On τ-tilting theory and higher Auslander-Reiten theory

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Contents

Introduction

The purpose of this thesis is to report on recent results obtained by the author in τ-tilting theory and higher Auslander-Reiten theory. These theories, introduced by Adachi-Iyama-Reiten in [**1**] respectively Iyama in [**56**], are generalizations of classical tilting theory and Auslander-Reiten theory.

Roughly speaking, Auslander-Reiten theory is concerned with the investigation of the structure of module categories, for which almost-split sequences are an important tool. In analogy, the main objects of study in higher Auslander-Reiten theory are n-almost-split sequences ($n \geq 1$), which are a special kind of exact sequences with $n+2$ terms.

On the other hand, the objective of classical tilting theory is the construction of derived equivalences of the form $\mathrm{D^b}(\mathcal{A}) \to \mathrm{D^b}(\mathsf{mod}\, A)$ where $\mathcal{A}$ is an abelian category and A is a finite dimensional algebra. Tilting modules over hereditary algebras have interesting combinatorics closely related to the mutation combinatorics of Fomin-Zelevinsky's cluster algebras [**54, 33**]. The introduction of τ-tilting theory obeys the desire to extend these combinatorics to arbitrary finite dimensional algebras.

In what follows, we explain the particular problems in these theories that are studied in this thesis. These are:

- The classification of the 2-representation-finite derived-canonical algebras, a problem in higher Auslander-Reiten theory.
- The introduction and development of the basic properties of n-abelian and n-exact categories, motivated by the concept of n-cluster-tilting which is central in higher Auslander-Reiten theory.
- τ-tilting reduction, which is a combinatorial problem in τ-tilting theory.

Classification of the 2-representation-finite derived-canonical algebras

From the homological viewpoint, the simplest class of algebras is that of *hereditary algebras*, *i.e.* algebras of global dimension at most 1. Note that a basic finite dimensional algebra A is hereditary if there exist an acyclic quiver such that A is isomorphic to the path algebra KQ. On the other hand, an algebra A is *representation-finite* if there are only finitely many isomorphism classes of indecomposable A-modules. The basic representation-finite hereditary algebras were classified by Gabriel in [**35**]; these are precisely the path algebras of quivers whose underlying graph is a disjoint union of Dynkin diagrams.

Recall that an A-module M is *n-cluster-tilting* if

$$\begin{aligned}\mathsf{add}\, M &= \left\{ N \in \mathsf{mod}\, A \mid \mathsf{Ext}_A^k(M,N) = 0,\ k \in \{1,\dots,n-1\} \right\} \\ &= \left\{ N \in \mathsf{mod}\, A \mid \mathsf{Ext}_A^k(N,M) = 0,\ k \in \{1,\dots,n-1\} \right\}.\end{aligned}$$

From the point of view of higher Auslander-Reiten theory, n-representation-finite algebras were introduced by Iyama-Oppermann in [**59**]. These are algebras A of global dimension at most n and such that there exist an n-cluster-tilting A-module.

Observe that 1-representation-finite algebras are exactly the representation-finite hereditary algebras.

A natural problem is then to classify the n-representation-finite algebras. Although the general problem seems rather difficult, Herschend-Iyama have obtained several structure theorems for 2-representation-finite algebras [**51**]. In particular, the 2-representation-finite algebras can be characterized as certain quotients associated with selfinjective quivers with potentials.

Recall that an algebra A is *piecewise hereditary* if there exists a hereditary abelian category $\mathcal{H}$ such that $\mathrm{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ and $\mathrm{D}^{\mathrm{b}}(\mathcal{H})$ are equivalent as triangulated categories. Moreover, due to a celebrated result of Happel [**45**], we can assume that $\mathcal{H}$ is either $\mathsf{mod}\,KQ$ for some acyclic quiver Q (in which case A is said to be *iterated-tilted*), or $\mathcal{H}$ is $\mathsf{coh}\,\mathbb{X}$, the category of coherent sheaves over some *weighted projective line* (in which case we say that A is derived-canonical) [**36**]. The 2-representation-finite iterated-tilted algebras have been classified by Iyama-Oppermann in [**60**]. Hence, it remains to classify the 2-representation-finite derived-canonical algebras to complete the classification of the 2-representation-finite piecewise hereditary algebras. This classification is obtained in Chapter 1.

Observe that, as a consequence of Happel's result, we need to classify the endomorphism algebras of tilting complexes in $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$ which are 2-representation-finite. Also, recall that there exist a distinguished autoequivalence $\tau\colon \mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X}) \to \mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$ such that $\tau[1]$ is the Serre functor of $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$. Finally, we say that an algebra is *cluster-tilted of canonical type* if it is isomorphic to the endomorphism algebra of a cluster-tilting object in the cluster category $\mathcal{C}_{\mathbb{X}}$ associated with the hereditary category $\mathsf{coh}\,\mathbb{X}$. We show the following, see 1.3.1 and 1.3.3:

- A tilting sheaf $T \in \mathsf{coh}\,\mathbb{X}$ has 2-representation-finite endomorphism algebra if and only if $\tau^2 T \cong T$. We call sheaves satisfying this last property *τ^2-stable*.
- A tilting complex in $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$ has 2-representation-finite endomorphism algebra if and only if it can be obtain by a process called "iterated 2-APR-tilting" from a τ^2-stable tilting sheaf in $\mathsf{coh}\,\mathbb{X}$. We note that the effect of 2-APR-tilting on these algebras can be easily described combinatorially in terms of their quivers with relations.
- A cluster-tilting object $T \in \mathcal{C}_{\mathbb{X}}$ has selfinjective endomorphism algebra if and only if $T[2] \cong T$.

Our main result is the following.

THEOREM (see Theorems 1.1.1, 1.1.2 and 1.1.3 for details). *The following objects can be classified:*

(i) The 2-representation-finite derived-canonical algebras.
(ii) The τ^2-stable tilting complexes in $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$.
(iii) The selfinjective cluster-tilted algebras of canonical type.

Moreover, these objects exist if and only if the corresponding weighted projective line has tubular type $(2,2,2,2;\boldsymbol{\lambda})$, $(2,4,4)$ or $(2,3,6)$.

It is worth noting that the last item in the theorem complements Ringel's classification of the selfinjective cluster-tilted algebras [**81**]. The contents of this chapter are available in [**65**].

n-abelian categories and n-exact categories

Recently, the concept of *cluster-tilting subcategory* was introduced in representation theory of algebras; the n-representation-finite algebras described in the previous section provide examples of these subcategories.

Cluster-tilting objects in triangulated categories where introduced by Buan-Marsh-Reineke-Reiten-Todorov in [**27**] as the crucial concept in the additive categorification of cluster algebras. More generally, n-cluster-tilting objects in triangulated categories where introduced by Iyama-Yoshino in [**62**]. Moreover, Geiß-Keller-Oppermann introduced a new class of additive categories, called $(n+2)$-angulated categories [**39**], which are higher analogs of Grothendieck-Verdier's triangulated categories. The main examples of $(n+2)$-angulated categories are obtained from n-cluster-tilting subcategories of triangulated categories which are closed under the n-th power of the shift functor. We also mention that $(n+2)$-angulated categories have been investigated by Bergh-Thaule in [**21, 22, 23**].

From a different perspective, n-cluster-tilting subcategories of certain abelian and exact categories where introduced by Iyama in [**56**] as the playground of higher Auslander-Reiten theory. In addition, n-cluster-tilting subcategories have also been constructed in algebro-geometric contexts. Consider the following example, due to Herschend-Iyama-Minamoto-Oppermann [**52**]. Let $\mathbb{P}^n$ be the n-dimensional projective space. Then, the category $\mathsf{vect}\,\mathbb{P}^n_K$ of vector bundles over $\mathbb{P}^n$ is an exact subcategory of the category of coherent sheaves over $\mathbb{P}^n$. The category $\mathsf{add}\,\{\mathcal{O}(i) \mid i \in \mathbb{Z}\}$ of direct sums of line bundles is an n-cluster-tilting subcategory of $\mathsf{vect}\,\mathbb{P}^n_K$.

Although the ambient categories in which n-cluster-tilting subcategories arise vary, n-cluster-tilting subcategories have several homological properties in common. In Chapter 2 we introduce n-abelian and n-exact categories to serve as a categorical framework for the investigation of the intrinsic homological properties of n-cluster-tilting subcategories. These are higher generalizations of abelian respectively exact categories with regard to the length of exact sequences. The following result shows that our main goal is achieved.

THEOREM (see Theorems 2.3.16 and 2.4.14 for details). *n-cluster-tilting subcategories of abelian (resp. exact) categories are n-abelian (resp. n-exact).*

As a partial converse, we show that certain n-abelian categories can be realized as n-cluster-tilting subcategories of abelian categories. More precisely, we prove the following theorem.

THEOREM (see Theorem 2.3.20 for details). *Let $\mathcal{M}$ be a small projectively generated n-abelian category, and $\mathcal{P}$ the category of projective objects in $\mathcal{M}$. If $\mathsf{mod}\,\mathcal{P}$ is injectively cogenerated, then $\mathcal{M}$ is equivalent to an n-cluster-tilting subcategory of $\mathsf{mod}\,\mathcal{P}$.*

To prove this theorem, we show that projective objects in n-abelian categories satisfy the following strong property, which is obvious in the case of abelian categories.

THEOREM (see Theorem 2.3.12 for details). *Let $\mathcal{M}$ be an n-abelian category and $P \in \mathcal{M}$ a projective object. Then, for every morphism $f\colon L \to M$ and every weak cokernel $g\colon M \to N$ of f, the following sequence is exact:*

$$\mathcal{M}(P,L) \xrightarrow{?\cdot f} \mathcal{M}(P,M) \xrightarrow{?\cdot g} \mathcal{M}(P,N).$$

An important result of Happel shows that the stable category of a Frobenius exact category has a natural structure of a triangulated category. In order to extend this result, we introduce Frobenius n-exact categories and prove the following result.

THEOREM (see Theorem 2.5.11 for details). *The stable category of a Frobenius n-exact category has a natural structure of a $(n+2)$-angulated category.*

Finally, we prove the following result also in the direction of Frobenius n-exact categories.

THEOREM (see Theorem 2.5.16 for details). *Let $\mathcal{M}$ be an n-cluster-tilting subcategory of a Frobenius exact category $\mathcal{E}$, and suppose that $\mathcal{M}$ is closed under taking n-th cosyzygies. Then, $\mathcal{M}$ is a Frobenius n-exact category.*

This theorem is closely related to the results of Geiß-Keller-Oppermann. The relation between both approaches to construct $(n+2)$-angulated categories is explained in Theorem 2.5.16.

At the end of the chapter we provide a collection of examples which suggest that the theory we present is far from abstract nonsense. The contents of this chapter are available in [**64**].

Reduction of τ-tilting modules

Let A be a finite dimensional algebra. We recall that an A-module M with $\mathsf{p.dim.}\, M \geq 1$ is *tilting* if $\mathsf{Ext}^1_A(M,M) = 0$ and the number of pairwise non-isomorphic indecomposable direct summands of M (which we denote by $|M|$) equals the number of simple A-modules or, equivalently, equals $|A|$. More generally, M is *support tilting* if there exists an idempotent $e \in A$ such that M is a tilting $(A/\langle e\rangle)$-module. Ingalls-Thomas showed that support tilting modules have nice mutation combinatorics in the case of hereditary algebras [**54**]. More precisely, they showed that if M is a basic A-module with $\mathsf{p.dim.}\, M \geq 1$ and $\mathsf{Ext}^1_A(M,M)$ satisfying moreover $|M| = |A| - 1$, then there exist exactly two basic support tilting modules having M as a direct summand. We note that this result does not hold for arbitrary algebras.

With the motivation of obtaining an analog of Ingalls-Thomas' result for arbitrary algebras, Adachi-Iyama-Reiten introduced τ-tilting modules in [**1**]. We say that an A-module M is *τ-rigid* if $\mathsf{Hom}_A(M, \tau M) = 0$ where τM is the Auslander-Reiten translate of M. Then, we say that a τ-rigid A-module M is *τ-tilting* if $|M| = |A|$. More generally, we say that M is *support τ-tilting* if there exists an idempotent $e \in A$ such that M is a τ-tilting $(A/\langle e\rangle)$-module. From the Auslander-Reiten formulas it follows that the class of support τ-tilting A-modules contains that of tilting A-modules; moreover, it is easy to see that these two classes coincide when A is a hereditary algebra.

One of the main results in [**1**] is that support τ-tilting modules have nice mutation combinatorics, for any finite dimensional algebra A. Namely, if M is a basic τ-rigid A-module satisfying $|M| = |A| - 1$, then there exist exactly two basic support τ-tilting modules having M as a direct summand.

For an algebra A we denote by $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$ the set of isomorphism classes of basic support τ-tilting A-modules. Let U be a basic τ-rigid A-module and

$$\mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A := \{M \in \mathrm{s}\tau\text{-}\mathsf{tilt}\, A \mid U \text{ is a direct summand of } M\}.$$

It is important to describe this set in order to enhance our understanding of the mutation combinatorics of τ-tilting modules. We do so in Chapter 3 where we prove the following result, which we call τ-tilting reduction. We note that a similar result regarding tilting modules over hereditary algebras was obtained by Happel-Unger in [**48**].

THEOREM (see Theorem 3.1.1 for details). *Let A be a finite dimensional algebra and U a basic τ-rigid A-module. Then, there exist a finite dimensional algebra C and a bijection*

$$\mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A \longrightarrow \mathrm{s}\tau\text{-}\mathsf{tilt}\, C.$$

We note that given an A-module U as in the theorem above, Adachi-Iyama-Reiten constructed a τ-tilting A-module T which is a maximum in $\mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A$ with respect to a natural partial order. This A-module T is called the *Bongartz completion of U in* $\mathsf{mod}\,A$. The algebra C in the theorem is obtained as the quotient of $\mathsf{End}_A(T)$ by the ideal generated by the idempotent corresponding to the projective $\mathsf{End}_A(T)$-module $\mathsf{Hom}_A(T,U)$. Also, we observe that when $|U| = |A| - 1$ we recover Adachi-Iyama-Reiten's result, as local algebras have only two basic support τ-tilting modules up to isomorphism.

In the proof of the theorem above, the following result is of crucial importance.

THEOREM (see Theorem 3.1.4 for details and definitions). *With the hypotheses of the previous theorem, let T be the Bongartz completion of U in* $\mathsf{mod}\,A$. *Then, the functor* $\mathsf{Hom}_A(T,-) : \mathsf{mod}\,A \to \mathsf{mod}(\mathsf{End}_A(T_U))$ *induces an equivalence of exact categories*

$$F : {}^{\perp}(\tau U) \cap U^{\perp} \longrightarrow \mathsf{mod}\,C.$$

In the case of hereditary algebras, the category ${}^{\perp}(\tau U) \cap U^{\perp}$ is precisely the right perpendicular category associated with U in the sense of [**37**].

Finally, we mention that τ-tilting reduction is compatible with silting reduction and 2-Calabi-Reduction, see Theorems 3.4.12 and 3.4.23. The contents of this chapter are available in [**63**].

CHAPTER 1

τ^2-stable tilting complexes over weighted projective lines

Let $\mathbb{X}$ be a weighted projective line and $\operatorname{coh}\mathbb{X}$ the associated category of coherent sheaves. We classify the tilting complexes T in $\mathrm{D^b}(\operatorname{coh}\mathbb{X})$ such that $\tau^2 T \cong T$, where τ is the Auslander-Reiten translation in $\mathrm{D^b}(\operatorname{coh}\mathbb{X})$. As an application of this result, we classify the 2-representation-finite algebras which are derived-equivalent to a canonical algebra. This complements Iyama-Oppermann's classification of the iterated-tilted 2-representation-finite algebras. By passing to 3-preprojective algebras, we obtain a classification of the selfinjective cluster-tilted algebras of canonical-type. This complements Ringel's classification of the selfinjective cluster-tilted algebras. This article originated from a note by Prof. H. Lenzing on the classification of τ^2-stable tilting sheaves (Theorem 1.3.5). The contents of this chapter are available in preprint form in [**65**].

1.1. Introduction

Let $\mathbb{X}$ be a weighted projective line over an algebraically closed field and

$$\tau\colon \mathrm{D^b}(\operatorname{coh}\mathbb{X}) \to \mathrm{D^b}(\operatorname{coh}\mathbb{X})$$

be the Auslander-Reiten translation in the bounded derived category of $\operatorname{coh}\mathbb{X}$, see [**36**] for definitions. The following objects, which are closely related to each other, are classified in this chapter:

(i) The τ^2-stable tilting complexes in $\mathrm{D^b}(\operatorname{coh}\mathbb{X})$,
(ii) the 2-representation-finite algebras which are derived equivalent to $\operatorname{coh}\mathbb{X}$ and
(iii) the selfinjective cluster-tilted algebras of canonical type.

The interest in classifying the objects above has its origin in higher Auslander-Reiten theory which was introduced by Iyama in [**56**]. As the name suggests, it is a higher-dimensional analog of classical Auslander-Reiten theory for finite dimensional algebras. Higher Auslander-Reiten theory can be developed in distinguished subcategories of $\operatorname{mod}\Lambda$, nowadays called n-cluster-tilting subcategories. A subcategory $\mathcal{M}$ of $\operatorname{mod}\Lambda$ is an *n-cluster-tilting subcategory* if

$$\begin{aligned}\mathcal{M} &= \left\{N \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(-,N)|_{\mathcal{M}} = 0 \text{ for } i \in \{1,\dots,n-1\}\right\}\\ &= \left\{N \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(N,-)|_{\mathcal{M}} = 0 \text{ for } i \in \{1,\dots,n-1\}\right\}.\end{aligned}$$

One of the most remarkable features of higher Auslander-Reiten theory is the existence of a functor $\tau_n\colon \mathcal{M} \to \mathcal{M}$ together with a natural isomorphism

$$\operatorname{Ext}^n_\Lambda(X,Y) \cong D\overline{\operatorname{Hom}}_\Lambda(Y,\tau_n X) \quad \text{for all} \quad X, Y \in \mathcal{M},$$

which is a higher analog of usual Auslander-Reiten duality.

The simplest class of algebras which have an n-cluster-tilting subcategory are the so-called n-representation-finite algebras, which where introduced by Iyama and Oppermann in [**59**]. A finite dimensional algebra Λ is said to be *n-representation-finite* if Λ has global dimension n and there exists a Λ-module M such that $\operatorname{add} M$

is an n-cluster-tilting subcategory (in this case M is called a *n-cluster-tilting module*). For example, 1-representation-finite algebras are precisely representation-finite hereditary algebras. In this sense, n-representation-finite algebras may be regarded as a higher analog of representation-finite hereditary algebras.

Now we explain what are the objects that we classify in this chapter, and how do they relate to each other. The 1-representation-finite algebras were classified by Gabriel in [**35**]: they are precisely the algebras which are Morita-equivalent to the path algebras of quivers whose underlying graph is a Dynkin diagram of simply-laced type. It is then natural to study 2-representation-finite algebras. Important structural results regarding 2-representation finite algebras in terms of selfinjective quivers with potential have been obtained by Herschend and Iyama in [**51**] where they also have provided large classes of examples of such algebras. Following [**46**], we say that a finite dimensional algebra is *piecewise hereditary* if it is derived equivalent to a hereditary category $\mathcal{H}$ or, equivalently, if it is isomorphic to the endomorphism algebra of a tilting complex in $\mathrm{D^b}(\mathcal{H})$. From a homological point of view, the simplest kind of 2-representation-finite algebras are the ones which are piecewise hereditary.

By a celebrated result of Happel [**45**, Thm. 3.1], it is known that there are only two kinds of hereditary categories (satisfying suitable finiteness conditions) which have a tilting object: the ones which are derived equivalent to $\mathsf{mod}\, H$ where H is a finite dimensional hereditary algebra, and the ones which are derived equivalent to $\mathsf{coh}\,\mathbb{X}$ where $\mathbb{X}$ is a weighted projective line. We distinguish between piecewise hereditary algebras as follows: We say that a finite dimensional algebra Λ is *iterated tilted* if $\mathsf{mod}\,\Lambda$ is derived equivalent to $\mathsf{mod}\, H$ where H is a finite dimensional hereditary algebra. Similarly, we say that Λ is *derived-canonical* if $\mathsf{mod}\,\Lambda$ is derived equivalent to $\mathsf{coh}\,\mathbb{X}$ for some weighted projective line $\mathbb{X}$.

Taking advantage of Ringel's classification of the selfinjective cluster-tilted algebras [**81**], the 2-representation-finite algebras which are iterated tilted were classified by Iyama and Oppermann in [**60**, Thm. 3.12]. Note that these algebras are derived equivalent to representation-finite hereditary algebras whose underlying quiver is of Dynkin type D. In particular, there are no 2-representation-finite algebras which are derived equivalent to a tame or wild hereditary algebra.

The following result is the main result of this chapter. It gives a classification of the 2-representation-finite derived canonical algebras, and thus complements Iyama-Oppermann's classification [**60**, Thm. 3.12].

THEOREM 1.1.1 (see Theorem 1.3.6). *The complete list of all the basic 2-representation-finite derived-canonical algebras is given in Figures 1.1.1, 1.1.2 and 1.1.3. In this case, the corresponding weighted projective line has tubular type* $(2,2,2,2;\lambda)$*,* $(2,4,4)$ *or* $(2,3,6)$*.*

Note that there are no 2-representation-finite algebras which are derived equivalent to $\mathsf{coh}\,\mathbb{X}$ for a weighted projective line $\mathbb{X}$ of wild type. It is important to note that in the case $(2,2,2,2;\lambda)$ all derived-canonical algebras are 2-representation-finite. The classification of all derived-canonical algebras of type $(2,2,2,2;\lambda)$ is known, see for example Skowroński [**83**, Ex. 3.3], Barot-de la Peña [**15**, Fig. 1] and Meltzer in [**76**, Thm. 10.4.1]. Also, part 1 of Figure 1.1.2 already appeared in [**51**, Fig. 1].

We mention that there exist a notion of 2-APR-(co)tilting, which is a higher analog of classical APR-(co)tilting, and that it preserves 2-representation-finiteness, see Definition 1.2.14. The algebras in Figures 1.1.1, 1.1.2 and 1.1.3 are related by 2-APR-(co)tilting as indicated.

Let $\tau\colon \mathrm{D^b}(\mathsf{coh}\,\mathbb{X}) \to \mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$ be the Auslander-Reiten translation. We say that a sheaf $X \in \mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$ is *τ^2-stable* if $\tau^2 X \cong X$. Theorem 1.1.1 is a consequence of the following result, which gives a classification of the τ^2-stable tilting sheaves over a weighted projective line.

THEOREM 1.1.2 (see Theorem 1.3.7). *Let* $\mathbb{X}$ *be a weighted projective line and* T *a basic tilting complex in* $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$. *Then* T *is* τ^2*-stable if and only if* $\mathsf{End}_{\mathrm{D}^{\mathrm{b}}(\mathbb{X})}(T)$ *is isomorphic to one of the algebras in Figures 1.1.1, 1.1.4 and 1.1.5. Moreover, this determines* T *up to an autoequivalence of* $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$. *In this case, the corresponding weighted projective line has tubular type* $(2,2,2,2;\lambda)$, $(2,4,4)$ *or* $(2,3,6)$.

A finite dimensional algebra is *cluster-tilted of canonical type* if it is isomorphic to the endomorphism algebra of a cluster-tilting object in the cluster category $\mathcal{C}_{\mathbb{X}}$ associated to a weighted projective line $\mathbb{X}$, see Section 1.2.4 for definitions.

By results of Keller [**71**] and Amiot [**4**], the basic cluster-tilted algebras of canonical type are 3-preprojective algebras of basic derived canonical algebras of global dimension at most 2. Moreover, they are Jacobian algebras of quivers with potential, see Section 1.2.4. As a consequence of Theorem 1.1.1, we obtain a classification of the selfinjective cluster-tilted algebras of canonical type. This complements Ringel's classification [**81**].

THEOREM 1.1.3 (see Theorem 1.3.8). *The complete list of all basic selfinjective cluster-tilted algebras of canonical type is given by the Jacobian algebras of the quivers with potential in Figures 1.1.6, 1.1.7 and 1.1.8. In this case, the corresponding weighted projective line has tubular type* $(2,2,2,2;\lambda)$, $(2,4,4)$ *or* $(2,3,6)$.

The algebras listed in Theorem 1.1.3 already appeared in related contexts: Figure 1.1.6 is precisely the exchange graph of endomorphism algebras of cluster-tilting objects the cluster category associated to a weighted projective line of type $(2,2,2,2;\lambda)$, see [**16**]. In addition, Figures 1.1.7 and 1.1.8 appeared in [**51**, Figs. 3 and 2] respectively.

1.2. Preliminaries

We begin by fixing our conventions and notation. Throughout this chapter we work over an algebraically closed field K. If Λ is a finite dimensional K-algebra, we denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated right Λ-modules. If Λ is a basic algebra, we denote its Gabriel quiver by Q_Λ. More generally, if X is a basic object in a Krull-Schmidt K-linear category $\mathcal{A}$, we denote by Q_X the Gabriel quiver of the algebra $\mathsf{End}_{\mathcal{A}}(X)$. If $\mathcal{A}$ is an abelian category, we denote by $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of $\mathcal{A}$ and we identify $\mathcal{A}$ with the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ given by the complexes concentrated in degree zero.

1.2.1. Coherent sheaves over a weighted projective line. We recall the construction of the category of coherent sheaves over a weighted projective line together with its basic properties. We follow the exposition of [**73**].

Choose a *parameter sequence* $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_t)$ of pairwise distinct points of $\mathbb{P}^1_K$ and a *weight sequence* $\mathbf{p} = (p_1, \dots, p_t)$ of positive integers. Without loss of generality, we assume that $t \geq 3$ and that for each $i \in \{1, \dots, t\}$ we have $p_i \geq 1$. Moreover, we may also assume that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$. For convenience, we set $p := \mathsf{lcm}(p_1, \dots, p_t)$. We call the triple $\mathbb{X} = (\mathbb{P}^1_K, \boldsymbol{\lambda}, \mathbf{p})$ a *weighted projective line of weight type* $\mathbf{p}$.

The category $\mathsf{coh}\,\mathbb{X}$ of *coherent sheaves over* $\mathbb{X}$ is defined as follows. Consider the rank 1 abelian group $\mathbb{L} = \mathbb{L}(\mathbf{p})$ with generators $\vec{x}_1, \dots, \vec{x}_t, \vec{c}$ subject to the relations

$$p_1\vec{x}_1 = \cdots = p_t\vec{x}_t = \vec{c}.$$

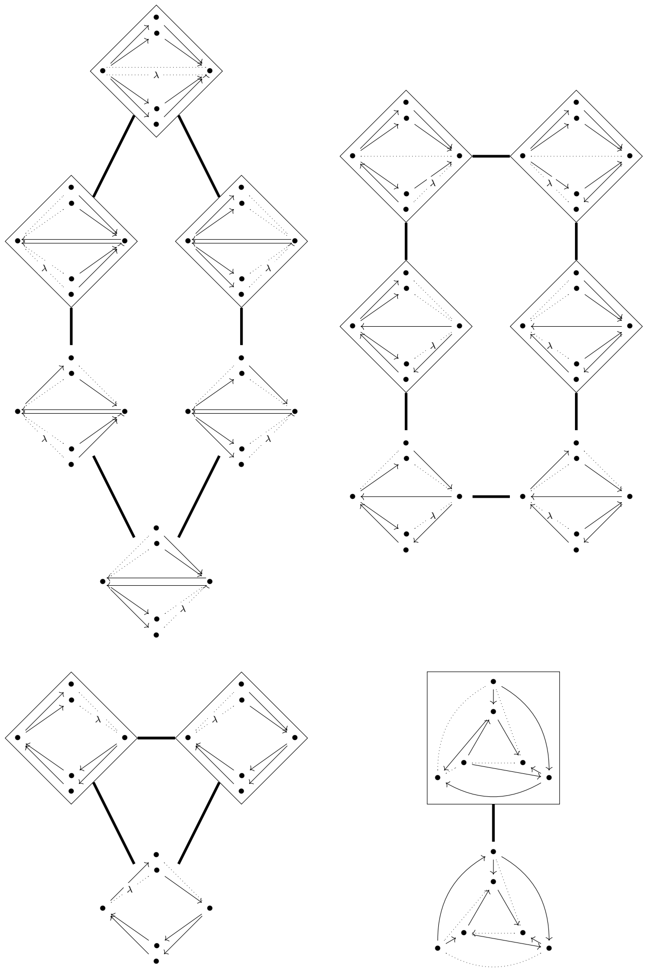


FIGURE 1.1.1. Endomorphism algebras of basic tilting complexes in $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$ for type $(2,2,2,2;\lambda)$. All complexes are τ^2-stable since τ^2 is the identity on $\mathsf{coh}\,\mathbb{X}$. The relations are induced by the quivers with potential in Figure 1.1.6; those with label λ correspond to relations involving the distinguished parameter. Thick lines indicate 2-APR-(co)tilting. The algebras that arise as endomorphism algebras of tilting sheaves in $\mathsf{coh}\,\mathbb{X}$ are enclosed in a frame.

The element $\vec{c}$ is called the *canonical element of* $\mathbb{L}$. It follows that every $\vec{x} \in \mathbb{L}$ can be written uniquely in the form

$$\vec{x} = m\vec{c} + \sum_{i=1}^{t} m_i \vec{x}_i$$

where $m \in \mathbb{Z}$ and $0 \leq m_i < p_i$ for each $i \in \{1, \dots, t\}$. Hence $\mathbb{L}$ is an ordered group with positive cone $\sum_{i=1}^{t} \mathbb{N}x_i$, and that for every $\vec{x} \in \mathbb{L}$ we have either $0 \leq \vec{x}$ or $\vec{x} \leq \vec{c} + \vec{\omega}$ where

$$\vec{\omega} := (t-2)\vec{c} - \sum_{i=1}^{t} x_i$$

is the *dualizing element of* $\mathbb{X}$.

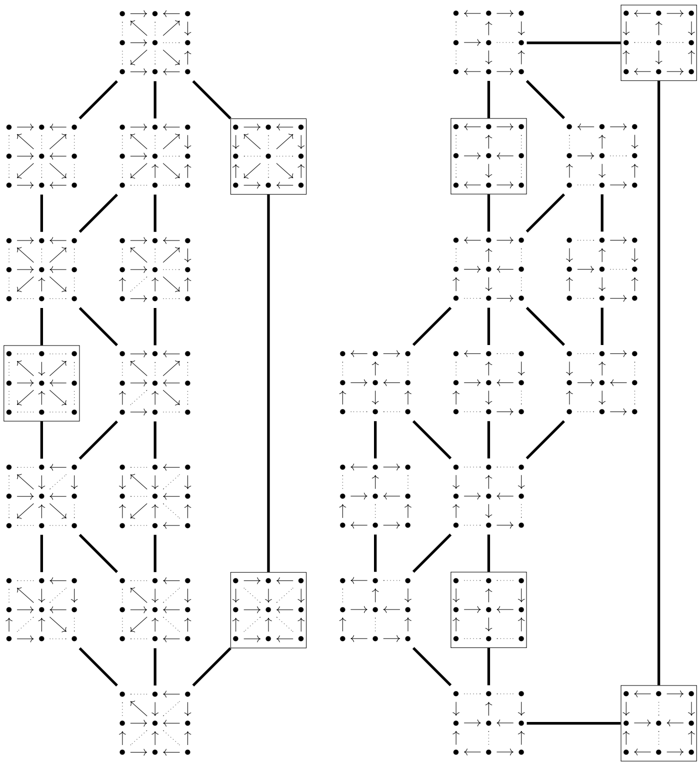


FIGURE 1.1.2. (Part 1 of 2) The basic 2-representation-finite derived-canonical algebras of type (2,4,4). Thick lines indicate 2-APR-(co)tilting. The algebras that arise as endomorphism algebras of tilting sheaves in $\mathsf{coh}\,\mathbb{X}$ are enclosed in a frame.

Next, consider the $\mathbb{L}$-graded algebra $K[x_1, \dots, x_t]$ where $\mathsf{deg}\, x_i = \vec{x}_i$ for each $i \in \{1, \dots, t\}$. When $t = 3$, we write $x = x_1$, $y = x_2$, $z = x_3$ and relabel the generators of $\mathbb{L}$ accordingly. Let $I = (f_3, \dots, f_t)$ be the homogeneous ideal of $K[x_1, \dots, x_t]$ generated by all the *canonical relations*

$$f_i = x_i^{p_i} - \lambda_i' x_2^{p_2} - \lambda_i'' x_1^{p_1}.$$

Consequently, we obtain an $\mathbb{L}$-graded algebra $R = R(\boldsymbol{\lambda}, \mathbf{p}) := K[x_1, \dots, x_t]/I$. Note that the group $\mathbb{L}$ acts by degree shift on the category $\mathsf{gr}^{\mathbb{L}} R$ of finitely generated $\mathbb{L}$-graded R-modules. Namely, given an $\mathbb{L}$-graded R-module M and $\vec{x} \in \mathbb{L}$ we denote by $M(\vec{x})$ the R-module with grading $M(\vec{x})_{\vec{y}} := M_{\vec{x}+\vec{y}}$.

Let $\mathsf{gr}^{\mathbb{L}} R$ be te category of finitely generated $\mathbb{L}$-graded R-modules. Note that $\mathbb{L}$ acts on $\mathsf{gr}^{\mathbb{L}} R$ by degree shift: given $\vec{x} \in \mathbb{L}$ and $M \in \mathsf{gr}^{\mathbb{L}} R$, we define $M(\vec{x}) \in \mathsf{gr}^{\mathbb{L}} R$ to be the R-module with M with new grading $M(\vec{x})_{\vec{y}} := M_{\vec{x}+\vec{y}}$. The category $\mathsf{coh}\,\mathbb{X}$ is defined as the localization $\mathsf{qgr}^{\mathbb{L}} R$ of $\mathsf{gr}^{\mathbb{L}} R$ by its Serre subcategory $\mathsf{gr}_0^{\mathbb{L}} R$ of finite dimensional $\mathbb{L}$-graded R-modules. We denote the image of a module M under the canonical quotient functor $\mathsf{gr}^{\mathbb{L}} R \to \mathsf{qgr}^{\mathbb{L}} R$ by $\widetilde{M}$. It follows that the action of $\mathbb{L}$ on $\mathsf{gr}^{\mathbb{L}} R$ induces an action on $\mathsf{coh}\,\mathbb{X}$ given by $\widetilde{M}(\vec{x}) := (M(\vec{x}))^{\sim}$. We call $\mathcal{O} = \mathcal{O}_{\mathbb{X}} := \widetilde{R}$ the *structure sheaf of* $\mathbb{X}$.

THEOREM 1.2.1. [**36**] [**73**, Thm. 2.2] *The category* $\mathsf{coh}\,\mathbb{X}$ *is connected,* Hom-*finite, K-linear and abelian. Moreover we have the following:*

(i) The category $\mathsf{coh}\,\mathbb{X}$ *is hereditary,* i.e. *we have* $\mathsf{Ext}^i_{\mathbb{X}}(-,-) = 0$ *for all* $i \geq 2$.

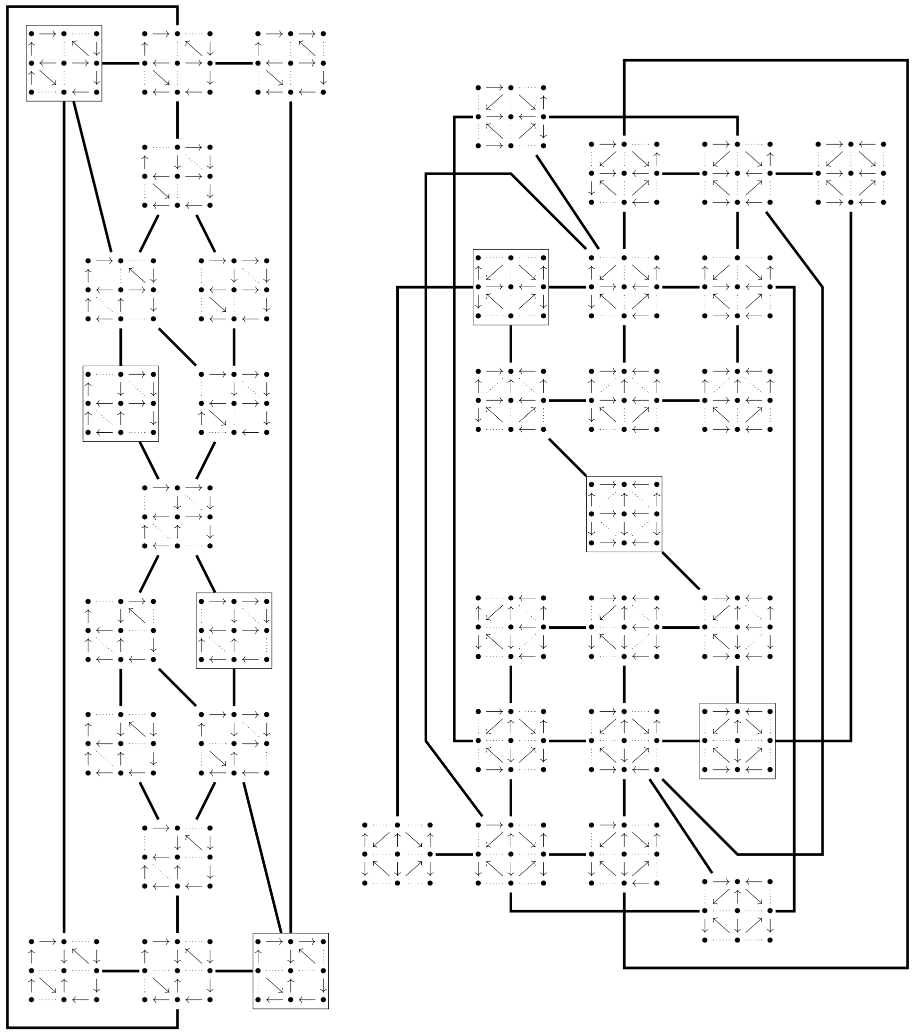


FIGURE 1.1.2. (Part 2 of 2) The basic 2-representation-finite derived-canonical algebras of type (2,4,4). Thick lines indicate 2-APR-(co)tilting. The algebras that arise as endomorphism algebras of tilting sheaves in $\mathsf{coh}\,\mathbb{X}$ are enclosed in a frame.

(ii) *(Serre duality) Let* $\tau : \mathsf{coh}\,\mathbb{X} \to \mathsf{coh}\,\mathbb{X}$ *be the autoequivalence given by* $E \mapsto E(\vec{\omega})$*. Then, there is a bifunctorial isomorphism*

$$D\,\mathsf{Ext}^1_{\mathbb{X}}(X,Y) \cong \mathsf{Hom}_{\mathbb{X}}(Y,\tau X).$$

We call τ *the* Auslander-Reiten translation *of* $\mathsf{coh}\,\mathbb{X}$*.*

(iii) *Let* $\mathsf{coh}_0\,\mathbb{X}$ *be the full subcategory of* $\mathsf{coh}\,\mathbb{X}$ *of sheaves of finite length (*=torsion sheaves*). Also, let* $\mathsf{vect}\,\mathbb{X}$ *be the full subcategory of* $\mathsf{coh}\,\mathbb{X}$ *of sheaves with no non-zero torsion subsheaves (*=vector bundles*). Then, each* $X \in \mathsf{coh}\,\mathbb{X}$ *has a unique decomposition* $X = X_+ \oplus X_0$ *where* $X_+ \in \mathsf{vect}\,\mathbb{X}$ *and* $X_0 \in \mathsf{coh}_0\,\mathbb{X}$*.*

(iv) *The simple objects in* $\mathsf{coh}_0\,\mathbb{X}$ *are parametrized by* $\mathbb{P}^1_K$ *as follows: For each* $\lambda \in \mathbb{P}^1_K \setminus \boldsymbol{\lambda}$ *there exist a unique simple sheaf* S_λ *called the* ordinary simple concentrated at λ*, and for each* $\lambda_i \in \boldsymbol{\lambda}$ *there exist* p_i exceptional *(*i.e. *not*

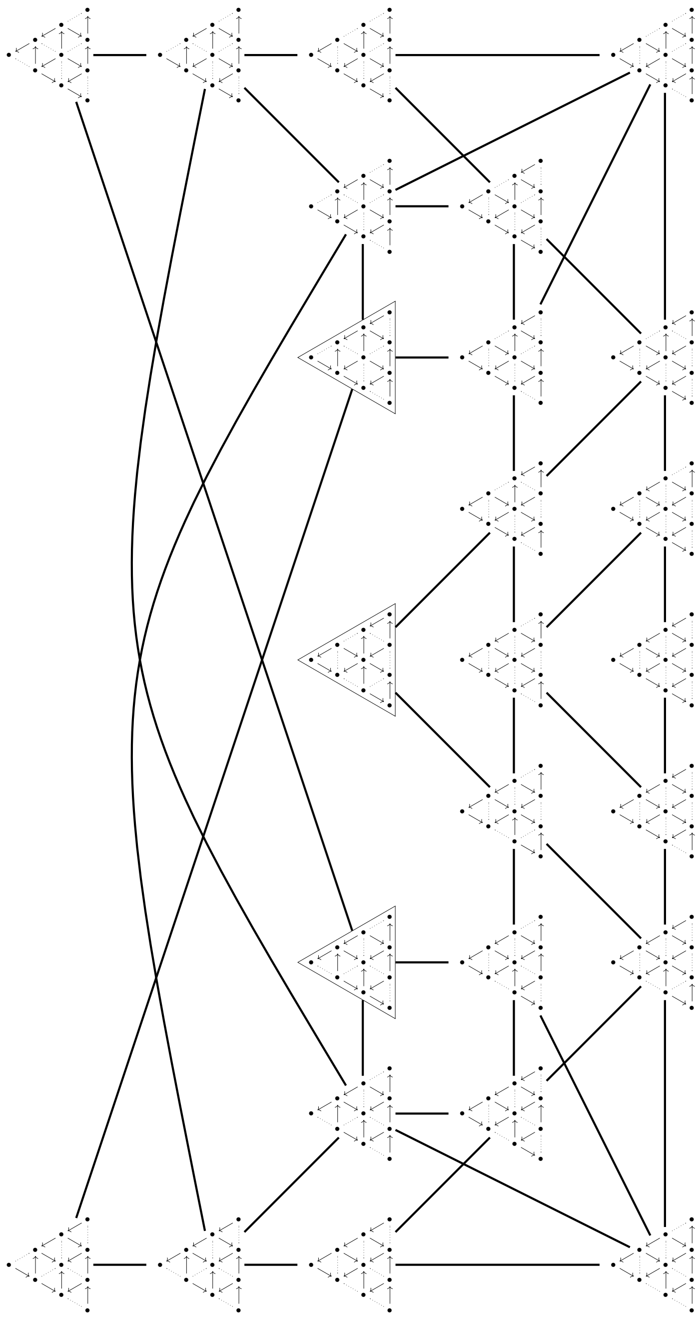


FIGURE 1.1.3. (Part 1 of 2) The basic 2-representation-finite derived-canonical algebras of type (2,3,6). Thick lines indicate 2-APR-(co)tilting. The algebras that arise as endomorphism algebras of tilting sheaves in $\operatorname{coh}\mathbb{X}$ are enclosed in a frame.

ordinary) simple sheaves $S_{\lambda,1},\dots,S_{\lambda,p_i}$ *defined by a short exact sequence*

$$0 \longrightarrow \mathcal{O}(-m\vec{x}_i) \longrightarrow \mathcal{O}((1-m)\vec{x}_i) \longrightarrow S_{\lambda_i,m} \longrightarrow 0$$

for $i \in \{1,\dots,t\}$ *and* $m \in \{1,\dots,p_i\}$.

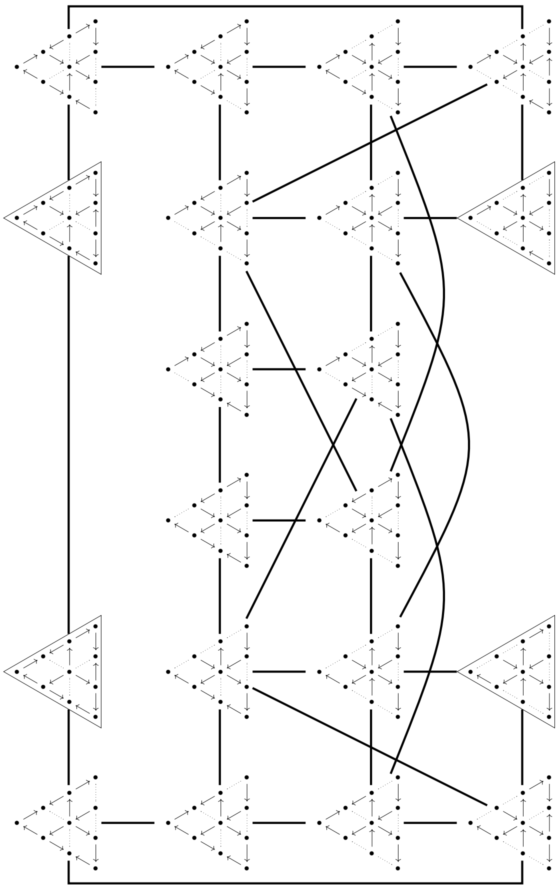


FIGURE 1.1.3. (Part 2 of 2) The basic 2-representation-finite derived-canonical algebras of type (2,3,6). Thick lines indicate 2-APR-(co)tilting. The algebras that arise as endomorphism algebras of tilting sheaves in $\operatorname{\mathsf{coh}} \mathbb{X}$ are enclosed in a frame.

(v) *For each simple sheaf S we have $\operatorname{\mathsf{End}}_{\mathbb{X}}(S) \cong K$. If S is an ordinary simple sheaf, then $\operatorname{\mathsf{Ext}}^1_{\mathbb{X}}(S,S) \cong K$. If S is an exceptional simple sheaf, then $\operatorname{\mathsf{Ext}}^1_{\mathbb{X}}(S,S) = 0$.*

(vi) *Let $\lambda \in \mathbb{P}^1_K$. The category $\mathcal{T}(\lambda)$ of all sheaves which have a finite filtration by simple sheaves concentrated at λ form a standard tube. If $\lambda \notin \boldsymbol{\lambda}$ then $\mathcal{T}(\lambda)$ has rank 1; if $\lambda = \lambda_i$, then $\mathcal{T}(\lambda)$ has rank p_i .*

(vii) *Let $\vec{a}, \vec{b} \in \mathbb{L}$. Then $\operatorname{\mathsf{Hom}}_{\mathbb{X}}(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{b})) = R_{\vec{b}-\vec{a}}$. In particular, there is a non-zero morphism $\mathcal{O}(\vec{a}) \to \mathcal{O}(\vec{b})$ if and only if $\vec{b} - \vec{a} \geq 0$.*

The complexity of the classification of indecomposable sheaves $\operatorname{\mathsf{coh}} \mathbb{X}$ is controlled by its *Euler characteristic*

$$\chi(\mathbb{X}) := 2 - \sum_{i=1}^{t} \left(1 - \frac{1}{p_i}\right).$$

Weighted projective lines of Euler characteristic zero will turn out to be our main concern in this chapter. An easy calculation shows that $\chi(\mathbb{X}) = 0$ if and only if

$$\mathbf{p} \in \{(2,2,2,2), (3,3,3), (2,4,4), (2,3,6)\},$$

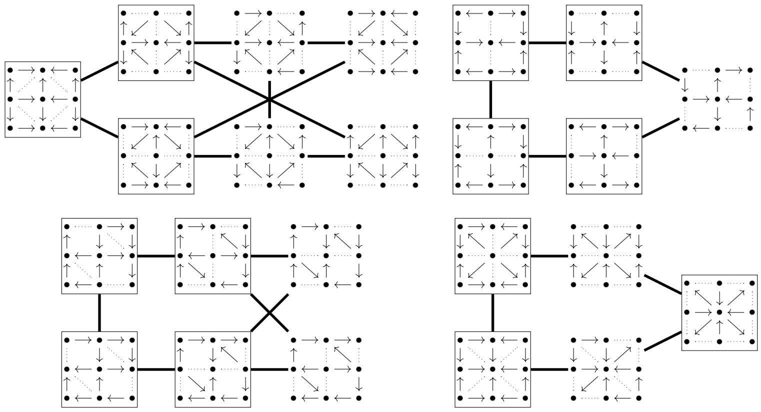


FIGURE 1.1.4. Endomorphism algebras of τ^2-stable basic tilting complexes in $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$ for type $(2,4,4)$. All relations are commutativity or zero relations, *cf.* Figure 1.1.7. Thick lines indicate 2-APR-(co)tilting along orbits of the action of τ^2, which is given by rotation by π. The algebras that arise as endomorphism algebras of tilting sheaves in $\mathsf{coh}\,\mathbb{X}$ are enclosed in a frame.

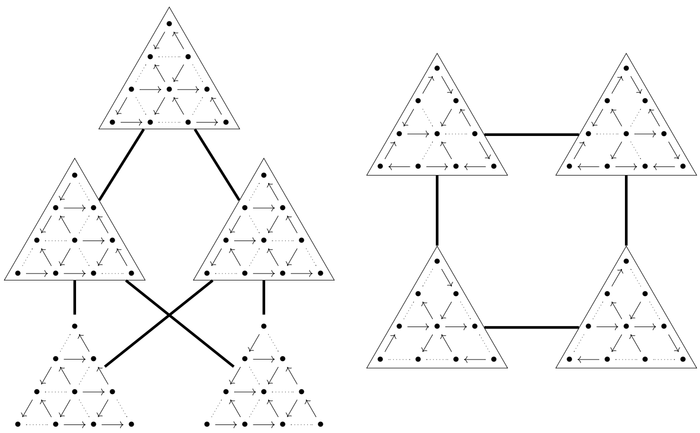


FIGURE 1.1.5. Endomorphism algebras of τ^2-stable basic tilting complexes in $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$ for type $(2,3,6)$. All relations are commutativity or zero relations, *cf.* Figure 1.1.8. Thick lines indicate 2-APR-(co)tilting along orbits of the action of τ^2, which is given by counter-clockwise rotation by $2\pi/3$. The algebras that arise as endomorphism algebras of tilting sheaves in $\mathsf{coh}\,\mathbb{X}$ are enclosed in a frame.

if and only if the dualizing element $\vec{\omega}$ has finite order $p = \mathsf{lcm}(p_1, \ldots, p_t)$ in $\mathbb{L}$. In this case, we say that $\mathbb{X}$ has *tubular type* and it follows that τ acts periodically on each connected component of the Auslander-Reiten quiver of $\mathsf{coh}\,\mathbb{X}$.

Let $\mathrm{K}_0(\mathbb{X})$ be the Grothendieck group of $\mathsf{coh}\,\mathbb{X}$. There are two important linear forms rk and deg on $\mathrm{K}_0(\mathbb{X})$. We refer the reader to [**73**, Sec. 2.2] for information on these numerical invariants. The rank $\mathsf{rk}\colon \mathrm{K}_0(\mathbb{X}) \to \mathbb{Z}$ is characterized by the property $\mathsf{rk}(\mathcal{O}(\vec{x})) = 1$ for each $\vec{x}$ in $\mathbb{L}$. The degree $\mathsf{deg}\colon \mathrm{K}_0(\mathbb{X}) \to \mathbb{Z}$ is characterized by the property $\mathsf{deg}(\mathcal{O}(\vec{x})) = \delta(\vec{x})$ where $\delta\colon \mathbb{L} \to \mathbb{Z}$ is the unique group homomorphism

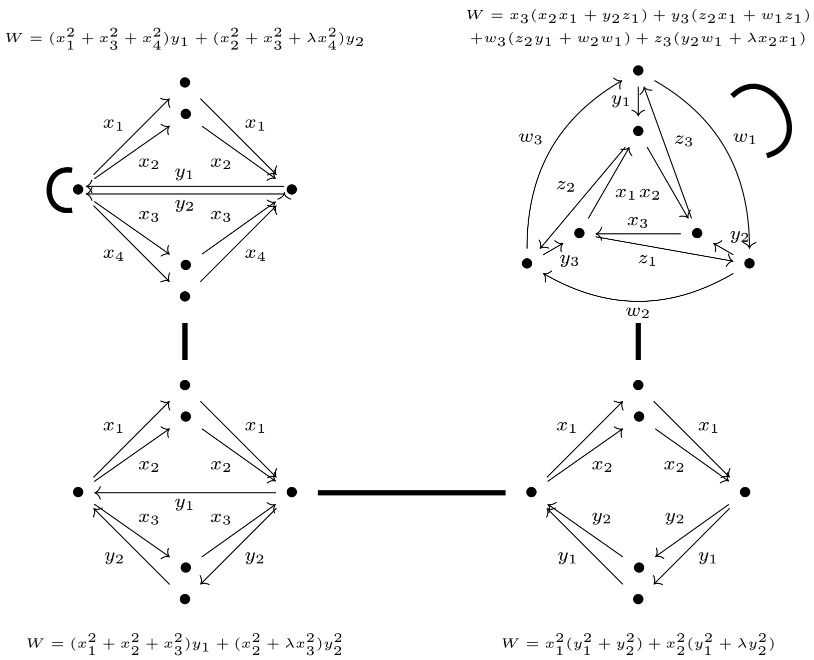


FIGURE 1.1.6. The quivers with potential associated to the basic selfinjective cluster-tilted algebras of type $\mathbf{p} = (2, 2, 2, 2; \lambda)$. All cluster-tilted algebras are selfinjective since $\tau^2 = 1_{\mathbb{X}}$. Thick edges indicate mutation of quivers with potential along the orbits of the Nakayama permutation, which is trivial in this case. Note that $\lambda \neq 0, 1$, and that we may replace λ by $1 - \lambda$, $\frac{1}{\lambda}$, $\frac{1}{1-\lambda}$, $\frac{\lambda}{1-\lambda}$ or $\frac{\lambda-1}{\lambda}$ without changing the isomorphism class of the associated Jacobian algebra.

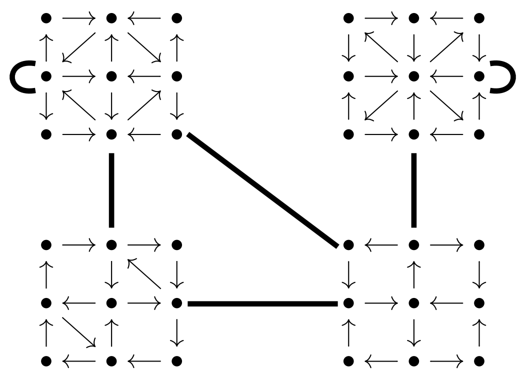


FIGURE 1.1.7. The quivers with potential associated to the basic selfinjective cluster-tilted algebras of type $\mathbf{p} = (2, 4, 4)$. For each quiver, the potential is given by $W = \sum(\text{clockwise cycles}) - \sum(\text{counter-clockwise cycles})$. Thick edges indicate mutation of quivers with potential along the orbits of the Nakayama permutation, which is given by rotation by π.

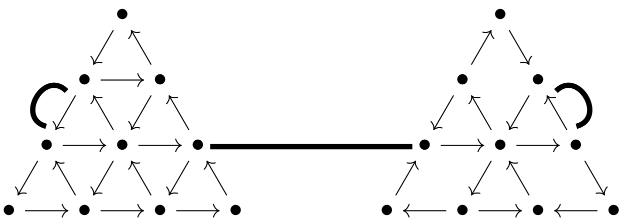


FIGURE 1.1.8. The quivers with potential associated to the basic selfinjective cluster-tilted algebras of type $\mathbf{p} = (2, 3, 6)$. For each quiver, the potential is given by $W = \sum(\text{clockwise cycles}) - \sum(\text{counter-clockwise cycles})$. Thick edges indicate mutation of quivers with potential along the orbits of the Nakayama permutation, which is given by counter-clockwise rotation by $2\pi/3$.

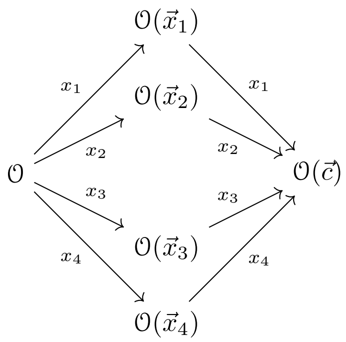


FIGURE 1.2.1. The Gabriel quiver of the endomorphism algebra of the canonical tilting bundle for $\mathbf{p} = (2, 2, 2, 2; \lambda)$.

sending each $\vec{x}_i$ to p/p_i. Note that we have

$$\chi(\mathbb{X}) = \frac{-\delta(\vec{\omega})}{p}. \tag{1.2.1}$$

Moreover, if a sheaf $X \in \mathsf{coh}\,\mathbb{X}$ satisfies $\mathsf{deg}(X) = \mathsf{rk}(X) = 0$, then $X = 0$. The slope of a non-zero sheaf $X$is defined as $\mathsf{S}(X) := \mathsf{rk}(X)/\,\mathsf{deg}(X) \in \mathbb{Q} \cup \{\infty\}$.

PROPOSITION 1.2.2. [**73**, Lemma. 2.5] *For each non-zero $X \in \mathsf{vect}\,X$ we have* $\mathsf{S}(\tau X) = \mathsf{S}(X) + \delta(\vec{\omega})$.

A complex T in $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$ is called a *tilting complex* if $\mathsf{Ext}^i_{\mathrm{D^b}(\mathbb{X})}(T, T) = 0$ for all $i \neq 0$ and if the conditions $\mathsf{Ext}^i_{\mathrm{D^b}(\mathbb{X})}(T, X) = 0$ for all $i \in \mathbb{Z}$ imply that $X = 0$. Equivalently, T is a tilting complex if and only if T is rigid and the number of pairwise non-isomorphic indecomposable direct summands of T equals $2 + \sum_{i=1}^{t}(p_i - 1)$, the rank of $\mathrm{K}_0(\mathbb{X})$.

The vector bundle

$$T = T_{\mathcal{O}} := \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$$

is a tilting sheaf whose endomorphism algebra is precisely $\Lambda = \Lambda(\boldsymbol{\lambda}, \mathbf{p})$, the *canonical algebra of type* $(\boldsymbol{\lambda}, \mathbf{p})$, see Figure 1.2.1 for an example. It follows that the bounded derived categories $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$ and $\mathrm{D^b}(\mathsf{mod}\,\Lambda)$ are equivalent as triangulated categories.

PROPOSITION 1.2.3. [**74**, Cor. 3.5]. *Let $\mathbb{X}$ be a weighted projective line of tubular type and T a tilting sheaf in $\mathsf{coh}\,\mathbb{X}$. Then there exists an automorphism $F\colon \mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X}) \to \mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$ such that $FT \in \mathsf{coh}\,\mathbb{X}$ has a simple sheaf as a direct summand.*

Let $\mathbb{X}$ be a weighted projective line of tubular type and T a tilting sheaf in $\mathsf{coh}\,\mathbb{X}$. We say that T is in *normal position* if $T_0 \neq 0$, see Theorem 1.2.1(iii) and Theorem 1.2.3.

