ON THE SPEED OF CONVERGENCE TO LIMIT DISTRIBUTIONS FOR HECKE L-FUNCTIONS ASSOCIATED WITH IDEAL CLASS CHARACTERS

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1. Introduction

Let K be an algebraic number field of degree ℓ, $L = \max\{\ell, 2\}$, $s = \sigma + it$ a complex variable, and $\zeta_K(s)$ the Dedekind zeta-function attached to K. In [6] [7] [8], the value-distribution of $\log \zeta_K(s)$ in the half-plane $\sigma = \Re s > 1 - L^{-1}$ has been studied.

The definition of $\log \zeta_K(\sigma + it)$ is clear for $\sigma > 1$, and for $1 - L^{-1} < \sigma \leq 1$ we define this function by analytic continuation along the horizontal line segment from $2 + it$. In case there exists a zero or a pole of $\zeta_K(s)$ on this line segment, we do not define $\log \zeta_K(s)$.

Let R be any fixed closed rectangle in the complex plane $\mathbb{C}$ with the edges parallel to the axes. We write $\mu_n(\cdot)$ for the n-dimensional Lebesgue measure. For any fixed $\sigma > 1 - L^{-1}$, let

$$V_K(T; R) = \mu_1(\{t \in [1, T] \mid \log \zeta_K(\sigma + it) \in R\}).$$

Then there exists the limit

$$W_K(R) = \lim_{T \to \infty} \frac{1}{T} V_K(T; R). \tag{1.1}$$

This was proved by Bohr and Jessen [1] [2] for the Riemann zeta-function $\zeta(s)$, and by the author [6] [7] for general case.

In [8], the author studied the speed of convergence on the right-hand side of (1.1), and proved

$$W_K(R) - \frac{1}{T} V_K(T; R) = O\left((\mu_2(R) + 1)(\log T)^{-C(\sigma)+\varepsilon}\right) \tag{1.2}$$

for any $\sigma > 1 - L^{-1}$ and any $\varepsilon > 0$, where

$$C(\sigma) = \begin{cases} (\sigma - 1)/(3 + 2\sigma) & (\sigma > 1), \\ 2(2\sigma - 1)/(21 + 8\sigma) & (1 \geq \sigma > 1 - L^{-1}). \end{cases} \tag{1.3}$$

In the case of $\zeta(s)$, the estimate (1.2) was first proved in a joint paper of Harman and the author [3]. This paper [3] gives an improvement of former weaker results proved in the author's previous papers [4] [5] [7]. In [7], such a weaker result was also shown for $\zeta_K(s)$ when K is a Galois extension of the rational number field $\mathbb{Q}$. In [3] it is mentioned without proof that (1.2) can be shown for $\zeta_K(s)$

of any Galois number field. Finally in [8], the proof of (1.2) for any (Galois or non-Galois) number field has been given.

The purpose of the present paper is to generalize (1.2) to the case of Hecke L-functions associated with ideal class characters. Recently, the value-distribution of Hecke L-functions of number fields has been studied extensively by Mishou [10] [11] [12] [13] [14] [15] (partly with Koyama). The present paper is another contribution to this topic.

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2. Statement of the result

First we recall the definition and basic properties of Hecke L-functions.

Let K be as in Section 1, $\mathcal{O}_K$ the ring of integers of K, r_1 the number of real places of K, and $2r_2$ the number of complex places of K. Denote by I the set of all ideals of $\mathcal{O}_K$, and by J the set of all fractional ideals of K. Fix an ideal $\mathfrak{f} \in I$, and define

$$J(\mathfrak{f}) = \{\mathfrak{a} \in J \mid (\mathfrak{a}, \mathfrak{f}) = 1\},$$
$$P(\mathfrak{f}) = \{(\alpha) \mid \alpha \in K, \alpha \equiv 1 \bmod \tilde{\mathfrak{f}}\},$$

where (α) denotes the principal ideal generated by α, and $\alpha \equiv 1 \bmod \tilde{\mathfrak{f}}$ means that α is totally positive and if we write $\alpha = a/b$, $a, b \in \mathcal{O}_K$, $(a, b) = 1$, then $a - b \in \mathfrak{f}$. Then $P(\mathfrak{f})$ is a subgroup of $J(\mathfrak{f})$ and the quotient

$$Cl(\mathfrak{f}) = J(\mathfrak{f})/P(\mathfrak{f}),$$

the ideal class group modulo $\mathfrak{f}$, is a finite Abelian group. Denote the projection map by π.

