

Multiple zeta values and zeta-functions of root systems

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Abstract: We propose the viewpoint that the r -ple zeta-function of Euler-Zagier type can be regarded as a specialization of the zeta-function associated with the root system of C_r type. From this viewpoint, we can see that Zagier's well-known formula for multiple zeta values (MZVs) coincides with Witten's volume formula associated with a sub-root system of C_r type. Based on this observation, we generalize Zagier's formula and also give analogous results which correspond to a sub-root system of B_r type. We announce those results as well as some relevant results for partial multiple zeta values.

Key words: Multiple zeta-values; root systems; witten zeta-functions.

1. Zeta-functions of root systems. The aim of this article is to announce our theory based on the observation that the Euler-Zagier r -ple sum

$$\zeta_r(s_1, \dots, s_r) = \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{s_1} m_2^{s_2} \dots m_r^{s_r}}$$

(where s_1, \dots, s_r are complex variables; see Hoffman [3], Zagier [20]) can be regarded as a specialization of the zeta-function of the root system of C_r type. The details will appear elsewhere.

First we prepare notations. For the details of basic facts about root systems and Weyl groups, see [2,4,5].

Let V be an r -dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. The dual space V^* is identified with V via the inner product of V . Let Δ be a finite irreducible reduced root system, and $\Psi = \{\alpha_1, \dots, \alpha_r\}$ its fundamental system. We fix Δ_+ and Δ_- as the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta = \Delta_+ \amalg \Delta_-$. Let $Q = Q(\Delta)$ be the root lattice, Q^\vee the coroot lattice, $P = P(\Delta)$ the weight lattice, P^\vee the coweight lattice, and P_{++} the set of integral strongly dominant weights respectively defined by

$$\begin{aligned} Q &= \bigoplus_{i=1}^r \mathbf{Z} \alpha_i, & Q^\vee &= \bigoplus_{i=1}^r \mathbf{Z} \alpha_i^\vee, \\ P &= \bigoplus_{i=1}^r \mathbf{Z} \lambda_i, & P^\vee &= \bigoplus_{i=1}^r \mathbf{Z} \lambda_i^\vee, \\ P_{++} &= \bigoplus_{i=1}^r \mathbf{N} \lambda_i, \end{aligned}$$

where the fundamental weights $\{\lambda_j\}_{j=1}^r$ and the fundamental coweights $\{\lambda_j^\vee\}_{j=1}^r$ are the dual bases of Ψ^\vee and Ψ satisfying $\langle \alpha_i^\vee, \lambda_j \rangle = \delta_{ij}$ (Kronecker's delta) and $\langle \lambda_i^\vee, \alpha_j \rangle = \delta_{ij}$ respectively.

Let $\sigma_\alpha : V \rightarrow V$ be the reflection with respect to a root $\alpha \in \Delta$ defined by

$$\sigma_\alpha : v \mapsto v - \langle \alpha^\vee, v \rangle \alpha.$$

For a subset $A \subset \Delta$, let $W(A)$ be the group generated by reflections σ_α for all $\alpha \in A$. In particular, $W = W(\Delta)$ is the Weyl group, and $\{\sigma_j := \sigma_{\alpha_j} \mid 1 \leq j \leq r\}$ generates W . For $w \in W$, denote $\Delta_w = \Delta_+ \cap w^{-1} \Delta_-$. The zeta-function associated with Δ is defined by

$$\zeta_r(\mathbf{s}, \mathbf{y}; \Delta) = \sum_{\lambda \in P_{++}} e^{2\pi i \langle \mathbf{y}, \lambda \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}},$$

where $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbf{C}^{|\Delta_+|}$ and $\mathbf{y} \in V$ (for the details, see [7–15]). This can be regarded as a multi-variable version of Witten zeta-functions formulated by Zagier [20] based on the work of Witten [18].

2. Fundamental formulas. In this section, we state several fundamental formulas which are certain extensions of our previous results given in [9,10,15].

