CONVOLUTIONS OF THE VON MANGOLDT FUNCTION AND RELATED DIRICHLET SERIES

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Abstract. In this paper, we first give a brief survey on the theory of meromorphic continuation and natural boundaries of multiple Dirichlet series. Then we consider the double Dirichlet series $\Phi_2(s)$ defined by the convolution of logarithmic derivatives of the Riemann zeta-function. Especially we propose the conjecture that $\Phi_2(s)$ would have the natural boundary on $\Re s = 1$, and give a supportive evidence. We further present an application of $\Phi_2(s)$ to the Riesz mean, and discuss its multiple analogues.

1. The analytic continuation of multiple Dirichlet series

Let $s = \sigma + it$ be a complex variable, and $P(X_1, \ldots, X_r)$ a polynomial of complex coefficients. The multiple zeta-function

$$\zeta_r(s; P) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} P(m_1, \ldots, m_r)^{-s} \tag{1.1}$$

was first studied by Mellin [29], [30], and independently by Barnes [5], [6] for P a linear form, at the beginning of the 20th century. Mellin proved the meromorphic continuation of (1.1) to the whole complex plane $\mathbb{C}$ if all the coefficients of P have positive real parts. Several mathematicians after Mellin proved the meromorphic continuation of (1.1) under weaker assumptions. At present, the assumption $(H_0 S)$ introduced by Essouabri [12] is the weakest. Essouabri [11] also pointed out that the multi-variable generalization

$$\begin{aligned} \zeta_r(s_1, \ldots, s_n; P_1, \ldots, P_n) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} & P_1(m_1, \ldots, m_r)^{-s_1} \\ & \times \cdots \times P_n(m_1, \ldots, m_r)^{-s_n} \end{aligned} \tag{1.2}$$

of (1.1), where $s_1, \ldots, s_n \in \mathbb{C}$ and $P_1, \ldots, P_n \in \mathbb{C}[X_1, \ldots, X_r]$, can be continued meromorphically to the whole space $\mathbb{C}^n$ under the same type of assumption.

A special type of multi-variable multiple series

$$\zeta_{EZ,r}(s_1,\ldots,s_r)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}m_1^{-s_1}(m_1+m_2)^{-s_2}\times\cdots\times(m_1+\cdots+m_r)^{-s_r}, \tag{1.3}$$

which is called the Euler-Zagier r-fold sum, has been studied extensively in recent years. The meromorphic continuation of (1.3) to $\mathbb{C}^r$ is included in the above theorem of Essouabri [11], but [11] is unpublished. Various different proofs of the continuation were published by Arakawa and Kaneko [3], Zhao [37], Akiyama, Egami and Tanigawa [1], and the second-named author [27]. The method in [27] is based on the Mellin-Barnes integral formula

$$(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}\lambda^{-z}dz \tag{1.4}$$

(where $s,\lambda\in\mathbb{C}$, $\lambda\neq 0$, $|\arg\lambda|<\pi$, $\Re s>0$, $0<c<\Re s$, and the path of integration is the vertical line from $c-i\infty$ to $c+i\infty$), which was already used in Mellin's papers [29], [30].

For arithmetical applications, it is important to consider various multiple Dirichlet series with arithmetical coefficients. Peter [32] discussed the analytic continuation of the series

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{a_1(m_1)\cdots a_r(m_r)}{P(m_1,\ldots,m_r)^s}, \tag{1.5}$$

where $a_k(m_k)$ $(1\leq k\leq r)$ are complex numbers. Actually he treated the more general situation that $P(m_1,\ldots,m_r)$ in the denominator is replaced by $P(\lambda_1(m_1),\ldots,\lambda_r(m_r))$, where $\lambda_k(m)$ are complex numbers in a certain fixed cone on $\mathbb{C}$ satisfying $\lim_{m\to\infty}|\lambda_k(m)|=\infty$ $(1\leq k\leq r)$. The multi-variable series

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{f(m_1,\ldots,m_r)}{m_1^{s_1}\cdots m_r^{s_r}}, \tag{1.6}$$

where $f(m_1,\ldots,m_r)$ is a non-negative arithmetical function, was studied by de la Bretéche [8].

In connection with sums of the Euler-Zagier type, multiple L-series defined by twisting (1.3) by Dirichlet characters have been investigated by Goncharov [19], Arakawa and Kaneko [3], [4], Akiyama and Ishikawa [2], and Ishikawa [21], [22].

More generally, we may claim that if Dirichlet series

$$\varphi_k(s) = \sum_{m=1}^{\infty} \frac{a_k(m)}{m^s} \qquad (1 \le k \le r) \tag{1.7}$$

behave nicely, then we can show that the multiple Dirichet series of the form

$$\Phi_r(s_1, \ldots, s_r; \varphi_1, \ldots, \varphi_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)}{m_1^{s_1}} \frac{a_2(m_2)}{(m_1+m_2)^{s_2}} \times \cdots \times \frac{a_r(m_r)}{(m_1+\cdots+m_r)^{s_r}} \tag{1.8}$$

also behaves nicely. In fact, the following theorem was proved in Matsumoto and Tanigawa [28].

Theorem 1.1. ([28]) *Assume that $\varphi_k(s)$ $(1 \le k \le r)$ are absolutely convergent for $\sigma > \alpha_k (> 0)$, can be continued meromorphically to the whole plane $\mathbb{C}$, holomorphic except for a possible pole (of order at most 1) at $s = \alpha_k$, and of polynomial order in any fixed strip $\sigma_1 \le \sigma \le \sigma_2$. Then $\Phi_r(s_1, \ldots, s_r; \varphi_1, \ldots, \varphi_r)$ can be continued meromorphically to the whole space $\mathbb{C}^r$, and the location of its possible singularities can be described explicitly. In particular, if all $\varphi_k(s)$ are entire, then $\Phi_r(s_1, \ldots, s_r; \varphi_1, \ldots, \varphi_r)$ is also entire.*

The proof of the above theorem is an analogue of the second-named author's proof of the meromorphic continuation of (1.3) given in [27], whose basic tool is the Mellin-Barnes formula (1.4). The idea of applying formula (1.4) in such a situation had been already mentioned by the first-named author [10] in the one-variable case.

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2. An example of double Dirichlet series with a natural boundary

In Theorem 1.1, there is the condition that $\varphi_k(s)$ are holomorphic except for only one possible pole. Actually it is possible to prove a result of similar type under the weaker condition that each $\varphi_k(s)$ has finitely many poles.

However, if some of $\varphi_k(s)$ has infinitely many poles, the behaviour of the multiple series $\Phi_r(s_1,\ldots,s_r;\varphi_1,\ldots,\varphi_r)$ may be quite different. The following simple example illustrates this phenomenon. Let $\Lambda(n)$ be the von Mangoldt function, and

$$M(s)=-\frac{\zeta'}{\zeta}(s)=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}, \tag{2.1}$$

where $\zeta(s)$ is the Riemann zeta-function. Then $M(s)$ is meromorphic in the whole plane, and has infinitely many poles because all zeros of $\zeta(s)$ are the poles of $M(s)$. In fact it is known that

$$N(T)\sim\frac{1}{2\pi}T\log T \qquad (T\geq 2) \tag{2.2}$$

(Theorem 9.4 of Titchmarsh [36]), where $N(T)$ is the number of zeros (counted with multiplicity) of $\zeta(s)$ in the region $0<\sigma<1$, $0<t\leq T$, which is expected to be equal to the number of poles of $M(s)$ in the same region because all zeros of $\zeta(s)$ are conjectured to be simple.

