Šiauliai Math. Semin.,
3 (11), 2008, 189–205

ŠIAULIAI
MATHEMATICAL
SEMINAR
http://siauliaims.su.lt

FUNCTIONAL RELATIONS AMONG CERTAIN DOUBLE POLYLOGARITHMS AND THEIR CHARACTER ANALOGUES

Kohji MATSUMOTO[1]**, Hirofumi TSUMURA**[2]

[1]Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan;
e-mail: kohjimat@math.nagoya-u.ac.jp

[2]Tokyo Metropolitan University, 1-1, Minami-Ohsawa, Hachioji,
Tokyo 192-0397, Japan; e-mail: tsumura@tmu.ac.jp

Abstract. In this paper, we consider certain double polylogarithms and the ordinary polylogarithm of complex variables, and introduce a method of producing functional relations among them. Furthermore we consider χ-analogues of them. These functional relations can be regarded as polylogarithmic generalizations of the known relations between the Mordell-Tornheim double zeta-function and the Riemann zeta-function, and their character analogues, which were studied by the authors.

Key words and phrases: Dirichlet L-functions, double L-series, double zeta values, Mordell-Tornheim double zeta-functions, Riemann zeta-function, polylogarithms.

2000 Mathematics Subject Classification: 40B05, 11M06, 33E20.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

The polylogarithm $\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} x^n n^{-k}$ and their multiple analogues play important roles in various fields of mathematics, for example, number theory, theory of special functions, infinite analysis and related mathematical physics.

As fascinating formulas, it is known that (see [7], [8]), for example,

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{1}{2}\zeta(2) - \frac{1}{2}(\log 2)^2, \tag{1.1}$$

$$\begin{aligned}\mathrm{Li}_3\left(\frac{3-\sqrt{5}}{2}\right) &= \frac{4}{5}\zeta(3) + \frac{1}{5}\zeta(2)\log\left(\frac{3-\sqrt{5}}{2}\right)\\ &\quad -\frac{1}{12}\left[\log\left(\frac{3-\sqrt{5}}{2}\right)\right]^3,\end{aligned} \tag{1.2}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta-function.

In this paper, we consider the polylogarithm of a complex variable and its double analogues, defined by $\mathcal{P}_1(s;x) = \sum_{n=1}^{\infty} x^n n^{-s}$ and

$$\mathcal{P}_{*,2}(s_1,s_2,s_3;x) = \sum_{m,n=1}^{\infty} \frac{x^n}{m^{s_1} n^{s_2} (m+n)^{s_3}}, \tag{1.3}$$

$$\mathcal{P}_{ш,2}(s_1,s_2,s_3;x) = \sum_{m,n=1}^{\infty} \frac{x^{m+n}}{m^{s_1} n^{s_2} (m+n)^{s_3}}, \tag{1.4}$$

for $s, s_1, s_2, s_3 \in \mathbb{C}$ and $x \in \mathbb{C}$ with $|x| \leqslant 1$. Note that $\mathcal{P}_1(k;x) = \mathrm{Li}_k(x)$, $k \in \mathbb{N}$, and that $\mathcal{P}_{*,2}(0,k,l;x)$ and $\mathcal{P}_{ш,2}(0,k,l;x)$, $k, l \in \mathbb{N}$, are called the double polylogarithms which have been studied as special cases of multiple polylogarithms in recent research (see, for example, [2], [3], [4]).

On the other hand, $\mathcal{P}_1(s;x)$ can be viewed as a special case of Hurwitz-Lerch zeta-functions. Furthermore $\mathcal{P}_{*,2}(s_1,s_2,s_3;1) = \mathcal{P}_{ш,2}(s_1,s_2,s_3;1)$ is called the Mordell-Tornheim double zeta-function denoted by $\zeta_{MT,2}(s_1,s_2,s_3)$ or is called the Witten zeta-function associated with $\mathfrak{sl}(3)$ studied by the first named author in [9], [10] (see also [12]). He proved the meromorphic continuation of $\zeta_{MT,2}$ using the Mellin-Barnes formula. The name of this function is derived from the classical results of Tornheim and Mordell (see [14], [16]). They studied the values of $\zeta_{MT,2}$ at positive integers and gave some relation formulas. Recently, their results were generalized by the second named author in [17], [19].

The main aim of this paper is to give functional relations among $\mathcal{P}_{*,2}$, $\mathcal{P}_{ш,2}$ and $\mathcal{P}_1$ (see Theorem 2.5), for example,

$$2\mathcal{P}_{*,2}(1,s,1;x) - \mathcal{P}_{ш,2}(1,1,s;x) = 2\mathcal{P}_1(s+2;x)$$

for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 0$ and $x \in \mathbb{C}$ with $|x| \leqslant 1$ (see Theorem 2.1). In particular, when $x = 1$, we obtain the known functional relation between $\zeta_{MT,2}$ and the Riemann zeta-function $\zeta(s)$ (see [13], [19]) which includes the well-known sum formula for the double zeta values given by Euler. On the other hand, as relations among the values at positive integers, we give some relation formulas for the double polylogarithms, which include a polylogarithmic generalization of the Mordell-Zagier formula ([14], [23]) given by the second named author in [20].

Additionally we give χ-analogues of these facts for any Dirichlet character χ (see Theorem 3.1). In particular, when $x = 1$, we obtain functional relations between double L-functions and Dirichlet L-functions (see [18]). On the other hand, as a relation among the values at positive integers, we give, for example,

$$\begin{aligned}\sum_{m,n=1}^{\infty} \frac{\chi_4(n)\left(2-\sqrt{3}\right)^n}{m(m+n)} \quad &= \quad \frac{2}{3}L(2;\chi_4) + \frac{1}{3}L(1;\chi_4)\log\left(2-\sqrt{3}\right) \\ &\quad + \frac{1}{4i}\Big[\big\{\log(1-(2-\sqrt{3})i)\big\}^2 \\ &\quad - \big\{\log(1+(2-\sqrt{3})i)\big\}^2\Big],\end{aligned}$$

where χ_4 is the unique primitive Dirichlet character of conductor 4 and $i = \sqrt{-1}$.

