ZETA-FUNCTIONS OF ROOT SYSTEMS

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Abstract. In this paper, we introduce multi-variable zeta-functions of roots, and prove the analytic continuation of them. For the root systems associated with Lie algebras, these functions are also called Witten zeta-functions associated with Lie algebras which can be regarded as several variable generalizations of Witten zeta-functions defined by Zagier. In the case of type A_r, we have already studied some analytic properties in our previous paper. In the present paper, we prove certain functional relations among these functions of types A_r ($r = 1, 2, 3$) which include what is called Witten's volume formulas. Moreover we mention some structural background of the theory of functional relations in terms of Weyl groups.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

Let $\mathfrak{g}$ be any semisimple Lie algebra. In [25], Zagier defined the Witten zeta-function associated with $\mathfrak{g}$ by

$$\zeta_W(s; \mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s} \quad (s \in \mathbb{C}), \tag{1.1}$$

where φ runs over all finite dimensional irreducible representations of $\mathfrak{g}$. Using Witten's result [24], Zagier noted that

$$\zeta_W(2k; \mathfrak{g}) \in \mathbb{Q}\, \pi^{2kl} \tag{1.2}$$

for $k \in \mathbb{N}$, where l is the number of positive roots of $\mathfrak{g}$, which is called *Witten's volume formula* (see [25, Section 7]). In particular, $\zeta_W(s; \mathfrak{sl}(3))$ was already studied by Tornheim [19] and Mordell [15] in the 1950's. The above result for $\zeta_W(s; \mathfrak{sl}(3))$ was showed by Mordell and its explicit formula was given by Subbarao and Sitaramachandrarao in [18]. Recently Gunnells and Sczech [3] evaluated $\zeta_W(2k; \mathfrak{g})$ for $k \in \mathbb{N}$ by means of the generalized higher-dimensional Dedekind sums, and gave certain evaluation formulas for $\zeta_W(2k; \mathfrak{sl}(3))$ and $\zeta_W(2k; \mathfrak{sl}(4))$ for $k \in \mathbb{N}$ in terms of $\zeta(2j)$ for $j \in \mathbb{N}$, where $\zeta(s)$ is the Riemann zeta-function.

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In [10], the second-named author defined multi-variable Witten zeta-functions by

$$\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3)=\sum_{m,n=1}^{\infty} m^{-s_1}n^{-s_2}(m+n)^{-s_3}, \tag{1.3}$$

$$\zeta_{\mathfrak{so}(5)}(s_1,s_2,s_3,s_4)=\sum_{m,n=1}^{\infty} m^{-s_1}n^{-s_2}(m+n)^{-s_3}(m+2n)^{-s_4}. \tag{1.4}$$

Note that $\zeta_W(s;\mathfrak{sl}(3))=2^s\zeta_{\mathfrak{sl}(3)}(s,s,s)$ and $\zeta_W(s;\mathfrak{so}(5))=6^s\zeta_{\mathfrak{so}(5)}(s,s,s,s)$. The second-named author proved the analytic continuation of them by the method using the Mellin-Barnes integral formula (see [7, 8, 9]) and calculated their possible singularities. The function $\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3)$ is also called the Tornheim double sum or the Mordell-Tornheim double zeta-function denoted by $T(s_1,s_2,s_3)$ or $\zeta_{MT,2}(s_1,s_2,s_3)$, because special values $\zeta_{\mathfrak{sl}(3)}(a_1,a_2,a_3)$ for $a_j\in\mathbb{N}$ $(j=1,2,3)$ were studied by Tornheim in [19].

Recently the third-named author [23] has proved certain functional relations between $\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3)$ and $\zeta(s)$. These can be regarded as certain continuous relations including the known formulas between Tornheim's double sums and values of $\zeta(s)$ at positive integers obtained in [15, 19]. For example, we proved that

$$2\zeta_{\mathfrak{sl}(3)}(2,s,2)+\zeta_{\mathfrak{sl}(3)}(2,2,s)=4\zeta(2)\zeta(s+2)-6\zeta(s+4) \tag{1.5}$$

for any $s\in\mathbb{C}$ except for singularities of each function on the both sides. In particular, putting $s=2$ in (1.5), we obtain Witten's volume formula $\zeta_{\mathfrak{sl}(3)}(2,2,2)=\pi^6/2835$ given by Mordell (see [15]).

More recently, in [12], the second-named and the third-named authors defined multi-variable Witten zeta-functions $\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s})$ $(r\in\mathbb{N})$ associated with $\mathfrak{sl}(r+1)$ and proved the analytic continuation of them. Furthermore, for $r\le 3$, true singularities of them were determined, certain functional relations for them were given and new evaluation formulas for their special values were introduced. However, in that paper, we were unable to prove functional relations which can be regarded as continuous relations including Witten's volume formulas (1.2).

It seems possible to generalize these results for $\mathfrak{sl}(r+1)$ in [12] to those for any general semisimple Lie algebra $\mathfrak{g}$. Indeed, in [4], we define the Witten zeta-function associated with $\mathfrak{g}$ denoted by $\zeta_r(\mathbf{s};\mathfrak{g})$ or $\zeta_r(\mathbf{s};X_r)$ if $\mathfrak{g}$ is of type X_r $(X=A,B,C,D,E,F,G)$, for $\mathbf{s}\in\mathbb{C}^n$, where n is the number of positive roots of $\mathfrak{g}$. For example, $\zeta_2(\mathbf{s};A_2)$ and $\zeta_2(\mathbf{s};B_2)$ coincide with $\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3)$ defined by (1.3) and $\zeta_{\mathfrak{so}(5)}(s_1,s_2,s_3,s_4)$ defined by (1.4), respectively, except for the order of variables. More generally we further define zeta-functions of roots. Considering their properties, we study the analytic continuation of $\zeta_r(\mathbf{s};\mathfrak{g})$, and certain recursive structures among them. In [5], we prove certain functional relations among them, in particular, of types B_r and C_r $(r\le 3)$, which include Witten's volume formulas. Furthermore, in [6], we will study $\zeta_2(\mathbf{s};G_2)$ associated with the exceptional Lie algebra of type G_2. Indeed, we will construct a certain functional relation for $\zeta_2(\mathbf{s};G_2)$ which includes Witten's volume formula.

In the present paper, we give an overview of our previous result [12] and our recent results [4, 5], and prove some related facts. In Section 2, we introduce the definition of zeta-functions of roots which include the multi-variable Witten zeta-functions. We further study certain recursive relations for them. In Section 3, we study some structural background of functional relations for these functions. As its concrete examples, in Section 4, we explicitly give certain functional relations for $\zeta_r(\mathbf{s}; A_r)$ $(r \leq 3)$ which can be regarded as continuous relations including Witten's volume formula. For example, we can give a functional relation including

$$\begin{aligned} (1.6) \qquad & \zeta_3(2,2,2,2,2,2;A_3) \\ & = \sum_{l,m,n=1}^{\infty} \frac{1}{l^2m^2n^2(l+m)^2(m+n)^2(l+m+n)^2} = \frac{23}{2554051500}\pi^{12} \end{aligned}$$

(see [3]) as a special value-relation. As mentioned above, we were unable to treat Witten's volume formulas in our previous paper ([12]), because we were only able to consider some limited types of relations for $\zeta_3(s_1,s_2,s_3,s_4,0,s_5;A_3)$ and $\zeta_3(s_1,s_2,s_3,0,s_4,s_5;A_3)$, that is, the case one-variable is equal to 0. In order to remove this limitation, we will study certain multiple polylogarithms of type A_3 (see (4.18) and (4.19)). By making use of them, we will be able to construct some functional relations including Witten's volume formula. This method can be applied to zeta-functions of root systems of other types (see [5, 6]).

2. Zeta functions of root systems

In this section, we give an overview of [12] and [4]. First, following [4], we introduce the definition of zeta-functions of root systems associated with semisimple Lie algebras, and more generally, zeta-functions of roots.