Let

$$\Phi_2(s)=\Phi_2(0,s;M,M)=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Lambda(k)\Lambda(m)}{(k+m)^s}. \tag{2.3}$$

This can be rewritten as

$$\Phi_2(s)=\sum_{n=1}^{\infty}\frac{G_2(n)}{n^s}, \tag{2.4}$$

where

$$G_2(n)=\sum_{k+m=n}\Lambda(k)\Lambda(m). \tag{2.5}$$

The series on the right-hand side of (2.3), (2.4) is absolutely convergent for $\Re s>2$, because

$$G_2(n)\leq\sum_{k=1}^{n-1}\log k\cdot\log(n-k)\leq n(\log n)^2. \tag{2.6}$$

In the present paper we will show, under the assumption of certain conjectures, that $\Phi_2(s)$ has the natural boundary on the line $\Re s=1$ (Theorem 2.2 below). Therefore it seems that the behaviour of $\Phi_2(s)$ is completely different from that of multiple series studied in [28].

The history of the investigation of natural boundaries of Dirichlet series also goes back to the beginning of the 20th century. The analytic continuation and the natural boundary of the function $\sum_p p^{-s}$ (p runs over primes) were studied by Kluyver [23], Landau [25], and Landau and Walfisz [26]. In 1928, Estermann published two papers [13], [14] on natural boundaries of Dirichlet series. In the former paper [13], he considered a certain class of Dirichlet series which have Euler products, and gave a criterion when the series can be continued to the whole plane and when it has the natural boundary. The continuation and natural boundaries of Euler products were further studied in more general situations by several mathematicians such as Dahlquist [9], Kurokawa [24]. A multi-variable generalization was recently discussed by Bhowmik, Essouabri and Lichtin [7].

The results in the present paper give a different direction of research on natural boundaries of Dirichlet series. A part of the present work was already announced on the occasion of a conference on number theory (in honour of Professor Akio Fujii) held at Rikkyo University, Tokyo, in January 2005. On the other hand, independently of the present work, Tanigawa and Zhai [35] have considered Dirichlet series which are more general than ours, and have discussed the same type of problems (except for the Riesz mean). Their proof of the claim on natural boundaries (Theorem 1.3 of [35]) seems incomplete; some condition similar to our (B) below seems to be necessary to verify their argument.

We mention here the number-theoretic motivation of the study of $\Phi_2(s)$. The function $G_2(n)$ defined by (2.5) is a classical subject matter of number theory, because it is connected with the famous conjecture of C. Goldbach (that is, any even integer (≥ 4) can be expressed as a sum of two primes); in fact, the conjecture implies that $G_2(n) > 0$ for all even $n \geq 4$. Fujii [15] studied the mean value of $G_2(n)$ and proved that, if we assume the Riemann hypothesis (RH) for $\zeta(s)$, then

$$\sum_{n\leq X} G_2(n) = \frac{1}{2}X^2 + O(X^{3/2}) \tag{2.7}$$

for any large positive X. In [16], Fujii improved his result to obtain

$$\sum_{n\leq X} G_2(n) = \frac{1}{2}X^2 - H(X) + O((X\log X)^{4/3}) \tag{2.8}$$

under RH. Here

$$H(X) = 2\sum_{\rho} \frac{X^{1+\rho}}{\rho(1+\rho)},$$

where ρ runs over the non-trivial zeros of $\zeta(s)$, counted with multiplicity.

From the work [20] of Hardy and Littlewood it is expected that $G_2(n)$ for even n is approximated by $nS_2(n)$, where

$$S_2(n) = \prod_{p|n}\left(1+\frac{1}{p-1}\right)\prod_{(p,n)=1}\left(1-\frac{1}{(p-1)^2}\right). \tag{2.9}$$

Moreover it follows from Lemma 1 of Montgomery and Vaughan [31] that

$$\sum_{n\leq X} nS_2(n) = \frac{1}{2}X^2 + O(X\log X). \tag{2.10}$$

From this viewpoint, Fujii [16] reformulated his formula (2.8) into

$$\sum_{n\leq X}(G_2(n) - nS_2(n)) = -H(X) + O((X\log X)^{4/3}). \tag{2.11}$$

Hence the term $H(X)$ represents the main oscillation in the above formulation of Goldbach's problem. Some properties of $H(X)$ have been studied in Fujii [17].

By (2.4) and Perron's formula we have

$$\sum_{n\leq X} G_2(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\Phi_2(s)\frac{X^s}{s}ds + O(T^{-1}X^{2+\varepsilon}) \tag{2.12}$$

with $c > 2$. Therefore the study of $\Phi_2(s)$ will be useful to understand the behaviour of $G_2(n)$. In the next section we will prove the following:

Theorem 2.1. (under RH) *The function $\Phi_2(s)$ can be continued meromorphically to the half-plane $\Re s > 1$, and holomorphic except for the simple poles at $s = 2$ (with residue 1) and $s = 1+\rho$ (with residue $-2n(\rho)/\rho$) for any non-trivial zero ρ of $\zeta(s)$, where $n(\rho)$ is the multiplicity of ρ.*

By this theorem, we can shift (under RH) the path of integration on the right-hand side of (2.12) to $\Re s = 1+\varepsilon$. We encounter the poles $s = 2$ and $s = 1+\rho$, and the sum of their residues is $(1/2)X^2 - H(X)$, which coincides with the explicit terms on the right-hand side of (2.8). In particular, we find that the properties of $H(X)$ are closely connected with the behaviour of $\Phi_2(s)$ on the line $\Re s = 3/2$.

Next we consider the behaviour of $\Phi_2(s)$ on the line $\Re s = 1$. We propose the following:

Conjecture 2.1. *The line $\Re s = 1$ is the natural boundary of $\Phi_2(s)$.*

In the present paper we will show an evidence which supports the above conjecture. Let $\mathcal{I}$ be the set of all imaginary parts of non-trivial zeros of $\zeta(s)$. A well-known conjecture speculates that the positive elements of $\mathcal{I}$ would be linearly independent over the rationals. The following statement is a special case of this conjecture:

(A) If $\gamma_j \in \mathcal{I}$ $(1 \leq j \leq 4)$ and $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 (\neq 0)$, then (γ_3, γ_4) equals (γ_1, γ_2) or (γ_2, γ_1).

These conjectures were mentioned on p.50 of Fujii [18]. In that paper Fujii made an extensive study on additive properties of the zeros of $\zeta(s)$. For instance he proved that the set

$$\{\gamma_1 + \gamma_2 \mid \gamma_1, \gamma_2 \in \mathcal{I}, \gamma_1 > 0, \gamma_2 > 0\}$$

is uniformly distributed mod 1 (Corollary 3 of [18]).

Here we introduce the following quantitative version of (A):

(B) There exists a constant α, with $0 < \alpha < \pi/2$, such that if $\gamma_j \in \mathcal{I}$ $(1 \leq j \leq 4)$, $\gamma_1 + \gamma_2 \neq 0$, and (γ_3, γ_4) is neither equal to (γ_1, γ_2) nor to (γ_2, γ_1), then

$$|(\gamma_1 + \gamma_2) - (\gamma_3 + \gamma_4)| \geq \exp\left(-\alpha(|\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4|)\right). \tag{2.13}$$

Clearly (B) implies (A).

In Section 4 of the present paper we will prove that, under RH, the set

$$\mathcal{K} = \{\kappa \mid \kappa = \gamma_1 + \gamma_2 \text{ for some } \gamma_1, \gamma_2 \in \mathcal{I}\} \setminus \{0\} \tag{2.14}$$

is dense in the whole set of real numbers $\mathbf{R}$. This result will yield the following theorem.