2. Double polylogarithms

First we recall some known results (see, for example, [17]).

Let

$$\phi(s) = \mathcal{P}_1(s;-1) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^s} = (2^{1-s}-1)\zeta(s).$$

From Lemma 2 of [17], we see that

$$\sum_{m=1}^{\infty} \frac{(-1)^m \sin(m\theta)}{m^{2k+1}} = \sum_{j=0}^{k} \phi(2k-2j)\, \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \tag{2.1}$$

and

$$\sum_{m=1}^{\infty} \frac{(-1)^m \cos(m\theta)}{m^{2l}} = \sum_{j=0}^{l} \phi(2l-2j)\, \frac{(-1)^j \theta^{2j}}{(2j)!} \tag{2.2}$$

for $k \in \mathbb{N}_0$, $l \in \mathbb{N}$ and $\theta \in (-\pi, \pi) \subset \mathbb{R}$. Note that $\phi(0) = \zeta(0) = -\frac{1}{2}$.

We fix $x \in \mathbb{C}$ with $|x| \leqslant 1$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Let

$$G(\theta; s; x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(n\theta)}{n^s}, \quad \theta \in \mathbb{R}. \tag{2.3}$$

Note that $G(\theta; s; x)$ is uniformly convergent with respect to $\theta \in (-\pi, \pi)$, because $|x| \leqslant 1$ and $\operatorname{Re} s > 1$. Furthermore, for $\theta \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we let

$$\begin{aligned} & F_1(\theta; 2k+1; x) \\ & \quad = 2\left(\sum_{m=1}^{\infty} \frac{(-1)^m \sin(m\theta)}{m^{2k+1}} - \sum_{j=0}^{k} \phi(2k-2j)\, \frac{(-1)^j \theta^{2j+1}}{(2j+1)!}\right) \\ & \qquad \times G(\theta; s; x). \end{aligned} \tag{2.4}$$

It follows from (2.1) that $F_1(\theta; 2k+1; x) = 0$ for $\theta \in (-\pi, \pi)$.

Now we prove the following.

Theorem 2.1. *For $x \in \mathbb{C}$ with $|x| \leqslant 1$ and $s \in \mathbb{C}$ with* $\operatorname{Re} s > 0$,

$$2\mathcal{P}_{*,2}(1, s, 1; x) - \mathcal{P}_{ш,2}(1, 1, s; x) = 2\mathcal{P}_1(s+2; x). \tag{2.5}$$

We consider (2.4) in the case $k = 0$, namely

$$F_1(\theta; 1; x) = 2\left(\sum_{m=1}^{\infty} \frac{(-1)^m \sin(m\theta)}{m} + \frac{\theta}{2}\right) \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(n\theta)}{n^s}. \tag{2.6}$$

First we assume $\operatorname{Re} s > 1$. From Lemma 2.1 of [13], the first term in the parentheses on the right-hand side of (2.6) is uniformly convergent on any compact subset in $(-\pi, \pi)$. Hence

$$\begin{aligned} F_1(\theta; 1; x) &= 2\sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} x^n \sin(m\theta)\cos(n\theta)}{mn^s} + \theta \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(n\theta)}{n^s} \\ &= \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} x^n \{\sin((m-n)\theta) + \sin((m+n)\theta)\}}{mn^s} \\ &\quad + \theta \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(n\theta)}{n^s} = 0 \end{aligned}$$

for $\theta \in (-\pi, \pi)$. This means, by integrating the both sides, that

$$\sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{(-1)^{m+n} x^n \cos((m-n)\theta)}{mn^s(m-n)} + \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} x^n \cos((m+n)\theta)}{mn^s(m+n)}$$

$$-\theta \sum_{n=1}^{\infty} \frac{(-1)^n x^n \sin(n\theta)}{n^{s+1}} - \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(n\theta)}{n^{s+2}} = C_0, \tag{2.7}$$

where C_0 is a constant which depends on x and s. In order to determine this constant C_0, we further consider the indefinite integral of (2.7). By the uniform convergence of (2.7), we can justify the term-by-term integration, namely

$$\begin{aligned}\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} &\frac{(-1)^{m+n} x^n \sin((m-n)\theta)}{mn^s(m-n)^2} + \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} x^n \sin((m+n)\theta)}{mn^s(m+n)^2} \\ &+\theta \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(n\theta)}{n^{s+2}} - 2\sum_{n=1}^{\infty} \frac{(-1)^n x^n \sin(n\theta)}{n^{s+3}} = D_0 + C_0\theta \end{aligned}\tag{2.8}$$

for $\theta \in (-\pi, \pi)$. Letting $\theta = 0$, we see that $D_0 = 0$. Since each side of (2.8) is continuous with respect to θ on $[-\pi, \pi]$, (2.8) holds for $\theta = \pi$. Since $\cos(k\pi) = (-1)^k$ for $k \in \mathbb{Z}$, we have

$$C_0 = \mathcal{P}_1(s+2; x). \tag{2.9}$$

Putting $l = m - n$ and $j = n - m$ in the first term on the left-hand side of (2.7) according as $m > n$ and $m < n$ respectively, and using (2.9), we can write (2.7) as

$$\begin{aligned}\sum_{n,l=1}^{\infty} &\frac{(-1)^l x^n \cos(l\theta)}{ln^s(l+n)} - \sum_{m,j=1}^{\infty} \frac{(-1)^j x^{j+m} \cos(j\theta)}{jm(j+m)^s} \\ &+ \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} x^n \cos((m+n)\theta)}{mn^s(m+n)} - \theta \sum_{n=1}^{\infty} \frac{(-1)^n x^n \sin(n\theta)}{n^{s+1}} \\ &- \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(n\theta)}{n^{s+2}} = \mathcal{P}_1(s+2; x). \end{aligned}\tag{2.10}$$