The result below collects further properties of $\mathsf{coh}\,\mathbb{X}$ which are needed to prove Theorem 1.3.5.

THEOREM 1.2.4. [**37, 73**] *Let $\mathbb{X}$ be a weighted projective line of weight type $(p_1, \ldots, p_t)$ and T a tilting sheaf in normal position. Then, the following statements hold:*

(i) Let q_i be the number of indecomposable direct summands of T_0 in $\mathcal{T}(\lambda_i)$. Then, the perpendicular category

$$T_0^{\perp} := \left\{ X \in \mathsf{coh}\,\mathbb{X} \mid \mathsf{Hom}_{\mathbb{X}}(T_0, X) = 0 \text{ and } \mathsf{Ext}^1_{\mathbb{X}}(T_0, X) = 0 \right\}$$

is equivalent to $\mathsf{coh}\,\mathbb{Y}$ where $\mathbb{Y}$ is a weighted projective line of weight type $(p_1 - q_1, \ldots, p_t - q_t)$.

(ii) If $\mathbb{X}$ has tubular type, then $\chi(\mathbb{Y}) > 0$. In this case, $\mathsf{coh}\,\mathbb{Y}$ is derived equivalent to $\mathsf{mod}\,H$ for a tame herediary algebra of extended Dynkin type Δ, and the Auslander-Reiten quiver of $\mathsf{vect}\,\mathbb{Y}$ has shape $\mathbb{Z}\Delta$.

(iii) The embedding $\mathsf{coh}\,\mathbb{Y} \cong T_0^{\perp} \subset \mathsf{coh}\,\mathbb{X}$ preserves line bundles and torsion sheaves. That is, we have $\mathsf{vect}\,\mathbb{Y} \cong (\mathsf{vect}\,\mathbb{X} \cap T_0^{\perp})$ and $\mathsf{coh}_0\,\mathbb{Y} \cong (\mathsf{coh}_0\,\mathbb{X} \cap T_0^{\perp})$.

(iv) Let $\vec{x} \in \mathbb{L}$ be such that $T(\vec{x}) \cong T$. Then the functor $?(\vec{x})\colon \mathsf{coh}\,\mathbb{X} \to \mathsf{coh}\,\mathbb{X}$ induces an action on $\mathsf{coh}\,\mathbb{Y}$ which acts freely on line bundles in $\mathsf{coh}\,\mathbb{Y}$.

(v) The sheaf T_+ is a tilting bundle in $\mathsf{coh}\,\mathbb{Y}$. If $\mathbb{X}$ has tubular type, then T_+ contains a line bundle as a direct summand.

PROOF. Statements (i) and (iii) are shown in [**37**, Thm. 9.5 and Prop. 9.6].

(ii) The first claim is a straightforward computation. The remaining statements are shown for example in [**73**, Thm. 3.5, Cor. 3.6].

(iv) First, note that the group $\mathbb{L}$ acts freely on line bundles in $\mathsf{coh}\,\mathbb{X}$ by degree shift. Moreover, this action preserves $\mathsf{vect}\,\mathbb{X}$ and $\mathsf{coh}_0\,\mathbb{X}$. Let $\vec{x} \in \mathbb{L}$ be such that $T(\vec{x}) \cong T$. Since we have $T_0(\vec{x}) \cong T_0$, it follows that $(\vec{x})$ induces an action on $T_0^{\perp} \cong \mathsf{coh}\,\mathbb{Y}$. By part (iii) this action acts freely on line bundles in $\mathsf{coh}\,\mathbb{Y}$.

(v) The first claim follows since T_+ is also rigid in $T_0^{\perp} \subset \mathsf{coh}\,\mathbb{X}$ and the number of indecomposable direct summands of T_+ coincides with the rank of $\mathrm{K}_0(\mathbb{Y})$. The second claim follows since $\chi(\mathbb{Y}) > 0$, and hence every tilting bundle in $\mathsf{coh}\,\mathbb{Y}$ contains a line bundle as a direct summand, see [**73**, Cor. 3.7]. □

We have the following simple observation regarding τ^2-stable rigid sheaves.

LEMMA 1.2.5. *Let $\mathbb{X}$ be a weighted projective line and $X \in \mathsf{coh}\,\mathbb{X}$ be a τ^2-stable rigid sheaf. Then, each indecomposable direct summand of X is an exceptional simple sheaf.*

PROOF. Firstly, by Theorem 1.2.1(vii) there are no rigid sheaves in an exceptional tube of rank 1. Secondly, since X is a rigid τ^2-stable sheaf, we have that

$$\mathsf{Ext}^1_{\mathbb{X}}(X, X) \cong D\,\mathsf{Hom}_{\mathbb{X}}(X, \tau X) \cong D\,\mathsf{Hom}_{\mathbb{X}}(\tau X, X) = 0.$$

Let Y be an indecomposable direct summand of X. Then we have $\mathsf{Hom}_{\mathbb{X}}(\tau Y, Y) = 0$. This is happens if and only if Y is an exceptional simple sheaf. □

The Auslander-Reiten translation of $\mathsf{coh}\,\mathbb{X}$ extends to an autoequivalence

$$\tau\colon \mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X}) \to \mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X}).$$

Moreover, the autoequivalence $\nu := \tau[1]$ gives a Serre functor of $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$.

DEFINITION 1.2.6. A complex X in $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$ is *τ^2-stable* if $\tau^2 X \cong X$.

The following result is a particular case of [**74**, Thm. 3.1]. It allows us to compute the endomorphism algebra of a tilting sheaf in a given weighted projective line in terms of a weighted projective line of smaller weights.

THEOREM 1.2.7. [**74**, Thm. 3.1] *Let $\mathbb{X}$ be a weighted projective line of type $(p_1, \dots, p_t)$. Let T be a τ^2-stable tilting sheaf in $\mathsf{coh}\,\mathbb{X}$, and suppose that the indecomposable direct summands of T_0 are exceptional simple sheaves at the points $\lambda_{i_i}, \dots, \lambda_{i_k} \in \boldsymbol{\lambda}$. We make the identification $\mathsf{coh}\,\mathbb{Y} \cong T_0^{\perp}$, see Proposition 1.2.4*(i)*. Finally, let $E \in \mathsf{coh}\,\mathbb{Y}$ be the direct sum of all exceptional simple sheaves at the points $\lambda_{i_1}, \dots, \lambda_{i_k}$. Then, there is an isomorphism of algebras*

$$\mathsf{End}_{\mathbb{X}}(T) \cong \mathsf{End}_{\mathbb{Y}}(T_+ \oplus E) \cong \begin{bmatrix} \mathsf{End}_{\mathbb{Y}}(T_+) & \mathsf{Hom}_{\mathbb{Y}}(T_+, E) \\ 0 & \mathsf{End}_{\mathbb{Y}}(E) \end{bmatrix}.$$

PROOF. Let $r\colon \mathsf{coh}\,\mathbb{X} \to \mathsf{coh}\,\mathbb{Y}$ be the right adjoint of the inclusion $\mathsf{coh}\,\mathbb{Y} \cong T_0^{\perp}$. It is easy to see that r induces a bijection between the indecomposable direct summands of T_0 and the exceptional simple sheaves in $\mathsf{coh}\,\mathbb{Y}$ at the points $\lambda_{i_1}, \dots, \lambda_{i_k}$. Then the result follows immediately from the proof of [**74**, Thm. 3.1]. $\square$

1.2.2. Graded quivers with potential and their mutations. Quivers with potentials and their Jacobian algebras where introduced in [**31**] as a tool to prove several of the conjectures of [**34**] about cluster algebras in a rather general setting, see [**32**]. Their graded version was introduced in [**6**] in order to describe the effect of mutation of cluster tilting objects in generalized cluster categories at the level of the corresponding derived category.

Let $Q = (Q_0, Q_1)$ be a finite quiver without loops or 2-cycles and $d : Q_1 \to \mathbb{Z}$ a map called a *degree function* on the set of arrows of Q. Then d induces a $\mathbb{Z}$-grading on the complete path algebra $\widehat{kQ}$ in an obvious way. We endow $\widehat{KQ}$ with the $\mathfrak{J}$-adic topology where $\mathfrak{J}$ is the radical of $\widehat{kQ}$. A *potential* xin Q is a formal linear combination of cyclic paths in Q; we are only interested in potentials which are homogeneous elements of $\widehat{kQ}$. For a cyclic path $a_1 \cdots a_d$ in Q and $a \in Q_1$, let

$$\partial_a(a_1 \cdots a_d) = \sum_{a_i = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}$$

and extend it linearly and continuously to an arbitrary potential in Q. The maps ∂_a are called *cyclic derivatives.*

DEFINITION 1.2.8. A *graded quiver with potential* is a triple (Q, W, d) where (Q, d) is a $\mathbb{Z}$-graded finite quiver without loops and 2-cycles and W is a homogeneous potential for Q. The *graded Jacobian algebra* of (Q, W, d) is the $\mathbb{Z}$-graded algebra

$$\mathsf{Jac}(Q, W, d) \cong \frac{\widehat{kQ}}{\partial(W)}$$

where $\partial(W)$ is the closure in $\widehat{kQ}$ of the ideal generated by the subset

$$\{\partial_a(W) \mid a \in Q_1\}.$$

For each vertex of Q there is a pair of well defined operations on the right-equivalence classes of graded quivers with potential called *left and right mutations* (see [**31**, Def. 4.2] for the definition of right-equivalence). Note that right equivalent quivers with potential have isomorphic Jacobian algebras.

Let (Q,W,d) be graded quiver with potential with W homogeneous of degree $d(W)$ and $k \in Q_0$. The *non-reduced left mutation at* k of (Q,W,d) is the graded quiver with potential $\tilde{\mu}_k^L(Q,W,d) = (Q',W',d')$ defined as follows:

(i) The quivers Q and Q' have the same set of vertices.
(ii) All arrows of Q which are not adjacent to k are also arrows of Q' and of the same degree.
(iii) Each path $i \xrightarrow{a} k \xrightarrow{b} j$ in Q creates an arrow $[ba] : i \to j$ of degree $d(a)+d(b)$ in Q'.
(iv) Each arrow $a : i \to k$ of Q is replaced in Q' by an arrow $a^* : k \to i$ of degree $-d(a)+d(W)$.
(v) Each arrow $b : k \to j$ of Q is replaced in Q' by an arrow $b^* : j \to k$ of degree $-d(b)$.
(vi) The new potential is given by
$$W' = [W] + \sum_{i \xrightarrow{a} k \xrightarrow{b} j} [ba]a^*b^*$$
where $[W]$ is the potential obtained from W by replacing each path $i \xrightarrow{a} k \xrightarrow{b} j$ which appears in W with the corresponding arrow $[ba]$ of Q'.

By [**6**, Thm. 6.6], there exists a so-called *reduced* graded quiver with potential $(Q'_{\mathrm{red}}, W'_{\mathrm{red}}, d'_{\mathrm{red}})$ which is right equivalent to (Q',W',d'). The *left mutation at* k of (Q',W',d') is then defined as
$$\mu_k^L(Q,W,d) := (Q'_{\mathrm{red}}, W'_{\mathrm{red}}, d').$$
The *right mutation at* k of (Q,W,d) is defined almost identically, just by replacing (d) and (e) above by

(d) Each arrow $a : i \to k$ of Q is replaced in Q' by an arrow $a^* : k \to i$ of degree $-d(a)$.
(e) Each arrow $b : k \to j$ of Q is replaced in Q' by an arrow $b^* : j \to k$ of degree $-d(b)+d(W)$.

Finally, the following definition is very convenient for our purposes.

Definition 1.2.9. [**51**, Sec. 3] Let (Q,W,d) be a graded quiver with potential with $d(W)$. Then the *truncated Jacobian algebra* is the degree zero part of $\mathsf{Jac}(Q,W,d)$, which is given by the factor algebra
$$\mathsf{Jac}(Q,W,d) := \mathsf{Jac}(Q,W)/\overline{\langle a \in Q \mid d(a)=1\rangle} = \widehat{kQ}/\overline{\langle \partial_a(W) \mid d(a)=1\rangle}.$$
Also, we say that (Q,W,d) is *algebraic* if $\mathsf{Jac}(Q,W,d)$ has global dimension at most 2 and the set
$$\{\partial_a(W) \mid d(a)=1\}$$
is a minimal set of generators of the ideal $\overline{\langle \partial_a(W) \in Q \mid d(a)=1\rangle}$ of $\widehat{kQ}$.

Note that left and right mutation differ from each other at the level of the grading only.

1.2.3. 2-representation-finite algebras and 2-APR-tilting. Let Λ be a finite dimensional algebra of global dimension at most 2. Following [**59**, Def. 2.2], we say that Λ is *2-representation-finite* if there exist a finite dimensional Λ-module M such that
$$\mathsf{add}\, M = \left\{X \in \mathsf{mod}\,\Lambda \mid \mathsf{Ext}^1_\Lambda(M,X)=0\right\} = \left\{X \in \mathsf{mod}\,\Lambda \mid \mathsf{Ext}^1_\Lambda(X,M)=0\right\}.$$
Such Λ-module M is called a *2-cluster-tilting module.* The functors
$$\tau_2 := D\,\mathsf{Ext}^2_\Lambda(-,\Lambda) : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda$$

and

$$\nu_2 := \nu[-2] : \mathrm{D^b}(\mathsf{mod}\,\Lambda) \to \mathrm{D^b}(\mathsf{mod}\,\Lambda),$$

where $\nu : - \otimes^{\mathbf{L}}_{\Lambda} D\Lambda : \mathrm{D^b}(\mathsf{mod}\,\Lambda) \to \mathrm{D^b}(\mathsf{mod}\,\Lambda)$ is the Nakayama functor of $\mathrm{D^b}(\mathsf{mod}\,\Lambda)$ play an important role in the theory of 2-representation-finite algebras. Moreover, they are related by a functorial isomorphism $\tau_2 \cong H^0(\nu_2-)$. Note that τ_2 induces a bijection between indecomposable non-projective objects in $\mathsf{add}\,M$ and indecomposable non-injective objects in $\mathsf{add}\,M$.

Definition 1.2.10. [**50**, Def. 1.2] Let Λ be a 2-representation-finite algebra. We say that Λ is *2-homogeneous* if each τ_2-orbit of indecomposable objects in $\mathsf{add}\,M$ consists of precisely two objects. This is equivalent to $\nu_2^{-1}(\Lambda)$ being an injective Λ-module.

The class of 2-representation-finite algebras can be characterized in terms of the so-called 3-preprojective algebras.

Definition 1.2.11. [**71**] Let Λ be a finite dimensional algebra of global dimension at most 2. The *complete 3-preprojective algebra of* Λ is the tensor algebra

$$\Pi_3(\Lambda) := \prod_{d\geq 0} \mathsf{Ext}^2_{\Lambda}(D\Lambda, \Lambda)^{\otimes d}.$$

We have the following characterization of 2-representation-finite algebras.

Proposition 1.2.12. [**51**, Prop. 3.9] *Let Λ be a finite dimensional algebra of global dimension at most 2. Then Λ is 2-representation-finite if and only if $\Pi_3(\Lambda)$ is a finite dimensional selfinjective algebra.*

Following [**71**], $\Pi_3(\Lambda)$ can be presented as a graded Jacobian algebra for some quiver with potential obtained from Λ. For this, let Q be the Gabriel quiver of Λ and let

$$\Lambda \cong \widehat{kQ}/\overline{\langle r_1, \dots r_s\rangle}$$

where $\{r_1, \dots, r_s\}$ is a minimal set of relations for Λ. Consider the extended quiver

$$\tilde{Q} = Q \amalg \{r_i^* : t(r_i) \to s(r_i) \mid r_i : s(r_i) \dashrightarrow t(r_i)\}_{1\leq i\leq s},$$

i.e. $\tilde{Q}$ is obtained from Q by adding an arrow in the opposite direction for each relation in Λ. We consider $\tilde{Q}$ as a graded quiver where the arrows in Q_1 have degree zero and the arrows r_i^* have degree one. Then we can define a homogeneous potential W in $\tilde{Q}$ of degree one by

$$W := \sum_{i=1}^{s} r_i r_i^*.$$

Theorem 1.2.13. [**71**, Thm. 6.10] *Let Λ be a finite dimensional algebra of global dimension at most 2. Then there is an isomorphism of graded algebras between $\mathsf{Jac}(\tilde{Q}, W, d)$ and $\Pi_3(\Lambda)$.*

A useful tool to construct 2-representation-finite algebras which are derived equivalent to a given one is 2-APR-tilting, which is a higher analog of usual APR-tilting. The notion of 2-APR-co-tilting is defined dually.

Definition 1.2.14. [**59**, Def. 3.14] Let Λ be a finite dimensional algebra of global dimension at most 2 and $\Lambda = P \oplus Q$ any direct summand decomposition of Λ such that

(i) $\mathsf{Hom}_{\Lambda}(Q, P) = 0$.
(ii) $\mathsf{Ext}^i_{\Lambda}(\nu Q, P) = 0$ for any $0 < i \neq 2$.

We call the complex

$$T := (\nu_2^{-1}P) \oplus Q \in \mathrm{D^b}(\mathsf{mod}\,\Lambda)$$

the *2-APR-tilting complex associated with P.*

In analogy with APR-tilting for hereditary algebras, 2-APR-tilting preserves 2-representation-finiteness.

THEOREM 1.2.15. [**59**, Thm. 4.7] *Let Λ be a 2-representation-finite algebra and T a 2-APR-tilting complex in $\mathrm{D^b}(\mathsf{mod}\,\Lambda)$. Then the algebra $\mathsf{End}_{\mathrm{D^b}(\Lambda)}(T)$ is also 2-representation-finite.*

We can describe the effect of 2-APR-tilting using Theorem 1.2.13 as follows:

THEOREM 1.2.16. [**59**, Sec. 3.3] *Let Λ be a 2-representation-finite algebra and P an indecomposable projective Λ-module which corresponds to a sink k in the Gabriel quiver of Λ and let T be the associated 2-APR-tilting Λ-module. Also, let $(\tilde{Q}, W, d)$ be the graded quiver with potential associated to $\Pi_3(\Lambda)$, see Theorem 1.2.13. Then there is an isomorphism of graded algebras*

$$\mathsf{End}_\Lambda(T) \cong \mathsf{Jac}(\tilde{Q}, W, d')$$

where d' coincides with d on arrows not incident to k, for an arrow $a \in \tilde{Q}$ incident to k we have $d'(a) = 1$ if $d(a) = 0$, and we have $d'(a) = 0$ if $d(a) = 1$.

1.2.4. The cluster category of $\mathsf{coh}\,\mathbb{X}$. Cluster categories associated with hereditary algebras were introduced in [**27**] in order to categorify the combinatorics of acyclic cluster algebras. The cluster category of a weighted projective line was studied in [**18**], [**16**] and [**17**]. For the point of view of this chapter, they arise as the categorical environment of 3-preprojective algebras of endomorphism algebras of tilting sheaves in $\mathsf{coh}\,\mathbb{X}$.

The cluster category associated with $\mathsf{coh}\,\mathbb{X}$ is by definition the orbit category

$$\mathcal{C} = \mathcal{C}_\mathbb{X} := \mathrm{D^b}(\mathsf{coh}\,\mathbb{X})/(\tau[-1]).$$

Thus, the objects of $\mathcal{C}$ are bounded complexes of coherent sheaves over $\mathbb{X}$ and the morphism spaces are given by

$$\mathsf{Hom}_\mathcal{C}(X, Y) := \bigoplus_{i\in\mathbb{Z}} \mathsf{Hom}_{\mathrm{D^b}(\mathbb{X})}(X, \tau^i Y[-i])$$

with the obvious composition rule. Note that $\mathsf{Hom}_\mathcal{C}(X, Y)$ has a natural $\mathbb{Z}$-grading. It is known [**18**] that $\mathcal{C}$ is a Hom-finite, Krull-Schmidt, K-linear triangulated category with the 2-Calabi-Yau property: There is a natural isomorphism

$$D\,\mathsf{Hom}_\mathcal{C}(X, Y) \cong \mathsf{Hom}_\mathcal{C}(Y, X[2])$$

for every X, Y in $\mathcal{C}$. It follows from [**18**, Prop. 2.1] that $\mathsf{coh}\,\mathbb{X}$ is a complete system of representatives of isomorphism classes in $\mathcal{C}$ and that we have a natural isomorphism

$$\mathsf{Hom}_\mathcal{C}(X, Y) \cong \mathsf{Hom}_\mathbb{X}(X, Y) \oplus \mathsf{Ext}^1_\mathbb{X}(X, \tau^{-1}Y)$$

for $X, Y \in \mathsf{coh}\,\mathbb{X}$. Recall that an object T in $\mathcal{C}$ is said to be *rigid* provided that $\mathsf{Hom}_\mathcal{C}(T, T[1]) = 0$; more strongly, if we have that

$$\mathsf{add}\,T = \{X \in \mathcal{C} \mid \mathsf{Hom}_\mathcal{C}(X, T[1]) = 0\},$$

then T is called a *cluster-tilting object*. Identifying isomorphism classes in $\mathsf{coh}\,\mathbb{X}$ with those in $\mathcal{C}_\mathbb{X}$, it follows that tilting (resp. rigid) sheaves in $\mathsf{coh}\,\mathbb{X}$ are precisely cluster-tilting (resp. rigid) objects in $\mathcal{C}$. Moreover, we have the following description of the endomorphism algebras of cluster-tilting objects in $\mathcal{C}$.

PROPOSITION 1.2.17. [**4**, Prop. 4.7] *Let T be a tilting sheaf in $\mathsf{coh}\,\mathbb{X}$. Then there is an isomorphism of graded algebras between $\mathsf{End}_\mathcal{C}(T)$ and $\Pi_3(\mathsf{End}_\mathbb{X}(T))$.*

The category $\mathcal{C}$ has a cluster structure in the sense of [**27**]. Moreover, mutation of cluster-tilting objects is compatible with mutations of tilting sheaves and mutation of Jacobian algebras, see [**38**, Secs. 1, 2.5] for example.

Finally, we have the following characterization of cluster-tilting objects with selfinjective endomorphism algebra.

PROPOSITION 1.2.18. [**51**, Prop. 4.4] *Let T be a cluster-tilting object in $\mathcal{C}$. Then $T \cong T[2]$ if and only if $\mathsf{End}_{\mathcal{C}}(T)$ is a selfinjective algebra.*

1.3. Proofs of the main results

In this section we give the proofs of the main results of this chapter, see Theorems 1.3.6, 1.3.7 and 1.3.8.

Note that by the definition of the cluster category associated to $\mathbb{X}$, we have a commutative diagram of functors

$$\begin{tikzcd} \mathsf{coh}\,\mathbb{X} \arrow[r, "\tau"] \arrow[d] & \mathsf{coh}\,\mathbb{X} \arrow[d] \\ \mathcal{C}_{\mathbb{X}} \arrow[r, "\tau"] & \mathcal{C}_{\mathbb{X}} \end{tikzcd}$$

where the vertical arrows correspond to the canonical projection functor. The following characterization can be easily deduced from known results.

PROPOSITION 1.3.1. *Let T be a tilting complex in $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$. Then, the following conditions are equivalent:*

(i) The algebra $\mathsf{End}_{\mathbb{X}}(T)$ is a 2-representation-finite algebra.
(ii) The algebra $\mathsf{End}_{\mathcal{C}}(T)$ is a finite-dimensional selfinjective algebra.
(iii) We have $T[2] \cong T$ in $\mathcal{C}_{\mathbb{X}}$ and $\mathsf{End}_{\mathrm{D^b}(\mathbb{X})}(T)$ has global dimension at most 2.

Moreover, if $T \in \mathsf{coh}\,\mathbb{X}$, then any of the three equivalent conditions above is equivalent to T being τ^2-stable.

PROOF. (i) is equivalent to (ii). Let $\Lambda := \mathsf{End}_{\mathbb{X}}(T)$. By Proposition 1.2.12, the algebra Λ is 2-representation-finite if and only if $\Pi_3(\Lambda)$ is a selfinjective finite dimensional algebra. Moreover, Proposition 1.2.17 yields an isomorphism between $\Pi_3(\Lambda)$ and $\mathsf{End}_{\mathcal{C}}(T)$. The claim follows. The equivalence between (ii) and (iii) is shown in Proposition 1.2.18.

Finally, let $T \in \mathsf{coh}\,\mathbb{X}$. We show that (iii) is equivalent to T being τ^2-stable. Note that, by the definition of $\mathcal{C}$, the functors $[1]\colon \mathcal{C} \to \mathcal{C}$ and $\tau\colon \mathcal{C} \to \mathcal{C}$ are naturally isomorphic. Hence, we have $T[2] \cong \tau^2 T$ as objects of $\mathcal{C}$ and, since isomorphism classes in $\mathcal{C}$ and $\mathsf{coh}\,\mathbb{X}$ coincide, we have $T[2] \cong \tau^2 T$ in $\mathsf{coh}\,\mathbb{X}$. The claim follows. $\square$

In the case of τ^2-stable tilting sheaves we obtain further restrictions on their endomorphism algebras.

PROPOSITION 1.3.2. *Let T be a tilting complex in $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$. Then, T is τ^2-stable if and only if $\mathsf{End}_{\mathrm{D^b}(\mathbb{X})}(T)$ is a 2-homogeneous 2-representation-finite algebra.*

PROOF. Let T be a tilting complex in $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$ and set $\Lambda := \mathsf{End}_{\mathrm{D^b}(\mathbb{X})}(T)$. Then, we have $\tau^2 T \cong T$ if and only if $(\nu[-1])^2(\Lambda) \cong \Lambda$ which is equivalent to $D\Lambda = \nu\Lambda \cong \nu_2^{-1}\Lambda$. Hence, to show that Λ is a 2-homogeneous 2-representation-finite algebra, see Definition 1.2.10, we only need to show that if T is τ^2-stable then Λ has global dimension at most 2. Indeed, for each $i \geq 3$ we have

$$\mathsf{Ext}^i_\Lambda(D\Lambda, \Lambda) \cong \mathsf{Hom}_{\mathrm{D^b}(\Lambda)}(\nu^{-1}\Lambda[2], \Lambda[i]) \cong D\,\mathsf{Hom}_{\mathrm{D^b}(\Lambda)}(\Lambda[i-2], \Lambda) = 0.$$

Thus Λ has global dimension at most 2 as required. $\square$

The following result is crucial in our approach, as it allows to pass from τ^2-stable tilting complex to τ^2-stable tilting sheaves using 2-APR-(co)tilting. Recall that the effect of 2-APR-(co)tilting on the endomorphism algebras of basic tilting complexes can be described using mutations of graded quivers with potential, see Theorem 1.2.16.

PROPOSITION 1.3.3. *Let T be a basic tilting complex in $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$ such that $\mathsf{End}_{\mathrm{D^b}(\mathbb{X})}(T)$ is a 2-representation-finite algebra. Then, there exists a τ^2-stable tilting sheaf $E \in \mathsf{coh}\,\mathbb{X}$ obtained by iterated 2-APR-tilting from T.*

PROOF. Since shifting does not change endomorphism algebras, we can assume that T is concentrated in degrees $-\ell, \dots, -1, 0$. Since $\mathsf{coh}\,\mathbb{X}$ is hereditary, we have $T \cong T_\ell[-\ell] \oplus \cdots \oplus T_1[-1] \oplus T_0$ where each T_i is a non-zero sheaf. We proceed by induction on ℓ. The case $\ell = 0$ follows immediately from Proposition 1.3.1, so let $\ell > 0$. We claim that the complex

$$T' := (\tau^{-1}T_\ell)[1-\ell] \oplus T_{\ell-1}[1-\ell] \oplus \cdots \oplus T_1[-1] \oplus T_0$$

is a 2-APR-tilting complex. Indeed, since there are no negative extensions between objects of $\mathsf{coh}\,\mathbb{X}$, we have

$$\bigoplus_{i=0}^{\ell-1} \mathsf{Hom}_{\mathrm{D^b}(\mathbb{X})}(T_i[-i], T_\ell[-\ell]) = \bigoplus_{i=0}^{\ell-1} \mathsf{Hom}_{\mathrm{D^b}(\mathbb{X})}(T_i, T_\ell[i-\ell]) = 0.$$

Moreover, using the identity $\nu = \tau[1]$, we obtain

$$\begin{aligned} \bigoplus_{i=0}^{\ell-1} \mathsf{Ext}^1_{\mathrm{D^b}(\mathbb{X})}(\nu T_i[-i], T_\ell[-\ell]) &\cong \bigoplus_{i=0}^{\ell-1} \mathsf{Hom}_{\mathrm{D^b}(\mathbb{X})}(\tau T_i[-i], T_\ell[-\ell]) \\ &\cong \bigoplus_{i=0}^{\ell-1} \mathsf{Hom}_{\mathrm{D^b}(\mathbb{X})}(\tau T_i, T_\ell[i-\ell]) = 0. \end{aligned}$$

Finally, since $\mathsf{End}_{\mathrm{D^b}(\mathbb{X})}(T)$ has global dimension 2 we have that

$$\bigoplus_{i=0}^{\ell-1} \mathsf{Ext}^j_{\mathrm{D^b}(\mathbb{X})}(\nu T_i[-i], T_\ell[-\ell]) = 0$$

for all $j \geq 3$. This shows that T' is a 2-APR tilting complex and, by Theorem 1.2.15, we have that $\mathsf{End}_{\mathrm{D^b}(\mathbb{X})}(T')$ is a 2-representation-finite algebra. Hence, by the induction hypothesis, by iterated 2-APR-tilting we can construct a τ^2-stable tilting sheaf E from T. □

Next, we determine which weighted projective lines can have τ^2-stable tilting sheaves.

PROPOSITION 1.3.4. *Let $T \in \mathsf{coh}\,\mathbb{X}$ be a τ^2-stable tilting sheaf. Then $\mathbb{X}$ has tubular weight type $(2,2,2,2)$, $(2,4,4)$ or $(2,3,6)$.*

PROOF. Since there are no tilting sheaves of finite length, we have $\mathsf{S}(T) \in \mathbb{Q}$. Moreover, as we have $\tau^2 T \cong T$, it follows from Proposition 1.2.2 that $\delta(\vec{\omega}) = 0$. Then, using equation (1.2.1), we have that $\chi(\mathbb{X}) = 0$ hence $\mathbb{X}$ has tubular type.

We recall if $\mathbb{X}$ has tubular type, then the full subcategory of $\mathsf{coh}\,\mathbb{X}$ given of all sheaves of a fixed slope is equivalent to the category $\mathsf{coh}_0\,\mathbb{X}$ of torsion sheaves over $\mathbb{X}$ [**73**, Thm. 3.10]. Assume now that X is an indecomposable summand of T which belongs to a tube of odd period $2a+1$, so we have $X \cong \tau(\tau^{2a}X)$. By hypothesis, $\tau^{2a}X$ is a direct summand of T. Hence, by Serre duality we have

$$0 = \mathsf{Ext}^1_{\mathbb{X}}(\tau^{2a}X, X) \cong D\,\mathsf{Hom}_{\mathbb{X}}(X, \tau(\tau^{2a}X)) = D\,\mathsf{Hom}_{\mathbb{X}}(X, X),$$

$$\begin{tikzcd}
1 \arrow[r, gray] \arrow[d, gray] \arrow[dr, dotted, no head] & 1 \arrow[d] & 1 \arrow[l, gray] \arrow[d, gray] \arrow[dl, dotted, no head] \\
1 \arrow[r] & 0 & 1 \arrow[l] \\
1 \arrow[r, gray] \arrow[u, gray] \arrow[ur, dotted, no head] & 1 \arrow[u] & 1 \arrow[l, gray] \arrow[u, gray] \arrow[ul, dotted, no head]
\end{tikzcd}$$

$\mathbb{Y}(1,4,4):\ C_1$

$$\begin{tikzcd}
1 \arrow[d, gray] & 1 \arrow[l, gray] \arrow[r, gray] \arrow[d, dotted, no head] & 1 \arrow[d, gray] \\
1 \arrow[r] & 0 & 1 \arrow[l] \\
1 \arrow[u, gray] & 1 \arrow[l, gray] \arrow[r, gray] \arrow[u, dotted, no head] & 1 \arrow[u, gray]
\end{tikzcd}$$

$\mathbb{Y}(1,4,4):\ C_2$

$$\begin{tikzcd}
1 \arrow[r, gray] & 1 \arrow[r] \arrow[d] & 0 \arrow[d, dotted, no head] \\
1 \arrow[u, gray] \arrow[r, dotted, no head] \arrow[d, dotted, no head] \arrow[dr, gray] & 2 \arrow[r, dotted, no head] & 1 \arrow[ul, gray] \arrow[d, gray] \\
0 & 1 \arrow[l] \arrow[u] \arrow[r, gray] & 1
\end{tikzcd}$$

$\mathbb{Y}(1,2,4):\ C_3$

$$\begin{tikzcd}
0 \arrow[r, dotted, no head] & 1 \arrow[r, dotted, no head] \arrow[dl, gray] \arrow[dr, gray] \arrow[d, dotted, no head] & 0 \\
1 \arrow[u] \arrow[r] \arrow[d] & 0 \arrow[d, dotted, no head] & 1 \arrow[l] \arrow[u] \arrow[d] \\
0 \arrow[r, dotted, no head] & 1 \arrow[r, dotted, no head] \arrow[ul, gray] \arrow[ur, gray] & 0
\end{tikzcd}$$

$\mathbb{Y}(1,2,2):\ C_4$

$$\begin{tikzcd}
0 \arrow[r, dotted, no head] & 1 \arrow[r, dotted, no head] \arrow[d, gray] & 0 \\
1 \arrow[u] \arrow[d] & 2 \arrow[l, gray] \arrow[r, gray] & 1 \arrow[u] \arrow[d] \\
0 \arrow[r, dotted, no head] & 1 \arrow[r, dotted, no head] \arrow[u, gray] & 0
\end{tikzcd}$$

$\mathbb{Y}(2,2,2):\ C_5$

$$\begin{tikzcd}
0 \arrow[r, dotted, no head] \arrow[d, dotted, no head] & 1 \arrow[r, dotted, no head] & 0 \arrow[d, dotted, no head] \\
1 \arrow[r, gray] \arrow[d, dotted, no head] & 2 \arrow[ul] \arrow[ur] \arrow[u, gray, leftarrow] \arrow[dl] \arrow[dr] & 1 \arrow[l, gray] \arrow[d, dotted, no head] \\
0 \arrow[r, dotted, no head] & 1 \arrow[r, dotted, no head] \arrow[u, gray] & 0
\end{tikzcd}$$

$\mathbb{Y}(2,2,2):\ C_6$

$$\begin{tikzcd}
1 \arrow[r] & 0 \arrow[r, dotted, no head] & 1 \arrow[d, gray] \\
2 \arrow[u, gray] \arrow[r, gray] \arrow[ur, dotted, no head] & 2 \arrow[u] \arrow[d] & 2 \arrow[l, gray] \arrow[d, gray] \arrow[dl, dotted, no head] \\
1 \arrow[u, gray] \arrow[r, dotted, no head] & 0 & 1 \arrow[l]
\end{tikzcd}$$

$\mathbb{Y}(2,2,4):\ C_7$

$$\begin{tikzcd}
1 \arrow[r] & 0 & 1 \arrow[l] \\
2 \arrow[u, gray] \arrow[r, gray] \arrow[ur, dotted, no head] \arrow[d, gray] \arrow[dr, dotted, no head] & 2 \arrow[u] \arrow[d] & 2 \arrow[l, gray] \arrow[u, gray] \arrow[ul, dotted, no head] \arrow[d, gray] \arrow[dl, dotted, no head] \\
1 \arrow[r] & 0 & 1 \arrow[l]
\end{tikzcd}$$

$\mathbb{Y}(2,2,4):\ C_8$

$$\begin{tikzcd}
0 & 1 \arrow[l] \arrow[r] \arrow[dl, dotted, no head] \arrow[dr, dotted, no head] & 0 \\
1 \arrow[u] \arrow[d] & 2 \arrow[l, gray] \arrow[r, gray] \arrow[u, gray] \arrow[d, gray] \arrow[dl, dotted, no head] \arrow[dr, dotted, no head] & 1 \arrow[u] \arrow[d] \\
0 & 1 \arrow[l] \arrow[r] & 0
\end{tikzcd}$$

$\mathbb{Y}(2,2,2):\ C_1^{\mathrm{op}}$

$$\begin{tikzcd}
1 \arrow[r] & 0 \arrow[d, dotted, no head] & 1 \arrow[l] \\
2 \arrow[u, gray] \arrow[d, gray] & 2 \arrow[l, gray] \arrow[r, gray] \arrow[d, dotted, no head] & 2 \arrow[u, gray] \arrow[d, gray] \\
1 \arrow[r] & 0 & 1 \arrow[l]
\end{tikzcd}$$

$\mathbb{Y}(2,2,4):\ C_2^{\mathrm{op}}$

$$\begin{tikzcd}
1 \arrow[r] & 0 \arrow[r, dotted, no head] \arrow[d, dotted, no head] & 1 \arrow[d, gray] \\
2 \arrow[u, gray] \arrow[dr] & 2 \arrow[l, gray] \arrow[r, gray] \arrow[d, dotted, no head] & 2 \arrow[ul] \arrow[d, gray] \\
1 \arrow[u, gray] \arrow[r, dotted, no head] & 0 & 1 \arrow[l]
\end{tikzcd}$$

$\mathbb{Y}(2,2,4):\ C_3^{\mathrm{op}}$

$$\begin{tikzcd}
1 \arrow[r, dotted, no head] \arrow[d, gray] & 0 \arrow[r, dotted, no head] \arrow[d, dotted, no head] & 1 \arrow[d, gray] \\
2 \arrow[ur] \arrow[dr] & 2 \arrow[l, gray] \arrow[r, gray] \arrow[d, dotted, no head] & 2 \arrow[ul] \arrow[dl] \\
1 \arrow[u, gray] \arrow[r, dotted, no head] & 0 \arrow[r, dotted, no head] & 1 \arrow[u, gray]
\end{tikzcd}$$

$\mathbb{Y}(2,2,4):\ C_4^{\mathrm{op}}$

$$\begin{tikzcd}
1 \arrow[r, dotted, no head] \arrow[d, gray] & 0 \arrow[r, dotted, no head] & 1 \arrow[d, gray] \\
2 \arrow[r, gray] & 2 \arrow[u] \arrow[d] & 2 \arrow[l, gray] \\
1 \arrow[u, gray] \arrow[r, dotted, no head] & 0 \arrow[r, dotted, no head] & 1 \arrow[u, gray]
\end{tikzcd}$$

$\mathbb{Y}(2,2,4):\ C_5^{\mathrm{op}}$

$$\begin{tikzcd}
1 \arrow[r, gray] & 1 \arrow[r] & 0 \arrow[d, dotted, no head] \\
1 \arrow[u, gray] \arrow[ur, dotted, no head, gray] \arrow[r, gray] \arrow[d, dotted, no head] & 2 \arrow[u, gray] \arrow[d, gray] & 1 \arrow[l, gray] \arrow[d, gray] \arrow[dl, dotted, no head, gray] \\
0 & 1 \arrow[l] & 1 \arrow[l, gray]
\end{tikzcd}$$

$\mathbb{Y}(2,2,4):\ C_7^{\mathrm{op}}$

FIGURE 1.3.1. Endomorphism algebras of τ^2-stable tilting sheaves in normal position over a weighted projective line of weight type $(2,4,4)$. We have indicated $\mathsf{End}_{\mathbb{X}}(T_+)$ by gray arrows. Note that τ^2 acts on each configuration by rotation by π and that C_6 and C_8 are self-opposite. The weight type of the reduced weighted projective line $\mathbb{Y}$ is indicated for reference.

a contradiction. Hence every indecomposable summand of T belongs to a tube of even period. This rules out weight type $(3,3,3)$. Therefore must have tubular weight type $(2,2,2,2)$, $(2,4,4)$ or $(2,3,6)$. $\square$

The following result gives a classification of the endomorphism algebras of basic τ^2-stable tilting sheaves in $\mathsf{coh}\,\mathbb{X}$.

THEOREM 1.3.5. *Let T be a basic τ^2-stable tilting sheaf in $\mathsf{coh}\,\mathbb{X}$. Then $\mathsf{End}_{\mathbb{X}}(T)$ is isomorphic to one of the algebras indicated in Figures 1.1.1, 1.3.1 or 1.3.2.*

PROOF. First, suppose that $\mathbb{X}$ has type $(2,2,2,2;\lambda)$. Since τ^2 is the identity in $\mathsf{coh}\,\mathbb{X}$, all tilting sheaves are τ^2-stable in this case. Their endomorphism algebras are known, see Skowroński [**83**, Ex. 3.3] (see also Figure 1.1.1).

For the other cases, weight types $(2,4,4)$ and $(2,3,6)$, we rely on the following argument. Let T be a τ^2-stable tilting sheaf in $\mathsf{coh}\,\mathbb{X}$. By Proposition 1.2.3, we can assume that T is in normal position. By Lemma 1.2.5 every indecomposable direct summand of T_0 is an exceptional simple sheaf. Also, the perpendicular category $T_0^{\perp}$ is equivalent to a category of the form $\mathsf{coh}\,\mathbb{Y}$ where $\mathbb{Y}$ is a weighted projective line with $\chi(\mathbb{Y}) > 0$. Moreover, there exist a finite dimensional algebra H of extended

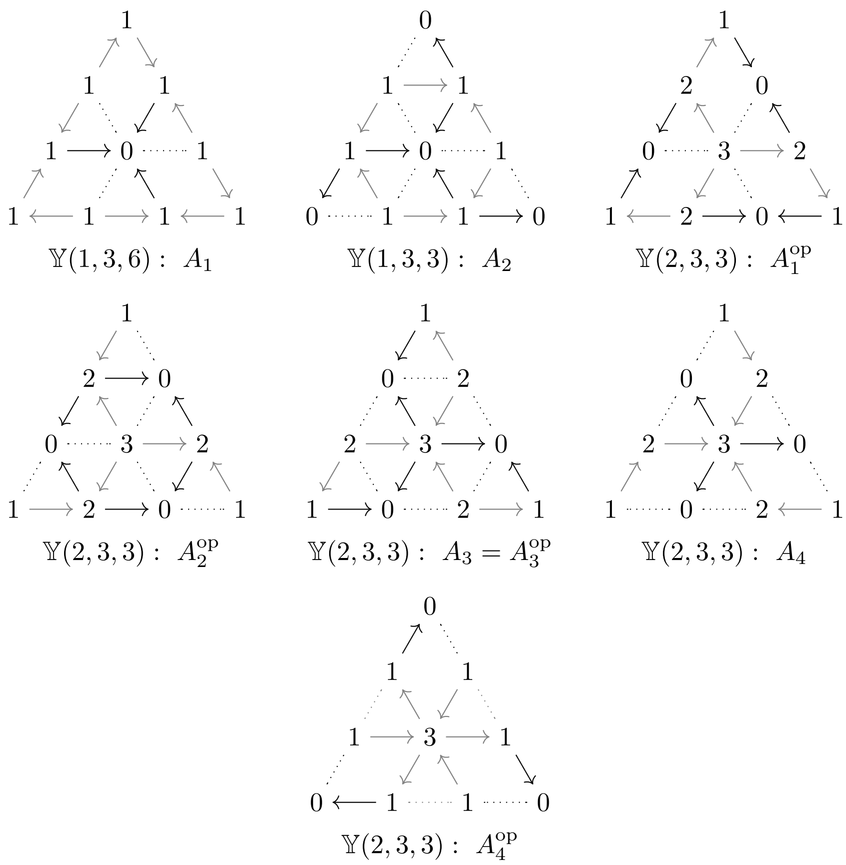


FIGURE 1.3.2. Endomorphism algebras of τ^2-stable tilting sheaves in normal position over a weighted projective line $\mathbb{X}$ of weight type $(2,3,6)$. We have indicated $\mathsf{End}_{\mathbb{X}}(T_+)$ by gray arrows. Note that τ^2 acts on each configuration by left rotation by $\pi/3$. The weight type of the reduced weighted projective line $\mathbb{Y}$ is indicated for reference.

Dynkin type Δ such that $\mathsf{coh}\,\mathbb{Y}$ is derived equivalent to $\mathsf{coh}\,\mathbb{Y}$. In addition, we have $T_+ \in \mathsf{vect}\,\mathbb{Y}$, see Proposition 1.2.4. It follows that $\mathsf{End}_{\mathbb{X}}(T_+) = \mathsf{End}_{\mathbb{Y}}(T_+)$ is isomorphic to the endomorphism algebra of a preprojective H-module. Note that $\mathsf{End}_{\mathbb{Y}}(T_+)$ must admit an action of order $p/2$ which does not fix any line bundle summands of T_+, see by Proposition 1.2.4(iv). Finally, Proposition 1.2.7 yields an isomorphism of algebras

$$\mathsf{End}_{\mathbb{X}}(T) \cong \mathsf{End}_{\mathbb{Y}}(T_+ \oplus E) \cong \begin{bmatrix} \mathsf{End}_{\mathbb{Y}}(T_+) & \mathsf{End}_{\mathbb{Y}}(T_+, E) \\ 0 & \mathsf{End}_{\mathbb{Y}}(E) \end{bmatrix}$$

where E is the direct sum of all regular simple modules in $\mathsf{coh}\,\mathbb{Y}$ in the exceptional tubes concentrated in the λ_i's such that $q_i \neq 0$ (note that here q_i must be either zero or $p_i/2$). It follows that $\mathsf{End}_{\mathbb{Y}}(E)$ is a semisimple algebra with $q_1 + \cdots + q_t$ simple modules.

Hence, to prove the theorem we only need to do the following:

(i) Take a vector bundle in $T_+ \in \mathsf{coh}\,\mathbb{Y}$ whose endomorphism algebra admits a symmetry of order 2 for $\mathbb{X}$ of type (2,4,4) or order 3 for $\mathbb{X}$ of type (2,3,6) not fixing any line bundles.

$\mathbb{X}(2,4,4)$	
$\mathbb{Y}$	Δ
(1,4,4)	$\tilde{A}_{4,4}$
(1,2,4)	$\tilde{A}_{2,4}$
(1,2,2)	$\tilde{A}_{2,2}$
(2,2,2)	$\tilde{D}_4$
(2,2,4)	$\tilde{D}_6$

$\mathbb{X}(2,3,6)$	
$\mathbb{Y}$	Δ
(1,3,6)	$\tilde{A}(3,6)$
(1,3,3)	$\tilde{A}(3,3)$
(2,3,3)	$\tilde{E}_6$

TABLE 1.3.1. Possible weight types for $\mathbb{Y}$ and the extended Dynkin type Δ of the associated hereditary algebra.

(ii) Compute the algebra $\mathsf{End}_{\mathbb{X}}(T_+ \oplus E)$.
(iii) Check if $\mathsf{End}_{\mathbb{X}}(T_+ \oplus E)$ is a 2-representation-finite algebra, see Proposition 1.3.1.

This process, although lengthy, is straightforward. We illustrate part of it for $\mathbb{X}$ of weight type $(2,4,4)$. The case were $\mathbb{X}$ has type $(2,3,6)$ is completely analogous. The cases we need to deal with are stated in Table 1.3.1.

$\mathbb{Y}(1,4,4)$ In this case we have $\Delta = \tilde{A}_{4,4}$. The only possibility for the Gabriel quiver of $\mathsf{End}_{\mathbb{Y}}(T_+)$ is a non-oriented cycle with 8 vertices. Moreover, it must have 4 arrows pointing in clockwise direction and 4 arrows pointing in counterclockwise direction. These are the quivers highlighted in the algebras C_1 and C_2 Figure 1.3.1. All of these algebras are 2-representation-finite algebras, as can be readily verified by checking that their 3-preprojective algebras are selfinjective, see Proposition 1.2.12. The reader can verify that they indeed arise by the procedure described in Theorem 1.2.7.

The cases $\mathbb{Y}(1,2,4)$ and $\mathbb{Y}(1,2,2)$ are completely analogous, The resulting 2-representation finite algebras correspond to C_3 and C_4 respectively in Figure 1.3.1.

$\mathbb{Y}(2,2,2)$ In this case we have $\Delta = \tilde{D}_4$. The only possible endomorphism algebras of preprojective tilting H-modules are orientations of the Dynkin diagram of type $\tilde{D}_6$

$$\begin{array}{ccccc} & & 1 & & \\ & & | & & \\ 1 & — & 2 & — & 1 \\ & & | & & \\ & & 1 & & \end{array}$$

or the canonical algebra of type $(2,2,2)$, see Happel-Vossieck's list [**49**]. The only quivers which admit an action of order 2 which does not fixes any line bundle summand of T_+ are the ones highlighted in algebras C_1^{op}, C_5 and C_6 in Figure 1.3.1, corresponding to symmetric orientations of the Dynkin diagram above.

$\mathbb{Y}(2,2,4)$ We have $\Delta = \tilde{D}_6$. In this case (and only in this case), there are algebras in Happel-Vossieck's list which have an action of order $p/2$ which do not extend to a 2-representation-finite algebra. One way to rule out these algebras before doing any computation is to determine the action induced by τ^2 on the Auslander-Reiten quiver of $\mathsf{vect}\,\mathbb{Y}$, which has shape $\mathbb{Z}\tilde{D}_6$, see Theorem 1.2.4(ii) and [**73**, Table 1]. We prove below that this action is given by rotation along the horizontal axis of $\mathsf{vect}\,\mathbb{Y}$, corresponding to the action given by degree shift by $\vec{y} + 2\vec{\omega}_{\mathbb{Y}} \in \mathbb{L}(2,2,4)$. Taking this into account, according to [**49**] the possible endomorphism algebras of preprojective tilting H-modules are given in Figure 1.3.3. The only quivers in Figure 1.3.3 which are stable under rotation by π along the

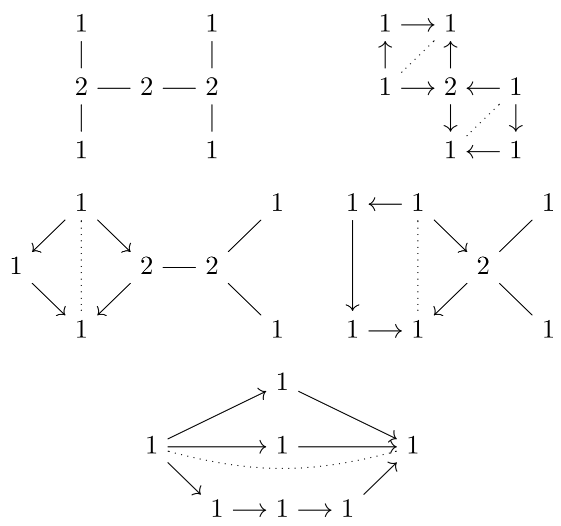


FIGURE 1.3.3. Endomorphism algebras of preprojective tilting modules of type $\tilde{D}_6$ with dimension vectors. The orientation of simple edges can be chosen arbitrarily and the relations indicate that the sum of all paths with the corresponding endpoints is zero.

horizontal axis of the Auslander-Reiten quiver of $\mathsf{vect}\,\mathbb{Y}$ are the ones highlighted in algebras C_2^{op}, C_3^{op}, C_4^{op}, C_5^{op}, C_7, C_7^{op} and C_8 in Figure 1.3.1.

Finally, let us prove that the action induced by τ^2 on $\mathsf{vect}\,\mathbb{Y}$ is indeed given by rotation along the horizontal axis. This also serves as an example of the method to compute $\mathsf{End}_{\mathbb{X}}(T)$ using Theorem 1.2.7.

Let $\mathbb{X}$ be a weighted projective line of tubular type $(2, 4, 4)$. Let X be an exceptional simple sheaf concentrated at λ_2 and set $T_0 := X \oplus \tau^2 X$. We write $\mathbb{L}(2, 2, 4) = \langle \vec{x}, \vec{y}, \vec{z}, \vec{c} \mid 2\vec{x} = 2\vec{y} = 4\vec{z} = \vec{c} \rangle$ and $\vec{\omega} = \vec{\omega}_{\mathbb{Y}}$. Also, we put $R := R(2, 2, 4)$

Let $T_+ \in \mathsf{vect}\,\mathbb{Y}$ be the tilting bundle indicated in Figure 1.3.4 and $E = S \oplus S'$ be the direct sum of the two exceptional simple sheaves in $\mathsf{coh}\,\mathbb{Y}$ concentrated at the point λ_2. Put $T := T_+ \oplus T_0$. By Theorem 1.2.7 we have an isomorphism of K-algebras

$$\mathsf{End}_{\mathbb{X}}(T) \cong \begin{bmatrix} \mathsf{End}_{\mathbb{Y}}(T_+) & \mathsf{Hom}_{\mathbb{Y}}(T_+, E) \\ 0 & \mathsf{End}_{\mathbb{Y}}(E) \cong K \times K \end{bmatrix}.$$

We need to compute $\mathsf{End}_{\mathbb{Y}}(T_+, E)$. For this, recall from Theorem 1.2.1(iv) that we have short exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{Y}}(-\vec{y}) \longrightarrow \mathcal{O}_{\mathbb{Y}} \longrightarrow S \longrightarrow 0 \tag{1.3.1}$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{Y}}(-2\vec{y}) \longrightarrow \mathcal{O}_{\mathbb{Y}}(-\vec{y}) \longrightarrow S' \longrightarrow 0. \tag{1.3.2}$$

We shall compute $\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}), S)$ and $\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}), S')$ by applying the functor $\mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}), -)$. Before that, it its convenient to make some preliminary calculations.

Firstly, by Theorem 1.2.1(vii) we have $\mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}), \mathcal{O}) = 0$ and using Serre duality we obtain

$$\mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}), E) \cong D\,\mathsf{Hom}_{\mathbb{X}}(E, \mathcal{O}(\vec{z} + \vec{\omega})) = 0,$$

$$
\begin{tikzcd}[column sep=small]
& \mathcal{O}(\vec{y}+3\vec{\omega}) \arrow[dr] & & \boxed{\mathcal{O}(\vec{y}+2\vec{\omega})} \arrow[dr] & & \mathcal{O}(\vec{y}+\vec{\omega}) \arrow[dr] & \\
\bullet \arrow[r] \arrow[ur] \arrow[dr] & \boxed{\mathcal{O}(\vec{x}+3\vec{\omega})} \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \mathcal{O}(\vec{x}+2\vec{\omega}) \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \boxed{\mathcal{O}(\vec{x}+\vec{\omega})} \arrow[r] & \bullet \\
\cdots & \bullet \arrow[ur] \arrow[dr] & & \boxed{U} \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & \cdots \\
\bullet \arrow[r] \arrow[ur] \arrow[dr] & \boxed{\mathcal{O}(\vec{z}+2\vec{\omega})} \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \mathcal{O}(\vec{z}+\vec{\omega}) \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \boxed{\mathcal{O}(\vec{z})} \arrow[r] & \bullet \\
& \mathcal{O}(\vec{\omega}) \arrow[ur] & & \boxed{\mathcal{O}} \arrow[ur] & & \mathcal{O}(-\vec{\omega}) \arrow[ur] &
\end{tikzcd}
$$

FIGURE 1.3.4. Tilting bundle in $\mathsf{vect}\,\mathbb{Y}$. Black vertices indicate vector bundles of rank 2.

since there are no non-zero morphisms from a torsion sheaf to a vector bundle. Secondly, again by Theorem 1.2.1(vii) and Serre duality we have

$$\mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}) \cong D\,\mathsf{Hom}_{\mathbb{Y}}(\mathcal{O},\mathcal{O}(\vec{z}+\vec{\omega})) \cong R_{\vec{x}-\vec{y}} = 0.$$

Simlarly, we have

$$\mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}(-\vec{y})) \cong D\,\mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(-\vec{y}),\mathcal{O}(\vec{z}+\vec{\omega})) = R_{\vec{\omega}+\vec{y}+\vec{z}} = R_{\vec{x}}.$$

In addition, we have

$$\mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}) = R_{-\vec{z}} = 0 \quad \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}(-\vec{y})) = R_{-\vec{y}-\vec{z}} = 0.$$

Hence, applying the functor $\mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}_{\mathbb{Y}}(\vec{z}),-)$ to the sequences (1.3.1) and (1.3.2) yields exact sequences

$$0 \longrightarrow \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}),S) \longrightarrow \mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}(-\vec{y})) \longrightarrow 0$$

and

$$0 \longrightarrow \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}),S') \longrightarrow \mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}(-2\vec{y})) \longrightarrow \mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}(-\vec{y})) \longrightarrow 0$$

Then, by Theorem 1.2.1(vii) we have

$$\begin{aligned}
\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}),S) &= \dim \mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}(-\vec{y})) \\
&= \dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(-\vec{y}),\mathcal{O}(\vec{z}+\vec{\omega})) \\
&= \dim R_{\vec{\omega}+\vec{z}+\vec{y}} \\
&= \dim R_{\vec{x}} = 1
\end{aligned}$$

and

$$\begin{aligned}
\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}),S') &= \dim \mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}(-2\vec{y})) - \dim \mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}),\mathcal{O}(-\vec{y})) \\
&= \dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(-2\vec{y}),\mathcal{O}(\vec{z}+\vec{\omega})) - 1 \\
&= \dim R_{2\vec{y}+\vec{z}+\vec{\omega}} - 1 \\
&= \dim R_{\vec{x}+\vec{y}} - 1 = 0.
\end{aligned}$$

A similar argument shows that

$$\mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{x}+\vec{\omega}), S) = 0$$

and

$$\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{x}+\vec{\omega}), S') = 1.$$

Proceeding in the same fashion, the reader can verify the equalities

$$\begin{aligned}
\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{x}+3\vec{\omega}), S) &= 0, \quad \dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{x}+3\vec{\omega}), S') = 1,\\
\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{y}+2\vec{\omega}), S) &= 0, \quad \dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{y}+2\vec{\omega}), S') = 1,\\
\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}+2\vec{\omega}), S) &= 1, \quad \dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}+2\vec{\omega}), S') = 0,\\
\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}, S) &= 1, \quad \dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}, S') = 0.
\end{aligned}$$

It remains to compute $\dim \mathsf{Hom}_{\mathbb{Y}}(U, S)$ and $\dim \mathsf{Hom}_{\mathbb{Y}}(U, S')$. We have an exact sequence

$$0 \longrightarrow \mathcal{O}(\vec{\omega}) \longrightarrow U \longrightarrow \mathcal{O}(\vec{z}) \longrightarrow 0.$$

Applying the contravariant functor $\mathsf{Hom}_{\mathbb{Y}}(-, S)$ yields an exact sequence

$$0 \to \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}), S) \to \mathsf{Hom}_{\mathbb{Y}}(U, S) \to \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{\omega}), S) \to \mathsf{Ext}^1_{\mathbb{Y}}(\mathcal{O}(\vec{z}), S) = 0.$$

We already know that $\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{z}), S) = 1$. Proceeding as before, applying the functor $\mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{\omega}), -)$ to the short exact sequence (1.3.1), we can show that $\dim \mathsf{Hom}_{\mathbb{Y}}(\mathcal{O}(\vec{\omega}), S) = 0$. Hence $\dim \mathsf{Hom}_{\mathbb{Y}}(U, S) = 1$. We can show that $\dim \mathsf{Hom}_{\mathbb{Y}}(U, S') = 1$ in a similar manner.