Theorem 2.2. (under RH) *If we assume that* (B) *is true, then Conjecture* 1 *is true.*

Hence the continuation achieved by Theorem 1.1 seems to be best-possible. It is therefore not rash to propose the following

Conjecture 2.2. *The error term on the right-hand side of* (2.8) *is to be* $O(X^{1+\varepsilon})$ *and* $\Omega(X)$, *where* $\Omega(X)$ *means that it is not* $o(X)$.

3. Proof of Theorem 2.1

In this section we prove Theorem 2.1. First we assume $\Re s > 2+2\varepsilon$. Then we have

$$\begin{aligned}\Phi_2(s) &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Lambda(k)\Lambda(m)}{(k+m)^s} \\ &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Lambda(k)\Lambda(m)}{k^s}\left(1+\frac{m}{k}\right)^{-s}.\end{aligned} \tag{3.1}$$

We apply the Mellin-Barnes formula (1.4) with $\lambda = m/k$ to (3.1) to obtain

$$\begin{aligned}\Phi_2(s) &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Lambda(k)\Lambda(m)}{k^s}\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}\left(\frac{m}{k}\right)^{-z}dz \\ &= \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}\sum_{k=1}^{\infty}\Lambda(k)k^{-s+z}\sum_{m=1}^{\infty}\Lambda(m)m^{-z}dz.\end{aligned} \tag{3.2}$$

Two infinite series in the integrand are convergent when $\sigma - c > 1$ and $c > 1$. These conditions, and also the condition $0 < c < \sigma$ (which is necessary to apply (1.4)), are satisfied by the choice $c = 1+\varepsilon$. Under this choice of c, we have

$$\Phi_2(s) = \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}M(s-z)M(z)dz. \tag{3.3}$$

The next step is to shift the path of integration from $\Re z = c = 1+\varepsilon$ to $\Re z = -\varepsilon$. First we have to show that this shifting is possible. It is known that

$$N(T+1) - N(T) \ll \log T \tag{3.4}$$

for any $T \geq 2$, where $f \ll g$ means $f = O(g)$ (Theorem 9.2 of Titchmarsh [36]). Hence we can find an arbitrarily large T such that

$$|T-\gamma| \gg (\log T)^{-1} \tag{3.5}$$

for any $\gamma \in \mathcal{I}$. Combining (3.5) with the formula

$$M(z) = -\sum_{|y-\gamma|<1}\frac{1}{z-\rho} + O(\log(|y|+2)) \qquad (y = \Im z, \gamma = \Im \rho), \tag{3.6}$$

which is known to hold uniformly for $-1 \leq x = \Re z \leq 2$ (Theorem 9.6(A) of [36]), we see that, if T satisfies (3.5), then

$$M(z) \ll (\log T)^2 \tag{3.7}$$

for $z = x+iT$, $-1 \le x \le 2$. Also, $M(s-z) = O(1)$ for $-\varepsilon \le x \le 1+\varepsilon$. Hence, using Stirling's formula, we have

$$\begin{aligned}(3.8)\quad \int_{-\varepsilon+iT}^{1+\varepsilon+iT} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} M(s-z)M(z)dz &\ll e^{-(\pi/2)(T+|t-T|-|t|)} \\ &\times \frac{(|t-T|+1)^{\sigma-1/2}(\log T)^2}{(|t|+1)^{\sigma-1/2}T^{1/2}} \int_{-\varepsilon}^{1+\varepsilon} (|t-T|+1)^{-x}T^x dx\end{aligned}$$

for any T satisfying (3.5), and (3.8) tends to 0 as T tends to infinity. This implies that the above shifting is possible.

In the course of this shifting, we encounter the poles $z = 1$, $z = \rho$ for any non-trivial zero, and $z = 0$. The residues of the integrand at those poles are

$$\frac{M(s-1)}{s-1}, \qquad -n(\rho)\frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)}M(s-\rho)$$

and

$$M(s)M(0) = -M(s)\log 2\pi,$$

respectively. Hence we obtain

$$\begin{aligned}(3.9)\quad \Phi_2(s) = \frac{M(s-1)}{s-1} &- \sum_{\rho} \frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)}M(s-\rho) - M(s)\log 2\pi \\ &+ \frac{1}{2\pi i}\int_{(-\varepsilon)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}M(s-z)M(z)dz.\end{aligned}$$

Now we continue $\Phi_2(s)$ meromorphically by using (3.9). The first and the third terms on the right-hand side of (3.9) are clearly meromorphic on the whole plane. The poles of the third term coincide with the poles of $M(s)$, which are $s = 1$ and $s = \rho$ (non-trivial zeros). The poles of the first term are $s = 2$, $s = \rho + 1$, and $s = 1$. The residues of the first term at $s = 2$ and $s = \rho + 1$ are 1 and $-n(\rho)/\rho$, respectively.

The integral on the right-hand side of (3.9) is convergent uniformly in any compact subset of the half-plane $\Re s > 1-\varepsilon$, and hence holomorphic in that half-plane. Actually it is possible to continue this integral meromorphically to the whole plane, by shifting the path further to the left.

The most difficult part is the second term

$$(3.10)\qquad B_2(s) = -\sum_{\rho} \frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)}M(s-\rho).$$

The factor $\Gamma(s-\rho)$ has poles at $s = \rho - \ell$ ($\ell = 0, 1, 2, \ldots$), while the factor $M(s-\rho)$ has poles at $s = \rho + 1$ and at $s = \rho + \rho'$, where ρ' denotes the non-trivial zeros of $\zeta(s)$. In order to control this situation, we now assume RH (to the end of this section). Then the only poles of $B_2(s)$ in the region $\Re s > 1$ are $s = \rho + 1$ for non-trivial zeros ρ, and the residue there is

$$-n(\rho)\frac{\Gamma(1)\Gamma(\rho)}{\Gamma(\rho+1)} = -\frac{n(\rho)}{\rho}.$$

These poles are isolated singularities, and hence $B_2(s)$ can be continued to $\Re s > 1$. This implies the meromorphic continuation of $\Phi_2(s)$ to $\Re s > 1$. The residue of $\Phi_2(s)$ at $s = 2$ is 1, and at $s = 1 + \rho$ is

$$-\frac{n(\rho)}{\rho} - \frac{n(\rho)}{\rho} = -\frac{2n(\rho)}{\rho}.$$

Now the proof of Theorem 2.1 is complete.

4. Proof of Theorem 2.2

To prove Theorem 2.2, we use the classical explicit formula

$$M(s) = b + \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{4.1}$$

where $b = 1 + (C_0/2) - \log 2\pi$ and C_0 is Euler's constant (formula (2.12.7) of [36]). Substituting this into (3.10), for $\Re s > 1$ we obtain

(4.2)
$$\begin{aligned} B_2(s) =& -\sum_{\rho}\frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)}\left\{b + \frac{1}{s-\rho-1} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s-\rho}{2}+1\right)\right\} \\ &+ \sum_{\rho}\sum_{\rho'}\frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)}\left(\frac{1}{s-\rho-\rho'} + \frac{1}{\rho'}\right) \\ =& B_{21}(s) + B_{22}(s), \end{aligned}$$

say. Clearly $B_{21}(s)$ is meromorphic on the whole plane, and has no pole on the line $\Re s = 1$. To investigate $B_{22}(s)$, we assume RH (to the end of this section), and rewrite $\rho = \rho_1 = 1/2 + i\gamma_1$ and $\rho' = \rho_2 = 1/2 + i\gamma_2$ to obtain

$$B_{22}(s) = \frac{1}{\Gamma(s)}\sum_{\rho_1}\sum_{\rho_2}\frac{\Gamma(s+1-\rho_1)\Gamma(\rho_1)}{(s-\rho_1-\rho_2)\rho_2} \qquad (\Re s > 1). \tag{4.3}$$

Therefore $B_{22}(s)$ may behave singularly as s tends to $\rho_1 + \rho_2$, that is, any point of the form $1 + i\kappa$ with $\kappa \in \mathcal{K}$ (where $\mathcal{K}$ is the set defined by (2.14)). Before studying this phenomenon closely, we first prove

Lemma 4.1. (under RH) *The set $\mathcal{K}$ is dense in* $\mathbf{R}$.

