Previdi's delooping conjecture and the classification theorem for torsors over the sheaf of K-theory spaces

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Abstract

We give a classification theorem for certain geometric objects, called *torsors over the sheaf of K-theory spaces*, showing that their moduli space is equivalent to the K-theory of Tate vector bundles. This allows us to present a very natural and simple construction of a canonical ∞-categorical central extension of the automorphism group of a Tate vector bundle, which we expect to truncate to Drinfeld's categorical Tate central extension. The proof relies on two technical results, one of which is a delooping theorem for non-connective K-theory spectra by Beilinson's exact category of locally compact objects, conjectured by L. Previdi and proved by the author in this thesis. The other one is a theorem of Drinfeld that the first negative K-group vanishes Nisnevich locally. The recently developed theory of ∞-topoi, which has highly elaborated and convenient concepts of higher categorical groups, actions and torsors, allows us to combine these two results to obtain the classification theorem.

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Contents

1 Introduction

We present a classification theorem for certain geometric objects, called *torsors over the sheaf of K-theory spaces*, showing that their moduli space is equivalent to the K-theory of Tate vector bundles. In order to prove this we first show that Beilinson's construction on exact categories gives rise to a delooping of non-connective K-theory spectra, affirming a conjecture posed by L. Previdi. We then combine this result, using the theory of ∞-topoi, with Drinfeld's theorem on the Nisnevich local vanishing of the first negative K-group to obtain the classification theorem.

Background and aim

The work presented here is related to the infinite-dimensional linear algebra of Tate vector spaces. Let k be a field, considered as a topological field with the discrete topology. A topological vector space over k is called a *Tate vector space* if it is isomorphic to the direct sum of a discrete vector space and the dual of a discrete vector space. (We refer as the *dual* of a discrete vector space V to the space of linear functionals from V to k, endowed with the subspace topology in the product topology of k^V.) A typical example of a Tate vector space is the vector space $k((t))$ of formal Laurent series with the t-adic topology, which is the direct sum of the discrete vector space $t^{-1}k[t^{-1}]$ and the dual $k[[t]]$ of the discrete vector space $k[t]$. Tate vector spaces provide a framework for the representation theory of loop groups. Indeed, if G is an algebraic group and V a finite dimensional representation then there is an induced natural representation of the corresponding loop group $G((t))$ on a Tate vector space $V((t))$.

Let $\mathbb{V}$ be a Tate vector space and $GL(\mathbb{V})$ the group of continuous automorphisms (which is considered as a sheaf of groups on an appropriate site; we will be more precise in the next subsection). Then there is a canonical central extension

$$1 \to \mathbb{G}_m \to \widehat{GL(\mathbb{V})} \to GL(\mathbb{V}) \to 1,$$

which is called the *Tate central extension*. Corresponding to a group-stack map $GL(\mathbb{V}) \to B\mathbb{G}_m$, the Tate central extension can be seen as an analogue of the determinant in finite dimensional linear algebra, which is a map of group-sheaves $GL(V) \to \mathbb{G}_m$ for a finite dimensional V. The Tate central extension plays important roles in the geometry of curves, representation theory of loop groups, and in the geometric Langlands program. We refer the reader to an excellent expository note [8], for more detailed background discussions on Tate vector spaces and the canonical central extensions of their automorphism groups.

More generally, Drinfeld [9] formulates a notion of a Tate vector bundle on a scheme as follows. Let R be a commutative ring with the discrete topology. An *elementary Tate R-module* is a topological R-module isomorphic to the direct sum of a discrete R-module and the dual of a discrete R-module. (We refer as the *dual* of a discrete R-module P to the R-module of R-linear maps from P to R, endowed with the subspace topology in the product topology of R^P.) In general, the additive category of elementary Tate R-modules is not idempotent complete, and the idempotent completion of it is defined to be the category of *Tate R-modules*, i.e. a topological R-module is a Tate R-module if it is isomorphic to a direct summand of an elementary Tate R-module. A very important and crucial fact on Tate R-modules due to Drinfeld ([9], Theorem 3.4) is that every Tate R-module is Nisnevich locally elementary. Using this property, he generalizes the definition of the Tate central extension to Tate R-modules.