This also holds for $\theta = \pi$. Hence (2.5) holds for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Additionally we see that $\mathcal{P}_{ш,2}(1,1,s;x)$ and $\mathcal{P}_{*,2}(1,s,1;x)$ are convergent absolutely for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$ and $|x| \leqslant 1$. In fact, letting $\sigma = \operatorname{Re} s > 0$, we obtain $|\mathcal{P}_{ш,2}(1,1,s;x)| \leqslant 2^{-\sigma}\zeta(1+\frac{\sigma}{2})^2$ from the relation $m+n \geqslant 2\sqrt{mn}$. Next we can easily see that $ab \ll a^p + b^q$ for $a, b > 0$, where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Putting $(a,b) = (m^{\frac{1}{p}}, n^{\frac{1}{q}})$ in this inequality, we have $m^{\frac{1}{p}} n^{\frac{1}{q}} \ll m + n$. Therefore, we have $mn^\sigma(m+n) \gg m^{1+\frac{1}{p}} n^{\sigma+\frac{1}{q}}$. We can find $q > 1$ which satisfies $\sigma + \frac{1}{q} > 1$. Hence we can see that $\mathcal{P}_{*,2}(1,s,1;x)$ is convergent absolutely for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$ and $|x| \leqslant 1$. Thus, (2.5) holds for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$ and $|x| \leqslant 1$. This completes the proof.

Example 2.2. Letting $x=1$ in (2.5), we obtain the functional relation

$$2\zeta_{MT,2}(1,s,1)-\zeta_{MT,2}(1,1,s)=2\zeta(s+2),$$

which was given in [13], [19]. Letting $s=1$ and using the relation

$$\frac{1}{mn}=\frac{1}{m+n}\left(\frac{1}{m}+\frac{1}{n}\right),$$

we obtain Euler's well-known formula

$$\sum_{m,n=1}^{\infty}\frac{1}{m(m+n)^2}=\frac{1}{2}\zeta_{MT,2}(1,1,1)=\zeta(3).$$

On the other hand, letting $s=1$ in (2.5), we have

$$\sum_{m,n=1}^{\infty}\frac{2x^n-x^{m+n}}{mn(m+n)}=2\mathrm{Li}_3(x),\quad |x|\leqslant 1,$$

which was given in [20]. Putting $x=\frac{3-\sqrt{5}}{2}$ and using (1.2), we have

$$\begin{aligned}&\sum_{m,n=1}^{\infty}\frac{\left(\frac{3-\sqrt{5}}{2}\right)^n\left\{2-\left(\frac{3-\sqrt{5}}{2}\right)^m\right\}}{mn(m+n)}\\ &=\frac{8}{5}\zeta(3)+\frac{2}{5}\zeta(2)\log\left(\frac{3-\sqrt{5}}{2}\right)-\frac{1}{6}\left[\log\left(\frac{3-\sqrt{5}}{2}\right)\right]^3.\end{aligned}\tag{2.11}$$

Now we proceed to the next step. By repeating the procedure used in the proof of Theorem 2.1, we can inductively prove that

$$\begin{aligned}&\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{(-1)^{m+n}x^n\cos((m-n)\theta)}{mn^s(m-n)^{2d+1}}+\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^n\cos((m+n)\theta)}{mn^s(m+n)^{2d+1}}\\ &\qquad -(2d+1)\sum_{n=1}^{\infty}\frac{(-1)^n x^n\cos(n\theta)}{n^{s+2d+2}}-\theta\sum_{n=1}^{\infty}\frac{(-1)^n x^n\sin(n\theta)}{n^{s+2d+1}}\\ &=\sum_{j=0}^{d}C_{d-j}\frac{(-1)^j\theta^{2j}}{(2j)!},\end{aligned}\tag{2.12}$$

and

$$\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{(-1)^{m+n}x^n\sin((m-n)\theta)}{mn^s(m-n)^{2d+2}}+\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^n\sin((m+n)\theta)}{mn^s(m+n)^{2d+2}}$$

$$-(2d+2)\sum_{n=1}^{\infty}\frac{(-1)^n x^n \sin(n\theta)}{n^{s+2d+3}} + \theta\sum_{n=1}^{\infty}\frac{(-1)^n x^n \cos(n\theta)}{n^{s+2d+2}}$$
$$= \sum_{j=0}^{d} C_{d-j}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!} \tag{2.13}$$

for $d \in \mathbb{N}_0$ and $\theta \in (-\pi, \pi)$, where $\{C_0, C_1, \ldots, C_d\}$ can be determined inductively. Note that C_0 has been determined by (2.9). Since we can let $\theta \to \pi$ on both sides of (2.12) and (2.13), we obtain

$$\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{x^n}{mn^s(m-n)^{2d+1}} + \sum_{m,n=1}^{\infty}\frac{x^n}{mn^s(m+n)^{2d+1}}$$
$$-(2d+1)\sum_{n=1}^{\infty}\frac{x^n}{n^{s+2d+2}} = \sum_{j=0}^{d} C_{d-j}\frac{(-1)^j\pi^{2j}}{(2j)!}, \tag{2.14}$$

and

$$\pi\sum_{n=1}^{\infty}\frac{x^n}{n^{s+2d+2}} = \sum_{j=0}^{d} C_{d-j}\frac{(-1)^j\pi^{2j+1}}{(2j+1)!}. \tag{2.15}$$

Now we recall the following lemma.

Lemma 2.3 ([19], Lemma 4.4). *Let $\{\alpha_{2d}\}_{d\in\mathbb{N}_0}$, $\{\beta_{2d}\}_{d\in\mathbb{N}_0}$, $\{\gamma_{2d}\}_{d\in\mathbb{N}_0}$ be sequences satisfying that*

$$\alpha_{2d} = \sum_{j=0}^{d}\gamma_{2d-2j}\frac{(-1)^j\pi^{2j}}{(2j)!}, \quad \beta_{2d} = \sum_{j=0}^{d}\gamma_{2d-2j}\frac{(-1)^j\pi^{2j}}{(2j+1)!}$$

for any $d \in \mathbb{N}_0$. Then

$$\alpha_{2d} = -2\sum_{\nu=0}^{d}\beta_{2\nu}\zeta(2d-2\nu)$$

for any $d \in \mathbb{N}_0$.