Proof. It is classically known that

$$N(T) = \frac{1}{2\pi} T \log T - C_1 T + O(\log T) \qquad \left(C_1 = \frac{1 + \log 2\pi}{2\pi} \right)$$

(Theorem 9.4 of [36]), and, under RH, the error term in the above formula can be replaced by $O(\log T / \log\log T)$ (Theorem 14.13 of [36]). Therefore, for any fixed $h \in \mathbf{R}$, the number of zeros on the interval $(1/2 + iT, 1/2 + i(T+h)]$ is

(4.4)

$$\begin{aligned}
&\frac{1}{2\pi}(T+h)\log(T+h) - C_1(T+h) \\
&\qquad - \frac{1}{2\pi} T \log T + C_1 T + O\left(\frac{\log T}{\log\log T}\right) \\
&= \frac{1}{2\pi} T \left\{ \log T + \log\left(1 + \frac{h}{T}\right)\right\} + \frac{1}{2\pi} h \left\{ \log T + \log\left(1 + \frac{h}{T}\right)\right\} \\
&\qquad - C_1 h - \frac{1}{2\pi} T \log T + O\left(\frac{\log T}{\log\log T}\right) \\
&= \frac{h}{2\pi} \log T + O\left(\frac{\log T}{\log\log T}\right).
\end{aligned}$$

There exists a sufficiently large $T_0 = T_0(h)$, such that the right-hand side of (4.4) is positive for any $T \geq T_0$. Let α be any non-zero real number, and ε be arbitrarily small. Then, by using this positivity, we can find a sufficiently large $T = T(\alpha, \varepsilon)$ and γ_1, $\gamma_2 \in \mathcal{I}$, satisfying

$$\gamma_1 \in (T + \alpha - \varepsilon/2, T + \alpha + \varepsilon/2], \qquad \gamma_2 \in (-T - \varepsilon/2, -T + \varepsilon/2].$$

Hence $|\alpha - (\gamma_1 + \gamma_2)| < \varepsilon$. Moreover, if $\varepsilon < |\alpha|$, then $\gamma_2 \neq -\gamma_1$, so $\gamma_1 + \gamma_2 \in \mathcal{K}$. Thus we conclude the assertion of the lemma.

In view of the above lemma we now know that the points of the form $1 + i\kappa$ ($\kappa \in \mathcal{K}$) are dense on the line $\Re s = 1$. Now we assume (B), and prove the following

Lemma 4.2. (under RH and (B)) *For any $\kappa \in \mathcal{K}$, the function $B_{22}(s)$ tends to infinity as s tends to $1 + i\kappa$ from the right.*

Proof. By (A) we see that there is only one pair (γ_1^0, γ_2^0) (and its reverse-ordered pair (γ_2^0, γ_1^0)) satisfying $\gamma_1^0 + \gamma_2^0 = \kappa$. Put $\rho_1^0 = (1/2) +$

$i\gamma_1^0$, $\rho_2^0=(1/2)+i\gamma_2^0$. Then

$$\begin{aligned} B_{22}(s) &= \frac{n(\rho_1^0)n(\rho_2^0)}{\Gamma(s)}\left\{\frac{\Gamma(s+(1/2)-i\gamma_1^0)\Gamma((1/2)+i\gamma_1^0)}{(s-1-i\gamma_1^0-i\gamma_2^0)((1/2)+i\gamma_2^0)}\right. \\ &\qquad\left.+\frac{\Gamma(s+(1/2)-i\gamma_2^0)\Gamma((1/2)+i\gamma_2^0)}{(s-1-i\gamma_1^0-i\gamma_2^0)((1/2)+i\gamma_1^0)}\right\} \\ &\quad+\frac{1}{\Gamma(s)}\sum_{\gamma_1}\sum_{\gamma_2}{}^{*}\frac{\Gamma(s+1-\rho_1)\Gamma(\rho_1)}{(s-\rho_1-\rho_2)\rho_2} \\ &= B_{22}^{*}(s)+B_{22}^{**}(s), \end{aligned}$$

say, where the symbol $\sum\sum^{*}$ means the sum over all (γ_1,γ_2) satisfying $(\gamma_1,\gamma_2)\neq(\gamma_1^0,\gamma_2^0),(\gamma_2^0,\gamma_1^0)$. Then $B_{22}^{*}(s)$ is meromorphic on the whole plane, and its residue at $s=1+i\kappa=1+i(\gamma_1^0+\gamma_2^0)$ is

$$\begin{aligned} &\frac{n(\rho_1^0)n(\rho_2^0)}{\Gamma(1+i\kappa)}\left\{\frac{\Gamma((3/2)+i(\kappa-\gamma_1^0))\Gamma((1/2)+i\gamma_1^0)}{(1/2)+i\gamma_2^0}\right. \\ &\qquad\left.+\frac{\Gamma((3/2)+i(\kappa-\gamma_2^0))\Gamma((1/2)+i\gamma_2^0)}{(1/2)+i\gamma_1^0}\right\} \\ &=\frac{2n(\rho_1^0)n(\rho_2^0)}{\Gamma(1+i\kappa)}\Gamma(\rho_1^0)\Gamma(\rho_2^0), \end{aligned}\tag{4.5}$$

which does not vanish. That is, $B_{22}^{*}(s)\to\infty$ as $s\to 1+i\kappa$. Therefore the remaining task is to show that $B_{22}^{**}(s)$ remains finite as $s\to 1+i\kappa$. Putting $s=1+\eta+i\kappa$ ($\eta\geq 0$, small), we have

$$\begin{aligned} B_{22}^{**}(1+\eta+i\kappa) &= \frac{1}{\Gamma(1+\eta+i\kappa)} \\ &\times\sum_{\gamma_1}\sum_{\gamma_2}{}^{*}\frac{\Gamma((3/2)+\eta+i(\kappa-\gamma_1))\Gamma((1/2)+i\gamma_1)}{(\eta+i(\kappa-\gamma_1-\gamma_2)((1/2)+i\gamma_2)}. \end{aligned}\tag{4.6}$$

To prove the lemma, it is enough to show that the right-hand side of (4.6) is absolutely convergent, uniformly in η. By using Stirling's formula we have

$$\begin{aligned} B_{22}^{**}(1+\eta+i\kappa) &\ll \frac{1}{\Gamma(1+\eta+i\kappa)}\sum_{\gamma_1}(|\kappa-\gamma_1|+1)^{1+\eta} \\ &\times e^{-(\pi/2)(|\kappa-\gamma_1|+|\gamma_1|)}\sum_{\gamma_2}{}^{*}\frac{1}{|\kappa-\gamma_1-\gamma_2|(1+|\gamma_2|)}. \end{aligned}\tag{4.7}$$

