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Barnes multiple zeta-functions, Ramanujan's formula, and relevant series involving hyperbolic functions

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Abstract. In the former part of this paper, we give functional equations for Barnes multiple zeta-functions and consider some relevant results. In particular, we show that Ramanujan's classical formula for the Riemann zeta values can be derived from functional equations for Barnes zeta-functions. In the latter half part, we generalize some evaluation formulas for certain series involving hyperbolic functions in terms of Bernoulli polynomials. The original formulas were classically given by Cauchy, Mellin, Ramanujan, and later recovered and reformulated by Berndt. From our consideration, we give multiple versions of these known formulas.

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1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We begin with the classical work of Cauchy [10] who studied the series defined by

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m}{\sinh(m\pi)m^s} \quad (s \in \mathbb{Z}), \tag{1.1}$$

where $\sinh x = (e^x - e^{-x})/2$. He showed that several values at $s = 4k + 3$ ($k \in \mathbb{N}_0$) can be written in terms of π. After his work, this series was considered by Mellin, Ramanujan, and several other authors (see [8,9,15,16]), and the following fascinating formula was proved:

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m}{\sinh(m\pi)m^{4k+3}} = (2\pi)^{4k+3} \times \sum_{j=0}^{2k+2} (-1)^{j+1} \frac{B_{2j}(1/2)}{(2j)!} \frac{B_{4k+4-2j}(1/2)}{(4k+4-2j)!} \tag{1.2}$$

for $k \in \mathbb{N}_0$, where $B_j(y)$ is the jth Bernoulli polynomial defined by

$$F(t, y) = \frac{te^{ty}}{e^t - 1} = \sum_{j=0}^{\infty} B_j(y)\frac{t^j}{j!} \tag{1.3}$$

(see [11]). As a result related to (1.2), it is also known that

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\coth(m\pi)}{m^{4k+3}} = (2\pi)^{4k+3} \sum_{j=0}^{2k+2} (-1)^{j+1} \frac{B_{2j}(0)}{(2j)!} \frac{B_{4k+4-2j}(0)}{(4k+4-2j)!} \tag{1.4}$$

for $k \in \mathbb{N}_0$, which is written in Ramanujan's notebooks (see Berndt [8, (25.3) p. 293]), where $\coth x = (e^x + e^{-x})/(e^x - e^{-x})$. In fact, (1.4) can be easily derived from Ramanujan's famous formula (see Berndt [8, p. 275]):

$$\begin{aligned}
&\alpha^{-N} \left\{ \frac{1}{2}\zeta(2N+1) + \sum_{k=1}^{\infty} \frac{1}{\left(e^{2k\alpha} - 1\right) k^{2N+1}} \right\} \\
&= (-\beta)^{-N} \left\{ \frac{1}{2}\zeta(2N+1) + \sum_{k=1}^{\infty} \frac{1}{\left(e^{2k\beta} - 1\right) k^{2N+1}} \right\} \\
&\quad - 2^{2N} \sum_{k=0}^{N+1} (-1)^k \frac{B_{2k}(0)}{(2k)!} \frac{B_{2N+2-2k}(0)}{(2N+2-2k)!} \alpha^{N+1-k} \beta^k,
\end{aligned} \tag{1.5}$$

where N is any non-zero integer, α and β are positive numbers such that $\alpha\beta = \pi^2$ and $\zeta(s)$ is the Riemann zeta-function.

In the 1970's, Berndt [4,5] studied generalized Eisenstein series and proved transformation formulas for them. Using those results, he gave a family of evaluation formulas for certain Dirichlet series in [6,7], including (1.2), (1.4) and (1.5) (see also Remark 6.6).

What is the meaning of the above infinite series involving hyperbolic functions? We can find that they are connected with Barnes multiple zeta-functions. In fact, in the former half part of this paper, we first show functional equations for Barnes zeta-functions. Those functional equations imply that some infinite series involving hyperbolic functions are the "dual" of Barnes zeta-functions, in the sense that they appear on the right-hand side of functional equations for Barnes zeta-functions (see (2.7)).

Then we show two expressions of the Barnes zeta-functions or their residues at integers. We observe that Ramanujan's formula (1.5) (and hence (1.4)) can be deduced by combining these two expressions in the double case. Hence in the multiple cases, the combination of these expressions may be regarded as generalizations of Ramanujan's formula (see Corollary 2.4).

Motivated by this observation, in the latter half part, we first give a very general form of evaluation formulas (see Theorem 5.1), which is out of the frame of Barnes zeta-functions. From this form, we deduce a certain explicit evaluation formula with a parameter $y \in [0, 1]$ (see Theorem 6.1) which may be regarded as a relation of several Barnes zeta-functions at non-positive integers. This formula especially implies (1.2) and (1.4) (see Corollaries 6.2 and 6.3) and also implies a lot of presumably new formulas, for example,

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{\sinh(m\pi i/\rho)^2 m^4} = -\frac{1}{2835}\pi^4, \tag{1.6}$$

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\coth(m\pi i/\rho)^2}{m^4} = \frac{62}{2835}\pi^4, \tag{1.7}$$

where $i = \sqrt{-1}$ and $\rho = (-1+\sqrt{-3})/2$, the cube root of unity, and the same type of formulas including higher power roots of unity (see Corollary 6.4, Example 6.5).

2. Functional equations for Barnes zeta-functions

For $\theta \in \mathbb{R}$ let $H(\theta) = \{z = re^{i(\theta+\phi)} \in \mathbb{C} \mid r > 0, -\pi/2 < \phi < \pi/2\}$ be the open half plane whose normal vector is $e^{i\theta}$. We recall the Barnes zeta-function defined by the following multiple Dirichlet series:

$$\zeta_n(s, a; \omega_1, \ldots, \omega_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{1}{(a + \omega_1 m_1 + \cdots + \omega_n m_n)^s}, \tag{2.1}$$

where all $a, \omega_1, \ldots, \omega_n \in H(\theta)$ for some θ. Then it is known that this Dirichlet series converges absolutely uniformly on any compact subset in $\Re s > n$.