We quote some notation and results about semisimple Lie algebras from [2, 17]. Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank r. We denote by $\Delta = \Delta(\mathfrak{g})$ the set of all roots of $\mathfrak{g}$, by $\Delta_+ = \Delta_+(\mathfrak{g})$ (resp. $\Delta_- = \Delta_-(\mathfrak{g})$) the set of all positive roots (resp. negative roots) of $\mathfrak{g}$, and by $\Psi = \Psi(\mathfrak{g}) = \{\alpha_1,\ldots,\alpha_r\}$ the fundamental system of Δ. For any $\alpha \in \Delta$, we denote by $\alpha^\vee$ the associated coroot. Let $\lambda_1,\ldots,\lambda_r$ be the fundamental weights satisfying $\langle \alpha_i^\vee, \lambda_j\rangle = \lambda_j(\alpha_i^\vee) = \delta_{ij}$ (Kronecker's delta). Then we see that any dominant weight can be written as

$$\lambda = n_1\lambda_1 + \cdots + n_r\lambda_r \qquad (n_1,\ldots,n_r \in \mathbb{N}_0). \tag{2.1}$$

In particular, the lowest strongly dominant form is $\rho = \lambda_1 + \cdots + \lambda_r$.

Let d_λ be the dimension of the representation space corresponding to the dominant weight λ. Then Weyl's dimension formula (see [17, Section 3.8]) gives that

$$d_\lambda = \prod_{\alpha\in\Delta_+} \frac{\langle \alpha^\vee, \lambda+\rho\rangle}{\langle \alpha^\vee, \rho\rangle} = \prod_{\alpha\in\Delta_+} \frac{\langle \alpha^\vee, (n_1+1)\lambda_1 + \cdots + (n_r+1)\lambda_r\rangle}{\langle \alpha^\vee, \lambda_1+\cdots+\lambda_r\rangle}.$$

Putting $m_j = n_j + 1$, it follows from (1.1) that

$$\begin{aligned} (2.2) \qquad \zeta_W(s;\mathfrak{g}) &= \sum_{\lambda} \prod_{\alpha\in\Delta_+} \left(\frac{\langle \alpha^\vee, m_1\lambda_1+\cdots+m_r\lambda_r\rangle}{\langle \alpha^\vee, \lambda_1+\cdots+\lambda_r\rangle} \right)^{-s} \\ &= K(\mathfrak{g})^s \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, m_1\lambda_1+\cdots+m_r\lambda_r\rangle^{-s}, \end{aligned}$$

where the sum on the second member of the above runs over all dominant weights of the form (2.1), and

$$K(\mathfrak{g}) = \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r \rangle. \tag{2.3}$$

Let Δ^* be a subset of $\Delta_+ = \Delta_+(\mathfrak{g})$ which satisfies the condition that for any λ_j $(1 \le j \le r)$, there exists an element $\alpha \in \Delta^*$ such that $\langle \alpha^\vee, \lambda_j \rangle \neq 0$. Then we define the zeta-function of Δ^* by

$$\zeta_r(\mathbf{s}; \Delta^*) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\alpha\in\Delta^*} \langle \alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r \rangle^{-s_\alpha}, \tag{2.4}$$

where $\mathbf{s} = \mathbf{s}(\Delta^*) = (s_\alpha)_{\alpha\in\Delta^*} \in \mathbb{C}^{n^*}$ with $n^* = |\Delta^*|$. In particular when $\Delta^* = \Delta_+(\mathfrak{g})$, we denote $\zeta_r(\mathbf{s}; \Delta^+(\mathfrak{g}))$ by $\zeta_r(\mathbf{s}; \mathfrak{g})$ or $\zeta_r(\mathbf{s}; X_r)$ if $\mathfrak{g}$ is of type X_r, and call it the multi-variable Witten zeta-function associated with $\mathfrak{g}$ or X_r. Namely we have

$$\zeta_r(\mathbf{s}; \mathfrak{g}) = \zeta_r(\mathbf{s}; X_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r \rangle^{-s_\alpha}, \tag{2.5}$$

where $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^n$ and $n = |\Delta_+|$ is the number of positive roots of $\mathfrak{g}$. Note that $K(\mathfrak{g})^s \zeta_r(s, \ldots, s; \mathfrak{g})$ coincides with the one-variable Witten zeta-function $\zeta_W(s; \mathfrak{g})$ defined by (1.1). The series (2.4) and (2.5) can be continued meromorphically to the whole space (see [10, Theorem 3]). In the case $X_r = A_r$, we see that

$$\zeta_r(\mathbf{s}; A_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{1\le i<j\le r+1} (m_i + \cdots + m_{j-1})^{-s_{ij}}, \tag{2.6}$$

where $\mathbf{s} = \mathbf{s}(A_r) = (s_{ij}) \in \mathbb{C}^{r(r+1)/2}$ (see [4, (2.3)] or [12, (1.5)]). In Section 4, we prove some functional relations among $\zeta_r(\mathbf{s}; A_r)$ $(r \le 3)$ which can be regarded as continuous relations including Witten's volume formulas for $\mathfrak{sl}(4)$. Moreover, in [5], we give some functional relations of types B_r and C_r for $r \le 3$.

Now we consider $\mathfrak{g} = \mathfrak{sl}(r+1)$. Let ε_j be the j-th coordinate function. It is well-known that positive roots are

$$\Delta_+ = \Delta_+(A_r) = \left\{ \varepsilon_i - \varepsilon_j = \sum_{i\le k<j} \alpha_k \;\middle|\; 1 \le i < j \le r+1 \right\}. \tag{2.7}$$

(see the list at the end of [2]). We further define

$$\Delta^*(A_r) = \{\varepsilon_i - \varepsilon_j \mid 2 \le i < j \le r+1\}; \tag{2.8}$$

$$\Delta^*_h(A_r) = \{\varepsilon_1 - \varepsilon_j \mid 2 \le j \le h\} \cup \Delta^*(A_r) \tag{2.9}$$

$(2 \le h \le r+1)$ which are subsets of $\Delta_+(A_r)$. For example, putting $r = 3$, we have the relation $\Delta^*(A_3) \subset \Delta^*_2(A_3) \subset \Delta^*_3(A_3) \subset \Delta^*_4(A_3) = \Delta_+(A_3)$. We can see that $\mathbf{s}(\Delta^*_h(A_3)) = (s_{12}, \ldots, s_{1h}, s_{23}, s_{24}, s_{34})$ $(h = 2, 3, 4)$.

We recall the Mellin-Barnes integral formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi\sqrt{-1}} \int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \lambda^z dz, \tag{2.10}$$

where s, λ are complex numbers with $\Re s > 0$, $|\arg \lambda| < \pi$, $\lambda \neq 0$, c is real with $-\Re s < c < 0$, and the path (c) of integration is the vertical line $\Re z = c$. Using this formula, we have

$$\begin{aligned}(2.11)\qquad & \zeta_3(s_{12},\ldots,s_{1h},s_{23},s_{24},s_{34};\Delta_h^*(A_3)) \\ &= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}\prod_{2\le i<j\le 4}(m_i+\cdots+m_{j-1})^{-s_{ij}} \\ &\quad \times \prod_{2\le j<h}(m_1+\cdots+m_{j-1})^{-s_{1j}}\frac{1}{2\pi\sqrt{-1}}\int_{(c_h)}\frac{\Gamma(s_{1h}+z_h)\Gamma(-z_h)}{\Gamma(s_{1h})} \\ &\quad \times (m_1+\cdots+m_{h-2})^{-s_{1h}-z_h}m_{h-1}^{z_h}dz_h \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{(c_h)}\frac{\Gamma(s_{1h}+z_h)\Gamma(-z_h)}{\Gamma(s_{1h})}\zeta_3(\mathbf{s}^*(A_3,z_h);\Delta_{h-1}^*(A_3))dz_h\end{aligned}$$

for $h = 3, 4$, where

$$\begin{aligned}\mathbf{s}^*(A_3,z_3) &= (s_{12}+s_{13}+z_3, s_{23}-z_3, s_{24}, s_{34}); \\ \mathbf{s}^*(A_3,z_4) &= (s_{12}, s_{13}+s_{14}+z_4, s_{23}, s_{24}, s_{34}-z_4).\end{aligned}$$

Hence we find the recursive structure which we denote by

$$(2.12)\qquad \zeta_3(\cdot\,;A_3) = \zeta_3(\cdot\,;\Delta_4^*(A_3)) \to \zeta_3(\cdot\,;\Delta_3^*(A_3)) \to \zeta_3(\cdot\,;\Delta_2^*(A_3)).$$