The inner sum on the right-hand side of (4.7) can be divided into

$$\sum_{0<|\gamma_2-\lambda|\leq 1}+\sum_{|\gamma_2-\lambda|>1}=\Sigma_1+\Sigma_2,$$

say, where $\lambda = \kappa - \gamma_1$. If $\lambda = 0$, then obviously $\Sigma_2 = O(1)$. If $\lambda > 0$, we divide Σ_2 as

$$\Sigma_2 = \sum_{\gamma_2 > \lambda+1} + \sum_{o<\gamma_2<\lambda-1} + \sum_{\gamma_2<0} = \Sigma_{21} + \Sigma_{22} + \Sigma_{23},$$

say. The sum Σ_{23} is clearly $O(1)$, while by using partial summation and (2.2) we can easily show that Σ_{21}, Σ_{22} are $O(\log(\lambda+1))$. The case $\lambda < 0$ can be treated similarly. The conclusion is that

$$\Sigma_2 = O(\log(|\kappa - \gamma_1| + 1)). \tag{4.8}$$

Next consider Σ_1. Since $0 < |\gamma_2 - \lambda| \le 1$, we have $1 + |\gamma_2| \gg 1 + |\lambda|$. Hence

$$\Sigma_1 \ll \frac{1}{1+|\lambda|} \sum_{0<|\gamma_2-\lambda|\le 1} \frac{1}{|\kappa - \gamma_1 - \gamma_2|}.$$

Then, since $\kappa = \gamma_1^0 + \gamma_2^0$, applying assumption (B) we have

$$\begin{aligned}\Sigma_1 &\ll \frac{1}{1+|\lambda|} \sum_{0<|\gamma_2-\lambda|\le 1} \exp\left(\alpha(|\gamma_1| + |\gamma_2| + |\gamma_1^0| + |\gamma_2^0|)\right) \\ &\ll \frac{1}{1+|\lambda|} \exp\left(\alpha(|\gamma_1| + |\lambda| + |\gamma_1^0| + |\gamma_2^0|)\right) \sum_{0<|\gamma_2-\lambda|\le 1} 1,\end{aligned}$$

where we have used $|\gamma_2| \le |\lambda| + 1$. Applying (3.4) we obtain

$$\Sigma_1 \ll \frac{\log(1+|\lambda|)}{1+|\lambda|} \exp\left(\alpha(|\gamma_1| + |\lambda| + |\gamma_1^0| + |\gamma_2^0|)\right). \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7), we find that

$$\begin{aligned}B_{22}^{**}(1+\eta+i\kappa) &\ll \frac{\exp\left(\alpha(|\gamma_1^0| + |\gamma_2^0|)\right)}{\Gamma(1+\eta+i\kappa)} \sum_{\gamma_1} (|\kappa - \gamma_1| + 1)^{1+\eta} \\ &\quad \times \log(|\kappa - \gamma_1| + 1) \exp\left(\left(\alpha - \frac{\pi}{2}\right)(|\gamma_1| + |\kappa - \gamma_1|)\right),\end{aligned} \tag{4.10}$$

which is absolutely convergent because $\alpha < \pi/2$. Hence the assertion of Lemma 4.2 follows.

From Lemma 4.1 and Lemma 4.2 we can conclude that singular points of $\Phi_2(s)$ are distributed densely on the line $\Re s = 1$. In fact, (4.5) implies

$$\Phi_2(s) \sim \frac{2}{\Gamma(1+i\kappa)} n(\rho_1^0) n(\rho_2^0) \Gamma(\rho_1^0) \Gamma(\rho_2^0) \frac{1}{s - (1+i\kappa)} \tag{4.11}$$

as $s \to 1 + i\kappa$. This completes the proof of Theorem 2.2.

5. An application to the Riesz mean

At present the authors have no idea how to prove the desired estimate $O(X^{1+\varepsilon})$ of Conjecture 2.2. Instead, in this section we consider the Riesz mean of $G_2(n)$, that is,

$$\mathcal{D}_{2,a}(X) = \frac{1}{\Gamma(a+1)} \sum_{n \le X} (X-n)^a G_2(n) \tag{5.1}$$

where $a > 0$. The treatment of $\mathcal{D}_{2,a}(X)$ becomes easier when a becomes larger. The aim of this section is to prove the following theorem.

Theorem 5.1. (under RH) *For any $a > 1/2$, the asymptotic formula*

$$\begin{aligned} \mathcal{D}_{2,a}(X) = & \frac{1}{\Gamma(3+a)} X^{2+a} - \sum_{\rho} \frac{2\Gamma(1+\rho)}{\rho\Gamma(2+\rho+a)} X^{1+\rho+a} \\ & + O(X^{1+a+\varepsilon}) \end{aligned} \tag{5.2}$$

holds.

If (5.2) would hold for $a = 0$, then it would give (2.8) with the desired error estimate $O(X^{1+\varepsilon})$.

The basic tool for the proof of Theorem 5.1 is the Mellin transformation formula

$$\mathcal{D}_{2,a}(X) = \frac{1}{2\pi i} \int_{(2+\varepsilon)} \frac{\Gamma(s)}{\Gamma(s+a+1)} \Phi_2(s) X^{s+a} ds. \tag{5.3}$$

In order to obtain (5.2), we shift the path of integration on the right-hand side of (5.3) to the left. By Theorem 2.1 we know that $\Phi_2(s)$ can be continued to $\Re s > 1$. Therefore we shift the path to $\Re s = 1 + \varepsilon$. To check that this shifting procedure is possible, we need the following lemma.

Lemma 5.1. (under RH) *There exists an arbitrarily large T for which the estimate*

$$\Phi_2(s) \ll T^{1/2} (\log T)^2 \tag{5.4}$$

holds for $s = \sigma + iT$, $1 + \varepsilon \le \sigma \le 2 + \varepsilon$.

Proof. We estimate each term on the right-hand side of (3.9) for $1 + \varepsilon \le \sigma \le 2 + \varepsilon$.

First we do not assume RH, and let $T = \Im s$ be any sufficiently large positive number. Since $M(s)$ is absolutely convergent for $\sigma \geq 1+\varepsilon$, we have

$$M(s)\log 2\pi = O(1) \qquad (1+\varepsilon \leq \sigma \leq 2+\varepsilon). \tag{5.5}$$

Next, denote the integral term on the right-hand side of (3.9) by $I_2(s)$. From (3.6) we have

$$M(-\varepsilon+iy) \ll \sum_{|y-\gamma|<1} 1 + O(\log(|y|+2)) \ll \log(|y|+2), \tag{5.6}$$

where in the second inequality we have used (3.4). It is clear that $M(s-(-\varepsilon+iy)) = O(1)$ for $1+\varepsilon \leq \sigma \leq 2+\varepsilon$. Using these estimates and Stirling's formula, we find that

$$\begin{aligned} I_2(s) \ll & e^{(\pi/2)T} T^{1/2-\sigma} \int_{-\infty}^{\infty} e^{-(\pi/2)(|y|+|T-y|)} \\ & \times (|T-y|+1)^{\sigma+\varepsilon-1/2}(|y|+1)^{-\varepsilon-1/2}\log(|y|+1)dy \\ & = e^{(\pi/2)T}T^{1/2-\sigma}\left(\int_{-\infty}^{0} + \int_{0}^{T} + \int_{T}^{\infty}\right) \\ & = e^{(\pi/2)T}T^{1/2-\sigma}(J_1+J_2+J_3), \end{aligned} \tag{5.7}$$

say. By changing y by $-y$, we see that

$$J_1 = e^{-(\pi/2)T}\int_0^\infty e^{-\pi y}(T+y+1)^{\sigma+\varepsilon-1/2}(y+1)^{-\varepsilon-1/2}\log(y+1)dy.$$

Divide the last integral into two parts at $y = T$. In the interval $[0,T]$ we use $T+y+1 \asymp T$ to conclude that the integral from 0 to T is $O(T^{\sigma+\varepsilon-1/2})$. The integral from T to ∞ is of exponential decay. Hence $J_1 \ll e^{-(\pi/2)T}T^{\sigma+\varepsilon-1/2}$. As for J_3, changing $y-T$ by y we obtain

$$J_3 = e^{-(\pi/2)T}\int_0^\infty e^{-\pi y}(y+1)^{\sigma+\varepsilon-1/2}(T+y+1)^{-\varepsilon-1/2}\log(T+y+1)dy.$$