Assume at first that $\Re s > n$. For $x \in H(\theta)$, we have the formula for the gamma function

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^{e^{-i\theta}\infty} e^{-xt} t^{s-1} dt. \tag{2.2}$$

Since

$$a + \omega_1 m_1 + \cdots + \omega_n m_n \in H(\theta) \tag{2.3}$$

for $m_1, \ldots, m_n \in \mathbb{N}_0$, we can apply (2.2) to each term in (2.1) to get

$$\begin{aligned}
&\zeta_n(s, a; \omega_1, \ldots, \omega_n) \\
&= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{1}{\Gamma(s)} \int_0^{e^{-i\theta}\infty} e^{-(a+\omega_1 m_1 + \cdots + \omega_n m_n)t} t^{s-1} dt \\
&= \frac{1}{\Gamma(s)} \int_0^{e^{-i\theta}\infty} \frac{e^{(\omega_1 + \cdots + \omega_n - a)t}}{(e^{\omega_1 t} - 1) \cdots (e^{\omega_n t} - 1)} t^{s-1} dt \\
&= \frac{1}{\Gamma(s)(e^{2\pi i s} - 1)} \int_{C(\theta)} \frac{e^{(\omega_1 + \cdots + \omega_n - a)t}}{(e^{\omega_1 t} - 1) \cdots (e^{\omega_n t} - 1)} t^{s-1} dt,
\end{aligned} \tag{2.4}$$

where the argument of t is taken in $-\theta \le \arg t \le -\theta + 2\pi$ and $C(\theta)$ is a contour which starts at $e^{-i\theta}\infty$, goes counterclockwise around the origin with sufficiently small radius, and ends at $e^{-i\theta}\infty$. Let $0 \le y_1, \ldots, y_n < 1$ and put

$$a = a(y_1, \ldots, y_n) = \omega_1(1 - y_1) + \cdots + \omega_n(1 - y_n) \in H(\theta).$$

Then

$$\begin{aligned}
&\zeta_n(s, a(y_1, \ldots, y_n); \omega_1, \ldots, \omega_n) \\
&= \frac{1}{\Gamma(s)(e^{2\pi i s} - 1)} \int_{C(\theta)} \frac{e^{(\omega_1 y_1 + \cdots + \omega_n y_n)t}}{(e^{\omega_1 t} - 1) \cdots (e^{\omega_n t} - 1)} t^{s-1} dt \\
&= \frac{\prod_{j=1}^n \omega_j^{-1}}{\Gamma(s)(e^{2\pi i s} - 1)} \int_{C(\theta)} \left(\prod_{j=1}^n F(\omega_j t, y_j) \right) t^{s-n-1} dt.
\end{aligned} \tag{2.5}$$

If $t \in C(\theta)$ is sufficiently far from the origin, then $\Re(\omega_j t) > 0$ $(1 \le j \le n)$. Therefore the integral on the rightmost side converges absolutely uniformly on the whole space $\mathbb{C}$, so (2.5) gives the meromorphic continuation of $\zeta_n(s, a(y_1, \ldots, y_n); \omega_1, \ldots, \omega_n)$ to the whole space $\mathbb{C}$.

In the following, we assume that $n \geq 2$ and $\Im(\omega_j/\omega_k) \neq 0$ for any pair (j,k) with $j \neq k$. From the above integral expression we obtain the following functional equations for Barnes zeta-functions. When $y_1 = \cdots = y_n = y$, we write $a(y,\ldots,y) = a(y)$ for brevity.

Theorem 2.1 (functional equations). *We have*

$$\begin{aligned}
&\zeta_n(s, a(y); \omega_1, \ldots, \omega_n) \\
&\quad = -\frac{2\pi i}{\Gamma(s)(e^{2\pi i s}-1)} \\
&\qquad \times \sum_{k=1}^{n} \sum_{m \in \mathbb{Z}\setminus\{0\}} \omega_k^{-1} \Bigg(\Bigg(\prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{e^{(2m\pi i \omega_j/\omega_k)y}}{e^{2m\pi i \omega_j/\omega_k}-1}\Bigg)\Bigg) (2m\pi i \omega_k^{-1})^{s-1} e^{2m\pi i y},
\end{aligned} \tag{2.6}$$

where $-\theta < \arg(2m\pi i \omega_k^{-1}) < -\theta + \pi$ *and the right-hand side converges absolutely uniformly on the whole space* $\mathbb{C}$ *if* $0 < y < 1$, *and on the region* $\Re s < 0$ *if* $y = 0$.

In particular, if $y = 1/2$, *we have*

$$\begin{aligned}
&\zeta_n(s, (\omega_1 + \cdots + \omega_n)/2; \omega_1, \ldots, \omega_n) \\
&\quad = -\frac{1}{2^{n-1}} \frac{2\pi i}{\Gamma(s)(e^{2\pi i s}-1)} \\
&\qquad \times \sum_{k=1}^{n} \sum_{m \in \mathbb{Z}\setminus\{0\}} \omega_k^{-1} (-1)^m \Bigg(\prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{\sinh(m\pi i \omega_j/\omega_k)}\Bigg) (2m\pi i \omega_k^{-1})^{s-1}.
\end{aligned} \tag{2.7}$$

In the case $y = 0$ of Theorem 2.1, the series expression (2.6) is valid only for $\Re s < 0$. In order to remove this restriction, we decompose the series into the terms involving the Riemann zeta-function and the remaining parts. For $k \in \{1, \ldots, n\}$, let $I_k^+ = \{j \in \{1, \ldots, n\}\setminus\{k\} | \Im(\omega_j/\omega_k) > 0\}$ and $I_k^- = \{j \in \{1, \ldots, n\}\setminus\{k\} | \Im(\omega_j/\omega_k) < 0\}$. Let

$$\delta(J) = \begin{cases} 0 & (J \neq \emptyset) \\ (-1)^{n+1} & (J = \emptyset) \end{cases} \tag{2.8}$$

for $J \subset \{1, \ldots, n\}$.

Corollary 2.2. *We have*

$$\zeta_n(s, a(0); \omega_1, \ldots, \omega_n) = -\frac{2\pi i}{\Gamma(s)(e^{2\pi i s} - 1)} \times \sum_{k=1}^{n} \omega_k^{-1} \Bigg\{ \sum_{m>0} (2m\pi i \omega_k^{-1})^{s-1} \Bigg(\Bigg(\prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{e^{2m\pi i \omega_j/\omega_k} - 1} \Bigg) - \delta(I_k^-) \Bigg) + \sum_{m>0} (-2m\pi i \omega_k^{-1})^{s-1} \Bigg(\Bigg(\prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{e^{-2m\pi i \omega_j/\omega_k} - 1} \Bigg) - \delta(I_k^+) \Bigg) + \delta(I_k^-)(2\pi i \omega_k^{-1})^{s-1}\zeta(1-s) + \delta(I_k^+)(-2\pi i \omega_k^{-1})^{s-1}\zeta(1-s) \Bigg\}, \tag{2.9}$$

where the series on the right-hand side converge absolutely uniformly on the whole space $\mathbb{C}$.

Proofs of Theorem 2.1 and Corollary 2.2 will be given in Section 4.

In the following, the empty sum should be understood as 0.