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Let \mathcal{V} be the set of all bases $\mathbf{V} \subset \Delta_+$. Let $\mathbf{V}^* = \{\mu_\beta^{\mathbf{V}}\}_{\beta \in \mathbf{V}}$ be the dual basis of $\mathbf{V}^\vee = \{\beta^\vee\}_{\beta \in \mathbf{V}}$. Let $L(\mathbf{V}^\vee) = \bigoplus_{\beta \in \mathbf{V}} \mathbf{Z}\beta^\vee$. Then we have $|Q^\vee/L(\mathbf{V}^\vee)| < \infty$. Fix $\phi \in V$ such that $\langle \phi, \mu_\beta^{\mathbf{V}} \rangle \neq 0$ for all $\mathbf{V} \in \mathcal{V}$ and $\beta \in \mathbf{V}$. If the root system Δ is of A_1 type, then we choose $\phi = \alpha_1^\vee$. We define a multiple generalization of the fractional part as

$$\{\mathbf{y}\}_{\mathbf{V}, \beta} = \begin{cases} \{\langle \mathbf{y}, \mu_\beta^{\mathbf{V}} \rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}} \rangle > 0), \\ 1 - \{-\langle \mathbf{y}, \mu_\beta^{\mathbf{V}} \rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}} \rangle < 0). \end{cases}$$

Let $\mathbf{T} = \{t \in \mathbf{C} \mid |t| < 2\pi\}^{|\Delta_+|}$.

Definition 2.1. For $\mathbf{t} = (t_\alpha)_{\alpha \in \Delta_+} \in \mathbf{T}$ and $\mathbf{y} \in V$, we define

$$\begin{aligned} F(\mathbf{t}, \mathbf{y}; \Delta) &= \sum_{\mathbf{V} \in \mathcal{V}} \left(\prod_{\gamma \in \Delta_+ \setminus \mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta \in \mathbf{V}} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}} \rangle} \right) \\ &\times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \\ &\times \sum_{q \in Q^\vee/L(\mathbf{V}^\vee)} \left(\prod_{\beta \in \mathbf{V}} \frac{t_\beta \exp(t_\beta \{\mathbf{y} + q\}_{\mathbf{V}, \beta})}{e^{t_\beta} - 1} \right), \end{aligned}$$

which is independent of choice of ϕ .

Remark 2.2. In [10], $F(\mathbf{t}, \mathbf{y}; \Delta)$ is defined in a different way. The above is [10, Theorem 4.1].

For $\mathbf{v} \in V$, and a differentiable function f on V , let

$$(\partial_{\mathbf{v}} f)(\mathbf{y}) = \lim_{h \rightarrow 0} \frac{f(\mathbf{y} + h\mathbf{v}) - f(\mathbf{y})}{h}$$

and for $\alpha \in \Delta_+$,

$$\mathfrak{D}_\alpha = \frac{\partial}{\partial t_\alpha} \Big|_{t_\alpha=0} \partial_{\alpha^\vee}.$$

Let $A = \{\nu_1, \dots, \nu_N\} \subset \Delta_+$, and define

$$\mathfrak{D}_A = \mathfrak{D}_{\nu_N} \cdots \mathfrak{D}_{\nu_1}.$$

Further, let $A_j = \{\nu_1, \dots, \nu_j\}$ ($1 \leq j \leq N-1$), $A_0 = \emptyset$, and

$$\mathcal{V}_A = \{\mathbf{V} \in \mathcal{V} \mid \nu_{j+1} \notin \langle \mathbf{V} \cap A_j \rangle \mid (0 \leq j \leq N-1)\},$$

where $\langle \cdot \rangle$ denotes the linear span.

Theorem 2.3. For $A = \{\nu_1, \dots, \nu_N\} \subset \Delta_+$ and $\mathbf{t}_{\Delta_+ \setminus A} = \{t_\alpha\}_{\alpha \in \Delta_+ \setminus A}$, we have

$$\begin{aligned} (\mathfrak{D}_A F)(\mathbf{t}_{\Delta_+ \setminus A}, \mathbf{y}; \Delta) &= \sum_{\mathbf{V} \in \mathcal{V}_A} (-1)^{|A \setminus \mathbf{V}|} \\ &\times \left(\prod_{\gamma \in \Delta_+ \setminus (\mathbf{V} \cup A)} \frac{t_\gamma}{t_\gamma - \sum_{\beta \in \mathbf{V} \setminus A} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}} \rangle} \right) \end{aligned}$$

$$\begin{aligned} &\times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \\ &\times \sum_{q \in Q^\vee/L(\mathbf{V}^\vee)} \left(\prod_{\beta \in \mathbf{V} \setminus A} \frac{t_\beta \exp(t_\beta \{\mathbf{y} + q\}_{\mathbf{V}, \beta})}{e^{t_\beta} - 1} \right), \end{aligned}$$

which is independent of choice of the order of A . This function is holomorphic with respect to $\mathbf{t}_{\Delta_+ \setminus A}$ around the origin.