The classical construction (see, e.g. [12], section 1.3), as well as its direct generalization by Drinfeld ([9], section 3.6), of the Tate central extension is performed by forming a $\mathbb{G}_m$-gerbe for each Tate vector space or Tate R-module, equipped with a canonical action of the automorphism group. This construction of the gerbe, however, is not nicely compatible with direct sums of Tate R-modules. This led Drinfeld [9] and Beilinson et al. [4] to introduce the notion of a *torsor over a sheaf of Picard groupoids*, and they enriched the classical $\mathbb{G}_m$-gerbe to a $\mathrm{Pic}^{\mathbb{Z}}$-torsor, where $\mathrm{Pic}^{\mathbb{Z}}$ denotes the stack of $\mathbb{Z}$-graded line bundles. The $\mathrm{Pic}^{\mathbb{Z}}$-torsor is regarded as classifying the *categorical Tate central extension* of the automorphism group by $\mathrm{Pic}^{\mathbb{Z}}$. See [4], section 2, and [9], secion 5, for details.

Their construction of the $\mathrm{Pic}^{\mathbb{Z}}$-torsor is given by a direct analogy of the classical construction of the plain $\mathbb{G}_m$-gerbe as in [12], requiring to deal with a bunch of 2-categorical data and conditions. However, Drinfeld proposes in section 5.5 of [9] an interesting idea, attributed to Beilinson, which vastly simplifies the construction and gives rise to a clarified perspective. The starting observation is that the stack of graded line bundles $\mathrm{Pic}^{\mathbb{Z}}$ can be interpreted as a truncated K-theory: Namely, assuming the existence of a precise formulation and proof of the equivalence between the 2-categories of stacks of groupoids and of sheaves of homotopy 1-types, $\mathrm{Pic}^{\mathbb{Z}}$ should correspond to the sheaf of 1-truncated K-theory spaces. Their idea, posed as a "somewhat vague picture," roughly says that, under the identification of $\mathrm{Pic}^{\mathbb{Z}}$ with the 1-truncated K-theory sheaf, the $\mathrm{Pic}^{\mathbb{Z}}$-torsor classifying the categorical Tate central extension admits a homotopical interpretation in terms of algebraic K-theory. Drinfeld's description of their idea remains in a sketchy state (which is why it is called a "vague picture"), and he leaves it as a problem to make it precise. See section 5.5 of [9] for details.

The aim of this thesis is to propose and prove a more precisely and more comprehensively formulated version of Beilinson-Drinfeld's picture. We show that, behind the fact that each Tate vector bundles defines a canonical categorical central extension of its automorphism group, there is a delooping theorem of algebraic K-theory, which gives rise in the geometric setting to a description of the moduli space of objects called *torsors over the sheaf of K-theory spaces*. Although Drinfeld [9] described their picture in the language of 2-categories, the more powerful and general framework of ∞-*categories* is available today, with the well-developed and established foundation thanks to the recent works of Lurie [13] et al. In particular, the theory of ∞-*topoi*, as developed in [13], which treats the ∞-categories of sheaves of (not necessarily truncated) spaces, makes it possible to regard the whole sheaf of K-theory spaces as a group object, allowing us to meaningfully speak of torsors over it. We prove that the moduli space of those torsors is equivalent to the K-theory sheaf of Tate vector bundles (Theorem 1.10), as a geometric incarnation of an abstract delooping theorem for non-connective K-theory spectra of exact categories (Theorem 1.2), with the aid of Drinfeld's theorem that the first negative K-group vanishes Nisnevich locally ([9], Theorem 3.7). This directly leads to a canonical construction of a torsor over the sheaf of K-theory spaces for each Tate vector bundle. The torsor thus obtained admits a canonical action by the sheaf of automorphisms of the Tate vector bundle (Theorem 1.13), thereby resulting an object that should be regarded as classifying a canonical ∞-*categorical central extension* of the automorphism group of the Tate vector bundle by the sheaf of K-theory spaces.