Let χ be a character of $Cl(\mathfrak{f})$. Define the mapping $\chi : I \setminus \{0\} \to \mathbb{C}$ (ideal class character) by

$$\chi(\mathfrak{a}) = \begin{cases} \chi(\pi(\mathfrak{a})) & \text{if } \mathfrak{a} \in I \cap J(\mathfrak{f}), \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{B}$ the set of all values taken by the ideal class character χ. Clearly this is a finite set.

The Hecke L-function associated with χ is

$$L_K(s, \chi) = \sum_{\mathfrak{a} \in I \setminus \{0\}} \chi(\mathfrak{a})(N\mathfrak{a})^{-s}, \tag{2.1}$$

where $N\mathfrak{a}$ denotes the norm of $\mathfrak{a}$. This series is convergent absolutely for $\sigma > 1$, and can be continued meromorphically to the whole plane $\mathbb{C}$. The functional equation is

$$\xi_K(1-s,\bar{\chi}) = \xi_K(s,\chi), \tag{2.2}$$

where

$$\xi_K(s,\chi) = d(\mathfrak{f})^s \Gamma(s)^{r_2} \prod_{m=1}^{r_1} \Gamma\left(\frac{s+a_m}{2}\right) L_K(s,\chi)$$

with a constant $d(\mathfrak{f})$ depending only on $\mathfrak{f}$ and $a_m \in \{0,1\}$ $(1 \le m \le r_1)$. Hence the critical strip is $0 \le \sigma \le 1$, and the critical line is $\sigma = 1/2$.

Define $\log L_K(s,\chi)$ for $\sigma > 1 - L^{-1}$ as in the case of $\log \zeta_K(s)$ explained in Section 1. Let

$$V_K(T;R,\chi) = \mu_1(\{t \in [1,T] \mid \log L_K(\sigma+it,\chi) \in R\})$$

for any fixed $\sigma > 1 - L^{-1}$. Then the existence of the limit

$$W_K(R,\chi) = \lim_{T\to\infty} \frac{1}{T} V_K(T;R,\chi) \tag{2.3}$$

can be established. This is a special case of a general limit theorem proved in [6]. The restriction $\sigma > 1 - L^{-1}$ comes from the fact that, at present, we can prove the mean square estimate

$$\int_1^T |L_K(\sigma+it,\chi)|^2 dt = O(T) \tag{2.4}$$

only for $\sigma > 1 - L^{-1}$. This follows from the functional equation (2.2) and Potter's general result [17]. The estimate (2.4) is necessary to apply the result of [6].

In the present paper we will prove the following generalization of (1.2).

Theorem. *Let K be an algebraic number field of degree ℓ, and let $L = \max\{\ell, 2\}$. Then, for any $\varepsilon > 0$, we have*

$$W_K(R,\chi) - \frac{1}{T} V_K(T;R,\chi) = O\left((\mu_2(R)+1)(\log T)^{-C(\sigma)+\varepsilon}\right) \tag{2.5}$$

for $\sigma > 1 - L^{-1}$, where $C(\sigma)$ is given by (1.3).

The basic structure of the proof is the same as in [8], so we omit the details, only describing several key points of the argument in the following three sections.

It is desirable to generalize the above theorem further to the case of Hecke L-functions associated with any Grössencharacters, but it seems that the argument in the present paper is not sufficient for that purpose.

3. Limit distributions for finite truncations

It is well known that $L_K(s,\chi)$ has the Euler product expansion

$$L_K(s,\chi) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{(N\mathfrak{p})^s}\right)^{-1} \qquad (\sigma > 1), \tag{3.1}$$

where $\mathfrak{p}$ runs over all prime ideals. Let p_n be the n-th prime number, and $\mathfrak{p}_n^{(1)}, \ldots, \mathfrak{p}_n^{(g(n))}$ the prime divisors of p_n with norm $N\mathfrak{p}_n^{(j)} = p_n^{f(j,n)}$ $(1 \leq j \leq g(n))$. Then

$$L_K(s,\chi) = \prod_{n=1}^{\infty} \prod_{j=1}^{g(n)} \left(1 - \chi(\mathfrak{p}_n^{(j)})(N\mathfrak{p}_n^{(j)})^{-s}\right)^{-1}$$
$$= \prod_{n=1}^{\infty} \prod_{j=1}^{g(n)} \left(1 - \chi(\mathfrak{p}_n^{(j)}) p_n^{-f(j,n)\sigma} \exp(-if(j,n)t \log p_n)\right)^{-1}.$$