Dividing the integral at $y = T$ and proceeding similarly to the case of J_1, we find that $J_3 \ll e^{-(\pi/2)T}T^{-\varepsilon-1/2}\log T$. Lastly, dividing at $y = T/2$ we find that

$$\begin{aligned} J_2 \ll & e^{-(\pi/2)T}\left\{\int_0^{T/2} T^{\sigma+\varepsilon-1/2}(y+1)^{-\varepsilon-1/2}\log(y+1)dy \right. \\ & \left. + \int_{T/2}^{T}(T-y+1)^{\sigma+\varepsilon-1/2}T^{-\varepsilon-1/2}\log T dy\right\} \\ \ll & e^{-(\pi/2)T}T^{\sigma}\log T. \end{aligned}$$

Substituting these estimates into (5.7), we obtain

$$I_2(s) = O(T^{1/2}\log T). \tag{5.8}$$

In order to treat the second term $B_2(s)$ on the right-hand side of (3.9), we now assume RH (to the end of this section). Then from (3.6) we have

(5.9)
$$\begin{aligned}
M(s-\rho) &= M\left(\sigma - \frac{1}{2} + I(T-\gamma)\right)\\
&= -\sum_{\substack{\gamma' \\ |T-\gamma-\gamma'|<1}} \frac{1}{\sigma - 1 + i(T-\gamma-\gamma')} + O(\log(|T-\gamma|+1))\\
&\ll \sum_{\substack{\gamma' \\ |T-\gamma-\gamma'|<1}} 1 + O(\log(|T-\gamma|+1))\\
&\ll \log(|T-\gamma|+1)
\end{aligned}$$

(using (3.4) for the last inequality) for $1+\varepsilon \leq \sigma \leq 2+\varepsilon$, where γ' runs over all imaginary parts of non-trivial zeros of $\zeta(s)$ satisfying $|T-\gamma-\gamma'|<1$. By using Stirling's formula and (5.9), we have

$$\begin{aligned}
B_2(s) &\ll e^{(\pi/2)T}T^{1/2-\sigma}\\
&\quad \times \sum_{\rho} e^{-(\pi/2)(|\gamma|+|T-\gamma|)}(|T-\gamma|+1)^{\sigma-1}\log(|T-\gamma|+1)\\
&= e^{(\pi/2)T}T^{1/2-\sigma}\left(\sum_{\gamma<0} + \sum_{0<\gamma\leq T} + \sum_{\gamma>T}\right)\\
&= e^{(\pi/2)T}T^{1/2-\sigma}(C_1+C_2+C_3),
\end{aligned} \tag{5.10}$$

say. We can estimate C_j analogously to the case of J_j $(j=1,2,3)$. As for C_1, changing γ by $-\gamma$, we obtain

$$C_1 = e^{-(\pi/2)T}\sum_{\gamma>0} e^{-\pi\gamma}(T+\gamma+1)^{\sigma-1}\log(T+\gamma+1).$$

Divide the last sum into two parts corresponding to $0<\gamma\leq T$ and $\gamma>T$, respectively. Using partial summation and (2.2), we find that the first sum is $O(T^{\sigma-1}\log T)$ and the second sum is of exponential decay. Hence $C_1 \ll e^{-(\pi/2)T}T^{\sigma-1}\log T$. The treatment of C_3 is even simpler than that of J_3; by partial summation and (2.2) we have

$$\begin{aligned} C_3 &= e^{(\pi/2)T}\sum_{\gamma>T} e^{-\pi\gamma}(\gamma-T+1)^{\sigma-1}\log(T-\gamma+1) \\ &\ll e^{(\pi/2)T}\int_T^\infty \xi\log\xi\cdot e^{-\pi\xi}(\xi-T+1)^{\sigma-1}\log(\xi-T+1)d\xi \\ &\ll e^{-(\pi/2)T}T^\sigma(\log T)^2. \end{aligned}$$

Lastly, dividing C_2 into two parts corresponding to $0<\gamma\le T/2$ and $T/2<\gamma\le T$, and applying partial summation and (2.2), we obtain $C_2 \ll e^{-(\pi/2)T}T^\sigma(\log T)^2$. Therefore

$$B_2(s) = O(T^{1/2}(\log T)^2). \tag{5.11}$$

Now the only remaining term on the right-hand side of (3.9) is the first term $M(s-1)/(s-1)$. For the purpose of estimating this term suitably, we now specify T; we choose the same T as in (3.5), and put $s=\sigma+iT$. Then from (3.6) we obtain

$$M(s-1) = M(\sigma-1+iT) = O((\log T)^2). \tag{5.12}$$

From (5.5), (5.8), (5.11) and (5.12), the assertion of Lemma 5.1 follows.

In the above proof, the special choice of T is necessary only for obtaining (3.5). But (3.5) is required only when $\sigma-1$ is near $1/2$. In fact, if $\sigma\notin(3/2-\varepsilon, 3/2+\varepsilon)$, then just using (3.6) we obtain $M(s-1)\ll\log T$ for any T (under RH). Therefore we obtain

Lemma 5.2. (under RH) *The estimate* (5.4) *of Lemma 5.1 is valid for any T if $1+\varepsilon\le\sigma\le 3/2-\varepsilon$ or $3/2+\varepsilon\le\sigma\le 2+\varepsilon$.*

Remark 1. The estimate of Lemma 5.1 can be improved for $3/2+\varepsilon\le\sigma\le 2+\varepsilon$. In fact, from Lemma 5.2 we have

$$\Phi_2((3/2)+\varepsilon+iT) \ll T^{1/2}(\log T)^2$$

for any T. It is clear that $\Phi_2(2+\varepsilon+iT)=O(1)$. Moreover from Theorem 2.1 we know that $\Phi_2(s)$ is holomorphic (except for $s=2$) in the region $3/2+\varepsilon\le\sigma\le 2+\varepsilon$. Therefore by the Phragmén-Lindelöf convexity principle we obtain

$$\Phi_2(\sigma+iT) \ll \left(T^{1/2}(\log T)^2\right)^{2(2+\varepsilon-\sigma} \ll T^{2-\sigma+\varepsilon} \tag{5.13}$$

for any T when $3/2+\varepsilon\le\sigma\le 2+\varepsilon$ (under RH).

Now we complete the proof of Theorem 5.1. Choose T as in Lemma 5.1, and write (5.3) as

$$\mathcal{D}_{2,a}(X) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{2+\varepsilon-iT}^{2+\varepsilon+iT} \frac{\Gamma(s)}{\Gamma(s+a+1)}\Phi_2(s)X^{s+a}ds. \tag{5.14}$$

Deform the path into the oriented polygonal path joining $2+\varepsilon-iT$, $1+\varepsilon-iT$, $1+\varepsilon+iT$, and $2+\varepsilon+iT$. By using Lemma 5.1 and Stirling's formula we see that the integrand is $O(T^{-a-1/2}(\log T)^2X^{\sigma+a})$ on the horizontal segments, and hence the integrals on these segments vanish as $T\to\infty$. On the line $\Re s = 1+\varepsilon$, estimate (5.4) is valid for any T by Lemma 5.2. Hence, as $T\to\infty$, the integral on the line $\Re s = 1+\varepsilon$ is

$$\ll \int_{-\infty}^{\infty} T^{-a-1/2}(\log T)^2 X^{1+\varepsilon+a}dT \ll X^{1+\varepsilon+a}$$

if $a > 1/2$. Therefore we can shift the path of integration on the right-hand side of (5.3) to $\Re s = 1+\varepsilon$ if $a>1/2$. The relevant poles are at $s=2$ and $s=1+\rho$, the residues at which are

$$\frac{1}{\Gamma(3+a)}X^{2+a}, \qquad -\frac{\Gamma(1+\rho)}{\Gamma(2+\rho+a)}\cdot\frac{2n(\rho)}{\rho}\cdot X^{1+\rho+a},$$

respectively. The assertion of Theorem 5.1 now follows.