Combining (2.14), (2.15) and Lemma 2.3, we obtain

$$\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{x^n}{mn^s(m-n)^{2d+1}} + \sum_{m,n=1}^{\infty}\frac{x^n}{mn^s(m+n)^{2d+1}}$$
$$-(2d+1)\sum_{n=1}^{\infty}\frac{x^n}{n^{s+2d+2}}$$

$$= -2\sum_{j=0}^{d}\zeta(2d-2j)\mathcal{P}_1(s+2j+2;x). \tag{2.16}$$

Putting $l=m-n$ and $j=n-m$ in the first term on the left-hand side of (2.16) according as $m>n$ and $m<n$ respectively, we obtain the following.

THEOREM 2.4. *For $d\in\mathbb{N}_0$, $x\in\mathbb{C}$ with $|x|\leqslant 1$ and $s\in\mathbb{C}$ with* $\operatorname{Re}s>0$,

$$\begin{aligned}&\mathcal{P}_{*,2}(2d+1,s,1;x)+\mathcal{P}_{*,2}(1,s,2d+1;x)-\mathcal{P}_{ш,2}(2d+1,1,s;x)\\&\quad=(2d+1)\mathcal{P}_1(s+2d+2;x)\\&\qquad-2\sum_{j=0}^{d}\zeta(2d-2j)\mathcal{P}_1(s+2j+2;x).\end{aligned} \tag{2.17}$$

Theorem 2.1 is a special case of this theorem.

Next we present a further generalization. Instead of (2.6), we consider $F_1(\theta;2k+1;x)$, and

$$\begin{aligned}F_2(\theta;2k+1;x)\;&=\;2\left(\sum_{m=1}^{\infty}\frac{(-1)^m\sin(m\theta)}{m^{2k+1}}-\sum_{j=0}^{k}\phi(2k-2j)\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\right)\\&\qquad\times\sum_{n=1}^{\infty}\frac{(-1)^nx^n\sin(n\theta)}{n^s},\end{aligned} \tag{2.18}$$

$$\begin{aligned}F_3(\theta;2l;x)\;&=\;2\left(\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\theta)}{m^{2l}}-\sum_{j=0}^{l}\phi(2l-2j)\frac{(-1)^j\theta^{2j}}{(2j)!}\right)\\&\qquad\times\sum_{n=1}^{\infty}\frac{(-1)^nx^n\cos(n\theta)}{n^s}\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}F_4(\theta;2l;x)\;&=\;2\left(\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\theta)}{m^{2l}}-\sum_{j=0}^{l}\;\phi(2l-2j)\;\frac{(-1)^j\theta^{2j}}{(2j)!}\right)\\&\qquad\times\sum_{n=1}^{\infty}\frac{(-1)^nx^n\sin(n\theta)}{n^s}\end{aligned} \tag{2.20}$$

for $k,l\in\mathbb{N}$. Then, by (2.1) and (2.2), we see that these are equal to 0 for $\theta\in(-\pi,\pi)$. Hence, by the same argument as above, we can prove the following.

THEOREM 2.5. *For $p, q \in \mathbb{N}$, $x \in \mathbb{C}$ with $|x| \leqslant 1$ and $s \in \mathbb{C}$ with* $\operatorname{Re} s > 0$,

$$\begin{aligned}
&(-1)^p \mathcal{P}_{*,2}(p,s,q;x) + (-1)^q \mathcal{P}_{*,2}(q,s,p;x) + \mathcal{P}_{ш,2}(p,q,s;x) \\
&\quad = 2 \sum_{\substack{j=0 \\ j \equiv p \ (2)}}^{p} \left(2^{1-p+j}-1\right) \zeta(p-j) \\
&\qquad \times \sum_{\mu=0}^{[\frac{j}{2}]} \frac{(-1)^\mu \pi^{2\mu}}{(2\mu)!} \binom{q-1+j-2\mu}{j-2\mu} \mathcal{P}_1(s+q+j-2\mu;x) \\
&\qquad -4 \sum_{\substack{j=0 \\ j \equiv p \ (2)}}^{p} \left(2^{1-p+j}-1\right) \zeta(p-j) \sum_{\mu=0}^{[\frac{j-1}{2}]} \frac{(-1)^\mu \pi^{2\mu}}{(2\mu+1)!} \sum_{\substack{\nu=0 \\ \nu \equiv q \ (2)}}^{q} \zeta(q-\nu) \\
&\qquad \times \binom{\nu-1+j-2\mu}{j-2\mu-1} \mathcal{P}_1(s+\nu+j-2\mu;x). \qquad (2.21)
\end{aligned}$$

Note that if $|x| < 1$ then (2.21) *holds for all $s \in \mathbb{C}$.*

We give the proof in the case $(p,q) = (2k+1, 2d+1)$ for $k, d \in \mathbb{N}_0$. Note that, in the case $k = 0$, we have already given the proof in Theorem 2.4.