It follows, by a suitable change of basis of $\mathsf{End}_{\mathbb{Y}}(T_+)$, that the quiver with relations of $\mathsf{End}_{\mathbb{X}}(T)^{\mathrm{op}}$ is given by

$$\begin{tikzcd}
\mathcal{O} \arrow[r] & \mathcal{O}(\vec{z}) \arrow[r] \arrow[dl, dotted, no head] & S \arrow[d, dotted, no head] \\
\mathcal{O}(\vec{z}+2\vec{\omega}) \arrow[u] \arrow[r] \arrow[d, dotted, no head] & U \arrow[u] \arrow[d] & \mathcal{O}(\vec{x}+3\vec{\omega}) \arrow[l] \arrow[d] \arrow[dl, dotted, no head] \\
S' & \mathcal{O}(\vec{x}+\vec{\omega}) \arrow[l] & \mathcal{O}(\vec{y}+2\vec{\omega}) \arrow[l]
\end{tikzcd}$$

where each relation is a zero relation or a commutative relation.

Using Proposition 1.2.12 it is easy to check that $\mathsf{End}_{\mathbb{X}}(T)$ is a 2-representation-finite algebra. Therefore T is a τ^2-stable tilting sheaf, see Proposition 1.3.1. Then, we can see in Figure 1.3.4 that the only action of order 2 on the Auslander-Reiten quiver of $\mathsf{vect}\,\mathbb{Y}$ which fixes T_+ given by rotation by π along the horizontal axis, which can be interpreted as degree shift by $\vec{y}+2\vec{\omega}_{\mathbb{Y}}$. This concludes the proof of the theorem. □

As a consequence of Theorem 1.3.5 we obtain the following classification results.

THEOREM 1.3.6. *Let $\mathbb{X}$ be a weighted projective line and basic T a tilting complex in $\mathrm{D^b}(\mathbb{X}) = \mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$. Then, $\mathsf{End}_{\mathrm{D^b}(\mathbb{X})}(T)$ is a 2-representation-finite algebra if and only if $\mathsf{End}_{\mathrm{D^b}(\mathbb{X})}(T)$ is one of the algebras in Figures 1.1.1, 1.1.2 and 1.1.3. Moreover, this determines T up to an autoequivalence of $\mathrm{D^b}(\mathsf{coh}\,\mathbb{X})$.*

PROOF. The first claim follows immediately from Proposition 1.3.3 and Theorem 1.3.5, since 1.1.2 and 1.1.3 are all algebras that can be obtained by 2-APR-(co)tilting from the algebras in Figures 1.1.4 and 1.1.5. The second claim is a standard application of [**75**, Thm. 3.2]. □

THEOREM 1.3.7. *Let T be a basic complex in* $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$. *Then, T is τ^2-stable if and only if* $\mathsf{End}_{\mathrm{D}^{\mathrm{b}}(\mathbb{X})}(T)$ *is one of the algebras in Figures 1.1.1, 1.1.4, or 1.1.5. Moreover, this determines T up to an autoequivalence of* $\mathrm{D}^{\mathrm{b}}(\mathsf{coh}\,\mathbb{X})$.

PROOF. By Theorem 1.3.6, the algebra $\mathsf{End}_{\mathbb{X}}(T)$ can be obtained from one of the algebras in Figures 1.1.1, 1.3.1 or 1.3.2 by iterated 2-APR-(co)-tilting. By Proposition 1.3.1, we have that T is τ^2-stable is a 2-homogeneous 2-representation-finite algebra. These are precisely the algebras in Figures 1.1.1, 1.1.4, or 1.1.5. □

THEOREM 1.3.8. *Let T be a basic τ^2-stable tilting sheaf in* $\mathsf{coh}\,\mathbb{X}$. *Then the cluster-tilted algebra* $\mathsf{End}_{\mathcal{C}}(T)$ *is isomorphic to the Jacobian algebra associated to one of the quivers with potential in Figures 1.1.6, 1.1.7 or 1.1.8, and all of the Jacobian algebras associated to one of these quivers with potential arise in this way.*

PROOF. Let T be a basic τ^2-stable tilting sheaf in $\mathsf{coh}\,\mathbb{X}$ and set $\Lambda = \mathsf{End}_{\mathbb{X}}(T)$. It follows from Theorem 1.3.5 that Λ is isomorphic to one of the algebras in Figures 1.1.1, 1.3.1 or 1.3.2. Then, by Proposition 1.2.17 there exist an isomorphism $\mathsf{End}_{\mathcal{C}}(T) \cong \Pi_3(\Lambda)$. By Theorem 1.2.13, we have that $\Pi_3(\Lambda)$ is isomorphic to the Jacobian algebra to one of the quivers with potential in Figures 1.1.6, 1.1.7 or 1.1.8.

Conversely, each Jacobian algebra associated to one of the quivers with potential in Figures 1.1.6, 1.1.7 or 1.1.8 is of the form $\Pi_3(\Lambda)$ for some Λ in Figures 1.1.1, 1.3.1 or 1.3.2, see [**51**, Secs. 5.1, 9.2 and 9.3]. The theorem follows. □

CHAPTER 2

n-abelian and n-exact categories

We introduce n-abelian and n-exact categories, these are analogs of abelian and exact categories from the point of view of higher homological algebra. We show that n-cluster-tilting subcategories of abelian (resp. exact) categories are n-abelian (resp. n-exact). These results allow to construct several examples of n-abelian and n-exact categories. Conversely, we prove that n-abelian categories satisfying certain mild assumptions can be realized as n-cluster-tilting subcategories of abelian categories. In analogy with a classical result of Happel, we show that the stable category of a Frobenius n-exact category has a natural $(n+2)$-angulated structure in the sense of Geiß-Keller-Oppermann. We give several examples of n-abelian and n-exact categories which have appeared in representation theory, commutative algebra, commutative and non-commutative algebraic geometry. The contents of this chapter are available in preprint form in [**64**].

2.1. Introduction

Let n be a positive integer. In this article we introduce n-abelian and n-exact categories, these are higher analogs of abelian and exact categories from the viewpoint of higher homological algebra. Throughout we use the comparative adjective "higher" in relation to the length of exact sequences and *not* in the sense of higher category theory.

Abelian categories were introduced by Grothendieck in [**43**] to axiomatize the properties of the category of modules over a ring and of the category of sheaves over a scheme. It is often the case that interesting additive categories are not abelian but still have good homological properties with respect with a restricted class of short exact sequences. Exact categories were introduced by Quillen in [**79**] from this perspective to axiomatize extension-closed subcategories of abelian categories.

Derived categories play an important role in the study of the homological properties of abelian and exact categories. Their properties are captured by the notion of triangulated categories, introduced by Grothendieck-Verdier in [**84**]. By a result of Happel, the stable category of a Frobenius exact category has a natural structure of a triangulated category, see [**44**, Thm. I.2.6]. Triangulated categories arising in this way have been called *algebraic* by Keller in [**68**]. Algebraic triangulated categories have a natural *dg*-enhancement in the sense of Bondal-Kapranov [**25**], thus are often considered as a more reasonable class than that of general triangulated categories.

Recently, a new class of additive categories appeared in representation theory. The 2-cluster-tilting subcategories were introduced by Buan-Marsh-Reiten-Reineke-Todorov in [**27**] as the key concept involved in the additive categorification of the mutation combinatorics of Fomin-Zelevinsky's cluster algebras [**33**] via 2-Calabi-Yau triangulated categories. It was then observed by Iyama-Yoshino [**62**] that the notion of mutation can be extended to the class of n-cluster-tilting subcategories of triangulated categories.

From a different perspective, n-cluster-tilting subcategories of certain exact categories were introduced by Iyama in [**56**] and further investigated in [**57, 55**] from the viewpoint of higher Auslander-Reiten theory. In this theory, the notion of n-almost-split sequence, which are certain exact sequences with $n+2$ terms, plays an important role.

With motivation coming from these examples in representation theory, the class of $(n+2)$-angulated categories was introduced by Geiß-Keller-Oppermann as categories "naturally inhabited by the shadows of exact sequences with $n+2$ terms", to paraphrase the authors. We note that the case $n=1$ corresponds to triangulated categories. Their main source of examples of $(n+2)$-angulated categories are n-cluster-tilting subcategories of triangulated categories which are closed under the n-th power of the shift functor [**39**, Thm. 1]. The properties of $(n+2)$-angulated categories have been investigated by Bergh-Thaule in [**21, 22, 23**].

The aim of this article is to introduce n-abelian categories which are categories inhabited by certain exact sequences with $n+2$ terms, called n-exact sequences. The case $n=1$ corresponds to the classical concepts of abelian categories. We do so by modifying the axioms of abelian categories in a suitable manner. We prove several basic properties of n-abelian categories, including the existence of n-pushout (resp. n-pullback) diagrams which are analogs of classical pushout (resp. pullback) diagrams, see Theorem 2.3.8.

An important source of examples of n-abelian categories are n-cluster-tilting subcategories. This is made precise by the following theorem.

THEOREM (see Theorem 2.3.16 for details). *Let $\mathcal{M}$ be an n-cluster-tilting subcategory of an abelian category. Then $\mathcal{M}$ is an n-abelian category.*

We introduce the notion of projective object in an n-abelian category, and study their properties. Remarkably, projective objects satisfy the following strong property which is obvious in the case of abelian categories.

THEOREM (see Theorem 2.3.12 for details). *Let $\mathcal{M}$ be an n-abelian category and $P \in \mathcal{M}$ a projective object. Then, for every morphism $f\colon L \to M$ and every weak cokernel $g\colon M \to N$ of f, the following sequence is exact:*

$$\mathcal{M}(P,L) \xrightarrow{?\cdot f} \mathcal{M}(P,M) \xrightarrow{?\cdot g} \mathcal{M}(P,N).$$

Using this result, we show that certain n-abelian categories can be realized as n-cluster-tilting subcategories of abelian categories. More precisely, we prove the following theorem.

THEOREM (see Theorem 2.3.20 for details). *Let $\mathcal{M}$ be a small projectively generated n-abelian category, and $\mathcal{P}$ the category of projective objects in $\mathcal{M}$. If $\operatorname{mod}\mathcal{P}$ is injectively cogenerated, then $\mathcal{M}$ is equivalent to an n-cluster-tilting subcategory of $\operatorname{mod}\mathcal{P}$.*

After introducing n-abelian categories, it is natural to introduce n-exact categories as higher analogs of exact categories. For this, we modify Keller-Quillen's axioms of exact categories. We prove that the class of n-exact categories contains that of n-abelian categories, see Theorem 2.4.5. Similarly to the case of n-abelian categories, we prove the following theorem.

THEOREM (see Theorem 2.4.14 for details). *Let $\mathcal{M}$ be an n-cluster-tilting subcategory of an exact category. Then $\mathcal{M}$ is an n-exact category.*

We also introduce Frobenius n-exact categories. These are n-exact categories with enough projectives and enough injectives, and such that these two classes of objects coincide. Frobenius n-exact categories are related to $(n+2)$-angulated categories as shown by the following theorem.

THEOREM (see Theorem 2.5.11 for details). *Let $\mathcal{M}$ be a Frobenius n-exact category. Then, the stable category $\underline{\mathcal{M}}$ has a natural structure of an $(n+2)$-angulated category.*

Finally, we prove the following result also in the direction of Frobenius n-exact categories.

THEOREM (see Theorem 2.5.16 for details). *Let $\mathcal{M}$ be an n-cluster-tilting subcategory of a Frobenius exact category $\mathcal{E}$, and suppose that $\mathcal{M}$ is closed under taking n-th cosyzygies. Then, $\mathcal{M}$ is a Frobenius n-exact category.*

This theorem is closely related to the results of Geiß-Keller-Oppermann. The relation between both approaches to construct $(n+2)$-angulated categories is explained in Theorem 2.5.16.

Now we explain the notion of 2-exact category with concrete examples. The first example is due to Herschend-Iyama-Minamoto-Oppermann [**52**] (see also Theorem 2.6.8 and the example after it). Let $\mathsf{coh}\,\mathbb{P}^2_K$ be the category of coherent sheaves over the projective plane over K, and denote the category of vector bundles over $\mathbb{P}^2_K$ by $\mathsf{vect}\,\mathbb{P}^2_K$. Note that $\mathsf{vect}\,\mathbb{P}^2_K$ is closed under extensions in $\mathsf{coh}\,\mathbb{P}^2_K$, and hence is an exact category. Then, the category

$$\mathcal{U} := \mathsf{add}\,\{\mathcal{O}(i) \mid i \in \mathbb{Z}\}$$

of direct sums of line bundles on $\mathbb{P}^2_K$ is a 2-cluster-tilting subcategory of $\mathsf{vect}\,\mathbb{P}^2_K$. In view of the previous theorem, the category $\mathcal{U}$ is a 2-exact category. An interesting consequence of the 2-cluster-tilting property is that for every exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0 \tag{2.1.1}$$

with terms in $\mathcal{U}$ the sequences of functors

$$0 \longrightarrow \mathsf{Hom}(-,A)|_{\mathcal{U}} \longrightarrow \mathsf{Hom}(-,B)|_{\mathcal{U}} \longrightarrow \mathsf{Hom}(-,C)|_{\mathcal{U}} \longrightarrow \mathsf{Hom}(-,D)|_{\mathcal{U}},$$

$$0 \longrightarrow \mathsf{Hom}(D,-)|_{\mathcal{U}} \longrightarrow \mathsf{Hom}(C,-)|_{\mathcal{U}} \longrightarrow \mathsf{Hom}(B,-)|_{\mathcal{U}} \longrightarrow \mathsf{Hom}(A,-)|_{\mathcal{U}}$$

are exact. In general, we call a sequence of the form (2.1.1) satisfying these properties a *2-exact sequence*. In this case, the Koszul complexes

$$0 \longrightarrow \mathcal{O}(i-3) \longrightarrow \mathcal{O}(i-2)^3 \longrightarrow \mathcal{O}(i-1)^3 \longrightarrow \mathcal{O}(i) \longrightarrow 0 \quad (i \in \mathbb{Z})$$

gives a special class of 2-exact sequences called 2-almost-split sequences in higher Auslander-Reiten theory.

Let us provide the reader with another example of a 2-exact category, following Iyama [**56**, Sec. 2.5]. Let K be an algebraically closed field and $S := K[\![x_0, x_1, x_2]\!]$ be the ring of power series three commuting variables. Also, let G be a finite subgroup of $\mathsf{SL}_3(K)$ and $R := S^G$ the associated invariant subring of S. Finally, we denote the category of Cohen-Macaulay R-modules by $\mathsf{CM}\,R$, see Section 2.6.4 for details and definitions. Note that $\mathsf{CM}\,R$ is a Frobenius exact category. Then, the category

$$\mathcal{S} := \mathsf{add}\,S = \{M \in \mathsf{CM}\,R \mid M \text{ is a direct summand of } S^m \text{ for some } m\}$$

is a 2-cluster-tilting subcategory of $\mathsf{CM}\,R$, *i.e.* we have

$$\mathcal{S} = \left\{M \in \mathsf{CM}\,R \mid \mathsf{Ext}^1_R(\mathcal{S}, M) = 0\right\} = \left\{M \in \mathsf{CM}\,R \mid \mathsf{Ext}^1_R(M, \mathcal{S}) = 0\right\}.$$

An important example of a 2-exact sequence in this case is given by the Koszul complex of S:

$$K(S)\colon 0 \longrightarrow S \longrightarrow S^3 \longrightarrow S^3 \longrightarrow S \longrightarrow K \longrightarrow 0.$$

As a complex of R-modules, $K(S)$ is the direct sum of 2-almost-split sequences and a 2-fundamental sequence.

Let us mention other examples of n-cluster-tilting subcategories, which give us then examples of n-abelian and n-exact categories.

Finite dimensional algebras of finite global dimension whose module category contains an n-cluster-tilting subcategory are one of the central objects of study of higher Auslander-Reiten theory. A distinguished class of such algebras, the so-called n-representation-finite algebras, were introduced by Iyama-Oppermann in [**59**] and have been studied in greater detail by Herschend-Iyama in the case $n = 2$, see [**51**].

In a parallel direction, 2-cluster-tilting subcategories of the module category of a preprojective algebra of Dynkin type, which has infinite global dimension, are central in Geiß-Leclerc-Schröer's categorification of cluster algebras arising in Lie theory, see [**41**] and the references therein.

Further examples of n-cluster-tilting subcategories of abelian and exact categories have been constructed by Amiot-Iyama-Reiten in the category of Cohen-Macaulay modules over an isolated singularity [**5**].

Finally, let us give a brief description of the contents of this article. In Section 2.2 we introduce the basic concepts behind the definitions of n-abelian and n-exact categories: n-cokernels, n-kernels, n-exact sequences, and n-pushout and n-pullback diagrams (the reader will forgive the author for his lack of inventiveness in naming these concepts). The class of n-abelian categories is introduced in Section 2.3, where we also give a characterization of semisimple categories in terms of n-abelian categories. In Theorems 2.3.16 and 2.3.20 we explore the connection between n-abelian categories and n-cluster-tilting subcategories of abelian categories. Later, in Section 2.4 we introduce n-exact categories and establish a connection with n-cluster-tilting subcategories of exact categories in Theorem 2.4.14. Frobenius n-exact categories and their main properties are introduced in Section 2.5. At last, in Section 2.6 we provide several examples to illustrate our results.

2.2. Preliminary concepts

We begin by fixing our conventions and notation, and by reminding the reader of basic concepts in homological algebra that we use freely in the remainder.

2.2.1. Conventions and notation. Throughout this article n always denotes a fixed positive integer. Let $\mathcal{C}$ be a category. If $A, B \in \mathcal{C}$, then we denote the set of morphisms $A \to B$ in $\mathcal{C}$ by $\mathcal{C}(A, B)$. We denote the identity morphism of an object $C \in \mathcal{C}$ by $1 = 1_C$. We denote composition of morphisms by concatenation: if $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, then $fg \in \mathcal{C}(A, C)$. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, then the *essential image of* F is the full subcategory of $\mathcal{D}$ given by

$$F\mathcal{C} := \{D \in \mathcal{D} \mid \exists C \in \mathcal{C} \text{ such that } FC \cong D\}.$$

A morphism $e \in \mathcal{C}(A, A)$ is *idempotent* if $e^2 = e$. We say that $\mathcal{C}$ is *idempotent complete* if for every idempotent $e \in \mathcal{C}(A, A)$ there exist an object B and morphisms $r \in \mathcal{C}(A, B)$ and $s \in \mathcal{C}(B, A)$ such that $rs = e$ and $sr = 1_B$.

Let $\mathcal{C}$ be an additive category in the sense of [**86**, Sec. A.4.1]. If $\mathcal{X}$ is a class of objects in $\mathcal{C}$, then we denote by $\mathsf{add}\,\mathcal{X}$ the full subcategory whose objects are direct summands of direct sums of objects in $\mathcal{X}$.

We denote the category of (cochain) complexes in $\mathcal{C}$ by $\mathsf{Ch}(\mathcal{C})$. Also, we denote the full subcategory of $\mathsf{Ch}(\mathcal{C})$ given by all complexes concentrated in non-negative (resp. non-positive) degrees by $\mathsf{Ch}^{\geq 0}(\mathcal{C})$ (resp. $\mathsf{Ch}^{\leq 0}(\mathcal{C})$). For convenience, we denote by $\mathsf{Ch}^{n}(\mathcal{C})$ the full subcategory of $\mathsf{Ch}(\mathcal{C})$ given by all complexes

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

which are concentrated in degrees $0, 1, \dots, n+1$. A morphism of complexes

$$\begin{tikzcd}
X \arrow[d, "f"] & \cdots \arrow[r, "d_X^{-1}"] & X^0 \arrow[r, "d_X^0"] \arrow[d, "f^0"] & X^1 \arrow[r, "d_X^1"] \arrow[d, "f^1"] & X^2 \arrow[r, "d_X^2"] \arrow[d, "f^2"] & \cdots \\
Y & \cdots \arrow[r, "d_Y^{-1}"] & Y^0 \arrow[r, "d_Y^0"] & Y^1 \arrow[r, "d_Y^1"] & Y^2 \arrow[r, "d_Y^2"] & \cdots
\end{tikzcd}$$

is *null-homotopic* if for all $k \in \mathbb{Z}$ there exists a morphism $h^k \colon X^k \to Y^{k-1}$ such that

$$f^k = h^k d_Y^{k-1} + d_X^k h^{k+1}.$$

In this case we say that $h = \big(h^k \mid k \in \mathbb{Z}\big)$ is a *null-homotopy.* We say that two morphisms of complexes $f \colon X \to Y$ and $g \colon X \to Y$ are *homotopic* if their difference is null-homotopic. A *homotopy* between f and g is a null-homotopy h of $f - g$ and we write $h \colon f \to g$. It is easily verified that being homotopic induces an equivalence relation on $\mathsf{Ch}(\mathcal{C})(X, Y)$. The *homotopy category of* $\mathcal{C}$, denoted by $\mathsf{H}(\mathcal{C})$, is the category with the same objects as $\mathsf{Ch}(\mathcal{C})$ and in which morphisms are given by morphisms of complexes modulo homotopy. For further information on chain complexes and the homotopy category we refer the reader to [**86**, Ch. 1].

We remind the reader of the notion of functorially finite subcategory of an additive category. Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a subcategory of $\mathcal{C}$. We say that $\mathcal{D}$ is *covariantly finite in* $\mathcal{C}$ if for every $C \in \mathcal{C}$ there exists an object $D \in \mathcal{D}$ and a morphism $f : C \to D$ such that, for all $D' \in \mathcal{D}$, the sequence of abelian groups

$$\mathcal{C}(D, D') \xrightarrow{f \cdot ?} \mathcal{C}(C, D') \longrightarrow 0$$

is exact. Such a morphism f is called a *left* $\mathcal{D}$*-approximation of* C. The notions of *contravariantly finite subcategory of* $\mathcal{C}$ and *right* $\mathcal{D}$*-approximation* are defined dually. A *functorially finite subcategory of* $\mathcal{C}$ is a subcategory which is both covariantly and contravariantly finite in $\mathcal{C}$. For further information on functorially finite subcategories we refer the reader to [**13, 12**].

2.2.2. n-cokernels, n-kernels, and n-exact sequences. Let $\mathcal{C}$ be an additive category and $f \colon A \to B$ a morphism in $\mathcal{C}$. A *weak cokernel of* f is a morphism $g \colon B \to C$ such that for all $C' \in \mathcal{C}$ the sequence of abelian groups

$$\mathcal{C}(C, C') \xrightarrow{g \cdot ?} \mathcal{C}(B, C') \xrightarrow{f \cdot ?} \mathcal{C}(A, C')$$

is exact. Equivalently, g is a weak cokernel of f if $fg = 0$ and for each morphism $h \colon B \to C'$ such that $fh = 0$ there exists a (not necessarily unique) morphism $p \colon C \to C'$ such that $h = gp$. These properties are subsumed in the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "g"] \arrow[d, "\forall h"] & C \arrow[dl, dotted, "\exists p"] \\
& C' &
\end{tikzcd}$$

Clearly, a weak cokernel g of f is a cokernel of f if and only if g is an epimorphism. The concept of *weak kernel* is defined dually.

The following general result, together with its dual, plays a central role in the sequel.

COMPARISON LEMMA 2.2.1. *Let $\mathcal{C}$ be an additive category and $X \in \mathsf{Ch}^{\geq 0}(\mathcal{C})$ a complex such that for all $k \geq 0$ the morphism d_X^{k+1} is a weak cokernel of d_X^k. If $f \colon X \to Y$ and $g \colon X \to Y$ are morphisms in $\mathsf{Ch}^{\geq 0}(\mathcal{C})$ such that $f^0 = g^0$, then there exists a homotopy $h \colon f \to g$ such that h^1 is the zero morphism.*

PROOF. Let $u := f - g$ and for all $k \leq 1$ let $h^k \colon X^k \to Y^{k-1}$ be the zero morphism. Note that $u^0 = 0$ by hypothesis. We proceed by induction on k. Let $k \geq 1$ and suppose that for all $\ell \leq k$ we have constructed a morphism

$$h^\ell \colon X^\ell \to Y^{\ell-1}$$

such that

$$u^{\ell-1} = h^{\ell-1} d_Y^{\ell-2} + d_X^{\ell-1} h^\ell.$$

Since u is a morphism of complexes, we have

$$\begin{aligned} d_X^{k-1}(u^k - h^k d_Y^{k-1}) =& d_X^{k-1} u^k + (h^{k-1} d_Y^{k-2} - u^{k-1}) d_Y^{k-1} \\ =& d_X^{k-1} u^k - u^{k-1} d_Y^{k-1} \\ =& 0. \end{aligned}$$

Hence, given that d_X^k is a weak cokernel of d_X^{k-1}, there exists a morphism

$$h^{k+1} \colon X^{k+1} \to Y^k$$

such that $u^k - h^k d_Y^{k-1} = d_X^k h^{k+1}$ or, equivalently,

$$u^k = h^k d_Y^{k-1} + d_X^k h^{k+1}.$$

This finishes the construction of the required null-homotopy $h \colon f - g \to 0$. □

The following terminology will prove convenient in the sequel.

DEFINITION 2.2.2. Let $\mathcal{C}$ be an additive category and $d^0 \colon X^0 \to X^1$ a morphism in $\mathcal{C}$. An *n-cokernel of d^0* is a sequence

$$\left(d^1, \ldots, d^n\right) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots \xrightarrow{d^n} X^{n+1}$$

such that for all $Y \in \mathcal{C}$ the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{C}(X^{n+1}, Y) \xrightarrow{d^n \cdot ?} \mathcal{C}(X^n, Y) \xrightarrow{d^{n-1} \cdot ?} \cdots \xrightarrow{d^1 \cdot ?} \mathcal{C}(X^1, Y) \xrightarrow{d^0 \cdot ?} \mathcal{C}(X^0, Y)$$

is exact. Equivalently, the sequence $\left(d^1, \ldots, d^n\right)$ is an n-cokernel of d^0 if for all $1 \leq k \leq n-1$ the morphism d^k is a weak cokernel of d^{k-1}, and d^n is moreover a cokernel of d^{n-1}. The concept of *n-kernel* of a morphism is defined dually.

REMARK 2.2.3. If $n \geq 2$, then n-cokernels are not unique in general. Indeed, for each object $C \in \mathcal{C}$ the sequence $0 \to C \xrightarrow{1} C$ is a 2-cokernel of the morphism $0 \to 0$. This shortcoming can be resolved if one considers n-cokernels up to isomorphism in $\mathsf{H}(\mathcal{C})$, see Proposition 2.2.7.

As explained in the Introduction, n-exact sequences, defined below, are the object of study of higher homological algebra. The investigation of their properties is our main concern for the rest of this article.

DEFINITION 2.2.4. Let $\mathcal{C}$ be an additive category. An *n-exact sequence in* $\mathcal{C}$ is a complex

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

in $\mathsf{Ch}^n(\mathcal{C})$ such that $(d^0, \dots, d^{n-1})$ is an n-kernel of d^n, and $(d^1, \dots, d^n)$ is an n-cokernel of d^0.

Let $\mathcal{C}$ be an additive category. We remind the reader that a complex $X \in \mathsf{Ch}(\mathcal{C})$ is *contractible* if the identity morphism of X is null-homotopic or, equivalently, X is isomorphic to the zero complex in $\mathsf{H}(\mathcal{C})$. As a first analogy with the classical theory, let us show that the class of n-exact sequences is closed under isomorphisms in $\mathsf{H}(\mathcal{C})$.

PROPOSITION 2.2.5. *Let $\mathcal{C}$ be an additive category and X and Y be complexes in $\mathsf{Ch}^n(\mathcal{C})$ which are isomorphic in $\mathsf{H}(\mathcal{C})$. Then the following statements hold.*

(i) The complex X is an n-exact sequence if and only if Y is an n-exact sequence.

(ii) Every contractible complex with $n+2$ terms is an n-exact sequence.

PROOF. Note that the second claim follows immediately from the first one since the zero complex in $\mathsf{Ch}^n(\mathcal{C})$ is clearly an n-exact sequence. Suppose that X is an n-exact sequence. By hypothesis, there exist morphisms of complexes

$$\begin{tikzcd}
Y \arrow[d, "f"] & Y^0 \arrow[r] \arrow[d] & Y^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Y^n \arrow[r] \arrow[d] & Y^{n+1} \arrow[d] \\
X \arrow[d, "g"] & X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[d] \\
Y & Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^n \arrow[r] & Y^{n+1}
\end{tikzcd}$$

together with a homotopy $h\colon fg \to 1_Y$. Hence, for all $k \in \{1, \dots, n\}$ we have

$$1_{Y^k} = f^k g^k - h^k d_Y^{k-1} - d_Y^k h^{k+1}. \tag{2.2.1}$$

In particular, we have

$$1_{Y^{n+1}} = f^{n+1} g^{n+1} - h^{n+1} d_Y^n. \tag{2.2.2}$$

We claim that for all $k \in \{1, \dots, n\}$ the morphism d_Y^k is a weak cokernel of d_Y^{k-1}. Indeed, let $k \in \{1, \dots, n\}$ and $u\colon Y^k \to C$ be a morphism such that $d_Y^{k-1} u = 0$. It follows that

$$(d_X^{k-1} g^k) u = g^{k-1}(d_Y^{k-1} u) = 0.$$

Since d_X^k is a weak cokernel of d_X^{k-1} there exists a morphism $v\colon X^{k+1} \to C$ such that $g^k u = d_X^k v$. Therefore,

$$f^k(g^k u) = (f^k d_X^k) v = d_Y^k f^{k+1} v. \tag{2.2.3}$$

By composing the identity (2.2.1) on the right with u and substituting the identity (2.2.3) we obtain

$$u = (f^k g^k) u - (d_Y^k h^{k+1}) u = d_Y^k (f^{k+1} v - h^{k+1} u).$$

Therefore u factors through d_Y^k. This shows that d_Y^k is a weak kernel of d_Y^{k-1}.

We need to show that d_Y^n is moreover a cokernel of d_Y^{n-1}. For this it is enough to show that d_Y^n is an epimorphism for we already know that it is a weak cokernel of d_Y^{n-1}. Let $w\colon Y^{n+1} \to C$ be a morphism such that $d_Y^n w = 0$. It follows that

$$d_X^n (g^{n+1} w) = g^n (d_Y^n w) = 0.$$

Given that d_X^n is an epimorphism we deduce that $g^{n+1}w = 0$. By composing (2.2.2) on the right with w, we obtain

$$w = f^{n+1}(g^{n+1}w) - h^{n+1}(d_Y^n w) = 0.$$

Therefore d_Y^n is an epimorphism. This shows that $\left(d_Y^1, \dots, d_Y^n\right)$ is an n-cokernel of d_Y^0. By duality, the sequence $\left(d_Y^0, \dots, d_Y^{n-1}\right)$ is an n-kernel of d_Y^n. Hence Y is an n-exact sequence. The converse implication is analogous. $\square$

We have the following useful characterization of contractible n-exact sequences.

Proposition 2.2.6. *Let $\mathcal{C}$ be an additive category and X a complex in $\mathsf{Ch}^n(\mathcal{C})$ such that $\left(d^1, \dots, d^n\right)$ is an n-cokernel of d^0. Then, d^0 is a split monomorphism if and only if X is a contractible n-exact sequence.*

Proof. Suppose that d^0 is a split monomorphism. Hence there exists a morphism $h^1 \colon X^1 \to X^0$ such that $d^0h^1 = 1_{X^0}$. We shall extend h^1 to a null-homotopy of 1_X. Inductively, let $k \in \{0, 1, \dots, n\}$ and suppose that for all $\ell \leq k$ we have constructed a morphism $h^\ell \colon X^\ell \to X^{\ell-1}$ such that

$$1_{X^{\ell-1}} = h^{\ell-1}d^{\ell-2} + d^{\ell-1}h^\ell.$$

Composing this identity, for $\ell = k$, on the left with d^{k-1} we obtain

$$d^{k-1} = (h^{k-1}d^{k-2} + d^{k-1}h^k)d^{k-1} = d^{k-1}(h^k d^{k-1}).$$

Since d^k is a weak cokernel of d^{k-1}, there exists a morphism $h^{k+1} \colon X^{k+1} \to X^k$ such that $d^k h^{k+1} = 1_{X^k} - h^k d^{k-1}$ or, equivalently,

$$1_{X^k} = h^k d^{k-1} + d^k h^{k+1}.$$

This finishes the induction step. It remains to show that $1_{X^{n+1}} = h^n d^n$. For this, let $k = n$ and note that composing the previous equality on the right by d^n yields

$$d^n = (h^n d^{n-1} + d^n h^{n+1})d^n = d^n(h^{n+1}d^n).$$

Since d^n is an epimorphism, we have $1_{X^{n+1}} = h^n d^n$, which is what we needed to show. This shows that X is a contractible complex, and so it is also an n-exact sequence. The converse implication is obvious. $\square$

The following result implies that n-cokernels and n-kernels are unique up to isomorphism in $\mathsf{H}(\mathcal{C})$.

Proposition 2.2.7. *Let $\mathcal{C}$ be an additive category and $f \colon X \to Y$ a morphism of n-exact sequences in $\mathcal{C}$ such that f^k and f^{k+1} are isomorphisms for some $k \in \{1, \dots, n\}$. Then, f induces an isomorphism in $\mathsf{H}(\mathcal{C})$.*

Proof. Using the factorization property of weak cokernels and weak kernels we can construct a morphism of n-exact sequences $g \colon Y \to X$ where g^k and g^{k+1} are the inverses of f^k and f^{k+1} respectively:

$$\begin{tikzcd}
X \arrow[d, "f"] & X^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{k-1} \arrow[r] \arrow[d] & X^k \arrow[r] \arrow[d, "f^k"] & X^{k+1} \arrow[r] \arrow[d, "f^{k+1}"] & X^{k+2} \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n+1} \arrow[d] \\
Y \arrow[d, "g"] & Y^0 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Y^{k-1} \arrow[r] \arrow[d, dotted] & Y^k \arrow[r] \arrow[d, "g^k"] & Y^{k+1} \arrow[r] \arrow[d, "g^{k+1}"] & Y^{k+2} \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Y^{n+1} \arrow[d, dotted] \\
X & X^0 \arrow[r] & \cdots \arrow[r] & X^{k-1} \arrow[r] & X^k \arrow[r] & X^{k+1} \arrow[r] & X^{k+2} \arrow[r] & \cdots \arrow[r] & X^{n+1}
\end{tikzcd}$$

Then, the Comparison Lemma 2.2.1 and its dual applied to diagrams

$$
\begin{tikzcd}
X^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{k-1} \arrow[r] \arrow[d] & X^k \arrow[d, "f^{k+1}"] \\
Y^0 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Y^{k-1} \arrow[r] \arrow[d, dotted] & Y^k \arrow[d, "g^{k+1}"] \\
X^0 \arrow[r] & \cdots \arrow[r] & X^{k-1} \arrow[r] & X^k
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X^{k+1} \arrow[r] \arrow[d, "f^{k+1}"] & X^{k+2} \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n+1} \arrow[d] \\
Y^{k+1} \arrow[r] \arrow[d, "g^{k+1}"] & Y^{k+2} \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Y^{n+1} \arrow[d, dotted] \\
X^{k+1} \arrow[r] & X^{k+2} \arrow[r] & \cdots \arrow[r] & X^{n+1}
\end{tikzcd}
$$

respectively imply that f and g induce mutually inverse isomorphisms in the homotopy category $\mathsf{H}(\mathcal{C})$. □

Remark 2.2.8. The statement of Proposition 2.2.7 can be interpreted as saying that each morphism in an n-exact sequence determines the others "up to homotopy". To prove that equivalences of n-exact sequences also induce isomorphisms in $\mathsf{H}(\mathcal{C})$ we need to impose a richer structure on the category $\mathcal{C}$, see Proposition 2.4.10.

Definition 2.2.9. Let $\mathcal{C}$ be an additive category. A *morphism of n-exact sequences in* $\mathcal{C}$ is a morphism of complexes

$$
\begin{tikzcd}
X \arrow[d, "f"] & X^0 \arrow[r] \arrow[d, "f^0"] & X^1 \arrow[r] \arrow[d, "f^1"] & \cdots \arrow[r] & X^n \arrow[r] \arrow[d, "f^n"] & X^{n+1} \arrow[d, "f^{n+1}"] \\
Y & Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^n \arrow[r] & Y^{n+1}
\end{tikzcd}
$$

in which each row is an n-exact sequence. We say that f is an *equivalence* if $f^0 = 1_{X^0}$ and $f^{n+1} = 1_{X^{n+1}}$.

Remark 2.2.10. In Proposition 2.4.10 we show that, in the case of n-exact categories, equivalences of n-exact sequences are in fact an equivalence relation on the class of n-exact sequences.

2.2.3. n-pushout diagrams and n-pullback diagrams. Let $\mathcal{C}$ be an additive category. A pushout diagram of a pair of morphisms

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"] & Z \\
Y &
\end{tikzcd}
$$

in $\mathcal{C}$ can be identified with a cokernel of the morphism $[-g\ f]^{\top} \colon X \to Z \oplus Y$. This motivates us to introduce the following concept.

Definition 2.2.11. Let $\mathcal{C}$ be an additive category, X a complex in $\mathsf{Ch}^{n-1}(\mathcal{C})$, and $f^0 \colon X^0 \to Y^0$ a morphism in $\mathcal{C}$. An *n-pushout diagram of X along f^0* is a morphism of complexes

$$
\begin{tikzcd}
X \arrow[d, "f"] & X^0 \arrow[r] \arrow[d, "f^0"] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & X^n \arrow[d] \\
Y & Y^0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n
\end{tikzcd}
$$

such that in the *mapping cone* $C = C(f)$

$$
X^0 \xrightarrow{d_C^{-1}} X^1 \oplus Y^0 \xrightarrow{d_C^0} \cdots \xrightarrow{d_C^{n-2}} X^n \oplus Y^{n-1} \xrightarrow{d_C^{n-1}} Y^n.
$$

the sequence $\left(d_C^0, \dots, d_C^{n-1}\right)$ is an n-cokernel of d_C^{-1}, where we define

$$d_C^k := \begin{bmatrix} -d_X^{k+1} & 0 \\ f^{k+1} & d_Y^k \end{bmatrix} : X^{k+1} \oplus Y^k \longrightarrow X^{k+2} \oplus Y^{k+1} \tag{2.2.4}$$

for each $k \in \{-1, 0, 1, \dots, n-1\}$. In particular,

$$d^{-1} = \begin{bmatrix} -d_X^0 \\ f^0 \end{bmatrix} \quad \text{and} \quad d^{n-1} = \begin{bmatrix} f^n & d_Y^{n-1} \end{bmatrix}.$$

Note that the fact that $C(f)$ is a complex encodes precisely that X and Y are complexes and that f is a morphism of complexes. The concept of *n-pullback diagram* is defined dually.

We now state some of general properties of n-pushout diagrams.

Proposition 2.2.12. *Let $\mathcal{C}$ be an additive category. Suppose that we are given an n-pushout diagram*

$$\begin{tikzcd} X \arrow[d, "f"] & X^0 \arrow[r] \arrow[d, "g^0"] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & X^n \arrow[d] \\ Y & Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n \end{tikzcd}$$

and let $k \in \{0, 1, \dots, n-2\}$. If d_Y^{k+1} is a weak cokernel of d_Y^k, then d_X^{k+1} is a weak cokernel of d_X^k.

Proof. Put $C := C(f)$ and let $u\colon X^{k+1} \to M$ be a morphism such that $d_X^k u = 0$. Consider the solid part of the following commutative diagram:

$$\begin{tikzcd} & X^k \arrow[r] \arrow[d, "f^k"] & X^{k+1} \arrow[r] \arrow[d, "f^{k+1}"'] \arrow[dd, bend left, "u"] & X^{k+2} \arrow[d, "f^{k+2}"] \arrow[ddl, dotted, "h"'] \\ Y^{k-1} \arrow[r] & Y^k \arrow[r] \arrow[dr, "0"'] & Y^{k+1} \arrow[r] \arrow[d, dotted, "v"] & Y^{k+2} \arrow[dl, dotted, "w"] \\ & & M & \end{tikzcd}$$

Given that $d_C^k\colon X^{k+1} \oplus Y^k \to X^{k+2} \oplus Y^{k+1}$ is a weak cokernel of $d_C^{-1}\colon X^k \oplus Y^{k-1} \to X^{k+1} \oplus Y^k$, there exist morphisms $v\colon Y^{k+1} \to M$ and $h\colon X^{k+2} \to M$ such that $d_Y^k v = 0$ and $u - f^{k+1} v = d_X^{k+1} h$. Since d_Y^{k+1} is a weak cokernel of d_Y^k there exists a morphism $w\colon Y^{k+2} \to M$ such that $v = d_Y^{k+1} w$. Therefore we have

$$\begin{aligned} u &= d_X^{k+1} h + f^{k+1} v \\ &= d_X^{k+1} h + f^{k+1}(d_Y^{k+1} w) \\ &= d_X^{k+1}(h + f^{k+2} w). \end{aligned}$$

This shows that d_X^{k+1} is a weak cokernel of d_X^k. □

Our choice of terminology in Definition 2.2.11 is justified by the following property.

Proposition 2.2.13. *Let $\mathcal{C}$ be an additive category, $g\colon X \to Z$ a morphism of complexes in $\mathsf{Ch}^{n-1}(\mathcal{C})$ and suppose there exists an n-pushout diagram of X along*

g^0

$$\begin{tikzcd} X \arrow[d, "f"] & X^0 \arrow[r] \arrow[d, "g^0"] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & X^n \arrow[d] \\ Y & Y^0 = Z^0 \arrow[r] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n \end{tikzcd}$$

Then, there exists a morphism of complexes $p\colon Y \to Z$ such that $p^0 = 1_{Z^0}$ and a homotopy $h\colon fp \to g$ with $h^1 = 0$. Moreover, these properties determine p uniquely up to homotopy.

Proof. Let $h^1\colon X^1 \to Z^0$ be the zero morphism, $p^0 = 1_{Z^0}$ and $C := C(f)$. Inductively, suppose that $0 \leq k \leq n$ and that for all $\ell \leq k$ we have constructed a morphism $p^\ell\colon Y^\ell \to Z^\ell$ such that

$$d_Y^{\ell-1}p^\ell = p^{\ell-1}d_Z^{\ell-1}$$

and a morphism $h^{\ell+1}\colon X^{\ell+1} \to Z^\ell$ such that

$$f^\ell p^\ell - g^\ell = h^\ell d_Z^{\ell-1} + d_X^\ell h^{\ell+1}.$$

We claim that the composition

$$X^k \oplus Y^{k-1} \xrightarrow{\begin{bmatrix} -d_X^k & 0 \\ f^k & d_Y^{k-1} \end{bmatrix}} X^{k+1} \oplus Y^k \xrightarrow{\begin{bmatrix} g^{k+1} - h^{k+1}d_Z^k & p^k d_Z^k \end{bmatrix}} Z^{k+1}$$

vanishes. Indeed, on one hand we have

$$f^k(p^k d_Z^k) = (g^k + d_X^k h^{k+1})d_Z^k = d_X^k(g^{k+1} - h^{k+1}d_Z^k).$$

On the other hand, we have

$$d_Y^{k-1}(p^k d_Z^k) = p^{k-1}d_Z^{k-1}d_Z^k = 0.$$

The claim follows.

Next, since d_C^k is a weak cokernel of d_C^{k-1}, there exists a morphism $p^{k+1}\colon Y^{k+1} \to Z^{k+1}$ such that

$$d_Y^k p^{k+1} = p^k d_Z^k$$

and a morphism $h^{k+2}\colon X^{k+2} \to Y^{k+1}$ such that

$$g^{k+1} + h^{k+1}d_Z^k = -d_X^{k+1}h^{k+2} + f^{k+1}p^{k+1}.$$

This finishes the induction step, and the construction of the required morphism $p\colon Y \to Z$. Moreover, $h\colon fp \to g$ is a homotopy (note that $h^{n+1} = 0$). The last claim follows immediately from the Comparison Lemma 2.2.1. □

Definition-Proposition 2.2.14. Let $\mathcal{C}$ be an additive category and $g^0\colon X^0 \to Z^0$ a morphism in $\mathcal{C}$. Suppose that there exists an n-pushout diagram of X along g^0

$$\begin{tikzcd} X \arrow[d, "f"] & X^0 \arrow[r] \arrow[d, "g^0"] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & X^n \arrow[d] \\ Y & Y^0 = Z^0 \arrow[r] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n \end{tikzcd}$$

Then, the following statements hold:

(i) There exists an n-pushout diagram $\tilde{f}\colon X \to \tilde{Y}$ of X along g^0 such that for every morphism $g\colon X \to Z$ of complexes lifting g^0 there exists a morphism of complexes $p\colon Y \to Z$ such that $p^0 = 1_{Z^0}$ and $\tilde{f}p = g$.
(ii) For each $2 \leq k \leq n$ the morphism $\tilde{f}^k$ is a split monomorphism.
(iii) We have $\tilde{Y} = Y \oplus X'$ for a contractible complex $X' \in \mathsf{Ch}^{n-1}(\mathcal{C})$.

We call the morphism $\tilde{f}\colon X \to \tilde{Y}$ a *good n-pushout diagram of X along g^0*.

PROOF. If $n = 1$ the result is trivial, so we may assume that $n \geq 2$. For $C \in \mathcal{C}$ and $k \in \mathbb{Z}$, let $i_k(C)$ be the complex with $d^k = 1_C$ and which is 0 in each degree different from k and $k+1$. We define

$$X' := \bigoplus_{k=2}^{n} i_{k-1}(X^k)$$

and $\tilde{Y} := Y \oplus X'$. Note that $\tilde{Y}^0 = Y^0$ and that X' is a contractible complex. It readily follows that the diagram

$$\begin{tikzcd}
X \arrow[d, "\tilde{f}"] & X^0 \arrow[r] \arrow[d, "f^0=g^0"] & X^1 \arrow[r] \arrow[d, "\begin{bmatrix} f^1 \\ d_{X^1} \end{bmatrix}"] & X^2 \arrow[r] \arrow[d, "\begin{bmatrix} f^2 \\ 1 \\ d_X^2 \end{bmatrix}"] & \cdots \arrow[r] & X^k \arrow[r] \arrow[d, "\begin{bmatrix} f^k \\ 1 \\ d_X^k \end{bmatrix}"] & \cdots \arrow[r] & X^n \arrow[d, "\begin{bmatrix} f^n \\ 1 \end{bmatrix}"] \\
\tilde{Y} & Y^0 \arrow[r] & \tilde{Y}^1 \arrow[r] & \tilde{Y}^2 \arrow[r] & \cdots \arrow[r] & \tilde{Y}^k \arrow[r] & \cdots \arrow[r] & \tilde{Y}^n
\end{tikzcd}$$

commutes. Observe that for each $2 \leq k \leq n$ the morphism $\tilde{f}^k$ is a split monomorphism. Using Proposition 2.2.13, it is easy to show that $\tilde{f}$ has the required factorization property; the details are left to the reader. $\square$

2.3. n-abelian categories

In this section we introduce n-abelian categories and establish their basic properties; we give a characterization of semisimple categories in terms of n-abelian categories. We also introduce projective objects in n-abelian categories and study their basic properties. Finally, we show that n-cluster-tilting subcategories of abelian categories are n-abelian; we give a partial converse in the case of n-abelian categories with enough projectives.

2.3.1. Definition and basic properties. The following definition is motivated by the axioms of abelian categories given in [**86**, Def. A.4.2].

DEFINITION 2.3.1. Let n be a positive integer. An *n-abelian category* is an additive category $\mathcal{M}$ which satisfies the following axioms:

(A0) The category $\mathcal{M}$ is idempotent complete.

(A1) Every morphism in $\mathcal{M}$ has an n-kernel and an n-cokernel.

(A2) For every monomorphism $f^0\colon X^0 \to X^1$ in $\mathcal{M}$ and for every n-cokernel $(f^1, \dots, f^n)$ of f^0, the following sequence is n-exact:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

(A2$^{\mathrm{op}}$) For every epimorphism $g^n\colon X^n \to X^{n+1}$ in $\mathcal{M}$ and for every n-kernel $(g^0, \dots, g^{n-1})$ of g^n, the following sequence is n-exact:

$$X^0 \xrightarrow{g^0} X^1 \xrightarrow{g^1} \cdots \xrightarrow{g^{n-1}} X^n \xrightarrow{g^n} X^{n+1}.$$

Let us give some important remarks regarding Definition 2.3.1.

REMARK 2.3.2. By Proposition 2.2.5 and Proposition 2.2.7 we can replace axiom (A2) by the following weaker version:

(A2') For every monomorphism $f^0\colon X^0 \to X^1$ in $\mathcal{M}$ there exists an n-exact sequence:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

Naturally, we can weaken axiom (A2$^{\mathrm{op}}$) in a dual manner.

REMARK 2.3.3. Let $\mathcal{M}$ be an n-abelian category. An immediate consequence of axioms (A1) and (A2) (resp. (A2$^{\mathrm{op}}$)) is that every monomorphism (resp. epimorphism) in $\mathcal{M}$ appears as the leftmost (resp. rightmost) morphism in some n-exact sequence.

REMARK 2.3.4. Let m and n be distinct positive integers. Note that the only categories which are both n-abelian and m-abelian are the semisimple categories, see Corollary 2.3.10.

Note that 1-abelian categories are precisely abelian categories in the usual sense. It is easy to see that abelian categories are idempotent complete; thus, if $n = 1$, then axiom (A0) in Definition 2.3.1 is redundant. However, if $n \geq 2$, then axiom (A0) is independent from the remaining axioms as shown by the following example.

EXAMPLE 2.3.5. Let $n \geq 2$ and K be a field. Consider the full subcategory $\mathcal{V}$ of $\mathsf{mod}\, K$ given by the finite dimensional K-vector spaces of dimension different from 1. Then, $\mathcal{V}$ is not idempotent complete but it satisfies axioms (A1), (A2) and (A2$^{\mathrm{op}}$).

PROOF. Firstly, note that $\mathcal{V}$ is an additive subcategory of $\mathsf{mod}\, K$. The fact that $\mathcal{V}$ is not idempotent complete is obvious (consider the idempotent $0 \oplus 1_K \colon K^2 \to K^2$ whose kernel is one-dimensional, for example). Let us show that $\mathcal{V}$ satisfies axiom (A1). Indeed, let $f \colon V \to W$ be a morphism in V. If $\mathsf{coker}\, f$ has dimension different from 1, then

$$V \longrightarrow W \longrightarrow \mathsf{coker}\, f \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

gives an n-cokernel of f in $\mathcal{V}$. If $\mathsf{coker}\, f$ has dimension 1, then we can construct an n-cokernel of f in $\mathcal{V}$ by a commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] & W \arrow[r] \arrow[d] & K^3 \arrow[r] & K^2 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \\ & \mathsf{coker}\, f \arrow[ur] & & & & & \end{tikzcd}$$

where $\mathsf{coker}\, f \to K^3 \to K^2$ is a kernel-cokernel pair. We can construct an n-kernel of f in a dual manner. This shows that $\mathcal{V}$ satisfies axiom (A1). That $\mathcal{V}$ satisfies axioms (A2) and (A2$^{\mathrm{op}}$) follows from Proposition 2.2.6 since contractible complexes with $n+2$ terms are in particular n-exact sequences by Proposition 2.2.5. $\square$

LEMMA 2.3.6. *Let $\mathcal{C}$ be an idempotent complete additive category and suppose that we are given a sequence of morphisms in $\mathcal{C}$ of the form*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

If g is a weak cokernel of f, and h is both a split epimorphism and a cokernel of g, then f admits a cokernel in $\mathcal{C}$.

PROOF. Since h is a split epimorphism there exists a morphism $i \colon M \to Y$ such that $ih = 1_D$. It follows that the morphism $e := 1_C - hi$ is idempotent. Since the category $\mathcal{C}$ is idempotent complete, there exists an object $E \in \mathcal{C}$ and morphisms $r \colon C \to E$ and $s \colon E \to C$ such that $sr = 1_E$ and $rs = e$. Note that this implies that r is an epimorphism and $sh = 0$ for we have

$$r(sh) = (1 - hi)h = h - h = 0.$$

We claim that gr is a cokernel of f. Indeed, let $u\colon B \to B'$ be a morphism such that $fu = 0$. Since g is a weak cokernel of f there exists a morphism $v\colon C \to B'$ such that $u = gv$. It follows that

$$u = gv = g(1 - hi)v = (gr)(sv).$$

This shows that gr is a weak cokernel of f. It remains to show that gr is an epimorphism. For this, let $w\colon E \to E'$ be a morphism such that $(gr)w = 0$. Since h is a cokernel of g there exists a morphism $x\colon D \to E'$ such that $rw = hx$. It follows that

$$w = (sr)w = s(hx) = 0.$$

This shows that gr is an epimorphism. Therefore gr is a cokernel of f. □

Proposition 2.3.7. *Let $\mathcal{M}$ be an additive category which satisfies axioms (A0) and (A1), and let X a complex in $\mathsf{Ch}^{n-1}(\mathcal{C})$. If for all $0 \leq k \leq n-1$ the morphism d^k is a weak cokernel of d^{k-1}, then d^{n-1} admits a cokernel in $\mathcal{M}$.*

Proof. If $n = 1$, then the result follows trivially from axiom (A1). Hence we may assume that $n \geq 2$. By axiom (A1) there exists an n-cokernel

$$\left(d^k\colon X^k \to X^{k+1} \mid n \leq k \leq 2n-1\right)$$

of d^{n-1}. Using axiom (A1) again together with the factorization property of weak cokernels we obtain a commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r, "d^0"] \arrow[d, equal] & X^1 \arrow[r, "d^1"] \arrow[d, equal] & X^2 \arrow[r, "d^2"] \arrow[d, dotted, "f^2"] & \cdots \arrow[r, "d^{n-1}"] & X^n \arrow[r, "d^n"] \arrow[d, dotted, "f^n"] & X^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, dotted, "f^{n+1}"] & X^{n+2} \arrow[r, "d^{n+2}"] \arrow[d, dotted] & \cdots \arrow[r, "d^{2n-1}"] & X^{2n} \arrow[d, dotted] \\
Y^0 \arrow[r] \arrow[d, equal] & Y^1 \arrow[r] \arrow[d, equal] & Y^2 \arrow[r] \arrow[d, dotted, "g^2"] & \cdots \arrow[r] & Y^n \arrow[r] \arrow[d, dotted, "g^n"] & Y^{n+1} \arrow[r] \arrow[d, dotted, "g^{n+1}"] & 0 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & 0 \arrow[d, dotted] \\
X^0 \arrow[r, "d^0"] & X^1 \arrow[r, "d^1"] & X^2 \arrow[r, "d^2"] & \cdots \arrow[r, "d^{n-1}"] & X^n \arrow[r, "d^n"] & X^{n+1} \arrow[r, "d^{n+1}"] & X^{n+2} \arrow[r, "d^{n+2}"] & \cdots \arrow[r, "d^{2n-1}"] & X^{2n}
\end{tikzcd}$$

in which the middle row gives an n-cokernel of d^0. The Comparison Lemma 2.2.1 implies that there exists a morphism $h\colon X^{2n} \to X^{2n-1}$ such that $hd^{2n-1} = 1$. Hence we may apply Lemma 2.3.6 and reduce the length of the n-cokernel of d^{n-1} by one morphism. Proceeding inductively, we deduce that d^{n-1} has an n-cokernel in $\mathcal{M}$. □

The importance of axiom (A0) becomes apparent in the following result, which asserts that n-abelian categories have n-pushout diagrams and n-pullback diagrams.

Theorem 2.3.8 (Existence of n-pushout diagrams). *Let $\mathcal{M}$ be an additive category which satisfies axioms (A0) and (A1). Let X be a complex in $\mathsf{Ch}^{n-1}(\mathcal{C})$, and a morphism $f\colon X^0 \to Y^0$. Then, the following statements hold:*

(i) Then, there exists an n-pushout diagram

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d, "f"] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & X^n \arrow[d] \\
Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n
\end{tikzcd}$$

(ii) Moreover, if $\mathcal{M}$ is n abelian and d_X^0 is a monomorphism, then so is d_Y^0.

Proof. (i) We shall construct the complex Y inductively. Set $f^0 := f$ and

$$d_C^{-1} = \begin{bmatrix} -d_X^0 \\ f^0 \end{bmatrix} : X^0 \longrightarrow X^1 \oplus Y^0.$$

Let $0 \leq k \leq n-2$ and suppose that for each $\ell \leq k$ we have constructed an object Y^ℓ and morphisms $f^\ell \colon X^\ell \to Y^\ell$ and $d_Y^{\ell-1} \colon Y^{\ell-1} \to Y^\ell$ such that $d_C^{\ell-2} d_C^{\ell-1} = 0$ where

$$d_C^{\ell-1} := \begin{bmatrix} -d_X^\ell & 0 \\ f^\ell & d_Y^{\ell-1} \end{bmatrix} \colon X^\ell \oplus Y^{\ell-1} \longrightarrow X^{\ell+1} \oplus Y^\ell$$

(compare with (2.2.4)). Then, by axiom (A1), the morphism d_C^{k-1} has a weak cokernel $g^k := [f^{k+1} \ d_Y^k] \colon X^{k+1} \oplus Y^k \to Y^{k+1}$. We claim that

$$d_C^k := \begin{bmatrix} -d_X^{k+1} & 0 \\ f^{k+1} & d_Y^k \end{bmatrix} \colon X^{k+1} \oplus Y^k \longrightarrow X^{k+2} \oplus Y^{k+1}$$

is also a weak cokernel of d_C^{k-1}. Indeed, it is readily verified that $d_C^{k-1} d_C^k = 0$. Let $u \colon X^{k+1} \oplus Y^k \to M$ be a morphism such that $d_C^{k-1} u = 0$. Since g^k is a weak cokernel of d_C^{k-1}, there exists a morphism $v \colon Y^{k+1} \to M$ such that $u = g^k v$. It follows that the following diagram is commutative:

$$\begin{tikzcd} X^{k+1} \oplus Y^k \arrow[r, "d_C^k"] \arrow[dr, "u"'] & X^{k+2} \oplus Y^{k+1} \arrow[d, "{[0 \ v]}"] \\ & M \end{tikzcd}$$

This shows that d_C^k is a weak cokernel of d_C^{k-1}. Finally, Proposition 2.3.7 implies that the morphism d_C^{n-2} admits a cokernel

$$d_C^{n-1} \colon X^n \oplus Y^{n-1} \to Y^n.$$

This shows that the tuple $\left(d_C^0, d_C^1, \dots, d_C^{n-1}\right)$ is a weak cokernel of d_C^{-1}. The existence of the required commutative diagram follows from the fact that C is a complex.

