equal to $E(X^n)$ at $t = 0$.

The one of most important property of the moment generating function is that the moment generating function of the sum of the independent discrete random variables can be expressed as the product of the individual moment generating functions of the these variables. For example, consider two independent discrete random variables X and Y, the moment generating function of X and Y can be expressed as

$$\begin{aligned} m_X(t) &= E(e^{tX}) \\ &= \sum_{i=0}^{k_x} e^{tx_i} p_{x_i} \end{aligned} \tag{A.11}$$

where the vector $x = (x_0, x_1, \ldots, x_{k_x})$ consisting of the possible values of X and the vector $p = (p_0, p_1, \ldots, p_{k_x})$ consisting of the corresponding probabilities $p_{x_i} = P\{X = x_i\}$ represent the probabilistic distribution of variable X.

$$\begin{aligned} m_Y(t) &= E(e^{tY}) \\ &= \sum_{j=0}^{k_y} e^{ty_j} p_{y_j} \end{aligned} \tag{A.12}$$

where the vector $y = (y_0, y_1, \ldots, y_{k_y})$ consisting of the possible values of Y and the vector $p = (p_0, p_1, \ldots, p_{k_y})$ consisting of the corresponding probabilities $p_{y_j} = P\{Y = y_j\}$ represent the probabilistic distribution of variable Y. Then, the moment generating function of $X + Y$, $m_{X+Y}(t)$, can be expressed as

$$\begin{aligned} m_{X+Y}(t) &= m_X(t) m_Y(t) \\ &= \sum_{i=0}^{k_x} e^{tx_i} p_{x_i} \sum_{j=0}^{k_y} e^{ty_j} p_{y_j} \\ &= \sum_{i=0}^{k_x} \sum_{j=0}^{k_y} e^{tx_i} e^{ty_j} p_{x_i} p_{y_j} \\ &= \sum_{i=0}^{k_x} \sum_{j=0}^{k_y} e^{t(x_i+y_j)} p_{x_i} p_{y_j} \end{aligned} \tag{A.13}$$

Generally, the moment generating function of the sum of the independent discrete random variables, X_1, X_2, ..., X_n, can be expressed as

$$m_{\sum_{i=1}^{n}}X_i(t) = \prod_{i=1}^{n} m_{X_i}(t) \tag{A.14}$$

A.2.3 z-transform

The z-transform[60)], which can be considered as the discrete-time counterpart of the Laplace transform, is an essential mathematical tool for the design, analysis and monitoring of systems. The general idea of the z-transform is as follows. Consider a simpler expression of Equation (A.6) by replacing the function e^t by the variable z. A function corresponding to discrete random variable X and its probability mass function (pmf) can be obtained as

$$\begin{aligned} \tau(z) &= E(z^X) \\ &= \sum_{i=0}^{k} z^{x_i} p_i \end{aligned} \tag{A.15}$$

The above function is called the z-transform of discrete random variable X.

Some basic properties of the moment generating function of discrete random variable X are similar to that of the z-transform of discrete random variable X. For example, the first derivative of $\tau(z)$ can be expressed as

$$\begin{aligned} \tau'(z) &= \frac{d}{dt}(\sum_{i=0}^{k} z^{x_i} p_i) \\ &= \sum_{i=0}^{k} x_i z^{x_i - 1} p_i \end{aligned} \tag{A.16}$$

Then, the first moment can be obtained as

$$\begin{aligned} \tau'(1) &= \sum_{i=0}^{k} x_i p_i \\ &= E(X) \end{aligned} \tag{A.17}$$

From the above Equation, one can easily find that the first moment of discrete random variable X is equal to the first derivative of $\tau(z)$ at $z = 1$.