Let N be a positive integer, $\sigma > 1 - L^{-1}$, and consider the finite truncation

$$L_{N,K}(s,\chi) = \prod_{n=1}^{N} \prod_{j=1}^{g(n)} \left(1 - \chi(\mathfrak{p}_n^{(j)}) p_n^{-f(j,n)\sigma} \exp(-if(j,n)t \log p_n)\right)^{-1}. \tag{3.2}$$

Let $Q_N = [0,1)^N$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N) \in Q_N$,

$$z_{n,K}(\theta_n,\chi) = -\sum_{j=1}^{g(n)} \log \left(1 - \chi(\mathfrak{p}_n^{(j)}) p_n^{-f(j,n)\sigma} \exp(2\pi i f(j,n)\theta_n)\right), \tag{3.3}$$

and

$$S_{N,K}(\boldsymbol{\theta},\chi) = \sum_{n=1}^{N} z_{N,K}(\theta_n,\chi). \tag{3.4}$$

Then

$$\log L_{N,K}(s,\chi) = S_{N,K}(\mathbf{x}(t),\chi), \tag{3.5}$$

where

$$\mathbf{x}(t) = \left(\left\{-\frac{t}{2\pi}\log p_1\right\}, \ldots, \left\{-\frac{t}{2\pi}\log p_N\right\}\right)$$

($\{x\} = x - [x]$ is the fractional part of x). Define

$$V_{N,K}(T;R,\chi) = \mu_1(\{t \in [1,T] \mid \log L_{N,K}(\sigma+it,\chi) \in R\}).$$

From (3.5) we see that $\log L_{N,K}(\sigma+it,\chi) \in R$ if and only if $\mathbf{x}(t) \in \Omega_N(R,\chi)$, where

$$\Omega_N(R,\chi) = \{\boldsymbol{\theta} \in Q_N \mid S_{N,K}(\boldsymbol{\theta},\chi) \in R\}.$$

The uniqueness of the decomposition of integers into prime factors implies that $\log p_1, \ldots, \log p_N$ are linearly independent over $\mathbb{Q}$. Hence, by the Kronecker-Weyl theorem, we can show the existence of the limit

$$W_{N,K}(R,\chi) = \lim_{T\to\infty} \frac{1}{T} V_{N,K}(T;R,\chi), \tag{3.6}$$

and moreover $W_{N,K}(R,\chi) = \mu_N(\Omega_N(R,\chi))$. The latter shows that $W_{N,K}$ is a probability measure on $\mathbb{C}$.

We evaluate the speed of convergence on the right-hand side of (3.6).

Proposition 1. *Let N be sufficiently large, and let m and r be large positive integers with $2rN \leq m$. Then*

$$\begin{aligned}&\left|W_{N,K}(R,\chi)-\frac{1}{T}V_{N,K}(T;R,\chi)\right|\\&\ll\frac{N^{1/2}}{r}+\frac{Nr}{m}+\frac{1}{T}(6r\log m)^N\exp(mN\log N).\end{aligned}\tag{3.7}$$

In the case of Dedekind zeta-functions, this is Proposition 1 of [8], whose main idea goes back to [3] (and even [4] [9]).

In [8], the proposition has been deduced from (5.1), (5.2), (5.3) and Lemma 3 of [8]. Hence our task here is to generalize those to our present situation.

Inequalities (5.1), (5.2) and (5.3) of [8] were first proved in [3] in the case of $\zeta(s)$, and the method in [3] can be applied without change to our present situation.