We fix a general $k \in \mathbb{N}$ and assume $\operatorname{Re} s > 1$. By (2.1), we see that $F_1(\theta; 2k+1; x) = 0$ for $\theta \in (-\pi, \pi)$. By using (2.4) and the relation $2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$, we have

$$\begin{aligned}
&F_1(\theta; 2k+1; x) \\
&\quad = \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{(-1)^{m+n} x^n \sin((m-n)\theta)}{m^{2k+1} n^s} + \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} x^n \sin((m+n)\theta)}{m^{2k+1} n^s} \\
&\qquad -2 \sum_{j=0}^{k} \phi(2k-2j) \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos(n\theta)}{n^s} = 0 \qquad (2.22)
\end{aligned}$$

for $\theta \in (-\pi, \pi)$. As well as in the proof of Theorem 2.1, by integrating the both sides of (2.22) and repeating the integration by parts, we have

$$\begin{aligned}
&\sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{(-1)^{m+n} x^n \cos((m-n)\theta)}{m^{2k+1} n^s (m-n)} + \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} x^n \cos((m+n)\theta)}{m^{2k+1} n^s (m+n)} \\
&\qquad -2 \sum_{j=0}^{k} \phi(2k-2j) \sum_{\nu=0}^{2j+1} \frac{(-\theta)^\nu}{\nu!} \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos^{(\nu)}(n\theta)}{n^{s+2j+2-\nu}} \\
&\quad = C_0(2k+1), \qquad (2.23)
\end{aligned}$$

where $C_0(2k+1)$ is a constant which depends on (k,x,s), $\cos^{(\nu)}(x)$ is the νth derivative of $\cos x$ and $\cos^{(\nu)}(\alpha) := \cos^{(\nu)}(x)\big|_{x=\alpha}$. Note that $C_0(1)$ is equal to C_0 defined in (2.7). Indeed, (2.23) in the case $k=0$ is equal to (2.7). By repeating the procedure used in the proof of Theorem 2.1 and of Theorem 2.4, we can prove that

$$\begin{aligned}
&\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{(-1)^{m+n}x^n\cos((m-n)\theta)}{m^{2k+1}n^s(m-n)^{2d+1}} + \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}x^n\cos((m+n)\theta)}{m^{2k+1}n^s(m+n)^{2d+1}}\\
&\quad -2\sum_{j=0}^{k} \phi(2k-2j) \sum_{\nu=0}^{2j+1} \binom{2d+1+2j-\nu}{2j+1-\nu} \frac{(-\theta)^\nu}{\nu!}\\
&\quad \times \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos^{(\nu)}(n\theta)}{n^{s+2d+2j+2-\nu}} = \sum_{j=0}^{d} C_{d-j}(2k+1)\frac{(-1)^j\theta^{2j}}{(2j)!}
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
&\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{(-1)^{m+n}x^n\sin((m-n)\theta)}{m^{2k+1}n^s(m-n)^{2d+2}} + \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}x^n\sin((m+n)\theta)}{m^{2k+1}n^s(m+n)^{2d+2}}\\
&\quad -2\sum_{j=0}^{k} \phi(2k-2j) \sum_{\nu=0}^{2j+1} \binom{2d+2+2j-\nu}{2j+1-\nu} \frac{(-\theta)^\nu}{\nu!}\\
&\quad \times \sum_{n=1}^{\infty} \frac{(-1)^n x^n \cos^{(\nu+1)}(n\theta)}{n^{s+2d+2j+3-\nu}}\\
&\quad = \sum_{j=0}^{d} C_{d-j}(2k+1)\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}, \qquad \theta\in(-\pi,\pi),
\end{aligned} \tag{2.25}$$

by induction on $d\in\mathbb{N}_0$, where $\{C_0(2k+1), C_1(2k+1), \ldots, C_d(2k+1)\}$ can be determined inductively. The both sides of (2.24) and (2.25) are continuous with respect to $\theta\in[-\pi,\pi]$. Hence we can let $\theta\to\pi$ on both sides of (2.24) and (2.25). By $\cos(n\pi)=(-1)^n$ and $\sin(n\pi)=0$ ($n\in\mathbb{Z}$), we obtain

$$\begin{aligned}
&\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{x^n}{m^{2k+1}n^s(m-n)^{2d+1}} + \sum_{m,n=1}^{\infty} \frac{x^n}{m^{2k+1}n^s(m+n)^{2d+1}}\\
&\quad -2\sum_{j=0}^{k} \phi(2k-2j) \sum_{\mu=0}^{j} \binom{2d+1+2j-2\mu}{2j+1-2\mu} \frac{(-1)^\mu\pi^{2\mu}}{(2\mu)!}\\
&\quad \times \sum_{n=1}^{\infty} \frac{x^n}{n^{s+2d+2j+2-2\mu}} = \sum_{j=0}^{d} C_{d-j}(2k+1)\frac{(-1)^j\pi^{2j}}{(2j)!}
\end{aligned} \tag{2.26}$$

and

$$-2\sum_{j=0}^{k}\ \phi(2k-2j)\ \sum_{\mu=0}^{j}\binom{2d+1+2j-2\mu}{2j-2\mu}\frac{(-1)^{\mu}\pi^{2\mu+1}}{(2\mu+1)!}$$
$$\times\sum_{n=1}^{\infty}\frac{x^n}{n^{s+2d+2j+2-2\mu}}=\sum_{j=0}^{d}C_{d-j}(2k+1)\frac{(-1)^j\pi^{2j+1}}{(2j+1)!}. \qquad (2.27)$$

Combining (2.26), (2.27) and Lemma 2.3, we obtain

$$\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{x^n}{m^{2k+1}n^s(m-n)^{2d+1}}+\sum_{m,n=1}^{\infty}\frac{x^n}{m^{2k+1}n^s(m+n)^{2d+1}}$$
$$-2\sum_{j=0}^{k}\ \phi(2k-2j)\ \sum_{\mu=0}^{j}\binom{2d+1+2j-2\mu}{2j+1-2\mu}\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu)!}$$
$$\times\sum_{n=1}^{\infty}\frac{x^n}{n^{s+2d+2j+2-2\mu}}$$
$$=4\sum_{\nu=0}^{d}\zeta(2d-2\nu)\sum_{j=0}^{k}\ \phi(2k-2j)\sum_{\mu=0}^{j}\binom{2\nu+1+2j-2\mu}{2j-2\mu}\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu+1)!}$$
$$\times\sum_{n=1}^{\infty}\frac{x^n}{n^{s+2\nu+2j+2-2\mu}}. \qquad (2.28)$$

Using $\phi(s)=\left(2^{1-s}-1\right)\zeta(s)$, and putting $l=m-n$ and $j=n-m$ in the first term on the left-hand side of (2.28) according as $m>n$ and $m<n$ respectively, we can obtain the proof of (2.21) in the case $(p,q)=(2k+1,2d+1)$. Note that if $|x|=1$ then the left-hand side of (2.21) is convergent absolutely for $\operatorname{Re}s>0$ which can be shown similarly to the argument at the end of the proof of Theorem 2.1. Hence (2.21) holds for $\operatorname{Re}s>0$. On the other hand, if $|x|<1$ then we can easily see that (2.21) holds for all $s\in\mathbb{C}$.