(ii) Finally, suppose that $\mathcal{M}$ is n-abelian and d_X^0 is a monomorphism. Note that this implies that d_C^{-1} is also a monomorphism. Then, axiom (A2) implies that the C is an n-exact sequence. In order to show that d_Y^0 is a monomorphism, let $u \colon M \to Y^0$ be a morphism such that $u d_Y^0 = 0$. It follows that the composition

$$M \xrightarrow{\begin{bmatrix} 0 \\ u \end{bmatrix}} X^1 \oplus Y^0 \xrightarrow{\begin{bmatrix} -d_X^1 & 0 \\ f^1 & d_Y^0 \end{bmatrix}} X^2 \oplus Y^1$$

vanishes. Given that d_C^{-1} is a kernel of d_C^0, there exists a morphism $v \colon M \to X^0$ such that $v d_X^0 = 0$ and $v f^0 = u$. Since d_X^0 is a monomorphism, we have $u = 0$. This shows that d_Y^0 is a monomorphism. $\square$

We remind the reader that an additive category $\mathcal{C}$ is *semisimple* if every morphism $f \colon A \to B$ in $\mathcal{C}$ factors as $f = pi$, where p is a split epimorphism and i is a split monomorphism. The following result characterizes semisimple categories in terms of n-abelian categories.

THEOREM 2.3.9. *Let $\mathcal{C}$ be an additive category and n a positive integer. Then, the n-abelian categories in which every n-exact sequence is contractible are precisely the semisimple categories.*

PROOF. Suppose that $\mathcal{C}$ is a semisimple category. We only show that $\mathcal{C}$ is idempotent complete. It is straightforward to verify that $\mathcal{C}$ satisfies the remaining axioms of n-abelian categories, the fact that every n-exact sequence in $\mathcal{C}$ is contractible follows immediately from Proposition 2.2.6. Let us show then that $\mathcal{C}$ is idempotent complete. Let $e\colon A \to A$ be an idempotent in $\mathcal{C}$. Since $\mathcal{C}$ is semisimple, e factors as $e = pi$ where $p\colon A \to B$ is a split epimorphism and $i\colon B \to A$ is a split monomorphism. We claim that $ip = 1_B$. Indeed, let $h\colon A \to B$ be a morphism such that $ih = 1_B$. Given that $e^2 = e$ we have

$$p = p(ih) = eh = e^2h = (pipi)h = p(ip).$$

Since p is an epimorphism we have $ip = 1_B$ as claimed. This shows that $\mathcal{C}$ is idempotent complete.

Conversely, suppose that $\mathcal{C}$ is an n-abelian category in which every n-exact sequence is contractible and let $f\colon A \to B$ be a morphism in $\mathcal{C}$. We claim that f admits both a kernel and a cokernel in $\mathcal{C}$. Indeed, by axiom (A1) there exists an n-cokernel $\left(f^1, \cdots, f^n\right)$ of f. By hypothesis, the epimorphism f^n must split. Then, Lemma 2.3.6 implies that f^{n-2} has a cokernel in $\mathcal{C}$. By applying this argument inductively we deduce that f admits a cokernel in $\mathcal{C}$. By duality, f also admits a kernel in $\mathcal{C}$. The remaining part of the proof is classical, compare for example with the proof of [**29**, Prop. 4.8].

We need to show that f factors as $f = pi$ where p is a split epimorphism and i is a split monomorphism. Given that f has both a kernel and a cokernel in $\mathcal{C}$, is easy to construct a commutative diagram

$$\begin{tikzcd}
K \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "p'"] & B \arrow[r, "p"] & C \\
& J \arrow[r, dotted, "g"] & I \arrow[u, "i'"] &
\end{tikzcd}$$

where i is a kernel of f, and p is a cokernel of f, and the sequences $K \to A \to J$ and $I \to B \to C$ are kernel-cokernel pairs. We claim that g is an isomorphism, for which it is enough to show that is both a monomorphism and an epimorphism as all such morphisms split by hypothesis. By duality we only need to show that g is an epimorphism.

Let $h\colon I \to I'$ be a morphism such that $gh = 0$. Firstly, by Theorem 2.3.8 there exists a commutative diagram

$$\begin{tikzcd}
I \arrow[r, "i'"] \arrow[d, "h"] & Y \arrow[r, "p"] \arrow[d, "h'"] & C \\
I' \arrow[r, "i''"] & B' &
\end{tikzcd}$$

where i'' is a monomorphism. Secondly, we claim that $fh' = 0$. Indeed, we have

$$fh' = (p'gi')h' = p'(gh)i'' = 0.$$

Therefore, since p is a cokernel of f, there exists a morphism $j\colon C \to B'$. It follows that

$$hi'' = i'h' = i'(pj) = 0.$$

Finally, since i'' is a monomorphism, we have $h = 0$. This shows that g is an epimorphism. $\square$

Corollary 2.3.10. *Let m and n be two distinct positive integers and $\mathcal{C}$ an additive category. If $\mathcal{C}$ is both m-abelian and n-abelian, then $\mathcal{C}$ is a semisimple category.*

Proof. Without loss of generality we may assume that $m < n$. By Theorem 2.3.9 and Proposition 2.2.6 it is enough to show that every monomorphism in $\mathcal{C}$ splits. Let $f^0\colon X^0 \to X^1$ be a monomorphism in $\mathcal{C}$ and let $(f^1, \dots, f^n)$ be an

n-cokernel of f^0, and $(g^1, \dots, g^m)$ be an m-cokernel of f^0. It follows that there exists a commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r, "f^0"] \arrow[d, equal] & X^1 \arrow[r, "f^1"] \arrow[d, equal] & X^2 \arrow[r, "f^2"] \arrow[d, dotted] & \cdots \arrow[r, "f^m"] & X^{m+1} \arrow[r, "f^{m+1}"] \arrow[d, dotted] & X^{m+2} \arrow[r, "f^{m+2}"] \arrow[d, dotted] & \cdots \arrow[r, "f^n"] & X^{n+1} \arrow[d, dotted] \\
Y^0 \arrow[r] \arrow[d, equal] & Y^1 \arrow[r, "g^1"] \arrow[d, equal] & Y^2 \arrow[r, "g^2"] \arrow[d, dotted] & \cdots \arrow[r, "g^n"] & Y^{n+1} \arrow[r] \arrow[d, dotted] & 0 \arrow[r] \arrow[d, dotted] & 0 \arrow[r] & 0 \arrow[d, dotted] \\
X^0 \arrow[r, "f^0"] & X^1 \arrow[r, "f^1"] & X^2 \arrow[r, "f^2"] & \cdots \arrow[r, "f^m"] & X^{m+1} \arrow[r, "f^{m+1}"] & X^{m+1} \arrow[r, "f^{m+2}"] & \cdots \arrow[r, "f^n"] & X^n
\end{tikzcd}$$

By the Comparison Lemma 2.2.1 there exists a morphism $h\colon X^{n+1} \to X^n$ such that $hf^n = 1$. Thus f^n is a split epimorphism. Then, as $(f^0, f^1, \dots, f^n)$ is an n-exact sequence by axiom (A2), the dual of Proposition 2.2.6 implies that f^0 is a split monomorphism. $\square$

2.3.2. Projective objects in n-abelian categories. We remind the reader of the following classical definition.

DEFINITION 2.3.11. Let $\mathcal{C}$ be an additive category. We say that $P \in \mathcal{C}$ is *projective* if for every epimorphism $f\colon A \to B$ the sequence of abelian groups

$$\mathcal{C}(P, A) \xrightarrow{?\cdot f} \mathcal{C}(P, B) \longrightarrow 0$$

is exact. The concept of *injective object in* $\mathcal{C}$ is defined dually.

Our aim is to prove the following important property of projective objects in an n-abelian category.

THEOREM 2.3.12. *Let $\mathcal{M}$ be an n-abelian category and $P \in \mathcal{M}$ a projective object. Then, for every morphism $f\colon L \to M$ and every weak cokernel $g\colon M \to N$ of f, the sequence of abelian groups*

$$\mathcal{M}(P, L) \xrightarrow{?\cdot f} \mathcal{M}(P, M) \xrightarrow{?\cdot g} \mathcal{M}(P, N) \tag{2.3.1}$$

is exact.

For the proof of Theorem 2.3.12 we need the following result which, albeit technical, is interesting in its own right.

PROPOSITION 2.3.13. *Let $\mathcal{M}$ be an n-abelian category, $f^0\colon X^0 \to X^1$ a morphism in $\mathcal{M}$ and $\left(f^k\colon X^k \to X^{k+1} \mid 1 \leq k \leq n\right)$ an n-cokernel of f^0. Then, for every $k \in \{0, 1, \dots, n\}$ there exists a commutative diagram*

$$\begin{tikzcd}
Y_k^n \arrow[r, "g_k^n"] \arrow[d] & Y_k^{n-1} \arrow[r, "g_k^{n-1}"] \arrow[d, "p_k^{n-1}"] & \cdots \arrow[r, "g_k^2"] & Y_k^1 \arrow[r, "g_k^1"] \arrow[d, "p_k^1"] & X^k \arrow[r, "f^k"] \arrow[d, "p_k^0"] & X^{k+1} \\
0 \arrow[r] & Y_{k+1}^n \arrow[r, "g_{k+1}^n"] & \cdots \arrow[r, "g_{k+1}^3"] & Y_{k+1}^2 \arrow[r, "g_{k+1}^2"] & Y_{k+1}^1 \arrow[ur, "g_{k+1}^1"'] &
\end{tikzcd} \tag{2.3.2}$$

satisfying the following properties:

(i) *The sequence $\left(g_k^n, \dots, g_k^1\right)$ is an n-kernel of f^k.*

(ii) *The diagram*

$$\begin{tikzcd}
Y_k^n \arrow[r, "g_k^n"] \arrow[d] & Y_k^{n-1} \arrow[r, "g_k^{n-1}"] \arrow[d, "p_k^{n-1}"] & \cdots \arrow[r, "g_k^2"] & Y_k^1 \arrow[r, "g_k^1"] \arrow[d, "p_k^1"] & X^k \arrow[d, "p_k^0"] \\
0 \arrow[r] & Y_{k+1}^n \arrow[r, "g_{k+1}^n"] & \cdots \arrow[r, "g_{k+1}^3"] & Y_{k+1}^2 \arrow[r, "g_{k+1}^2"] & Y_{k+1}^1
\end{tikzcd}$$

is both an n-pullback diagram and a good n-pushout diagram, see Definition-Proposition 2.2.14*. In particular, the morphism*

$$\begin{bmatrix} p_k^0 & g_{k+1}^2 \end{bmatrix} : X^k \oplus Y_{k+1}^2 \longrightarrow Y_{k+1}^1$$

is an epimorphism.

(iii) *If $k \neq 0$, then the sequence $\left(g_k^{k-1}, \dots, g_k^1, f^k, \dots, f^n\right)$ is an n-cokernel of the morphism g_k^k.*

Proof. We proceed by induction on k, beginning with the case $1 \neq k = n$. By axiom (A1) there exists an n-kernel $(g_n^n, \dots, g_1^n)$ of f^n and by axiom (A2$^{\mathrm{op}}$) the sequence $\left(g_n^n, \dots, g_1^1, f^n\right)$ is an n-exact sequence. Note that this implies that the diagram

$$\begin{tikzcd}
Y_n^n \arrow[r, "g_n^n"] \arrow[d] & Y_n^{n-1} \arrow[r, "g_n^{n-1}"] \arrow[d] & \cdots \arrow[r, "g_n^2"] & Y_n^1 \arrow[r, "g_n^1"] \arrow[d] & X^n \arrow[r, "f^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & X^n \arrow[ur, "1"'] & &
\end{tikzcd}$$

is a both an n-pullback diagram and an n-pushout diagram. By Definition-Proposition 2.2.14 we can replace this diagram by a good n-pushout diagram

$$\begin{tikzcd}
Y_n^n \arrow[r, "g_n^n"] \arrow[d] & Y_n^{n-1} \arrow[r, "g_n^{n-1}"] \arrow[d] & \cdots \arrow[r, "g_n^2"] & Y_n^1 \arrow[r, "g_n^1"] \arrow[d] & X^n \arrow[r, "f^n"] \arrow[d, "\begin{bmatrix} f^n \\ 1 \end{bmatrix}"] & X^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & Y_n^{n-2} \arrow[r] & \cdots \arrow[r] & Y_n^1 \oplus X^n \arrow[r] & X^n \oplus X^n \arrow[ur, "\begin{bmatrix} 1 & 0 \end{bmatrix}"'] & &
\end{tikzcd}$$

Note that this passage does not change the fact that the bottom row gives an n-kernel of f^n. This shows that the result is holds in this case.

Let $2 \leq k \leq n$ and suppose that we have constructed a commutative diagram of the form (2.3.2) with the required properties. Since $f^{k-1} f^k = 0$ and g_k^1 is a weak kernel of f^k there exists a morphism $p_{k-1}^0 \colon X^{k-1} \to Y_k^1$ such that $f^{k-1} = p_{k-1}^0 g_k^1$. We claim that the morphism

$$\begin{bmatrix} p_{k-1}^0 & g_k^2 \end{bmatrix} : X^{k-1} \oplus Y_k^2 \longrightarrow Y_k^1$$

is an epimorphism. Indeed, let $u \colon Y_k^1 \to M$ be a morphism such that $p_{k-1}^0 u = 0$ and $g_k^2 u = 0$. Given that g_k^1 is a weak cokernel of g_k^2 there exists a morphism $v \colon X^k \to M$ such that $u = g_k^1 v$. It follows that

$$f^{k-1} v = (p_{k-1}^0 g_k^1) v = p_{k-1}^0 u = 0.$$

Then, since f^k is a weak cokernel of f^{k-1}, there exists a morphism $w \colon X^{k+1} \to M$ such that $v = f^k w$. Thus, we have

$$u = g_k^1 v = g_k^1 (f^k w) = 0.$$

The claim follows. By Theorem 2.3.8 there exists an n-pullback diagram

$$\begin{tikzcd}
Y_{k-1}^n \arrow[r, "g_{k-1}^n"] \arrow[d] & Y_{k-1}^{n-1} \arrow[r, "g_{k-1}"] \arrow[d, "p_{k-1}^{n-1}"] & \cdots \arrow[r, "g_{k-1}^2"] & Y_{k-1}^1 \arrow[r, "g_{k-1}^1"] \arrow[d, "p_{k-1}^1"] & X^{k-1} \arrow[d, "p_{k-1}^0"] \\
0 \arrow[r] & Y_k^n \arrow[r, "g_k^n"] & \cdots \arrow[r, "g_k^3"] & Y_k^2 \arrow[r, "g_k^2"] & Y_k^1
\end{tikzcd}$$

Note that axiom (A2$^{\mathrm{op}}$) implies that it this diagram is also an n-pushout diagram, and that Proposition 2.2.12 implies that it has the required properties, except that it need not be a good n-pushout diagram. Using Definition-Proposition 2.2.14 we may replace this diagram by a good n-pushout diagram. Note that since the passage to a good n-pushout diagram amounts to adding a contractible n-exact sequence it does not alter the properties of the previously constructed diagrams.

Finally, let $k = 1$. We have a commutative diagram

$$\begin{tikzcd}
Y_0^n \arrow[r, "g_0^n"] \arrow[d] & Y_0^{n-1} \arrow[r, "g_0^{n-1}"] \arrow[d, "p_0^{n-1}"] & \cdots \arrow[r, "g_0^2"] & Y_0^1 \arrow[r, "g_0^1"] \arrow[d, "p_0^1"] & X^0 \arrow[r, "f^0"] \arrow[d, "p_0^0"] & X^1 \\
0 \arrow[r] & Y_1^n \arrow[r, "g_1^n"] & \cdots \arrow[r, "g_1^3"] & Y_1^2 \arrow[r, "g_1^2"] & Y_1^1 \arrow[ur, "g_1^1"'] &
\end{tikzcd}$$

where the leftmost n squares form an n-pullback diagram; we claim that they form moreover an n-pushout diagram. To show this, by axiom (A2$^{\mathrm{op}}$) it is enough to show that the morphism

$$\begin{bmatrix} p_0^0 & g_1^2 \end{bmatrix} : X^0 \oplus Y_1^2 \longrightarrow Y_1^1$$

is an epimorphism. Let $u\colon Y_1^1 \to M$ be a morphism such that $p_0^0 u = 0$ and $g_1^2 = 0$. Put $Y_0^0 := X^0$ and $u^0 := u$. Proceeding inductively, since the diagrams we constructed are good n-pushout diagrams, for each $k \in \{1, \dots, n-1\}$ we obtain a commutative diagram

$$\begin{tikzcd}
 & Y_k^{k+1} \arrow[r] \arrow[d] & Y_k^k \arrow[d, "p_k^k"] \arrow[ddr, bend left, "u^k"] & \\
Y_{k+1}^{k+2} \arrow[r] & Y_{k+1}^{k+2} \arrow[r] \arrow[drr, "0"'] & Y_{k+1}^{k+1} \arrow[dr, dotted, "u^{k+1}"] & \\
 & & & M
\end{tikzcd}$$

Moreover, by Theorem 2.3.8 there exists a commutative diagram

$$\begin{tikzcd}
Y_n^n \arrow[r] \arrow[d, "u^n"] & Y_n^{n-1} \arrow[d, "p_n^{n-1}"] \\
M \arrow[r, "v"] & N
\end{tikzcd}$$

where v is a monomorphism. It readily follows that

$$u = p_1^1 \cdots p_{n-1}^{n-1} u^n \quad \text{and} \quad uv = g_1^1 p_1^0 \cdots p_n^{n-1}.$$

Next, observe that

$$f^0(p_1^0 \cdots p_n^{n-1}) = p_0^0 g_1^1 p_1^0 \cdots p_n^{n-1} = p_0^0 uv = 0.$$

Given that f^1 is a weak cokernel of f^0, there exists a morphism $w\colon X^1 \to N$ such that $f^1 w = p_1^0 \cdots p_n^{n-1}$. Finally, we have

$$uv = g_1^1(p_1^0 \cdots p_n^{n-1}) = g_1^1 f^1 w = 0.$$

Since v is a monomorphism, we have $u = 0$ which is what we needed to show. $\square$

We are ready to give the proof of Theorem 2.3.12.

Proof of Theorem 2.3.12. By Proposition 2.3.13 there exists a commutative diagram

$$\begin{tikzcd}
& L \arrow[r, "f"] \arrow[d, "p"] & M \arrow[r, "g"] & N \\
K_2 \arrow[r, "q"] & K_1 \arrow[ur, "i"] & &
\end{tikzcd}$$

where i is a weak kernel of g, we have $qi = 0$, and

$$\begin{bmatrix} p & q \end{bmatrix} : L \oplus K_2 \longrightarrow K_1$$

is an epimorphism.

Let $h\colon P \to M$ be a morphism such that $hg = 0$. Since i is a weak kernel of g, there exists a morphism $j\colon P \to K_1$ such that $h = ji$. Given that P is projective and $L \oplus K_2 \to K_1$ is an epimorphism, there exist morphisms $j'\colon P \to L$ and $j''\colon P \to K_2$ such that $j = j'p + j''q$. It follows that

$$h = ji = (j'p + j''q)i = j'pi = j'f.$$

Hence h factors through f. This shows that the sequence (2.3.1) is exact. $\square$

2.3.3. n-abelian categories and cluster-tilting. Recall that a subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is *cogenerating* if for every object $X \in \mathcal{A}$ there exists an object $Y \in \mathcal{C}$ and a monomorphism $X \to Y$. The concept of *generating* subcategory is defined dually. We use the following variant of the definition of a cluster-tilting subcategory of an abelian category.

Definition 2.3.14. Let $\mathcal{A}$ be an abelian category and $\mathcal{M}$ a generating-cogenerating subcategory of $\mathcal{A}$. We say that $\mathcal{M}$ is an *n-cluster-tilting subcategory of* $\mathcal{A}$ if $\mathcal{M}$ is functorially finite (see Subsection 2.2.1) in $\mathcal{A}$ and

$$\begin{aligned}
\mathcal{M} &= \left\{ X \in \mathcal{A} \mid \forall i \in \{1, \dots, n-1\} \ \mathsf{Ext}^i_{\mathcal{A}}(X, \mathcal{M}) = 0 \right\} \\
&= \left\{ X \in \mathcal{A} \mid \forall i \in \{1, \dots, n-1\} \ \mathsf{Ext}^i_{\mathcal{A}}(\mathcal{M}, X) = 0 \right\}.
\end{aligned}$$

Note that $\mathcal{A}$ itself is the unique 1-cluster-tilting subcategory of $\mathcal{A}$.

Remark 2.3.15. Let $\mathcal{A}$ be an abelian category and $\mathcal{M}$ an n-cluster-tilting subcategory of $\mathcal{A}$. Since $\mathcal{M}$ is a cogenerating subcategory of $\mathcal{A}$, for all $A \in \mathcal{A}$ each left $\mathcal{M}$-approximation of A is a monomorphism.

The following result provides us with a way of recognizing n-abelian categories.

Theorem 2.3.16. *Let $\mathcal{A}$ be an abelian category and $\mathcal{M}$ an n-cluster-tilting subcategory of $\mathcal{A}$. Then, $\mathcal{M}$ is an n-abelian category.*

To prove Theorem 2.3.16 we need the following technical results.

Proposition 2.3.17. *Let $\mathcal{A}$ be an abelian category and $\mathcal{M}$ an n-cluster-tilting subcategory of $\mathcal{A}$. Then, for all $A \in \mathcal{A}$ there exists an exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[rr, "f^0"] \arrow[dr, equal] & & M^1 \arrow[r, "f^1"] & \cdots \arrow[rr, "f^{n-2}"] \arrow[dr, "g^{n-2}"'] & & M^{n-1} \arrow[rr, "f^{n-1}"] \arrow[dr, "g^{n-1}"] & & M^n \arrow[r] & 0 \\
& & C^1 \arrow[ur, "h^1"'] & & & C^{n-1} \arrow[ur, "h^{n-1}"] & & C^n \arrow[ur, equal] & &
\end{tikzcd}$$

satisfying the following properties:

(i) For each $k \in \{1, \dots, n\}$ we have $M^k \in \mathcal{M}$.
(ii) For each $k \in \{1, \dots, n-1\}$ the morphism $h^k\colon C^k \to M^k$ is a left $\mathcal{M}$-approximation.

(iii) For each $k \in \{1, \dots, n-1\}$ the morphism $g^k \colon M^k \to C^{k+1}$ is a cokernel of $h^k \colon C^k \to M^k$.

(iv) For all $M \in \mathcal{M}$ the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{A}(M^n, M) \longrightarrow \cdots \longrightarrow \mathcal{A}(M^1, M) \longrightarrow \mathcal{A}(A, M) \longrightarrow 0$$

is exact.

Proof. This proof is an adaptation of the proof of [**56**, Thm. 2.2.3]. Given that for all $k \in \{1, \dots, n-1\}$ the morphism $h^k \colon C^k \to M^k$ is a left $\mathcal{M}$-approximation, it readily follows that the sequence

$$0 \longrightarrow \mathcal{A}(M^n, M) \longrightarrow \cdots \longrightarrow \mathcal{A}(M^1, M) \longrightarrow \mathcal{A}(A, M) \longrightarrow 0$$

is exact. It remains to show that $C^n \in \mathcal{M}$.

We claim that for each $M \in \mathcal{M}$ and each $k \in \{2, \dots, n\}$ we have $\mathsf{Ext}^i_{\mathcal{A}}(C^k, M) = 0$ for all $1 \leq i \leq k-1$. We proceed by induction on k. First, note that for all $M \in \mathcal{M}$ applying the contravariant functor $\mathcal{A}(-, M)$ to the exact sequence $0 \to A \xrightarrow{f^0} M^2 \to C^2$ we have an exact sequence

$$\mathcal{A}(M^1, M) \xrightarrow{f^0 \cdot ?} \mathcal{A}(A, M) \longrightarrow \mathsf{Ext}^1_{\mathcal{A}}(C^2, M) \longrightarrow \mathsf{Ext}^1_{\mathcal{A}}(M^1, M) = 0.$$

Moreover, the morphism $\mathcal{A}(M^1, M) \to \mathcal{A}(A, M)$ is an epimorphism for f^0 is a left $\mathcal{M}$-approximation of $\mathcal{A}$. Thus we have $\mathsf{Ext}^1_{\mathcal{A}}(C^2, M) = 0$ as required.

Let $2 \leq k \leq n-1$ and suppose that the claim holds for all $\ell \leq k$. Note that since $\mathcal{M}$ is a cogenerating subcategory of $\mathcal{A}$, the morphism h^k is a monomorphism. In particular, we have that h^k is a kernel of g^k. For all $M \in \mathcal{M}$ and for each $2 \leq i \leq k$, applying the contravariant functor $\mathcal{A}(-, M)$ to the exact sequence $0 \to C^k \to M^k \to C^{k+1} \to 0$ yields an exact sequence of the form

$$0 = \mathsf{Ext}^{i-1}_{\mathcal{A}}(C^k, M) \longrightarrow \mathsf{Ext}^i_{\mathcal{A}}(C^{k+1}, M) \longrightarrow \mathsf{Ext}^i_{\mathcal{A}}(M^k, M) = 0.$$

Therefore $\mathsf{Ext}^i_{\mathcal{A}}(C^{k+1}, M) = 0$ for all $2 \leq i \leq k$. Moreover, since h^k is a left $\mathcal{M}$-approximation of Y^k, the induced morphism $\mathcal{A}(M^k, M) \to \mathcal{A}(C^k, M)$ is an epimorphism. Hence, applying the contravariant functor $\mathcal{A}(-, M)$ to the exact sequence $0 \to C^k \to M^k \to C^{k+1} \to 0$ yields an exact sequence

$$\mathcal{A}(M^k, M) \longrightarrow \mathcal{A}(C^k, M) \longrightarrow \mathsf{Ext}^1_{\mathcal{A}}(C^{k+1}, M) \longrightarrow \mathsf{Ext}^1_{\mathcal{A}}(M^k, M) = 0$$

where the left morphism is an epimorphism. Thus $\mathsf{Ext}^1_{\mathcal{A}}(C^{k+1}, M) = 0$. This finishes the induction step. We have shown that for all $M \in \mathcal{M}$ we have $\mathsf{Ext}^i_{\mathcal{A}}(C^n, M) = 0$ for all $i \in \{1, \dots, n-1\}$. Since $\mathcal{M}$ is an n-cluster-tilting subcategory of $\mathcal{M}$, this implies $C^n = M^n \in \mathcal{M}$ as required. □

Proposition 2.3.18. *Let $\mathcal{A}$ be an abelian category, $B \in \mathcal{A}$, and $\mathcal{M}$ a subcategory of $\mathcal{A}$ such that $\mathsf{Ext}^k_{\mathcal{A}}(\mathcal{M}, B) = 0$ for all $k \in \{1, \dots, n-1\}$. Consider an exact sequence*

$$M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow A \longrightarrow 0$$

in $\mathcal{A}$ such that $M_k \in \mathcal{M}$ for all $k \in \{0, 1, \dots, n-1\}$. Then, for each $k \in \{1, \dots, n-1\}$ there is an isomorphism between $\mathsf{Ext}^k_{\mathcal{A}}(A, B)$ and the cohomology of the induced *complex*

$$\mathcal{A}(M_0,B) \longrightarrow \mathcal{A}(M_1,B) \longrightarrow \cdots \longrightarrow \mathcal{A}(M_{n-1},B) \longrightarrow \mathcal{A}(M_n,B) \tag{2.3.3}$$

at $\mathcal{A}(M_k,B)$.

Proof. Let $A_k := \mathsf{coker}(M_{k+1} \to M_k)$. Note that $A_0 = A$. Firstly, let us show that for each $k \in \{1,\dots,n-1\}$ there exist isomorphisms

$$\mathsf{Ext}^k_{\mathcal{A}}(A_0,B) \cong \mathsf{Ext}^{k-1}_{\mathcal{A}}(A_1,B) \cong \cdots \cong \mathsf{Ext}^1_{\mathcal{A}}(A_{k-1},B).$$

The case $k=1$ is obvious. If $2 \leq k \leq n-1$, then for each $2 \leq \ell \leq k$ applying the functor $\mathcal{A}(-,B)$ to the exact sequence $0 \to A_{k-\ell+1} \to M_{k-\ell} \to A_{k-\ell} \to 0$ yields an exact sequence

$$0 = {}^{\ell-1}_{\mathcal{A}}(M_{k-\ell},B) \to {}^{\ell-1}_{\mathcal{A}}(A_{k-\ell+1},B) \to {}^{\ell}_{\mathcal{A}}(A_{k-\ell},B) \to {}^{\ell}_{\mathcal{A}}(M_{k-\ell},B) = 0$$

where we omitted $\mathsf{Ext}_{\mathcal{A}}$ because of lack of space. The claim follows.

Secondly, let us show that $\mathsf{Ext}^1_{\mathcal{A}}(A_{k-1},B)$ is isomorphic to the cohomology of the complex (2.3.3) at $\mathcal{A}(M_k,B)$. This follows by definition from the commutative diagram

$$\begin{tikzcd}
\mathcal{A}(M_{k-1},B) \arrow[rr] \arrow[dr] & & \mathcal{A}(M_k,B) \arrow[r] & \mathcal{A}(M_{k+1},B) \\
& \mathcal{A}(A_k,B) \arrow[ur] \arrow[dr] & & \\
0 \arrow[ur] & & \mathsf{Ext}^1_{\mathcal{A}}(A_{k-1},B) \arrow[dr] & \\
& & & \mathsf{Ext}^1_{\mathcal{A}}(M_{k-1},B) = 0
\end{tikzcd}$$

which is the glueing of two exact sequences. This concludes the proof. □

Now we give the proof of Theorem 2.3.16.

Proof of Theorem 2.3.16. We shall show that $\mathcal{M}$ satisfies the axioms of Definition 2.3.1.

(A0) Since the abelian category $\mathcal{A}$ is idempotent complete, it follows immediately from the definition of n-cluster-tilting subcategory that $\mathcal{M}$ also is idempotent complete.

(A1) Let $f\colon A \to B$ be a morphism in $\mathcal{M}$. Let $B \to C$ be a cokernel of f, applying Proposition 2.3.17 to C gives the desired n-cokernel of f. By duality, f has an n-cokernel.

(A2) Let $f^0\colon X^0 \to X^1$ be a monomorphism in $\mathcal{A}$ such that $X^0, X^1 \in \mathcal{M}$ and let $\left(f^k\colon X^k \to X^{k+1} \mid 1 \leq k \leq n\right)$ be an n-cokernel of f^0 in $\mathcal{M}$ obtained as in the previous paragraph. Applying the dual of Proposition 2.3.18 to the exact sequence

$$0 \longrightarrow X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1},$$

we obtain that for all $Y \in \mathcal{M}$ and for all $k \in \{1,\dots,n-1\}$ the cohomology of the induced complex

$$\mathcal{A}(Y,X^1) \longrightarrow \cdots \longrightarrow \mathcal{A}(Y,X^n) \longrightarrow \mathcal{A}(Y,X^{n+1})$$

at $\mathcal{A}(Y,X^{k+1})$ is isomorphic to $\mathsf{Ext}^k_{\mathcal{A}}(Y,X^0)$ which vanishes since $\mathcal{M}$ is an n-cluster-tilting subcategory of $\mathcal{A}$. This shows that $(f^0,\dots,f^{n-1})$ is an n-kernel of f^n in $\mathcal{M}$. That $\mathcal{M}$ also satisfies axiom (A2$^{\mathrm{op}}$) now follows by duality. This concludes the proof of the theorem. □

DEFINITION 2.3.19. Let $\mathcal{M}$ be an n-abelian category. We say that $\mathcal{M}$ *is projectively generated* if for every object $M \in \mathcal{M}$ there exists a projective object $P \in \mathcal{M}$ and an epimorphism $P \to M$. The notion of *injectively cogenerated n-abelian category* is defined dually.

Our next aim is to show that a partial converse of Theorem 2.3.16 holds for projectively generated n-abelian categories. For this, we remind the reader of the notion of coherent modules on an additive category.

Let $\mathcal{C}$ be a *small* additive category. A *$\mathcal{C}$-module* is a contravariant functor $F\colon \mathcal{C} \to \mathsf{Mod}\,\mathbb{Z}$. The category $\mathsf{Mod}\,\mathcal{C}$ of $\mathcal{C}$-modules is an abelian category. Morphisms in $\mathsf{Mod}\,\mathcal{C}$ are natural transformations of contravariant functors. If $M, N \in \mathsf{mod}\,\mathcal{C}$, then we denote the set of natural transformations $M \to N$ by $\mathsf{Hom}_{\mathcal{C}}(M,N)$. As a consequence of Yoneda's lemma representable, functors are projective objects in $\mathsf{Mod}\,\mathcal{C}$. The *category of coherent $\mathcal{C}$-modules*, denoted by $\mathsf{mod}\,\mathcal{C}$, is the full subcategory of $\mathsf{Mod}\,\mathcal{C}$ whose objects are the $\mathcal{C}$-modules F such that there exists a morphism $f\colon X \to Y$ in $\mathcal{C}$ and an exact sequence of functors

$$\mathcal{C}(-,X) \xrightarrow{?\cdot f} \mathcal{C}(-,Y) \longrightarrow F \longrightarrow 0.$$

Note that $\mathsf{mod}\,\mathcal{C}$ is closed under cokernels and extensions in $\mathsf{Mod}\,\mathcal{C}$. Moreover, $\mathsf{mod}\,\mathcal{C}$ is closed under kernels in $\mathsf{Mod}\,\mathcal{C}$ if and only if $\mathcal{C}$ has weak kernels, in which case $\mathsf{mod}\,\mathcal{C}$ is an abelian category. For further information on coherent $\mathcal{C}$-modules we refer the reader to [**9**].

Our aim is to prove the following theorem.

THEOREM 2.3.20. *Let $\mathcal{M}$ be a small projectively generated n-abelian category. Let $\mathcal{P}$ be the category of projective objects in $\mathcal{M}$ and $F\colon \mathcal{M} \to \mathsf{mod}\,\mathcal{P}$ the functor defined by $FX := \mathcal{M}(-,X)|_{\mathcal{P}}$. Also, let*

$$F\mathcal{M} := \{M \in \mathsf{mod}\,\mathcal{P} \mid \exists X \in \mathcal{M} \text{ such that } M \cong FX\}$$

be the essential image of F. If $\mathsf{mod}\,\mathcal{P}$ is injectively cogenerated, then $F\mathcal{M}$ is an n-cluster-tilting subcategory of $\mathsf{mod}\,\mathcal{P}$.

REMARK 2.3.21. The requirement in Theorem 2.3.20 of $\mathsf{mod}\,\mathcal{P}$ being injectively cogenerated is satisfied, for example, if $\mathcal{P}$ is a dualizing variety in the sense of [**11**].

In fact, instead of proving Theorem 2.3.20, we prove the following more general statement.

LEMMA 2.3.22. *Let $\mathcal{M}$ be a small projectively generated n-abelian category. Let $\mathcal{P}$ be the category of projective objects in $\mathcal{M}$ and $F\colon \mathcal{M} \to \mathsf{mod}\,\mathcal{P}$ the functor defined by $FX := \mathcal{M}(-,X)|_{\mathcal{P}}$. Also, let*

$$F\mathcal{M} := \{M \in \mathsf{mod}\,\mathcal{P} \mid \exists X \in \mathcal{M} \text{ such that } M \cong FX\}$$

be the essential image of F. Then, the following statements hold:

(i) *The category $\mathsf{mod}\,\mathcal{P}$ is abelian.*
(ii) *The functor $F\colon \mathcal{M} \to \mathsf{mod}\,\mathcal{P}$ is fully faithful.*
(iii) *For all $k \in \{1, \dots, n-1\}$ we have $\mathsf{Ext}^k_{\mathcal{P}}(F\mathcal{M}, F\mathcal{M}) = 0$.*
(iv) *We have*

$$F\mathcal{M} = \Big\{X \in \mathsf{mod}\,\mathcal{P} \mid \forall k \in \{1, \dots, n-1\} \;\; \mathsf{Ext}^k_{\mathcal{P}}(X, \mathcal{M}) = 0\Big\}.$$

(v) *We have*

$$F\mathcal{M} = \Big\{X \in \mathsf{mod}\,\mathcal{P} \mid \forall k \in \{1, \dots, n-1\} \;\; \mathsf{Ext}^k_{\mathcal{P}}(\mathcal{M}, X) = 0\Big\}.$$

(vi) *The subcategory $F\mathcal{M}$ is contravariantly finite in $\mathsf{mod}\,\mathcal{P}$.*

(vii) If $\mathsf{mod}\,\mathcal{P}$ *is injectively cogenerated, then* $F\mathcal{M}$ *is covariantly finite in* $\mathsf{mod}\,\mathcal{P}$*. Hence,* $F\mathcal{M}$ *is a functorially finite subcategory of* $\mathsf{mod}\,\mathcal{P}$ *in this case.*

(viii) If $\mathsf{mod}\,\mathcal{P}$ *is injectively cogenerated, then* $F\mathcal{M}$ *is a generating-cogenerating subcategory of* $\mathsf{mod}\,\mathcal{P}$*.*

Proof. (i) This statements follows from the fact that $\mathcal{M}$ has weak kernels [**9**].

(ii) Let $M \in \mathcal{M}$. Since $\mathcal{M}$ is projectively generated, there exists an n-exact sequence

$$K_n \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 \xrightarrow{u} M. \tag{2.3.4}$$

where K^0 is a projective object. For the same reason, there exist a projective object P^1 and an epimorphism $v\colon P^1 \to K_1$. Let $f = vu$ and put $P^0 := K_0$. It follows that the sequence

$$FP^1 \xrightarrow{Ff} FP^0 \longrightarrow FM \longrightarrow 0$$

is exact in $\mathsf{Mod}\,\mathcal{P}$. This shows that $FM \in \mathsf{mod}\,\mathcal{P}$. That F is fully faithful follows from Yoneda's lemma and the existence of a sequence of the form $FP^1 \xrightarrow{Ff} FP^0 \to FM \to 0$ with $P^0, P^1 \in \mathcal{M}$ for each $M \in \mathcal{M}$.

(iii) Let us show that for every $M, N \in \mathcal{M}$ we have $\mathsf{Ext}^k_{\mathcal{P}}(FM, FN) = 0$ for all $k \in \{1, \dots, n-1\}$. Consider an n-exact sequence of the form (2.3.4). Applying F to (2.3.4) yields an exact sequence

$$0 \longrightarrow FK_n \longrightarrow FK_{n-1} \longrightarrow \cdots \longrightarrow FK_1 \longrightarrow FK_0 \longrightarrow FM \longrightarrow 0. \tag{2.3.5}$$

Since F is fully faithful there is an isomorphism of complexes

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{P}}(FK_0, FN) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathsf{Hom}_{\mathcal{P}}(FK_{n-1}, FN) \arrow[r] \arrow[d] & \mathsf{Hom}_{\mathcal{P}}(FK_n, FN) \arrow[d] \\
\mathcal{A}(K_0, N) \arrow[r] & \cdots \arrow[r] & \mathcal{A}(K_{n-1}, N) \arrow[r] & \mathcal{A}(K_n, N)
\end{tikzcd}$$

Note that the bottom row is exact by the property of n-exact sequences, hence the top row is also exact. Put $C_\ell := \mathsf{coker}(FK_{\ell+1} \to FK_\ell)$. Note that $C_0 := FM$. There is an exact sequence

$$\mathsf{Hom}_{\mathcal{P}}(FK_0, FN) \to \mathsf{Hom}_{\mathcal{P}}(C_1, FN) \to \mathsf{Ext}^1_{\mathcal{P}}(FM, FN) \to \mathsf{Ext}^1_{\mathcal{P}}(FK_0, FN) = 0$$

(recall that FK_0 is projective in $\mathsf{mod}\,\mathcal{P}$). Since the top row of the above diagram is exact, this implies that $\mathsf{Ext}^1_{\mathcal{P}}(FM, FN) = 0$. Hence we have $\mathsf{Ext}^1_{\mathcal{P}}(F\mathcal{M}, F\mathcal{M}) = 0$. We shall that we have a sequence of isomorphisms

$$\mathsf{Ext}^\ell_{\mathcal{P}}(C_0, FN) \cong \mathsf{Ext}^{\ell-1}_{\mathcal{P}}(C_1, FN) \cong \cdots \cong \mathsf{Ext}^1_{\mathcal{P}}(C_{\ell-1}, FN) = 0$$

for all $\ell \in \{1, \dots, n-1\}$. If $\ell = 1$, then the claim is trivial. Inductively, suppose that we have shown that $\mathsf{Ext}^m_{\mathcal{P}}(F\mathcal{M}, F\mathcal{M}) = 0$ for all $1 \le m \le \ell - 1$. Firstly, note that applying the functor $\mathsf{Hom}_{\mathcal{P}}(-, FN)$ to the exact sequence $0 \to C_\ell \to FK_{\ell-1} \to C_{\ell-1} \to 0$ gives an exact sequence

$$\mathsf{Hom}_{\mathcal{P}}(FK_{\ell-1}, FN) \to \mathsf{Hom}_{\mathcal{P}}(C_\ell, FN) \to \mathsf{Ext}^1_{\mathcal{P}}(C_{\ell-1}, FN) \to \mathsf{Ext}^1_{\mathcal{P}}(FK_{\ell-1}, FN) = 0,$$

Since the top row of the above diagram is exact, this implies that $\mathsf{Ext}^1_{\mathcal{P}}(C_{\ell-1}, FN) = 0$. Secondly, by the induction hypothesis, for each $1 \le m \le \ell - 1$ applying the functor $\mathsf{Hom}_{\mathcal{P}}(-, FN)$ to the exact sequence $0 \to C_{\ell-m} \to FK_{\ell-m-1} \to C_{\ell-m-1} \to 0$ yields an exact sequence

$$0 = {}^{m}_{\mathcal{P}}(FK_{\ell-m-1}, FN) \to {}^{m}_{\mathcal{P}}(C_{\ell-m}, FN) \to {}^{m+1}_{\mathcal{P}}(C_{\ell-m-1}, FN) \to {}^{m+1}_{\mathcal{P}}(FK_{\ell-m-1}, FN) = 0$$

where we omitted Ext because of lack of space (for $m = \ell - 1$, recall that FK_0 is projective in $\mathsf{mod}\,\mathcal{P}$). The claim follows.

(iv) Let $G \in \mathsf{mod}\,\mathcal{P}$ be such that for all $N \in \mathcal{M}$ and for all $k \in \{1, \dots, n-1\}$ we have $\mathsf{Ext}^k_{\mathcal{P}}(G, FN) = 0$. We need to show that there exists $M \in \mathcal{M}$ such that G and FM are isomorphic. For this, let

$$FP_1 \xrightarrow{Ff_0} FP_0 \longrightarrow G \longrightarrow 0$$

be a projective presentation of G in $\mathsf{mod}\,\mathcal{P}$ and let

$$X_n \xrightarrow{f_n} \cdots \longrightarrow X_1 \xrightarrow{f_1} X_0$$

be an n-kernel of f_0 (by convention, $X_0 := P_1$). Let $N \in \mathcal{M}$. Applying the functor $\mathsf{Hom}_{\mathcal{P}}(F(-), FN)$ to the sequence $(f_n, \dots, f_1, f_0)$ together with the fact that F is fully faithful yields a commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{P}}(FP_0, FN) \arrow[r] \arrow[d] & \mathsf{Hom}_{\mathcal{P}}(FX_0, FN) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathsf{Hom}_{\mathcal{P}}(FX_n, FN) \arrow[d] \\
\mathcal{A}(P_0, N) \arrow[r] & \mathcal{A}(X_0, N) \arrow[r] & \cdots \arrow[r] & \mathcal{A}(X_n, N)
\end{tikzcd}$$

where the vertical arrows are isomorphisms. By what we showed in the previous paragraph and Proposition 2.3.18, for all $k \in \{1, \dots, n-1\}$ the homology of the top row at $\mathsf{Hom}_{\mathcal{P}}(FX_{k-1}, FN)$ is isomorphic to $\mathsf{Ext}^k_{\mathcal{P}}(G, FN)$ which vanishes by hypothesis. It follows that the bottom row is an exact sequence. By applying Proposition 2.3.7 to f_{n-1} and the sequence $(f_{n-2}, \dots, f_1, f_0)$ we deduce that f_0 admits a cokernel $P_0 \to M$ in $\mathcal{M}$. Finally, Theorem 2.3.12 implies that the sequence

$$FP_1 \xrightarrow{Ff_0} FP_0 \longrightarrow FM \longrightarrow 0$$

is exact in $\mathsf{mod}\,\mathcal{P}$. Therefore G is isomorphic to FM which is what we needed to show.

(v) Let us show that if $G \in \mathsf{mod}\,\mathcal{P}$ is such that for all $M \in \mathcal{M}$ and for all $k \in \{1, \dots, n-1\}$ we have $\mathsf{Ext}^k_{\mathcal{P}}(FM, G) = 0$, then $G \in F\mathcal{M}$. Indeed, let

$$FP_1 \xrightarrow{Ff^0} FP_0 \longrightarrow G \longrightarrow 0$$

be a projective presentation of G in $\mathsf{mod}\,\mathcal{P}$. By axiom (A1), there exists an n-cokernel $\big(f^k \colon M^{k-1} \to M^k \mid 1 \leq k \leq n\big)$ of f^0 in $\mathcal{M}$ (by convention, $M^0 := P^0$). Then, Theorem 2.3.12 implies that the sequence

$$FP^1 \xrightarrow{Ff^0} FP^0 \xrightarrow{Ff^1} FM^1 \xrightarrow{Ff^2} \cdots \xrightarrow{Ff^n} FM^n \longrightarrow 0$$

is exact in $\mathsf{mod}\,\mathcal{P}$. It follows that there is an exact sequence

$$0 \longrightarrow G \longrightarrow FM^1 \xrightarrow{Ff^2} \cdots \xrightarrow{Ff^n} FM^n \longrightarrow 0.$$

For each $k \in \{1, \dots, n-1\}$ let $G^k := \ker Ff^{k+1}$; we claim that $G^k \in F\mathcal{M}$. Note that $\mathsf{Ext}^1_{\mathcal{P}}(FM^k, G^k) = 0$ by hypothesis. In particular, G^{n-1} is a direct summand of FM^{n-1}. Since $\mathcal{M}$ is idempotent complete, there exists an object $L \in \mathcal{M}$ such that $G^{n-1} \cong FL \in F\mathcal{M}$. Inductively, we deduce that $G^1 = G \in F\mathcal{M}$.

(vi) Let $G \in \mathsf{mod}\,\mathcal{P}$ and take a projective presentation

$$FP^1 \xrightarrow{Ff} FP^0 \xrightarrow{p} G \longrightarrow 0$$

of G in $\mathsf{mod}\,\mathcal{P}$. Let $g\colon P^0 \to M$ be a weak cokernel of f in $\mathcal{M}$. We obtain the solid part of the following commutative diagram:

$$\begin{tikzcd}
FP^1 \arrow[r, "Ff"] & FP^0 \arrow[r, "Fg"] \arrow[d, "p"] & FM \\
& G \arrow[ur] \arrow[r, dotted, "h"] & FN
\end{tikzcd}$$

Let $h\colon G \to FN$ be a morphism in $\mathsf{mod}\,\mathcal{P}$. Since g is a weak cokernel there exists a morphism $? \cdot q\colon FM \to FN$ such that the diagram

$$\begin{tikzcd}
FP^0 \arrow[r, "Fg"] \arrow[d, "p"] & FM \arrow[d, "Fq"] \\
G \arrow[r, "h"] & FN
\end{tikzcd}$$

is commutative. Finally, given that p is an epimorphism, we conclude that the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[dr, "h"] & FM \arrow[d, "Fq"] \\
& FN
\end{tikzcd}$$

commutes. This shows that $G \to FM$ is a right $F\mathcal{M}$-approximation of G.

(vii) and (viii) Suppose that $\mathsf{mod}\,\mathcal{P}$ is injectively cogenerated. From part (iii) we deduce that for every injective object $I \in \mathsf{mod}\,\mathcal{P}$ we have $I \in F\mathcal{M}$. Hence (vii) follows by duality from part (vi). Also, (viii) follows since it is now clear that $F\mathcal{M}$ is a cogenerating subcategory of $\mathsf{mod}\,\mathcal{P}$. □

2.4. n-exact categories

In this section we introduce n-exact categories and establish their basic properties. We show that n-cluster-tilting subcategories of exact categories have a natural n-exact structure.

2.4.1. Definition and basic properties. The treatment of this section is parallel to Bühler's exposition of the basics of the theory of exact categories given in [**29**, Sec. 2].

Let $\mathcal{C}$ be an additive category. If $\mathcal{X}$ is a class of n-exact sequences in $\mathcal{C}$, then we call its members $\mathcal{X}$-*admissible n-exact sequences*. Furthermore, if

$$X^0 \overset{d^0}{\rightarrowtail} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \overset{d^n}{\twoheadrightarrow} X^{n+1}$$

is an $\mathcal{X}$-admissible n-exact sequence, we say that d^0 is an $\mathcal{X}$-*admissible monomorphism* and that d^n is an $\mathcal{X}$-*admissible epimorphism*. In analogy with [**29**], we depict $\mathcal{X}$-admissible monomorphisms by $\rightarrowtail$ and $\mathcal{X}$-admissible epimorphisms by $\twoheadrightarrow$. A sequence $\rightarrowtail\to\cdots\to\twoheadrightarrow$ of $n+1$ morphisms always denotes an $\mathcal{X}$-admissible n-exact sequence. When the class $\mathcal{X}$ is clear from the context, we write "admissible" instead of "$\mathcal{X}$-admissible".

Definition 2.4.1. We say that a morphism $f\colon X \to Y$ of n-exact sequences in $\mathsf{Ch}^n(\mathcal{C})$ is a *weak isomorphism* if f^k and f^{k+1} are isomorphisms for some $k \in \{0, 1, \dots, n+1\}$ with $n+2 := 0$ (this terminology is borrowed from the theory of $(n+2)$-angulated categories, see Section 2.5.1). Note that weak isomorphisms induce isomorphisms in $\mathsf{H}(\mathcal{C})$ by Proposition 2.2.7.

Definition 2.4.2. Let n be a positive integer and $\mathcal{M}$ an additive category. An *n-exact structure on* $\mathcal{M}$ is a class $\mathcal{X}$ of n-exact sequences in $\mathcal{M}$, closed under weak isomorphisms of n-exact sequences, and which satisfies the following axioms:

(E0) The sequence $0 \rightarrowtail 0 \to \cdots \to 0 \twoheadrightarrow 0$ is an $\mathcal{X}$-admissible n-exact sequence.

(E1) The class of $\mathcal{X}$-admissible monomorphisms is closed under composition.

(E1$^{\mathrm{op}}$) The class of $\mathcal{X}$-admissible epimorphisms is closed under composition.

(E2) For each $\mathcal{X}$-admissible n-exact sequence X and each morphism $f\colon X^0 \to Y^0$, there exists an n-pushout diagram of $(d_X^0, \dots, d_X^{n-1})$ along f such that d_Y^0 is an $\mathcal{X}$-admissible monomorphism. The situation is illustrated in the following commutative diagram:

$$\begin{tikzcd}
X^0 \arrow[r, tail, "d_X^0"] \arrow[d, "f"] & X^1 \arrow[r, "d_X^1"] \arrow[d, dotted] & \cdots \arrow[r, "d_X^{n-1}"] & X^n(\arrow[d, dotted] \arrow[r, two heads, "d_X^{n+1}"] & X^{n+1}) \\
Y^0 \arrow[r, tail, dotted, "d_Y^0"] & Y^1 \arrow[r, dotted, "d_Y^1"] & \cdots \arrow[r, dotted, "d_Y^{n-1}"] & Y^n &
\end{tikzcd}$$

(E2$^{\mathrm{op}}$) For each $\mathcal{X}$-admissible n-exact sequence X and each morphism $g\colon Y^{n+1} \to X^{n+1}$, there exists an n-pullback diagram of $(d_X^1, \dots, d_X^n)$ along g such that d_Y^n is an $\mathcal{X}$-admissible epimorphism. The situation is illustrated in the following commutative diagram:

$$\begin{tikzcd}
& Y^1 \arrow[r, dotted, "d_Y^1"] \arrow[d, dotted] & \cdots \arrow[r, dotted, "d_Y^{n^1}"] & Y^n \arrow[r, dotted, two heads, "d_Y^n"] \arrow[d, dotted] & Y^{n+1} \arrow[d, "g"] \\
(X^0 \arrow[r, tail, "d_X^0"] &)X^1 \arrow[r, "d_X^1"] & \cdots \arrow[r, "d_X^{n-1}"] & X^n \arrow[r, two heads, "d_X^n"] & X^{n+1}
\end{tikzcd}$$

An *n-exact category* is a pair $(\mathcal{M}, \mathcal{X})$ where $\mathcal{M}$ is an additive category and $\mathcal{X}$ is an n-exact structure on $\mathcal{M}$. If the class $\mathcal{X}$ is clear from the context, we identify $\mathcal{M}$ with the pair $(\mathcal{M}, \mathcal{X})$.

Remark 2.4.3. Our choice of axioms for n-exact categories is inspired by Keller's minimal list of axioms for exact categories [**66**, App. A], although we opt for a more convenient self-dual collection. In particular, we point out to the reader who is more familiar with Quillen's axioms that the so-called "obscure axiom", axiom c) of [**79**, Sec. 2], is redundant, see [**29**, p. 4, Prop. 2.16].

Remark 2.4.4. Let $(\mathcal{M}, \mathcal{X})$ be an n-exact category. The fact that $\mathcal{X}$ is closed under isomorphisms in $\mathsf{H}(\mathcal{M})$ together with axiom (E0) implies that all contractible n-exact sequences are admissible. Moreover, it is easy to show that the class of all contractible n-exact sequences in an additive category $\mathcal{M}$ is an n-exact structure; in fact, this is the smallest n-exact structure on $\mathcal{M}$. In particular, every additive category can be considered as an n-exact category with a "contractible n-exact structure". integer n.

The following result shows that n-abelian categories have a canonical n-exact structure. Therefore the class of n-exact categories contains the class of n-abelian categories.

Theorem 2.4.5. *Let $\mathcal{M}$ be an n-abelian category and $\mathcal{X}$ the class of all n-exact sequences in $\mathcal{M}$. Then, $(\mathcal{M}, \mathcal{X})$ is an n-exact category.*

PROOF. We shall show that $(\mathcal{M},\mathcal{X})$ satisfies the axioms of Definition 2.4.2. It is obvious that the class $\mathcal{X}$ is closed under weak isomorphisms and that axiom (E0) is satisfied. By axioms (A1) and (A2) in Definition 2.3.1, every monomorphism in $\mathcal{M}$ is the leftmost morphism in an n-exact sequence, see Remark 2.3.3. Since the composition of two monomorphisms is again a monomorphism, axiom (E1) is satisfied. That axiom (E1$^{\mathrm{op}}$) is also satisfied then follows by duality. Finally, Theorem 2.3.8 and its dual show that axioms (E2) and (E2$^{\mathrm{op}}$) are satisfied. This shows that $(\mathcal{M},\mathcal{X})$ is an n-exact category. □

We begin our investigation of the properties of n-exact categories with a simple but useful observation.

LEMMA 2.4.6. *Let $(\mathcal{M},\mathcal{X})$ be an n-exact category and $X^0 \rightarrowtail X^1$ and admissible monomorphism. If the sequence $\big(X^k \to X^{k+1} \mid 1 \leq k \leq n\big)$ is an n-cokernel of f^0, then the sequence*

$$X : X^0 \rightarrowtail X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow X^{n+1}$$

is an admissible n-exact sequence.

PROOF. Since $X^0 \rightarrowtail X^1$ is an admissible monomorphism, there exists an admissible n-exact sequence Y whose first morphism is $X^0 \rightarrowtail X^1$. By the factorization property of n-cokernels, there exists a weak isomorphism $X \to Y$. Then, $X \in \mathcal{X}$ since the class $\mathcal{X}$ is closed under weak isomorphisms. □

The next result shows that the n-exact structure of an n-exact category is closed under direct sums.

PROPOSITION 2.4.7. *Let $(\mathcal{M},\mathcal{X})$ be an n-exact category, and X and Y be admissible n-exact sequences. Then, their direct sum $X \oplus Y$ is an admissible n-exact sequence.*

PROOF. This is an adaptation of the proof of [**29**, Prop. 2.9]. Clearly, $X \oplus Y$ is an n-exact sequence. We claim that $d_X^0 \oplus 1_{Y^0}$ is an admissible monomorphism. Indeed, the sequence

$$X^0 \oplus Y^0 \xrightarrow{d_X^0 \oplus 1_{Y^0}} X^1 \oplus Y^0 \xrightarrow{(d_X^1\ 0)} X^2 \xrightarrow{d_X^2} \cdots \xrightarrow{d_X^n} \twoheadrightarrow X^{n+1}$$

is an n-exact sequence. Since d_X^n is an admissible epimorphism, it follows from the dual of Lemma 2.4.6 that this sequence is moreover an admissible n-exact sequence and therefore $d_X^0 \oplus 1_{Y^0}$ is an admissible monomorphism. We can show that $1_{X^1} \oplus d_Y^0$ is an admissible monomorphism with a similar argument. Next, observe that

$$d_X^0 \oplus d_Y^0 = (d_X^0 \oplus 1_{Y^0}) \cdot (1_{X^1} \oplus d_Y^0).$$

By axiom (E1) we have that $d_X^0 \oplus d_Y^0 \colon X^0 \oplus Y^0 \to X^1 \oplus Y^1$ is an admissible monomorphism. Since $X \oplus Y$ is an admissible n-exact sequence, the claim follows from Lemma 2.4.6. □

The following characterization of n-pushout diagrams of n-exact sequences should be compared with [**29**, Prop. 2.12].