Corollary 2.3. *For* $l \in \mathbb{Z}$ *(or* $l > n$ *if* $y = 0$*), we have*

$$\sum_{k=1}^{n} \sum_{m\in\mathbb{Z}\setminus\{0\}} \omega_k^{-1} \Bigg(\prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{e^{(2m\pi i \omega_j/\omega_k)y}}{e^{2m\pi i \omega_j/\omega_k} - 1} \Bigg) (2m\pi i \omega_k^{-1})^{n-l-1} e^{2m\pi i y} = \begin{cases} \frac{(-1)^{l-n+1}}{(l-n)!} \zeta_n(n-l, a(y); \omega_1, \ldots, \omega_n) & (l \geq n) \\ -(n-l-1)! \operatorname{Res}_{s=n-l} \zeta_n(s, a(y); \omega_1, \ldots, \omega_n) & (l < n). \end{cases} \tag{2.10}$$

On the other hand, this is equal to

$$-\sum_{\substack{m_1,\ldots,m_n \geq 0 \\ m_1+\cdots+m_n=l}} \prod_{j=1}^{n} \frac{B_{m_j}(y)}{m_j!} \omega_j^{m_j-1}. \tag{2.11}$$

Proof. For $k \in \mathbb{Z}$, the expansion

$$\frac{\Gamma(s)(e^{2\pi i s} - 1)}{2\pi i} = \begin{cases} \frac{(-1)^k}{(-k)!} + O(|s-k|) & (k \leq 0) \\ (k-1)!(s-k) + O((|s-k|^2)) & (k > 0) \end{cases} \tag{2.12}$$

holds when s is near to k. Using this and Theorem 2.1, we obtain (2.10). On the other hand, by use of the integral representation (2.5), we see that the left-hand side of (2.10) is equal to

$$-\left(\prod_{j=1}^{n}\omega_j^{-1}\right)\operatorname*{Res}_{t=0}\left\{\left(\prod_{j=1}^{n}F(\omega_j t,y)\right)t^{-l-1}\right\}, \tag{2.13}$$

which yields (2.11). □

Similarly we have the following.

Corollary 2.4. *For $l\in\mathbb{Z}\setminus\{n\}$, we have*

$$\begin{aligned}
&\sum_{k=1}^{n}\omega_k^{-1}\Biggl(\sum_{m>0}(2m\pi i\omega_k^{-1})^{n-l-1}\Biggl(\Biggl(\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{e^{2m\pi i\omega_j/\omega_k}-1}\Biggr)-\delta(I_k^-)\Biggr)\\
&\quad+\sum_{m>0}(-2m\pi i\omega_k^{-1})^{n-l-1}\Biggl(\Biggl(\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{e^{-2m\pi i\omega_j/\omega_k}-1}\Biggr)-\delta(I_k^+)\Biggr)\\
&\quad+\delta(I_k^-)(2\pi i\omega_k^{-1})^{n-l-1}\zeta(1-n+l)\\
&\quad+\delta(I_k^+)(-2\pi i\omega_k^{-1})^{n-l-1}\zeta(1-n+l)\Biggr)\\
&=\begin{cases}\frac{(-1)^{l-n+1}}{(l-n)!}\zeta_n(n-l,a(0);\omega_1,\ldots,\omega_n) & (l>n)\\ -(n-l-1)!\operatorname{Res}_{s=n-l}\zeta_n(s,a(0);\omega_1,\ldots,\omega_n) & (l<n)\end{cases}\\
&=-\sum_{\substack{m_1,\ldots,m_n\geq 0\\ m_1+\cdots+m_n=l}}\prod_{j=1}^{n}\frac{B_{m_j}(0)}{m_j!}\omega_j^{m_j-1}.
\end{aligned}\tag{2.14}$$

In the next section we will show that Ramanujan's formula (1.5) is a consequence of the case $n=2$ of (2.14). Therefore Corollary 2.4 can be regarded as generalizations of Ramanujan's formula.

Here we give historical remarks. A kind of functional equations for the Barnes zeta-functions was first proved by Hardy and Littlewood [13] in the case $n=2$, and a generalization to the case of general n was discussed in Egami's lecture note [12]. Our proof of Theorem 2.1 is essentially the same as those of them. On the other hand, by calculating explicitly the residue (2.13), we showed an expression of $\zeta_n(n-l,a(y);\omega_1,\ldots,\omega_n)$ or its residues in terms of Bernoulli polynomials. This type of results is also classical, already studied by Barnes himself [2], [3] (see also [1], [14]). In this sense, both of the two equalities in (2.10) and (2.11) are classical. The novel point in the present paper is to combine these two equalities. A consequence of such a combination is the observation concerning Ramanujan's formula in the next section.

3. Ramanujan's formula

In this section, we show that Ramanujan's formula (1.5) can be obtained by combining two equalities given in Corollary 2.4. In Corollary 2.4, consider the case $n = 2$, $\omega_1 = \alpha^{1/2}$, $\omega_2 = i\beta^{1/2}$ with $\alpha, \beta \in \mathbb{R}$. Let $N \in \mathbb{Z}\setminus\{0\}$.

The last member of (2.14) is

$$C_l = -(i\alpha^{1/2}\beta^{1/2})^{-1} \sum_{k=0}^{l} i^k \frac{B_{l-k}(0)}{(l-k)!} \frac{B_k(0)}{k!} \alpha^{(l-k)/2} \beta^{k/2}. \tag{3.1}$$

In particular, for $l = 2N + 2$, we have

$$\begin{aligned} C_{2N+2} &= -(i\alpha^{1/2}\beta^{1/2})^{-1} \sum_{j=0}^{2N+2} i^j \frac{B_{2N+2-j}(0)}{(2N+2-j)!} \frac{B_j(0)}{j!} \alpha^{(2N+2-j)/2} \beta^{j/2} \\ &= -(i\alpha^{1/2}\beta^{1/2})^{-1} \sum_{k=0}^{N+1} (-1)^k \frac{B_{2N+2-2k}(0)}{(2N+2-2k)!} \frac{B_{2k}(0)}{(2k)!} \alpha^{N+1-k} \beta^k, \end{aligned} \tag{3.2}$$

where we have used $B_{2j+1}(0) = 0$ (for $j \geq 1$).