Definition 2.4. For $A = \{\nu_1, \dots, \nu_N\} \subset \Delta_+$ and $\mathbf{t}_{\Delta_+ \setminus A} = \{t_\alpha\}_{\alpha \in \Delta_+ \setminus A}$, we define $\mathcal{P}_{\Delta_+ \setminus A}(\mathbf{k}_{\Delta_+ \setminus A}, \mathbf{y}; \Delta)$ by

$$\begin{aligned} &(\mathfrak{D}_A F)(\mathbf{t}_{\Delta_+ \setminus A}, \mathbf{y}; \Delta) \\ &= \sum_{\mathbf{k}_{\Delta_+ \setminus A} \in \mathbf{Z}_{\geq 0}^{|\Delta_+ \setminus A|}} \mathcal{P}_{\Delta_+ \setminus A}(\mathbf{k}_{\Delta_+ \setminus A}, \mathbf{y}; \Delta) \prod_{\alpha \in \Delta_+ \setminus A} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}. \end{aligned}$$

Theorem 2.5. For $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha \in \Delta_+}$ with $k_\alpha \in \mathbf{Z}_{\geq 1}$ ($\alpha \in \Delta_+ \setminus A$), $k_\alpha = 0$ ($\alpha \in A$), we have

$$\begin{aligned} (2.1) \quad &\sum_{w \in W} \left(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{k_\alpha} \right) \zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y}; \Delta) \\ &= (-1)^{|\Delta_+|} \mathcal{P}_{\Delta_+ \setminus A}(\mathbf{k}_{\Delta_+ \setminus A}, \mathbf{y}; \Delta) \\ &\times \left(\prod_{\alpha \in \Delta_+ \setminus A} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \right) \end{aligned}$$

provided all the series on the left-hand side absolutely converge.

Assume that Δ is not simply-laced. Then we have the disjoint union $\Delta = \Delta_l \cup \Delta_s$, where Δ_l is the set of all long roots and Δ_s is the set of all short roots. By applying Theorem 2.5 to $A = \Delta_l$ or Δ_s , we obtain the following theorem immediately, which is a generalization of the explicit volume formula proved in [15, Theorem 4.6].

Theorem 2.6. Let $\Delta_1 = \Delta_l$ (resp. Δ_s), $\Delta_2 = \Delta_s$ (resp. Δ_l), and $\Delta_{j+} = \Delta_j \cap \Delta_+$ ($j = 1, 2$). For $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha \in \Delta_+}$ with $k_\alpha = k \in 2\mathbf{Z}_{\geq 1}$ ($\alpha \in \Delta_{1+}$), $k_\alpha = 0$ ($\alpha \in \Delta_{2+}$), and $\nu \in P^\vee/Q^\vee$, we have

$$\begin{aligned} &\zeta_r(\mathbf{k}, \nu; \Delta) \\ &= \frac{(-1)^{|\Delta_+|}}{|W|} \mathcal{P}_{\Delta_{1+}}(\mathbf{k}_{\Delta_{1+}}, \nu; \Delta) \left(\prod_{\alpha \in \Delta_{1+}} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \right). \end{aligned}$$

3. Multiple zeta values. Special values of Euler-Zagier sums when s_1, \dots, s_r are positive integers are usually called multiple zeta values (MZVs), and have been studied extensively. In the study of MZVs, the main target is to give non-trivial relations among them, in order to investigate

the structure of the algebra generated by them (for the details, see Kaneko [6]). Here, we study MZVs from the viewpoint of zeta-functions of root systems. In our previous paper [14], we regarded MZVs as special values of zeta-functions of A_r type, and clarified the structure of the shuffle product procedure for MZVs. In this article, we regard MZVs as special values of zeta-functions of C_r type.

