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Log-Linear Demand Systems with Differentiated Products Are Inconsistent with the Representative Consumer Approach*

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Abstract

We argue that log-linear demands with differentiated products, which are viewed as useful modelling from an empirical standpoint, are generated from the representative consumer's utility only in a restrictive form.

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1 Introduction

Log-linear demands are viewed as useful modelling from an empirical standpoint: in the case of single-product monopoly and homogeneous-product oligopoly, the coefficient for the log of the own price term is interpreted as own price elasticity. However, as products are more or less differentiated in reality, researchers often consider a system of log-linear demand for that also includes separate terms of the logs of other products' prices. The coefficient for such a term is then interpreted as the cross price elasticity. However, log-linear demands with differentiated products are not only intractable as the number of products becomes large,[1] but also they are conceptually flawed even if the number of differentiated products is the smallest (i.e., two). As for the former point, Jaffe and Kominers (2012), a subsequent work of Jaffe and Weyl (2010), show that market demand that is additively separable in own price (log-linear demand is one) cannot be generated from discrete choice modelling, which is ubiquitous in empirical studies of industrial organization.[2] This note argues the latter point: log-linear demands cannot be generated from the representative consumer's utility, either. Thus, under log-linear demands, consumer surplus, defined by the integral of the positive difference between the inverse demand and the price, can have no welfare basis from individual's utility either by the discrete choice approach or the representative consumer approach.[3] More specifically, we point out

[1]In the empirical industrial organization literature, this is known as the J^2 problem. Suppose that consumers face J products. If one starts with (linear or log linear) market demand function for product j as a function of (among others) other rival products' prices as well as j's own price, then the number of parameters to be estimated is J^2. Instead, one can think of consumers gaining utility from a product as a bundle of product characteristics, and then each consumer's probability of demanding for a particular product is aggregated to construct demand function for the product (this is often called the *product characteristics approach*). See, e.g., Nevo (2001), Davis and Garcés (2010), Aguirregabiria and Nevo (2013), and Aguirregabiria (2014) for excellent surveys on the product characteristics approach.

[2]In relation to these two papers, Armstrong and Vickers (2014) provide a necessary and sufficient condition for a multi-product demand system to be consistent with discrete choice.

[3]For instance, in the study of third-degree price discrimination, Vaian (1985) and Schwartz (1990) employ the representative consumer approach. Adachi (2004) argues that, in response to Bertoletti (2004), a representative consumer approach and a discrete choice approach can generate different conclusions on welfare effects of monopolistic third-degree price discrimination. See Vives (1999, Chapter 6) for a general exposition on the representative consumer approach. Anderson, de Palma and Thisse (1992) study the relationships between the two approaches.

that the representative consumer approach can generate log-linear market demands only for the case of complements with the sum of own and cross price elasticities being unity.

2 Main Argument

Consider the system of demand functions for two symmetric firms, A and B, each of which produces a differentiated product:

$$\begin{cases} q^A = a(p^A)^{-\varepsilon}(p^B)^{\sigma} \\ q^B = a(p^B)^{-\varepsilon}(p^A)^{\sigma}, \end{cases}$$

where $a > 0$, $\varepsilon > 1$, and $\varepsilon > \sigma > 0$. This demand system is log-linear in the sense that

$$\begin{cases} \ln q^A = \ln a - \varepsilon \ln p^A + \sigma \ln p^B \\ \ln q^B = \ln a - \varepsilon \ln p^B + \sigma \ln p^A. \end{cases}$$

In particular, both own and cross price elasticities are constant because

$$-\frac{\partial q^A}{\partial p^A}\frac{p^A}{q^A} = \varepsilon$$

and

$$\frac{\partial q^A}{\partial p^B}\frac{p^B}{q^A} = \sigma.$$

Solving the demand system for p^A and p^B yields the following inverse demand functions:

$$\begin{cases} p^A = a^{\frac{1}{\varepsilon-\sigma}}(q^A)^{\frac{-\varepsilon}{\varepsilon^2-\sigma^2}}(q^B)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}} \\ p^B = a^{\frac{1}{\varepsilon-\sigma}}(q^B)^{\frac{-\varepsilon}{\varepsilon^2-\sigma^2}}(q^A)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}}. \end{cases}$$