Remark 2. From the case $a=1$ of Theorem 5.1, with the aid of (2.6), we can deduce

$$\sum_{n\le X} G_2(n) = \frac{1}{2}X^2 - H(X) + O(X^{3/2+\varepsilon}) \tag{5.15}$$

(under RH) by the standard difference-operator argument. However this is weaker than Fujii's result (2.8).

6. The multiple case

So far we have mainly discussed the double Dirichlet series $\Phi_2(s)$, but it is possible to consider the multiple case in an analogous manner. Let

$$G_r(n) = \sum_{k_1+\cdots+k_r=n} \Lambda(k_1)\cdots\Lambda(k_r) \qquad (r\ge 2) \tag{6.1}$$

and

$$\Phi_r(s) = \sum_{n=1}^{\infty}\frac{G_r(n)}{n^s} = \sum_{k_1=1}^{\infty}\cdots\sum_{k_r=1}^{\infty}\frac{\Lambda(k_1)\cdots\Lambda(k_r)}{(k_1+\cdots+k_r)^s}. \tag{6.2}$$

Similarly to (2.6) we have $G_r(n) \le n^{r-1}(\log n)^r$, and hence the series (6.2) is absolutely convergent for $\Re s > r$. In this section we prove some analytic properties of $\Phi_r(s)$.

It is to be noted that in the paper [32] quoted in Section 1, Peter considered the more general series

$$S_r(s) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \frac{\Lambda(k_1)\cdots\Lambda(k_r)}{P(k_1,\ldots,k_r)^s}$$

for the purpose of evaluating

$$\sum_{P(k_1,\ldots,k_r)\le X} \Lambda(k_1)\cdots\Lambda(k_r). \tag{6.3}$$

The reason why Peter's method can treat $S_r(s)$ is that his method is based on the idea of Sargos [33], [34], which can be applied to the case when the associated single series has infinitely many poles. By using the Tauberian theorem of Ikehara, Peter proved a certain asymptotic formula for (6.3).

The first purpose of this section is to show the following:

Theorem 6.1. (under RH) *The function $\Phi_r(s)$ $(r \ge 2)$ can be continued meromorphically to $\Re s > r-1$, and holoporphic there except for the simple poles at $s = r$ and $s = r-1+\rho$ for all non-trivial zeros ρ of $\zeta(s)$. The residues at $s = r$ and $s = r-1+\rho$ are*

$$\frac{1}{(r-1)!}, \qquad -\frac{r\cdot n(\rho)}{\rho(1+\rho)\cdots(r-2+\rho)},$$

respectively.

We prove this theorem by induction on r. When $r = 2$, this theorem is exactly Theorem 2.1. Assume that the theorem is true for $r-1$. Applying (1.4) to (6.2), we obtain

$$\Phi_r(s) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} \Phi_{r-1}(s-z)M(z)dz \tag{6.4}$$

for $\Re s > r$, where $1 < c < \Re s - (r-1)$. Shift the path of integration to $\Re z = -\varepsilon$. By using the same T as in (3.5), we can show that this shifting is possible. (Note that in the strip $-\varepsilon \le \Re z \le c$ the factor

$\Phi_{r-1}(s-z)$ is in the domain of its absolute convergence, hence is $O(1)$.) The result is that

$$
(6.5)\quad \Phi_r(s)=\frac{\Phi_{r-1}(s-1)}{s-1}-\sum_{\rho}\frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)}\Phi_{r-1}(s-\rho)
$$
$$
-\Phi_{r-1}(s)\log 2\pi+\frac{1}{2\pi i}\int_{(-\varepsilon)}\frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}\Phi_{r-1}(s-z)M(z)dz.
$$

Under the induction assumption, this expression gives the continuation of $\Phi_r(s)$ to $\Re s>r-1$. Moreover, the residues of $\Phi_{r-1}(s-1)/(s-1)$ at $s=r$, $s=r-1+\rho$ are

$$
\frac{1}{(r-1)!},\qquad -\frac{(r-1)n(\rho)}{\rho(1+\rho)\cdots(r-2+\rho)},
$$

respectively, while the residue of

$$
B_r(s)=-\sum_{\rho}\frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)}\Phi_{r-1}(s-\rho)
$$

at $s=r-1+\rho$ is

$$
-n(\rho)\cdot\frac{\Gamma(r-1)\Gamma(\rho)}{\Gamma(r-1+\rho)}\cdot\frac{1}{(r-2)!}=-\frac{n(\rho)}{\rho(1+\rho)\cdots(r-2+\rho)}.
$$

Hence the assertion of Theorem 6.1 follows.

The function $\Phi_{r-1}(s-\rho)$ is singular at $s=r-2+\rho+\rho'$ for any non-trivial zero ρ'. Hence, in view of Lemma 4.1, it is natural to raise the following:

Conjecture 6.1. *The line $\Re s=r-1$ is the natural boundary of $\Phi_r(s)$.*

In fact, under a certain assumption, we can show that

$$
(6.6)\qquad \Phi_r(s)\sim\frac{r(r-1)}{\Gamma(r-1+i\kappa)}n(\rho_1^0)n(\rho_2^0)\Gamma(\rho_1^0)\Gamma(\rho_2^0)\frac{1}{s-(r-1+i\kappa)}
$$

as $s\to r-1+i\kappa$ for any $\kappa=\gamma_1^0+\gamma_2^0\in\mathcal{K}$. This implies, as in the proof of Theorem 2.2, that Conjecture 6.1 is true.

When $r=2$, (6.6) is nothing but (4.11), which has been shown under RH and (B).

We prove (6.6) for general r by induction. When $s \to r-1+i\kappa$, we have

$$\begin{aligned}\text{(6.7)}\qquad \frac{\Phi_{r-1}(s-1)}{s-1} &\sim \frac{(r-1)(r-2)}{\Gamma(r-2+i\kappa)} n(\rho_1^0)n(\rho_2^0)\Gamma(\rho_1^0)\Gamma(\rho_2^0) \\ &\quad \times \frac{1}{r-2+i\kappa}\frac{1}{(s-1)-(r-2+i\kappa)} \\ &= \frac{(r-1)(r-2)}{\Gamma(r-1+i\kappa)} n(\rho_1^0)n(\rho_2^0)\Gamma(\rho_1^0)\Gamma(\rho_2^0)\frac{1}{s-(r-1+i\kappa)}\end{aligned}$$

by induction assumption. Next, we divide $B_r(s)$ as

$$\begin{aligned}\text{(6.8)}\qquad B_r(s) &= -n(\rho_1^0)\frac{\Gamma(s-\rho_1^0)\Gamma(\rho_1^0)}{\Gamma(s)}\Phi_{r-1}(s-\rho_1^0) \\ &\quad - n(\rho_2^0)\frac{\Gamma(s-\rho_2^0)\Gamma(\rho_2^0)}{\Gamma(s)}\Phi_{r-1}(s-\rho_2^0) \\ &\quad - \sum_{\rho\neq\rho_1^0,\rho_2^0}\frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)}\Phi_{r-1}(s-\rho) \\ &= B_{r1}(s)+B_{r2}(s)+B_{r3}(s),\end{aligned}$$