Also, the z-transform preserves the most important property of the moment generating function, i.e., the z-transform of the sum of the independent discrete random variables can be expressed as the product of the individual z-transform of the these variables. For example, consider two independent discrete random variables X and Y, the z-transform of X and Y can be expressed as

$$\begin{aligned} \tau_X(z) &= E(z^X) \\ &= \sum_{i=0}^{k_x} z^{x_i} p_{x_i} \end{aligned} \tag{A.18}$$

and

$$\begin{aligned} \tau_Y(z) &= E(z^Y) \\ &= \sum_{j=0}^{k_y} z^{y_j} p_{y_j} \end{aligned} \tag{A.19}$$

Then, the moment generating function of $X+Y$, $\tau_{X+Y}(t)$, can be expressed as

$$\begin{aligned} \tau_{X+Y}(z) &= \tau_X(z)\tau_Y(z) \\ &= \sum_{i=0}^{k_x} z^{x_i} p_{x_i} \sum_{j=0}^{k_y} z^{y_j} p_{y_j} \\ &= \sum_{i=0}^{k_x} \sum_{j=0}^{k_y} z^{x_i} z^{y_j} p_{x_i} p_{y_j} \\ &= \sum_{i=0}^{k_x} \sum_{j=0}^{k_y} z^{(x_i+y_j)} p_{x_i} p_{y_j} \end{aligned} \tag{A.20}$$

Generally, the z-transform of the sum of the independent discrete random variables, X_1, X_2, $\ldots$, X_n, can be expressed as

$$\tau_{\sum_{i=1}^{n}} X_i(z) = \prod_{i=1}^{n} m_{X_i}(z) \tag{A.21}$$

The more details of z-transform can be found at the book by Grimmett and Stirzaker[61)] and Ross[62)].

A.2.4 Definition of Universal Generating Function

Consider n independent discrete random variables X_1, X_2, $\ldots$, X_n and assume that each variable X_i has a pmf represented by the vectors $x_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,k_i})$, $p_i = (p_{i,0}, p_{i,1}, \ldots, p_{i,k_i})$. Let $f(X_1, X_2, \ldots, X_n)$ denotes the pmf of an arbitrary function of

X_1, X_2, ..., X_n. All of the combinations are mutually exclusive, and the total number of combinations of the possible values of the function $f(X_1, X_2, \ldots, X_n)$ is

$$N = \prod_{i=1}^{n}(k_i + 1) \tag{A.22}$$

where $k_i + 1$ is the number of possible values of X_i. The probability of the jth possible value of X_1, X_2, ..., X_n can be expressed as

$$q_j = \prod_{i=1}^{n} p_{i,j_i} \tag{A.23}$$

By introduce a general composition operator $\underset{f}{\otimes}$ over z-transform, the z-transform of the arbitrary function $f(X_1, X_2, \ldots, X_n)$ can be obtained

$$\underset{f}{\otimes}(\sum_{j_i=0}^{k_i} p_{ij_i} z^{x_{ij_i}}) = \sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k_2}\cdots\sum_{j_n=0}^{k_n}(\prod_{i=0}^{n} p_{i,j_i} z^{f(x_{i,j_1},\ldots,x_{n,j_n})}) \tag{A.24}$$

By replacing the z-transform of random variable X_i by the $u_i(z)$, the above Equation can be expressed as

$$U(z) = \underset{f}{\otimes}(u_1(z), u_2(z), \ldots, u_n(z)) \tag{A.25}$$

The above technique using z-transform and composition operators $\underset{f}{\otimes}$ is called the universal z-transform or universal (moment) generating function (UGF) technique, and $U(z)$ represents the UGF of X_1, X_2, ..., X_n.