Note that

$$(2.13)\qquad \zeta_3(\mathbf{s}^*(A_3,z_3);\Delta_2^*(A_3)) = \zeta_2(s_{23}-z_3, s_{24}, s_{34};\Delta^*(A_3))\zeta(s_{12}+s_{13}+z_3).$$

Substituting (2.13) into (2.11), we have

$$\begin{aligned}(2.14)\qquad \zeta_3(\mathbf{s};\Delta_3^*(A_3)) &= \frac{1}{2\pi\sqrt{-1}}\int_{(c_3)}\frac{\Gamma(s_{13}+z_3)\Gamma(-z_3)}{\Gamma(s_{13})} \\ &\qquad \times \zeta_2(s_{23}-z_3, s_{24}, s_{34};\Delta^*(A_3))\zeta(s_{12}+s_{13}+z_3)dz_3.\end{aligned}$$

Therefore we can rewrite (2.12) as

$$(2.15)\qquad \zeta_3(\cdot\,;A_3) = \zeta_3(\cdot\,;\Delta_4^*(A_3)) \to \zeta_3(\cdot\,;\Delta_3^*(A_3)) \to \zeta_2(\cdot\,;\Delta^*(A_3)),$$

by neglecting the Riemann zeta factor. Renaming ε_i as ε_{i-1} $(2 \le i \le r+1)$, we find that $\zeta_2(\cdot\,;\Delta^*(A_3))$ is equal to $\zeta_2(\cdot\,;A_2)$. Consequently we can write (2.15) as

$$(2.16)\qquad \zeta_3(\cdot\,;A_3) \to \zeta_3(\cdot\,;\Delta_3^*(A_3)) \to \zeta_2(\cdot\,;A_2).$$

The above argument can be applied to A_r and A_{r-1} for any $r \ge 2$. Hence we can obtain the following result which can be viewed as a refinement of [12, Theorem 2.2].

Theorem 2.1 ([4, Theorem 1]). *Between $\zeta_r(\cdot\,;A_r)$ and $\zeta_{r-1}(\cdot\,;A_{r-1})$ (for each $r \ge 2$) there is the recursive relation like (2.16), which can be summarizingly written as*

$$(2.17)\qquad \zeta_r(\cdot\,;A_r) \to \zeta_{r-1}(\cdot\,;A_{r-1}) \to \cdots \to \zeta_2(\cdot\,;A_2) \to \zeta_1(\cdot\,;A_1) = \zeta.$$

For zeta-functions of other types, we can also give certain recursive relations similar to (2.17). For the details, see [4, Sections 4 and 5].

3. Structural background of functional relations

Hereafter we discuss the topic on functional relations for zeta-functions of root systems. First, in this section, we study the structural background in a general framework. For the details, see [5].

As we mentioned in Section 1, we have already obtained some functional relations for $\zeta_2(s_1, s_2, s_3; A_2)$ like (1.5) (For the details, see [23], also [13, 14]). Recently, in [16], Nakamura gave an alternative proof of functional relations given in [23]. Inspired by Nakamura's method, we aim to generalize these functional relations of type A_2 to those for zeta-functions of general root systems.

We use the same notation as in Section 2. Let $\mathfrak{g}$ be a complex simple Lie algebra of type X_r and V be the vector space generated by the roots of $\mathfrak{g}$. Let W be the Weyl group acting on V. We denote by P and P_+ the lattice of weights and the set of dominant weights, respectively. Let

$$C := \{v \in V \mid \langle v, \alpha_i^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta\} \tag{3.1}$$

be a fundamental domain of W, which we call the fundamental Weyl chamber. Let

$$H_\Delta = V \setminus \bigcup_{w \in W} w\,\mathring{C},$$

where by $\mathring{C}$ we denote the interior of C, and

$$\begin{aligned} S\left((s_\alpha)_{\alpha\in\Delta_+}; X_r\right) &:= \sum_{\lambda \in P \setminus H_\Delta} \prod_{\alpha \in \Delta_+} \langle \alpha^\vee, \lambda \rangle^{-s_\alpha} \\ &= \sum_{w \in W} \sum_{\lambda \in P_+} \prod_{\alpha \in \Delta_+} \langle \alpha^\vee, w(\lambda+\rho) \rangle^{-s_\alpha} \\ &= \sum_{w \in W} \sum_{\lambda \in P_+} \prod_{w\alpha \in \Delta_+} \langle \alpha^\vee, \lambda+\rho \rangle^{-s_{w\alpha}}, \end{aligned} \tag{3.2}$$

where s_α for $\alpha \in \Delta_-$ is defined by $s_\alpha = s_{-\alpha}$ and ρ is the lowest strongly dominant form. Let $\Delta_w = \Delta_+ \cap w^{-1}\,\Delta_-$. Divide the product on the right-hand side of (3.2) into two parts according to $\alpha \in \Delta_+$ or $\alpha \in \Delta_-$, and in the latter part replace α by $-\alpha$. Then we have

$$\begin{aligned} S\left((s_\alpha); X_r\right) &= \sum_{w \in W} \Big(\prod_{\alpha \in \Delta_w} (-1)^{-s_{w\alpha}} \Big) \sum_{\lambda \in P_+} \prod_{\alpha \in \Delta_+} \langle \alpha^\vee, \lambda+\rho \rangle^{-s_{w\alpha}} \\ &= \sum_{w \in W} \Big(\prod_{\alpha \in \Delta_w} (-1)^{-s_{w\alpha}} \Big) \zeta_r\left((s_{w\alpha}); X_r\right), \end{aligned} \tag{3.3}$$

which is a 'Weyl group symmetric' linear combination of our zeta-function.

Now we consider the Lerch-type series

$$L_s(x) = -\frac{\Gamma(s+1)}{(2\pi i)^s} \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{e^{2\pi i m x}}{m^s} \tag{3.4}$$

for $s \in \mathbb{C}$ with $\Re s > 1$, $x \in \mathbb{R}$ and $i = \sqrt{-1}$. Then we obtain the following general form of functional relations for zeta-functions of root systems (for the proof, see [5]).

Theorem 3.1.

$$
\begin{aligned}
(3.5) \qquad S\left((s_\alpha);X_r\right) &= \sum_{w\in W}\Big(\prod_{\alpha\in\Delta_w}(-1)^{-s_{w\alpha}}\Big)\zeta_r\left((s_{w\alpha});X_r\right)\\
&= (-1)^{|\Delta_+|}\sum_{\lambda\in P}\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{s_\alpha}}{\Gamma(s_\alpha+1)}\int_0^1 L_{s_\alpha}(x_\alpha)e^{-2\pi i\langle\alpha^\vee,\lambda\rangle x_\alpha}dx_\alpha\\
&= (-1)^{|\Delta_+|}\Big(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{s_\alpha}}{\Gamma(s_\alpha+1)}\Big)\underbrace{\int_0^1\cdots\int_0^1}_{|\Delta_+\setminus\Psi|\ \textit{times}}\Big(\prod_{\alpha\in\Delta_+\setminus\Psi}L_{s_\alpha}(x_\alpha)\Big)\\
&\qquad\times\Big(\prod_{i=1}^r L_{s_{\alpha_i}}\Big(-\sum_{\alpha\in\Delta_+\setminus\Psi}x_\alpha\langle\alpha^\vee,\lambda_i\rangle\Big)\Big)\prod_{\alpha\in\Delta_+\setminus\Psi}dx_\alpha
\end{aligned}
$$

holds for any s_α in a certain region.

Remark 3.2. When $s_\alpha\in\mathbb{N}$, we see that $L_{s_\alpha}(x_\alpha)=B_{s_\alpha}(\{x_\alpha\})$, where $\{x\}=x-[x]$ denotes the fractional part of x and $B_n(\,\cdot\,)$ is the Bernoulli polynomial (see [1]). When $X_r=A_2$, the idea of Theorem 3.1 is essentially included in Nakamura's proof of certain functional relations for $\zeta_2(s_1,s_2,s_3;A_2)$ ([16, Theorem 1.2]) which coincide with those given by the third-named author in [23]. For example, see (1.5).