say. The factor $\Phi_{r-1}(s-\rho_1^0)$ has a pole at $s=r-1+i\kappa=r-2+\rho_1^0+\rho_2^0$, whose residue is given by Theorem 6.1. Therefore

$$\begin{aligned}\text{(6.9)}\qquad B_{r1}(s) &\sim -n(\rho_1^0)\frac{\Gamma(r-2+\rho_2^0)\Gamma(\rho_1^0)}{\Gamma(r-1+i\kappa)} \\ &\quad \times \left(-\frac{(r-1)n(\rho_2^0)}{\rho_2^0(1+\rho_2^0)\cdots(r-3+\rho_2^0)}\right)\frac{1}{s-(r-1+i\kappa)} \\ &= \frac{r-1}{\Gamma(r-1+i\kappa)}n(\rho_1^0)n(\rho_2^0)\Gamma(\rho_1^0)\Gamma(\rho_2^0)\frac{1}{s-(r-1+i\kappa)}\end{aligned}$$

as $s \to r-1+i\kappa$. The asymptotic behaviour of $B_{r2}(s)$ when $s \to r-1+i\kappa$ is exactly the same. Therefore, if we assume

$(\mathrm{C})_r$ The sum $B_{r3}(s)$ remains finite when $s \to r-1+i\kappa$,

then we have

$$\text{(6.10)}\qquad B_r(s) \sim \frac{2(r-1)}{\Gamma(r-1+i\kappa)}n(\rho_1^0)n(\rho_2^0)\Gamma(\rho_1^0)\Gamma(\rho_2^0)\frac{1}{s-(r-1+i\kappa)}$$

as $s \to r-1+i\kappa$. From (6.5), (6.7) and (6.10), we obtain (6.6), which implies the following:

Theorem 6.2. (under RH) *If we assume that* (B) *and* $(\mathrm{C})_k$ $(k \le r)$ *are true, then Conjecture 6.1 is true.*

References

[1] S. Akiyama, S. Egami and Y. Tanigawa, Analytic continuation of multiple zeta functions and their values at non-positive integers, Acta Arith. **98** (2001), 107-116.

[2] S. Akiyama and H. Ishikawa, On analytic continuation of multiple L-functions and related zeta-functions, in "Analytic Number Theory", C. Jia and K. Matsumoto (eds.), Dev. Math. Vol.6, Kluwer, 2002, pp.1-16.

[3] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, Nagoya Math. J. **153** (1999), 189-209.

[4] T. Arakawa and M. Kaneko, On multiple L-values, J. Math. Soc. Japan **56** (2004), 967-991.

[5] E. W. Barnes, The theory of the double gamma function, Philos. Trans. Roy. Soc. (A) **196** (1901), 265-387.

[6] E. W. Barnes, On the theory of the multiple gamma function, Trans. Cambridge Philos. Soc. **19** (1904), 374-425.

[7] G. Bhowmik, D. Essouabri and B. Lichtin, Meromorphic continuation of multivariable Euler products, Forum Math., to appear.

[8] R. de la Bretèche, Estimation de sommes multiples de fonctions arithmétiques, Compositio Math. **128** (2001), 261-298.

[9] G. Dahlquist, On the analytic continuation of Eulerian products, Ark. Mat. **1** (1952), 533-555.

[10] S. Egami, Some curious Dirichlet series, Sūrikaisekikenkyūsho Kōkyūroku **1091**, RIMS Kyoto Univ. (1999), 172-174.

[11] D. Essouabri, Singularités des séries de Dirichlet associées à des polynômes de plusieurs variables et applications à la théorie analytique des nombres, Thése, Univ. Henri Poincaré - Nancy I, 1995.

[12] D. Essouabri, Singularités des séries de Dirichlet associées à des polynômes de plusieurs variables et applications en théorie analytique des nombres, Ann. Inst. Fourier **47** (1997), 429-483.

[13] T. Estermann, On certain functions represented by Dirichlet series, Proc. London Math. Soc. (2)**27** (1928), 435-448.

[14] T. Estermann, On a problem of analytic continuation, ibid., 471-482.

[15] A. Fujii, An additive problem of prime numbers, Acta Arith. **58** (1991), 173-179.

[16] A. Fujii, An additive problem of prime numbers II, Proc. Japan Acad. **67A** (1991), 248-252.

[17] A. Fujii, An additive problem of prime numbers III, ibid., 278-283.

[18] A. Fujii, An additive theory of the zeros of the Riemann zeta function, Comment. Math. Univ. St. Pauli **45** (1996), 49-116.

[19] A. B. Goncharov, Multiple polylogarithms, cyclotomy and modular complexes, Math. Res. Letters **5** (1998), 497-516.

[20] G. H. Hardy and J. E. Littlewood, Some problems of "partitio numerorum" (V): A further contribution to the study of Goldbach's problem, Proc. London Math. Soc. (2)**22** (1924), 46-56.

[21] H. Ishikawa, On analytic properties of a multiple L-function, in "Analytic Extension Formulas and their Applications", S. Saitoh et al. (eds.), Soc. Anal. Appl. Comput. Vol.9, Kluwer, 2001, pp.105-122.

[22] H. Ishikawa, A multiple character sum and a multiple L-function, Arch. Math. **79** (2002), 439-448.

[23] J. C. Kluyver, Benaderingsformules betreffende de priemgetallen beneden eene gegeven grens, Koninglijke Akad. Wet. Amsterdam, Versl. Gew. Verg. Wis- en Natuur. Afd. **8** (1900), 672-682.

[24] N. Kurokawa, On the meromorphy of Euler products (I), Proc. London Math. Soc. (3)**53** (1986), 1-47; (II), ibid., 209-236.

[25] E. Landau, Üder die Multiplikation Dirichlet'scher Reihen, Rend. Circ. Mat. Palermo **24** (1907), 81-160.

[26] E. Landau and A. Walfisz, Üder die Nichtfortsetzbarkeit einiger durch Dirichletsche Reihen definierter Funktionen, ibid. **44** (1920), 82-86.

[27] K. Matsumoto, Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series, Nagoya Math. J. **172** (2003), 59-102.

[28] K. Matsumoto and Y. Tanigawa, The analytic continuation and the order estimate of multiple Dirichlet series, J. Théorie des Nombres de Bordeaux **15** (2003), 267-274.

[29] H. Mellin, Eine Formel für den Logarithmus transcendenter Funktionen von endlichem Geschlecht, Acta Soc. Sci. Fenn. **29** (1900), no.4.

[30] H. Mellin, Die Dirichlet'schen Reihen, die zahlentheoretischen Funktionen und die unendlichen Produkte von endlichem Geschlecht, Acta Math. **28** (1904), 37-64.

[31] H. L. Montgomery and R. C. Vaughan, Error terms in additive prime number theory, Quart. J. Math. Oxford (2)**24** (1973), 207-216.

[32] M. Peter, Dirichlet series associated with polynomials, Acta Arith. **84** (1998), 245-278.

[33] P. Sargos, Prolongement méromorphe des séries de Dirichlet associées à des fractions rationnelles de plusieurs variables, Ann. Inst. Fourier **34**(3) (1984), 83-123.

[34] P. Sargos, Croissance de certaines séries de Dirichlet et applications, J. Reine Angew. Math. **367** (1986), 139-154.

[35] Y. Tanigawa and W. Zhai, Dirichlet series associated with polynomials and applications, J. Number Theory, to appear.

[36] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Oxford, 1951.

[37] J. Zhao, Analytic continuation of multiple zeta-functions, Proc. Amer. Math. Soc. **128** (2000), 1275-1283.

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