Next we compute the first member of (2.14). Since $\delta(I_1^+) = \delta(I_2^-) = 0$ and $\delta(I_1^-) = \delta(I_2^+) = -1$, this is equal to

$$\begin{aligned} &\alpha^{-1/2} \sum_{m>0} (2m\pi i\alpha^{-1/2})^{1-l} \left(\frac{1}{e^{-2m\pi\beta^{1/2}\alpha^{-1/2}} - 1} + 1 \right) \\ &\quad + \alpha^{-1/2} \sum_{m>0} (-2m\pi i\alpha^{-1/2})^{1-l} \frac{1}{e^{2m\pi\beta^{1/2}\alpha^{-1/2}} - 1} \\ &\quad - i\beta^{-1/2} \sum_{m>0} (2m\pi\beta^{-1/2})^{1-l} \frac{1}{e^{2m\pi\alpha^{1/2}\beta^{-1/2}} - 1} \\ &\quad - i\beta^{-1/2} \sum_{m>0} (-2m\pi\beta^{-1/2})^{1-l} \left(\frac{1}{e^{-2m\pi\alpha^{1/2}\beta^{-1/2}} - 1} + 1 \right) \\ &\quad - \alpha^{-1/2}(2\pi i\alpha^{-1/2})^{1-l}\zeta(l-1) + i\beta^{-1/2}(-2\pi\beta^{-1/2})^{1-l}\zeta(l-1). \end{aligned} \tag{3.3}$$

In particular, in the case $\alpha^{1/2}\beta^{1/2}=\pi$, we see that this is equal to

$$
\begin{aligned}
&-\alpha^{-1/2}\sum_{m>0}(2m\pi i\alpha^{-1/2})^{1-l}\frac{1}{e^{2m\beta}-1}\\
&\quad+\alpha^{-1/2}\sum_{m>0}(-2m\pi i\alpha^{-1/2})^{1-l}\frac{1}{e^{2m\beta}-1}\\
&\quad-i\beta^{-1/2}\sum_{m>0}(2m\pi\beta^{-1/2})^{1-l}\frac{1}{e^{2m\alpha}-1}\\
&\quad+i\beta^{-1/2}\sum_{m>0}(-2m\pi\beta^{-1/2})^{1-l}\frac{1}{e^{2m\alpha}-1}\\
&\quad-\alpha^{-1/2}(2\pi i\alpha^{-1/2})^{1-l}\zeta(l-1)+i\beta^{-1/2}(-2\pi\beta^{-1/2})^{1-l}\zeta(l-1)\\
&=(2\pi)^{1-l}i^{1-l}\alpha^{(l-2)/2}\\
&\quad\times\left(-\zeta(l-1)+((-1)^{l-1}-1)\sum_{m=1}^{\infty}\frac{1}{(e^{2m\beta}-1)m^{l-1}}\right)\\
&\quad-(2\pi)^{1-l}i\beta^{(l-2)/2}\\
&\quad\times\left((-1)^{l}\zeta(l-1)+(1-(-1)^{l-1})\sum_{m=1}^{\infty}\frac{1}{(e^{2m\alpha}-1)m^{l-1}}\right). \qquad (3.4)
\end{aligned}
$$

Further in the case $l=2N+2$, we see that (3.4) reduces to

$$
\begin{aligned}
&(\pi i)^{-1}(2\pi)^{-2N}(-\alpha)^{N}\left(-\frac{1}{2}\zeta(2N+1)-\sum_{m=1}^{\infty}\frac{1}{(e^{2m\beta}-1)m^{2N+1}}\right)\\
&\quad+(\pi i)^{-1}(2\pi)^{-2N}\beta^{N}\left(\frac{1}{2}\zeta(2N+1)+\sum_{m=1}^{\infty}\frac{1}{(e^{2m\alpha}-1)m^{2N+1}}\right). \qquad (3.5)
\end{aligned}
$$

By equating (3.2) and (3.5), we finally obtain

$$
\begin{aligned}
&-2^{2N}\sum_{k=0}^{N+1}(-1)^{k}\frac{B_{2N+2-2k}(0)}{(2N+2-2k)!}\frac{B_{2k}(0)}{(2k)!}\alpha^{N+1-k}\beta^{k}\\
&\quad=(-\beta)^{-N}\left(-\frac{1}{2}\zeta(2N+1)-\sum_{m=1}^{\infty}\frac{1}{(e^{2m\beta}-1)m^{2N+1}}\right)\\
&\qquad+\alpha^{-N}\left(\frac{1}{2}\zeta(2N+1)+\sum_{m=1}^{\infty}\frac{1}{(e^{2m\alpha}-1)m^{2N+1}}\right), \qquad (3.6)
\end{aligned}
$$

which recovers (1.5).

4. Proofs

Proof of Theorem 2.1. For $z \in \mathbb{C}$ and $\varepsilon > 0$, let $D(z, \varepsilon)$ be the closed disk whose center is z with radius ε. We use the notation $x_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$. We first note that for any $\varepsilon > 0$, there exists $M' = M'(\varepsilon) > 0$ such that for $t \in \mathbb{C}\setminus \bigcup_{m\in\mathbb{Z}} D(2m\pi i, \varepsilon)$ the inequality

$$\left|\frac{1}{e^t - 1}\right| \le M' e^{-(\Re t)_+} \tag{4.1}$$

holds. Hence for $t \in \mathbb{C}\setminus \bigcup_{j=1}^{n} \bigcup_{m\in\mathbb{Z}} D(2m\pi i\omega_j^{-1}, \varepsilon)$, we have

$$\left|\prod_{j=1}^{n} F(\omega_j t, y)\right| \le M|t|^n e^{\sum_{j=1}^{n}(\Re(\omega_j t)y - \Re(\omega_j t)_+)} \tag{4.2}$$

with a certain $M = M(\varepsilon, \omega_1, \ldots, \omega_n) > 0$. Since

$$\Re(\omega_j t)y - \Re(\omega_j t)_+ \le 0 \tag{4.3}$$

for $1 \le j \le n$, we see that there exists $T = T(\varepsilon) \ge 0$ such that for all $t \in \mathbb{C}$ with $|t| = 1$,

$$\sum_{j=1}^{n}((\Re\omega_j t)y - (\Re\omega_j t)_+) \le -T. \tag{4.4}$$

Hence we see that for all $t \in \mathbb{C}\setminus \bigcup_{j=1}^{n} \bigcup_{m\in\mathbb{Z}} D(2m\pi i\omega_j^{-1}, \varepsilon)$,

$$\left|\prod_{j=1}^{n} F(\omega_j t, y)\right| \le M|t|^n e^{-T|t|}. \tag{4.5}$$

If $0 < y < 1$, then we can choose $T > 0$. In fact, since $\Im(\omega_j/\omega_k) \ne 0$ for $j \ne k$ and $n \ge 2$, for any t with $|t| = 1$ we find at least one j for which $\Re(\omega_j t) \ne 0$ holds. Then, using $0 < y < 1$ we see that $\Re(\omega_j t)y - \Re(\omega_j t)_+ < 0$, from which $T > 0$ easily follows.