PROPOSITION 2.4.8. *Let $(\mathcal{M},\mathcal{X})$ be an n-exact category. Suppose that we are given a commutative diagram*

$$\begin{tikzcd} X^0 \arrow[r, tail, "d_X^0"] \arrow[d, "f^0"] & X^1 \arrow[r, "d_X^1"] \arrow[d, "f^1"] & \cdots \arrow[r, "d_X^{n-1}"] & X^n \arrow[d, "f^n"] (\arrow[r, two heads, "d_X^n"] & X^{n+1}) \\ Y^0 \arrow[r, tail, "d_Y^0"] & Y^1 \arrow[r, "d_Y^1"] & \cdots \arrow[r, "d_Y^{n-1}"] & Y^n \end{tikzcd} \tag{2.4.1}$$

in which the top row is an admissible n-exact sequence and d_Y^0 is an admissible monomorphism. Then the following statements are equivalent:

(i) *Diagram* (2.4.1) *is an n-pushout diagram.*
(ii) *The mapping cone of diagram* (2.4.1) *is an admissible n-exact sequence.*
(iii) *Diagram* (2.4.1) *is both an n-pushout and an n-pullback diagram.*
(iv) *There exists a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"] & X^0 \arrow[r, tail, "d_X^0"] \arrow[d, "f^0"] & X^1 \arrow[r, "d_X^1"] \arrow[d, "f^1"] & \cdots \arrow[r, "d_X^{n-1}"] & X^n \arrow[r, two heads, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[d, equal] \\ Y & Y^0 \arrow[r, tail, "d_Y^0"] & Y^1 \arrow[r, "d_Y^1"] & \cdots \arrow[r, "d_Y^{n-1}"] & Y^n \arrow[r, two heads, dotted, "d_Y^n"] & X^{n+1} \end{tikzcd}$$

whose rows are admissible n-exact sequences.

Proof. (i) implies (ii). This is an adaptation of the proof of [**29**, Prop. 2.12]. Since the leftmost n squares in (2.4.1) form an n-pushout diagram, by definition its mapping cone gives an n-cokernel of the morphism

$$d_C^{-1} = [-d_X^0 \ f^0]^\top : X^0 \to X^1 \oplus Y^0.$$

Hence, by Lemma 2.4.6, it is sufficient to show that d_C^{-1} is an admissible monomorphism. For this, observe that d_C^{-1} equals the composition

$$X^0 \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} X^0 \oplus Y^0 \xrightarrow[\sim]{\begin{bmatrix} -1 & 0 \\ f^0 & 1 \end{bmatrix}} X^0 \oplus Y^0 \xrightarrow{\begin{bmatrix} d_X^0 & 0 \\ 0 & 1 \end{bmatrix}} X^1 \oplus Y^0$$

where the rightmost morphism is an admissible monomorphism by Proposition 2.4.7 and the remaining morphisms are admissible monomorphisms by Remark 2.4.4. Thus d_C^{-1} is an admissible monomorphism by axiom (E1).

That (ii) implies (iii) follows directly from the definitions of n-pushout and n-pullback diagrams. That (iii) implies (i) is obvious. Therefore statements (i), (ii) and (iii) are equivalent.

(i) implies (iv). We begin by constructing the morphism $d_Y^n : Y^n \to X^{n+1}$. Since d_C^{n-1} is a cokernel of d_C^{n-2}, there exists a unique morphism $d_Y^n : Y^n \to X^{n+1}$ such that $d_X^n = f^n d_Y^n$ and $d_Y^{n-1} d_Y^n = 0$, see (2.2.4). Since d_X^n is an epimorphism, it follows immediately that so is d_Y^n. It remains to show that d_Y^n is a cokernel of d_Y^{n-1}. For this, let $u : Y^n \to M$ be a morphism such that $d_Y^{n-1} u = 0$. Then we have that

$$d_X^{n-1}(f^n u) = (f^{n-1} d_Y^{n-1}) u = 0.$$

Since d_X^n is a cokernel of d_X^{n-1}, there exists a morphism $v : X^{n+1} \to M$ such that $f^n u = d_X^n v$. It follows that

$$f^n u = d_X^n v = f^n (d_Y^n v)$$

and

$$d_Y^{n-1} u = 0 = d_Y^{n-1}(d_Y^n v).$$

Since d_C^{n-1} is a cokernel of d_C^{n-2}, we have $u = d_Y^n v$. This shows that the epimorphism d_Y^n is a cokernel of d_Y^{n-1}.

Let $2 \leq k \leq n$. We need to show that d_Y^k is a weak cokernel of d_Y^{k-1}. Let $u : Y^k \to M$ be a morphism such that $d_Y^{k-1} u = 0$. Then we have

$$d_X^{k-1}(f^k u) = (f^{k-1} d_Y^{k-1}) u = 0.$$

Since d_X^k is a weak cokernel of d_X^{k-1}, there exists a morphism $v : X^{k+1} \to M$ such that $f^k u = d_X^k v$. Hence, given that d_C^{k-1} is a weak cokernel of d_C^{k-2}, there exists a morphism $w : Y^{k+1} \to M$ such that $d_Y^k w = u$, see (2.2.4). Therefore d_Y^k is a weak cokernel of d_Y^{k-1}, as required. This shows that $(d_Y^1, \dots, d_Y^n)$ is an n-cokernel of d_Y^0, so we have finished the construction of the required commutative diagram. Moreover, by Lemma 2.4.6, the sequence Y is an admissible n-exact sequence.

(iv) implies (ii). We assume that $n \geq 2$. The case $n = 1$ can be shown by combining the arguments below, and is easily found in the literature, see for example [**29**, Prop. 2.12]. By definition, we need to show that in the mapping cone $C = C(f)$ we have that $(d_C^0, d_C^1, \dots, d_C^n)$ is an n-cokernel of d_C^{-1}.

Let $2 \leq k \leq n$. We shall show that d_C^{k-1} is a weak cokernel of d_C^{k-2}, see (2.2.4). Indeed, let $u\colon Y^{k-1} \to M$ and $v\colon X^k \to M$ be morphisms such that $d_Y^{k-2}u = 0$ and $d_X^{k-1}v = f^{k-1}u$. Since d_Y^{k-1} is a weak cokernel of d_Y^{k-2}, there exists a morphism $w\colon Y^k \to M$ such that $u = d_Y^{k-1}w$. Moreover, note that

$$d_X^{k-1}v = f^{k-1}u = f^{k-1}(d_Y^{k-1}w) = d_X^{k-1}(f^k w)$$

for f is a morphism of complexes. Given that $d_X^k\colon X^k \to X^{k+1}$ is a weak cokernel of $d_X^{k-1}\colon X^{k-1} \to X^k$, there exists a morphism $h^{k+1}\colon X^{k+1} \to M$ such that $f^k w - v = d_X^k h^{k+1}$. If $k \neq n$, then the claim follows. If $k = n$, then let $w' := w - d_Y^n h^{n+1}$. It follows that $d_Y^{n-1}w' = d_Y^{n-1}w = u$ and

$$f^n w' = f^n w - f^n d_Y^n h^{n+1} = v + d_X^n h^{k+1} - d^n h^{k+1} = v.$$

This shows that d_C^{n-1} is a weak cokernel of d_C^{n-2}.

We need to show that d_C^{n-1} is a cokernel of d_C^{n-2}. For this it is enough to show that d_C^{n-1} is an epimorphism. Let $p\colon Y^n \to M$ be a morphism such that $d_Y^{n-1}p = 0$ and $f^n p = 0$. Then, since d_Y^n is a cokernel of d_Y^{n-1}, there exists a morphism $q\colon X^{n+1} \to M$ such that $p = d_Y^n q$. Thus,

$$d_X^n q = (f^n d_Y^n)q = f^n p = 0.$$

Since d_X^n is an epimorphism, we have $q = 0$ which implies that $p = 0$. This shows that d_C^{n-1} is an epimorphism.

It remains to show that d_C^0 is a weak cokernel of d_C^{-1}. Let $y\colon Y^0 \to Z^0$ and $v\colon X^0 \to Z^0$ be morphisms such that $f^0 u = d_X^0 v$. By axiom (E2) there exists an n-pushout diagram of $(d_Y^0, \dots, d_Y^{n-1})$ along u. Moreover, since we have shown the implication from (i) to (iv), we can construct a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & X^0 \arrow[r, tail] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^n \arrow[r, two heads] \arrow[d] & X^{n+1} \arrow[d, equal] \\
Y \arrow[d, "g"] & Y^0 \arrow[r, tail] \arrow[d, "u"] & Y^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Y^n \arrow[r, two heads] \arrow[d] & X^{n+1} \arrow[d, equal] \\
Z & Z^0 \arrow[r, tail] & Z^1 \arrow[r] & \cdots \arrow[r] & Z^n \arrow[r, two heads, dotted] & X^{n+1}
\end{tikzcd}$$

where the leftmost n squares of the two bottom rows form a pushout diagram. It follows that the following diagram commutes

$$\begin{tikzcd}
X^0 \arrow[r, tail, "d_X^0"] \arrow[d, "0"] & X^1 \arrow[r, "d_X^1"] \arrow[d, "f^0u - vd_Z^0"] & X^2 \arrow[r] \arrow[d, "f^2g^2"] & \cdots \arrow[r] & X^n \arrow[r, two heads, "d_X^n"] \arrow[d, "f^n\gamma^n"] & X^{n+1} \arrow[d, equal] \\
Z^0 \arrow[r, tail, "d_Z^0"] & Z^1 \arrow[r, "d_Z^1"] & Z^2 \arrow[r] & \cdots \arrow[r] & Z^n \arrow[r, two heads] & X^{n+1}
\end{tikzcd}$$

Then, by the Comparison Lemma 2.2.1, there exists a morphism $h\colon X^{n+1} \to Z^n$ such that $hd_Z^n = 1_{X^{n+1}}$. Therefore d_Z^{n+1} is a split epimorphism. From the dual of Proposition 2.2.6 we conclude that d_Z^0 is a split monomorphism. It follows that there exists a morphism $w\colon Z^1 \to Z^0$ such that $d_Z^0 w = 1_{Z^0}$. Finally,

$$d_Y^0(g^1 w) = u(d_Z^0 w) = u$$

and

$$f^0(uw) - d_X^0(vw) = (f^0 u - d_X^0 v)w = (v d_Z^0)w = v$$

which is what we needed to show. This concludes the proof. □

The following property is a refinement of Proposition 2.2.13.

PROPOSITION 2.4.9. *Let $(\mathcal{M}, \mathcal{X})$ be an n-exact category and $g\colon X \to Z$ a morphism of admissible n-exact sequences. Then, for every morphism of admissible n-exact sequences $g\colon X \to Y$ there exists a commutative diagram*

$$\begin{tikzcd}
X \arrow[d, "f"] & X^0 \arrow[r, tail] \arrow[d, "g^0"] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^n \arrow[r, two heads] \arrow[d] & X^{n+1} \arrow[d, equal] \\
Y \arrow[d, "p"] & Y^0 \arrow[r, tail] \arrow[d, equal] & Y^1 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Y^n \arrow[r, two heads] \arrow[d, dotted] & Y^{n+1} \arrow[d, "g^{n+1}"] \\
Z & Z^0 \arrow[r, tail] & Z^1 \arrow[r] & \cdots \arrow[r] & Z^n \arrow[r, two heads] & Z^{n+1}
\end{tikzcd}$$

where $f^0 = g^0$ and $p^{n+1} = g^{n+1}$. Moreover, there exists a homotopy $h\colon fp \to g$ with $h^1 = 0$ and $h^{n+1} = 0$.

PROOF. By Propositions 2.2.13 and 2.4.8, we only need to show that $p^n d_Z^n = d_Y^n g^{n+1}$. Indeed, on one hand we have

$$f^n(g^n d_Z^n) = d_X^n g^{n+1} = f^n(d_Y^n g^{n+1}).$$

On the other hand,

$$d_Y^{n-1}(g^n d_Z^n) = g^{n-1}(d_Z^{n-1} d_Z^n) = 0 = d_Y^{n-1}(d_Y^n g^{n+1}).$$

Since in the mapping cone $C(f)$ we have that d_C^{n-1} is a cokernel of d_C^{n-2}, we have $p^n d_Z^n = d_Y^n g^{n+1}$, see (2.2.4). This concludes the proof. □

The next result shows that, in an n-exact category, equivalences of admissible n-exact sequences induce isomorphisms in the homotopy category, *cf.* Proposition 2.2.7 and the remark after it.

PROPOSITION 2.4.10. *Let $(\mathcal{M}, \mathcal{X})$ be an n-exact category, $f\colon X \to Y$ an equivalence of admissible n-exact sequences. Then, there exists an equivalence of n-exact sequences $g\colon Y \to X$ such that f and g are mutually inverse isomorphisms in $\mathsf{H}(\mathcal{M})$.*

PROOF. By Proposition 2.4.8, the mapping cone $C = C(f)$ of the diagram

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d, equal] & X^1 \arrow[r] \arrow[d, "f^1"] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "f^{n-1}"] & X^n \arrow[d, "f^n"] \\
Y^0 \arrow[r, tail] & Y_1 \arrow[r] & \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n
\end{tikzcd}$$

is an admissible n-exact sequence. Since $d_C^{-1}\colon X^0 \to X^1 \oplus X^0$ is a split monomorphism, C is a contractible n-exact sequence, see Proposition 2.2.6. Hence there exists a null-homotopy $h\colon 1_C \to 0$. It is straightforward to verify that h induces an equivalence of admissible n-exact sequences $g\colon Y \to X$. Finally, the Comparison Lemma 2.2.1 implies that f and g induce mutually inverse isomorphisms in $\mathsf{H}(\mathcal{M})$. □

By axiom (E1), the class of admissible morphisms in an n-exact category is closed under composition. The next result, an analog of Quillen's obscure axiom for exact categories, shows that a partial converse also holds, *cf.* [**29**, Prop. 2.16].

Proposition 2.4.11 (Obscure axiom). *Let $(\mathcal{M}, \mathcal{X})$ be an n-exact category. Suppose there is a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"] & X^0 \arrow[r] \arrow[d, equal] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[d] \\ Y & Y^0 \arrow[r, tail] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^n \arrow[r, two heads] & Y^{n+1} \end{tikzcd} \tag{2.4.2}$$

where the bottom row is an admissible n-exact sequence and $(d_X^1, \dots, d_X^n)$ is an n-cokernel of d_X^0. Then, the top row is also an admissible n-exact sequence.

Proof. This is an adaptation of the proof of [**29**, Prop. 2.16], which is due to Keller. Since a pushout of the bottom row along d_X^0 exists, by Proposition 2.4.8 the morphism $[-d_Y^0 \; d_X^0]^\top : X^0 \to Y^1 \oplus X^1$ is an admissible monomorphism. Moreover, the diagram

$$\begin{tikzcd} X^0 \arrow[r, tail, "{\begin{bmatrix} -d_Y^0 & d_X^0 \end{bmatrix}^\top}"] \arrow[d, equal] & Y^1 \oplus X^1 \arrow[d, tail, "{\begin{bmatrix} 1 & f^1 \\ 0 & 1 \end{bmatrix}}"] \\ X^0 \arrow[r, "{\begin{bmatrix} 0 & d_X^0 \end{bmatrix}^\top}"] & Y^1 \oplus X^1 \end{tikzcd}$$

commutes. Hence $[0 \; d_X^0]^\top : X^0 \to Y^1 \oplus X^1$ is the composition of an admissible monomorphism with an isomorphism and, by axiom (E1) and Remark 2.4.4, we conclude that itself is an admissible monomorphism. Using the factorization property of weak cokernels we can construct a commutative diagram

$$\begin{tikzcd} X \arrow[d, "p"] & X^0 \arrow[r, "d_X^0"] \arrow[d, equal] & X^1 \arrow[r] \arrow[d, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"] & X^3 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & X^{n+1} \arrow[d, dotted] \\ Z \arrow[d, "q"] & X^0 \arrow[r, tail, "{\begin{bmatrix} 0 & d_X^0 \end{bmatrix}^\top}"] \arrow[d, equal] & Y^1 \oplus X^1 \arrow[r] \arrow[d, "{\begin{bmatrix} 0 & 1 \end{bmatrix}}"] & Z^3 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r, two heads] & Z^{n+1} \arrow[d, dotted] \\ X & X^0 \arrow[r, "d_X^0"] & X^2 \arrow[r] & X^3 \arrow[r] & \cdots \arrow[r] & X^{n+1} \end{tikzcd}$$

where the middle row is an admissible n-exact sequence. By the Comparison Lemma 2.2.1, the morphisms p and q induce mutually inverse isomorphisms in $\mathsf{H}(\mathcal{M})$. By Proposition 2.2.5 we have that X is an n-exact sequence. Then, Proposition 2.4.9 gives an equivalence between X and an admissible n-exact which is obtain by n-pullback from Y along f^{n+1}. Since $\mathcal{X}$ is closed under weak isomorphisms (in particular, equivalences) of n-exact sequences, we have that X is moreover an admissible n-exact sequence. $\square$

The following result shows that the class of n-exact sequences in an n-exact category is closed under direct summands.

Proposition 2.4.12. *Let $(\mathcal{M}, \mathcal{X})$ be an n-exact category, and X and Y complexes in $\mathcal{C}^n(\mathcal{C})$. If $X \oplus Y$ is an n-exact sequence, then so are X and Y.*

PROOF. This is an adaptation of the proof of [**29**, Cor. 2.18]. Clearly, X and Y are n-exact sequences. In particular, d_X^0 admits an n-cokernel. Moreover, we have a commutative diagram

$$\begin{tikzcd}
X^0 \arrow[rr, "d_X^0"] \arrow[d, equal] & & X^1 \arrow[d, "\begin{bmatrix} 1_{X^1} \\ 0 \end{bmatrix}"] \\
X_1 \arrow[r, tail, "\begin{bmatrix} 1_{X^0} \\ 0 \end{bmatrix}"] & X^0 \oplus X^1 \arrow[r, tail, "d_X^0 \oplus d_Y^0"] & X^1 \oplus Y^1
\end{tikzcd}$$

We conclude from Proposition 2.4.11 that X is an admissible n-exact sequence. Analogously one can show that Y is an admissible n-exact sequence. $\square$

2.4.2. n-exact categories and cluster-tilting. Let $(\mathcal{E}, \mathcal{X})$ be an exact category. Recall that a morphism $X \to Y$ in $\mathcal{E}$ is *proper* if it has a factorization $X \twoheadrightarrow Z \rightarrowtail Y$ through an $\mathcal{X}$-admissible epimorphism and an $\mathcal{X}$-admissible monomorphism. A sequence of proper morphisms

$$\begin{tikzcd}
\cdots \arrow[r] & X^{k-1} \arrow[rr] \arrow[dr, two heads] & & X^k \arrow[rr] \arrow[dr, two heads] & & X^{k+1} \arrow[r] & \cdots \\
& & Y^{k-1} \arrow[ur, tail] & & Y^k \arrow[ur, tail] & &
\end{tikzcd}$$

is an $\mathcal{X}$-*acyclic complex* if for each $k \in \mathbb{Z}$ the sequence $Y^{k-1} \rightarrowtail X^k \twoheadrightarrow Y^k$ is an $\mathcal{X}$-admissible exact sequence. If the class $\mathcal{X}$ is clear from the complex, then we may write “acyclic” instead of “$\mathcal{X}$-acyclic”.

Following Neeman [**78**], the derived category $\mathcal{D} = \mathsf{D}(\mathcal{E}, \mathcal{X})$ is by definition the Verdier quotient $\mathsf{H}(\mathcal{E})/\,\mathsf{thick}(\mathsf{Ac}(\mathcal{X}))$. Then, for all $k \geq 1$ and for all $E \in \mathcal{E}$ we can define the functor $\mathsf{Ext}^k_{\mathcal{X}}(E, -)\colon \mathcal{E} \to \mathsf{Mod}\,\mathbb{Z}$ by $F \mapsto \mathsf{Hom}_{\mathcal{D}}(E, F[k])$.

We use the following variant of the definition of n-cluster-tilting subcategory of an exact category. Note that in the case of abelian categories this definition agrees with Definition 2.3.14, see Remark 2.3.15.

DEFINITION 2.4.13. Let $(\mathcal{E}, \mathcal{X})$ be a small exact category and $\mathcal{M}$ a subcategory of $\mathcal{E}$. We say that $\mathcal{M}$ is an *n-cluster-tilting subcategory of* $(\mathcal{E}, \mathcal{X})$ if the following conditions are satisfied:

(i) Every object $E \in \mathcal{E}$ has a left $\mathcal{M}$-approximation by an $\mathcal{X}$-admissible monomorphism $E \rightarrowtail M$.
(ii) Every object $E \in \mathcal{E}$ has a right $\mathcal{M}$-approximation by an $\mathcal{X}$-admissible epimorphism $M' \twoheadrightarrow E$.
(iii) We have

$$\begin{aligned}\mathcal{M} &= \left\{E \in \mathcal{E} \mid \forall i \in \{1, \ldots, n-1\}\ \mathsf{Ext}^i_{\mathcal{X}}(E, \mathcal{M}) = 0\right\} \\ &= \left\{E \in \mathcal{E} \mid \forall i \in \{1, \ldots, n-1\}\ \mathsf{Ext}^i_{\mathcal{X}}(\mathcal{M}, E) = 0\right\}.\end{aligned}$$

Note that $\mathcal{E}$ itself is the unique 1-cluster-tilting subcategory of $\mathcal{E}$.

Our aim in this section is to prove the following result, which is analogous to Theorem 2.3.16 in the case of exact categories.

THEOREM 2.4.14. *Let $(\mathcal{E}, \mathcal{X})$ be an exact category and $\mathcal{M}$ an n-cluster-tilting subcategory of $(\mathcal{E}, \mathcal{X})$. Let $\mathcal{Y} = \mathcal{Y}(\mathcal{M}, \mathcal{X})$ be the class of all $\mathcal{X}$-acyclic complexes*

$$X^0 \rightarrowtail X^1 \longrightarrow \cdots \longrightarrow X^n \twoheadrightarrow X^{n+1} \tag{2.4.3}$$

such that for all $k \in \{0, 1, \ldots, n+1\}$ we have $X^k \in \mathcal{M}$. Then $(\mathcal{M}, \mathcal{Y})$ is an n-exact category.

To prove Theorem 2.4.14, we need the following result, whose proof is completely analogous to the proof of Proposition 2.3.17.

PROPOSITION 2.4.15. *Let $(\mathcal{E},\mathcal{X})$ be an exact category and $\mathcal{M}$ an n-cluster-tilting subcategory of $\mathcal{E}$. Then, for each $E \in \mathcal{E}$ there exists an acyclic complex sequence*

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, tail, "f^0"] \arrow[dr, equal] & M^1 \arrow[r, "f^1"] & \cdots \arrow[rr, "f^{n-2}"] \arrow[dr, two heads, "g^{n-2}"'] & & M^{n-1} \arrow[rr, two heads, "f^{n-1}"] \arrow[dr, two heads, "g^{n-1}"] & & M^n \arrow[r] & 0 \\
& & C^1 \arrow[u, tail, "h^1"'] & & C^{n-1} \arrow[ur, tail, "h^{n-1}"'] & & C^n \arrow[ur, equal] & &
\end{tikzcd}$$

satisfying the following properties:

(i) For each $k \in \{1,\dots,n\}$ we have $M^k \in \mathcal{M}$.

(ii) For each $k \in \{1,\dots,n-1\}$ the morphism $h^k \colon C^k \rightarrowtail M^k$ is an $\mathcal{X}$-admissible monomorphism and a left $\mathcal{M}$-approximation of C^k.

(iii) For each $k \in \{1,\dots,n-1\}$ the pair $C^k \rightarrowtail M^k \twoheadrightarrow C^{k+1}$ is an $\mathcal{X}$-admissible exact sequence.

We now give the proof of Theorem 2.4.14.

PROOF OF THEOREM 2.4.14. If $n=1$, then the result is trivial. Let $n \geq 2$. We observe that the class $\mathcal{Y}$ indeed consists of n-exact sequences. To see this, given $X \in \mathcal{Y}$ and $M \in \mathcal{M}$, apply the functor $\mathcal{E}(M,-)$ to X to obtain a sequence

$$0 \longrightarrow \mathcal{E}(M,X^0) \longrightarrow \mathcal{E}(M,X^1) \longrightarrow \cdots \longrightarrow \mathcal{E}(M,X^n) \longrightarrow \mathcal{E}(M,X^{n+1})$$

which is exact at $\mathcal{E}(M,X^0)$ and $\mathcal{E}(M,X^1)$. By the version of the dual of Proposition 2.3.18 for exact categories, for each $k \in \{1,\dots,n-1\}$ the homology of this complex at $\mathcal{E}(M,X^{k+1})$ is isomorphic to $\mathsf{Ext}^k_{\mathcal{X}}(M,X^0)$ which vanishes by assumption. Combining with the dual argument, this shows that X is an n-exact sequence.

We only show that $(\mathcal{M},\mathcal{Y})$ satisfies axioms (E1) and (E2) in Definition 2.4.2, since the remaining axioms are dual. Note that axiom (E0) is obviously satisfied.

The pair $(\mathcal{M},\mathcal{Y})$ satisfies axiom (E1): Let $M \rightarrowtail N$ be an $\mathcal{X}$-admissible monomorphism such that $M,N \in \mathcal{M}$ and $N \twoheadrightarrow E$ a cokernel of $M \rightarrowtail N$. Applying Proposition 2.4.15 to N yields an acyclic complex in $\mathcal{Y}$ with $M \rightarrowtail N$ as the leftmost morphism. This shows that the class of $\mathcal{Y}$-admissible monomorphisms coincides with the class of $\mathcal{X}$-admissible monomorphisms. Therefore $(\mathcal{M},\mathcal{Y})$ satisfies axiom (E1). That $(\mathcal{M},\mathcal{Y})$ satisfies axiom (E1$^{\mathrm{op}}$) then follows by duality.

The pair $(\mathcal{M},\mathcal{Y})$ satisfies axiom (E2): Let X be a $\mathcal{Y}$-admissible n-exact sequence and $f^0 \colon X^0 \to Y^0$ a morphism in $\mathcal{M}$. We need to construct a commutative diagram

(2.4.4)
$$\begin{tikzcd}
X^0 \arrow[r, tail, "d_X^0"] \arrow[d, "f^0"] & X^1 \arrow[r, "d_X^1"] \arrow[d, "f^1"] & \cdots \arrow[r, "d_X^{n-1}"] & X^n \arrow[r, two heads, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \\
Y^0 \arrow[r, tail, "d_Y^0"] & Y^1 \arrow[r, "d_Y^1"] & \cdots \arrow[r, "d_Y^{n-1}"] & Y^n &
\end{tikzcd}$$

such that d_Y^0 is a $\mathcal{Y}$-admissible monomorphism or, equivalently by the previous paragraph, an $\mathcal{X}$-admissible monomorphism.

Step 1: We claim that there is a $\mathcal{Y}$-admissible n-exact sequence

(2.4.5)
$$C: \qquad X^0 \rightarrowtail X^1 \oplus Y^0 \longrightarrow \cdots \longrightarrow X^n \oplus Y^{n-1} \twoheadrightarrow Y^n$$

with differentials

$$d_C^{k-1} := \begin{bmatrix} -d_X^k & 0 \\ f^k & d_Y^{k-1} \end{bmatrix} : X^k \oplus Y^{k-1} \longrightarrow X^{k+1} \oplus Y^k.$$

We shall construct C inductively. Set $Z^0 := X^0$, $u^0 := 1_{X^0}$, $p^0 := -d_X^0$ and $q^0 := f^0$. Let $0 \leq k \leq n-2$ and suppose that we have constructed a commutative diagram

$$\begin{tikzcd}
X^k \oplus Y^{k-1} \arrow[rr, "{\begin{bmatrix} -d_X^k & 0 \\ f^k & d_Y^{k-1} \end{bmatrix}}"] \arrow[dr, two heads, "{\begin{bmatrix} u^k & v^k \end{bmatrix}}"'] & & X^{k+1} \oplus Y^k \\
& Z^k \arrow[ur, tail, "{\begin{bmatrix} p^k \\ q^k \end{bmatrix}}"'] &
\end{tikzcd}$$

Since $Z^k \rightarrowtail X^{k+1} \oplus Y^k$ is an $\mathcal{X}$-admissible monomorphism, there exist morphisms $u^{k+1} \colon X^{k+1} \to Z^{k+1}$ and $v^{k+1} \colon Y^k \to Z^{k+1}$ such that $[u^{k+1}\ v^{k+1}] \colon X^{k+1} \oplus Y^k \twoheadrightarrow Z^{k+1}$. Therefore the square

$$\begin{tikzcd}
Z^k \arrow[r, "p^k"] \arrow[d, "q^k"] & X^{k+1} \arrow[d, "u^{k+1}"] \\
Y^k \arrow[r, "v^{k+1}"] & Z^{k+1}
\end{tikzcd} \tag{2.4.6}$$

is a pushout diagram.

It is readily verified that the composition

$$\begin{tikzcd}
X^k \oplus Y^{k-1} \arrow[r, "{\begin{bmatrix} -d_X^k & 0 \\ f^k & d_Y^{k-1} \end{bmatrix}}"] & X^{k+1} \oplus Y^k \arrow[r, "{\begin{bmatrix} d_X^{k+1} & 0 \end{bmatrix}}"] & X^{k+2}
\end{tikzcd}$$

is zero. Then, given that $X^k \oplus Y^{k-1} \twoheadrightarrow Z^k$ is an epimorphism, the composition

$$\begin{tikzcd}
Z^k \arrow[r, tail, "{\begin{bmatrix} p^k \\ q^k \end{bmatrix}}"] & X^{k+1} \oplus Y^k \arrow[r, "{\begin{bmatrix} d_X^{k+1} & 0 \end{bmatrix}}"] & X^{k+2}
\end{tikzcd}$$

is also zero. Hence we have $p^k d_X^0 = 0$.

Since (2.4.6) is a pushout diagram, there exists a morphism $p^{k+1} \colon Z^{k+1} \to X^{k+2}$ such that $u^{k+1}p^{k+1} = -d_X^{k+1}$ and $v^{k+1}p^{k+1} = 0$. Let $q^{k+1} \colon Z^{k+1} \rightarrowtail Y^{k+1}$ be a left $\mathcal{M}$-approximation of Z^{k+1}. Given that $-q^{k+1}$ is an $\mathcal{X}$-admissible monomorphism, Proposition 2.4.8 applied to the exact category $(\mathcal{E}, \mathcal{X})$ implies that the morphism $[q^{k+1}\ p^{k+1}] \colon Z^{k+1} \rightarrowtail X^{k+2} \oplus Y^{k+1}$ is an $\mathcal{X}$-admissible monomorphism. Set $f^{k+1} := u^{k+1}q^{k+1}$ and $d_Y^k := v^{k+1}q^{k+1}$. It is readily verified that the following diagram commutes:

$$\begin{tikzcd}
d_C^k : X^{k+1} \oplus Y^k \arrow[rr, "{\begin{bmatrix} -d_X^{k+1} & 0 \\ f^{k+1} & d_Y^k \end{bmatrix}}"] \arrow[dr, two heads, "{\begin{bmatrix} u^{k+1} & v^{k+1} \end{bmatrix}}"'] & & X^{k+2} \oplus Y^{k+1} \\
& Z^{k+1} \arrow[ur, tail, "{\begin{bmatrix} p^{k+1} \\ q^{k+1} \end{bmatrix}}"'] &
\end{tikzcd}$$

Finally, let $d_C^{n-1} \colon X^n \oplus Y^{n-1} \to Y^n$ be a cokernel of $Z^{n-1} \rightarrowtail X^n \oplus Y^{n-1}$. It follows that $C \in \mathcal{Y}$, and hence is an n-exact sequence.

Step 2: We claim that the morphism d_Y^0 is a $\mathcal{Y}$-admissible monomorphism. Indeed, we have $d_Y^0 = v^1 q^1$. Moreover, v^1 is an $\mathcal{X}$-admissible monomorphism for it is defined by the following pushout diagram in the exact category $(\mathcal{E}, \mathcal{X})$:

$$\begin{tikzcd} X^0 \arrow[r, tail, "d_X^0"] \arrow[d, "f^0"] & X^1 \arrow[d, "u^1"] \\ Y^0 \arrow[r, tail, "v^1"] & Z^1 \end{tikzcd}$$

Therefore d_Y^0 is the composition of two $\mathcal{X}$-admissible monomorphisms, hence itself is an $\mathcal{X}$-admissible monomorphism. Since $\mathcal{X}$-admissible monomorphisms and $\mathcal{Y}$-admissible morphisms coincide, the claim follows. The existence of a commutative diagram of the form (2.4.4) follows since C is a complex. This shows that $(\mathcal{M}, \mathcal{Y})$ satisfies axiom (E2). That $(\mathcal{M}, \mathcal{Y})$ satisfies axiom (E2$^{\mathrm{op}}$) then follows by duality. $\square$

2.5. Frobenius n-exact categories

In this section we introduce Frobenius n-exact categories and show that their stable categories have the structure of an $(n+2)$-angulated category; this allows us to introduce algebraic $(n+2)$-angulated categories. We give a method to construct Frobenius n-exact categories (and thus also algebraic n-angulated categories) from certain n-cluster-tilting subcategories of Frobenius exact categories. We show that our construction is closely related to the standard construction of $(n+2)$-angulated categories given in [**39**, Thm. 1], see Theorem 2.5.16.

2.5.1. Reminder on $(n+2)$-angulated categories. We follow the exposition of [**39**, Sec. 2] although our conventions on indexation are different.

Let n be a positive integer and $\mathcal{F}$ an additive category equipped with an automorphism $\Sigma\colon \mathcal{F} \to \mathcal{F}$. An *$n$-$\Sigma$-sequence* in $\mathcal{F}$ is a sequence of morphisms

$$X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} X^2 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} X^{n+1} \xrightarrow{\alpha^{n+1}} \Sigma X^0. \tag{2.5.1}$$

Its *left rotation* is the n-Σ-sequence

$$X^1 \xrightarrow{\alpha^1} X^2 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} X^{n+1} \xrightarrow{\alpha^{n+1}} \Sigma X^0 \xrightarrow{(-1)^n \Sigma\alpha^0} \Sigma X^1$$

A *morphism of n-Σ-sequences* is a commutative diagram

$$\begin{tikzcd} X^0 \arrow[r, "\alpha^0"] \arrow[d, "\varphi^0"] & X^1 \arrow[r, "\alpha^1"] \arrow[d, "\varphi^1"] & X^2 \arrow[r, "\alpha^2"] \arrow[d, "\varphi^2"] & \cdots \arrow[r, "\alpha^n"] & X^{n+1} \arrow[r, "\alpha^{n+1}"] \arrow[d, "\varphi^{n+1}"] & \Sigma X^0 \arrow[d, "\Sigma\varphi^0"] \\ Y^0 \arrow[r, "\beta^0"] & Y^1 \arrow[r, "\beta^1"] & Y^2 \arrow[r, "\beta^2"] & \cdots \arrow[r, "\beta^n"] & Y^{n+1} \arrow[r, "\beta^{n+1}"] & \Sigma Y^0 \end{tikzcd}$$

where each row is an n-Σ-sequence. The mapping cone $C(\varphi)$ of the above morphism is the n-Σ-sequence

$$X^1 \oplus Y^0 \xrightarrow{\gamma^0} X^2 \oplus Y^1 \xrightarrow{\gamma^1} \cdots \xrightarrow{\gamma^n} \Sigma X^0 \oplus Y^{n+1} \xrightarrow{\gamma^{n+1}} \Sigma X^1 \oplus \Sigma Y^0$$

where for each $k \in \{0, \dots, n\}$ we define

$$\gamma^k := \begin{bmatrix} -\alpha^{k+1} & 0 \\ \varphi^{k+1} & \beta^k \end{bmatrix} : X^{k+1} \oplus Y^k \longrightarrow X^{k+2} \oplus Y^{k+1}$$

(by convention, $\alpha^{n+2} := \Sigma\alpha^0$, $\varphi^{n+2} := \Sigma\varphi^0$). A *weak isomorphism* is a morphism of n-Σ-sequences such that φ^k and φ^{k+1} are isomorphisms for some $k \in \{0, 1, \dots, n+1\}$. Abusing the terminology, we say that two n-Σ-sequences are *weakly isomorphic* if they are connected by a finite zigzag of weak isomorphisms.

DEFINITION 2.5.1. [**39**] A *pre-$(n+2)$-angulated category* is a triple $(\mathcal{F}, \Sigma, \mathcal{S})$ where $\mathcal{F}$ is an additive category, $\Sigma\colon \mathcal{F} \to \mathcal{F}$ is an automorphism[1], and $\mathcal{S}$ is a class of n-Σ-sequences (whose members we call $(n+2)$-angles) which satisfies the following axioms:

(F1) The class $\mathcal{S}$ is closed under taking direct summands and making direct sums. For all $X \in \mathcal{F}$ the *trivial sequence*

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma X$$

belongs to $\mathcal{S}$. For each morphism α in $\mathcal{M}$ there exists an n-angle whose first morphism is α.

(F2) An n-Σ-sequence of the form (2.5.1) is a $(n+2)$-angle if and only if its left rotation is a $(n+2)$-angle.

(F3) Each commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r, "\alpha^0"] \arrow[d, "\varphi^0"] & X^1 \arrow[r, "\alpha^1"] \arrow[d, "\varphi^1"] & X^2 \arrow[r, "\alpha^2"] \arrow[d, dotted, "\varphi^2"] & \cdots \arrow[r, "\alpha^n"] & X^{n+1} \arrow[r, "\alpha^{n+1}"] \arrow[d, dotted, "\varphi^{n+1}"] & \Sigma X^0 \arrow[d, "\Sigma\varphi^0"] \\
Y^0 \arrow[r, "\beta^0"] & Y^1 \arrow[r, "\beta^1"] & Y^2 \arrow[r, "\beta^2"] & \cdots \arrow[r, "\beta^n"] & Y^{n+1} \arrow[r, "\beta^{n+1}"] & \Sigma Y^0
\end{tikzcd}$$

whose rows are $(n+2)$-angles can be completed to a morphism of n-Σ-sequences.

If moreover the following axiom is satisfied, then $(\mathcal{F}, \Sigma, \mathcal{S})$ is called an *n-angulated category.*

(F4) In the situation of axiom (F3), the morphisms $\varphi^2, \dots, \varphi^{n+1}$ can be chosen in such a way that the mapping cone $C(\varphi)$ is a $(n+2)$-angle.

An n-Σ-sequence X is *exact* if for all $F \in \mathcal{F}$ and for all $k \in \mathbb{Z}$ the induced sequence of abelian groups

$$\mathcal{F}(F, \Sigma^{k-1}X^{n+1}) \longrightarrow \mathcal{F}(F, \Sigma^k X^0) \longrightarrow \cdots \longrightarrow \mathcal{F}(F, \Sigma^k X^{n+1}) \longrightarrow \mathcal{F}(F, \Sigma^{k+1}X^0)$$

is exact. We need the following result.

PROPOSITION 2.5.2. [**39**, Prop. 2.5(c)] *Let $(\mathcal{F}, \Sigma, \mathcal{S})$ be a pre-$(n+2)$-angulated category. If $\mathcal{S}'$ is a pre-$(n+2)$-angulation of $\mathcal{F}$ such that $\mathcal{S}' \subseteq S$, then $\mathcal{S} = \mathcal{S}'$.*

Let $(\mathcal{F}, \Sigma_{\mathcal{F}}, \mathcal{S}_{\mathcal{F}})$ and $(\mathcal{G}, \Sigma_{\mathcal{G}}, \mathcal{S}_{\mathcal{G}})$ be $(n+2)$-angulated categories. We say that a functor $F\colon \mathcal{F} \to \mathcal{G}$ is *exact* if there exists a natural transformation $\eta\colon F\Sigma_{\mathcal{F}} \to \Sigma_{\mathcal{G}}F$ such that for every $(n+2)$-angle in $\mathcal{F}$

$$X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} X^2 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} X^{n+1} \xrightarrow{\alpha^{n+1}} \Sigma_{\mathcal{F}} X^0.$$

we have that

$$FX^0 \xrightarrow{F\alpha^0} FX^1 \xrightarrow{F\alpha^1} FX^2 \xrightarrow{F\alpha^2} \cdots \xrightarrow{F\alpha^n} FX^{n+1} \xrightarrow{\beta} \Sigma_{\mathcal{G}} FX^0.$$

is a $(n+2)$-angle in $\mathcal{G}$, where $\beta := (F\alpha^{n+1})\eta_{X^0}$.

[1] One may consider the more general case when $\Sigma\colon \mathcal{F} \to \mathcal{F}$ is an autoequivalence. As mentioned in [**39**, 2.2 Rmks.], it can be shown that the assumption of Σ being invertible is but a mild sacrifice, *cf.* [**72**, Sec. 2].

2.5.2. Frobenius n-exact categories and algebraic $(n+2)$-angulated categories. Our approach in this subsection is analogous to [**44**, Sec. I.2].

DEFINITION 2.5.3. Let $(\mathcal{M},\mathcal{X})$ be an n-exact category. An object $I \in \mathcal{M}$ is $\mathcal{X}$-injective if for every admissible monomorphism $f\colon M \rightarrowtail N$ the sequence of abelian groups

$$\mathcal{M}(N,I) \xrightarrow{f\cdot ?} \mathcal{M}(M,I) \longrightarrow 0$$

is exact. We say that $(\mathcal{M},\mathcal{X})$ *has enough $\mathcal{X}$-injectives* if for every object $M \in \mathcal{M}$ there exists $\mathcal{X}$-injective objects $I^1,\dots,I^n$ and an admissible n-exact sequence

$$M \rightarrowtail I^1 \longrightarrow \cdots \longrightarrow I^n \twoheadrightarrow N$$

The notion of *having enough $\mathcal{X}$-projectives* is defined dually.

REMARK 2.5.4. If $n=1$, then the notions of $\mathcal{X}$-injectively cogenerated exact category and of an exact category with enough $\mathcal{X}$-injectives coincide. On the other hand, if $n \geq 2$, then there are n-exact categories which are $\mathcal{X}$-injectively cogenerated but do not have enough $\mathcal{X}$-injectives. Indeed, let Λ be a finite-dimensional selfinjective algebra. By Theorem 2.3.16 we know that an n-cluster-tilting subcategories $\mathcal{M}$ of $\operatorname{mod}\Lambda$ is an injectively cogenerated n-abelian category. However, $\mathcal{M}$ has enough injectives if and only if $\mathcal{M}$ is stable under taking n-th cosyzygies, *cf.* Theorem 2.5.16.

Let $(\mathcal{M},\mathcal{X})$ be an n-exact category. For objects $M,N \in \mathcal{M}$, we denote by $I(M,N)$ the subgroup of $\mathcal{M}(M,N)$ of morphisms which factor through an $\mathcal{X}$-injective object. The *$\mathcal{X}$-injectively stable category of* $\mathcal{M}$, denoted by $\overline{\mathcal{M}}$, is the category with the same objects as $\mathcal{M}$ and with morphisms groups defined by

$$\overline{\mathcal{M}}(M,N) := \mathcal{M}(M,N)/I(M,N).$$

If $\alpha\colon M \to N$ is a morphism in $\mathcal{M}$, we denote its equivalence class in $\overline{\mathcal{M}}(M,N)$ by $\overline{\alpha}$. It easy to see that $\overline{\mathcal{M}}$ is also an additive category. The *$\mathcal{X}$-projectively stable category of* $\mathcal{M}$, denoted by $\underline{\mathcal{M}}$, is defined dually.

DEFINITION 2.5.5. We say that an n-exact category $(\mathcal{M},\mathcal{X})$ is *Frobenius* if it has enough $\mathcal{X}$-injectives, enough $\mathcal{X}$-projectives, and if $\mathcal{X}$-injective and $\mathcal{X}$-projective objects coincide. In this case one has $\overline{\mathcal{M}} = \underline{\mathcal{M}}$, and we refer to this category as the *stable category of* $\mathcal{M}$.

REMARK 2.5.6. In keeping the convention of the classical theory, if $(\mathcal{M},\mathcal{X})$ is a Frobenius n-exact category, then we denote its stable category by $\underline{\mathcal{M}}$.

Our aim is to show that the stable category of a Frobenius n-exact category, has a natural structure of a $(n+2)$-angulated category. We begin with the construction of an autoequivalence $\Sigma\colon \underline{\mathcal{M}} \to \underline{\mathcal{M}}$. The following result should be compared with [**44**, Lemma I.2.2].

LEMMA 2.5.7. *Let $(\mathcal{M},\mathcal{X})$ be an n-exact category. Suppose that we are given two admissible n-exact sequences X and Y such that $X^0 = Y^0$ and, for $k \in \{1,n\}$, the objects X^k and Y^k are $\mathcal{X}$-injective. Then, X^{n+1} and Y^{n+1} are isomorphic in $\overline{\mathcal{M}}$.*

PROOF. Since X^1 and Y^1 are $\mathcal{X}$-injective and X and Y are admissible n-exact sequences, we can construct a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & X^0 \arrow[r, tail] \arrow[d, equal] & X^1 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & X^n \arrow[r, two heads] \arrow[d, dotted] & X^{n+1} \arrow[d, dotted] \\
Y \arrow[d, "g"] & Y^0 \arrow[r, tail] \arrow[d, equal] & Y^1 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Y^n \arrow[r, two heads] \arrow[d, dotted] & Y^{n+1} \arrow[d, dotted] \\
X & X^0 \arrow[r, tail] & X^1 \arrow[r] & \cdots \arrow[r] & X^n \arrow[r, two heads] & X^{n+1}
\end{tikzcd}$$

By the Comparison Lemma 2.2.1 there exists a morphism $h\colon X^{n+1} \to X^n$ such that

$$f^{n+1}g^{n+1} - 1 = hd_X^n.$$

Since X^n is $\mathcal{X}$-injective, we have $\overline{f^{n+1}g^{n+1}} = \overline{1}$. A similar argument shows that $\overline{g^{n+1}f^{n+1}} = \overline{1}$. Therefore X^{n+1} and Y^{n+1} are isomorphic in $\overline{\mathcal{M}}$. $\square$

Let $(\mathcal{M}, \mathcal{X})$ be an n-exact category with enough $\mathcal{X}$-injectives. For each $M \in \mathcal{M}$ we choose an admissible n-exact sequence

$$I(M)\colon M \rightarrowtail I^1(M) \longrightarrow \cdots \longrightarrow I^n(M) \twoheadrightarrow SM.$$

such that for each $k \in \{1, \dots, n\}$ the objects $I^k(M)$ are $\mathcal{X}$-injective. It follows from Lemma 2.5.7 that the isomorphism class of SM in $\underline{\mathcal{M}}$ does not depend on the choice of $I(M)$.

Let $f\colon M \to N$ be a morphism in $\mathcal{M}$. Since $I^1(N)$ is $\mathcal{X}$-injective, there is a commutative diagram of admissible n-exact sequences

$$\begin{tikzcd}
I(M) \arrow[d, "I(f)"] & M \arrow[r, tail] \arrow[d, "f"] & I^1(M) \arrow[r] \arrow[d, "I^1(f)"] & \cdots \arrow[r] & I^n(M) \arrow[r, two heads] \arrow[d, "I^n(f)"] & SM \arrow[d, "Sf"] \\
I(N) & N \arrow[r, tail] & I^1(N) \arrow[r] & \cdots \arrow[r] & I^n(N) \arrow[r, two heads] & SY
\end{tikzcd}$$

The Comparison Lemma 2.2.1 implies that $\overline{Sf}$ does not depend on the choices of $I^1(f), \dots, I^n(f)$. It is readily verified that the correspondences $M \mapsto SM$ and $f \mapsto \overline{Sf}$ define a functor $\Sigma\colon \overline{\mathcal{M}} \to \overline{\mathcal{M}}$.

The proof of the following result is straightforward, *cf.* [**44**, Prop. 2.2.]. We leave the details to the reader.

PROPOSITION 2.5.8. *Let $(\mathcal{M}, \mathcal{X})$ be a Frobenius n-exact category. Then $\Sigma\colon \underline{\mathcal{M}} \mapsto \underline{\mathcal{M}}$ is an autoequivalence. Moreover, any two choices of assignments $M \mapsto SM$ and $M \mapsto S'M$ yield isomorphic functors.*

REMARK 2.5.9. Let $(\mathcal{M}, \mathcal{X})$ be a Frobenius n-exact category. In analogy with [**39**, Rmk. 2.2(d)] we assume that $\Sigma\colon \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ is not only an autoequivalence but an automorphism of $\underline{\mathcal{M}}$ (see also [**72**, Sec. 2]).

Let $(\mathcal{M}, \mathcal{X})$ be a Frobenius n-exact category. We define a class $\mathcal{S} = \mathcal{S}(\mathcal{X})$ of n-Σ-sequences in $\underline{\mathcal{M}}$ as follows. Let $\alpha^0\colon X^0 \to X^1$ be a morphism in $\mathcal{M}$. Then, for every morphism of n-exact sequences of the form

$$\begin{tikzcd}
X^0 \arrow[r, tail] \arrow[d, "\alpha^0"] & I^1(X^0) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads] \arrow[d] & SX^0 \arrow[d, equal] \\
X^1 \arrow[r, tail, "\alpha^1"] & X^2 \arrow[r, "\alpha^2"] & \cdots \arrow[r, "\alpha^n"] & X^{n+1} \arrow[r, two heads, "\alpha^{n+1}"] & SX^0
\end{tikzcd}$$

the sequence

$$X^0 \xrightarrow{\overline{\alpha^0}} X^1 \xrightarrow{\overline{\alpha^1}} X^2 \xrightarrow{\overline{\alpha^2}} \cdots \xrightarrow{\overline{\alpha^n}} X^{n+1} \xrightarrow{\overline{\alpha^{n+1}}} \Sigma X^0$$

is called a *standard* $(n+2)$*-angle.* An n-Σ-sequence Y in $\underline{\mathcal{M}}$ belongs to $\mathcal{S}$ if and only if it is isomorphic to a standard $(n+2)$-angle.

We need the following result, which shows how admissible n-exact sequences give rise to standard $(n+2)$-angles, *cf.* [**44**, Lemma I.2.7].

LEMMA 2.5.10. *Let* $(\mathcal{M},\mathcal{X})$ *be a Frobenius* n*-exact category and* X *an admissible* n*-exact sequence in* $\mathcal{M}$*. The following statements hold:*

(i) There exists a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"] & X^0 \arrow[r, tail] \arrow[d, equal] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^n \arrow[r, two heads] \arrow[d] & X^{n+1} \arrow[d] \\ I(X^0) & X^0 \arrow[r, tail] & I^1(X^0) \arrow[r] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads] & SX^0 \end{tikzcd} \tag{2.5.2}$$

(ii) The sequence

$$X^0 \xrightarrow{\overline{d^0}} X^1 \xrightarrow{\overline{d^1}} \cdots \xrightarrow{\overline{d^n}} X^{n+1} \xrightarrow{(-1)^n\overline{f^{n+1}}} \Sigma X^0$$

is a standard $(n+2)$*-angle.*

PROOF. (i) The existence of the required commutative diagram follows from the fact that $I^1(M)$ is $\mathcal{X}$-injective and X is an n-exact sequence.

(ii) The dual of Proposition 2.4.8 implies that the mapping cone $C(f)$ is an admissible n-exact sequence. For each $k\in\{1,\dots,n\}$ we define

$$g^k := \begin{bmatrix} 0 & (-1)^{k-1}\end{bmatrix}^\top : I^k(X^0) \longrightarrow X^{k+1}\oplus I^k(X^0).$$

It readily follows that the diagram

$$\begin{tikzcd} X^0 \arrow[r, tail] \arrow[d, "\alpha^0"] & I^1(X^0) \arrow[r] \arrow[d, "g^1"] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads] \arrow[d, "g^n"] & SX^0 \arrow[d, equal] \\ X^1 \arrow[r, tail, "d_C^{-1}"] & X^2\oplus I^1(X^0) \arrow[r, "d_C^0"] & \cdots \arrow[r, "d_C^{n-2}"] & X^{n+1}\oplus I^n(X^0) \arrow[r, two heads, "(-1)^n d_C^{n-1}"] & SX^0 \end{tikzcd}$$

is commutative, and that it gives rise to the standard $(n+2)$-angle

$$X^0 \xrightarrow{\overline{d^0}} X^1 \xrightarrow{\overline{d^1}} \cdots \xrightarrow{\overline{d^n}} X^{n+1} \xrightarrow{(-1)^n\overline{f^{n+1}}} \Sigma X^0. \quad \square$$

The following result is a higher analog of [**44**, Thm. I.2.6].

THEOREM 2.5.11. *Let* $(\mathcal{M},\mathcal{X})$ *a Frobenius* n*-exact category. Then,* $(\underline{\mathcal{M}},\Sigma,\mathcal{S}(\mathcal{X}))$ *is an* $(n+2)$*-angulated category.*

PROOF. We need to show that $(\underline{\mathcal{M}},\Sigma,\mathcal{S}(\mathcal{X}))$ satisfies the axioms of $(n+2)$-angulated categories, see Definition 2.5.1. For axioms (F1), (F2) and (F3), our proof is an adaptation of the proof of [**44**, Thm. I.2.6].

(F1) Firstly, recall that $\mathcal{X}$ is closed under direct sums and direct summands, see Propositions 2.4.7 and 2.4.12. It is easy to show that this implies that the same is true for $\mathcal{S}$. Secondly, the diagram

$$\begin{tikzcd}
M \arrow[r, tail] \arrow[d, equal] & I^1(M) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & I^n(M) \arrow[r, two heads] \arrow[d, equal] & SM \arrow[d, equal] \\
X \arrow[r, tail] & I^1(M) \arrow[r] & \cdots \arrow[r] & I^n(M) \arrow[r, two heads] & SM
\end{tikzcd}$$

shows that the n-Σ-sequence

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma X.$$

is a standard $(n+2)$-angle. Finally, by the definition of the class $\mathcal{S}$, every morphism is the first morphism of some standard $(n+2)$-angle. This shows that $(\underline{\mathcal{M}}, \Sigma, \mathcal{S}(\mathcal{X}))$ satisfies axiom (F1).

(F2) It suffices to consider the case of standard $(n+2)$-angles. Let

$$\begin{tikzcd}
I(X^0) \arrow[d, "f"] & X^0 \arrow[r, tail] \arrow[d, "\alpha^0"] & I^1(X^0) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads] \arrow[d] & SX^0 \arrow[d, equal] \\
X & X^1 \arrow[r, tail, "\alpha^1"] & X^2 \arrow[r, "\alpha^2"] & \cdots \arrow[r, "\alpha^n"] & X^{n+1} \arrow[r, two heads, "\alpha^{n+1}"] & SX^0
\end{tikzcd} \tag{2.5.3}$$

be a commutative diagram giving rise to the standard $(n+2)$-angle

$$X^0 \xrightarrow{\overline{\alpha^0}} X^1 \xrightarrow{\overline{\alpha^1}} \cdots \xrightarrow{\overline{\alpha^n}} X^{n+1} \xrightarrow{\overline{\alpha^{n+1}}} \Sigma X^0. \tag{2.5.4}$$

We need to show that its left rotation is a standard $(n+2)$-angle.

Firstly, by the definition of $\Sigma\colon \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ we have a commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r, tail] \arrow[d, "\alpha^0"] & I^1(X^0) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads] \arrow[d] & SX^0 \arrow[d, "S(\alpha^0)"] \\
X^1 \arrow[r, tail] & I^1(X^1) \arrow[r] & \cdots \arrow[r] & I^n(X^1) \arrow[r, two heads] & SX^1
\end{tikzcd}$$

Secondly, Proposition 2.4.9 yields a commutative diagram

$$\begin{tikzcd}
I(X^0) \arrow[d, "f"] & X^0 \arrow[r, tail] \arrow[d, "\alpha^0"] & I^1(X^0) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads] \arrow[d] & SX^0 \arrow[d, equal] \\
X \arrow[d, "p"] & X^1 \arrow[r, tail, "\alpha^1"] \arrow[d, equal] & X^2 \arrow[r, "\alpha^2"] \arrow[d] & \cdots \arrow[r] & X^{n+1} \arrow[r, two heads, "\alpha^{n+1}"] \arrow[d] & SX^0 \arrow[d, "S\alpha^0"] \\
I(X^1) & X^1 \arrow[r, tail] & I^1(X^1) \arrow[r] & \cdots \arrow[r] & I^n(X^1) \arrow[r, two heads] & SX^1
\end{tikzcd}$$

By the dual of Proposition 2.4.8, the mapping cone $C = C(p)$ is an admissible exact sequence. Thirdly, for each $k \in \{1, \ldots, n\}$ we define

$$g^k := \begin{bmatrix} 0 & (-1)^{k-1} \end{bmatrix}^\top : I^k(X^1) \longrightarrow X^{k+2} \oplus I^k(X^1)$$

(by convention, $X^{n+1} := \Sigma X^0$). It follows that the diagram

$$\begin{tikzcd}
X^1 \arrow[r, tail] \arrow[d, "\alpha^2"] & I^1(X^1) \arrow[r] \arrow[d, "g^1"] & \cdots \arrow[r] & I^n(X^1) \arrow[r, two heads] \arrow[d, "g^n"] & SX^1 \arrow[d, equal] \\
X^2 \arrow[r, tail, "d_C^{-1}"] & X^3 \oplus I^1(X^1) \arrow[r, "d_C^0"] & \cdots \arrow[r, "d_C^{n-2}"] & SX^0 \oplus I^n(X^1) \arrow[r, two heads, "(-1)^n d_C^{n-1}"] & SX^1
\end{tikzcd}$$

is commutative. Note that the bottom row is an admissible n-exact sequence for it is isomorphic to C. Finally, the standard $(n+2)$-angle induced by this diagram is isomorphic in $\underline{\mathcal{M}}$ to the left rotation of (2.5.4).