Example 3.3. In the case of type $B_2(=C_2)$, we can verify that $\zeta_2(s_1,s_2,s_3,s_4;B_2)$ coincides with $\zeta_{\mathfrak{so}(5)}(s_2,s_3,s_4,s_1)$ (see (1.4) and (2.5)). As well as in [16], direct calculations on the right-hand side of (3.5) for $\mathfrak{g}=\mathfrak{so}(5)$ show that, for example,

$$
\begin{aligned}
(3.6) \qquad &\zeta_2(1,2,s,2;B_2)+\zeta_2(1,2,2,s;B_2)+\zeta_2(2,1,2,s;B_2)-\zeta_2(2,1,s,2;B_2)\\
&\qquad = 3\zeta(2)\zeta(s+3)-\frac{39}{8}\zeta(s+5);\\
(3.7) \qquad &\zeta_2(2,2,s,2;B_2)+\zeta_2(2,2,2,s;B_2)=\frac{3}{2}\zeta(2)\zeta(s+4)-\frac{39}{16}\zeta(s+6).
\end{aligned}
$$

Putting $s=2$ in (3.6) and (3.7), we have

$$
(3.8) \qquad \zeta_2(1,2,2,2;B_2)=\frac{3}{2}\zeta(2)\zeta(5)-\frac{39}{16}\zeta(7);
$$

$$
(3.9) \qquad \zeta_2(2,2,2,2;B_2)=\frac{3}{4}\zeta(2)\zeta(6)-\frac{39}{32}\zeta(8)=\frac{\pi^8}{302400}.
$$

Note that (3.8) has been obtained in [22] by a different method, and (3.9) is just equal to Witten's volume formula (1.2) for $\mathfrak{g}=\mathfrak{so}(5)$. However it seems to be hard to calculate (3.5) for $\zeta_r(\cdot;X_r)$ directly if r is large. Hence we need another technique. In [5], we introduced a certain technique using multiple polylogarithms (which may be called the 'polylogathm technique') to prove general formulas for zeta-functions of types B_2, B_3 and C_3 including (3.8) and (3.9). In Section 4, using this technique, we treat $\zeta_r(\cdot;A_r)$ and prove certain functional relations for them.

At the end of this section, we consider a generating function of values of $S((s_\alpha);X_r)$ at positive integers. Let $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{N}_0^{|\Delta_+|}$ and $\mathbf{t}=(t_\alpha)_{\alpha\in\Delta_+}\in\mathbb{C}^{|\Delta_+|}$ with $|t_\alpha|<2\pi$ for each α. Then we define

$$F(\mathbf{t};X_r):=(-1)^{|\Delta_+|}\sum_{\mathbf{k}}S(\mathbf{k};X_r)\prod_{\alpha\in\Delta_+}\left(\frac{t_\alpha}{2\pi i}\right)^{k_\alpha}, \tag{3.10}$$

where

$$S\left(\mathbf{k};X_r\right)=(-1)^{|\Delta_+|}\Bigg(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigg)\underbrace{\int_0^1\cdots\int_0^1}_{|\Delta_+\backslash\Psi|\text{ times}}\Bigg(\prod_{\alpha\in\Delta_+\backslash\Psi}B_{k_\alpha}(x_\alpha)\Bigg)\times$$
$$\Bigg(\prod_{i=1}^{r}B_{k_{\alpha_i}}\Big(\Big\{-\sum_{\alpha\in\Delta_+\backslash\Psi}x_\alpha\langle\alpha^\vee,\lambda_i\rangle\Big\}\Big)\Bigg)\prod_{\alpha\in\Delta_+\backslash\Psi}dx_\alpha$$

(see (3.5) and Remark 3.2). For example, we can explicitly calculate that

$$F(t_1,t_2,t_3,t_4;B_2)=\left(\prod_{j=1}^{4}\frac{t_j}{e^{t_j}-1}\right)\left(P(\mathbf{t})+Q(\mathbf{t})-R(\mathbf{t})\right), \tag{3.11}$$

where $t_1=t_{\alpha_1}$, $t_2=t_{\alpha_2}$, $t_3=t_{\alpha_1+\alpha_2}$, $t_4=t_{\alpha_1+2\alpha_2}$ and

$$P(\mathbf{t})=\frac{2(e^{t_1}-1)(1+e^{(t_2+t_3)/2})(e^{(t_2+t_3)/2}-e^{t_4})}{(2t_1+t_2-t_3)(t_2+t_3-2t_4)};$$
$$Q(\mathbf{t})=\frac{(e^{t_2}-1)(e^{t_1+t_4}-e^{t_3})}{(2t_1+t_2-t_3)(t_1-t_3+t_4)};\ R(\mathbf{t})=\frac{(e^{t_3}-1)(e^{t_1+t_2}-e^{t_4})}{(2t_1+t_2-t_3)(t_1+t_2-t_4)}.$$

Hence we can calculate the Taylor expansion as

$$\begin{aligned} F(t_1,t_2,t_3,t_4;B_2) &= 1+\frac{1}{2880}(4t_1t_2t_3^2+2t_1t_2t_4^2-4t_1t_2^2t_3+t_1t_2^2t_4-2t_1t_3t_4^2-t_1t_3^2t_4\\ &\qquad +4t_1^2t_2t_3-2t_1^2t_2t_4-2t_1^2t_3t_4-4t_2t_3t_4^2-4t_2t_3^2t_4-4t_2^2t_3t_4)\\ &\quad +\frac{1}{241920}(3t_1t_2t_3^2t_4^2-3t_1t_2^2t_3t_4^2-3t_1^2t_2t_3^2t_4-3t_1^2t_2^2t_3t_4+8t_1^2t_2^2t_3^2+2t_1^2t_2^2t_4^2\\ &\qquad +2t_1^2t_3^2t_4^2+8t_2^2t_3^2t_4^2)\\ &\quad +\frac{1}{9676800}t_1^2t_2^2t_3^2t_4^2+\cdots \end{aligned} \tag{3.12}$$

Comparing the coefficients of $t_1^2t_2^2t_3^2t_4^2$ in (3.10) and in (3.12), we can also obtain (3.9), namely

$$\zeta(2,2,2,2;B_2)=(-1)^4\frac{(2\pi i)^8}{2!\cdot 2^2}\frac{1}{9676800}=\frac{\pi^8}{302400}.$$

However it is also hard to evaluate the zeta-function of type X_r by this method when r is large.

4. Functional relations for $\zeta_3(\mathbf{s};A_3)$

In this section, we consider zeta-functions of type A_3. Though we have already considered certain special cases of this type in our previous paper ([12]), we will study this type more generally and in detail, by considering a certain multiple polylogarithm of this type.

From (2.5), we have

$$\zeta_2(s_{12},s_{13},s_{23};A_2)=\sum_{m_1,m_2=1}^{\infty}\frac{1}{m_1^{s_{12}}(m_1+m_2)^{s_{13}}m_2^{s_{23}}};$$

$$\zeta_3(s_{12},s_{13},s_{14},s_{23},s_{24},s_{34};A_3)$$
$$=\sum_{m_1,m_2,m_3=1}^{\infty}\frac{1}{m_1^{s_{12}}(m_1+m_2)^{s_{13}}(m_1+m_2+m_3)^{s_{14}}m_2^{s_{23}}(m_2+m_3)^{s_{24}}m_3^{s_{34}}}.$$

In our previous paper [12], we proved a closed form of functional relation for

$$(-1)^a\zeta_3(s_1,s_3,b,s_2,0,a;A_3)+(-1)^b\zeta_3(s_1,s_3,a,s_2,0,b;A_3)$$
$$+\zeta_3(a,s_1,s_3,0,b,s_2;A_3)+\zeta_3(s_1,a,s_3,0,s_2,b;A_3)$$

in terms of $\zeta_2(s_1,s_2,s_3;A_3)$ and $\zeta_1(s;A_2)=\zeta(s)$, where $a,b\in\mathbb{N}_0$ and $s_1,s_2,s_3\in\mathbb{C}$. For example, we have

$$\begin{aligned}&-\zeta_2(s_1,s_3,2,s_2,0,1;A_3)+\zeta_2(s_1,s_3,1,s_2,0,2;A_3)\\&\quad+\zeta_2(1,s_1,s_3,0,2,s_2;A_3)+\zeta_2(s_1,1,s_3,0,s_2,2;A_3)\\&\quad=-3\zeta_2(s_1,s_3+3,s_2;A_2)-\zeta_2(s_1+1,s_3,s_2+2;A_2)\\&\qquad+2\zeta(2)\zeta_2(s_1,s_3+1,s_2;A_2)\end{aligned}$$

(see [12, Theorems 5.9 and 5.10]). However our previous result does not include Witten's volume formula (for example, (1.6)).