We propose this equivalence between the moduli space of torsors over the sheaf of K-theory spaces and the K-theory sheaf of Tate vector bundles as our refined version of Beilinson-Drinfeld's picture. Although we do not know for the moment whether our canonical torsor is compatible with (i.e. truncates to) Drinfeld's $\mathrm{Pic}^{\mathbb{Z}}$-torsor constructed in terms of infinite dimensional Grassmannians, we believe that our approach via a delooping theorem of K-theory, or via its consequent description of the moduli space of torsors, is the most comprehensive and conceptually appropriate way of treating categorical central extensions of the automorphism groups of Tate vector bundles. We leave it as a problem to compare these two approaches to the central extensions of the automorphism groups of Tate vector bundles. It is also an interesting question to ask how the picture presented in this thesis can be generalized to more higher dimensional contexts of *higher Tate vector bundles*. We sketch in the last part of section 5 an idea towards such possible generalizations.

Summary of the results

Let us give here a more detailed and precise summary of our results.

Write Π for the filtered category of pairs (i,j) of integers with $i \leq j$, where there is a unique morphism $(i,j) \to (i',j')$ if $i \leq i'$ and $j \leq j'$. For an exact category $\mathcal{A}$ in the sense of Definition 2.2-1. below, let $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ be the full subcategory of $\operatorname{Ind}\operatorname{Pro}\mathcal{A}$ consisting of ind-pro-objects $X = (X_{i,j})_{(i,j)\in\Pi}$, indexed by Π, satisfying that for every $i \leq j \leq k$ the sequence

$$0 \to X_{i,j} \to X_{i,k} \to X_{j,k} \to 0$$

is a short exact sequence in $\mathcal{A}$. If the exact category $\mathcal{A}$ is an extension-closed, full additive subcategory of an abelian category $\mathcal{F}$, then $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ is an extension-closed, full additive subcategory of the abelian category $\operatorname{Ind}\operatorname{Pro}\mathcal{F}$, so that $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ is endowed with a structure of an exact category. We give a more detailed discussion on the exact category $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ in section 3 below, following [3], A.3, and [17].

In his thesis [16], L. Previdi investigated categorical properties of the exact category $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$, especially its behaviour with respect to K-theoretical constructions under a certain assumption on $\mathcal{A}$. Call the exact category $\mathcal{A}$ to be *partially abelian* if for every pair of admissible monomorphism with common target $a \hookrightarrow b \hookleftarrow b$, there is a pullback

$$\begin{array}{ccc} d & \longrightarrow & c \\ \downarrow & & \downarrow \\ a & \longrightarrow & b, \end{array}$$

and if for every pair of admissible epimorphisms with common source $a \twoheadleftarrow d \twoheadrightarrow c$, there is a pushout

$$\begin{array}{ccc} d & \longrightarrow & c \\ \downarrow & & \downarrow \\ a & \longrightarrow & b. \end{array}$$

Let $S(\mathcal{A})$ be the Waldhausen space of $\mathcal{A}$ (see Definition 2.12 below), which is defined as the geometric realization of the simplicial category $iS_\bullet(\mathcal{A})$ given by Waldhausen's $S_\bullet$-construction [25]. The homotopy groups of its loop space are the algebraic K-theory groups of the exact category $\mathcal{A}$. (Definition 2.13.) The following is the concluding conjecture of Previdi's thesis [16].