Conversely, suppose that there is a commutative diagram

$$\begin{tikzcd}
I(X^1) \arrow[d, "f"] & X^1 \arrow[r, tail] \arrow[d, "\alpha^1"] & I^1(X^1) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^n(X^1) \arrow[r, two heads] \arrow[d] & SX^1 \arrow[d, equal] \\
Y & X^2 \arrow[r, tail, "\alpha^2"] & X^3 \arrow[r, "\alpha^3"] & \cdots \arrow[r, "\alpha^{n+1}"] & SX^0 \arrow[r, two heads, "(-1)^d S\alpha^0"] & SX^1
\end{tikzcd}$$

which gives rise to a standard $(n+2)$-angle of the form

$$X^1 \xrightarrow{\overline{\alpha^1}} X^2 \xrightarrow{\overline{\alpha^2}} \cdots \xrightarrow{\overline{\alpha^n}} X^{n+1} \xrightarrow{\overline{\alpha^{n+1}}} \Sigma X^0 \xrightarrow{(-1)^n \Sigma\overline{\alpha^0}} \Sigma X^1$$

On one hand, the definition of $\Sigma\colon \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ yields the top two rows in the following commutative diagram:

$$\begin{tikzcd}
I(X^0) \arrow[d, "I(\alpha^0)"] & X^0 \arrow[r, tail] \arrow[d, "\alpha^0"] & I^1(X^0) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads, "d^n"] \arrow[d] & SX^0 \arrow[d, "S\alpha^0"] \\
I(X^1) \arrow[d, "f"] & X^1 \arrow[r, tail] \arrow[d, "\alpha^1"] & I^1(X^1) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^n(X^1) \arrow[r, two heads] \arrow[d] & SX^1 \arrow[d, equal] \\
Y & X^2 \arrow[r, tail, "\alpha^2"] & X^3 \arrow[r, "\alpha^3"] & \cdots \arrow[r, "\alpha^{n+1}"] & SX^0 \arrow[r, two heads, "(-1)^d S\alpha^0"] & SX^1
\end{tikzcd}$$

On the other hand, we have a commutative diagram

$$\begin{tikzcd}
I(X^0) \arrow[d, "q"] & X^0 \arrow[r, tail] \arrow[d, "0"] & \cdots \arrow[r] & I^{n-1}(X^0) \arrow[r] \arrow[d, "0"] & I^n(X^0) \arrow[r, two heads, "d^n"] \arrow[d, "(-1)^n d^n"] & SX^0 \arrow[d, "S\alpha^0"] \\
Y & X^2 \arrow[r, tail, "\alpha^2"] & \cdots \arrow[r, "\alpha^n"] & X^{n+1} \arrow[r, "\alpha^{n+1}"] & SX^0 \arrow[r, two heads, "(-1)^n S\alpha^0"] & SX^1
\end{tikzcd}$$

Then, the dual of the Comparison Lemma 2.2.1 implies the existence of a homotopy $h\colon I(\alpha^0)f \to q$. For each $k \in \{1, \dots, n\}$ we define

$$g^k := \begin{bmatrix}(-1)^k I^k(\alpha^0) & (-1)^{k-1} h^k\end{bmatrix}^\top : I^k(X^0) \longrightarrow I^k(X^k) \oplus X^{k+1}.$$

It is straightforward to verify that the diagram

$$\begin{tikzcd}
X^0 \arrow[r, tail] \arrow[d, "\alpha^0"] & I^1(X^0) \arrow[r] \arrow[d, "g^1"] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads] \arrow[d, "g^n"] & SX^0 \arrow[d, equal] \\
X^1 \arrow[r, tail, "d_C^{-1}"] & I^1(X^1) \oplus X^2 \arrow[r, "d_C^0"] & \cdots \arrow[r, "d_C^{n-2}"] & I^n(X^1) \oplus X^{n+1} \arrow[r, two heads, "d_C^{n-1}"] & SX^0
\end{tikzcd}$$

commutes, where the bottom row is given by $C(f)$. Finally, the standard $(n+2)$-angle induced by this diagram is isomorphic to the n-Σ-sequence

$$X^0 \xrightarrow{\overline{\alpha^0}} X^1 \xrightarrow{\overline{\alpha^1}} \cdots \xrightarrow{\overline{\alpha^n}} X^{n+1} \xrightarrow{\overline{\alpha^{n+1}}} \Sigma X^0.$$

This shows that $(\underline{\mathcal{M}}, \mathcal{X}, \mathcal{S}(\mathcal{X}))$ satisfies axiom (F2).

(F3) for standard $(n+2)$-angles. Let

$$\begin{tikzcd}
I(X^0) \arrow[d, "f"] & X^0 \arrow[r, tail, "d^0_{IX}"] \arrow[d, "\alpha^0"] & I^1(X^0) \arrow[r, "d^1_{IX}"] \arrow[d] & \cdots \arrow[r, "d^{n-1}_{IX}"] & I^n(X^0) \arrow[r, two heads, "d^n_{IX}"] \arrow[d] & SX^0 \arrow[d, equal] \\
X & X^1 \arrow[r, tail, "\alpha^1"] & X^2 \arrow[r, "\alpha^2"] & \cdots \arrow[r, "\alpha^n"] & X^{n+1} \arrow[r, two heads, "\alpha^{n+1}"] & SX^0
\end{tikzcd}$$

and

$$\begin{tikzcd}
I(Y^0) \arrow[d, "g"] & Y^0 \arrow[r, tail, "d^0_{IY}"] \arrow[d, "\beta^0"] & I^1(Y^0) \arrow[r, "d^1_{IY}"] \arrow[d, "g^1"] & \cdots \arrow[r, "d^{n-1}_{IY}"] & I^n(X^0) \arrow[r, two heads, "d^n_{IY}"] \arrow[d] & SY^0 \arrow[d, equal] \\
Y & Y^1 \arrow[r, tail, "\beta^1"] & Y^2 \arrow[r, "\beta^2"] & \cdots \arrow[r, "\beta^n"] & Y^{n+1} \arrow[r, two heads, "\beta^{n+1}"] & SY^0
\end{tikzcd}$$

be pushout diagrams in $\mathcal{M}$. We set $C := C(f)$.

Also, let $\varphi^0\colon X^0 \to Y^0$ and $\varphi^1\colon X^1 \to Y^1$ be morphisms such that $\overline{\varphi^0\beta^0} = \overline{\alpha^0\varphi^1}$. Thus, we have a diagram

$$\begin{tikzcd}
X^0 \arrow[r, "\overline{\alpha^0}"] \arrow[d, "\overline{\varphi^0}"] & X^1 \arrow[r, "\overline{\alpha^1}"] \arrow[d, "\overline{\varphi^1}"] & X^2 \arrow[r, "\overline{\alpha^2}"] & \cdots \arrow[r, "\overline{\alpha^n}"] & X^{n+1} \arrow[r, "\overline{\alpha^{n+1}}"] & \Sigma X^0 \arrow[d, "\Sigma\overline{\varphi^0}"] \\
Y^0 \arrow[r, "\overline{\beta^0}"] & Y^1 \arrow[r, "\overline{\beta^1}"] & Y^2 \arrow[r, "\overline{\beta^2}"] & \cdots \arrow[r, "\overline{\beta^n}"] & Y^{n+1} \arrow[r, "\overline{\beta^{n+1}}"] & \Sigma Y^0
\end{tikzcd} \tag{2.5.5}$$

whose rows are standard $(n+2)$-angles. Recall that by the definition of $\Sigma\colon \underline{\mathcal{M}} \to \underline{\mathcal{M}}$, there is a commutative diagram

$$\begin{tikzcd}
I(X^0) \arrow[d, "I(\varphi^0)"] & X^0 \arrow[r, tail] \arrow[d, "\varphi^0"] & I^1(X^0) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads] \arrow[d] & SX^0 \arrow[d, "S\varphi^0"] \\
I(Y^0) & Y^0 \arrow[r, tail] & I^1(Y^0) \arrow[r] & \cdots \arrow[r] & I^n(Y^0) \arrow[r, two heads] & SY^0
\end{tikzcd}$$

We shall construct a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\varphi"] & X^1 \arrow[r, tail, "\alpha^1"] \arrow[d, "\varphi^1"] & X^2 \arrow[r, "\alpha^2"] \arrow[d, dotted, "\varphi^2"] & \cdots \arrow[r, "\alpha^n"] & X^{n+1} \arrow[r, two heads, "\alpha^{n+1}"] \arrow[d, dotted, "\varphi^{n+1}"] & SX^0 \arrow[d, "S\varphi^0"] \\
Y & Y^1 \arrow[r, tail, "\beta^1"] & Y^2 \arrow[r, "\beta^2"] & \cdots \arrow[r, "\beta^n"] & Y^{n+1} \arrow[r, two heads, "\beta^{n+1}"] & SY^0
\end{tikzcd}$$

together with a homotopy $h\colon f\varphi \to I(\varphi^0)g$ such that $h^{n+1}\colon SX^0 \to Y^{n+1}$ is the zero morphism. Note that this gives the required completion of diagram (2.5.5).

We begin with the construction of h^1 and φ^2. Since $\overline{\varphi^0\beta^0} = \overline{\alpha^0\varphi^1}$, there exists an $\mathcal{X}$-injective object $I \in \mathcal{M}$ and morphisms $u\colon X^0 \to I$ and $v\colon I \to Y^1$ such that $\alpha^0\varphi^1 - \varphi^0\beta^0 = uv$. Then, given that d^0_{IX} is an admissible monomorphism and I is $\mathcal{X}$-injective, we can construct a commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r, tail, "d^0_{IX}"] \arrow[dr, "u"'] & I^1(X^0) \arrow[d, dotted] \arrow[dr, dotted, "h^1"] & \\
& I \arrow[r, "v"] & Y^1
\end{tikzcd}$$

Hence $\alpha^0\varphi^1 - \varphi^0\beta^0 = d^0_{IX}h^1$, as required. Then we have

$$\begin{aligned}
d^0_{IX}(I(\varphi^0)^1g^1 + h^1\beta^1) =& \varphi^0 d^0_{IY}g^1 + (\alpha^0\varphi^1 - \varphi^0 g^1)\beta^1 \\
=& \alpha^0\varphi^1\beta^1.
\end{aligned}$$

Since d_C^0 is a weak cokernel of d_C^{-1}, there exists morphisms $\varphi^2 \colon X^2 \to Y^2$ and $h^2 \colon I^2(X^0) \to Y^2$ such that $\varphi^1\beta^1 = \alpha^1\varphi^2$ and $f^1\varphi^2 - (I(\varphi^0)^1 g^1 + h^1\beta^1) = d_{IX}^2 h^2$ or, equivalently,

$$f^1\varphi^2 - I(\varphi^0)^1 g^1 = h^1\beta^1 + d_{IX}^1 h^2$$

Let $2 \leq k \leq n$ and suppose that for each $\ell \leq k$ we have constructed morphisms $\varphi^\ell \colon X^\ell \to Y^\ell$ and $h^\ell \colon I^\ell(X^0) \to Y^\ell$ such that $\alpha^{\ell-1}\varphi^\ell = \varphi^{\ell-1}\beta^{\ell-1}$ and

$$f^{\ell-1}\varphi^\ell - I(\varphi^0)^{\ell-1} g^{\ell-1} = h^{\ell-1}\beta^{\ell-1} + d_{IX}^{\ell-1} h^\ell.$$

Then we have

$$\begin{aligned} d_{IX}^{k-1}(I(\varphi^0)^k g^k + h^k\beta^k) =& I(\varphi^0)^{k-1} d_{IY}^{k-1} g^k + (f^{k-1}\varphi^k - I(\varphi^0)^{k-1} g^{k-1} - h^{k-1}\beta^{k-1})\beta^k \\ =& f^{k-1}\varphi^k\beta^k. \end{aligned}$$

Moreover, $\alpha^{k-1}(\varphi^k\beta^k) =$ Since d_C^{k-1} is a weak cokernel of d_C^{k-2}, there exists morphisms $\varphi^{k+1} \colon X^{k+1} \to Y^{k+1}$ and $h^{k+1} \colon I^{k+1}(X^0) \to Y^{k+1}$ such that $\alpha^k\varphi^{k+1} = \varphi^k\beta^k$ and

$$f^k\varphi^{k+1} - I(\varphi^0)^k g^k = h^k\beta^k + d_{IX}^k h^{k+1}.$$

This finishes the induction step.

It remains to show that $\alpha^{n+1}S\varphi^0 = \varphi^{n+1}\beta^{n+1}$. Indeed, we have $\alpha^n(\alpha^{n+1}S\varphi^0) = 0$ and $\alpha^n(\varphi^{n+1}\beta^{n+1}) = \varphi^n(\beta^n\beta^{n+1}) = 0$. Moreover,

$$f^{n+1}(\varphi^{n+1}\beta^{n+1}) = (I^n(\varphi^0)g^n - h^n\beta^n)\beta^{n+1} = d_{IX}^n S(\varphi^0) = f^{n+1}(\alpha^{n+1}S\varphi^0).$$

Since d_C^{n-1} is a cokernel of d_C^{n-2}, we have $\alpha^{n+1}S(\varphi^0) = \varphi^{n+1}\beta^{n+1}$, as required.

This shows that $(\underline{\mathcal{M}}, \Sigma, \mathcal{S}(\mathcal{X}))$ satisfies axiom (F3) in the case of standard $(n+2)$-angles. The general case is left to the reader.

(F4) for standard $(n+2)$-angles. We shall show that the mapping cone of the morphism of standard $(n+2)$-angles that we constructed in the proof of axiom (F3) is a $(n+2)$-angle. We keep the notation and morphisms of the previous paragraphs.

For each $k \in \{0, 1 \ldots, n-1\}$ we define

$$r^k := [\varphi^{k+1} \ g^k] \colon X^{k+1} \oplus I^k(Y^0) \longrightarrow Y^{k+1}.$$

Also, we define

$$r^n := \begin{bmatrix} \alpha^{n+1} & 0 \\ \varphi^{n+1} & g^n \end{bmatrix} : X^{n+1} \longrightarrow I^n(Y^0) \to Y^{n+1}$$

and

$$r^{n+1} := \begin{bmatrix} 1_{SX^0} & 0 \\ S\varphi^0 & 1_{SY^0} \end{bmatrix} : SX^0 \oplus SY^0 \longrightarrow SX^0 \oplus SY^0.$$

It is straightforward to check that this defines a morphism of admissible n-exact sequences

$$r \colon X \oplus I(Y^0) \longrightarrow T(SX^0, 0) \oplus Y.$$

(recall that the direct sum of two admissible n-exact sequences is again an admissible n-exact sequence, see Proposition 2.4.7). Since r^{n+1} is an isomorphism, we have that the mapping cone $C(r)$ is an admissible n-exact sequence, see Proposition 2.4.8.

Next, by the definition of $\Sigma\colon \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ there is a commutative diagram

$$\begin{tikzcd}
I(X^0) \arrow[d, "I(\alpha^0)"] & X^0 \arrow[r, tail] \arrow[d, "\alpha^0"] & I^1(X^0) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^n(X^0) \arrow[r, two heads] \arrow[d] & SX^0 \arrow[d, "S\alpha^0"] \\
I(X^1) & X^1 \arrow[r, tail] & I^1(X^1) \arrow[r] & \cdots \arrow[r] & I^n(X^1) \arrow[r, two heads] & SX^1
\end{tikzcd}$$

Then, by Proposition 2.4.9 and the dual of Proposition 2.4.8 there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "s"] & X^1 \arrow[r, tail] \arrow[d, equal] & X^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n+1} \arrow[r, two heads] \arrow[d] & SX^0 \arrow[d, "S\alpha^0"] \\
I(X^1) & X^1 \arrow[r] & I^1(X^1) \arrow[r] & \cdots \arrow[r] & I^n(X^1) \arrow[r] & SX^1
\end{tikzcd}$$

Let $t := 1_{X^1} \oplus 1_{Y^0}$. For each $k \in \{1, \dots, n\}$ we define

$$t^k := \begin{bmatrix} s^k & 0 & 0 \\ 0 & 1_{I^k(Y^0)} & 0 \end{bmatrix} : X^{k+1} \oplus I^k(Y^0) \oplus Y^k \longrightarrow I^k(X^1) \oplus I^1(Y^0)$$

and

$$t^{n+1} := (-1)^n \begin{bmatrix} S\alpha^0 & 0 \\ -S\varphi^0 & \beta^{n+1} \end{bmatrix} : SX^0 \oplus Y^{n+1} \longrightarrow SX^1 \oplus SY^0.$$

It is readily verified that these morphisms define a morphism of admissible n-exact sequences

$$t\colon C(r) \longrightarrow I(X^1) \oplus I(Y^0).$$

Finally, applying Lemma 2.5.10 to the morphism t yields that the sequence

$$X^1 \oplus Y^0 \xrightarrow{\gamma^0} X^2 \oplus Y^1 \xrightarrow{\gamma^1} \cdots \xrightarrow{\gamma^n} \Sigma X^0 \oplus Y^{n+1} \xrightarrow{\gamma^{n+1}} \Sigma X^1 \oplus \Sigma Y^0$$

where for each $k \in \{0, \dots, n\}$ we have

$$\gamma^k = \begin{bmatrix} -\overline{\alpha}^{k+1} & 0 \\ \overline{\varphi}^{k+1} & \overline{\beta}^k \end{bmatrix} : X^{k+1} \oplus Y^k \longrightarrow X^{k+2} \oplus Y^{k+1}$$

is a standard $(n+2)$-angle.

This shows that $(\underline{\mathcal{M}}, \Sigma, \mathcal{S}(\mathcal{X}))$ satisfies axiom (F4) in the case of standard $(n+2)$-angles. The general case is left to the reader. $\square$

Theorem 2.5.11 allows us to define the following class of $(n+2)$-angulated categories.

Definition 2.5.12. We say that a $(n+2)$-angulated category $(\mathcal{F}, \Sigma_{\mathcal{F}}, \mathcal{S})$ is *algebraic* if there exists a Frobenius n-exact category $(\mathcal{M}, \mathcal{X})$ together with an equivalence of $(n+2)$-angulated categories between $(\underline{\mathcal{M}}, \Sigma_{\mathcal{M}}, \mathcal{S}(\mathcal{X}))$ and $(\mathcal{F}, \Sigma_{\mathcal{F}}, \mathcal{S})$.

2.5.3. Standard construction. We remind the reader of the definition of an n-cluster-tilting subcategory of a triangulated category.

Definition 2.5.13. Let $(\mathcal{T}, \Sigma, \mathcal{S})$ be a triangulated category and $\mathcal{M}$ a subcategory of $\mathcal{T}$. We say that $\mathcal{M}$ is an *n-cluster-tilting subcategory of* $\mathcal{T}$ if $\mathcal{M}$ is functorially finite (see subsection 2.2.1) in $\mathcal{T}$ and

$$\begin{aligned} \mathcal{M} &= \left\{X \in \mathcal{T} \mid \forall i \in \{1, \dots, n-1\} \ \mathsf{Ext}^i_{\mathcal{T}}(X, \mathcal{M}) = 0\right\} \\ &= \left\{X \in \mathcal{T} \mid \forall i \in \{1, \dots, n-1\} \ \mathsf{Ext}^i_{\mathcal{T}}(\mathcal{M}, X) = 0\right\}. \end{aligned}$$

In [**39**, Sec. 4], Geiß-Keller-Oppermann give a standard construction of $(n+2)$-angulated categories from n-cluster-tilting categories of a triangulated category which are closed under the n-th power of the suspension functor. More precisely, they prove the following theorem.

Theorem 2.5.14. [**39**, Thm. 1] *Let $(\mathcal{T}, \Sigma_3, \mathcal{S})$ be a triangulated category with an n-cluster-tilting subcategory $\mathcal{C}$ such that $\Sigma_3^n(\mathcal{C}) \subseteq \mathcal{C}$. Then, $(\mathcal{C}, \Sigma_3^n, \mathcal{S}(\mathcal{C}))$ is an $(n+2)$-angulated category where $\mathcal{S}(\mathcal{C})$ is the class of all sequences*

$$X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} X^2 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} X^{n+1} \xrightarrow{\alpha^{n+1}} \Sigma X^0.$$

such that there exists a diagram

$$\begin{tikzcd} & X^1 \arrow[rr, "\alpha^1"] \arrow[dr] & & X^2 \arrow[dr] & & \cdots & & X^n \arrow[dr, "\alpha^n"] & \\ X^0 \arrow[ur, "\alpha^0"] & & X^{1.5} \arrow[ll, "|" marking] \arrow[ur] & & X^{2.5} \arrow[ll, "|" marking] & \cdots & X^{n-1.5} \arrow[ur] & & X^{n+1} \arrow[ll, "|" marking] \end{tikzcd}$$

with $X^k \in \mathcal{C}$ for all $k \in \mathbb{Z}$ such that all oriented triangles are triangles in $\mathcal{T}$, all non-oriented triangles commute, and α^{n+1} is the composition along the lower edge of the diagram.

Our aim is to give an analogous construction for Frobenius n-exact categories. For this, we need some terminology.

Let $(\mathcal{E}, \mathcal{X})$ be a Frobenius exact category and $E \in \mathcal{E}$. An *n-th cosyzygy* $\mho^n(E)$ *of* E is defined by an acyclic complex

$$E \rightarrowtail I^1 \longrightarrow \cdots \longrightarrow I^n \twoheadrightarrow \mho^n(E)$$

where for all $k \in \{1, \dots, n\}$ the object I^k is $\mathcal{X}$-injective.

Proposition 2.5.15. *Let $(\mathcal{E}, \mathcal{X})$ be an exact category, $\mathcal{M}$ an n-cluster-tilting subcategory of $\mathcal{E}$ and $M \in \mathcal{M}$. If an n-th cosyzygy $\mho^n(M)$ of M satisfies $\mho^n(M) \in \mathcal{M}$, then so does any other n-th cosyzygy $\tilde{\mho}^n(M)$ of M.*

Proof. Note that for all $k \in \{1, \dots, n-1\}$ we have

$$\mathsf{Ext}^k_{\mathcal{E}}(-, \mho^n(E)) \cong \mathsf{Ext}^{k+n}_{\mathcal{E}}(-, E) \cong \mathsf{Ext}^k_{\mathcal{P}}(-, \tilde{\mho}^n(E)).$$

Then, it follows from the definition of n-cluster-tilting subcategory that $\mho^n(E) \in \mathcal{M}$ if and only if $\tilde{\mho}^n(E) \in \mathcal{M}$. $\square$

Let $(\mathcal{E}, \mathcal{X})$ be a Frobenius exact category and for each $E \in \mathcal{E}$ fix a choice of n-th cosyzygy $\mho^n(E)$ of E. This defines a map on objects $\mho^n \colon \mathsf{Obj}(\mathcal{E}) \to \mathsf{Obj}(\mathcal{E})$. Note that if $\mathcal{M}$ is an n-cluster-tilting subcategory of $\mathcal{E}$, then Proposition 2.5.15 shows that the condition $\mho^n(\mathcal{M}) \subseteq \mathcal{M}$ is independent of the choice of $\mho^n$. We have the following result, which is closely related to Theorem 2.5.14.

THEOREM 2.5.16. *Let $(\mathcal{E}, \mathcal{X})$ be a Frobenius exact category with an n-cluster-tilting subcategory $\mathcal{M}$ such that $\mho^n(\mathcal{M}) \subseteq \mathcal{M}$, and let $(\mathcal{M}, \mathcal{Y})$ be the n-exact structure on $\mathcal{M}$ given in Theorem 2.4.14. Then, the following statements hold:*

(i) The pair $(\mathcal{M}, \mathcal{Y})$ is a Frobenius n-exact category.
(ii) Let $(\underline{\mathcal{E}}, \Sigma_{\underline{\mathcal{E}}}, \mathcal{S}(\mathcal{X}))$ be the standard triangulated structure of $\underline{\mathcal{E}}$. Then, $\underline{\mathcal{M}}$ is an n-cluster-tilting subcategory of $\underline{\mathcal{E}}$.
(iii) Let $(\underline{\mathcal{M}}, \Sigma, \mathcal{S}(\underline{\mathcal{M}}))$ be the standard $(n+2)$-angulated structure of $\underline{\mathcal{M}}$. Then, we have an equivalence of $(n+2)$-angulated categories between $(\underline{\mathcal{M}}, \Sigma, \mathcal{S}(\underline{\mathcal{M}}))$ and $(\underline{\mathcal{M}}, \Sigma^n_{\underline{\mathcal{E}}}, \mathcal{S}(\underline{\mathcal{M}}))$.

PROOF. (i) By Theorem 2.4.14 we have that $(\mathcal{M}, \mathcal{Y})$ is an n-exact category; thus we only need to show that it is Frobenius. Indeed, note that the definition of n-cluster-tilting subcategory implies that $\mathcal{M}$ contains all $\mathcal{X}$-injective objects. Moreover, since $\mathcal{X}$-admissible monomorphisms with terms in $\mathcal{M}$ are precisely the $\mathcal{Y}$-admissible monomorphisms, all $\mathcal{X}$-injectives are also $\mathcal{Y}$-injectives. Finally, the condition $\mho(\mathcal{M}) \subseteq \mathcal{M}$ implies that $(\mathcal{M}, \mathcal{Y})$ has enough $\mathcal{Y}$-injectives. By duality, $(\mathcal{M}, \mathcal{Y})$ has enough $\mathcal{Y}$-projectives (and they are the $\mathcal{X}$-projectives). Since $\mathcal{X}$-projectives and $\mathcal{X}$-projectives coincide, this shows that $(\mathcal{M}, \mathcal{Y})$ is a Frobenius n-exact category.

(ii) This statement follows readily from the definitions.

(iii) For simplicity, we assume that $\Sigma_{\underline{\mathcal{M}}} = \Sigma^n_{\underline{\mathcal{E}}}$. By Proposition 2.5.2 it is enough to show that $\mathcal{S}(\underline{\mathcal{M}}) \subseteq \mathcal{S}(\mathcal{Y})$. For this, recall that a standard triangle $A \xrightarrow{\overline{u}} B \xrightarrow{\overline{v}} C \xrightarrow{\overline{w}} \Sigma_{\underline{\mathcal{E}}} A$ in $\mathcal{S}(\mathcal{X})$ is given by a morphism of admissible $\mathcal{X}$-exact sequences

$$\begin{tikzcd}
A \arrow[r, tail, "u'"] \arrow[d, "u"] & I(A) \arrow[r, two heads, "u"] \arrow[d, "v'"] & SA \arrow[d, equal] \\
B \arrow[r, "v"] & C \arrow[r, two heads, "w"] & SA
\end{tikzcd}$$

where $I(A)$ is an $\mathcal{X}$-injective object. By Proposition 2.4.8 this gives rise to an $\mathcal{X}$-admissible exact sequence

$$A \rightarrowtail^{\begin{bmatrix} u' \\ u \end{bmatrix}} I(A) \oplus B \xrightarrow{\begin{bmatrix} v' & v \end{bmatrix}} \!\!\!\!\twoheadrightarrow C.$$

Consider a $(n+2)$-angle $X \in \mathcal{S}(\underline{\mathcal{M}})$

$$\begin{tikzcd}[column sep=small]
X\colon & & X^1 \arrow[rr, "\overline{\alpha^1}"] \arrow[dr] & & X^2 \arrow[dr] & & \cdots & & X^n \arrow[dr, "\overline{\alpha^n}"] & \\
& X^0 \arrow[ur, "\overline{\alpha^0}"] & & X^{1.5} \arrow[ll, "|" marking] \arrow[ur] & & X^{2.5} \arrow[ll, "|" marking] & \cdots & X^{n-1.5} \arrow[ur] & & X^{n+1} \arrow[ll, "|" marking]
\end{tikzcd}$$

such that each of the involved triangles is a standard triangle in $(\underline{\mathcal{E}}, \mathcal{S}(\mathcal{X}))$. By gluing the associated $\mathcal{X}$-admissible exact sequences associated to each of these triangle we obtain a $\mathcal{Y}$-admissible n-exact sequence

$$X^0 \rightarrowtail I(X^0) \oplus X^1 \longrightarrow \cdots \longrightarrow I(X^{n-1.5}) \oplus X^n \twoheadrightarrow X^{n+1}.$$

Lemma 2.5.10 implies that this n-exact sequence induces a standard $(n+2)$-angle $X' \in \mathcal{S}(\mathcal{Y})$

$$X'\colon \quad X^0 \xrightarrow{\overline{\alpha^0}} X^1 \xrightarrow{\overline{\alpha^1}} \cdots \xrightarrow{\overline{\alpha^n}} X^{n+1} \longrightarrow \Sigma_{\underline{\mathcal{M}}} X^0.$$

Finally, a straightforward verification shows that one can take $X^{n+1} \to \Sigma_{\underline{\mathcal{M}}} X^0$ in X' equal to $\overline{\alpha^{n+1}}$ showing that $X = X'$ and the result follows. For example, for

$n = 2$ the last claim follows from Lemma 2.5.10 and the existence of a commutative diagram

$$
\begin{tikzcd}
X^0 \arrow[r, tail] \arrow[dd, equal] & I(X^0) \oplus X^1 \arrow[rr] \arrow[dd, equal] \arrow[dr, two heads] & & I(X^{1.5}) \oplus X^2 \arrow[r, two heads] \arrow[dd] & X^3 \arrow[dd, "-\beta^3"] \\
& & X^{1.5} \arrow[ur, tail] \arrow[dd, equal] & & \\
X^0 \arrow[r, tail] \arrow[dd, equal] & I(X^0) \oplus X^1 \arrow[rr] \arrow[dd] \arrow[dr, two heads] & & I(X^{1.5}) \arrow[r, two heads] \arrow[dd] & \Sigma X^{1.5} \arrow[dd, "-\Sigma_{\underline{\mathcal{M}}}\beta^{1.5}"] \\
& & X^{1.5} \arrow[ur, tail] \arrow[dd, "-\beta^{1.5}"] & & \\
X^0 \arrow[r, tail] & I(X^0) \arrow[rr] \arrow[dr, two heads] & & I(\Sigma_{\underline{\mathcal{M}}} X^0) \arrow[r, two heads] & \Sigma_{\underline{\mathcal{M}}}^2 X^0 \\
& & \Sigma_{\underline{\mathcal{M}}} X^0 \arrow[ur, tail] & &
\end{tikzcd}
$$

The diagram needed for the general case can be easily inferred from the diagram above. $\square$

2.6. Examples

We conclude this article with a collection of examples of n-abelian, n-exact categories and algebraic $(n+2)$-angulated categories. Most of the examples we present are known, and all of them arise as n-cluster-tilting subcategories in different contexts. Our main tools in this section are Theorems 2.3.16, 2.4.14, and 2.5.16. In the remainder, K denotes an algebraically closed field and all algebras are finite dimensional over K.

2.6.1. n-representation finite algebras. The class of n-representation finite algebras was introduced by Iyama-Oppermann in [**59**] in the context of higher Auslander-Reiten theory as higher analogs of representation-finite algebras.

Definition 2.6.1. [**59**, Def. 2.2] Let Λ be a finite dimensional algebra over a field K.

(i) A Λ-module $M \in \mathsf{mod}\,\Lambda$ is an *n-cluster-tilting module* if $\mathsf{add}\,M$ is an n-cluster-tilting subcategory of $\mathsf{mod}\,\Lambda$. Note that Theorem 2.3.16 implies that $\mathsf{add}\,M$ is an n-abelian category.
(ii) We say that Λ is *n-representation-finite* if $\mathsf{gl.dim.}\,\Lambda = n$ and there exists an n-cluster-tilting Λ-module.

The following result, observed jointly with Martin Herschend, gives examples of n-abelian categories for every positive integer n.

Proposition 2.6.2. *Let $n \geq 1$ and $m \geq 0$. Also, let $\vec{A}_{nm+1}$ be the linearly oriented quiver of Dynkin type A with $nm+1$ vertices, J be the Jacobson radical of the path algebra $K\vec{A}_{nm+1}$, and $\Lambda := K\vec{A}_{nm+1}/J^2$. Then, the following statements hold:*

(i) There exists a unique basic n-cluster-tilting Λ-module M.
(ii) The category $\mathsf{add}\,M \subseteq \mathsf{mod}\,\Lambda$ is n-abelian.

PROOF. We assume that $\vec{A}_{nm+1}$ has vertices $0, 1, \dots, nm$. Let S_i be the simple module concentrated at the vertex i, and P_i the indecomposable projective Λ-module with top P_i. The Auslander-Reiten quiver of $\mathsf{mod}\,\Lambda$ is given by

$$\begin{tikzcd}
& P_1 \arrow[dr] & & P_2 \arrow[dr] & & & & P_{nm} \arrow[dr] & \\
P_0 = S_0 \arrow[ur] \arrow[rr, dotted, no head] & & S_1 \arrow[ur] \arrow[rr, dotted, no head] & & S_2 & \cdots & S_{nm-1} \arrow[ur] \arrow[rr, dotted, no head] & & S_{nm}
\end{tikzcd}$$

It is straightforward to verify that the module

$$M := \Lambda \oplus S_n \oplus S_{2n} \oplus \cdots \oplus S_{(m-1)n} \oplus S_{nm}$$

is the unique basic n-cluster-tilting Λ-module. $\square$

For a finite dimensional algebra Λ of global dimension n and define

$$\tau_n := D\,\mathsf{Ext}^n_\Lambda(-,\Lambda)\colon\ \mathsf{mod}\,\Lambda \longrightarrow \mathsf{mod}\,\Lambda.$$

The following result gives concrete information on the n-cluster-tilting Λ-modules when Λ is n-representation-finite.

PROPOSITION 2.6.3. [**57**, Prop. 1.3(b)] *Let Λ be an n-representation-finite algebra and let $I_1, \dots, I_d$ be a complete set of representatives of the isomorphism classes of indecomposable injective Λ-modules. Then, the following statements hold:*

(i) There exists a permutation $\sigma \in \mathfrak{S}_d$ and positive integers $\ell_1, \dots, \ell_d$ such that for all k we have $\tau_n^{\ell_k}(I_k) \cong P_{\sigma(k)}$.

(ii) There exists a unique basic n-cluster-tilting Λ-module M. Moreover, M is given by the direct sum of the following pairwise non-isomorphic indecomposable Λ-modules:

$$\begin{array}{llllll}
I_1, & \tau_n(I_1), & \tau_n^2(I_1), & \cdots & \tau_n^{\ell_1-1}(I_1), & \tau_n^{\ell_1}(I_1) \cong P_{\sigma(1)} \\
I_2, & \tau_n(I_2), & \tau_n^2(I_2), & \cdots & \tau_n^{\ell_2-1}(I_2), & \tau_n^{\ell_2}(I_2) \cong P_{\sigma(2)} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
I_d, & \tau_n(I_d), & \tau_n^2(I_d), & \cdots & \tau_n^{\ell_d-1}(I_d), & \tau_n^{\ell_d}(I_d) \cong P_{\sigma(d)}
\end{array}$$

Let Λ be a finite dimensional algebra such that $\mathsf{gl.dim.}\,\Lambda \leq n$. Following [**71**], the $(n+1)$*-preprojective algebra of* Λ is defined as the tensor algebra

$$\Pi_{n+1}(\Lambda) := \bigoplus_{d \geq 0} \mathsf{Ext}^n_\Lambda(D\Lambda, \Lambda)^{\otimes_\Lambda d}.$$

The following is a structure theorem for 2-representation-finite algebras; it allows to produce examples of such algebras rather easily. We refer the reader to [**51, 31**] for details and definitions.

THEOREM 2.6.4. [**51**, Thm. 3.11] *Let Λ be a finite dimensional algebra such that $\mathsf{gl.dim.}\,\Lambda = 2$. Then, the following statements are equivalent:*

(i) The algebra Λ is 2-representation-finite.

(ii) The algebra $\Pi_3(\Lambda)$ is a finite dimensional selfinjective algebra.

(iii) There exists a quiver with potential with a cut $(Q, W; C)$ such that the Jacobian algebra $J(Q, W)$ is a finite dimensional selfinjective algebra and the truncated Jacobian algebra $J(Q, W; C)$ is isomorphic to Λ.

Let Λ be a representation-finite algebra, *i.e.* such that the set of isomorphism classes of indecomposable Λ-modules is finite. We remind the reader that the *Auslander algebra associated to* Λ is the endomorphism algebra of a basic Λ-module M such that $\mathsf{add}\,M = \mathsf{mod}\,\Lambda$.

EXAMPLE 2.6.5. Typical examples of 2-representation-finite algebras, hence sources of 2-abelian categories, are the Auslander algebras associated to KQ where Q is a Dynkin quiver $\vec{A}_m$, see [**51**, Sec. 9.2].

2.6.2. n-representation infinite algebras. The class of n-representation-infinite algebras was introduced by Herschend-Iyama-Oppermann in [**53**] as a higher analog of representation-infinite hereditary algebras from the viewpoint of higher Auslander-Reiten theory. These class of algebras complements that of n-representation-finite algebras.

Let Λ be a finite dimensional algebra with finite global dimension. Then, $\mathrm{D^b}(\mathsf{mod}\,\Lambda)$, the bounded derived category of $\mathsf{mod}\,\Lambda$, has a Serre functor

$$\nu := - \otimes_{\Lambda}^{L} D\Lambda \colon \mathrm{D^b}(\mathsf{mod}\,\Lambda) \longrightarrow \mathrm{D^b}(\mathsf{mod}\,\Lambda).$$

We define $\nu_n := \nu[-n]$.

DEFINITION 2.6.6. [**53**, Def. 2.7] Let Λ be a finite dimensional algebra such that $\mathsf{gl.dim.}\,\Lambda = n$. We say that Λ is *n-representation-infinite* if for all $i \geq 0$ we have $\nu_n^{-i}(\Lambda) \in \mathsf{mod}\,\Lambda$.

Let $\mathcal{T}$ be a triangulated category. Following [**20**], a *t-structure on* $\mathcal{T}$ is a pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of strictly full subcategories of $\mathcal{T}$ which satisfies the following properties:

(i) We have $\Sigma\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 0}$ and $\Sigma^{-1}\mathcal{T}^{\geq 0} \subseteq \mathcal{T}^{\geq 0}$.
(ii) For all $X \in \mathcal{T}^{\leq 0}$ and for all $Y \in \mathcal{T}^{\geq 0}$ we have $\mathcal{T}(X, \Sigma^{-1}Y) = 0$.
(iii) For each $X \in \mathcal{T}$ there exists a triangle $X' \to X \to X'' \to \Sigma X'$ with $X' \in \mathcal{T}^{\leq 0}$ and $X'' \in \Sigma^{-1}\mathcal{T}^{\geq 0}$.

The *heart* of $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is by definition $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$. The heart of a t-structure is always an abelian category.

Note that $\mathrm{D^b}(\mathsf{mod}\,\Lambda)$ has a standard t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defined by

$$\mathcal{D}^{\leq 0} := \left\{ X \in \mathrm{D^b}(\mathsf{mod}\,\Lambda) \mid \forall i > 0 \; H^i(X) = 0 \right\},$$

$$\mathcal{D}^{\geq 0} := \left\{ X \in \mathrm{D^b}(\mathsf{mod}\,\Lambda) \mid \forall i < 0 \; H^i(X) = 0 \right\}.$$

The heart of $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is precisely $\mathsf{mod}\,\Lambda$.

THEOREM 2.6.7. [**77**, Thm. 3.7] *Let Λ be an n-representation infinite algebra such that $\Pi_{n+1}(\Lambda)$ is noetherian. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be the standard t-structure of $\mathrm{D^b}(\mathsf{mod}\,\Lambda)$ and define*

$$\mathcal{X}^{\leq 0} := \left\{ X \in \mathrm{D^b}(\mathsf{mod}\,\Lambda) \mid \nu_n^{-i}(X) \in \mathcal{D}^{\leq 0} \, \forall i \gg 0 \right\}$$

$$\mathcal{X}^{\geq 0} := \left\{ X \in \mathrm{D^b}(\mathsf{mod}\,\Lambda) \mid \nu_n^{-i}(X) \in \mathcal{D}^{\geq 0} \; \forall i \gg 0 \right\}.$$

Then, the pair $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$ is a t-structure in $\mathrm{D^b}(\mathsf{mod}\,\Lambda)$. Moreover, the heart of this t-structure is equivalent to the non-commutative projective scheme $\mathsf{qgr}\,\Pi_{n+1}(\Lambda)$, see [**7**] *for the definition.*

The following result gives examples of n-exact categories.

THEOREM 2.6.8. [**52**] *Let Λ be an n-representation infinite algebra such that $\Pi_{n+1}(\Lambda)$ is noetherian. Let $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$ be the t-structure defined in Theorem* 2.6.7 *and $\mathcal{H}$ be its heart. Then, the following statements hold:*

(i) The category

$$\mathcal{E} := \left\{ X \in \mathcal{H} \mid \nu_n^{i}(X) \in (\mathsf{mod}\,\Lambda)[-n] \; \forall i \gg 0 \right\}.$$

is an extension closed subcategory of $\mathcal{H}$.

(ii) *The category*
$$\mathcal{U} := \mathsf{add}\left\{\nu_n^{-i}(\Lambda) \mid i \in \mathbb{Z}\right\}$$
is an n-cluster-tilting subcategory of $\mathcal{E}$.

With the notation of Theorem 2.6.8, note that $\mathcal{E}$ is an exact category hence Theorem 2.4.14 implies that $\mathcal{U}$ is an n-exact category.

We now give a concrete example of an n-exact category constructed using Theorem 2.6.8.

EXAMPLE 2.6.9. Let $\mathsf{coh}\,\mathbb{P}^n_K$ be the category of coherent sheaves over the projective n-space over K, and let Λ be the endomorphism algebra of the tilting bundle $\mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(n)$, see [**19**]. It is known that Λ is an n-representation-infinite algebra and that $\Pi_{n+1}(\Lambda)$ is noetherian, see [**53**, Ex. 2.15]. Moreover, there is an equivalence of triangulated categories
$$\mathrm{D^b}(\mathsf{mod}\,\Lambda) \cong \mathrm{D^b}(\mathsf{coh}\,\mathbb{P}^n_K).$$
This equivalence induces an equivalence of exact categories between the category $\mathcal{E}$ given in Theorem 2.6.8 and $\mathsf{vect}\,\mathbb{P}^n_K$, the category of vector bundles over $\mathbb{P}^n_K$. Also, it induces an equivalence of additive categories
$$\mathcal{U} \cong \mathsf{add}\left\{\mathcal{O}(i) \mid i \in \mathbb{Z}\right\}.$$
Finally, Theorem 2.4.14 implies that $\mathcal{U}$ is an n-exact category with respect to the class of all exact sequences
$$0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow 0$$
with all terms in $\mathcal{U}$.

2.6.3. Relative n-cluster-tilting subcategories. Let Λ be a finite-dimensional algebra and T a Λ-module. It is easy to see that the perpendicular category
$$T^{\perp_{>0}} := \left\{M \in \mathsf{mod}\,\Lambda \mid \forall k > 0,\ \mathsf{Ext}^k_\Lambda(T, M) = 0\right\}$$
is exact for it is an extension closed subcategory of $\mathsf{mod}\,\Lambda$. We have the following result.

PROPOSITION 2.6.10. [**57**, Cor. 1.15] *Let Q be a Dynkin quiver. Then, there exists a tilting KQ-module T of projective dimension 1 such that $T^{\perp_{>0}}$ contains a 2-cluster-tilting subcategory $\mathcal{M}$.*

REMARK 2.6.11. With the notation of Proposition 2.6.10, the category $\mathcal{M}$ is 2-exact by Theorem 2.4.14.

More generally, in [**57**, Cor. 1.16] for each n an algebra of global dimension at most n such that there exists a tilting Λ-module of finite T projective dimension and $T^{\perp_{>0}}$ has an n-cluster-titling subcategory was constructed.

2.6.4. Isolated singularities. Let R be a commutative complete Gorenstein ring of dimension n with residue field K. The category of *Cohen-Macaulay R-modules* is by definition
$$\mathsf{CM}\,R := \{M \in \mathsf{mod}\,R \mid \mathsf{depth}\,M = n\}.$$
Note that $\mathsf{CM}\,R$ is a Frobenius exact category.

We remind the reader that R is an *isolated singularity* if R is not regular and for all non-maximal prime ideals $\mathfrak{p} \subset R$ we have that $R_{\mathfrak{p}}$ is a regular ring. In this case, $\mathsf{CM}\,R$ has almost-split sequences [**14, 87**].

THEOREM 2.6.12. [**56**, Thm. 2.5] *and* [**62**, Cor. 8.2] *Let K be an algebraically closed field of characteristic 0 and set $S := K[\![x_0, x_1, \dots, x_n]\!]$. Also, let G be a finite subgroup of $\mathsf{SL}_{n+1}(K)$ such that every element $\sigma \neq 1$ of G does not have eigenvalue 1. Then G acts on S in a natural way and we define*

$$S^G := \{s \in S \mid \forall g \in G,\ g \cdot s = s\}.$$

Then, the following statements hold:

(i) *The ring S^G is an isolated singularity.*
(ii) *We have $S \in \mathsf{CM}\, S^G$.*
(iii) *The category $\mathsf{add}\, S$ is an n-cluster-tilting subcategory of $\mathsf{CM}\, S^G$.*

With the notation of Theorem 2.6.12, note that Theorem 2.4.14 implies that $\mathsf{add}\, S$ is an n-exact category with respect to the class of all exact sequences

$$0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow 0$$

in $\mathsf{mod}\, S^G$ with terms in $\mathsf{add}\, S$.

2.6.5. Algebraic $(n{+}2)$-angulated categories. In this subsection we revisit the examples of [**39**, Sec. 6.3] from the viewpoint of algebraic $(n+2)$-angulated categories. We remind the reader that we say that an object T in a triangulated category $\mathcal{C}$ is *n-cluster-tilting* if $\mathsf{add}\, T$ is an n-cluster-tilting subcategory of $\mathcal{C}$.

Let $\mathcal{C}$ be a algebraic triangulated category. Hence, there exists a Frobenius exact category $(\mathcal{E}, \mathcal{X})$ such that $\underline{\mathcal{E}}$ is equivalent to $\mathcal{C}$ as triangulated categories. It is easy to see that each n-cluster-tilting subcategory of $\mathcal{C}$ lifts to an n-cluster-tilting subcategory of $\mathcal{E}$ by including all the $\mathcal{X}$-injective objects in $\mathcal{E}$. By Theorem 2.5.16, every $(n+2)$-angulated category constructed using Theorem 2.5.14 from an algebraic triangulated category is in turn an algebraic $(n+2)$-angulated category. Known examples of algebraic $(n+2)$-angulated categories arising in this way are the following:

- Let $\mathcal{C} = \mathcal{C}_Q$ be the cluster category associated with an acyclic quiver Q, see [**27**] for details. It is known that $\mathcal{C}$ is an algebraic triangulated category [**70**]. Moreover, a basic 2-cluster-tilting object $T \in \mathcal{C}$ satisfies $\Sigma^2 T \cong T$ if and only if $\mathcal{C}(T, T)$ is a selfinjective algebra, see [**60**, Cor. 3.8]. All such algebras were classified by Ringel in [**81**]. In particular, such algebras exist only if Q is a Dynkin quiver of type D (including $D_3 = A_3$). Hence, if T is a 2-cluster-tilting object in $\mathcal{C}$ such that $\Sigma^2 T \cong T$, then $\mathsf{add}\, T \subseteq \mathcal{C}$ is an algebraic 4-angulated category.
- Let $\mathcal{C} = \mathcal{C}_{\mathbb{X}}$ be the cluster category associated with a weighted projective line, see [**18**] for details. As in the previous case, a 2-cluster-tilting object $T \in \mathcal{C}$ satisfies $\Sigma^2 T \cong T$ if and only if $\mathcal{C}(T, T)$ is a selfinjective algebra. All such algebras are classified in [**65**, Thm. 1.3]. Such algebras exist if and only if $\mathbb{X}$ has tubular weight type $(2, 2, 2, 2)$, $(2, 4, 4)$, or $(2, 3, 6)$. If T is a 2-cluster-tilting object in $\mathcal{C}$ such that $\Sigma^2 T \cong T$, then $\mathsf{add}\, T \subseteq \mathcal{C}$ is an algebraic 4-angulated category.
- Let Λ be the preprojective algebra of type A_n. Recall that $\mathsf{mod}\, \Lambda$ is a Frobenius abelian category hence the stable category $\underline{\mathsf{mod}}\Lambda$ is triangulated. It is known that the standard 2-cluster-tilting Λ-module T corresponding to the linear orientation of A_n satisfies $\mho^2(T) \cong T$, see [**40**]. It follows that $\mathsf{add}\, T \subseteq \mathsf{mod}\, \Lambda$ is a Frobenius 2-exact category and thus $\underline{\mathsf{add}}\, T \subseteq \underline{\mathsf{mod}}\Lambda$ is an algebraic 4-angulated category.
- Let Λ be an n-representation-finite algebra. Then, [**60**, Cor. 3.7] implies that the canonical n-cluster-tilting object $\pi\Lambda$ in the associated Amiot n-cluster category $\mathcal{C}$ is stable under the Serre functor Σ^n. It is known

that $\mathcal{C}$ is an algebraic triangulated category, see [**60**, Thm. 4.15], hence $\mathsf{add}\,\pi\Lambda \subseteq \mathcal{C}$ is an algebraic $(n+2)$-angulated category.

We refer the reader to [**39**] for more details.

CHAPTER 3

Reduction of τ-tilting modules and torsion classes

The class of support τ-tilting modules was introduced recently by Adachi, Iyama and Reiten. These modules complete the class of tilting modules from the point of view of mutations. Given a finite dimensional algebra A, we study all basic support τ-tilting A-modules which have a given basic τ-rigid A-module as a direct summand. We show that there exist an algebra C such that there exists a bijection between these modules and all basic support τ-tilting C-modules; we call this process τ-tilting reduction. An important step in this process is the formation of τ-perpendicular categories which are analogs of ordinary perpendicular categories. We give several examples to illustrate this procedure. Finally, we show that τ-tilting reduction is compatible with silting reduction in triangulated categories (satisfying suitable finiteness conditions) with a silting object and Calabi-Yau reduction in 2-Calabi-Yau categories with a cluster-tilting object. The contents of this chapter are available in preprint form in [**63**].

3.1. Introduction

Let A be a finite dimensional algebra over a field. Recently, Adachi, Iyama and Reiten introduced in [**1**] a generalization of classical tilting theory, which they called *τ-tilting theory.* Motivation to study τ-tilting theory comes from various sources, the most important one is mutation of tilting modules. Mutation of tilting modules has its origin in Bernstein-Gelfand-Ponomarev reflection functors [**24**], which were later generalized by Auslander, Reiten and Platzeck with the introduction of APR-tilting modules [**10**], which are obtained by replacing a simple direct summand of the tilting A-module A. Mutation of tilting modules was introduced in full generality by Riedtmann and Schofield in their combinatorial study of tilting modules [**80**]. Also, Happel and Unger showed in [**47**] that tilting mutation is intimately related to the partial order of tilting modules induced by the inclusion of the associated torsion classes.

We note that one limitation of mutation of tilting modules is that it is not always possible. This is the motivation for the introduction of τ-tilting theory. Support τ-tilting (resp. τ-rigid) A-modules are a generalization of tilting (resp. partial-tilting) A-modules defined in terms of the Auslander-Reiten translation, see Definition 3.2.8. Support τ-tilting modules can be regarded as a “completion” of the class of tilting modules from the point of view of mutation. In fact, it is shown in [**1**, Thm. 2.17] that a basic almost-complete support τ-tilting A-module is the direct summand of exactly two basic support τ-tilting A-modules. This means that mutation of support τ-tilting A-modules is always possible.

It is then natural to consider more generally all support τ-tilting A-modules which have a given τ-rigid A-module U as a direct summand. Our main result is the following bijection:

Theorem 3.1.1 (see Theorem 3.3.15 for details). *Let U be a basic τ-rigid A-module. Then there exists a finite dimensional algebra C such that there is an order-preserving bijection between the set of isomorphism classes of basic support τ-tilting*

A-modules which have U as a direct summand and the set of isomorphism classes of all basic support τ-tilting C-modules. We call this process τ-tilting reduction*.*

As a special case of Theorem 3.1.1 we obtain an independent proof of [**1**, Thm. 2.17].

Corollary 3.1.2 (Corollary 3.3.17). *Every almost-complete support τ-tilting A-module is the direct summand of exactly two support τ-tilting A-modules.*

If we restrict ourselves to hereditary algebras, then Theorem 3.1.1 gives the following improvement of [**48**, Thm. 3.4], where U is assumed to be faithful.

Corollary 3.1.3 (Corollary 3.3.18). *Let A be a hereditary algebra and U be a basic partial-tilting A-module. Then there exists a hereditary algebra C such that there is an order-preserving bijection between the set of isomorphism classes of basic support tilting A-modules which have U as a direct summand and the set of isomorphism classes of all basic support tilting C-modules.*

Now we explain a category equivalence which plays a fundamental role in the proof of Theorem 3.1.1, and which is of independent interest. Given a τ-rigid module U, there are two torsion pairs in $\mathsf{mod}\,A$ which are naturally associated to U. Namely, $(\mathsf{Fac}\,U, U^{\perp})$ and $({}^{\perp}(\tau U), \mathsf{Sub}(\tau U))$. We have the following result about the category ${}^{\perp}(\tau U) \cap U^{\perp}$, which is an analog of the perpendicular category associated with U in the sense of [**37**], see Example 3.3.4.

Theorem 3.1.4 (Theorem 3.3.8). *With the hypotheses of Theorem 3.1.1, let T_U be the Bongartz completion of U in $\mathsf{mod}\,A$. Then, the functor $\mathsf{Hom}_A(T_U, -)$: $\mathsf{mod}\,A \to \mathsf{mod}(\mathsf{End}_A(T_U))$ induces an equivalence of exact categories*

$$F : {}^{\perp}(\tau U) \cap U^{\perp} \longrightarrow \mathsf{mod}\,C.$$

It is shown in [**1**, Thm. 2.2] that basic support τ-tilting A-modules are precisely the Ext-progenerators of functorially finite torsion classes in $\mathsf{mod}\,A$. The proof of Theorem 3.1.1 makes heavy use of the bijection between functorially finite torsion classes in $\mathsf{mod}\,A$ and basic support τ-tilting A-modules. The following result extends the bijection given in Theorem 3.1.1, as the torsion classes involved do not need to be functorially finite:

Theorem 3.1.5 (Theorem 3.3.12). *With the hypotheses of Theorem 3.1.1, the map*

$$\mathcal{T} \mapsto F(\mathcal{T} \cap U^{\perp})$$

induces a bijection between torsion classes $\mathcal{T}$ in $\mathsf{mod}\,A$ such that $\mathsf{Fac}\,U \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)$ and torsion classes in $\mathsf{mod}\,C$, where F is the equivalence obtained in Theorem 3.1.4.

We would like to point out that support τ-tilting modules are related with important classes of objects in representation theory: silting objects in triangulated categories and cluster-tilting objects in 2-Calabi-Yau triangulated categories. On one hand, if $\mathcal{T}$ is a triangulated category satisfying suitable finiteness conditions with a silting object S, then there is a bijection between basic silting objects contained in the subcategory $S * S[1]$ of $\mathcal{T}$ and basic support τ-tilting $\mathsf{End}_{\mathcal{T}}(S)$-modules, see [**1**, Thm. 3.2] for a special case. On the other hand, if $\mathcal{C}$ is a 2-Calabi-Yau triangulated category with a cluster-tilting object T, then there is a bijection between basic cluster-tilting objects in $\mathcal{C}$ and basic support τ-tilting $\mathsf{End}_{\mathcal{C}}(T)$-modules, see [**1**, Thm. 4.1].

Reduction techniques exist both for silting objects and cluster-tilting objects, see [**3**, Thm. 2.37] and [**62**, Thm. 4.9] respectively. The following result shows that τ-tilting reduction fits nicely in these contexts.

THEOREM 3.1.6 (see Theorems 3.4.12 and 3.4.23 for details). *Let A be a finite dimensional algebra. Then we have the following:*

(i) τ-tilting reduction is compatible with silting reduction.
(ii) If A is 2-Calabi-Yau tilted, then τ-tilting reduction is compatible with 2-Calabi-Yau reduction.

These results enhance our understanding of the relationship between silting objects, cluster-tilting objects and support τ-tilting modules. We refer the reader to [**26**] for an in-depth survey of the relations between these objects and several other important concepts in representation theory.

3.2. Preliminaries

Let us fix our conventions and notations, which we kindly ask the reader to keep in mind for the remainder of this chapter.

CONVENTIONS 3.2.1. In what follows, A always denotes a (fixed) finite dimensional algebra over a field k. We denote by $\mathsf{mod}\,A$ the category of finite dimensional right A-modules. Whenever we consider a subcategory of $\mathsf{mod}\,A$ we assume that it is full and closed under isomorphisms. If M is an A-module, we denote by $\mathsf{Fac}\,M$ the subcategory of $\mathsf{mod}\,A$ which consists of all factor modules of direct sums of copies of M; the subcategory $\mathsf{Sub}\,M$ is defined dually. Given morphisms $f : X \to Y$ and $g : Y \to Z$ in some category $\mathcal{C}$, we denote their composition by $g \circ f = gf$. Given a subcategory $\mathcal{X}$ of an additive category $\mathcal{C}$, we denote by ${}^{\perp}\mathcal{X}$ the subcategory of $\mathcal{C}$ whose objects are all objects M in $\mathcal{C}$ such that $\mathsf{Hom}_{\mathcal{C}}(M, \mathcal{X}) = 0$; the category $\mathcal{X}^{\perp}$ is defined dually. Also, we denote by $[\mathcal{X}]$ the ideal of $\mathcal{C}$ of morphisms which factor through $\mathcal{X}$. For an object X of $\mathcal{C}$, we denote by $\mathsf{add}\,X$ the smallest additive subcategory of $\mathcal{C}$ containing X and closed under isomorphisms. If $\mathcal{X} = \mathsf{add}\,X$ for some object X in $\mathcal{C}$ we write ${}^{\perp}X$ instead of ${}^{\perp}\mathcal{X}$ and so on. If $\mathcal{C}$ is a k-linear category we denote by D the usual k-duality $\mathsf{Hom}_k(-, k)$.

There is a strong interplay between the classical concept of torsion class in $\mathsf{mod}\,A$ and the recently investigated class of support τ-tilting modules. In this section we collect the basic definitions and main results relating this two theories.

3.2.1. Torsion pairs. Recall that a subcategory $\mathcal{X}$ of an additive category $\mathcal{C}$ is said to be *contravariantly finite in* $\mathcal{C}$ if for every object M of $\mathcal{C}$ there exist some X in $\mathcal{X}$ and a morphism $f : X \to M$ such that for every X' in $\mathcal{X}$ the sequence

$$\mathsf{Hom}_{\mathcal{C}}(X', X) \xrightarrow{f\cdot} \mathsf{Hom}_{\mathcal{C}}(X', M) \to 0$$

is exact. In this case f is called a *right* $\mathcal{X}$*-approximation*. Dually we define *covariantly finite subcategories in* $\mathcal{C}$ and *left* $\mathcal{X}$*-approximations*. Furthermore, a subcategory of $\mathcal{C}$ is said to be *functorially finite in* $\mathcal{C}$ if it is both contravariantly and covariantly finite in $\mathcal{C}$.

A subcategory $\mathcal{T}$ of $\mathsf{mod}\,A$ is called a *torsion class* if it is closed under extensions and factor modules in $\mathsf{mod}\,A$. Dually, *torsion-free classes* are defined. An A-module M in $\mathcal{T}$ is said to be Ext*-projective in* $\mathcal{T}$ if $\mathsf{Ext}^1_A(M, \mathcal{T}) = 0$. If $\mathcal{T}$ is functorially finite in $\mathsf{mod}\,A$, then there are only finitely many indecomposable Ext-projective modules in $\mathcal{T}$ up to isomorphism, and we denote by $P(\mathcal{T})$ the direct sum of each one of them. For convenience, we will denote the set of all torsion classes in $\mathsf{mod}\,A$ by $\mathsf{tors}\,A$, and by $\mathsf{f\text{-}tors}\,A$ the subset of $\mathsf{tors}\,A$ consisting of all torsion classes which are functorially finite in $\mathsf{mod}\,A$.