It is seen below that σ cannot be a free parameter if the demand system is generated from the representative consumer's utility. It must be equal to $1 - \epsilon$. It is also seen below that $\epsilon > 1$ is necessary for a positive value for consumer surplus. However, it implies that the two products are *complements*. Thus, one cannot deal with the case of substitutes. To see these claims, notice that the representative consumer's utility $U(q^A, q^B)$ must satisfy

$$\frac{\partial U}{\partial q^A} = a^{\frac{1}{\varepsilon-\sigma}}(q^A)^{\frac{-\varepsilon}{\varepsilon^2-\sigma^2}}(q^B)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}} \tag{1}$$

and

$$\frac{\partial U}{\partial q^B} = a^{\frac{1}{\varepsilon-\sigma}} (q^B)^{\frac{-\varepsilon}{\varepsilon^2-\sigma^2}} (q^A)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}}. \tag{2}$$

From (1), it is derived that

$$\frac{\partial^2 U}{\partial q^B \partial q^A} = \frac{-\sigma}{\varepsilon^2-\sigma^2} a^{\frac{1}{\varepsilon-\sigma}} (q^A)^{\frac{-\varepsilon}{\varepsilon^2-\sigma^2}} \ (q^B)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}-1}$$

and from (2),

$$\frac{\partial^2 U}{\partial q^A \partial q^B} = \frac{-\sigma}{\varepsilon^2-\sigma^2} a^{\frac{1}{\varepsilon-\sigma}} (q^A)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}-1} (q^B)^{\frac{-\varepsilon}{\varepsilon^2-\sigma^2}}.$$

For the utility function to satisfy the symmetry in the cross partial derivatives, it must be that

$$\begin{aligned} \frac{-\sigma}{\varepsilon^2-\sigma^2} - 1 &= \frac{-\varepsilon}{\varepsilon^2-\sigma^2} \\ &\Leftrightarrow \\ \varepsilon + \sigma &= 1. \end{aligned}$$

Thus, if one wants to base welfare evaluation on the representative consumer's utility, she must confine her attention to the demand system such that

$$\begin{cases} q^A = a(p^A)^{-\varepsilon}(p^B)^{1-\epsilon} \\ q^B = a(p^B)^{-\varepsilon}(p^A)^{1-\epsilon}, \end{cases}$$

which is consistent with the following representative consumer's utility:

$$\begin{aligned} U(q^A, q^B) &= a^{\frac{1}{\varepsilon-\sigma}} \frac{\varepsilon^2-\sigma^2}{-\varepsilon+\varepsilon^2-\sigma^2} \left(q^A q^B\right)^{\frac{-\varepsilon+\varepsilon^2-\sigma^2}{\varepsilon^2-\sigma^2}} \\ &= a^{\frac{1}{2\varepsilon-1}} \frac{2\varepsilon-1}{\varepsilon-1} \left(q^A q^B\right)^{\frac{\varepsilon-1}{2\varepsilon-1}}. \end{aligned}$$

Consumer surplus based on the representative consumer approach is defined by

$$CS(q^A, q^B) = a^{\frac{1}{2\varepsilon-1}} \frac{2\varepsilon-1}{\varepsilon-1} \left(q^A q^B\right)^{\frac{\varepsilon-1}{2\varepsilon-1}} - p^A q^A - p^B q^B.$$

For $U(q^A, q^B)$ to be positive, it must be that $\varepsilon > 1$. Recall that if Firm A raises its price p^A by one percent, then it loses (with everything else held equal) its demand by ε percent. At the same time, the rival firm gains a $(1-\varepsilon)$ percent increase in its demand. Because $1-\varepsilon$ is negative, it means that Firm B loss a $|1-\varepsilon|$ percent of the consumers in response to the price increase by Firm A. Thus, the representative consumer approach is consistent with log-linear demands only for the case of *complements* with the sum of own and cross price elasticities being unity.

On the other hand, linear market demands can be generated from the representative consumer's utility, and thus they have welfare basis from individual's utility, although, as Jaffe and Weyl (2010) and Jaffe and Kominers (2012) show, they are not consistent with the discrete choice approach. To see this, consider the following utility function in a quadratic form,

$$U(q^A, q^B) = \alpha \cdot (q^A + q^B) - \frac{1}{2}\left(\beta[q^A]^2 + 2\gamma q^A q^B + \beta[q^B]^2\right),$$