Conjecture 1.1 (Previdi [16], 5.1.7) *If $\mathcal{A}$ is partially abelian, then there is a weak equivalence between $S(\mathcal{A})$ and $\Omega S(\underset{\longleftrightarrow}{\lim}\,\mathcal{A})$.*

Our first main result is the proof of Previdi's conjecture, in a more powerful and convenient form where the Waldhausen spaces are replaced by the non-connective K-theory spectra. We write $\mathbb{K}$ for Schlichting's non-connective K-theory spectrum of an exact category, introduced in [22], whose positive homotopy groups are the positive K-groups of the exact category, and whose 0-th homotopy group is the 0-th K-group of the idempotent completion of the exact category, and whose negative homotopy groups recover the classical negative K-groups when the exact category is the category of finitely generated projective modules over a ring or the category of vector bundles on a quasi-compact, quasi-separated scheme with an ample family of line bundles. See section 2.4 below for details. The following is our refined version of Previdi's conjecture, which we prove in section 4 below. Note that we impose no assumption on the exact category $\mathcal{A}$.

Theorem 1.2 *There is a weak equivalence of spectra between $\mathbb{K}(\mathcal{A})$ and $\Omega\mathbb{K}((\underset{\longleftrightarrow}{\lim}\mathcal{A})^\natural)$, where $(-)^\natural$ denotes the idempotent completion.*

Remark 1.3

1. *We also prove the original statement of Conjecture 1.1 in section 4.3, under the weaker assumption that $\mathcal{A}$ is idempotent complete.*
2. *In the case where $\mathcal{A}$ is the category of finitely generated projective R-modules, Drinfeld [9] observed a fact which is essentially the π_{-1}-part of this equivalence. That is, he observed an isomorphism between the first negative K-group of R and the 0-th K-group of the exact category of Tate R-modules in his sense. This is Theorem 3.6-(iii) of [9], and combines with Theorem 3.4 of loc. cit. to show that every element of the first negative K-group is Nisnevich locally trivial ([9], Theorem 3.7).*
3. *Recent work of Bräunling, Grochenig and Wolfson [7] provides an interpretation of this theorem as an algebraic analogue of the Atiyah-Janich theorem in topological K-theory.*

Let R be a commutative ring, and denote by $\mathcal{P}(R)$ the exact category of finitely generated projective R-modules. Then the idempotent-completed exact category $(\underset{\longleftrightarrow}{\lim}\mathcal{P}(R))^\natural$ is very close to the category $\mathrm{Tate}_R^{\mathrm{Dr}}$ of Tate R-modules in Drinfeld's sense (which is denoted by $\mathcal{T}_R$ in [9], 3.3.2). Indeed, if $(M_{i,j})_{i\leq j}$ is an object of $\underset{\longleftrightarrow}{\lim}\mathcal{P}(R)$, the R-module $\varinjlim_j \varprojlim_i M_{i,j}$ endowed with the topology induced from the discrete ones on $M_{i,j}$ is an elementary Tate R-module . Recent work by Bräunling, Grochenig, and Wolfson [6] shows this induces a fully faithful functor $(\underset{\longleftrightarrow}{\lim}\mathcal{P}(R))^\natural \hookrightarrow \mathrm{Tate}_R^{\mathrm{Dr}}$, which is an equivalence onto the full subcategory of Tate R-modules of countable type (that is, direct summands of elementary Tate R-modules $P \oplus Q^*$ where P and Q are countably generated discrete, projective R-modules). See [6], Theorem 5.22.

Definition 1.4 *We call $(\underset{\longleftrightarrow}{\lim}\mathcal{P}(R))^\natural$ the category of* Tate vector bundles *over the affine scheme* Spec R. *If R' is an R-algebra we write $(-)\otimes_R R'$ for the functor $(\underset{\longleftrightarrow}{\lim}\mathcal{P}(R))^\natural \to (\underset{\longleftrightarrow}{\lim}\mathcal{P}(R'))^\natural$ induced by idempotent completion from the functor $\underset{\longleftrightarrow}{\lim}\mathcal{P}(R) \to \underset{\longleftrightarrow}{\lim}\mathcal{P}(R')$ given by $(M_{i,j})_{i\leq j} \mapsto (M_{i,j}\otimes_R R')_{i\leq j}$.*