In the root system of C_r type, for $\Delta = \Delta(C_r)$, we have the disjoint union $\Delta_+^\vee = (\Delta_{l+})^\vee \cup (\Delta_{s+})^\vee$, where $\Delta_{l+} = \Delta_l \cap \Delta_+$, $\Delta_{s+} = \Delta_s \cap \Delta_+$, and

$$(\Delta_{l+})^\vee = \{\alpha_r^\vee, \alpha_{r-1}^\vee + \alpha_r^\vee, \dots, \alpha_2^\vee + \dots + \alpha_r^\vee, \alpha_1^\vee + \dots + \alpha_r^\vee\}.$$

Therefore by setting $s_\alpha = 0$ for $\alpha \in \Delta_{s+}$, we have

$$\zeta_r(\mathbf{s}, \mathbf{0}; \Delta) = \sum_{m_1, \dots, m_r=1}^{\infty} \prod_{i=1}^r \frac{1}{(\sum_{j=r-i+1}^r m_j + m_r)^{s_i}},$$

which is exactly the Euler-Zagier sum $\zeta_r(s_1, \dots, s_r)$. It is to be noted that some authors use the opposite order of summation in the definition of $\zeta_r(s_1, \dots, s_r)$.

Corollary 3.1. *Let $\Delta = \Delta(C_r)$ and $2\mathbf{k}_{\Delta_{l+}} = (2k, \dots, 2k) \in \mathbf{N}^r$ for any $k \in \mathbf{N}$,*

$$\begin{aligned} & \zeta_r(2k, 2k, \dots, 2k) \\ &= \frac{(-1)^r}{2^r r!} \mathcal{P}_{\Delta_{l+}}(2\mathbf{k}_{\Delta_{l+}}, \mathbf{0}; \Delta) \frac{(2\pi i)^{2kr}}{\{(2k)!\}^r} \in \mathbf{Q} \cdot \pi^{2kr}. \end{aligned}$$

Remark 3.2. The fact that $\zeta_r(2k, \dots, 2k) \in \mathbf{Q} \cdot \pi^{2kr}$ was first proved by Zagier [20]. We emphasize that the above formula can be regarded as a kind of Witten's volume formula.

Let $\Delta = \Delta(C_2)$ be the root system of C_2 type. By Theorem 2.3, we have

$$\begin{aligned} & (\mathfrak{D}_{\Delta_{s+}} F)(t_1, t_2, y_1, y_2; \Delta) \\ &= 1 + \frac{t_1 t_2 e^{\{y_2\}t_1}}{(e^{t_1} - 1)(t_1 - t_2)} \\ &+ \frac{t_1 t_2 e^{\{y_2\}t_2}}{(e^{t_2} - 1)(-t_1 + t_2)} + \frac{t_1 t_2 e^{(1 - \{y_1 - y_2\})t_1 + \{y_1\}t_2}}{(e^{t_1} - 1)(e^{t_2} - 1)} \\ &- \frac{t_1 t_2 e^{(1 - \{2y_1 - y_2\})t_1}}{(e^{t_1} - 1)(t_1 + t_2)} - \frac{t_1 t_2 e^{\{2y_1 - y_2\}t_2}}{(e^{t_2} - 1)(t_1 + t_2)} \\ &= \sum_{k_1, k_2=1}^{\infty} \mathcal{P}_{\Delta_{l+}}(k_1, k_2, y_1, y_2; \Delta) \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}. \end{aligned}$$

Hence, by Corollary 3.1, we can compute $\zeta_2(2k, 2k)$ for $k \in \mathbf{N}$, though in this case we can also compute $\zeta_2(2k, 2k)$ by using the harmonic product formula for double zeta values

$$\zeta(s)\zeta(t) = \zeta_2(s, t) + \zeta_2(t, s) + \zeta(s+t).$$

In the general C_r case, considering the expansion of $(\mathfrak{D}_{\Delta_{s+}} F)(\mathbf{t}_{\Delta_{l+}}, \mathbf{0}; \Delta(C_r))$ similarly, we can systematically compute $\zeta_r(2k, \dots, 2k)$. Moreover, considering the case $\nu \neq \mathbf{0}$ for $\zeta_r(\mathbf{s}, \nu; \Delta(C_r))$, we can give character analogues of Corollary 3.1 for multiple L -values, which were first proved by Yamasaki [19].