A.3 Properties of Universal Generating Function

The main interested property of UGF is the properties of composition operator $\underset{f}{\otimes}$, and the properties of the function $f(X_1, X_2, \ldots, X_n)$ decide the properties of composition operator $\underset{f}{\otimes}$. There are four main properties[60)]:

- Consecutive property:

If

$$f(X_1, X_2, \ldots, X_n) = f(f(X_1, X_2, \ldots, X_{n-1}), X_n) \tag{A.26}$$

then the corresponding UGF can be expressed as

$$\begin{aligned} U(z) &= \underset{f}{\otimes}(u_1(z), u_2(z), \ldots, u_n(z)) \\ &= \underset{f}{\otimes}(\underset{f}{\otimes}(u_1(z), u_2(z), \ldots, u_{n-1}(z)), u_n(z)) \end{aligned} \tag{A.27}$$

By assuming $U_j(z) = \underset{f}{\otimes}(u_1(z), u_2(z), \ldots, u_j(z))$, the above Equation can be expressed as

$$\begin{aligned} U_j(z) &= \underset{f}{\otimes}(U_{j-1}(z), u_j(z)) \\ 2 &\leq j \leq n \end{aligned} \tag{A.28}$$

- Associative property:

If

$$\begin{aligned} & f(X_1, X_2, \ldots, X_j, X_{j+1}, \ldots, X_n) \\ = \; & f(f(X_1, X_2, \ldots, X_j), f(X_{j+1}, \ldots, X_n)) \end{aligned} \tag{A.29}$$

then the corresponding UGF can be expressed as

$$\begin{aligned} & \underset{f}{\otimes}(u_1(z), u_2(z), \ldots, u_n(z)) \\ = \; & \underset{f}{\otimes}(\underset{f}{\otimes}(u_1(z), u_2(z), \ldots, u_{j-1}(z)), \underset{f}{\otimes}(u_j(z), \ldots, u_n(z))) \end{aligned} \tag{A.30}$$

- Commutative property:

If

$$\begin{aligned} & f(X_1, X_2, \ldots, X_j, X_{j+1}, \ldots, X_n) \\ = \; & f(f(X_1, X_2, \ldots, X_{j+1}, X_j, \ldots, X_n) \end{aligned} \tag{A.31}$$

then the corresponding UGF can be expressed as

$$\begin{aligned} & \underset{f}{\otimes}(u_1(z), u_2(z), \ldots, u_j(z), u_{j+1}(z), \ldots, u_n(z)) \\ = \; & \underset{f}{\otimes}(u_1(z), u_2(z), \ldots, u_{j+1}(z), u_j(z), \ldots, u_n(z)) \end{aligned} \tag{A.32}$$

- Recursive property:

If a function takes the recursive form

$$f(f_1(X_1, \ldots, X_j), f_2(X_{j+1}, \ldots, X_h), \ldots, f_m(X_l, \ldots, X_n)) \tag{A.33}$$

then the corresponding UGF can be expressed as

$$\underset{f}{\otimes}(\underset{f_1}{\otimes}(u_1(z), \ldots, u_j(z)), \underset{f_2}{\otimes}(u_{j+1}(z), \ldots, u_h(z)), \ldots, \underset{f_m}{\otimes}(u_l(z), \ldots, u_n(z))) \tag{A.34}$$

Let us take a simple example, consider the variables X_1, X_2, X_3 with pmf x_1 = (3, 6, 9), p_1 =(0.4, 0.3, 0.3), x_2 = (6, 10), p_2 =(0.6, 0.4), and x_3 = (0, 1), p_3 =(0.5, 0.5), respectively. The UGF of X_1, X_2, X_3 can be expressed as

$$\begin{aligned} u_1(z) &= 0.4z^3 + 0.3z^6 + 0.3z^9 \\ u_2(z) &= 0.6z^6 + 0.4z^{10} \\ u_3(z) &= 0.5z^0 + 0.5z^1 \end{aligned}$$