We write $\operatorname{Spec} R_{\mathrm{Nis}}$ for the site whose underlying category is the opposite category of étale R-algebras and R-homomorphisms, and whose notion of a covering is given as follows. A collection of étale morphisms $\{\operatorname{Spec} R'_\alpha \to \operatorname{Spec} R'\}_{\alpha\in A}$ over $\operatorname{Spec} R$ is a covering in $\operatorname{Spec} R_{\mathrm{Nis}}$ if it is the opposite of a family of étale R-homomorphisms $\{\phi_\alpha : R' \to R'_\alpha\}_{\alpha\in A}$ for which there exists a finite sequence of elements $a_1, \ldots, a_n \in R'$ such that $(a_1, \ldots, a_n) = R'$ and for every $1 \leq i \leq n$ there exists an $\alpha \in A$ and an R-homomorphism $\psi : R'_\alpha \to R'[\frac{1}{a_i}]/(a_1, \ldots, a_{i-1})$ whose composition with $\phi_\alpha : R' \to R'_\alpha$ equals the canonical map $R' \to R'[\frac{1}{a_i}]/(a_1, \ldots, a_{i-1})$. (See [14], section 1, for details.)

Definition 1.5 *We refer to* Spec R_{Nis} *as the* small Nisnevich site *of the affine scheme* Spec R.

Let Set_Δ denote the category of simplicial sets, which is a combinatorial, simplicial model category with the Kan model structure. Recall that there is a Quillen equivalence

$$|-| : \mathrm{Set}_\Delta \rightleftarrows \mathcal{CG} : \mathrm{Sing},$$

where $\mathcal{CG}$ is the category of compactly generated weakly Hausdorff spaces with the Serre model structure, and $|-|$ and Sing the geometric realization and singular complex functors, respectively. We write $\mathrm{Set}_\Delta^{\mathrm{Spec}\, R_{\mathrm{Nis}}^{\mathrm{op}}}$ for the combinatorial, simplicial model category of simplicial presheaves on the underlying category of Spec R_{Nis} with the injective model structure, and $(\mathrm{Set}_\Delta^{\mathrm{Spec}\, R_{\mathrm{Nis}}^{\mathrm{op}}})^\circ$ for the full subcategory of fibrant-cofibrant objects. By Proposition 4.2.4.4 of [13], there is an equivalence of ∞-categories

$$\theta : N(\mathrm{Set}_\Delta^{\mathrm{Spec}\, R_{\mathrm{Nis}}^{\mathrm{op}}})^\circ \xrightarrow{\sim} \mathrm{Fun}(N\,\mathrm{Spec}\, R_{\mathrm{Nis}}^{\mathrm{op}}, (\mathrm{Spaces})) = \mathrm{Preshv}_{(\mathrm{Spaces})}(N\,\mathrm{Spec}\, R_{\mathrm{Nis}}),$$

where N denotes the simplicial nerve and Fun the ∞-category of functors (see section A.1) and (Spaces) is the ∞-category of spaces, which is by definition the simplicial nerve of the simplicial category of Kan complexes (section A.2). (We note that, in our notation, the ∞-category of presheaves of spaces on an ∞-category $\mathcal{C}$ is denoted by $\mathrm{Preshv}_{(\mathrm{Spaces})}(\mathcal{C})$.) Let $\mathrm{Set}_{\Delta,\mathrm{loc}}^{\mathrm{Spec}\, R_{\mathrm{Nis}}^{\mathrm{op}}}$ denote the combinatorial, simplicial model category of simplicial presheaves on the site Spec R_{Nis} with respect to Jardine's local model structure [11], and $(\mathrm{Set}_{\Delta,\mathrm{loc}}^{\mathrm{Spec}\, R_{\mathrm{Nis}}^{\mathrm{op}}})^\circ$ the full subcategory of fibrant-cofibrant objects. Then Proposition 6.5.2.14 of [13], which is recalled as Proposition A.9 in the Appendix, section A.5 below, shows that the above equivalence θ restricts to the equivalence