By considering $F_2(\theta;2k+1;x)$, $F_3(\theta;2l;x)$ and $F_4(\theta;2l;x)$ instead of $F_1(\theta;2k+1;x)$, we can similarly obtain the proofs of (2.21) in the cases $(p,q)=(2k+1,2e)$, $(2l,2e)$ and $(2l,2d+1)$ for $k,d\in\mathbb{N}_0$ and $l,e\in\mathbb{N}$.

Example 2.6. Applying (2.21) in the case $(p,q)=(2,2)$, we obtain

$$2\mathcal{P}_{*,2}(2,s,2;x)+\mathcal{P}_{ш,2}(2,2,s;x)=4\zeta(2)\mathcal{P}_1(s+2;x)-6\mathcal{P}_1(s+4;x). \qquad (2.29)$$

Letting $s=2$, we obtain the formula given in Proposition 2 of [20]. Furthermore, letting $x=1$, we obtain Mordell's formula ([14])

$$\zeta_{MT,2}(2,2,2)=\frac{4}{3}\zeta(2)\zeta(4)-2\zeta(6).$$

As a next example, put $(s,x)=(2.34,0.8)$ in (2.29). Then we can approximately calculate that

$$\begin{aligned}
&\mathcal{P}_{*,2}(2,2.34,2\,;0.8)=0.253330048\ldots,\\
&\mathcal{P}_{ш,2}(2,2,2.34\,;0.8)=0.152764587\ldots,\\
&\mathcal{P}_{1}(4.34\,;0.8)=0.837450277\ldots,\\
&\mathcal{P}_{1}(6.34\,;0.8)=0.808462890\ldots\ .
\end{aligned}$$

Hence we can numerically check the validity of (2.29) for $(s,x)=(2.34,0.8)$.

3. Character analogues of double polylogarithms

Let χ be any primitive Dirichlet character of conductor f. We define

$$\mathcal{P}_1(s;x;\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)x^n}{n^s}, \tag{3.1}$$

$$\mathcal{P}_{*,2}(s_1,s_2,s_3;x;\chi) = \sum_{m,n=1}^{\infty}\frac{\chi(n)x^n}{m^{s_1}n^{s_2}(m+n)^{s_3}}, \tag{3.2}$$

$$\mathcal{P}_{ш,2}(s_1,s_2,s_3;x;\chi) = \sum_{m,n=1}^{\infty}\frac{\chi(m+n)x^{m+n}}{m^{s_1}n^{s_2}(m+n)^{s_3}}. \tag{3.3}$$

In particular, when $x=1$, these coincide with the Dirichlet L-function $L(s;\chi)$ and the Mordell-Tornheim double L-functions (see [18], [22], see also [1]).

It is well known that

$$\chi(b)\tau(\overline{\chi})=\sum_{a=1}^{f}\overline{\chi}(a)\mathrm{e}^{\frac{2\pi iab}{f}},\quad b\in\mathbb{N},$$

where $\tau(\chi)=\sum_{a=1}^{f}\chi(a)\mathrm{e}^{\frac{2\pi ia}{f}}$ is the Gauss sum and $\overline{\chi}=\chi^{-1}$ (see [21], Lemma 4.7). Hence we see that

$$\chi(n)x^n=\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{f}\overline{\chi}(a)\left(x\mathrm{e}^{\frac{2\pi ia}{f}}\right)^n,\quad n\in\mathbb{N}. \tag{3.4}$$

Therefore, we have

$$(-1)^p\mathcal{P}_{*,2}(p,s,q;x;\chi)+(-1)^q\mathcal{P}_{*,2}(q,s,p;x;\chi)+\mathcal{P}_{ш,2}(p,q,s;x;\chi)$$

$$
\begin{aligned}
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{f} \overline{\chi}(a) \sum_{m,n=1}^{\infty} \Biggl\{ (-1)^p \frac{\left(x e^{\frac{2\pi i a}{f}}\right)^n}{m^p n^s (m+n)^q} \\
&\quad + (-1)^q \frac{\left(x e^{\frac{2\pi i a}{f}}\right)^n}{m^q n^s (m+n)^p} + \frac{\left(x e^{\frac{2\pi i a}{f}}\right)^{m+n}}{m^p n^q (m+n)^s} \Biggr\} \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{f} \overline{\chi}(a) \Biggl\{ (-1)^p \mathcal{P}_{*,2}\bigl(p, s, q; x e^{\frac{2\pi i a}{f}}\bigr) \\
&\quad + (-1)^q \mathcal{P}_{*,2}\bigl(q, s, p; x e^{\frac{2\pi i a}{f}}\bigr) + \mathcal{P}_{ш,2}\bigl(p, q, s; x e^{\frac{2\pi i a}{f}}\bigr) \Biggr\} \qquad (3.5)
\end{aligned}
$$

for $p, q \in \mathbb{N}$. Substituting (2.21) into the right-hand side of (3.5), and using (3.4) again, we obtain the following.