The function $Y = \min(X_1, X_2, X_3)$ having both commutative and associative properties can be expressed as

$$\min(X_1, X_2, X_3) = \min(\min(X_1, X_2), X_3) \tag{A.35}$$

then,

$$\begin{aligned} u_4(z) &= u_1(z) \underset{\min}{\otimes} u_1(z) \\ &= (0.4z^3 + 0.3z^6 + 0.3z^9) \underset{\min}{\otimes} (0.6z^6 + 0.4z^{10}) \\ &= 0.24z^{\min(3,6)} + 0.18z^{\min(6,6)} + 0.18z^{\min(9,6)} + \\ &\quad 0.16z^{\min(3,10)} + 0.12z^{\min(6,10)} + 0.12z^{\min(9,10)} \\ &= 0.40z^3 + 0.48z^6 + 0.12z^9 \end{aligned} \tag{A.36}$$

$$\begin{aligned} U_Y(z) &= u_4(z) \underset{\min}{\otimes} u_3(z) \\ &= (0.40z^3 + 0.48z^6 + 0.12z^9) \underset{\min}{\otimes} (0.5z^0 + 0.5z^1) \\ &= 0.20z^{\min(3,0)} + 0.24z^{\min(6,0)} + 0.06z^{\min(9,0)} + \\ &\quad 0.20z^{\min(3,1)} + 0.24z^{\min(6,1)} + 0.06z^{\min(9,1)} \\ &= 0.50z^0 + 0.50z^1 \end{aligned} \tag{A.37}$$

The function $Y = \min(X_1, X_2, X_3)$ can also be expressed as

$$\min(X_1, X_2, X_3) = \min(\min(X_1, X_3), X_2) \tag{A.38}$$

then,

$$\begin{aligned}
u_4(z) &= u_1(z) \underset{\min}{\otimes} u_3(z) \\
&= (0.4z^3 + 0.3z^6 + 0.3z^9) \underset{\min}{\otimes} (0.5z^0 + 0.5z^1) \\
&= 0.20z^{\min(3,0)} + 0.15z^{\min(6,0)} + 0.15z^{\min(9,0)} + \\
&\quad 0.20z^{\min(3,1)} + 0.15z^{\min(6,1)} + 0.15z^{\min(9,1)} \\
&= 0.50z^0 + 0.50z^1
\end{aligned} \tag{A.39}$$

$$\begin{aligned}
U_Y(z) &= u_4(z) \underset{\min}{\otimes} u_2(z) \\
&= (0.5z^0 + 0.5z^1) \underset{\min}{\otimes} (0.6z^6 + 0.4z^{10}) \\
&= 0.30z^{\min(0,6)} + 0.30z^{\min(1,6)} \\
&\quad 0.20z^{\min(0,10)} + 0.20z^{\min(1,10)} \\
&= 0.50z^0 + 0.50z^1
\end{aligned} \tag{A.40}$$

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ACKNOWLEDGEMENTS

I would like to express my deep appreciation to everyone who has made this thesis possible. The special thank goes to my supervisor Professor Yasuhiro Mori, Department of Environmental Engineering and Architecture, Graduate School of Environmental Studies, Nagoya University, for his insightful guidance, passionate advice, constant support, effective teaching, and constant encouragement. He has been my perfect role model as a researcher, teacher, advisor, and mentor. He has been my inspiration as I hurdle all the obstacles in the completion of my doctoral thesis.

Thanks to Professor Hiroshi Ohmori, Professor Hiroshi Tagawa, Professor Fuminobu Ozaki, Professor Hideki Itoda, for their valuable opinions and comments on my research work. To my predecessor, Doctor Hejin Gao, Master Keiko Shirai, Master Kota Ibuki, Master Shinya Hirukawa, Master Xin Lu, Master Naoyuki Ogura, Master Yuki Kawasaki, Master Sayo Kojima, Master Mamiko Nakao, for their kind help. Thanks also go to all the rest members in our laboratory who helped me from time to time during my study in Japan. I sincerely thank my friends in Japan who gave me supports and help while I am living in Japan.

Finally, I would like to express my great thanks to my beloved family for their unselfish understanding and assistance.

Yi Chang

January 2014