A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathsf{mod}\,A$ is called a *torsion pair* if $\mathcal{F} = \mathcal{T}^{\perp}$ and $\mathcal{T} = {}^{\perp}\mathcal{F}$. In such case $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion-free class in $\mathsf{mod}\,A$. The following proposition characterizes torsion pairs in $\mathsf{mod}\,A$ consisting of functorially finite subcategories.

PROPOSITION 3.2.2. [**1**, Prop. 1.1] *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair in* $\mathsf{mod}\, A$. *The following properties are equivalent:*

(i) $\mathcal{F}$ *is functorially finite in* $\mathsf{mod}\, A$ *(or equivalently,* $\mathcal{F}$ *is contravariantly finite).*
(ii) $\mathcal{T}$ *is functorially finite in* $\mathsf{mod}\, A$ *(or equivalently,* $\mathcal{T}$ *is covariantly finite).*
(iii) $\mathcal{T} = \mathsf{Fac}\, P(\mathcal{T})$.
(iv) $P(\mathcal{T})$ *is a tilting* $(A/\operatorname{\mathsf{ann}} \mathcal{T})$*-module.*
(v) For every M *in* $\mathcal{T}$ *there exists a short exact sequence* $0 \to L \to T' \xrightarrow{f} M \to 0$ *where* f *is right* $(\mathsf{add}\, P(\mathcal{T}))$*-approximation and* L *is in* $\mathcal{T}$.

A torsion pair in $\mathsf{mod}\, A$ which has any of the equivalent properties of Proposition 3.2.2 is called a *functorially finite torsion pair.* In view of property (c), we call the A-module $P(\mathcal{T})$ the Ext*-progenerator* of $\mathcal{T}$.

3.2.2. τ-tilting theory. Now we recall the definition of support τ-tilting modules and the results relating such modules with functorially finite torsion classes in $\mathsf{mod}\, A$.

DEFINITION 3.2.3. [**1**, Def. 0.1(a)] Let A be a finite dimensional algebra. An A-module M is said to be *τ-rigid* if $\mathsf{Hom}_A(M, \tau M) = 0$ where τ is the Auslander-Reiten translation.

REMARK 3.2.4. By the Auslander-Reiten duality formula [**8**, Thm. IV.2.13], for every A-module M we have an isomorphism $D\,\overline{\mathsf{Hom}}_A(M, \tau M) \cong \mathsf{Ext}^1_A(M, M)$. Thus M is rigid (*i.e.* $\mathsf{Ext}^1_A(M, M) = 0$) provided M is τ-rigid.

The following classical result of Auslander and Smalø characterizes τ-rigid modules in terms of torsion classes.

PROPOSITION 3.2.5. [**13**, 5.8] *Let* M *and* N *be two* A*-modules. Then the following holds:*

(i) $\mathsf{Hom}_A(N, \tau M) = 0$ *if and only if* $\mathsf{Ext}^1_A(M, \mathsf{Fac}\, N) = 0$.
(ii) M *is* τ*-rigid if and only if* M *is* Ext*-projective in* $\mathsf{Fac}\, M$.
(iii) $\mathsf{Fac}\, M$ *is a functorially finite torsion class in* $\mathsf{mod}\, A$.

For an A-module M and an ideal I of A contained in $\operatorname{\mathsf{ann}} M$, the following proposition describes the relationship between M being τ-rigid as A-module and τ-rigid as (A/I)-module. We denote by $\tau_{A/I}$ the Auslander-Reiten translation in $\mathsf{mod}(A/I)$.

PROPOSITION 3.2.6. [**1**, Lemma 2.1] *Let* I *be an ideal of* A *and* M *and* N *two* (A/I)*-modules. Then we have the following:*

(i) If $\mathsf{Hom}_A(M, \tau N) = 0$, *then* $\mathsf{Hom}_{A/I}(M, \tau_{A/I} N) = 0$.
(ii) If $I = \langle e \rangle$ *for some idempotent* $e \in A$, *then* $\mathsf{Hom}_A(M, \tau N) = 0$ *if and only if* $\mathsf{Hom}_{A/I}(M, \tau_{A/I} N) = 0$.

The following lemma, which is an analog of Wakamatsu's Lemma, *cf.* [**12**, Lemma 1.3], often comes handy.

LEMMA 3.2.7. [**1**, Lemma 2.5] *Let* $0 \to L \to M \xrightarrow{f} N$ *be an exact sequence. If* f *is a right* $(\mathsf{add}\, M)$*-approximation of* N *and* M *is* τ*-rigid, then* L *is in* ${}^{\perp}(\tau M)$.

We denote the number of pairwise non-isomorphic indecomposable summands of an A-module M by $|M|$. Thus $|A|$ equals the rank of the Grothendieck group of $\mathsf{mod}\, A$.

DEFINITION 3.2.8. [**1**, Defs. 0.1(b), 0.3] Let M be a τ-rigid A-module. We say that M is a *τ-tilting A-module* if $|M| = |A|$. More generally, we say that M is a *support τ-tilting A-module* if there exists an idempotent $e \in A$ such that M is a τ-tilting $(A/\langle e\rangle)$-module. Support tilting A-modules are defined analogously, see [**54**].

REMARK 3.2.9. Note that the zero-module is a support τ-tilting module (take $e = 1_A$ in Definition 3.2.8). Thus every non-zero finite dimensional algebra A admits at least two support τ-tilting A-modules: 0 and A.

The following observation follows immediately from the Auslander-Reiten formulas and Definition 3.2.8.

PROPOSITION 3.2.10. [**1**] *Let A be a hereditary algebra and M an A-module. Then M is a τ-rigid (resp. τ-tilting) A-module if and only if M is a rigid (resp. tilting) A-module.*

We also need the following result:

PROPOSITION 3.2.11. [**1**, Prop. 2.2] *Let A be a finite dimensional algebra. The following statements hold:*

(i) τ-tilting A-modules are precisely sincere support τ-tilting A-modules.
(ii) Tilting A-modules are precisely faithful support τ-tilting A-modules.
(iii) Any τ-tilting (resp. τ-rigid) A-module M is a tilting (resp. partial tilting) $(A/\operatorname{\mathsf{ann}} T)$-module.

The following result provides the conceptual framework for the main results of this chapter. It says that basic support τ-tilting A-modules are precisely the Ext-progenerators of functorially finite torsion classes in $\mathsf{mod}\,A$.

THEOREM 3.2.12. [**1**, Thm. 2.2] *There is a bijection*

$$\mathsf{f}\text{-}\mathsf{tors}\,A \longrightarrow \mathrm{s}\tau\text{-}\mathsf{tilt}\,A$$

given by $\mathcal{T} \mapsto P(\mathcal{T})$ with inverse $M \mapsto \mathsf{Fac}\,M$.

REMARK 3.2.13. Observe that the inclusion of subcategories gives a partial order in $\mathsf{tors}\,A$. Thus the bijection of Theorem 3.2.12 induces a partial order in $\mathrm{s}\tau\text{-}\mathsf{tilt}\,A$. Namely, if M and N are support τ-tilting A-modules, then

$$M \leq N \quad \text{if and only if} \quad \mathsf{Fac}\,M \subseteq \mathsf{Fac}\,N.$$

Hence, as with every partially ordered set, we can associate to $\mathrm{s}\tau\text{-}\mathsf{tilt}\,A$ a *Hasse quiver* $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\,A)$ whose set of vertices is $\mathrm{s}\tau\text{-}\mathsf{tilt}\,A$ and there is an arrow $M \to N$ if and only if $M > N$ and there is no $L \in \mathrm{s}\tau\text{-}\mathsf{tilt}\,A$ such that $M > L > N$.

The following proposition is a generalization of Bongartz completion of tilting modules, see [**8**, Lemma VI.2.4]. It plays an important role in the sequel.

PROPOSITION 3.2.14. [**1**, Prop. 2.9] *Let U be a τ-rigid A-module. Then the following holds:*

(i) ${}^{\perp}(\tau U)$ is a functorially finite torsion class which contains U.
(ii) U is Ext-projective in ${}^{\perp}(\tau U)$, that is $U \in \mathsf{add}\,P({}^{\perp}(\tau U))$.
(iii) $T_U := P({}^{\perp}(\tau U))$ is a τ-tilting A-module.

The module T_U is called the Bongartz completion of U in $\mathsf{mod}\,A$.

Recall that, by definition, a partial-tilting A-module T is a tilting A-module if and only if there exists a short exact sequence $0 \to A \to T' \to T'' \to 0$ with $T', T'' \in \mathsf{add}\,T$. The following proposition gives a similar criterion for a τ-rigid A-module to be a support τ-tilting A-module.

PROPOSITION 3.2.15. *Let M be a τ-rigid A-module. Then M is a support τ-tilting A-module if and only if there exists an exact sequence*

$$A \xrightarrow{f} M' \xrightarrow{g} M'' \to 0 \tag{3.2.1}$$

with $M', M'' \in \operatorname{\mathsf{add}} M$ and f a left $(\operatorname{\mathsf{add}} M)$-approximation of A.

PROOF. The necessity is shown in [**1**, Prop. 2.22]. For the sufficiency, suppose there exists an exact sequence of the form (3.2.1). Let $I = \operatorname{\mathsf{ann}} M$, we only need to find an idempotent $e \in A$ such that $e \in I$ and $|M| = |A/\langle e\rangle|$. By Proposition 3.2.11(c) M is a partial-tilting (A/I)-module. Moreover, f induces a morphism $\bar{f} : A/I \to M'$. We claim that the sequence

$$0 \to A/I \xrightarrow{\bar{f}} M' \xrightarrow{g} M'' \to 0$$

is exact, for which we only need to show that the induced morphism $\bar{f}$ is injective. It is easy to see that $\bar{f} : A/I \to M'$ is a left $(\operatorname{\mathsf{add}} M)$-approximation of A/I. Since M is a faithful (A/I)-module, by [**8**, VI.2.2] we have that $\bar{f}$ is injective, and the claim follows. Thus M is a tilting (A/I)-module, and we have $|M| = |A/I|$. Let e be a maximal idempotent in A such that $e \in I$. Then by the choice of e we have that $|M| = |A/I| = |A/\langle e\rangle|$. □

The following result justifies the claim that support τ-tilting modules complete the class of tilting modules from the point of view of mutation. We say that a basic τ-rigid A-module U is *almost-complete* if $|U| = |A| - 1$.

THEOREM 3.2.16. [**1**, Thm. 2.17] *Let U be an almost-complete τ-tilting A-module. Then there exist exactly two basic support τ-tilting A-modules having U as a direct summand.*

DEFINITION 3.2.17. It follows from Theorem 3.2.16 that we can associate with $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$ an *exchange graph* whose vertices are basic support τ-tilting A-modules and there is an edge between two non-isomorphic support τ-tilting A-modules M and N if and only if the following holds:

- There exists an idempotent $e \in A$ such that $M, N \in \mathsf{mod}(A/\langle e\rangle)$.
- There exists an almost-complete τ-tilting $(A/\langle e\rangle)$-module U such that $U \in \operatorname{\mathsf{add}} M$ and $U \in \operatorname{\mathsf{add}} N$.

In this case we say that M and N are obtained from each other by mutation. Note that this exchange graph is n-regular, where $|A| = n$ is the number of simple A-modules. It is shown in [**1**, Cor. 2.31] that the underlying graph of $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\, A)$ coincides with the exchange graph of $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$.

We conclude this section with some examples of support τ-tiling modules.

EXAMPLE 3.2.18. Let A be a hereditary algebra. By Proposition 3.2.11 support τ-tilting A-modules are precisely support tilting A-modules. For example, let A be the path algebra of the quiver $2 \leftarrow 1$. The Auslander-Reiten quiver of $\mathsf{mod}\, A$ is given by

$$\begin{tikzcd} & {\begin{smallmatrix}1\\2\end{smallmatrix}} \arrow[dr] & \\ 2 \arrow[ur] \arrow[rr, dotted, no head] & & 1 \end{tikzcd}$$

where modules are represented by their radical filtration. Then $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\, A)$ is given by

$$
\begin{tikzcd}[column sep=small]
& 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dl] \arrow[dr] & \\
\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \arrow[d] & & 2 \arrow[d] \\
1 \arrow[rr] & & 0
\end{tikzcd}
$$

Note that the only τ-tilting A-modules are $2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1$, and since A is hereditary they are also tilting A-modules.

EXAMPLE 3.2.19. Let A be the algebra given by the quiver

$$
\begin{tikzcd}
2 \arrow[r, bend right, "y"'] & 1 \arrow[l, bend right, "x"']
\end{tikzcd}
$$

subject to the relation $yx = 0$. The Auslander-Reiten quiver of $\mathsf{mod}\, A$ is given by

$$
\begin{tikzcd}[column sep=small]
& & \begin{smallmatrix}1\\2\\1\end{smallmatrix} \arrow[dr] & & \\
& \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
1 \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[ur] \arrow[rr, dotted, no head] & & 1
\end{tikzcd}
$$

where the two copies of $S_1 = 1$ are to be identified. Then $Q(\mathsf{s}\tau\text{-}\mathsf{tilt}\, A)$ is given as follows:

$$
\begin{tikzcd}[column sep=small]
& \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dl] \arrow[dr] & \\
2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[d] & & \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus 1 \arrow[d] \\
2 \arrow[dr] & & 1 \arrow[dl] \\
& 0 &
\end{tikzcd}
$$

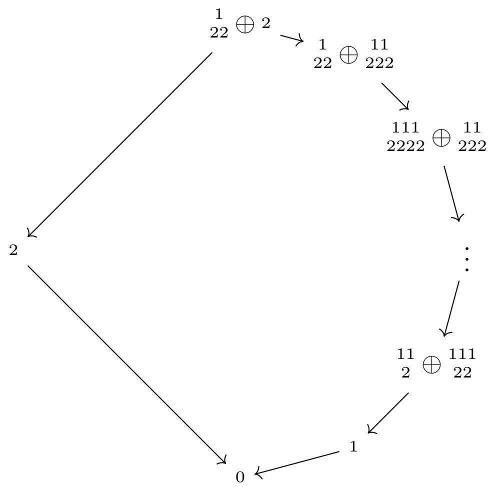
EXAMPLE 3.2.20. Let A be a self-injective algebra. Then the only basic tilting A-module is A. On the other hand, in general there are many basic support τ-tilting modules. For example, let A be the path algebra of the quiver

$$
\begin{tikzcd}[column sep=small]
& 1 \arrow[dl, "x"'] & \\
2 \arrow[rr, "y"'] & & 3 \arrow[ul, "z"']
\end{tikzcd}
$$

subject to the relations $xy = 0$, $yz = 0$ and $zx = 0$. Thus A is a self-injective cluster-tilted algebra of type A_3, see [**28, 81**]. It follows from [**1**, Thm. 4.1] that basic support τ-tilting A-modules correspond bijectively with basic cluster-tilting objects in the cluster category of type A_3. Hence there are 14 support τ-tilting A-modules, see [**27**, Fig. 4].

The following example gives an algebra with infinitely many support τ-tilting modules.

Example 3.2.21. Let A be the Kronecker algebra, *i.e.* the path algebra of the quiver $2 \leftleftarrows 1$. Then $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\, A)$ is the following quiver, where each module is represented by its radical filtration:



3.3. Main results

This section is devoted to prove the main results of this chapter. First, let us fix the setting of our results.

Setting 3.3.1. We fix a finite dimensional algebra A and a basic τ-rigid A-module U. Let $T = T_U$ be the Bongartz completion of U in $\mathsf{mod}\, A$, see Proposition 3.2.14. The algebras

$$B = B_U := \mathsf{End}_A(T_U) \quad \text{and} \quad C = C_U := B_U/\langle e_U \rangle$$

play an important role in the sequel, where e_U is the idempotent corresponding to the projective B-module $\mathsf{Hom}_A(T_U, U)$. We regard $\mathsf{mod}\, C$ as a full subcategory of $\mathsf{mod}\, B$ *via* the canonical embedding.

In this section we study the subset of $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$ given by

$$\mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A := \{M \in \mathrm{s}\tau\text{-}\mathsf{tilt}\, A \mid U \in \mathsf{add}\, M\}\,.$$

In Theorem 3.3.15 we will show that there is an order-preserving bijection between $\mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A$ and $\mathrm{s}\tau\text{-}\mathsf{tilt}\, C$.

3.3.1. The τ-perpendicular category. The following observation allows us to describe $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$ in terms of the partial order in $\mathsf{tors}\, A$.

Proposition 3.3.2. [**1**, Prop. 2.8] *Let U be a τ-rigid A-module and M a support τ-tilting A-module. Then, $U \in \mathsf{add}\, M$ if and only if*

$$\mathsf{Fac}\, U \subseteq \mathsf{Fac}\, M \subseteq {}^{\perp}(\tau U).$$

Recall that we have $M \leqslant N$ for two basic support τ-tilting A-modules if and only if $\mathsf{Fac}\, M \subseteq \mathsf{Fac}\, N$, see Remark 3.2.13. Hence it follows from Proposition 3.3.2 that $\mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A$ is an interval in $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$, *i.e.* we have that

$$\mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A = \{M \in \mathrm{s}\tau\text{-}\mathsf{tilt}\, A \mid P(\mathsf{Fac}\, U) \leqslant M \leqslant T_U\} \tag{3.3.1}$$

In particular there are two distinguished functorially finite torsion pairs associated with $P(\mathsf{Fac}\, U)$ and T_U. Namely,

$$(\mathsf{Fac}\, U, U^{\perp}) \quad \text{and} \quad ({}^{\perp}(\tau U), \mathsf{Sub}\, \tau U)$$

which satisfy $\mathsf{Fac}\, U \subseteq {}^{\perp}(\tau U)$ and $\mathsf{Sub}\, \tau U \subseteq U^{\perp}$.

DEFINITION 3.3.3. The *τ-perpendicular category associated to U* is the subcategory of $\mathsf{mod}\, A$ given by $\mathcal{U} := {}^{\perp}(\tau U) \cap U^{\perp}$.

The choice of terminology in Definition 3.3.3 is justified by the following example.

EXAMPLE 3.3.4. Suppose that U is a partial-tilting A-module. Since U has projective dimension less or equal than 1. Then, by the Auslander-Reiten formulas, for every A-module M we have that $\mathsf{Hom}_A(M, \tau U) = 0$ if and only if $\mathsf{Ext}^1_A(U, M) = 0$. Then

$$\mathcal{U} = \left\{ M \in \mathsf{mod}\, A \mid \mathsf{Hom}(U, M) = 0 \text{ and } \mathsf{Ext}^1_A(U, M) = 0 \right\}.$$

Thus $\mathcal{U}$ is exactly the right perpendicular category associated to U in the sense of [**37**].

We need a simple observation which is a consequence of a theorem of Brenner and Butler.

PROPOSITION 3.3.5. *With the hypotheses of Setting 3.3.1, the functors*

$$F := \mathsf{Hom}_A(T, -) : \ \mathsf{mod}\, A \to \mathsf{mod}\, B \quad \textit{and} \tag{3.3.2}$$

$$G := - \otimes_B T : \ \mathsf{mod}\, B \to \mathsf{mod}\, A \tag{3.3.3}$$

induce mutually quasi-inverse equivalences $F : \mathsf{Fac}\, T \to \mathsf{Sub}\, DT$ and $G : \mathsf{Sub}\, DT \to \mathsf{Fac}\, T$. Moreover, these equivalences are exact, i.e. *F sends short exact sequences in $\mathsf{mod}\, A$ with terms in $\mathsf{Fac}\, T$ to short exact sequences in $\mathsf{mod}\, B$, and so does G.*

PROOF. In view of Proposition 3.2.11, we have that T is a tilting $(A/\mathsf{ann}\, T)$-module. Then it follows from [**8**, Thm. VI.3.8] that $F : \mathsf{Fac}\, T \to \mathsf{Sub}\, DT$ is an equivalence with quasi-inverse $G : \mathsf{Sub}\, DT \to \mathsf{Fac}\, T$.

Now we show that both F and G are exact. For this, let $0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathsf{mod}\, A$ with terms in $\mathsf{Fac}\, T$. Then F induces an exact sequence

$$0 \to FL \to FM \to FN \to \mathsf{Ext}^1_A(T, L) = 0$$

as T is Ext-projective in ${}^{\perp}(\tau U)$. Consequently F is exact.

Next, let $0 \to L' \to M' \to N' \to 0$ be a short exact sequence in $\mathsf{mod}\, A$ with terms in $\mathsf{Sub}\, DT$, then there is an exact sequence

$$0 = \mathsf{Tor}^B_1(N', T) \to GL' \to GM' \to GN' \to 0.$$

as $N' \in \mathsf{Sub}\, DT = \ker \mathsf{Tor}^B_1(-, T)$, see [**8**, Cor. VI.3.9(i)]; hence G is also exact. $\square$

The following proposition gives us a basic property of $\mathcal{U}$.

PROPOSITION 3.3.6. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathsf{mod}\, A$. If any two of L, M and N belong to $\mathcal{U}$, then the third one also belongs to $\mathcal{U}$.*

PROOF. First, $\mathcal{U}$ is closed under extensions since both ${}^{\perp}(\tau U)$ and $U^{\perp}$ are closed under extensions in $\mathsf{mod}\, A$. Thus if L and N belong to $\mathcal{U}$, then so does M.

Secondly, suppose that L and M belong to $\mathcal{U}$. Since ${}^{\perp}(\tau U)$ is closed under factor modules we only need to show that $\mathsf{Hom}_A(U, N) = 0$. In this case we have an exact sequence

$$0 = \mathsf{Hom}_A(U, M) \to \mathsf{Hom}_A(U, N) \to \mathsf{Ext}^1_A(U, L) = 0$$

since by Proposition 3.2.14(b) we have that U is Ext-projective in ${}^{\perp}(\tau U)$, hence N is in $\mathcal{U}$.

Finally, suppose that M and N belong to $\mathcal{U}$. Since $U^{\perp}$ is closed under submodules, we only need to show that $\mathsf{Hom}_A(L, \tau U) = 0$. We have an exact sequence

$$0 = \mathsf{Hom}_A(M, \tau U) \to \mathsf{Hom}_A(L, \tau U) \to \mathsf{Ext}^1_A(N, \tau U).$$

By the dual of Proposition 3.2.14(b) we have that τU is Ext-injective in $U^{\perp}$, so we have $\mathsf{Ext}^1_A(N, \tau U) = 0$, hence $\mathsf{Hom}_A(L, \tau U) = 0$ and thus L is in $\mathcal{U}$. $\square$

REMARK 3.3.7. Since $\mathcal{U}$ is closed under extensions in $\mathsf{mod}\, A$, it has a natural structure of an exact category, see [**79, 67**]. Then Proposition 3.3.6 says that admissible epimorphisms (resp. admissible monomorphisms) in $\mathcal{U}$ are exactly epimorphisms (resp. monomorphisms) in $\mathsf{mod}\, A$ between modules in $\mathcal{U}$.

The next result is the main result of this subsection. It is the first step towards τ-tilting reduction.

THEOREM 3.3.8. *With the hypotheses of Setting 3.3.1, the functors F and G in Proposition 3.3.5 induce mutually quasi-inverse equivalences $F : \mathcal{U} \to \mathsf{mod}\, C$ and $G : \mathsf{mod}\, C \to \mathcal{U}$ which are exact.*

PROOF. By Proposition 3.3.5, the functors (3.3.2) and (3.3.3) induce mutually quasi-inverse equivalences between $\mathsf{Fac}\, T$ and $\mathsf{Sub}\, DT$. Hence by construction we have $F(\mathcal{U}) \subseteq (FU)^{\perp} = \mathsf{mod}\, C$. Thus, we only need to show that $F : \mathcal{U} \to \mathsf{mod}\, C$ is dense.

Let N be in $\mathsf{mod}\, C$ and take a projective presentation

$$FT_1 \xrightarrow{Ff} FT_0 \to N \to 0 \tag{3.3.4}$$

of the B-module N; hence we have $T_0, T_1 \in T_U$. Let $M = \mathsf{Coker}\, f$. We claim that $FM \cong N$ and that M is in $\mathcal{U}$. In fact, let $L = \mathrm{Im}\, f$ and $K = \mathsf{Ker}\, f$. Then we have short exact sequences

$$0 \to K \to T_1 \to L \to 0 \quad \text{and} \tag{3.3.5}$$

$$0 \to L \to T_0 \to M \to 0. \tag{3.3.6}$$

Then L is in ${}^{\perp}(\tau U)$ since ${}^{\perp}(\tau U)$ is closed under factor modules. In particular we have that $\mathsf{Ext}^1_A(T, L) = 0$. Apply the functor F to the short exact sequences (3.3.5) and (3.3.6) to obtain a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & FT_1 \arrow[r, "Ff"] \arrow[d] & FT_0 \arrow[r] \arrow[d, equal] & FM \arrow[d, equal] & \\
0 \arrow[r] & FL \arrow[r] \arrow[d] & FT_0 \arrow[r] & FM \arrow[r] & \mathsf{Ext}^1_A(T, L) = 0 \\
 & \mathsf{Ext}^1_A(T, K) & & &
\end{tikzcd}$$

To prove that $FM \cong N$ it remains to show that $\mathsf{Ext}^1_A(T, K) = 0$. Since T is Ext-projective in ${}^{\perp}(\tau U)$, it suffices to show that K is in ${}^{\perp}(\tau U)$. Applying the functor $\mathsf{Hom}_A(-, \tau U)$ to (3.3.5), we obtain an exact sequence

$$0 = \mathsf{Hom}_A(T_1, \tau U) \to \mathsf{Hom}_A(K, \tau U) \to \mathsf{Ext}^1_A(L, \tau U) \xrightarrow{\mathsf{Ext}^1_A(f, \tau U)} \mathsf{Ext}^1_A(T_1, \tau U).$$

Thus we only need to show that the map $\mathsf{Ext}^1_A(f, \tau U) : \mathsf{Ext}^1_A(L, \tau U) \to \mathsf{Ext}^1_A(T_1, \tau U)$ is a monomorphism. By Auslander-Reiten duality it suffices to show that the map

$$\underline{\mathsf{Hom}}_A(U, T_1) \xrightarrow{f \circ -} \underline{\mathsf{Hom}}_A(U, L)$$

is an epimorphism. For this, observe that we have a commutative diagram

$$\begin{array}{ccc} \mathsf{Hom}_A(U,T_1) & \xrightarrow{f\circ -} & \mathsf{Hom}_A(U,L) \\ \downarrow & & \downarrow \\ \underline{\mathsf{Hom}}_A(U,T_1) & \xrightarrow[f\circ -]{} & \underline{\mathsf{Hom}}_A(U,L) \end{array}$$

where the vertical maps are natural epimorphisms. Hence it is enough to show that the map

$$\mathsf{Hom}_A(U,T_1) \xrightarrow{f\circ -} \mathsf{Hom}_A(U,L) \tag{3.3.7}$$

is surjective. Applying the functor $\mathsf{Hom}_B(FU,-)$ to the sequence (3.3.4) we obtain an exact sequence

$$\mathsf{Hom}_B(FU,FT_1) \xrightarrow{\mathsf{Hom}_B(FU,Ff)} \mathsf{Hom}_B(FU,FT_0) \to \mathsf{Hom}_B(FU,N) = 0$$

since FU is a projective B-module and N is in $\mathsf{mod}\, C = (FU)^{\perp}$. Thus $\mathsf{Hom}_B(FU,Ff)$ is surjective, and also the map (3.3.7) is surjective by Proposition 3.3.5. Hence we have that K belongs to ${}^{\perp}(\tau U)$ as desired. This shows that $FM \cong N$.

Moreover, we have that

$$0 = \mathsf{Hom}_B(FU,N) \cong \mathsf{Hom}_B(FU,FM) \cong \mathsf{Hom}_A(U,M).$$

Hence M is in $\mathcal{U}$. This shows that $F : \mathcal{U} \to \mathsf{mod}\, C$ is dense, hence F is an equivalence with quasi-inverse G. The fact that this equivalences are exact follows immediately from Proposition 3.3.5. This concludes the proof of the theorem. □

Definition 3.3.9. We say that a full subcategory $\mathcal{G}$ of $\mathcal{U}$ is a *torsion class* in $\mathcal{U}$ if the following holds: Let $0 \to X \to Y \to Z \to 0$ be an admissible exact sequence in $\mathcal{U}$, see Remark 3.3.7:

(i) If X and Z are in $\mathcal{G}$, then Y is in $\mathcal{G}$.
(ii) If Y is in $\mathcal{G}$, then Z is in $\mathcal{G}$.

We denote the set of all torsion classes in $\mathcal{U}$ by $\mathsf{tors}\,\mathcal{U}$. We denote by $\mathrm{f}\text{-}\mathsf{tors}\,\mathcal{U}$ the subset of $\mathsf{tors}\,\mathcal{U}$ consisting of torsion classes which are functorially finite in $\mathcal{U}$.

Example 3.3.10. If A is hereditary and U a basic partial-tilting A-module then by [**37**] the algebra C is hereditary and Theorem 3.3.8 specializes to a well-known result from *op. cit.* .

The following corollary is an immediate consequence of Theorem 3.3.8.

Corollary 3.3.11. *The following holds:*

(i) The functors F and G induce mutually inverse bijections between $\mathsf{tors}\,\mathcal{U}$ and $\mathsf{tors}\, C$.
(ii) These bijections restrict to bijections between $\mathrm{f}\text{-}\mathsf{tors}\,\mathcal{U}$ and $\mathrm{f}\text{-}\mathsf{tors}\, C$.
(iii) These bijections above are isomorphisms of partially ordered sets.

Proof. It is shown in Theorem 3.3.8 that F and G give equivalences of exact categories between $\mathcal{U}$ and $\mathsf{mod}\, C$. Since the notion of torsion class depends only on the exact structure of the category, see Definition 3.3.9, part (a) follows. Now (b) and (c) are clear. □

3.3.2. Reduction of torsion classes and τ-tilting modules. Given two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathsf{mod}\, A$ we denote by $\mathcal{X} * \mathcal{Y}$ the full subcategory of $\mathsf{mod}\, A$ induced by all A-modules M such that there exist a short exact sequence

$$0 \to X \to M \to Y \to 0$$

with X in $\mathcal{X}$ and Y in $\mathcal{Y}$. Obviously we have $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{X} * \mathcal{Y}$. The following two results give us reductions at the level of torsion pairs.

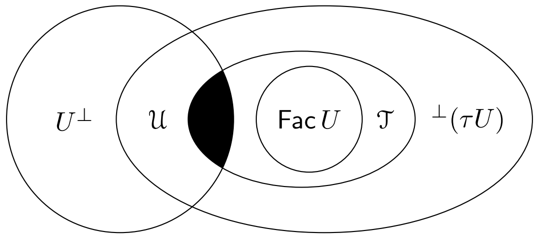


FIGURE 3.3.1. Reduction of torsion classes in $\mathsf{mod}\,A$ with respect to $\mathsf{Fac}\,U$, see Theorems 3.3.12 and 3.3.13.

THEOREM 3.3.12. *With the hypotheses of Setting 3.3.1, we have order-preserving bijections*

$$\left\{\mathcal{T} \in \mathsf{tors}\,A \mid \mathsf{Fac}\,U \subseteq \mathcal{T} \subseteq {}^\perp(\tau U)\right\} \xrightarrow{\mathsf{red}} \mathsf{tors}\,\mathcal{U} \xrightarrow{F} \mathsf{tors}\,C$$

where red *is given by* $\mathsf{red}(\mathcal{T}) := \mathcal{T} \cap U^\perp$ *with inverse* $\mathsf{red}^{-1}(\mathcal{G}) := (\mathsf{Fac}\,U) * \mathcal{G}$, *and* F *is given in Corollary 3.3.11.*

The situation of Theorem 3.3.12 is illustrated in Figure 3.3.1. Also, the following diagram is helpful to visualize this reduction procedure:

$$\begin{array}{ccc}
{}^\perp(\tau U) & \longmapsto & {}^\perp(\tau U) \cap U^\perp = \mathcal{U} \\
\cup\!\vert & & \cup\!\vert \\
\mathcal{T} & \longmapsto & \mathcal{T} \cap U^\perp \\
\cup\!\vert & & \cup\!\vert \\
\mathsf{Fac}\,U & \longmapsto & (\mathsf{Fac}\,U) \cap U^\perp = \{0\}
\end{array}$$

Moreover, we have the following bijections:

THEOREM 3.3.13. *The bijections of Theorem 3.3.12 restrict to order-preserving bijections*

$$\left\{\mathcal{T} \in \mathrm{f}\text{-}\mathsf{tors}\,A \mid \mathsf{Fac}\,U \subseteq \mathcal{T} \subseteq {}^\perp(\tau U)\right\} \xrightarrow{\mathsf{red}} \mathrm{f}\text{-}\mathsf{tors}\,\mathcal{U} \xrightarrow{F} \mathrm{f}\text{-}\mathsf{tors}\,C.$$

For readability purposes, the proofs of Theorems 3.3.12 and 3.3.13 are given in Section 3.3.3. First we use them to establish the bijection between $\mathrm{s}\tau\text{-}\mathsf{tilt}_U\,A$ and $\mathrm{s}\tau\text{-}\mathsf{tilt}\,C$.

Recall that the torsion pair $(\mathsf{Fac}\,U, U^\perp)$ gives functors $\mathrm{t} : \mathsf{mod}\,A \to \mathsf{Fac}\,U$ and $\mathrm{f} : \mathsf{mod}\,A \to U^\perp$ and natural transformations $\mathrm{t} \to 1_{\mathsf{mod}\,A} \to \mathrm{f}$ such that the sequence

$$0 \to \mathrm{t}M \to M \to \mathrm{f}M \to 0 \tag{3.3.8}$$

is exact for each A-module M. The sequence (3.3.8) is called a *canonical sequence*, and the functor t is called the *idempotent radical* associated to the torsion pair $(\mathsf{Fac}\,U, U^\perp)$. Any short exact sequence $0 \to L \to M \to N \to 0$ such that L is in $\mathsf{Fac}\,U$ and N is in $U^\perp$ is isomorphic to the canonical sequence of M, see [**8**, Prop. VI.1.5]. Note that since ${}^\perp(\tau U)$ is closed under factor modules we have

$$\mathrm{f}({}^\perp(\tau U)) \subseteq \mathcal{U}. \tag{3.3.9}$$

PROPOSITION 3.3.14. *Let* $\mathcal{T}$ *be a functorially finite torsion class in* $\mathsf{mod}\,A$. *Then* $\mathrm{f}P(\mathcal{T})$ *is* Ext*-projective in* $\mathcal{T} \cap U^\perp$ *and for every* A*-module* N *which is* Ext*-projective in* $\mathcal{T} \cap U^\perp$ *we have* $M \in \mathsf{add}(\mathrm{f}P(\mathcal{T}))$.

Proof. Applying the functor $\mathsf{Hom}_A(-, N)$ to (3.3.8), for any N in $\mathcal{T} \cap U^{\perp}$, we have an exact sequence

$$0 = \mathsf{Hom}_A(\mathrm{t}P(\mathcal{T}), N) \to \mathsf{Ext}^1_A(\mathrm{f}P(\mathcal{T}), N) \to \mathsf{Ext}^1_A(P(\mathcal{T}), N) = 0$$

This shows that $\mathrm{f}P(\mathcal{T})$ is Ext-projective in $\mathcal{T} \cap U^{\perp}$.

Now let N be Ext-projective in $\mathcal{T} \cap U^{\perp}$. Since $N \in \mathcal{T}$, by Proposition 3.2.2(e) there exist a short exact sequence $0 \to L \to M \to N \to 0$ with $M \in \mathsf{add}(P(\mathcal{T}))$ and $L \in \mathcal{T}$. Since $\mathrm{f}N = N$ as $N \in U^{\perp}$, by the functoriality of f we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r, "f"] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K := \mathsf{Ker}\,\mathrm{f}f \arrow[r] & \mathrm{f}M \arrow[r, "\mathrm{f}f"] & \mathrm{f}N = N \arrow[r] & 0
\end{tikzcd}$$

As the map $M \to \mathrm{f}M$ is surjective and the map $N \to \mathrm{f}N$ is bijective, by the snake lemma we have that the map $L \to K$ is surjective. Thus, since $L \in \mathcal{T}$, we have that K also belongs to $\mathcal{T}$. Moreover, K is a submodule of $\mathrm{f}M \in U^{\perp}$, hence K is also in $U^{\perp}$. Since N is Ext-projective in $\mathcal{T} \cap U^{\perp}$ and we have $K \in \mathcal{T} \cap U^{\perp}$, the lower sequence splits. Thus $N \in \mathsf{add}(\mathrm{f}P(\mathcal{T}))$. □

We are ready to state the main result of this chapter, which gives the procedure for τ-tilting reduction.

Theorem 3.3.15. *With the hypotheses of Setting 3.3.1, we have an order-preserving bijection*

$$\mathsf{red} : \mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A \longrightarrow \mathrm{s}\tau\text{-}\mathsf{tilt}\, C$$

given by $M \mapsto F(\mathrm{f}M)$ with inverse $N \mapsto P((\mathsf{Fac}\, U) * G(\mathsf{Fac}\, N))$. In particular, $\mathrm{s}\tau\text{-}\mathsf{tilt}\, C$ can be embedded as an interval in $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$.

Proof. By Theorems 3.2.12 and 3.3.13 we have a commutative diagram

$$\begin{tikzcd}
\left\{\mathcal{T} \in \mathsf{f}\text{-}\mathsf{tors}\, A \mid \mathsf{Fac}\, U \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)\right\} \arrow[r] & \mathsf{f}\text{-}\mathsf{tors}\, C \arrow[d] \\
\mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A \arrow[u] \arrow[r, dashed] & \mathrm{s}\tau\text{-}\mathsf{tilt}\, C
\end{tikzcd}$$

in which arrow is a bijection, and where the dashed arrow is given by $M \mapsto P((\mathsf{Fac}\, M) \cap U^{\perp})$ and the inverse is given by $N \mapsto P((\mathsf{Fac}\, U) * G(\mathsf{Fac}\, N))$. Hence to prove the theorem we only need to show that for any $M \in \mathrm{s}\tau\text{-}\mathsf{tilt}_U\, A$ we have $P(F(\mathsf{Fac}\, M \cap U^{\perp})) = F(\mathrm{f}M)$. Indeed, it follows from Proposition 3.3.14 that $\mathrm{f}M$ is the Ext-progenerator of $(\mathsf{Fac}\, M) \cap U^{\perp}$; and since F is an exact equivalence, see Theorem 3.3.8, we have that $F(\mathrm{f}M)$ is the Ext-progenerator of $F(\mathsf{Fac}\, M \cap U^{\perp})$ which is exactly what we needed to show. □

Corollary 3.3.16. *The bijection in Theorem 3.3.15 is compatible with mutation of support τ-tilting modules.*

Proof. It is shown in [**1**, Cor. 2.31] that the exchange graph of $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$ coincides with the Hasse diagram of $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$. Since τ-tilting reduction preserves the partial order in $\mathsf{f}\text{-}\mathsf{tors}$ (and hence in $\mathrm{s}\tau\text{-}\mathsf{tilt}$) the claim follows. □

As a special case of Theorem 3.3.15 we obtain an independent proof of [**1**, Thm. 2.17].

Corollary 3.3.17. *Every almost-complete support τ-tilting A-module U is the direct summand of exactly two support τ-tilting A-modules: $P(\mathsf{Fac}\, U)$ and T_U.*

PROOF. Let U be an almost complete support τ-tilting A-module. Clearly, we have $\{P(\mathsf{Fac}\,U), T_U\} \subseteq \mathrm{s}\tau\text{-}\mathsf{tilt}_U\,A$. On the other hand, $|U| = |A| - 1 = |T_U| - 1$ and thus $|C| = 1$, see Setting 3.3.1. By Theorem 3.3.15 we have a bijection between $\mathrm{s}\tau\text{-}\mathsf{tilt}_U\,A$ and $\mathrm{s}\tau\text{-}\mathsf{tilt}\,C$, and since $|C|$=1 we have that $\mathrm{s}\tau\text{-}\mathsf{tilt}\,C = \{0, C\}$, see [**1**, Ex. 6.1]. Thus $|\mathrm{s}\tau\text{-}\mathsf{tilt}_U\,A| = |\mathrm{s}\tau\text{-}\mathsf{tilt}\,C| = 2$ and we have the assertion. □

For a finite dimensional algebra A let $\mathrm{s}\text{-}\mathsf{tilt}\,A$ be the set of (isomorphism classes of) basic support tilting A-modules and, if U is a partial-tilting A-module, let $\mathrm{s}\text{-}\mathsf{tilt}_U\,A$ be the subset of $\mathrm{s}\text{-}\mathsf{tilt}\,A$ defined by

$$\mathrm{s}\text{-}\mathsf{tilt}_U\,A := \{M \in \mathrm{s}\text{-}\mathsf{tilt}\,A \mid U \in \mathsf{add}\,M\}\,.$$

If we restrict ourselves to hereditary algebras we obtain the following improvement of [**48**, Thm. 3.2].

COROLLARY 3.3.18. *Let A be a hereditary algebra. With the hypotheses of Setting 3.3.1, we have the following:*

(i) The algebra C is hereditary.
(ii) There is an order-preserving bijection

$$\mathsf{red} : \mathrm{s}\text{-}\mathsf{tilt}_U\,A \longrightarrow \mathrm{s}\text{-}\mathsf{tilt}\,C$$

given by $M \mapsto F(\mathrm{f}M)$ with inverse $N \mapsto P((\mathsf{Fac}\,U) * G(\mathsf{Fac}\,N))$.

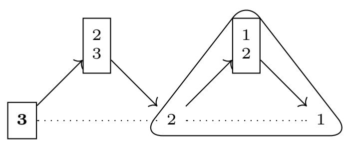
PROOF. Since A is hereditary, the τ-rigid module U is a partial-tilting module. Moreover, as explained in Example 3.3.10 we have that C is also a hereditary algebra. Then by Proposition 3.2.10 we have $\mathrm{s}\tau\text{-}\mathsf{tilt}_U\,A = \mathrm{s}\text{-}\mathsf{tilt}_U\,A$ and $\mathrm{s}\tau\text{-}\mathsf{tilt}\,C = \mathrm{s}\text{-}\mathsf{tilt}\,C$. Then the claim follows from Theorem 3.3.15. □

We conclude this section with some examples illustrating our results.

EXAMPLE 3.3.19. Let A be the algebra given by the quiver $3 \xleftarrow{y} 2 \xleftarrow{x} 1$ with the relation $xy = 0$, see [**1**, Ex. 6.4]. Consider the support τ-tilting A-module $U = P_3$. Then

$$\mathcal{U} = {}^{\perp}(\tau U) \cap U^{\perp} = (\mathsf{mod}\,A) \cap U^{\perp} = U^{\perp}.$$

Moreover, the Bongartz completion of U is given by $T = P_1 \oplus P_2 \oplus P_3$, which is the basic progenerator of ${}^{\perp}(\tau U) = \mathsf{mod}\,A$. Hence $C = \mathsf{End}_A(P_1 \oplus P_2) \cong k(\bullet \leftarrow \bullet)$, see Example 3.2.18. We may visualize this in the Auslander-Reiten quiver of $\mathsf{mod}\,A$, where each A-module is represented by its radical filtration:



The indecomposable summands of T are indicated with rectangles and $\mathcal{U}$ is enclosed in a triangle. Note that $\mathcal{U}$ is equivalent to $\mathsf{mod}\,C$ as shown in Theorem 3.3.8.

By Theorem 3.3.15 we have that $\mathrm{s}\tau\text{-}\mathsf{tilt}\,C$ can be embedded as an interval in $\mathrm{s}\tau\text{-}\mathsf{tilt}\,A$. We have indicated this embedding in $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\,A)$ in Figure 3.3.2 by enclosing the image of $\mathrm{s}\tau\text{-}\mathsf{tilt}\,C$ in rectangles.

EXAMPLE 3.3.20. Let A be the preprojective algebra of Dynkin type A_3, *i.e.* the algebra given by the quiver

$$\begin{tikzcd} 3 \arrow[r, bend right, "y_2"'] & 2 \arrow[l, bend right, "x_2"'] \arrow[r, bend right, "y_1"'] & 1 \arrow[l, bend right, "x_1"'] \end{tikzcd}$$

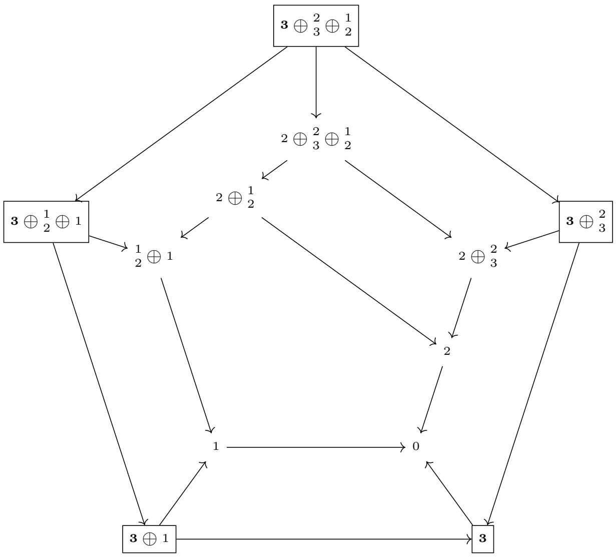


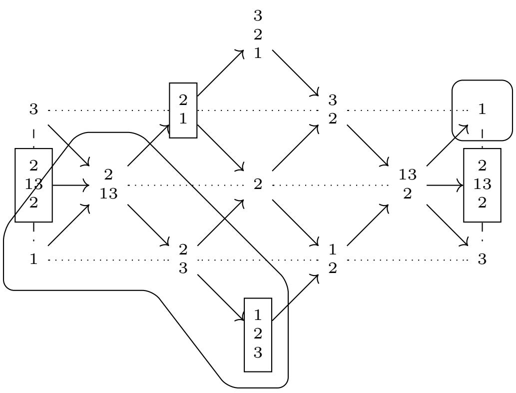
FIGURE 3.3.2. Embedding of sτ-$\mathsf{tilt}\, C$ in $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\, A)$, see Example 3.3.19.

with relations $x_1y_1 = 0$, $y_2x_2 = 0$ and $y_1x_1 = x_2y_2$.

Let $U = {2 \atop 1}$, then U is τ-rigid and not a support τ-tilting A-module. The Bongartz completion of U is given by $T = P_1 \oplus P_2 \oplus {2 \atop 1} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 13 \\ 2 \end{smallmatrix} \oplus {2 \atop 1}$; hence C is isomorphic to the path algebra given by the quiver

$$\begin{tikzcd} \bullet \arrow[r, bend right, "y"'] & \bullet \arrow[l, bend right, "x"'] \end{tikzcd}$$

with the relations $xy = 0$ and $yx = 0$, that is C is isomorphic to the preprojective algebra of type A_2. In this case ${}^{\perp}(\tau U)$ consists of all A-modules M such that $\tau U = S_3$ is not a direct summand of $\operatorname{top} M$. On the other hand, it is easy to see that the only indecomposable A-modules in ${}^{\perp}(\tau U)$ which do not belong to $U^{\perp}$ are U, P_2, S_2 and ${1 \atop 2}$. We can visualize this in the Auslander-Reiten quiver of $\mathsf{mod}\, A$ as follows (note that the dashed edges are to be identified to form a Möbius strip):



The indecomposable summands of T are indicated with rectangles and $\mathcal{U}$ is encircled. Note that $\mathcal{U}$ is equivalent to $\mathsf{mod}\, C$ as shown in Theorem 3.3.8. By Theorem 3.3.15 we have that sτ-$\mathsf{tilt}\, C$ can be embedded as an interval in sτ-$\mathsf{tilt}\, A$. We have indicated this embedding in $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\, A)$ in Figure 3.3.3 by enclosing the image of sτ-$\mathsf{tilt}\, C$ in rectangles. with double arrows.

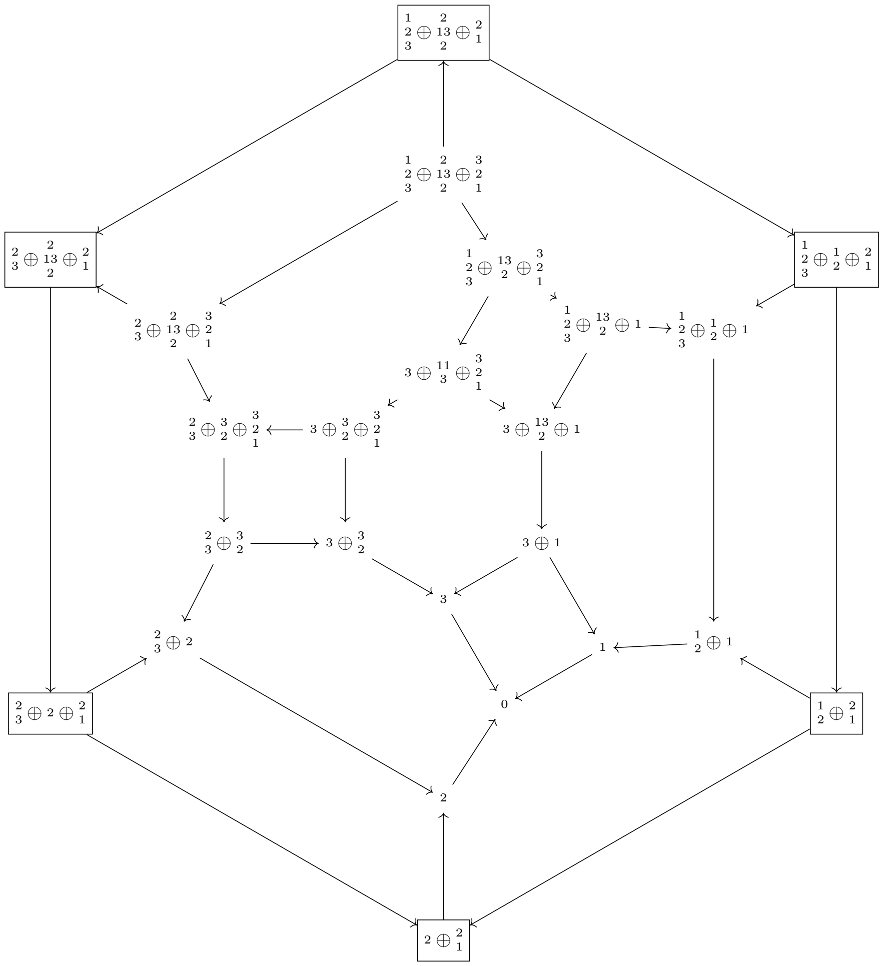
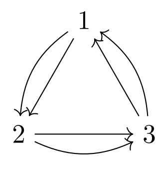


FIGURE 3.3.3. Embedding of $\mathrm{s}\tau\text{-}\mathsf{tilt}\, C$ in $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\, A)$, see Example 3.3.20.

EXAMPLE 3.3.21. Let A be the algebra given by the path algebra of the quiver



modulo the ideal generated by all paths of length two.

Let $U = \begin{smallmatrix}11\\222\end{smallmatrix}$. The Bongartz completion of U is given by $T = P_1 \oplus \begin{smallmatrix}11\\222\end{smallmatrix} \oplus P_3 = \begin{smallmatrix}1\\22\end{smallmatrix} \oplus \begin{smallmatrix}11\\222\end{smallmatrix} \oplus \begin{smallmatrix}3\\11\end{smallmatrix}$; hence $C \cong k \times k$. It is easy to see that $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\, C)$ is given by the quiver

$$\begin{tikzcd} & 1 \oplus 2 \arrow[dl] \arrow[dr] & \\ 1 \arrow[dr] & & 2 \arrow[dl] \\ & 0 & \end{tikzcd}$$

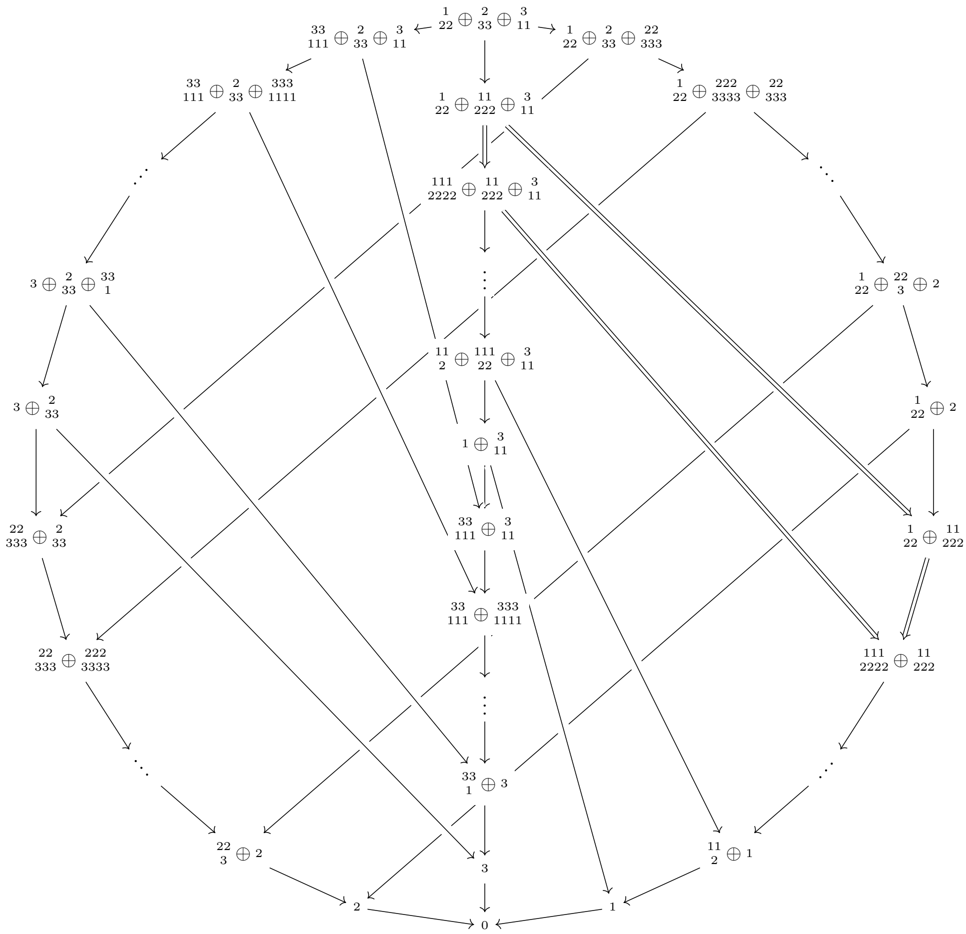


FIGURE 3.3.4. Embedding of $\mathrm{s}\tau\text{-}\mathsf{tilt}\,C$ in $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\,A)$, see Example 3.3.21.

By Theorem 3.3.15 we have that $\mathrm{s}\tau\text{-}\mathsf{tilt}\,C$ can be embedded as an interval in $\mathrm{s}\tau\text{-}\mathsf{tilt}\,A$. We have indicated this embedding in $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\,A)$ in Figure 3.3.4 by drawing $Q(\mathrm{s}\tau\text{-}\mathsf{tilt}\,C)$ with double arrows.

3.3.3. Proof of the main theorems. We begin with the proof of Theorem 3.3.12. The following proposition shows that the map $\mathcal{T} \mapsto \mathcal{T} \cap U^{\perp}$ in Theorem 3.3.12 is well defined.

PROPOSITION 3.3.22. *Let $\mathcal{T}$ be a torsion class in $\mathsf{mod}\,A$ such that $\mathsf{Fac}\,U \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)$. Then the following holds:*

(a) *$\mathcal{T} \cap U^{\perp}$ is in $\mathsf{tors}\,\mathcal{U}$.*
(b) *$\mathcal{T} \cap U^{\perp} = \mathrm{f}\mathcal{T}$.*

If in addition $\mathcal{T}$ is functorially finite in $\mathsf{mod}\,A$, then we have:

(c) *$\mathcal{T} \cap U^{\perp} = \mathsf{Fac}(\mathrm{f}P(\mathcal{T})) \cap U^{\perp}$.*
(d) *$F(\mathcal{T} \cap U^{\perp})$ is in $\mathrm{f}\text{-}\mathsf{tors}\,C$.*

PROOF. (a) $\mathcal{T} \cap U^{\perp}$ is closed under extensions since both $\mathcal{T}$ and $U^{\perp}$ are closed under extensions in $\mathsf{mod}\,A$. Now let $0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathsf{mod}\,A$ with terms in $\mathcal{U}$. If $M \in \mathcal{T}$, then $N \in \mathcal{T}$ since $\mathcal{T}$ is closed under factor modules. Thus $N \in \mathcal{T} \cap \mathcal{U} = \mathcal{T} \cap U^{\perp}$. This shows that $\mathcal{T} \cap U^{\perp}$ is a torsion class in $\mathcal{U}$.

(b) Since $\mathcal{T}$ is closed under factor modules in $\mathsf{mod}\, A$ we have that $\mathrm{f}\mathcal{T} \subseteq \mathcal{T} \cap U^{\perp}$, hence we only need to show the reverse inclusion. Let $M \in \mathcal{T} \cap U^{\perp}$. In particular we have that $M \in U^{\perp}$, hence $\mathrm{f}M = M$ and the claim follows.

(c) Since $\mathsf{Fac}(\mathrm{f}P(\mathcal{T})) \subseteq \mathcal{T}$, we have that $\mathsf{Fac}(\mathrm{f}P(\mathcal{T})) \cap U^{\perp} \subseteq \mathcal{T} \cap U^{\perp}$. Now we show the opposite inclusion. Let M be in $\mathcal{T} \cap U^{\perp}$, then there is an epimorphism $f : X \to M$ with X in $\mathsf{add}(P(\mathcal{T}))$. Since there are no non-zero morphisms from $\mathsf{Fac}\, U$ to $U^{\perp}$ we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{t}X \arrow[r] \arrow[dr, "0"'] & X \arrow[r] \arrow[d, two heads, "f"] & \mathrm{f}X \arrow[r] \arrow[dl, dashed] & 0 \\
 & & M & &
\end{tikzcd}$$

Hence M is in $\mathsf{Fac}(\mathrm{f}P(\mathcal{T})) \cap U^{\perp}$ and we have the equality $\mathcal{T} \cap U^{\perp} = \mathsf{Fac}(\mathrm{f}P(\mathcal{T})) \cap U^{\perp}$.

(d) By Proposition 3.3.14 we have that $\mathrm{f}P(\mathcal{T})$ is the Ext-progenerator of $\mathcal{T} \cap U^{\perp} \subseteq \mathcal{U}$, and since $F : \mathcal{U} \to \mathsf{mod}\, C$ is an exact equivalence, see Theorem 3.3.8, we have that $F(\mathrm{f}P(\mathcal{T}))$ is the Ext-progenerator of $F(\mathcal{T} \cap U^{\perp})$. Then by Proposition 3.2.2 we have that $F(\mathcal{T} \cap U^{\perp}) = \mathsf{Fac}(F(\mathrm{f}P(\mathcal{T})))$ is functorially finite in $\mathsf{mod}\, C$. $\square$

Now we consider the converse map $\mathcal{G} \mapsto (\mathsf{Fac}\, U) * \mathcal{G}$. We start with the following easy observation.