THEOREM 3.1. *For $p, q \in \mathbb{N}$, $x \in \mathbb{C}$ with $|x| \leqslant 1$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$,*

$$
\begin{aligned}
&(-1)^p \mathcal{P}_{*,2}(p, s, q; x; \chi) + (-1)^q \mathcal{P}_{*,2}(q, s, p; x; \chi) + \mathcal{P}_{ш,2}(p, q, s; x; \chi) \\
&= 2 \sum_{\substack{j=0 \\ j \equiv p\ (2)}}^{p} \left(2^{1-p+j} - 1\right) \zeta(p-j) \\
&\quad \times \sum_{\mu=0}^{[\frac{j}{2}]} \frac{(-1)^\mu \pi^{2\mu}}{(2\mu)!} \binom{q-1+j-2\mu}{j-2\mu} \mathcal{P}_1(s+q+j-2\mu; x; \chi) \\
&\quad -4 \sum_{\substack{j=0 \\ j \equiv p\ (2)}}^{p} \left(2^{1-p+j} - 1\right) \zeta(p-j) \sum_{\mu=0}^{[\frac{j-1}{2}]} \frac{(-1)^\mu \pi^{2\mu}}{(2\mu+1)!} \sum_{\substack{\nu=0 \\ \nu \equiv q\ (2)}}^{q} \zeta(q-\nu) \\
&\quad \times \binom{\nu-1+j-2\mu}{j-2\mu-1} \mathcal{P}_1(s+\nu+j-2\mu; x; \chi). \qquad (3.6)
\end{aligned}
$$

Note that if $|x| < 1$, then (3.6) *holds for all $s \in \mathbb{C}$.*

REMARK 3.2. The equation (3.6) in the case $x = 1$ has been essentially obtained by Nakamura in [15] by a totally different method. In fact, Nakamura's formulas are of more simple expressions, comparing with (3.6) in the case $x = 1$.

EXAMPLE 3.3. Applying Theorem 3.1 in the case $(p, q) = (1, 1)$, we have

$$
\begin{aligned}
&-2\mathcal{P}_{*,2}(1, s, 1; x; \chi) + \mathcal{P}_{ш,2}(1, 1, s; x; \chi) \\
&\quad = 2\zeta(0)\mathcal{P}_1(s+2; x; \chi) - 4\zeta^2(0)\mathcal{P}_1(s+2; x; \chi) \\
&\quad = -2\mathcal{P}_1(s+2; x; \chi), \qquad (3.7)
\end{aligned}
$$

because $\zeta(0)=-\frac{1}{2}$. If $|x|<1$, then (3.7) holds for $s=0$. Note that, for $|x|<1$,

$$\begin{aligned}\mathcal{P}_{ш,2}(1,1,0;x;\chi) &= \sum_{m,n=1}^{\infty}\frac{x^{m+n}}{mn}\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{f}\overline{\chi}(a)e^{\frac{2\pi ia(m+n)}{f}} \\ &= \frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{f}\overline{\chi}(a)\left\{\log\left(1-xe^{\frac{2\pi ia}{f}}\right)\right\}^2. \end{aligned} \tag{3.8}$$

From (2.29) in [8], we have

$$3\mathcal{P}_1(2;2-\sqrt{3};\chi_4)=2L(2;\chi_4)+L(1;\chi_4)\log(2-\sqrt{3}), \tag{3.9}$$

where χ_4 is the unique primitive Dirichlet character of conductor 4. By putting $x=2-\sqrt{3}$ and $s=0$ in (3.7) and using (3.8) and (3.9), we obtain

$$\begin{aligned}&\sum_{m,n=1}^{\infty}\frac{\chi_4(n)\left(2-\sqrt{3}\right)^n}{m(m+n)} \\ &= \mathcal{P}_1(2;2-\sqrt{3};\chi_4)+\frac{1}{2\tau(\overline{\chi})}\sum_{a=1}^{4}\overline{\chi}_4(a)\left\{\log\left(1-(2-\sqrt{3})e^{\frac{2\pi ia}{4}}\right)\right\}^2 \\ &= \frac{2}{3}L(2;\chi_4)+\frac{1}{3}L(1;\chi_4)\log\left(2-\sqrt{3}\right) \\ &\quad +\frac{1}{4i}\left[\left\{\log\left(1-(2-\sqrt{3})i\right)\right\}^2-\left\{\log\left(1+(2-\sqrt{3})i\right)\right\}^2\right]. \end{aligned} \tag{3.10}$$

Note that we can approximately calculate

$$\frac{1}{4i}\left[\left\{\log\left(1-(2-\sqrt{3})i\right)\right\}^2-\left\{\log\left(1+(2-\sqrt{3})i\right)\right\}^2\right]=-0.00907612\ldots.$$

The other terms in (3.10) can be easily calculated numerically, and hence we can numerically verify that (3.10) surely holds.

Remark 3.4. We will be able to generalize these results to more general multiple series. In fact the authors are studying Witten's type of multiple zeta-functions associated with semi-simple Lie algebras (see [12]). We would like to discuss functional relations for Witten's type of multiple polylogarithms in forthcoming papers.

Note added in the revised version. The first version of the present paper had already been completed in early 2006. After that, the authors noticed that the technique developed in the present paper is useful in the study of functional relations among multiple zeta-functions. In fact, under the name of the "polylogarithm technique", we applied the idea in the present paper to Witten zeta-functions in [5], [6], and proved various functional relations which are more general than those obtained in [12].