Therefore we aim to construct some functional relations for $\zeta_r(\mathbf{s};A_r)$ $(r\leq 3)$ which include Witten's volume formulas as value-relations.

Let

$$\phi(s)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m^s}=\left(2^{1-s}-1\right)\zeta(s). \tag{4.1}$$

We recall that

$$\sum_{l=1}^{\infty}\frac{(-1)^l\cos(l\theta)}{l^{2p}}=\sum_{\nu=0}^{p}\ \phi(2p-2\nu)\ \frac{(-1)^\nu\theta^{2\nu}}{(2\nu)!}; \tag{4.2}$$

$$\sum_{l=1}^{\infty}\frac{(-1)^l\sin(l\theta)}{l^{2q+1}}=\sum_{\nu=0}^{q}\ \phi(2q-2\nu)\ \frac{(-1)^\nu\theta^{2\nu+1}}{(2\nu+1)!} \tag{4.3}$$

for $p\in\mathbb{N}$, $q\in\mathbb{N}_0$ and $\theta\in(-\pi,\pi)\subset\mathbb{R}$ (see, for example, [20, Lemma 2]). Note that $\phi(0)=\zeta(0)=-\frac{1}{2}$. We see that the both sides of (4.2) and of (4.3) are continuous for $\theta\in[-\pi,\pi]$ in the case $p,q\in\mathbb{N}$.

Fix $p\in\mathbb{N}$ and $s,t,u,x,y\in\mathbb{C}$ with $\Re s>1$, $\Re t>1$, $\Re u>1$, $|x|\leq 1$ and $|y|\leq 1$. Put

$$F_0(\theta;2p,s,t,u;x,y)=2\left(\sum_{l=1}^{\infty}\frac{(-1)^l\cos(l\theta)}{l^{2p}}-\sum_{\nu=0}^{p}\ \phi(2p-2\nu)\ \frac{(-1)^\nu\theta^{2\nu}}{(2\nu)!}\right) \tag{4.4}$$
$$\times\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^my^ne^{i(m+n)\theta}}{m^sn^t(m+n)^u}.$$

By (4.2), we see that

$$F_0(\theta;2p,s,t,u;x,y)=0 \quad (-\pi\leq\theta\leq\pi). \tag{4.5}$$

By the assumptions on p,s,t,u, the right-hand side of (4.4) is absolutely convergent. Hence, by $\cos\theta=(e^{i\theta}+e^{-i\theta})/2$, we have

$$\begin{aligned}F_0(\theta;2p,s,t,u;x,y)=&\sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m+n}x^my^n\left(e^{i(l+m+n)\theta}+e^{i(-l+m+n)\theta}\right)}{l^{2p}m^sn^t(m+n)^u}\\&-2\sum_{\nu=0}^{p}\phi(2p-2\nu)\,\frac{(-1)^\nu\theta^{2\nu}}{(2\nu)!}\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^my^ne^{i(m+n)\theta}}{m^sn^t(m+n)^u}.\end{aligned}\tag{4.6}$$

Note that in our previous paper ([12]), we only considered the case $(x,y)=(1,1)$. Hence we were only able to treat some limited case of $\zeta_3(\cdot;A_3)$. In this paper, we consider $F_0(\theta;2p,s,t,u;x,y)$ which is a certain triple polylogarithm. Considering the integration of F_0 with respect to the variable θ in the numerator of each summand, we can produce a certain factor in the denominator of each summand, which corresponds to some relevant root. This is the 'polylogarithm technique' mentioned in the preceding section. We will execute this procedure as follows.

Corresponding to $F_0(\theta;2p,s,t,u;x,y)$, we define

$$\begin{aligned}G_d(\theta;2p,s,t,u;x,y)=\frac{1}{i^d}\Bigg[&\sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m+n}x^my^ne^{i(l+m+n)\theta}}{l^{2p}m^sn^t(m+n)^u(l+m+n)^d}\\&+\sum_{\substack{l,m,n=1\\ l\neq m+n}}^{\infty}\frac{(-1)^{l+m+n}x^my^ne^{i(-l+m+n)\theta}}{l^{2p}m^sn^t(m+n)^u(-l+m+n)^d}\\&-2\sum_{\nu=0}^{p}\phi(2p-2\nu)\sum_{\mu=0}^{2\nu}\binom{d-1+2\nu-\mu}{2\nu-\mu}\frac{(-i\theta)^\mu}{\mu!}\\&\times\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^my^ne^{i(m+n)\theta}}{m^sn^t(m+n)^{u+d+2\nu-\mu}}\Bigg]\end{aligned}\tag{4.7}$$

for $\theta\in[-\pi,\pi]$ and $d\in\mathbb{N}_0$, where

$$\binom{X}{a}=\begin{cases}\frac{X(X-1)\cdots(X-a+1)}{a!} & (a\in\mathbb{N});\\ 1 & (a=0).\end{cases}$$

In the case $d=0$, it follows from (4.5) that

$$G_0(\theta;2p,s,t,u;x,y)=F_0(\theta;2p,s,t,u;x,y)+C_0=C_0, \tag{4.8}$$

where

$$C_0=C_0(2p,s,t,u;x,y)=-\sum_{m,n=1}^{\infty}\frac{x^my^n}{m^sn^t(m+n)^{u+2p}}. \tag{4.9}$$

Note that the definition of $G_d(\theta;2p,s,t,u;x,y)$ seems to have been given abruptly. However, by using the same consideration as in our previous papers [12, 23], we can presume the expression of $G_d(\theta;2p,s,t,u;x,y)$ as (4.7).

Since we fix p,s,t,u,x,y, we regard $G_d(\theta;2p,s,t,u;x,y)$ as a continuous function for $\theta\in[-\pi,\pi]$. We can show

$$\frac{d}{d\theta}G_d(\theta;2p,s,t,u;x,y)=G_{d-1}(\theta;2p,s,t,u;x,y)\quad(d\in\mathbb{N}). \tag{4.10}$$

Hence, by integrating the both sides of (4.8) and multiplying by i on the both sides, we have

$$iG_1(\theta;2p,s,t,u;x,y)=C_0(i\theta)+C_1$$

for some $C_1=C_1(2p,s,t,u;x,y)$. By repeating the integration, and by (4.10), we obtain

Lemma 4.1. *With the above notation, for $d\in\mathbb{N}_0$, we have*

$$i^dG_d(\theta;2p,s,t,u;x,y)=\sum_{j=0}^{d}C_{d-j}\frac{(i\theta)^j}{j!}, \tag{4.11}$$

where $C_k=C_k(2p,s,t,u;x,y)$ $(0\le k\le d)$ are certain constants.