To prove Lemma 3 of [8], we used in [8] the Artin-Chebotarev density theorem to find a suitable rearrangement of the sequence of prime numbers. Here we apply the Artin-Chebotarev density theorem (see Proposition 7.15 of Narkiewicz [16]) in a slightly different way; by the Artin-Chebotarev theorem we see that there exist infinitely many primes p_n for which $g(n)=1$, $f(1,n)=\ell$ hold. Moreover, since there are only finitely many prime factors of $\mathfrak{f}$, we may assume that the above p_ns are coprime with $\mathfrak{f}$. Denote the first three of those primes by $p_{n(1)}$, $p_{n(2)}$ and $p_{n(3)}$, and define the rearrangement of primes, by using these $p_{n(\nu)}$ ($\nu=1,2,3$), similarly as in Section 5 of [8]. The curve $\Gamma_{n(\nu)}$ described by

$$z_{n(\nu),K}(\theta_{n(\nu)},\chi)=-\log\left(1-\chi(\mathfrak{p}_{n(\nu)}^{(1)})p_{n(\nu)}^{-\ell\sigma}\exp(2\pi i\ell\theta_{n(\nu)})\right)\tag{3.8}$$

($0\leq\theta_{n(\nu)}<1$) is convex ($\nu=1,2,3$). Since $(\mathfrak{p}_{n(\nu)}^{(1)},\mathfrak{f})=1$, we have $|\chi(\mathfrak{p}_{n(\nu)}^{(1)})|=1$. Write $\chi(\mathfrak{p}_{n(\nu)}^{(1)})=\exp(2\pi i\ell\varphi(\nu))$ with a certain real number $\varphi(\nu)$, and put $\theta'_{n(\nu)}=\ell(\theta_{n(\nu)}+\varphi(\nu))$. Then

$$z_{n(\nu),K}(\theta_{n(\nu)},\chi)=-\log\left(1-p_{n(\nu)}^{-\ell\sigma}\exp(2\pi i\theta'_{n(\nu)})\right),\tag{3.9}$$

and this describes the same curve as $\Gamma_{n(\nu)}$ when $\theta'_{n(\nu)}$ moves from 0 to ℓ. The difference from the argument in [8] is that, in the present case, $z_{n(\nu),K}(\theta_n(\nu),\chi)$ rounds ℓ-times along the curve $\Gamma_{n(\nu)}$ when $\theta'_{n(\nu)}$ moves from 0 to ℓ. When $\theta'_{n(\nu)}$ moves on the subinterval $[k,k+1)$ ($0\leq k\leq\ell-1$), we can show the analogue of Lemma 3 of [3] for $z_{n(\nu),K}(\theta_n(\nu),\chi)$. Hence the analogue of Lemma 4 of [3] can also be established for each k. Therefore, adding them, we find that the analogue of Lemma 4 of [3] is valid in our present situation.

To prove the analogue of Lemma 3 of [8], the remaining part of the proof is the same as in [8].

4. An application of Lévy's inversion formula

From (4.6) of [6] we have

$$\lim_{N\to\infty} W_{N,K}(R,\chi)=W_K(R,\chi) \tag{4.1}$$

for any rectangle R. In this section we evaluate the speed of this convergence to show, as a generalization of Proposition 2 of [8], the following

Proposition 2. *For any sufficiently large N, we have*

$$|W_K(R,\chi)-W_{N,K}(R,\chi)|\ll \mu_2(R)N^{1-2\sigma}(\log N)^{-2\sigma}. \tag{4.2}$$

The proof is based on Lévy's inversion formula. Consider the Fourier transform

$$\begin{aligned}\Lambda_{N,K}(w,\chi)&=\int_{\mathbb{C}} e^{i<z,w>}dW_{N,K}(z,\chi)\\ &=\int_{Q_N}\exp\left(i<S_{N,K}(\boldsymbol{\theta},\chi),w>\right)d\mu_N(\boldsymbol{\theta}),\end{aligned}$$

where $<z,w>=\Re z\Re w+\Im z\Im w$. Then the right-hand side is the product of

$$K_{n,K}(w,\chi)=\int_0^1\exp\left(i<z_{n,K}(\theta_n,\chi),w>\right)d\theta_n \qquad (1\leq n\leq N) \tag{4.3}$$

(see (3.4)).

In order to use Lévy's inversion formula, it is necessary to obtain a suitable upper bound of $|K_{n,K}(w,\chi)|$. For this purpose, in [8], we use the fact that there are only finitely many patterns of decomposition of primes into prime ideals of K. In the present case, we combine this fact with the finiteness of the set $\mathcal{B}$, introduced in Section 2.