where $|\gamma| < \beta$ denotes the degree of horizontal product differentiation. The two products are substitutes (complements) if $\gamma > 0$ ($\gamma < 0$). As Belleflamme and Peitz (2010, p.65) explain, $\gamma/\beta \in (-1, 1)$ is interpreted as a (normalized) measure of horizontal product differentiation: the greater γ/β, the greater the degree of substitution is (complementarity can be interpreted as negative substitutability).[4] Maximizing net utility, $U(q^A, q^B) - p^A q^A - p^B q^B$, with respect to q^A and q^B generates the inverse demand function for firm $j \in \{A, B\}$, $p^j(q^j, q^{-j}) = \alpha - \beta q^j - \gamma q^{-j}$. Thus, the market demands are given by

$$\begin{cases} q^A(p^A, p^B) = \dfrac{\alpha}{\beta+\gamma} - \dfrac{\beta}{\beta^2-\gamma^2}p^A + \dfrac{\gamma}{\beta^2-\gamma^2}p^B \\[2ex] q^B(p^A, p^B) = \dfrac{\alpha}{\beta+\gamma} - \dfrac{\beta}{\beta^2-\gamma^2}p^B + \dfrac{\gamma}{\beta^2-\gamma^2}p^A. \end{cases}$$

Finally, notice that if the market is governed by monopoly or homogenous-product oligopoly, the log-linear market demand is given by $q = a(p)^{-\varepsilon}$, where $\varepsilon > 1$ (Aguirre and Cowan (2013) use this market demand to study monopolistic third-degree price discrimination with constant elasticity). The inverse demand is

[4] For example, Adachi and Matsushima (2014) derive market demands from a representative consumer's utility when they provide a welfare analysis of oligopolistic third-degree price discrimination with differentiated products (including both substitutability and complementarity).

$p = a^{\frac{1}{\varepsilon}} q^{-\frac{1}{\varepsilon}}$. The representative consumer's utility is simply given by

$$U(Q; p) = \frac{1}{\varepsilon - 1} Q^{\frac{\varepsilon-1}{\varepsilon}} - pQ$$

where $Q = \sum_{i=1}^{n} q^i$ in the case of homogenous-product oligopoly.

3 Constructing Consumer Surplus with Log-Linear Market Demands

In the case of log-linear demands, a natural definition of consumer surplus for consumers who purchase from Firm A, given q^B, is

$$\begin{aligned} CS^A(q^A, q^B; p^A) &\equiv \int_0^{q^A} \left[a^{\frac{1}{\varepsilon-\sigma}} (\widetilde{q}^A)^{\frac{-\varepsilon}{\varepsilon^2-\sigma^2}} (q^B)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}} - p^A \right] d\widetilde{q}^A \\ &= \frac{(\varepsilon^2 - \sigma^2) a^{\frac{1}{\varepsilon-\sigma}}}{\varepsilon^2 - \varepsilon - \sigma^2} \left(q^A\right)^{\frac{\varepsilon^2-\varepsilon-\sigma^2}{\varepsilon^2-\sigma^2}} \left(q^B\right)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}} - p^A q^A \end{aligned} \tag{3}$$

and similarly, given q^A, consumer surplus for those purchasing from Firm B can be defined by

$$\begin{aligned} CS^B(q^A, q^B; p^B) &\equiv \int_0^{q^B} \left[a^{\frac{1}{\varepsilon-\sigma}} (\widetilde{q}^B)^{\frac{-\varepsilon}{\varepsilon^2-\sigma^2}} (q^A)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}} - p^B \right] d\widetilde{q}^B \\ &= \frac{(\varepsilon^2 - \sigma^2) a^{\frac{1}{\varepsilon-\sigma}}}{\varepsilon^2 - \varepsilon - \sigma^2} \left(q^A\right)^{\frac{-\sigma}{\varepsilon^2-\sigma^2}} \left(q^B\right)^{\frac{\varepsilon^2-\varepsilon-\sigma^2}{\varepsilon^2-\sigma^2}} - p^B q^B. \end{aligned} \tag{4}$$

Here, we implicitly assume that consumers are segmented into three groups: (i) those who purchase product A, (ii) those who purchase product B, and (iii) those who purchase nothing. Any consumers cannot purchase both A and B, as the standard discrete choice approach assumes. A useful point in the representative consumer approach is that one does not have to consider consumer segmentation explicitly as above. As a result, whether two products are substitutes or complements is defined parametrically. However, the representative consumer's utility consistent with the log-linear demands permits the perfect complementarity only: in this case, any consumers purchase either both products or nothing. Thus, if the market demands are log-linear, it is not possible to analyze the case of imperfect complements, as Adachi and Matsushima (2014) do, by considering the representative consumer's utility that generates linear market demands. If one wants to let σ be a free parameter, then the consumer surplus defined above has no welfare foundation based on individual's utility. Although consumer surpluses defined in (3) and (4) seem natural, one should be careful about that.