From (4.5), we see that the integral on the rightmost side of (2.5) converges to 0 when the radius of the contour goes to infinity if $0 < y < 1$ or, $y = 0$ with $\Re s < 0$. Namely, there is a sequence $R_l \to \infty$ such that

$$\lim_{l\to\infty} \int_{|t|=R_l} \left|\prod_{j=1}^{n} F(\omega_j t, y)\right| |t^{s-n-1}||dt| = 0. \tag{4.6}$$

Hence we can calculate the integral by counting all the residues on the whole space. Since by the assumption the poles of the integrand are all simple except the origin, we obtain

$$\begin{aligned}
&\zeta_n(s, a(y); \omega_1, \ldots, \omega_n) \\
&= \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \int_{C(\theta)} \frac{e^{(\omega_1 + \cdots + \omega_n)yt}}{(e^{\omega_1 t} - 1) \cdots (e^{\omega_n t} - 1)} t^{s-1} dt \\
&= -\frac{2\pi i}{\Gamma(s)(e^{2\pi is} - 1)} \\
&\times \sum_{k=1}^{n} \sum_{m \in \mathbb{Z} \setminus \{0\}} \omega_k^{-1} \Big(\prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{e^{(2m\pi i\omega_j/\omega_k)y}}{e^{2m\pi i\omega_j/\omega_k} - 1} \Big) (2m\pi i\omega_k^{-1})^{s-1} e^{2m\pi iy},
\end{aligned} \tag{4.7}$$

whose absolute and uniform convergence follows from the explicit form of the series. Therefore we obtain (2.6). □

Proof of Corollary 2.2. We observe that as $m \to +\infty$, $F(2\pi im\omega_j/\omega_k, 0) = O(m)$ if $j \in I_k^+$ while $F(2\pi im\omega_j/\omega_k, 0)$ decays exponentially if $j \in I_k^-$. Thus we see that if $I_k^- \neq \emptyset$, the series

$$\sum_{m>0} \Big(\prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{e^{2m\pi i\omega_j/\omega_k} - 1} \Big) (2m\pi i\omega_k^{-1})^{s-1}$$

converges absolutely uniformly for whole $s \in \mathbb{C}$.

Next consider the case $I_k^- = \emptyset$. We have

$$\begin{aligned}
&\sum_{m>0} \Big(\prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{e^{2m\pi i\omega_j/\omega_k} - 1} \Big) (2m\pi i\omega_k^{-1})^{s-1} \\
&= \sum_{m>0} (-1)^{n-1} (2m\pi i\omega_k^{-1})^{s-1} \Big(\prod_{\substack{j=1 \\ j \neq k}}^{n} \Big(1 + \frac{1}{e^{-2m\pi i\omega_j/\omega_k} - 1} \Big) \Big) \\
&= \sum_{m>0} (-1)^{n-1} (2m\pi i\omega_k^{-1})^{s-1} \Big(1 + \sum_{\substack{J \subset \{1,\ldots,n\} \setminus \{k\} \\ |J| \geq 1}} \Big(\prod_{j \in J} \frac{1}{e^{-2m\pi i\omega_j/\omega_k} - 1} \Big) \Big) \\
&= (-1)^{n-1} (2\pi i\omega_k^{-1})^{s-1} \zeta(1-s) \\
&\quad + (-1)^{n-1} \sum_{\substack{J \subset \{1,\ldots,n\} \setminus \{k\} \\ |J| \geq 1}} \sum_{m>0} (2m\pi i\omega_k^{-1})^{s-1} \Big(\prod_{j \in J} \frac{1}{e^{-2m\pi i\omega_j/\omega_k} - 1} \Big).
\end{aligned} \tag{4.8}$$

Since all $j \in I_k^+$, we see that the rightmost side of the above can be continued to the whole of $\mathbb{C}$, and is equal to

$$(-1)^{n-1}(2\pi i\omega_k^{-1})^{s-1}\zeta(1-s) + \sum_{m>0}(2m\pi i\omega_k^{-1})^{s-1}\Bigl(\Bigl(\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{e^{2m\pi i\omega_j/\omega_k}-1}\Bigr)-(-1)^{n-1}\Bigr). \tag{4.9}$$

For the series with $m < 0$, by exchanging the roles of I_k^+ and I_k^-, we have the same type of conclusions as follows: If $I_k^+ \neq \emptyset$, the series corresponding to $m < 0$ converges absolutely uniformly for whole $s \in \mathbb{C}$, and if $I_k^+ = \emptyset$,

$$\begin{aligned}\sum_{m<0}\Bigl(\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{e^{2m\pi i\omega_j/\omega_k}-1}\Bigr)(2m\pi i\omega_k^{-1})^{s-1} &= (-1)^{n-1}(-2\pi i\omega_k^{-1})^{s-1}\zeta(1-s)\\ &\quad + \sum_{m>0}(-2m\pi i\omega_k^{-1})^{s-1}\Bigl(\Bigl(\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{e^{-2m\pi i\omega_j/\omega_k}-1}\Bigr)-(-1)^{n-1}\Bigr).\end{aligned} \tag{4.10}$$

□

5. A general formulation

In the previous sections, we established some relations between Barnes zeta-functions and certain series involving hyperbolic functions (see (2.7)). In order to study this relationship further, it is convenient to introduce a general framework to evaluate more general series.

Let $g(t)$ be a meromorphic function on $\mathbb{C}$ which has possible poles only on $2\pi i\mathbb{Z}$. For example, we will consider $g(t) = (t/2)/\sinh(t/2)$ (see (6.8)).

Let $n \in \mathbb{N}$ with $n \geq 2$ and $\eta = e^{\pi i/n}$, that is, the primitive $2n$-th root of unity. Let

$$G(t) = \prod_{j=1}^{n} g(\eta^j t).$$

We assume that there exist real numbers γ_1, γ_2 with $\gamma_1 > 0$ and a small positive number ε such that

$$|G(t)| \leq \gamma_1 |t|^{-\gamma_2}$$

for all $t \in \mathbb{C}\setminus \bigcup_{l=0}^{n-1}\bigcup_{m\in\mathbb{Z}} D(2m\pi i\eta^l, \varepsilon)$. Then we have the following theorem, which is a simple consequence of residue calculus, but is a key result in the present paper.

Theorem 5.1. *For $h \in \mathbb{Z}$ with $h + \gamma_2 > 1$,*

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} (2m\pi i)^{-h} \Big(\sum_{l=0}^{n-1} \eta^{l(1-h)} \prod_{\substack{j=1\\ j+l\neq n}}^{n} g(2m\pi i \eta^{j+l})\Big) \operatorname*{Res}_{t=2m\pi i} g(-t)$$
$$= -\operatorname*{Res}_{t=0}\left\{t^{-h}\prod_{j=1}^{n} g(\eta^j t)\right\}. \tag{5.1}$$