Lemma 3.3.23. *Let $\mathcal{G}$ be in $\mathsf{tors}\, \mathcal{U}$ and M be an A-module. Then M is in $(\mathsf{Fac}\, U) * \mathcal{G}$ if and only if $\mathrm{f}M$ belongs to $\mathcal{G}$. In particular, if M is in $(\mathsf{Fac}\, U) * \mathcal{G}$ then $0 \to \mathrm{t}M \to M \to \mathrm{f}M \to 0$ is the unique way to express M as an extension of a module from $\mathsf{Fac}\, U$ by a module from $\mathcal{G}$.*

Proof. Both claims follow immediately from the uniqueness of the canonical sequence. $\square$

The following lemma gives canonical sequences in $\mathcal{U}$.

Lemma 3.3.24. *Let $\mathcal{G}$ be a torsion class in $\mathcal{U}$. Then there exist functors $\mathrm{t}_{\mathcal{G}} : \mathcal{U} \to \mathcal{G}$ and $\mathrm{f}_{\mathcal{G}} : \mathcal{U} \to \mathcal{G}^{\perp} \cap \mathcal{U}$ and natural transformations $\mathrm{t}_{\mathcal{G}} \to 1_{\mathcal{U}} \to \mathrm{f}_{\mathcal{G}}$ such that the sequence*

$$0 \to \mathrm{t}_{\mathcal{G}} M \to M \to \mathrm{f}_{\mathcal{G}} M \to 0$$

is exact in $\mathsf{mod}\, A$ for each M in $\mathcal{U}$.

Proof. Since $(F\mathcal{G}, F(\mathcal{G}^{\perp} \cap \mathcal{U}))$ is a torsion pair in $\mathsf{mod}\, C$ by Theorem 3.3.8, we have associated canonical sequences in $\mathsf{mod}\, C$. Applying the functor G, we get the desired functors. $\square$

The following proposition shows that the map $\mathcal{G} \mapsto (\mathsf{Fac}\, U) * \mathcal{G}$ in Theorem 3.3.13 is well-defined.

Proposition 3.3.25. *Let $\mathcal{G}$ be in $\mathsf{tors}\, \mathcal{U}$. Then $(\mathsf{Fac}\, U) * \mathcal{G}$ is a torsion class in $\mathsf{mod}\, A$ such that $\mathsf{Fac}\, U \subseteq (\mathsf{Fac}\, U) * \mathcal{G} \subseteq {}^{\perp}(\tau U)$.*

Proof. First, it is clear that $\mathsf{Fac}\, U \subseteq (\mathsf{Fac}\, U) * \mathcal{G} \subseteq {}^{\perp}(\tau U)$ since $\mathcal{G}$ and $\mathsf{Fac}\, U$ are subcategories of ${}^{\perp}(\tau U)$ and ${}^{\perp}(\tau U)$ is closed under extensions.

Now, let us show that $(\mathsf{Fac}\, U) * \mathcal{G}$ is closed under factor modules. Let M be in $(\mathsf{Fac}\, U) * \mathcal{G}$, so by Lemma 3.3.23 we have that $\mathrm{f}M$ is in $\mathcal{G}$, and let $f : M \to N$ be an epimorphism. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{t}M \arrow[r] \arrow[d, "\mathrm{t}f"] & M \arrow[r] \arrow[d, "f"] & \mathrm{f}M \arrow[r] \arrow[d, "\mathrm{f}f"] & 0 \\
0 \arrow[r] & \mathrm{t}N \arrow[r] & N \arrow[r] \arrow[d] & \mathrm{f}N \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. Since $\mathrm{f}M \in \mathcal{G} \subseteq {}^{\perp}(\tau U)$, we have that $\mathrm{f}N \in {}^{\perp}(\tau U)$. Thus $\mathrm{f}N \in {}^{\perp}(\tau U) \cap U^{\perp} = \mathcal{U}$. Since $\mathrm{f}M$ belong to $\mathcal{G}$ which is a torsion class in $\mathcal{U}$, we have that $\mathrm{f}N \in \mathcal{G}$. Finally, since $\mathrm{t}N \in \mathsf{Fac}\, U$ we have that $N \in (\mathsf{Fac}\, U) * \mathcal{G}$. The claim follows.

To show that $(\mathsf{Fac}\, U) * \mathcal{G}$ is closed under extensions it is sufficient to show that $\mathcal{G} * \mathsf{Fac}\, U \subseteq (\mathsf{Fac}\, U) * \mathcal{G}$ since this implies

$$\begin{aligned}((\mathsf{Fac}\, U) * \mathcal{G}) * ((\mathsf{Fac}\, U) * \mathcal{G}) =& (\mathsf{Fac}\, U) * (\mathcal{G} * (\mathsf{Fac}\, U)) * \mathcal{G} \\ \subseteq& (\mathsf{Fac}\, U) * (\mathsf{Fac}\, U) * \mathcal{G} * \mathcal{G} = (\mathsf{Fac}\, U) * \mathcal{G}\end{aligned}$$

by the associativity of the operation $*$. For this, let $0 \to N \to M \to L \to 0$ be a short exact sequence with N in $\mathcal{G}$ and L in $\mathsf{Fac}\, U$. We only have to show that $\mathrm{f}M \in \mathcal{G}$, or equivalently that $\mathrm{f}_{\mathcal{G}}(\mathrm{f}M) = 0$. Since $N \in \mathcal{G}$ and $\mathrm{f}_{\mathcal{G}}(\mathrm{f}M) \in \mathcal{G}^{\perp}$ we have $\mathsf{Hom}_A(N, \mathrm{f}_{\mathcal{G}}(\mathrm{f}M)) = 0$. Also, $\mathsf{Hom}_A(L, \mathrm{f}_{\mathcal{G}}(\mathrm{f}M)) = 0$ since $L \in \mathsf{Fac}\, U$ and $\mathrm{f}_{\mathcal{G}}(\mathrm{f}M) \in U^{\perp}$. Thus we have $\mathsf{Hom}_A(M, \mathrm{f}_{\mathcal{G}}(\mathrm{f}M)) = 0$. But $\mathrm{f}_{\mathcal{G}}(\mathrm{f}M)$ is a factor module of M so we have that $\mathrm{f}_{\mathcal{G}}(\mathrm{f}M) = 0$. Thus $\mathrm{f}M = \mathrm{t}_{\mathcal{G}}(\mathrm{f}M)$ belongs to $\mathcal{G}$. □

Now we give the proof Theorem 3.3.12.

Proof of Theorem 3.3.12. By Corollary 3.3.11 we have that the functors F and G induce mutually inverse bijections between $\mathsf{tors}\, \mathcal{U}$ and $\mathsf{tors}\, C$. It follows from Proposition 3.3.22(a) that the correspondence $\mathcal{T} \mapsto \mathcal{T} \cap U^{\perp}$ gives a well defined map

$$\left\{\mathcal{T} \in \mathsf{tors}\, A \mid \mathsf{Fac}\, U \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)\right\} \longrightarrow \mathsf{tors}\, \mathcal{U}.$$

On the other hand, it follows from Proposition 3.3.25 that the association $\mathcal{G} \mapsto (\mathsf{Fac}\, U) * \mathcal{G}$ gives a well defined map

$$\mathsf{tors}\, \mathcal{U} \longrightarrow \left\{\mathcal{T} \in \mathsf{tors}\, A \mid \mathsf{Fac}\, U \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)\right\}.$$

It remains to show that the maps

$$\mathcal{T} \mapsto \mathcal{T} \cap U^{\perp} \quad \text{and} \quad \mathcal{G} \mapsto \mathsf{Fac}\, U * \mathcal{G}$$

are inverse of each other. Let $\mathcal{T}$ be a torsion class in $\mathsf{mod}\, A$ such that $\mathsf{Fac}\, U \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)$. Since $\mathcal{T}$ is closed under extensions, we have that $(\mathsf{Fac}\, U) * (\mathcal{T} \cap U^{\perp}) \subseteq \mathcal{T}$. Thus we only need to show the opposite inclusion. Let M be in $\mathcal{T}$, then we have an exact sequence

$$0 \to \mathrm{t}M \to M \to \mathrm{f}M \to 0$$

with $\mathrm{t}M \in \mathsf{Fac}\, U$ and $\mathrm{f}M$ in $\mathcal{T} \cap U^{\perp}$ since $\mathcal{T}$ is closed under factor modules. Thus $M \in (\mathsf{Fac}\, U) * (\mathcal{T} \cap U^{\perp})$ holds and the claim follows.

On the other hand, let $\mathcal{G}$ be a torsion class in $\mathcal{U}$. It is clear that $\mathcal{G} \subseteq ((\mathsf{Fac}\, U) * \mathcal{G}) \cap U^{\perp}$, so we only need to show the opposite inclusion. But if M is in $((\mathsf{Fac}\, U) * \mathcal{G}) \cap U^{\perp}$, then $M \in U^{\perp}$ implies that $M \cong \mathrm{f}M$. Moreover, by Lemma 3.3.23 we have that $M \cong \mathrm{f}M$ belongs to $\mathcal{G}$. This finishes the proof of the theorem. □

Now we begin to prove Theorem 3.3.13. For this we need the following technical result:

Proposition 3.3.26. [**60**, Prop 5.33] *Let $\mathcal{X}$ and $\mathcal{Y}$ be covariantly finite subcategories of* $\mathsf{mod}\, A$. *Then $\mathcal{X} * \mathcal{Y}$ is also covariantly finite in* $\mathsf{mod}\, A$.

We also need the following observation.

LEMMA 3.3.27. *$\mathcal{U}$ is covariantly finite in ${}^{\perp}(\tau U)$.*

PROOF. Let M be in ${}^{\perp}(\tau U)$ and consider the canonical sequence

$$0 \to \mathrm{t}M \to M \xrightarrow{f} \mathrm{f}M \to 0.$$

Then $\mathrm{f}M$ is in $\mathcal{U}$ by (3.3.9) and clearly f is a left $\mathcal{U}$-approximation (mind that $\mathcal{U} \subseteq U^{\perp}$). Thus $\mathcal{U}$ is covariantly finite in ${}^{\perp}(\tau U)$ as required. □

The following proposition shows that the map $\mathcal{G} \mapsto (\mathsf{Fac}\, U) * \mathcal{G}$ in Theorem 3.3.13 preserves functorial finiteness, and thus is well defined.

PROPOSITION 3.3.28. *Let $\mathcal{G}$ be in $\mathrm{f}\text{-}\mathsf{tors}\,\mathcal{U}$. Then $(\mathsf{Fac}\, U) * \mathcal{G}$ is a functorially finite torsion class in $\mathsf{mod}\, A$ such that $\mathsf{Fac}\, U \subseteq (\mathsf{Fac}\, U) * \mathcal{G} \subseteq {}^{\perp}(\tau U)$.*

PROOF. By Proposition 3.3.25, we only need to show that $(\mathsf{Fac}\, U) * \mathcal{G}$ is covariantly finite in $\mathsf{mod}\, A$. Since $\mathsf{Fac}\, U$ is covariantly finite in $\mathsf{mod}\, A$, see Proposition 3.2.2(b), by Proposition 3.3.26 it is enough to show that $\mathcal{G}$ is covariantly finite in $\mathsf{mod}\, A$. By Lemma 3.3.27 we have that $\mathcal{U}$ is covariantly finite in ${}^{\perp}(\tau U)$. Since $\mathcal{G}$ is covariantly finite in $\mathcal{U}$ and ${}^{\perp}(\tau U)$ is covariantly finite in $\mathsf{mod}\, A$, see Proposition 3.2.2(b), we have that $\mathcal{G}$ is covariantly finite in $\mathsf{mod}\, A$. □

We are ready to give the proof of Theorem 3.3.13.

PROOF OF THEOREM 3.3.13. We only need to show that the bijections in Theorem 3.3.12 preserve functorial finiteness. But this follows immediately from Proposition 3.3.22(d) and Proposition 3.3.28. The theorem follows. □

3.4. Compatibility with other types of reduction

Let A be a finite dimensional algebra. Then support τ-tilting A-modules are in bijective correspondence with the so-called two-term silting complexes in $K^{\mathrm{b}}(\mathsf{proj}\, A)$, see [**1**, Thm. 3.2].

On the other hand, if A is a 2-Calabi-Yau-tilted algebra from a 2-Calabi-Yau category $\mathcal{C}$, then there is a bijection between $\mathrm{s}\tau\text{-}\mathsf{tilt}\, A$ and the set of isomorphism classes of basic cluster-tilting objects in $\mathcal{C}$, see [**1**, Thm. 4.1].

Reduction techniques were established (in greater generality) in [**62**, Thm. 4.9] for cluster-tilting objects and for silting objects in [**3**, Thm. 2.37] for a special case and in [**61**] for the general case. The aim of this section is to show that these reductions are compatible with τ-tilting reduction as established in Section 3.3.

Given two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of a triangulated category $\mathcal{T}$, we write $\mathcal{X} * \mathcal{Y}$ for the full subcategory of $\mathcal{T}$ consisting of all objects $Z \in \mathcal{T}$ such that there exists a triangle

$$X \to Z \to Y \to X[1]$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. For objects X and Y in $\mathcal{T}$ we define $X * Y := (\mathsf{add}\, X) * (\mathsf{add}\, Y)$.

3.4.1. Silting reduction. Let $\mathcal{T}$ be a Krull-Schmidt triangulated category and S an object in $\mathcal{T}$. Following [**3**, Def. 2.1], we say that M is a *presilting object in* $\mathcal{T}$ if

$$\mathsf{Hom}_{\mathcal{T}}(M, M[i]) = 0 \quad \text{for all } i > 0.$$

We call S a *silting object* if moreover $\mathsf{thick}(S) = \mathcal{T}$, where $\mathsf{thick}(S)$ is the smallest triangulated subcategory of $\mathcal{T}$ which contains S and is closed under direct summands and isomorphisms. We denote the set of isomorphism classes of all basic silting objects in $\mathcal{T}$ by $\mathsf{silt}\, \mathcal{T}$.

Let $M, N \in \mathsf{silt}\,\mathcal{T}$. We write $N \leqslant M$ if and only if $\mathsf{Hom}_{\mathcal{T}}(M, N[i]) = 0$ for each $i > 0$. Then $\leqslant$ is a partial order in $\mathsf{silt}\,\mathcal{T}$, see [**3**, Thm. 2.11].

SETTING 3.4.1. We fix a k-linear, Hom-finite, Krull-Schmidt triangulated category $\mathcal{T}$ with a silting object S, and let

$$A = A_S := \mathsf{End}_{\mathcal{T}}(S).$$

The subset $2_S\text{-}\,\mathsf{silt}\,\mathcal{T}$ of $\mathsf{silt}\,\mathcal{T}$ given by

$$2_S\text{-}\,\mathsf{silt}\,\mathcal{T} := \{M \in \mathsf{silt}\,\mathcal{T} \mid M \in S * (S[1])\}$$

plays an important role in the sequel. The notation $2_S\text{-}\,\mathsf{silt}\,\mathcal{T}$ is justified by the following remark.

REMARK 3.4.2. Let A be a finite dimensional algebra and $\mathcal{T} = K^{\mathrm{b}}(\mathsf{proj}\,A)$. Then A is a silting object in $\mathcal{T}$. In this case a silting complex M belongs to $2_A\text{-}\,\mathsf{silt}\,\mathcal{T}$ if and only if M is isomorphic to a complex concentrated in degrees -1 and 0, *i.e.* if M is a *two-term silting complex*.

Following [**3**, Sec. 2], we consider the subcategory of $\mathcal{T}$ given by

$$\mathcal{T}^{\leqslant 0} := \{M \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(S, M[i]) = 0 \text{ for all } i > 0\}\,.$$

We need the following generating properties of silting objects. Recall that a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathcal{T}$ is called a *torsion pair in* $\mathcal{T}$ if $\mathsf{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0$ and $\mathcal{T} = \mathcal{X} * \mathcal{Y}$.

PROPOSITION 3.4.3. [**3**, Prop. 2.23] *With the hypotheses of Setting 3.4.1, we have the following:*

$$\mathcal{T} = \bigcup_{\ell \geqslant 0} S[-\ell] * S[1-\ell] * \cdots * S[\ell],$$
$$\mathcal{T}^{\leqslant 0} = \bigcup_{\ell \geqslant 0} S * S[1] * \cdots * S[\ell],$$
$${}^{\perp}(\mathcal{T}^{\leqslant 0}) = \bigcup_{\ell > 0} S[-\ell] * S[1-\ell] * \cdots * S[-1].$$

Moreover, the pair $({}^{\perp}(\mathcal{T}^{\leqslant 0}), \mathcal{T}^{\leqslant 0})$ *is a torsion pair in* $\mathcal{T}$.

The following proposition describes $2_S\text{-}\,\mathsf{silt}\,\mathcal{T}$ in terms of the partial order in $\mathsf{silt}\,\mathcal{T}$. It is shown in [**2**, Prop. 2.9] in the case when $\mathcal{T} = K^{\mathrm{b}}(\mathsf{proj}\,A)$ and $S = A$.

PROPOSITION 3.4.4. *Let M be an object of $\mathcal{T}$. Then, $M \in S * S[1]$ if and only if $S[1] \leqslant M \leqslant S$.*

PROOF. Before starting the proof, let us make the following trivial observation: Given two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{T}$, for any object M of $\mathcal{T}$, we have that $\mathsf{Hom}_{\mathcal{T}}(\mathcal{X} * \mathcal{Y}, M) = 0$ if and only if $\mathsf{Hom}_{\mathcal{T}}(\mathcal{X}, M) = 0$ and $\mathsf{Hom}_{\mathcal{T}}(\mathcal{Y}, M) = 0$.

Now note that we have $M \leqslant S$ if and only if $\mathsf{Hom}_{\mathcal{T}}(S[-i], M) = 0$ for each $i > 0$, or equivalently by the above observation, $\mathsf{Hom}_{\mathcal{T}}(S[-\ell] * \cdots * S[-1], M) = 0$ for each $\ell > 0$. Thus, by Proposition 3.4.3 we have that

$$M \leqslant S \quad \text{if and only if} \quad \mathsf{Hom}_{\mathcal{T}}({}^{\perp}(\mathcal{T}^{\leqslant 0}), M) = 0 \tag{3.4.1}$$

or equivalently, since $({}^{\perp}(\mathcal{T}^{\leqslant 0}), \mathcal{T}^{\leqslant 0})$ is a torsion pair by Proposition 3.4.3, $M \in \mathcal{T}^{\leqslant 0} = (S * S[1]) * \mathcal{T}^{\leqslant 0}[2]$. By a similar argument, we have that

$$S[1] \leqslant M \quad \text{if and only if} \quad \mathsf{Hom}_{\mathcal{T}}(M, \mathcal{T}^{\leqslant 0}[2]) = 0. \tag{3.4.2}$$

Then it follows from (3.4.1) and (3.4.2) that $S[1] \leqslant M \leqslant S$ if and only if $M \in S * S[1]$. ☐

We need the following result:

Proposition 3.4.5. [**62**, Prop. 6.2(3)] *The functor*

$$\underline{(-)} = \mathsf{Hom}_{\mathcal{T}}(S, -) : S * S[1] \to \mathsf{mod}\, A \tag{3.4.3}$$

induces an equivalence of categories

$$\underline{(-)} : \frac{S * S[1]}{[S[1]]} \longrightarrow \mathsf{mod}\, A. \tag{3.4.4}$$

where $[S[1]]$ *is the ideal of* $\mathcal{T}$ *consisting of morphisms which factor through* $\mathsf{add}\, S[1]$.

Proof. Take $\mathcal{X} = \mathsf{add}\, S$, $\mathcal{Y} = \mathsf{add}\, S[1]$ and $\mathcal{Z} = \mathsf{add}\, S$ in [**62**, Prop. 6.2(3)]. □

In view of Proposition 3.4.5, for every $M, N \in S * S[1]$ we have a natural isomorphism

$$\frac{\mathsf{Hom}_{\mathcal{T}}(M, N)}{[S[1]](M, N)} \cong \mathsf{Hom}_A(\underline{M}, \underline{N}).$$

Setting 3.4.6. From now on, we fix a presilting object U in $\mathcal{T}$ contained in $S * S[1]$. For simplicity, we assume that U has no non-zero direct summands in $\mathsf{add}\, S[1]$. We are interested in the subset of $2_S\text{-}\mathsf{silt}\,\mathcal{T}$ given by

$$2_S\text{-}\mathsf{silt}_U\,\mathcal{T} := \{M \in 2_S\text{-}\mathsf{silt}\,\mathcal{T} \mid U \in \mathsf{add}\, S\}.$$

The following theorem is similar to [**1**, Thm. 3.2].

Theorem 3.4.7. [**58**, Thm. 4.5] *With the hypotheses of Setting 3.4.1, the functor* (3.4.3) *induces an order-preserving bijection*

$$\underline{(-)} : 2_S\text{-}\mathsf{silt}\,\mathcal{T} \longrightarrow \mathrm{s}\tau\text{-}\mathsf{tilt}\, A$$

which induces a bijection

$$\underline{(-)} : 2_S\text{-}\mathsf{silt}_U\,\mathcal{T} \longrightarrow \mathrm{s}\tau\text{-}\mathsf{tilt}_{\underline{U}}\, A.$$

Silting reduction was introduced in [**3**, Thm. 2.37] in a special case and [**61**] in the general case. We are interested in the following particular situation:

Theorem 3.4.8. [**61**] *Let* U *be a presilting object in* $\mathcal{T}$ *contained in* $S * S[1]$. *Then the canonical functor*

$$\mathcal{T} \longrightarrow \mathcal{U} := \frac{\mathcal{T}}{\mathsf{thick}(U)} \tag{3.4.5}$$

induces an order-preserving bijection

$$\mathsf{red} : \{M \in \mathsf{silt}\,\mathcal{T} \mid U \in \mathsf{add}\, M\} \longrightarrow \mathsf{silt}\,\mathcal{U}.$$

We need to consider the following analog of Bongartz completion for presilting objects in $S * S[1]$, *cf.* [**30**, Sec. 5] and [**85**, Prop. 6.1].

Definition-Proposition 3.4.9. [**2**, Prop. 2.16] Let $f : U' \to S[1]$ be a minimal right $(\mathsf{add}\, U)$-approximation of $S[1]$ in $\mathcal{T}$ and consider a triangle

$$S \to X_U \to U' \xrightarrow{f} S[1]. \tag{3.4.6}$$

Then $T_U := X_U \oplus U$ is in $2_S\text{-}\mathsf{silt}\,\mathcal{T}$ and moreover T_U has no non-zero direct summands in $\mathsf{add}\, S[1]$. We call T_U the *Bongartz completion of* U *in* $S * S[1]$.

Proof. It is shown in [**2**, Prop. 2.16] that T_U is a silting object in $\mathcal{T}$. Moreover, since $\mathsf{Hom}_{\mathcal{T}}(S, S[1]) = 0$ we have

$$T_U \in S * (S * S[1]) = (S * S) * S[1] = S * S[1],$$

hence $T_U \in 2_S\text{-}\mathsf{silt}\,\mathcal{T}$. Finally, since $\mathsf{Hom}_A(S, S[1]) = 0$ and U has no non-zero direct summands in $\mathsf{add}\, S[1]$, it follow from the triangle (3.4.6) that T_U has no non-zero direct summands in $\mathsf{add}\, S[1]$. □

We recall that by Proposition 3.3.2 we have that $\mathrm{s}\tau\text{-}\mathsf{tilt}_{\underline{U}}\, A$ equals the interval

$$\left\{M \in \mathrm{s}\tau\text{-}\mathsf{tilt}\, A \mid P(\mathsf{Fac}\, \underline{U}) \leqslant M \leqslant T_{\underline{U}}\right\} \subseteq \mathrm{s}\tau\text{-}\mathsf{tilt}\, A;$$

hence $T_{\underline{U}}$ is the unique maximal element in $\mathrm{s}\tau\text{-}\mathsf{tilt}_{\underline{U}}\, A$. The following proposition relates the Bongartz completion T_U of U in $S * S[1]$ with the Bongartz completion $T_{\underline{U}}$ of $\underline{U}$ in $\mathsf{mod}\, A$.

Proposition 3.4.10. *(i) T_U is the unique maximal element in $2_S\text{-}\mathsf{silt}_U\, \mathcal{T}$.*
(ii) $\overline{T_U} \cong T_{\overline{U}}$.

Proof. First, note that (b) follows easily from part (a). Indeed, since T_U has no non-zero direct summands in $\mathsf{add}\, S[1]$, see Definition-Proposition 3.4.9, we have that $|A| = |S| = |T_U| = |\overline{T_U}|$. By Theorem 3.4.7 we have that $\overline{T_U}$ is a τ-tilting A-module. Since $P({}^{\perp}(\tau\overline{U}))$ is the unique maximal element in $\mathrm{s}\tau\text{-}\mathsf{tilt}_{\overline{U}}\, A$, to show part (b), *i.e.* that $\overline{T_U} \cong P({}^{\perp}(\tau\overline{U}))$, we only need to show that T_U is the unique maximal element in $2_S\text{-}\mathsf{silt}_U\, \mathcal{T}$.

For this, let $M \in 2_S\text{-}\mathsf{silt}_U\, \mathcal{T}$ and fix $i > 0$. Applying the functor $\mathsf{Hom}_{\mathcal{T}}(-, M[i])$ to (3.4.6) we obtain an exact sequence

$$\mathsf{Hom}_{\mathcal{T}}(U', M[i]) \to \mathsf{Hom}_{\mathcal{T}}(X_U, M[i]) \to \mathsf{Hom}_{\mathcal{T}}(S, M[i]).$$

Now, since M is silting and $U' \in \mathsf{add}\, M$ we have that $\mathsf{Hom}_{\mathcal{T}}(U', M[i]) = 0$ for each $i > 0$. On the other hand, since $M \in S * S[1]$, by Proposition 3.4.4 we have $\mathsf{Hom}_{\mathcal{T}}(S, M[i]) = 0$ for each $i > 0$. Thus he have $\mathsf{Hom}_{\mathcal{T}}(X_U, M[i]) = 0$ for each $i > 0$. Since $T_U = X_U \oplus U$ we have $\mathsf{Hom}_{\mathcal{T}}(T_U, M[i]) = 0$ for each $i > 0$, hence $M \leqslant T_U$. The claim follows. □

From this we can deduce the following result:

Proposition 3.4.11. *Let $T_U \in 2_S\text{-}\mathsf{silt}\, \mathcal{T}$ be the Bongartz completion of U in $S * S[1]$. Then $T_U \cong S$ in $\mathcal{U}$ and the canonical functor* (3.4.5) *induces an order preserving map*

$$\mathsf{red} : 2_S\text{-}\mathsf{silt}_U\, \mathcal{T} \longrightarrow 2_{T_U}\text{-}\mathsf{silt}\, \mathcal{U}.$$

Proof. By (3.4.6) we have that $S \cong T_U$ in $\mathcal{U} = {}^{\perp}(\tau U) \cap U^{\perp}$, hence the canonical functor $\mathcal{T} \to \mathcal{U}$ restricts to a functor $S * S[1] \to T_U * T_U[1] \subset \mathcal{U}$. The claim now follows from Theorem 3.4.8. □

We are ready to state the main theorem of this section. We keep the notation of the above discussion.

Theorem 3.4.12. *With the hypotheses of Settings 3.4.1 and 3.4.6, we have the following:*

(i) The algebras $\mathsf{End}_{\mathcal{U}}(T_U)$ and $C = \mathsf{End}_A(\overline{T_U})/\langle e_{\overline{U}}\rangle$ are isomorphic, where $e_{\overline{U}}$ is the idempotent corresponding to the projective $\mathsf{End}_A(\overline{T_U})$-module $\mathsf{Hom}_A(\overline{T_U}, \overline{U})$.

(ii) We have a commutative diagram

$$\begin{tikzcd} 2_S\text{-}\mathsf{silt}_U\, \mathcal{T} \arrow[r, "\mathsf{Hom}_{\mathcal{T}}(S,-)"] \arrow[d, "\mathsf{red}"'] & \mathrm{s}\tau\text{-}\mathsf{tilt}_{\overline{U}}\, A \arrow[d, "\mathsf{red}"] \\ 2_{T_U}\text{-}\mathsf{silt}\, \mathcal{U} \arrow[r, "\mathsf{Hom}_{\mathcal{U}}(T_U,-)"'] & \mathrm{s}\tau\text{-}\mathsf{tilt}\, C \end{tikzcd}$$

in which each arrow is a bijection. The vertical maps are given in Proposition 3.4.11 and Theorem 3.3.15 respectively.

We begin by proving part (a) of Theorem 3.4.12. For this we need the following technical result.

Let $\mathcal{V}$ be the subcategory of $\mathcal{T}$ given by

$$\mathcal{V} := \{M \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(M, U[i]) = 0 \text{ and } \mathsf{Hom}_{\mathcal{T}}(U, M[i]) = 0 \text{ for each } i > 0\}.$$

Note that if M is an object in $\mathsf{silt}_U \mathcal{T}$ then $M \in \mathcal{V}$. The following theorem allows us to realize $\mathcal{U}$ as a subfactor category of $\mathcal{T}$.

THEOREM 3.4.13. [**61**] *The composition of canonical functors $\mathcal{V} \to \mathcal{T} \to \mathcal{U}$ induces an equivalence*

$$\frac{\mathcal{V}}{[U]} \cong \mathcal{U}$$

of additive categories. In particular, for every M in $\mathcal{V}$ there is a natural isomorphism

$$\frac{\mathsf{Hom}_{\mathcal{T}}(T_U, M)}{[U](T_U, M)} \cong \mathsf{Hom}_{\mathcal{U}}(T_U, M)$$

Now we can prove the following lemma:

LEMMA 3.4.14. *For each M in $\mathcal{V}$ we have a functorial isomorphism*

$$\frac{\mathsf{Hom}_A(\overline{T_U}, \overline{M})}{[\overline{U}](\overline{T_U}, \overline{M})} \cong \mathsf{Hom}_{\mathcal{U}}(T_U, M).$$

PROOF. By (3.4.4) we have the following functorial isomorphism

$$\frac{\mathsf{Hom}_A(\overline{T_U}, \overline{M})}{[\overline{U}](\overline{T_U}, \overline{M})} \cong \frac{\frac{\mathsf{Hom}_{\mathcal{C}}(T_U, M)}{[S[1]](T_U, M)]}}{\frac{[U](T_U, M) + [S[1]](T_U, M)}{[S[1]](T_U, M)}}.$$

We claim that $[S[1]](T_U, M) \subseteq [U](T_U, M)$, or equivalently

$$[S[1]](X_U, M) \subseteq [U](T_U, M)$$

since $T_U = X \oplus U$. Apply the contravariant functor $\mathsf{Hom}_{\mathcal{T}}(-, S[1])$ to the triangle (3.4.6) to obtain an exact sequence

$$\mathsf{Hom}_{\mathcal{T}}(U', S[1]) \to \mathsf{Hom}_{\mathcal{T}}(X_U, S[1]) \to \mathsf{Hom}_{\mathcal{T}}(S, S[1]) = 0.$$

Hence every morphism $X_U \to S[1]$ factors through U', and we have $[S[1]](X_U, M) \subseteq [U](X_U, M)$. Thus by Theorem 3.4.13 we have isomorphisms

$$\frac{\mathsf{Hom}_A(\overline{T_U}, \overline{M})}{[\overline{U}](\overline{T_U}, \overline{M})} \cong \frac{\mathsf{Hom}_{\mathcal{C}}(T_U, M)}{[U](T_U, M)} \cong \mathsf{Hom}_{\mathcal{U}}(T_U, M),$$

which shows the assertion. □

Now part (a) of Theorem 3.4.12 follows by putting $M = T_U$ in Lemma 3.4.14. In the remainder we prove Theorem 3.4.12(b).

For $X \in \mathsf{mod}\, A$ we denote by $0 \to \mathrm{t}X \to X \to \mathrm{f}X \to 0$ the canonical sequence of X with respect to the torsion pair $(\mathsf{Fac}\, \overline{U}, \overline{U}^{\perp})$ in $\mathsf{mod}\, A$.

PROPOSITION 3.4.15. *For each M in $\mathcal{V}$ there is an isomorphism of C-modules*

$$\mathsf{Hom}_A(\overline{T_U}, \mathrm{f}\overline{M}) \cong \mathsf{Hom}_{\mathcal{U}}(T_U, M)$$

PROOF. By Lemma 3.4.14 it is sufficient to show that

$$\frac{\mathsf{Hom}_A(\overline{T_U}, \overline{M})}{[\overline{U}](\overline{T_U}, \overline{M})} \cong \mathsf{Hom}_A(\overline{T_U}, \mathrm{f}\overline{M}).$$

Apply the functor $\mathsf{Hom}_A(\overline{T_U}, -)$ to the canonical sequence

$$0 \to \mathrm{t}\overline{M} \xrightarrow{i} \overline{M} \to \mathrm{f}\overline{M} \to 0$$

to obtain an exact sequence

$$0 \to \mathsf{Hom}_A(\overline{T_U}, \mathrm{t}\overline{M}) \xrightarrow{i\circ -} \mathsf{Hom}_A(\overline{T_U}, \overline{M}) \to \mathsf{Hom}_A(\overline{T_U}, \mathrm{f}\overline{M}) \to \mathsf{Ext}^1_A(\overline{T_U}, \mathrm{t}\overline{M}) = 0,$$

since $\overline{T_U}$ is Ext-projective in ${}^{\perp}(\tau\overline{U})$ by Proposition 3.4.10 and $\mathrm{t}\overline{M}$ is in $\mathsf{Fac}\,\overline{U} \subseteq {}^{\perp}(\tau\overline{U})$. Thus

$$\mathsf{Hom}_A(\overline{T_U}, \mathrm{f}\overline{M}) \cong \frac{\mathsf{Hom}_A(\overline{T_U}, \overline{M})}{i(\mathsf{Hom}_A(\overline{T_U}, \mathrm{t}\overline{M}))}.$$

Thus we only have to show the equality $\mathsf{Hom}_A(\overline{T_U}, \mathrm{t}\overline{M}) = i([U](\overline{T_U}, \mathrm{t}\overline{M}))$. First, $\mathsf{Hom}_A(\overline{T_U}, \mathrm{t}\overline{M}) \subseteq i([U](\overline{T_U}, \mathrm{t}\overline{M}))$ since i is a right $(\mathsf{Fac}\,U)$-approximation of $\overline{M}$. Next we show the reverse inclusion. It is enough to show that every map $\overline{T_U} \to \mathrm{t}\overline{M}$ factors through $\mathsf{add}\,\overline{U}$. Let $r : \overline{U'} \to \mathrm{t}\overline{M}$ be a right $(\mathsf{add}\,\overline{U})$-approximation. Since $t\overline{M} \in \mathsf{Fac}\,\overline{U}$ we have a short exact sequence

$$0 \to \overline{K} \to \overline{U'} \xrightarrow{r} \mathrm{t}\overline{M} \to 0.$$

Moreover, by Lemma 3.2.7, we have $\overline{K} \in {}^{\perp}(\tau\overline{U})$. Apply the functor $\mathsf{Hom}_A(\overline{T_U}, -)$ to the above sequence to obtain an exact sequence

$$\mathsf{Hom}_A(\overline{T_U}, \overline{U'}) \to \mathsf{Hom}_A(\overline{T_U}, \mathrm{t}\overline{M}) \to \mathsf{Ext}^1_A(\overline{T_U}, \overline{K}) = 0.$$

Thus the assertion follows. □

We are ready to prove Theorem 3.4.12(b).

Proof of Theorem 3.4.12(b). Let $M \in 2_S\text{-}\mathsf{silt}_U\,\mathcal{T}$. Then $M \in \mathcal{V}$. We only need to show that $\mathsf{Hom}_{\mathcal{U}}(T_U, M)$ coincides with the τ-tilting reduction of $\overline{M} \in \mathrm{s}\tau\text{-}\,\mathsf{tilt}\,A$ with respect to $\overline{U}$, which is given by $\mathsf{Hom}_A(\overline{T_U}, \mathrm{f}\overline{M})$, see Theorem 3.3.15. This is shown in Proposition 3.4.15. □

Corollary 3.4.16. *The map*

$$\mathsf{red} : \{M \in 2_S\text{-}\,\mathsf{silt}\,\mathcal{T} \mid U \in \mathsf{add}\,M\} \longrightarrow 2_{T_U}\text{-}\,\mathsf{silt}\,\mathcal{U}.$$

is bijective.

3.4.2. Calabi-Yau reduction. Let $\mathcal{C}$ be a Krull-Schmidt 2-Calabi-Yau triangulated category. Thus $\mathcal{C}$ is k-linear, Hom-finite and there is a bifunctorial isomorphism

$$\mathsf{Hom}_{\mathcal{C}}(M, N) \cong D\,\mathsf{Hom}_{\mathcal{C}}(N, M[2])$$

for every $M, N \in \mathcal{C}$, where $D = \mathsf{Hom}_k(-, k)$ is the usual k-duality. Recall that an object T in $\mathcal{C}$ is called *cluster-tilting* if

$$\mathsf{add}\,T = \{X \in \mathcal{C} \mid \mathsf{Hom}_{\mathcal{C}}(X, T[1]) = 0\}.$$

We denote by $\mathsf{c}\text{-}\,\mathsf{tilt}\,\mathcal{C}$ the set of isomorphism classes of all basic cluster-tilting objects in $\mathcal{C}$.

Setting 3.4.17. We fix a Krull-Schmidt 2-Calabi-Yau triangulated category $\mathcal{C}$ with a cluster-tilting object T. Also, we let

$$A = \mathsf{End}_{\mathcal{C}}(T).$$

The algebra A is said to be *2-Calabi-Yau tilted.*

Note that the functor

$$\overline{(-)} = \mathsf{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \longrightarrow \mathsf{mod}\,A \tag{3.4.7}$$

induces an equivalence of categories

$$\overline{(-)} : \frac{\mathcal{C}}{[T[1]]} \longrightarrow \mathsf{mod}\,A \tag{3.4.8}$$

where $[T[1]]$ is the ideal of $\mathcal{C}$ consisting of morphisms which factor through $\mathsf{add}\,T[1]$, [**69**, Prop. 2(c)]. Thus for every $M, N \in \mathcal{C}$ we have a natural isomorphism

$$\mathsf{Hom}_A(\overline{M}, \overline{N}) \cong \frac{\mathsf{Hom}_{\mathcal{C}}(M, N)}{[T[1]](M, N)}.$$

We have the following result:

THEOREM 3.4.18. [**1**, Thm. 4.1] *With the hypotheses of Setting 3.4.17, the functor* (3.4.7) *sends rigid object in* $\mathcal{C}$ *to* τ*-rigid objects in* $\mathsf{mod}\,A$*, and induces a bijection*

$$\overline{(-)} : \mathsf{c}\text{-}\mathsf{tilt}\,\mathcal{C} \longrightarrow \mathrm{s}\tau\text{-}\mathsf{tilt}\,A$$

which induces a bijection

$$\overline{(-)} : \mathsf{c}\text{-}\mathsf{tilt}_U\,\mathcal{C} \longrightarrow \mathrm{s}\tau\text{-}\mathsf{tilt}_U\,A.$$

SETTING 3.4.19. From now on we fix a rigid object U in $\mathcal{C}$, *i.e.* $\mathsf{Hom}_{\mathcal{C}}(U, U[1]) = 0$. To simplify the exposition, we assume that U has no non-zero direct summands in $\mathsf{add}\,T[1]$ although our results remain true in this case. We are interested in the subset of $\mathsf{c}\text{-}\mathsf{tilt}\,\mathcal{C}$ given by

$$\mathsf{c}\text{-}\mathsf{tilt}_U\,\mathcal{C} := \{M \in \mathsf{c}\text{-}\mathsf{tilt}\,\mathcal{C} \mid U \in \mathsf{add}\,M\}.$$

Calabi-Yau reduction was introduced in [**62**, Thm. 4.9]. We are interested in the following particular case:

THEOREM 3.4.20. [**62**, Sec. 4] *The category* $\mathcal{U}$ *has the structure of a 2-Calabi-Yau triangulated category and the canonical functor*

$$^{\perp}(U[1]) \longrightarrow \mathcal{U} := \frac{^{\perp}(U[1])}{[U]}$$

induces a bijection

$$\mathsf{red} : \mathsf{c}\text{-}\mathsf{tilt}_U\,\mathcal{C} \longrightarrow \mathsf{c}\text{-}\mathsf{tilt}\,\mathcal{U}.$$

We need to consider the following analog of Bongartz completion for rigid objects in $\mathcal{C}$.

DEFINITION-PROPOSITION 3.4.21. Let $f : U' \to T[1]$ be a minimal right $(\mathsf{add}\,U)$-approximation of $T[1]$ in $\mathcal{C}$ and consider a triangle

$$T \xrightarrow{h} X_U \xrightarrow{g} U' \xrightarrow{f} T[1]. \tag{3.4.9}$$

Then $T_U := X_U \oplus U$ is cluster-tilting in $\mathcal{C}$ and moreover T_U has no non-zero direct summands in $\mathsf{add}\,T[1]$. We call T_U the *Bongartz completion of* U *in* $\mathcal{C}$ *with respect to* T.

PROOF. (i) First we show that T_U is rigid. Our argument is a triangulated version of the proof of [**42**, Prop. 5.1]. We give the proof for the convenience of the reader.

Apply the functor $\mathsf{Hom}_{\mathcal{C}}(U, -)$ to the triangle (3.4.9) to obtain an exact sequence

$$\mathsf{Hom}_{\mathcal{C}}(U, U') \xrightarrow{f\circ -} \mathsf{Hom}_{\mathcal{C}}(U, T[1]) \to \mathsf{Hom}_{\mathcal{C}}(U, X_U[1]) \to \mathsf{Hom}_{\mathcal{C}}(U, U'[1]) = 0.$$

But $f : U' \to T[1]$ is a right $(\mathsf{add}\,U)$-approximation of $T[1]$, thus $f \circ -$ is an epimorphism and we have $\mathsf{Hom}_{\mathcal{C}}(U, X_U[1]) = 0$ and, by the 2-Calabi-Yau property, $\mathsf{Hom}_{\mathcal{C}}(X_U, U[1]) = 0$. It remains to show that $\mathsf{Hom}_{\mathcal{C}}(X_U, X_U[1]) = 0$. For this let $a : X_U \to X_U[1]$ be an arbitrary morphism. Since $\mathsf{Hom}_{\mathcal{C}}(X_U, U'[1]) = 0$ there exists a morphism $h : X_U \to T[1]$ such that the following diagram commutes:

$$\begin{tikzcd}
 & T \arrow[r] & X_U \arrow[r, "g"] \arrow[dl, dashed, "h"'] \arrow[d, "a"] & U' \arrow[r] & T[1] \\
U' \arrow[r] & T[1] \arrow[r, "h[1]"] & X_U[1] \arrow[r] & U'[1] &
\end{tikzcd}$$

Now, since $\mathsf{Hom}_{\mathcal{C}}(T, T[1]) = 0$ there exist a morphism $c : U' \to T[1]$ such that $h = cg$. Then we have

$$h[1](cg) = (h[1] \circ c)g = 0,$$

since $(h[1] \circ c) \in \mathsf{Hom}_{\mathcal{C}}(U', X_U[1]) = 0$. Hence $\mathsf{Hom}_{\mathcal{C}}(X_U, X_U[1]) = 0$ as required. Thus we have shown that T_U is rigid.

(ii) Now we show that T_U is cluster-tilting in $\mathcal{C}$. By the bijection in Theorem 3.4.18, we only need to show that $\overline{T_U}$ is a support τ-tilting A-module. Since T_U is rigid, we have that $\overline{T_U}$ is τ-rigid (see Theorem 3.4.18). Apply the functor (3.4.7) to the triangle (3.4.9) to obtain an exact sequence

$$A \xrightarrow{\bar{g}} \overline{X_U} \to \overline{U'} \to 0.$$

We claim that $\overline{g}$ is a left $(\mathsf{add}\,\overline{T_U})$-approximation of A. In fact, since we have $\mathsf{Hom}_{\mathcal{C}}(U[-1], T_U) = 0$, for every morphism $h : T \to T_U$ in $\mathcal{C}$ we obtain a commutative diagram

$$\begin{tikzcd}
U[-1] \arrow[r] \arrow[dr, dashed, "0"'] & T \arrow[r, "g"] \arrow[d, "h"] & X_U \arrow[r] \arrow[dl, dashed] & U \\
 & T_U & &
\end{tikzcd}$$

Thus g is a left $(\mathsf{add}\,T_U)$-approximation of T and then by the equivalence (3.4.8) we have that $\bar{g}$ is a left $(\mathsf{add}\,\overline{T_U})$-approximation of A. Then by Proposition 3.2.15 we have that $\overline{T_U}$ is a support τ-tilting A-module.

Finally, since $\mathsf{Hom}_A(T, T[1]) = 0$ and U has no non-zero direct summands in $\mathsf{add}\,T[1]$, it follow from the triangle (3.4.9) that T_U has no non-zero direct summands in $\mathsf{add}\,T[1]$. $\square$

The following proposition relates the Bongartz completion T_U of U in $\mathcal{C}$ with respect to T with the Bongartz completion $\overline{T_U}$ of $\overline{U}$ in $\mathsf{mod}\,A$. Recall that $T_{\overline{U}}$ is the unique maximal element in $\mathrm{s}\tau\text{-}\mathsf{tilt}_{\overline{U}}\,A$.

PROPOSITION 3.4.22. *We have* $\overline{T_U} \cong T_{\overline{U}}$.

PROOF. By Proposition 3.3.2, $P({}^{\perp}(\tau\overline{U}))$ is the unique maximal element in $\mathrm{s}\tau\text{-}\mathsf{tilt}_{\overline{U}}\,A$. Hence to show that $\overline{T_U} \cong P({}^{\perp}(\tau\overline{U}))$ we only need to show that if $\overline{M}$ is a support τ-tilting A-module such that $\overline{U} \in \mathsf{add}\,\overline{M}$, then $\overline{M} \in \mathsf{Fac}\,\overline{T_U}$. By definition, this is equivalent to show that there exists an exact sequence of A-modules

$$\mathsf{Hom}_{\mathcal{C}}(T, X) \to \mathsf{Hom}_{\mathcal{C}}(T, M) \to 0$$

with $X \in \mathsf{add}\,T_U$.

Let $f : T' \to M$ be a right $(\mathsf{add}\,T)$-approximation of M. By the definition of X_U, there exist a triangle

$$T' \xrightarrow{g} X \to U'' \to T[1]$$

where $X \in \mathsf{add}\, X_U$ and $U'' \in \mathsf{add}\, U$. Since $\mathsf{Hom}_{\mathcal{C}}(U'[-1], M) = D\,\mathsf{Hom}_{\mathcal{C}}(M, U'[1]) = 0$ by the 2-Calabi-Yau property of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
U''[-1] \arrow[r] \arrow[dr, "0"'] & T' \arrow[r, "g"] \arrow[d, "f"] & X \arrow[r] \arrow[dl, dotted, "h"] & U'' \\
& M & &
\end{tikzcd}$$

It follows that there exist a morphism $h : X \to M$ such that $f = hg$. Now it is easy to see that the sequence

$$\mathsf{Hom}_{\mathcal{C}}(T, X) \xrightarrow{h\cdot ?} \mathsf{Hom}_{\mathcal{C}}(T, M) \to 0$$

is exact. Indeed, let $u : T \to M$. since f is a right $(\mathsf{add}\, T)$-approximation of M, there exist a morphism $v : T \to T'$ such that $u = fv$. It follows that

$$u = fv = (hg)v = h(gv).$$

This shows that u factors through h, which is what we needed to show. $\square$

We are ready to state the main theorem of this section. We keep the notation of the above discussion.

THEOREM 3.4.23. *With the hypotheses of Settings 3.4.17 and 3.4.19, we have the following:*

(i) *The algebras* $\mathsf{End}_{\mathcal{U}}(T_U)$ *and* $C = \mathsf{End}_A(\overline{T_U})/\langle e_{\overline{U}}\rangle$ *are isomorphic, where* $e_{\overline{U}}$ *is the idempotent corresponding to the projective* $\mathsf{End}_A(\overline{T_U})$*-module* $\mathsf{Hom}_A(\overline{T_U}, \overline{U})$.

(ii) *We have a commutative diagram*

$$\begin{tikzcd}
\text{c-}\mathsf{tilt}_U\, \mathcal{C} \arrow[r, "\mathsf{Hom}_{\mathcal{C}}(T,-)"] \arrow[d, "\mathsf{red}"'] & \text{s}\tau\text{-}\mathsf{tilt}_{\overline{U}}\, A \arrow[d, "\mathsf{red}"] \\
\text{c-}\mathsf{tilt}\, \mathcal{U} \arrow[r, "\mathsf{Hom}_{\mathcal{U}}(T_U,-)"'] & \text{s}\tau\text{-}\mathsf{tilt}\, C
\end{tikzcd}$$

in which each arrow is a bijection. The vertical maps are given in Theorem 3.4.20 and Theorem 3.3.15.

We begin with the proof of part (a) of Theorem 3.4.23.

LEMMA 3.4.24. *For each M in ${}^{\perp}(U[1])$ we have an isomorphism of vector spaces*

$$\frac{\mathsf{Hom}_A(\overline{T_U}, \overline{M})}{[\overline{U}](\overline{T_U}, \overline{M})} \cong \mathsf{Hom}_{\mathcal{U}}(T_U, M).$$

PROOF. By the equivalence (3.4.8), we have the following isomorphism

$$\frac{\mathsf{Hom}_A(\overline{T_U}, \overline{M})}{[\overline{U}](\overline{T_U}, \overline{M})} \cong \frac{\frac{\mathsf{Hom}_{\mathcal{C}}(T_U, M)}{[T[1]](T_U, M)]}}{\frac{[U](T_U,M)+[T[1]](T_U,M)}{[S[1]](T_U,M)}}.$$

We claim that $[T[1]](T_U, M) \subseteq [U](T_U, M)$, or equivalently $[T[1]](X_U, M) \subseteq [U](X_U, M)$ since $T_U = X_U \oplus U$. Apply the contravariant functor $\mathsf{Hom}_{\mathcal{C}}(-, T[1])$ to the triangle (3.4.9) to obtain an exact sequence

$$\mathsf{Hom}_{\mathcal{C}}(U', T[1]) \to \mathsf{Hom}_{\mathcal{C}}(X_U, T[1]) \to \mathsf{Hom}_{\mathcal{C}}(T, T[1]) = 0.$$

Hence every morphism $X_U \to T[1]$ factors through U', and we have $[T[1]](X_U, M) \subseteq [U](X_U, M)$. Thus we have the required isomorphisms

$$\frac{\mathsf{Hom}_A(\overline{T_U}, \overline{M})}{[\overline{U}](\overline{T_U}, \overline{M})} \cong \frac{\mathsf{Hom}_{\mathcal{C}}(T_U, M)}{[U](T_U, M)} \cong \mathsf{Hom}_{\mathcal{U}}(T_U, M),$$

so the assertion follows. □

Now part (a) of Theorem 3.4.23 follows by putting $M = T_U$ in Lemma 3.4.24. In the remainder we prove Theorem 3.4.23(b).

For $X \in \operatorname{mod} A$ we denote by $0 \to \mathrm{t}X \to X \to \mathrm{f}X \to 0$ the canonical sequence of X with respect to the torsion pair $(\operatorname{Fac}\overline{U}, \overline{U}^{\perp})$ in $\operatorname{mod} A$.

PROPOSITION 3.4.25. *For each M in ${}^{\perp}(U[1])$ there is an isomorphism of C-modules*

$$\operatorname{Hom}_{\mathcal{U}}(T_U, M) \cong \operatorname{Hom}_A(\overline{T_U}, \mathrm{f}\overline{M}).$$

PROOF. By Lemma 3.4.24 it is enough to show that

$$\frac{\operatorname{Hom}_A(\overline{T_U}, \overline{M})}{[\overline{U}](\overline{T_U}, \overline{M})} \cong \operatorname{Hom}_A(\overline{T_U}, \mathrm{f}\overline{M}).$$

We can proceed exactly as in the proof of Proposition 3.4.15. □

We are ready to give the prove Theorem 3.4.23(b).

PROOF OF THEOREM 3.4.23(B). Let $M \in \text{c-tilt}\,\mathcal{C}$ be such that $U \in \operatorname{add} M$. Since τ-tilting reduction of $\overline{M}$ is given by $F(\mathrm{f}\overline{M}) = \operatorname{Hom}_A(T_{\overline{U}}, \mathrm{f}M)$, see Theorem 3.3.15, we only need to show that

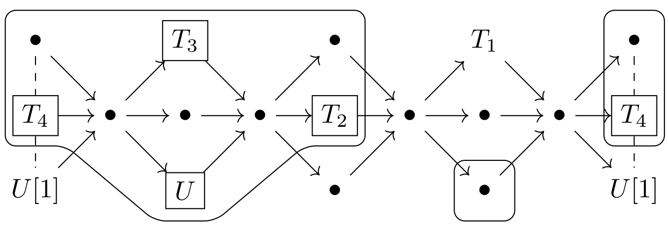
$$\operatorname{Hom}_{\mathcal{U}}(T_U, M) = \operatorname{Hom}_A(T_{\overline{U}}, \mathrm{f}M).$$

But this is precisely the statement of Proposition 3.4.25 since $\overline{T_U} \cong T_{\overline{U}}$ by Proposition 3.4.22. □

We conclude this section with an example illustrating the bijections in Theorem 3.4.23.

EXAMPLE 3.4.26. Let $\mathcal{C}$ be the cluster category of type D_4. Recall that $\mathcal{C}$ is a 2-Calabi-Yau triangulated category, see [**27**]. The Auslander-Reiten quiver of $\mathcal{C}$ is the following, where the dashed edges are to be identified to form a cylinder, see [**82**, 6.4]:

$\mathcal{C}$:



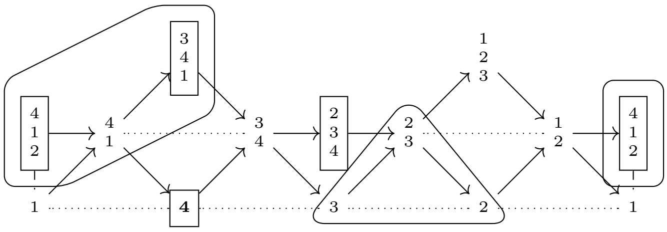
We have chosen a cluster-tilting object $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4$ and a rigid indecomposable object U in $\mathcal{C}$. The Bongartz completion of U with respect to T is given by $T_U = U \oplus T_2 \oplus T_3 \oplus T_4$, and is indicated with squares. Also, the ten indecomposable objects of the subcategory ${}^{\perp}(U[1])$ have been encircled.

On the other hand, let $A = \operatorname{End}_{\mathcal{C}}(T)$ and $\overline{(-)} = \operatorname{Hom}_{\mathcal{C}}(T, -)$. Then A is isomorphic to algebra given by the quiver

$$Q' \quad = \quad \begin{tikzcd} 2 \arrow[d, "x_2"'] & 1 \arrow[l, "x_1"'] \\ 3 \arrow[r, "x_3"'] & 4 \arrow[u, "x_4"'] \end{tikzcd}$$

with relations $x_1x_2x_3 = 0$, $x_2x_3x_4 = 0$, $x_3x_4x_1 = 0$ and $x_4x_1x_2 = 0$. Thus A is isomorphic to the Jacobian algebra of the quiver with potential $(Q', x_1x_2x_3x_4)$. The Auslander-Reiten quiver of $\operatorname{mod} A$ is the following:

$\mathsf{mod}\, A$:

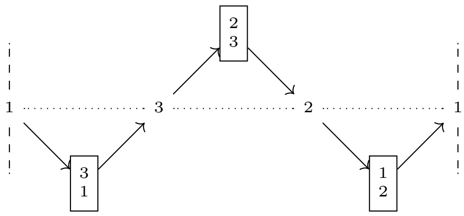


We have indicated the indecomposable direct summands of the Bongartz completion $\overline{T_U}$ of $\overline{U} = S_4$, with rectangles. The six indecomposable objects of the category ${}^{\perp}(\tau\overline{U}) \cap \overline{U}^{\perp}$ are encircled. Finally, let $C = \mathsf{End}_A(\overline{T_U})/\langle e_{\overline{U}}\rangle$. Not that C is isomorphic by the algebra by the quiver

$$Q \quad = \quad \begin{tikzcd} & 1 \arrow[dl, "x_1"'] & \\ 2 \arrow[rr, "x_2"'] & & 3 \arrow[ul, "x_3"'] \end{tikzcd}$$

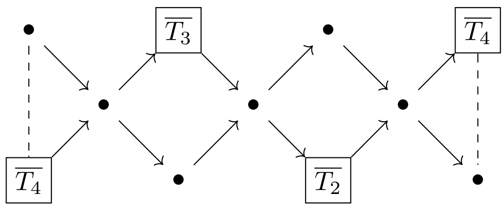
with relations $x_1x_2 = 0$, $x_2x_3 = 0$ and $x_3x_1 = 0$, see Example 3.2.20. Thus C is isomorphic to the Jacobian algebra of the quiver with potential $(Q, x_1x_2x_3)$, see [**31**] for definitions. By Theorem 3.3.8 we have that ${}^{\perp}(\tau\overline{U}) \cap \overline{U}^{\perp}$ is equivalent to $\mathsf{mod}\, C$. The Auslander-Reiten quiver of $\mathsf{mod}\, C$ is the following, where each C-module is represented by its radical filtration:

$\mathsf{mod}\, C$:



On the other hand, let $\mathcal{U} = {}^{\perp}(U[1])/[U]$. The Auslander-Reiten quiver of $\mathcal{U}$ is the following, note that the dashed edges are to be identified to form a Möbius strip:

$\mathcal{U}$:



Observe that $\mathcal{U}$ is equivalent to the cluster category of type A_3. Moreover, by Thorem 3.4.23(a) we have an isomorphism between $\mathsf{End}_{\mathcal{U}}(T_U)$ and C. By Theorem 3.4.23(b) we have a commutative diagram

$$\begin{tikzcd} \mathsf{c\text{-}tilt}_U\, \mathcal{C} \arrow[r, "\mathsf{Hom}_{\mathcal{C}}(T,-)"] \arrow[d, "\mathsf{red}"'] & \mathrm{s}\tau\text{-}\mathsf{tilt}_{\overline{U}}\, A \arrow[d, "\mathsf{red}"] \\ \mathsf{c\text{-}tilt}\, \mathcal{U} \arrow[r, "\mathsf{Hom}_{\mathcal{U}}(T_U,-)"'] & \mathrm{s}\tau\text{-}\mathsf{tilt}\, C \end{tikzcd}$$

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