We can determine $\{C_k\}$ as follows. Putting $\theta=\pm\pi$ in (4.11) with $d+1$, we have

$$\begin{aligned}&\frac{i^{d+1}}{2(i\pi)}\{G_{d+1}(\pi;2p,s,t,u;x,y)-G_{d+1}(-\pi;2p,s,t,u;x,y)\}\\&\quad=\sum_{\mu=0}^{[d/2]}C_{d-2\mu}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}.\end{aligned} \tag{4.12}$$

From (4.7), we see that the left-hand side of (4.12) is equal to

$$\begin{aligned}&2\sum_{\nu=0}^{p}\phi(2p-2\nu)\sum_{\sigma=0}^{\nu-1}\binom{d-1+2\nu-2\sigma}{2\nu-2\sigma-1}\frac{(i\pi)^{2\sigma}}{(2\sigma+1)!}\\&\qquad\times\sum_{m,n=1}^{\infty}\frac{x^my^n}{m^sn^t(m+n)^{u+d+2\nu-2\sigma}}.\end{aligned} \tag{4.13}$$

Applying [11, Lemma 2.1] to (4.13) with $a=2p$ and

$$g(X)=2\binom{d-1+X}{X-1}\sum_{m,n=1}^{\infty}\frac{x^my^n}{m^sn^t(m+n)^{u+d+X}},$$

we have

$$-\binom{2p+d-1}{2p-1}\sum_{m,n=1}^{\infty}\frac{x^my^n}{m^sn^t(m+n)^{u+2p+d}}=\sum_{\mu=0}^{[d/2]}C_{d-2\mu}\frac{(i\pi)^{2\mu}}{(2\mu+1)!} \tag{4.14}$$

for $d\in\mathbb{N}_0$. Applying [21, Lemma 3.3] to (4.14), we have

$$C_k=2\sum_{\nu=0}^{k}\phi(k-\nu)\lambda_{k-\nu}\binom{2p+\nu-1}{2p-1}\sum_{m,n=1}^{\infty}\frac{x^my^n}{m^sn^t(m+n)^{u+2p+\nu}} \tag{4.15}$$

for $k \in \mathbb{N}_0$, where $\lambda_j = \{1+(-1)^j\}/2$ $(j \in \mathbb{Z})$. Substitute (4.15) into (4.11). Then we have

$$\begin{aligned}
(4.16)\quad & i^d G_d(\theta;2p,s,t,u;x,y) \\
&= 2\sum_{j=0}^{d}\sum_{\nu=0}^{d-j}\phi(d-j-\nu)\lambda_{d-j-\nu}\binom{2p+\nu-1}{2p-1}\sum_{m,n=1}^{\infty}\frac{x^m y^n}{m^s n^t (m+n)^{u+2p+\nu}}\frac{(i\theta)^j}{j!} \\
&= 2\sum_{\eta=0}^{d}\phi(d-\eta)\lambda_{d-\eta}\sum_{j=0}^{\eta}\binom{2p+\eta-j-1}{2p-1}\sum_{m,n=1}^{\infty}\frac{x^m y^n}{m^s n^t (m+n)^{u+2p+\eta-j}}\frac{(i\theta)^j}{j!},
\end{aligned}$$

by putting $\eta = \nu + j$. Combining (4.7) and (4.16) with $d = 2q$ for $q \in \mathbb{N}$, we obtain

$$\begin{aligned}
(4.17)\quad & \sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m+n}x^m y^n e^{i(l+m+n)\theta}}{l^{2p}m^s n^t (m+n)^u (l+m+n)^{2q}} + \sum_{\substack{l,m,n=1\\ l\neq m+n}}^{\infty}\frac{(-1)^{l+m+n}x^m y^n e^{i(-l+m+n)\theta}}{l^{2p}m^s n^t (m+n)^u (-l+m+n)^{2q}} \\
&- 2\sum_{\nu=0}^{p}\phi(2p-2\nu)\sum_{\mu=0}^{2\nu}\binom{2q-1+2\nu-\mu}{2\nu-\mu}\frac{(-i\theta)^\mu}{\mu!}\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^m y^n e^{i(m+n)\theta}}{m^s n^t (m+n)^{u+2q+2\nu-\mu}} \\
&- 2\sum_{\nu=0}^{q}\phi(2q-2\nu)\sum_{\mu=0}^{2\nu}\binom{2p-1+2\nu-\mu}{2\nu-\mu}\frac{(i\theta)^\mu}{\mu!} \\
&\qquad \times \sum_{m,n=1}^{\infty}\frac{x^m y^n}{m^s n^t (m+n)^{u+2p+2\nu-\mu}} = 0 \quad (\theta \in [-\pi,\pi]).
\end{aligned}$$

Now we replace y by $-ye^{-i\theta}$ in (4.17). This is an important procedure to produce a factor in the denominator of each summand, which corresponds to the root. In fact, we consider

(4.18)

$$\begin{aligned}
& \widetilde{F}_0(\theta;2p,s,t,u,2q;x,y) \\
&:= \sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m}x^m y^n e^{i(l+m)\theta}}{l^{2p}m^s n^t (m+n)^u (l+m+n)^{2q}} + \sum_{\substack{l,m,n=1\\ l\neq m+n}}^{\infty}\frac{(-1)^{l+m}x^m y^n e^{i(-l+m)\theta}}{l^{2p}m^s n^t (m+n)^u (-l+m+n)^{2q}} \\
&- 2\sum_{\nu=0}^{p}\phi(2p-2\nu)\sum_{\mu=0}^{2\nu}\binom{2q-1+2\nu-\mu}{2\nu-\mu}\frac{(-i\theta)^\mu}{\mu!}\sum_{m,n=1}^{\infty}\frac{(-1)^{m}x^m y^n e^{im\theta}}{m^s n^t (m+n)^{u+2q+2\nu-\mu}} \\
&- 2\sum_{\nu=0}^{q}\phi(2q-2\nu)\sum_{\mu=0}^{2\nu}\binom{2p-1+2\nu-\mu}{2\nu-\mu}\frac{(i\theta)^\mu}{\mu!} \\
&\qquad \times \sum_{m,n=1}^{\infty}\frac{(-1)^n x^m y^n e^{-in\theta}}{m^s n^t (m+n)^{u+2p+2\nu-\mu}} = 0 \quad (\theta \in [-\pi,\pi]).
\end{aligned}$$

We can apply the same argument about F_0 as mentioned above to $\widetilde{F}_0$. In fact, for p,q,s,t,u,x,y as fixed above and for $d\in\mathbb{N}_0$, we put

$$
\begin{aligned}
(4.19)\quad & \widetilde{G}_d(\theta;2p,s,t,u,2q;x,y)\\
&:=\frac{1}{i^d}\Bigg[\sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m}x^my^ne^{i(l+m)\theta}}{l^{2p}m^sn^t(l+m)^d(m+n)^u(l+m+n)^{2q}}\\
&\quad+\sum_{\substack{l,m,n=1\\ l\neq m\\ l\neq m+n}}^{\infty}\frac{(-1)^{l+m}x^my^ne^{i(-l+m)\theta}}{l^{2p}m^sn^t(-l+m)^d(m+n)^u(-l+m+n)^{2q}}\\
&\quad-2\sum_{\nu=0}^{p}\phi(2p-2\nu)\sum_{\mu=0}^{2\nu}\binom{2q-1+2\nu-\mu}{2\nu-\mu}\sum_{\rho=0}^{\mu}\binom{d-1+\mu-\rho}{\mu-\rho}\\
&\qquad\times\frac{(-i\theta)^\rho}{\rho!}\sum_{m,n=1}^{\infty}\frac{(-1)^mx^my^ne^{im\theta}}{m^{s+d+\mu-\rho}n^t(m+n)^{u+2q+2\nu-\mu}}\\
&\quad-2(-1)^d\sum_{\nu=0}^{q}\phi(2q-2\nu)\sum_{\mu=0}^{2\nu}\binom{2p-1+2\nu-\mu}{2\nu-\mu}\sum_{\rho=0}^{\mu}\binom{d-1+\mu-\rho}{\mu-\rho}\\
&\qquad\times\frac{(i\theta)^\rho}{\rho!}\sum_{m,n=1}^{\infty}\frac{(-1)^nx^my^ne^{-in\theta}}{m^sn^{t+d+\mu-\rho}(m+n)^{u+2p+2\nu-\mu}}\Bigg]\quad(\theta\in[-\pi,\pi]).
\end{aligned}
$$

Let

$$
\widetilde{C}_0=\widetilde{C}_0(2p,s,t,u,2q;x,y)=-\sum_{m,n=1}^{\infty}\frac{x^my^n}{m^{s+2p}n^{t+2q}(m+n)^u}.
$$

Using $(d/d\theta)\widetilde{G}_d(\theta;2p,s,t,u,2q;x,y)=\widetilde{G}_{d-1}(\theta;2p,s,t,u,2q;x,y)$ $(d\in\mathbb{N})$ similarly to Lemma 4.1, we can obtain the following lemma.

Lemma 4.2. *With the above notation, for $d\in\mathbb{N}_0$, we have*

$$
i^d\widetilde{G}_d(\theta;2p,s,t,u,2q;x,y)=\sum_{j=0}^{d}\widetilde{C}_{d-j}\frac{(i\theta)^j}{j!},\tag{4.20}
$$

where $\widetilde{C}_k=\widetilde{C}_k(2p,s,t,u,2q;x,y)$ $(0\le k\le d)$ are certain constants.