In particular when g is an even function,

$$\mathcal{Z}_h \sum_{m\in\mathbb{Z}\setminus\{0\}} (2m\pi i)^{-h}\left(\prod_{j=1}^{n-1} g(2m\pi i\eta^j)\right)\operatorname*{Res}_{t=2m\pi i} g(t)$$
$$= -\operatorname*{Res}_{t=0}\left\{t^{-h}\prod_{j=0}^{n-1} g(\eta^j t)\right\}, \tag{5.2}$$

where

$$\mathcal{Z}_h = \sum_{j=0}^{n-1}\eta^{j(1-h)} = \begin{cases} n & \text{if } \ h \equiv 1 \pmod{2n},\\ 0 & \text{if } \ h \not\equiv 1 \pmod{2n} \ \text{and } 2 \nmid h,\\ \frac{2}{1-\eta^{1-h}} & \text{if } \ h \not\equiv 1 \pmod{2n} \ \text{and } 2 \mid h. \end{cases} \tag{5.3}$$

Proof. Let $h \in \mathbb{Z}$ with $h + \gamma_2 > 1$. For $R \in \mathbb{N}$, we have

$$\begin{aligned}\operatorname*{Res}_{t=0}\{G(t)t^{-h}\} &= \frac{1}{2\pi i}\int_{|t|=2\varepsilon} G(t)t^{-h}dt\\ &= -\sum_{l=0}^{n-1}\sum_{0<|m|\le R}\operatorname*{Res}_{t=2m\pi i\eta^l}\{G(t)t^{-h}\}\\ &\quad + \frac{1}{2\pi i}\int_{|t|=2\pi R+2\varepsilon} G(t)t^{-h}dt\\ &= -\sum_{l=0}^{n-1}\sum_{0<|m|\le R}\operatorname*{Res}_{t=2m\pi i}\{G(\eta^l t)(\eta^l t)^{-h}\eta^l\}\\ &\quad + \frac{1}{2\pi i}\int_{|t|=2\pi R+2\varepsilon} G(t)t^{-h}dt.\end{aligned} \tag{5.4}$$

Since $G(t)t^{-h} \le \gamma_1|t|^{-h-\gamma_2}$, we have

$$\left|\int_{|t|=2\pi R+2\varepsilon} G(t)t^{-h}dt\right| \le \int_{|t|=2\pi R+2\varepsilon} |G(t)t^{-h}||dt|$$
$$\le \gamma_1 \int_{|t|=2\pi R+2\varepsilon} |t|^{-h-\gamma_2}|dt|$$
$$\le 2\pi\gamma_1(2\pi R+2\varepsilon)^{-h-\gamma_2+1}$$
$$\to 0 \tag{5.5}$$

as $R \to \infty$. Hence by letting $R \to \infty$, we obtain

$$\operatorname*{Res}_{t=0}\{G(t)t^{-h}\} = -\sum_{l=0}^{n-1} \eta^{l(1-h)} \sum_{m\in\mathbb{Z}\setminus\{0\}} \operatorname*{Res}_{t=2m\pi i}\{G(\eta^l t)t^{-h}\} \tag{5.6}$$

because

$$\left|\operatorname*{Res}_{t=2m\pi i} G(\eta^l t)t^{-h}\right| \le \frac{1}{2\pi}\int_{|t-2m\pi i|=2\varepsilon} |G(\eta^l t)t^{-h}||dt|$$
$$\le \frac{\gamma_1}{2\pi}\int_{|t-2m\pi i|=2\varepsilon} |t|^{-h-\gamma_2}|dt|$$
$$\le 2\varepsilon\gamma_1(2\pi|m|-2\varepsilon)^{-h-\gamma_2} \tag{5.7}$$

and hence the convergence is absolute.

Since η is the primitive $2n$-th root of unity, we see that for $l \in \mathbb{Z}$ with $0 \le l \le n-1$, the residue of

$$t^{-h}G(\eta^l t) = t^{-h}\prod_{j=1}^{n} g(\eta^{j+l}t)$$

at $t = 2m\pi i$ is equal to

$$(2m\pi i)^{-h} \prod_{\substack{j=1\\ j+l\ne n}}^{n} g(2m\pi i\eta^{j+l}) \times \operatorname*{Res}_{t=2m\pi i} g(-t),$$

which gives (5.1).

In particular when g is even, we have $G(\eta^l t) = G(t)$ $(0 \le l \le n-1)$, because $g(\eta^r t) = g(-\eta^{r-n}t) = g(\eta^{r-n}t)$ for $n+1 \le r < 2n$. Therefore

$$\sum_{l=0}^{n-1}\eta^{l(1-h)} \sum_{m\in\mathbb{Z}\setminus\{0\}} \operatorname*{Res}_{t=2m\pi i}\{t^{-h}G(\eta^l t)\} = \mathcal{Z}_h \sum_{m\in\mathbb{Z}\setminus\{0\}} \operatorname*{Res}_{t=2m\pi i}\{t^{-h}G(t)\}.$$

This completes the proof. □

It is to be noted that the original method of Cauchy [10] is essentially similar.

6. Explicit formulas

We recall the Bernoulli polynomials $\{B_j(y)\}$ defined by (1.3):

$$F(t, y) = \frac{te^{ty}}{e^t - 1} = \sum_{j=0}^{\infty} B_j(y)\frac{t^j}{j!}.$$

For $y \in \mathbb{R}$, let

$$\begin{aligned} H(t, y) &= \frac{F(t, y) + F(-t, y)}{2} = \frac{t}{2}\frac{e^{t(y-1/2)} + e^{-t(y-1/2)}}{e^{t/2} - e^{-t/2}} \\ &= \frac{t}{2}\frac{\cosh(t(y-1/2))}{\sinh(t/2)}. \end{aligned} \tag{6.1}$$

Then we see that

$$H(t, y) = \sum_{m=0}^{\infty} B_{2m}(y)\frac{t^{2m}}{(2m)!}. \tag{6.2}$$

It follows from (6.1) that $H(t, y)$ has simple poles at $t = 2m\pi i$ $(m \in \mathbb{Z}\setminus\{0\})$ and its residue is

$$\operatorname*{Res}_{t=2m\pi i} H(t, y) = \frac{2m\pi i}{2}\frac{e^{2m\pi i(y-1/2)} + e^{-2m\pi i(y-1/2)}}{(-1)^m} = 2m\pi i\cos(2m\pi y). \tag{6.3}$$

By (5.2), we have the following result which includes the known formulas (1.2) and (1.4) given by Cauchy, Mellin, Ramanujan and so on (see Section 1.).