Finally, if both firms are symmetric, then the symmetric equilibrium realizes ($q^A = q^B \equiv q$ and $p^A = p^B \equiv p$), and the consumer surplus in each group m becomes

$$CS^m(q,p) \equiv CS^m(q,q;p) = \frac{(\varepsilon^2-\sigma^2)a^{\frac{1}{\varepsilon-\sigma}}}{\varepsilon^2-\varepsilon-\sigma^2}(q)^{\frac{\varepsilon-\sigma-1}{\varepsilon-\sigma}} - pq.$$

Then, letting $Q = 2q$ and the aggregate consumer surplus be defined by $CS(Q;p) \equiv CS^A(Q/2,p) + CS^B(Q/2,p)$ yields

$$CS(Q;p) \rightarrow \frac{(2a)^{\frac{1}{\varepsilon}}\varepsilon}{\varepsilon-1}(Q)^{\frac{\varepsilon-1}{\varepsilon}} - pQ$$

as $\sigma \rightarrow 0$. As the total output converges to the case of monopoly and homogenous oligopoly (and the equilibrium price always coincides with the monopolist's optimal price), $\lim_{\sigma\rightarrow 0} CS(Q,p)$ is not equal to $U(Q;p)$ unless $(2a)^{1/\varepsilon}\varepsilon = 1$. Thus, even in the limit of σ close to zero (i.e., two firms behave as monopolist), $CS(Q;p)$ is overvalued if a is sufficiently large that $\ln(2a) + \varepsilon \ln \varepsilon > 0$ (recall $\ln \varepsilon > 0$), and vice versa.

References

Adachi, Takanori. 2004. "Reply to Paolo Bertoletti, 'A Note on Third-Degree Price Discrimination and Output'." *Journal of Industrial Economics*, 52 (3), 457.

—, and Noriaki Matsushima. 2014. "The Welfare Effects of Third-Degree Price Discrimination in a Differentiated Oligopoly." *Economic Inquiry*, 52 (3), 1231-1244.

Aguirre, Iñaki, and Simon G. Cowan. 2013. "Monopoly Price Discrimination with Constant Elasticity Demand." Unpublished manuscript.

Aguirregabiria, Victor. 2014. *Empirical Industrial Organization: Models, Methods, and Applications.* Unpublished manuscript.

—, and Aviv Nevo. 2013. "Recent Developments in Empirical IO: Dynamic Demand and Dynamic Games." In Daron Acemoglu, Manuel Arellano, and Eddie Dekel (eds.), *Advances in Economics and Econometrics: Tenth World Congress*, Volume 3: Econometrics, Cambridge University Press.

Anderson, Simon P., André de Palma, and Jacques-François Thisse. 1992. *Discrete Choice Theory of Product Differentiation.* The MIT Press.

Armstrong, Mark, and John Vickers. 2014. "Which Demand Systems Can Be Generated By Discrete Choice?" Unpublished manuscript.

Belleflamme, Paul, and Martin Peitz. 2010. *Industrial Organization: Markets and Strategies.* Oxford University Press.

Bertoletti, Paolo. 2004. "A Note on Third-Degree Price Discrimination and Output." *Journal of Industrial Economics*, 52 (3), 457.

Davis, Peter, and Eliana Garcés. 2010. *Quantitative Techniques for Competition and Antitrust Analysis.* Princeton University Press.

Jaffe, Sonia, and Scott Dule Kominers. 2012. "Discrete Choice Cannot Generate Demand That Is Additively Separable in Own Price." *Economics Letters*, 116 (1), 129-132.

—, and E. Glen Weyl. 2010. "Linear Demand Systems Are Inconsistent With Discrete Choice." *B.E. Journal of Theoretical Economics* (Advances) 10.

Nevo, A., 2000. "A Practitioner's Guide to Estimation of Random-Coefficients Logit Models of Demand." *Journal of Economics & Management Strategy*, 9 (4), 513-548.

Vives, Xavier. 1999. *Oligopoly Pricing: Old Ideas and New Tools.* The MIT Press.