For any integer g satisfying $1\leq g\leq \ell$, let $\mathcal{F}_g(\chi)$ be the set of all integer vectors

$$(\mathbf{f},\mathbf{b})=(f(1),\ldots,f(g),b(1),\ldots,b(g)),$$

for which there exists an n such that $g=g(n)$, $f(j)=f(j,n)$ and $b(j)=\chi(\mathfrak{p}_n^{(j)})$ $(1\leq j\leq g)$ holds.Then $\mathcal{F}_g(\chi)$ is a finite set, because $\mathcal{B}$ is finite and

$$\sum_{j=1}^{g(n)} e(j,n)f(j,n)=\ell \tag{4.4}$$

holds, where $e(j,n)$ is the ramification index of $\mathfrak{p}_n^{(j)}$ over p_n. Hence

$$\mathcal{F}(\chi)=\bigcup_{1\leq g\leq\ell}\mathcal{F}_g(\chi)$$

is also finite.

For each $(\mathbf{f},\mathbf{b})\in\mathcal{F}(\chi)$, define

$$F_{(\mathbf{f},\mathbf{b})}(v)=-\sum_{j=1}^{g}\log\left(1-b(j)v^{f(j)}\right). \tag{4.5}$$

Let $\mathbb{N}$ be the set of all positive integers. For any $n \in \mathbb{N}$, we can find a unique $(\mathbf{f}, \mathbf{b}) \in \mathcal{F}(\chi)$ which satifsies

$$z_{n,K}(\theta_n, \chi) = F_{(\mathbf{f},\mathbf{b})}\left(p_n^{-\sigma} \exp(2\pi i \theta_n)\right). \tag{4.6}$$

Let $\mathcal{N}(\mathbf{f}, \mathbf{b})$ be the set of all n for which (4.6) holds. Then we have

$$\mathbb{N} = \bigcup_{(\mathbf{f},\mathbf{b}) \in \mathcal{F}(\chi)} \mathcal{N}(\mathbf{f}, \mathbf{b}). \tag{4.7}$$

This decomposition corresponds to (3.9) of [8], and then, by the same argument as in [8], we can show

$$K_{n,K}(w, \chi) = O\left(p_n^{\sigma\ell/2} |w|^{-1/2}\right). \tag{4.8}$$

In the procedure of proving (4.8), it is important that $\mathcal{F}(\chi)$ is a finite set.

Estimate (4.8) is exactly the same as (3.16) of [8], and from which we can deduce the assertion of Proposition 2.

5. Completion of the proof

Now we can combine Proposition 1 and Proposition 2 to complete the proof of our theorem, quite similarly to the argument in Section 6 of [8]. The main tools used in Section 6 of [8] are Lemma 4 of [8] and estimate (6.9) of [8]. Lemma 4 of [8] can be generalized to the present case, by using the rearrangement of primes defined in Section 3. Estimate (6.9) of [8] is based on Lemma 5 of [7]. Tha latter is a certain mean value estimate of Dedekind zeta-functions. This can be generalized to the present case, because of (2.4). Therefore the argument in Section 6 of [8] can be applied without change to $L_K(s, \chi)$. The proof of our theorem is now complete.

Remark. When K is a Galois extension of $\mathbb{Q}$, we have $f(1, n) = \cdots = f(g(n), n)$ $(= f(n)$, say), hence (3.3) is

$$z_{n,K}(\theta_n, \chi) = -\sum_{j=1}^{g(n)} \log\left(1 - \chi(\mathfrak{p}_n^{(j)}) p_n^{-f(n)\sigma} \exp(2\pi i f(n)\theta_n)\right). \tag{5.1}$$

In the case of the Dedekind zeta-function $\zeta_K(s)$, this is further reduced to

$$-g(n) \log(1 - p_n^{-f(n)\sigma} \exp(2\pi i f(n)\theta_n)),$$

which describes a convex curve. This is the reason why in [3] it is mentioned that estimate (1.2), proved for the Riemann zeta-function in that paper, can be generalized to $\zeta_K(s)$ of any Galois extension. However, for Hecke L-functions, the curve described by (5.1) is not always convex (because of the existence of $\chi(\mathfrak{p}_n^{(j)})$) even in the case of Galois extensions. Therefore the idea in [8], originally developed for the purpose of treating $\zeta_K(s)$ in the non-Galois case, is necessary even for Galois extensions when we consider Hecke L-functions.

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