Theorem 6.1. *Assume* $0 < y < 1$ *and* $p \in \mathbb{Z}$, *or* $y = 0, 1$ *and* $p > n/2$. *For* $n \in \mathbb{N}$ *with* $n \geq 2$ *and* $\eta = e^{\pi i/n}$,

$$\begin{aligned} &\mathcal{Z}_{2p+1} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\cos(2m\pi y)}{m^{2p+1-n}} \left(\prod_{j=1}^{n-1} \frac{\cosh(2m\pi i\eta^j(y-1/2))}{\sinh(m\pi i\eta^j)}\right) \\ &= -\frac{2^{n-1}(2\pi i)^{2p+1-n}}{\eta^{n(n-1)/2}} \sum_{\substack{m_1,\ldots,m_n\geq 0 \\ m_1+\cdots+m_n=p}} \prod_{\nu=1}^{n} \frac{B_{2m_\nu}(y)}{(2m_\nu)!}\eta^{2(\nu-1)m_\nu}, \end{aligned} \tag{6.4}$$

where $\mathcal{Z}_h$ *is defined by* (5.3). *Furthermore, assume* $0 < y < 1$ *and* $p \geq n/2$. *Then the both sides of* (6.4) *are also equal to*

$$\frac{(2\pi i)^{2p+1-n}(-1)^{1-n}}{2(2p-n)!} \times \sum_{y_0 \in \{y, 1-y\}} \cdots \sum_{y_{n-1} \in \{y, 1-y\}} \zeta_n\left(n-2p, \sum_{j=0}^{n-1} \eta^j y_j; 1, \eta, \ldots, \eta^{n-1}\right). \tag{6.5}$$

Proof. Since $H(-t, y) = H(t, y)$, we can apply (5.2) with $g(t) = H(t, y)$ and $h = 2p+1$. In this case γ_2 is arbitrarily large for $0 < y < 1$ and $\gamma_2 = -n$ for $y = 0, 1$, and hence the condition $h + \gamma_2 > 1$ is satisfied because we assume $p > n/2$ if $y = 0, 1$. By (6.2), we have

$$H(\eta^j t, y) = \sum_{m=0}^{\infty} B_{2m}(y) \frac{\eta^{2jm} t^{2m}}{(2m)!}.$$

Hence we obtain

$$\operatorname*{Res}_{t=0}\left\{t^{-2p-1} \prod_{j=0}^{n-1} H(\eta^j t, y)\right\} = \sum_{\substack{m_1, \ldots, m_n \geq 0 \\ m_1 + \cdots + m_n = p}} \prod_{\nu=1}^{n} \frac{B_{2m_\nu}(y)}{(2m_\nu)!} \eta^{2(\nu-1)m_\nu}.$$

Therefore, by using (6.1) and (6.3), we have

$$\begin{aligned}
&\mathcal{Z}_{2p+1} \sum_{m \in \mathbb{Z} \setminus \{0\}} (2m\pi i)^{-2p-1} \\
&\qquad \times \left(\prod_{j=1}^{n-1} m\pi i \eta^j \frac{\cosh(2m\pi i \eta^j (y - 1/2))}{\sinh(m\pi i \eta^j)}\right) 2m\pi i \cos(2m\pi y) \\
&\quad = -\sum_{\substack{m_1, \ldots, m_n \geq 0 \\ m_1 + \cdots + m_n = p}} \prod_{\nu=1}^{n} \frac{B_{2m_\nu}(y)}{(2m_\nu)!} \eta^{2(\nu-1)m_\nu}.
\end{aligned}$$

Thus we obtain (6.4).

Assume $0 < y < 1$ and $p \geq n/2$. Then $h = 2p + 1 \geq n + 1$. Note that

$$F(-t, y) = \frac{-te^{-ty}}{e^{-t} - 1} = \frac{te^{t(1-y)}}{e^t - 1} = F(t, 1 - y).$$

Then we have

$$G(t)=\prod_{j=0}^{n-1}H(\eta^j t,y)=\prod_{j=0}^{n-1}\frac{F(\eta^j t,y)+F(\eta^j t,1-y)}{2}$$
$$=2^{-n}\sum_{y_0\in\{y,1-y\}}\cdots\sum_{y_{n-1}\in\{y,1-y\}}\prod_{j=0}^{n-1}F(\eta^j t,y_j). \tag{6.6}$$

Hence

$$\operatorname*{Res}_{t=0}\{G(t)t^{-h}\}$$
$$=\frac{2^{-n}}{2\pi i}\sum_{y_0\in\{y,1-y\}}\cdots\sum_{y_{n-1}\in\{y,1-y\}}\int_{|t|=\varepsilon}\left(\prod_{j=0}^{n-1}F(\eta^j t,y_j)\right)t^{(n+1-h)-n-1}dt. \tag{6.7}$$

Since $0<y<1$, we see that $1,\eta,\ldots,\eta^{n-1}$ and $\sum_{j=0}^{n-1}\eta^j(1-y_j)$ are belonging to the half plane $H(\theta_n)$, where $\theta_n=\pi/2-\pi/(2n)$. Therefore, deforming the path $|t|=\varepsilon$ to $C(\theta_n)$, we find that each integral on the right-hand side of (6.7) is of the same form as the integral on the right-hand side of (2.5) with $s=n+1-h$. Using (2.5) and (2.12), we obtain that the right-hand side of (6.7) is equal to

$$\frac{2^{-n}\eta^{n(n-1)/2}}{2\pi i}\sum_{y_0\in\{y,1-y\}}\cdots\sum_{y_{n-1}\in\{y,1-y\}}\frac{2\pi i(-1)^{h-n-1}}{(h-n-1)!}$$
$$\times\zeta_n\left(n+1-h,\sum_{j=0}^{n-1}\eta^j(1-y_j);1,\eta,\ldots,\eta^{n-1}\right)$$
$$=\frac{2^{-n}(-1)^{h-n-1}\eta^{n(n-1)/2}}{(h-n-1)!}\sum_{y_0\in\{y,1-y\}}\cdots\sum_{y_{n-1}\in\{y,1-y\}}$$
$$\times\zeta_n\left(n+1-h,\sum_{j=0}^{n-1}\eta^j y_j;1,\eta,\ldots,\eta^{n-1}\right),$$

which implies (6.5). □

In particular when $y=\frac{1}{2}$ in (6.4) and (6.5), we have the following formula, which can be regarded as a multiple generalization of (1.2).

Corollary 6.2. *For $p \in \mathbb{Z}$, $n \in \mathbb{N}$ with $n \geq 2$ and $\eta = e^{\pi i/n}$,*

$$\mathcal{Z}_{2p+1} \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{(-1)^m}{\left(\prod_{j=1}^{n-1} \sinh(m\pi i \eta^j)\right) m^{2p+1-n}}$$

$$= -\frac{2^{n-1}(2\pi i)^{2p+1-n}}{\eta^{n(n-1)/2}} \sum_{\substack{m_1,\ldots,m_n \geq 0 \\ m_1+\cdots+m_n=p}} \prod_{\nu=1}^{n} \frac{B_{2m_\nu}(1/2)}{(2m_\nu)!} \eta^{2(\nu-1)m_\nu}. \tag{6.8}$$

Furthermore if $p \geq n/2$, then (6.8) *is equal to*

$$\frac{(2\pi i)^{2p+1-n}(-1)^{1-n}2^{n-1}}{2(2p-n)!} \zeta_n(n-2p, 1/(1-\eta); 1, \eta, \ldots, \eta^{n-1}). \tag{6.9}$$

Now we consider the case $n = 2$, $\eta = e^{\pi i/2} = i$ and $p = 2k+2$ in (6.4). Then we obtain the following.