Next, we consider more general situation. In Theorem 2.5, we considered the sum over W on the left-hand side of (2.1). Here we consider the sum over a certain set of minimal coset representatives on the left-hand side of (2.1). Then we obtain the following result in the case of C_2 .

Proposition 3.3. *For $p, q \in \mathbf{N}$, $p, q \geq 2$,*

$$\begin{aligned} & (1 + (-1)^p)\zeta_2(p, q) + (1 + (-1)^q)\zeta_2(q, p) \\ &= 2 \sum_{j=0}^{[p/2]} \binom{p+q-2j-1}{q-1} \zeta(2j)\zeta(p+q-2j) \\ &+ 2 \sum_{j=0}^{[q/2]} \binom{p+q-2j-1}{p-1} \zeta(2j)\zeta(p+q-2j) \\ &- \zeta(p+q). \end{aligned}$$

Actually this is a special case of the previous result for zeta-functions of A_2 type given by the third-named author [17, Theorem 4.5] (see also [12, Theorem 3.1]). In particular when p and q are of different parity, we see that $\zeta_2(p, q) \in \mathbf{Q}[\{\zeta(j+1) \mid j \in \mathbf{N}\}]$ which was first proved by Euler. For example, we have

$$\zeta_2(2, 3) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5).$$

On the other hand, considering the case of C_3 type, we have the following result which is not included in our previous result for zeta-functions of A_3 type (see [12, Theorem 7.1]).

Theorem 3.4. *For $p, q, u \in \mathbf{N}_{\geq 2}$,*

$$\begin{aligned} & (1 + (-1)^p)(1 + (-1)^u) \\ & \times \{\zeta_3(p, q, u) + \zeta_3(p, u, q) + \zeta_3(u, p, q)\} \\ & + (1 + (-1)^q)(1 + (-1)^u) \\ & \times \{\zeta_3(u, q, p) + \zeta_3(q, u, p) + \zeta_3(q, p, u)\} \\ & \in \mathbf{Q}[\{\zeta(j+1) \mid j \in \mathbf{N}\}]. \end{aligned}$$

In particular when p is odd and both q and u are even, then

$$(3.1) \quad \zeta_3(u, q, p) + \zeta_3(q, u, p) + \zeta_3(q, p, u) \\ \in \mathbf{Q}[\{\zeta(j+1) \mid j \in \mathbf{N}\}].$$

Remark 3.5. Combining (3.1) and the harmonic product formula for triple zeta values, we have

$$\zeta_3(p, q, u) - \zeta_3(u, q, p) \\ \in \mathbf{Q}[\{\zeta(j+1), \zeta_2(k, l+1) \mid j, k, l \in \mathbf{N}\}],$$

when p is odd and both q and u are even. This is a known fact given by Borwein et al. in the triple case (see [1, Theorem 3.1]).

4. The case of B_r type. As for the root system of B_r type, namely for $\Delta = \Delta(B_r)$, we see that

$$(\Delta_{s+})^\vee = \{\alpha_r^\vee, 2\alpha_{r-1}^\vee + \alpha_r^\vee, \dots, \\ 2\alpha_1^\vee + \dots + 2\alpha_{r-1}^\vee + \alpha_r^\vee\}.$$

By setting $s_\alpha = 0$ for all $\alpha \in \Delta_{l+}$, we have

$$\zeta_r(\mathbf{s}, \mathbf{0}; B_r) = \sum_{m_1, \dots, m_r=1}^{\infty} \prod_{i=1}^r \frac{1}{(2 \sum_{j=r-i+1}^{r-1} m_j + m_r)^{s_i}},$$

which is a partial sum of $\zeta_r(\mathbf{s})$. From the viewpoint of zeta-functions of root systems, values of this function at positive integers can be regarded as the objects dual to MZVs, in the sense that B_r and C_r are dual. For example,

$$\zeta_2((0, s_1, 0, s_2), \mathbf{0}; B_2) = \sum_{m, n=1}^{\infty} \frac{1}{n^{s_1} (2m+n)^{s_2}}, \\ \zeta_3((0, 0, s_1, 0, 0, s_2, 0, 0, s_3), \mathbf{0}; B_3) \\ = \sum_{l, m, n=1}^{\infty} \frac{1}{n^{s_1} (2m+n)^{s_2} (2l+2m+n)^{s_3}}.$$