In the third and the fourth terms on the right-hand side of (4.19), we change the running indices μ and ρ $(0\le\mu\le 2\nu,\ 0\le\rho\le\mu)$ to ρ and σ $(0\le\rho\le 2\mu,\ 0\le\sigma\le 2\nu-\rho)$, by putting $\sigma = 2\nu - \mu$. Then we have

$$
\begin{aligned}
(4.21)\qquad & \widetilde{G}_d(\theta; 2p, s, t, u, 2q; x, y) \\
&= \frac{1}{i^d}\Bigg[\sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m} x^m y^n e^{i(l+m)\theta}}{l^{2p} m^s n^t (l+m)^d (m+n)^u (l+m+n)^{2q}} \\
&\quad + \sum_{\substack{l,m,n=1 \\ l\neq m \\ l\neq m+n}}^{\infty} \frac{(-1)^{l+m} x^m y^n e^{i(-l+m)\theta}}{l^{2p} m^s n^t (-l+m)^d (m+n)^u (-l+m+n)^{2q}} \\
&\quad - 2\sum_{\nu=0}^{p} \phi(2p-2\nu) \sum_{\rho=0}^{2\nu}\sum_{\sigma=0}^{2\nu-\rho} \binom{2q-1+\sigma}{\sigma}\binom{d-1+2\nu-\sigma-\rho}{2\nu-\sigma-\rho} \\
&\qquad \times \frac{(-i\theta)^\rho}{\rho!} \sum_{m,n=1}^{\infty} \frac{(-1)^m x^m y^n e^{im\theta}}{m^{s+d+2\nu-\sigma-\rho} n^t (m+n)^{u+2q+\sigma}} \\
&\quad - 2(-1)^d \sum_{\nu=0}^{q} \phi(2q-2\nu) \sum_{\rho=0}^{2\nu}\sum_{\sigma=0}^{2\nu-\rho} \binom{2p-1+\sigma}{\sigma}\binom{d-1+2\nu-\sigma-\rho}{2\nu-\sigma-\rho} \\
&\qquad \times \frac{(i\theta)^\rho}{\rho!} \sum_{m,n=1}^{\infty} \frac{(-1)^n x^m y^n e^{-in\theta}}{m^s n^{t+d+2\nu-\sigma-\rho} (m+n)^{u+2p+\sigma}}\Bigg] \quad (\theta \in [-\pi, \pi]).
\end{aligned}
$$

For simplicity, we denote by $\zeta_2(s_1, s_2, s_3; x, y; A_2)$ the double polylogarithm of type A_2 defined by

$$
(4.22)\qquad \zeta_2(s_1, s_2, s_3; x, y; A_2) = \sum_{m,n=1}^{\infty} \frac{x^m y^n}{m^{s_1} n^{s_3} (m+n)^{s_2}}.
$$

By (4.20) and (4.21) with $d = 2r+1$ for $r \in \mathbb{N}_0$, we have

$$
\begin{aligned}
(4.23)\qquad & \frac{i^{2r+1}}{2(i\pi)}\Big\{\widetilde{G}_{2r+1}(\pi; 2p, s, t, u, 2q; x, y) - \widetilde{G}_{2r+1}(-\pi; 2p, s, t, u, 2q; x, y)\Big\} \\
&= 2\sum_{\nu=0}^{p} \phi(2p-2\nu) \sum_{\mu=0}^{\nu-1}\sum_{\sigma=0}^{2\nu-2\mu-1} \binom{2q-1+\sigma}{\sigma}\binom{2r-1+2\nu-\sigma-2\mu}{2\nu-\sigma-2\mu-1} \\
&\qquad \times \frac{(i\pi)^{2\mu}}{(2\mu+1)!}\, \zeta_2(s+2r+2\nu-\sigma-2\mu, u+2q+\sigma, t; x, y; A_2) \\
&\quad + 2\sum_{\nu=0}^{q} \phi(2q-2\nu) \sum_{\mu=0}^{\nu-1}\sum_{\sigma=0}^{2\nu-2\mu-1} \binom{2p-1+\sigma}{\sigma}\binom{2r-1+2\nu-\sigma-2\mu}{2\nu-\sigma-2\mu-1} \\
&\qquad \times \frac{(i\pi)^{2\mu}}{(2\mu+1)!}\, \zeta_2(s, u+2p+\sigma, t+2r+2\nu-\sigma-2\mu; x, y; A_2) \\
&= \sum_{\nu=0}^{r} \widetilde{C}_{2r-2\nu} \frac{(i\pi)^{2\nu}}{(2\nu+1)!}.
\end{aligned}
$$

Applying [11, Lemma 2.1] to the second member of (4.23), we have

$$
\begin{aligned}
(4.24)\quad & -\sum_{\sigma=0}^{2p-1}\binom{2q-1+\sigma}{\sigma}\binom{2p+2r-\sigma-1}{2p-\sigma-1}\zeta_2(s+2p+2r-\sigma,u+2q+\sigma,t;x,y;A_2)\\
& -\sum_{\sigma=0}^{2q-1}\binom{2p-1+\sigma}{\sigma}\binom{2q+2r-\sigma-1}{2q-\sigma-1}\zeta_2(s,u+2p+\sigma,t+2q+2r-\sigma;x,y;A_2)\\
& \qquad =\sum_{\nu=0}^{r}\widetilde{C}_{2r-2\nu}\frac{(i\pi)^{2\nu}}{(2\nu+1)!}.
\end{aligned}
$$

On the other hand, by (4.20) and (4.21) with $d=2r$, we have

$$
\begin{aligned}
(4.25)\quad & \frac{i^{2r}}{2}\left\{\widetilde{G}_{2r}(\pi;2p,s,t,u,2q;x,y)+\widetilde{G}_{2r}(-\pi;2p,s,t,u,2q;x,y)\right\}\\
& =\sum_{l,m,n=1}^{\infty}\frac{x^m y^n}{l^{2p}m^s n^t(l+m)^{2r}(m+n)^u(l+m+n)^{2q}}\\
& +\sum_{\substack{l,m,n=1\\ l\neq m\\ l\neq m+n}}^{\infty}\frac{x^m y^n}{l^{2p}m^s n^t(-l+m)^{2r}(m+n)^u(-l+m+n)^{2q}}\\
& -2\sum_{\nu=0}^{p}\phi(2p-2\nu)\sum_{\mu=0}^{\nu}\sum_{\sigma=0}^{2\nu-2\mu}\binom{2q-1+\sigma}{\sigma}\binom{2r-1+2\nu-\sigma-2\mu}{2\nu-\sigma-2\mu}\\
& \quad\times\frac{(i\pi)^{2\mu}}{(2\mu)!}\zeta_2(s+2r+2\nu-\sigma-2\mu,u+2q+\sigma,t;x,y;A_2)\\
& -2\sum_{\nu=0}^{q}\phi(2q-2\nu)\sum_{\mu=0}^{\nu}\sum_{\sigma=0}^{2\nu-2\mu}\binom{2p-1+\sigma}{\sigma}\binom{2r-1+2\nu-\sigma-2\mu}{2\nu-\sigma-2\mu}\\
& \quad\times\frac{(i\pi)^{2\mu}}{(2\mu)!}\zeta_2(s,u+2p+\sigma,t+2r+2\nu-\sigma-2\mu;x,y;A_2)=\sum_{\nu=0}^{r}\widetilde{C}_{2r-2\nu}\frac{(i\pi)^{2\nu}}{(2\nu)!}.
\end{aligned}
$$

Applying [11, Lemma 2.1] to the third and the fourth terms on the second member of (4.25), and applying [23, Lemma 4.4] to (4.24), (4.25), we obtain the following.