Corollary 6.3. *For $k \in \mathbb{N}_0$ and $0 \leq y \leq 1$,*

$$\sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{\cos(2m\pi y)}{m^{4k+3}} \left(\frac{\cosh(2m\pi(y-1/2))}{\sinh(m\pi)} \right)$$

$$= (2\pi)^{4k+3} \sum_{j=0}^{2k+2} (-1)^{j+1} \frac{B_{2j}(y)}{(2j)!} \frac{B_{4k+4-2j}(y)}{(4k+4-2j)!}. \tag{6.10}$$

In particular when $y = 0$ and $y = \frac{1}{2}$, we obtain (1.4) *and* (1.2)*, respectively.*

Next we consider the case $n = 3$. Let $\eta = e^{\pi i/3} = -\rho^2$ with $\rho = e^{2\pi i/3}$ and $p = 3k+3$. Then we have $\eta^{3(3-1)/2} = -1$ and $\mathcal{Z}_{6(k+1)+1} = 3$ by (5.3). From (6.4), we have

$$\sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{\cos(2m\pi y)}{m^{6k+4}} \left(\frac{\cosh(2m\pi i(-\rho^2)(y-1/2)) \cosh(2m\pi i\rho(y-1/2))}{\sinh(m\pi i(-\rho^2)) \sinh(m\pi i\rho)} \right)$$

$$= -\frac{2^2(2\pi i)^{6k+4}}{-3} \sum_{\substack{m_1,m_2,m_3 \geq 0 \\ m_1+m_2+m_3=3k+3}} \prod_{\nu=1}^{3} \frac{B_{2m_\nu}(y)}{(2m_\nu)!} \rho^{2(\nu-1)m_\nu}. \tag{6.11}$$

Note that $\rho^2 = 1/\rho$ and $\rho = -1/\rho - 1$. Hence, by using

$$\sinh(m\pi i\rho) = -(-1)^m \sinh(m\pi i/\rho),$$

we can rewrite (6.11) as follows.

Corollary 6.4. *For $k \in \mathbb{N}_0$ and $0 \le y \le 1$,*

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m \cos(2m\pi y)}{m^{6k+4}} \times \left(\frac{\cosh((2m\pi i/\rho)(y-1/2))\cosh(2m\pi i\rho(y-1/2))}{\sinh(m\pi i/\rho)^2}\right) = \frac{4(-1)^k(2\pi)^{6k+4}}{3} \sum_{\substack{m_1,m_2,m_3\ge 0\\ m_1+m_2+m_3=3k+3}} \frac{B_{2m_1}(y)}{(2m_1)!}\frac{B_{2m_2}(y)}{(2m_2)!}\frac{B_{2m_3}(y)}{(2m_3)!}\rho^{m_2+2m_3}. \tag{6.12}$$

In particular when $y=0$ and $y=\frac{1}{2}$, the following equations hold:

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\coth(m\pi i/\rho)^2}{m^{6k+4}} = \frac{4(-1)^k(2\pi)^{6k+4}}{3} \sum_{\substack{m_1,m_2,m_3\ge 0\\ m_1+m_2+m_3=3k+3}} \frac{B_{2m_1}(0)}{(2m_1)!}\frac{B_{2m_2}(0)}{(2m_2)!}\frac{B_{2m_3}(0)}{(2m_3)!}\rho^{m_2+2m_3}, \tag{6.13}$$

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{\sinh(m\pi i/\rho)^2 m^{6k+4}} = \frac{4(-1)^k(2\pi)^{6k+4}}{3} \sum_{\substack{m_1,m_2,m_3\ge 0\\ m_1+m_2+m_3=3k+3}} \frac{B_{2m_1}(1/2)}{(2m_1)!}\frac{B_{2m_2}(1/2)}{(2m_2)!}\frac{B_{2m_3}(1/2)}{(2m_3)!}\rho^{m_2+2m_3}. \tag{6.14}$$

Example 6.5. From (6.13) and (6.14), we obtain

$$\sum_{m\neq 0} \frac{\coth(m\pi i/\rho)^2}{m^4} = \frac{62}{2835}\pi^4, \tag{6.15}$$

$$\sum_{m\neq 0} \frac{\coth(m\pi i/\rho)^2}{m^{10}} = \frac{40247}{1915538625}\pi^{10}, \tag{6.16}$$

$$\sum_{m\neq 0} \frac{1}{\sinh(m\pi i/\rho)^2 m^4} = -\frac{1}{2835}\pi^4, \tag{6.17}$$

$$\sum_{m\neq 0} \frac{1}{\sinh(m\pi i/\rho)^2 m^{10}} = -\frac{703}{1915538625}\pi^{10}. \tag{6.18}$$

Additionally, by setting $(p, n) = (4, 4), (5, 5)$ in equation (6.8), we obtain

$$\sum_{m\neq 0} \frac{(-1)^m}{\sinh(m\pi)\sinh(m\pi i\zeta_8)\sinh(m\pi i\zeta_8^{-1})m^5} = \frac{1}{37800}\pi^5, \tag{6.19}$$

$$\sum_{m\neq 0} \frac{(-1)^m}{\sinh(m\pi i\zeta_5)\sinh(m\pi i\zeta_5^2)\sinh(m\pi i\zeta_5^3)\sinh(m\pi i\zeta_5^4)m^6} = -\frac{1}{467775}\pi^6, \tag{6.20}$$

where $\zeta_k = e^{2\pi i/k}$ $(k \in \mathbb{N})$.

Remark 6.6. By using the same method as introduced in this paper, we can recover the known formulas, for example,

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\cosh((2m+1)\pi/2)((2m+1)/2)^{4k+1}} = \frac{(2\pi)^{4k+1}}{8}\sum_{j=0}^{2k}(-1)^j \frac{E_{2j}(1/2)}{(2j)!}\frac{E_{4k-2j}(1/2)}{(4k-2j)!},$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\cosh((2m+1)\sqrt{3}\pi/2)((2m+1)/2)^{6k+1}} = \frac{(-1)^{k+1}(2\pi)^{6k+1}}{2}\sum_{j=0}^{3k}\frac{E_{2j+1}(0)}{(2j+1)!}\frac{B_{6k-2j}(0)}{(6k-2j)!}\cos\left(\frac{(2j+1)\pi}{3}\right)$$

for $k \in \mathbb{N}_0$ (see Watson [17] and Berndt [7]), where $\{E_n(x)\}$ are the Euler polynomials defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

More generally, we can give relevant analogues of these results like those in Example 6.5.

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