For simplicity, we denote $\zeta_2((0, s_1, 0, s_2), \mathbf{0}; B_2)$ by $\zeta_2^\sharp(s_1, s_2)$. Then, similarly to Proposition 3.3, we can prove the following

Proposition 4.1. For $p, q \in \mathbf{N}_{\geq 2}$,

$$(1 + (-1)^p) \zeta_2^\sharp(p, q) + (1 + (-1)^q) \zeta_2^\sharp(q, p) \\ = 2 \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{1}{2^{p+q-2j}} \binom{p+q-1-2j}{q-1} \zeta(2j) \zeta(p+q-2j) \\ + 2 \sum_{j=0}^{\lfloor q/2 \rfloor} \frac{1}{2^{p+q-2j}} \binom{p+q-1-2j}{p-1} \zeta(2j) \zeta(p+q-2j) \\ - \zeta(p+q).$$

Example 4.2. In Proposition 4.1, if p and q are of different parity, then

$$\zeta_2^\sharp(p, q) \in \mathbf{Q}[\{\zeta(j+1) \mid j \in \mathbf{N}\}].$$

For example, setting $(p, q) = (3, 2)$, we have

$$\zeta_2^\sharp(2, 3) = \sum_{m, n=1}^{\infty} \frac{1}{n^2 (2m+n)^3} \\ = -\frac{21}{32} \zeta(5) + \frac{3}{8} \zeta(2) \zeta(3).$$

Furthermore, similarly to Corollary 3.1, we obtain the following

Proposition 4.3. For $k \in \mathbf{N}$,

$$\sum_{m_1, \dots, m_r=1}^{\infty} \prod_{i=1}^r \frac{1}{(2 \sum_{j=r-i+1}^{r-1} m_j + m_r)^{2k}} \in \mathbf{Q} \cdot \pi^{2kr}.$$

5. Partial zeta values. In [16], we studied zeta-functions of weight lattices of compact connected semisimple Lie groups. We can prove analogues of Theorem 2.5 for those zeta-functions by a method similar to the above. For example, considering the cases of B_2 , C_2 , B_3 and C_3 types, we obtain the following results on partial double and triple zeta values.

Theorem 5.1. For $p \in \mathbf{N}$,

$$\sum_{\substack{m, n=1 \\ n \equiv 1 \pmod{2}}}^{\infty} \frac{1}{n^{2p} (2m+n)^{2p}} \in \mathbf{Q} \cdot \pi^{4p}, \\ \sum_{\substack{m, n=1 \\ m \equiv 1 \pmod{2}}}^{\infty} \frac{1}{n^{2p} (m+n)^{2p}} \in \mathbf{Q} \cdot \pi^{4p}, \\ \sum_{\substack{l, m, n=1 \\ n \equiv 1 \pmod{2}}}^{\infty} \frac{1}{n^{2p} (2m+n)^{2p} (2l+2m+n)^{2p}} \in \mathbf{Q} \cdot \pi^{6p}, \\ \sum_{\substack{l, m, n=1 \\ l \equiv n \pmod{2}}}^{\infty} \frac{1}{n^{2p} (m+n)^{2p} (l+m+n)^{2p}} \in \mathbf{Q} \cdot \pi^{6p}.$$

Example 5.2. We can explicitly compute, for example,

$$\sum_{\substack{m, n=1 \\ m \equiv 1 \pmod{2}}}^{\infty} \frac{1}{n^6 (m+n)^6} = \frac{1}{58060800} \pi^{12}, \\ \sum_{\substack{m, n=1 \\ m \equiv 1 \pmod{2}}}^{\infty} \frac{1}{n^8 (m+n)^8} = \frac{17}{390168576000} \pi^{16}.$$

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