After the first version of this paper had been completed, we obtained the following result (see [11], Lemma 2.1):

For arbitrary functions $f, g : \mathbb{N}_0 \to \mathbb{C}$ and $a \in \mathbb{N}$, we have

$$\sum_{\substack{j=0 \\ j\equiv a \ (\bmod\ 2)}}^{a} \left(2^{1-a+j}-1\right)\zeta(a-j)\sum_{\mu=0}^{[\frac{j}{2}]} f(j-2\mu)\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu)!}$$
$$=\sum_{\rho=0}^{[\frac{a}{2}]}\zeta(2\rho)f(a-2\rho), \tag{3.11}$$

and

$$\sum_{\substack{j=0 \\ j\equiv a \ (\bmod 2)}}^{a} \left(2^{1-a+j}-1\right)\zeta(a-j)\sum_{\mu=0}^{[\frac{j-1}{2}]} g(j-2\mu)\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu+1)!}$$
$$=-\frac{1}{2}g(a). \tag{3.12}$$

Using this result, we can rewrite (2.21) and (3.6) to

$$\begin{aligned}
&(-1)^{p}\mathcal{P}_{*,2}(p,s,q;x)+(-1)^{q}\mathcal{P}_{*,2}(q,s,p;x)+\mathcal{P}_{ш,2}(p,q,s;x)\\
&\quad=2\sum_{\rho=0}^{[\frac{p}{2}]}\binom{p+q-2\rho-1}{p-2\rho}\zeta(2\rho)\mathcal{P}_{1}(s+p+q-2\rho;x)\\
&\qquad+2\sum_{\rho=0}^{[\frac{q}{2}]}\binom{p+q-2\rho-1}{q-2\rho}\zeta(2\rho)\mathcal{P}_{1}(s+p+q-2\rho;x),
\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}
&(-1)^{p}\mathcal{P}_{*,2}(p,s,q;x;\chi)+(-1)^{q}\mathcal{P}_{*,2}(q,s,p;x;\chi)+\mathcal{P}_{ш,2}(p,q,s;x;\chi)\\
&\quad=2\sum_{\rho=0}^{[\frac{p}{2}]}\binom{p+q-2\rho-1}{p-2\rho}\zeta(2\rho)\mathcal{P}_{1}(s+p+q-2\rho;x;\chi)\\
&\qquad+2\sum_{\rho=0}^{[\frac{q}{2}]}\binom{p+q-2\rho-1}{q-2\rho}\zeta(2\rho)\mathcal{P}_{1}(s+p+q-2\rho;x;\chi),
\end{aligned}\tag{3.14}$$

respectively.

References

[1] T. Arakawa, M. Kaneko, On multiple L-values, *J. Math. Soc. Japan* **56** (2004), 967–992.

[2] D. Bowman, D. M. Bradley, Multiple polylogarithms: a brief survey, in: *Conf. on q-Series with Applications to Combinatorics, Number Theory, and Physics, Urbana, IL, 2000, Contemp. Math.* **291**, B. C. Berndt, K. Ono (Eds.), Amer. Math. Soc., Providence, RI, 71–92, 2001.

[3] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, P. Lisonek, Special values of multidimensional polylogarithms, *Trans. Amer. Math. Soc.* **353** (2001), 907–941.

[4] A. B. Goncharov, Multiple polylogarithms, cyclotomy, and modular complexes, *Math. Res. Lett.* **5** (1998), 497–516.

[5] Y. Komori, K. Matsumoto, H. Tsumura, Zeta-functions of root systems, in: *Conf. on L-Functions*, World Sci. Publ., Hackensack, NJ, 115–140, 2007.

[6] Y. Komori, K. Matsumoto, H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras. III, *Preprint* (submitted).

[7] L. Lewin, *Dilogarithms and Associated Functions*, Macdonald, London, 1958.

[8] L. Lewin, *Polylogarithms and Associated Functions*, North-Holland, New York, 1981.

[9] K. Matsumoto, On the analytic continuation of various multiple zeta-functions, in: *Number Theory for the Millennium II, Proc. of the Millennial Conf. on Number Theory, Urbana-Champaign, 2000*, M. A. Bennett et al. (Eds.), A. K. Peters, Natick, 417–440, 2002.

[10] K. Matsumoto, On Mordell-Tornheim and other multiple zeta-functions, in: *Proc. of the Session in Analytic Number Theory and Diophantine Equations, Bonn, 2002*, D. R. Heath-Brown, B. Z. Moroz (Eds.), Bonner Mathematische Schriften, Nr. 360, Bonn, **25**, 17 p.p., 2003.

[11] K. Matsumoto, T. Nakamura, H. Ochiai, H. Tsumura, On value-relations, functional relations and singularities of Mordell-Tornheim and related triple zeta-functions, *Acta Arith.* (to appear).

[12] K. Matsumoto, H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras. I, *Ann. Inst. Fourier* **56** (2006), 1457–1504.

[13] K. Matsumoto, H. Tsumura, A new method of producing functional relations among multiple zeta-functions, *Quart. J. Math.* **59** (2008), 55–83.

[14] L. J. Mordell, On the evaluation of some multiple series, *J. London Math. Soc.* **33** (1958), 368–371.

[15] T. Nakamura, Double Lerch series and their functional relations, *Aequationes Math.* (to appear).

[16] L. Tornheim, Harmonic double series, *Amer. J. Math.* **72** (1950), 303–314.

[17] H. Tsumura, On some combinatorial relations for Tornheim's double series, *Acta Arith.* **105** (2002), 239–252.

[18] H. Tsumura, On some functional relations between Mordell-Tornheim double L-functions and Dirichlet L-functions, *J. Number Theory* **120** (2006), 161–178.

[19] H. Tsumura, On functional relations between the Mordell-Tornheim double zeta functions and the Riemann zeta function, *Math. Proc. Cambridge Philos. Soc.* **142** (2007), 395–405.

[20] H. Tsumura, On certain polylogarithmic double series, *Arch. Math.* **88** (2007), 42–51.

[21] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer-Verlag, New York, Berlin, Heidelberg, 1997.

[22] M. Wu, *On Analytic Continuation of Mordell-Tornheim and Apostol-Vu L-Functions*, Master Thesis, Nagoya University, 2003 (in Japanese).

[23] D. Zagier, Values of zeta-functions and their applications, in: *First European Congress of Mathematics (ECM), Paris, 1992. Vol. II: Invited lectures (Part 2)*, A. Joseph et al. (Eds.), Birkhäuser. Prog. Math., Basel, 497–512, 1994.

Received
18 January 2008