Proposition 4.3. *With the above notation,*

$$
\begin{aligned}
(4.26)\qquad & \sum_{l,m,n=1}^{\infty}\frac{x^m y^n}{l^{2p}m^s n^t(l+m)^{2r}(m+n)^u(l+m+n)^{2q}}\\
& +\sum_{\substack{l,m,n=1\\ l\neq m\\ l\neq m+n}}^{\infty}\frac{x^m y^n}{l^{2p}m^s n^t(-l+m)^{2r}(m+n)^u(-l+m+n)^{2q}}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{\rho=0}^{p}\zeta(2\rho)\sum_{\sigma=0}^{2p-2\rho}\binom{2q-1+\sigma}{\sigma}\binom{2p+2r-2\rho-\sigma-1}{2p-2\rho-\sigma}\\
&\qquad\times\zeta_2(s+2p+2r-2\rho-\sigma,u+2q+\sigma,t;x,y;A_2)\\
&+2\sum_{\rho=0}^{q}\zeta(2\rho)\sum_{\sigma=0}^{2q-2\rho}\binom{2p-1+\sigma}{\sigma}\binom{2q+2r-2\rho-\sigma-1}{2q-2\rho-\sigma}\\
&\qquad\times\zeta_2(s,u+2q+\sigma,t+2q+2r-2\rho-\sigma;x,y;A_2)\\
&+2\sum_{\rho=0}^{r}\zeta(2\rho)\sum_{\sigma=0}^{2p-1}\binom{2q-1+\sigma}{\sigma}\binom{2p+2r-2\rho-\sigma-1}{2p-\sigma-1}\\
&\qquad\times\zeta_2(s+2p+2r-2\rho-\sigma,u+2q+\sigma,t;x,y;A_2)\\
&+2\sum_{\rho=0}^{r}\zeta(2\rho)\sum_{\sigma=0}^{2q-1}\binom{2p-1+\sigma}{\sigma}\binom{2q+2r-2\rho-\sigma-1}{2q-\sigma-1}\\
&\qquad\times\zeta_2(s,u+2q+\sigma,t+2q+2r-2\rho-\sigma;x,y;A_2).
\end{aligned}
$$

Let $(x,y)=(1,1)$ and calculate the second term on the left-hand side. Then we obtain the main result in this section.

Theorem 4.4. *For $p,q\in\mathbb{N}$ and $r\in\mathbb{N}_0$,*

$$
\begin{aligned}
&\zeta_3(2p,2r,2q,s,u,t;A_3)+\zeta_3(2p,s,u,2r,2q,t;A_3)\\
&\quad+\zeta_3(2q,2r,2p,t,u,s;A_3)+\zeta_3(2q,t,u,2r,2p,s;A_3)\\
&= 2\sum_{\rho=0}^{p}\zeta(2\rho)\sum_{\sigma=0}^{2p-2\rho}\binom{2q-1+\sigma}{\sigma}\binom{2p+2r-2\rho-\sigma-1}{2p-2\rho-\sigma}\\
&\qquad\times\zeta_2(s+2p+2r-2\rho-\sigma,u+2q+\sigma,t;A_2)\\
&+2\sum_{\rho=0}^{q}\zeta(2\rho)\sum_{\sigma=0}^{2q-2\rho}\binom{2p-1+\sigma}{\sigma}\binom{2q+2r-2\rho-\sigma-1}{2q-2\rho-\sigma}\\
&\qquad\times\zeta_2(s,u+2q+\sigma,t+2q+2r-2\rho-\sigma;A_2)\\
&+2\sum_{\rho=0}^{r}\zeta(2\rho)\sum_{\sigma=0}^{2p-1}\binom{2q-1+\sigma}{\sigma}\binom{2p+2r-2\rho-\sigma-1}{2p-\sigma-1}\\
&\qquad\times\zeta_2(s+2p+2r-2\rho-\sigma,u+2q+\sigma,t;A_2)\\
&+2\sum_{\rho=0}^{r}\zeta(2\rho)\sum_{\sigma=0}^{2q-1}\binom{2p-1+\sigma}{\sigma}\binom{2q+2r-2\rho-\sigma-1}{2q-\sigma-1}\\
&\qquad\times\zeta_2(s,u+2q+\sigma,t+2q+2r-2\rho-\sigma;A_2)
\end{aligned}
\tag{4.27}
$$

holds for all $(s,t,u)\in\mathbb{C}^3$ except for singularities of all functions on the both sides of (4.27).

Example 4.5. Putting $(p,q,r)=(1,1,1)$ and $(s,t,u)=(2,2,2)$ in (4.27), we have

$$
\begin{aligned}
4\zeta_3(2,2,2,2,2,2;A_3)&=8\zeta(2)\left\{\zeta_2(4,4,2;A_2)+\zeta_2(3,5,2;A_2)\right\}\\
&\quad-6\left\{2\zeta_2(6,4,2;A_2)+2\zeta_2(5,5,2;A_2)+\zeta_2(4,6,2;A_2)\right\}.
\end{aligned}
\tag{4.28}
$$

Using the relation $\zeta_2(k,m,l;A_2)=\zeta_2(k,m-1,l+1;A_2)-\zeta_2(k-1,m,l+1;A_2)$ (see [19]) repeatedly, we see that

$$\zeta_2(4,4,2;A_2)+\zeta_2(3,5,2;A_2)=\frac{1}{2}\left\{2\zeta_2(4,3,3;A_2)-\zeta_2(3,4,3;A_2)\right\}.$$

From [20, Theorem 2], the right-hand side is equal to $\pi^{10}/935550$. Similarly we have

$$\begin{aligned}&2\zeta_2(6,4,2;A_2)+2\zeta_2(5,5,2;A_2)+\zeta_2(4,6,2;A_2)\\&\quad=2\zeta_2(6,3,3;A_2)-\zeta_2(3,6,3;A_2)-\frac{1}{2}\zeta_2(4,4,4;A_2).\end{aligned}$$

From [20, Theorem 2] and [18, Theorem 4.1], we see that the right-hand side is equal to $887\pi^{12}/3831077250$. Therefore we obtain Witten's volume formula

$$\zeta_3(2,2,2,2,2,2;A_3)=\frac{23}{2554051500}\pi^{12}. \tag{4.29}$$

This implies that (4.27) can be regarded as a continuous relation including Witten's volume formula.

Remark 4.6. If we start from (4.3) instead of (4.2) in (4.4), then we can construct a general functional relation for

$$\begin{aligned}&\zeta_3(a,c,b,s,u,t;A_3)+(-1)^a\zeta_3(a,s,u,c,b,t;A_3)\\&\quad+(-1)^{a+b+c}\zeta_3(b,c,a,t,u,s:A_3)+(-1)^{a+c}\zeta_3(b,t,u,c,a,s;A_3)\end{aligned}$$

in terms of $\zeta_2(s_1,s_2,s_3;A_2)$ and $\zeta(s)$, where $a,b\in\mathbb{N}$, $c\in\mathbb{N}_0$ and $s,t,u\in\mathbb{C}$. For example, in the case $(a,b,c)=(1,3,0)$, we can obtain

$$\begin{aligned}&\zeta_3(1,0,3,s,u,t;A_3)-\zeta_3(1,s,u,0,3,t;A_3)+\zeta_3(3,0,1,t,u,s;A_3)-\zeta_3(3,t,u,0,1,s;A_3)\\&\quad=4\zeta_2(s,u+4,t;A_2)+\zeta_2(s+1,u,t+3;A_2)-2\zeta(2)\zeta_2(s,u+2,t;A_2).\end{aligned}$$

Remark 4.7. As described in this section, we produced relation formulas for $\zeta_3(\,\cdot\,;A_3)$ by uniting those for $\zeta_2(\,\cdot\,;A_2)$ and $\zeta_1(\,\cdot\,;A_1)$. This corresponds to the fact that the fundamental root system of A_3 can be produced by adding a fundamental root to that of A_2 in the Dynkin diagram. This strategy can be applied to various other cases. For instance, in [5], we deduced relation formulas for $\zeta_2(\,\cdot\,;B_2)$ from those of $\zeta_2(\,\cdot\,;A_2)$ and $\zeta_1(\,\cdot\,;A_1)$. Similarly, it is possible to deduce relation formulas for $\zeta_3(\,\cdot\,;B_3)$ and $\zeta_3(\,\cdot\,;C_3)$ from those of $\zeta_3(\,\cdot\,;A_3)$, or alternatively, from $\zeta_2(\,\cdot\,;B_2)$ and $\zeta_1(\,\cdot\,;A_1)$. In [5], the process of the latter way was described in detail. These correspondences between functional relations for zeta-functions and the structure of root systems seems to be fascinating.

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