$$\theta : N(\mathrm{Set}_{\Delta,\mathrm{loc}}^{\mathrm{Spec}\, R_{\mathrm{Nis}}^{\mathrm{op}}})^\circ \xrightarrow{\sim} \mathrm{Shv}_{(\mathrm{Spaces})}(N\,\mathrm{Spec}\, R_{\mathrm{Nis}})^\wedge \subset \mathrm{Shv}_{(\mathrm{Spaces})}(N\,\mathrm{Spec}\, R_{\mathrm{Nis}}),$$

where $\mathrm{Shv}_{(\mathrm{Spaces})}(N\,\mathrm{Spec}\, R_{\mathrm{Nis}}) \subset \mathrm{Preshv}_{(\mathrm{Spaces})}(N\,\mathrm{Spec}\, R_{\mathrm{Nis}})$ is the ∞-topos of sheaves of spaces on $N\,\mathrm{Spec}\, R_{\mathrm{Nis}}$ (see Definition A.7 and Example A.8 below), and $(-)^\wedge$ denotes its hypercompletion ([13], 6.5.2).

Suppose R is noetherian and of finite Krull dimension. Then, by Thomason's Nisnevich descent theorem of non-connective K-theory ([23], 10.8, which is recalled in section 2 below as Theorem 2.25), the simplicial presheaf on Spec R_{Nis} given by K-theory spaces

$$R' \mapsto \mathrm{Sing}\,\Omega^\infty \mathbb{K}(R')$$

is a fibrant object of $\mathrm{Set}_{\Delta,\mathrm{loc}}^{\mathrm{Spec}\, R_{\mathrm{Nis}}^{\mathrm{op}}}$, so that, by the above equivalence θ, it defines an object of the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(N\,\mathrm{Spec}\, R_{\mathrm{Nis}})$.

Definition 1.6 *We denote this object by*

$$\mathcal{K} = \theta(\mathrm{Sing}\,\Omega^\infty \mathbb{K}(-)) \in \mathrm{ob}\,\mathrm{Shv}_{(\mathrm{Spaces})}(N\,\mathrm{Spec}\, R_{\mathrm{Nis}}).$$

Note that a presheaf of spectra satisfies Nisnevich descent if and only if it sends elementary Nisnevich squares to homotopy pullback-pushout squares. Since the suspension functor Σ preserves homotopy pullback-pushout squares of spectra, we see that the Nisnevich descent of the non-connective K-theory $\mathbb{K}(-)$ implies the Nisnevich descent of $\Sigma\mathbb{K}(-)$, which is weakly equivalent by Theorem 1.2 to the presheaf $\mathbb{K}((\varprojlim\!\!\!\!\longleftrightarrow \mathcal{P}(-))^{\natural})$. Hence the simplicial presheaf on $\operatorname{Spec} R_{\mathrm{Nis}}$ given by

$$R' \mapsto \operatorname{Sing}\Omega^{\infty}\mathbb{K}((\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(R'))^{\natural})$$

is also fibrant in $\mathrm{Set}_{\Delta,\mathrm{loc}}^{\operatorname{Spec} R_{\mathrm{Nis}}^{\mathrm{op}}}$, and thus defines, via the equivalence θ, an object of the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(N\operatorname{Spec} R_{\mathrm{Nis}})$.

Definition 1.7 *We denote this object by*

$$\mathcal{K}_{\mathrm{Tate}} = \theta(\operatorname{Sing}\Omega^{\infty}\mathbb{K}((\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(-))^{\natural}) \in \operatorname{ob}\mathrm{Shv}_{(\mathrm{Spaces})}(N\operatorname{Spec} R_{\mathrm{Nis}}).$$

We make essential use of the notions of group objects, their actions, and torsors, in an ∞-topos. These notions we recall in section A.5, as Definitions A.12, A.14, and A.15, respectively, following [13] and [15].

Proposition 1.8 *The object $\mathcal{K}$ is a group object in the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(N\operatorname{Spec} R_{\mathrm{Nis}})$.*

Definition 1.9 *We refer as a* torsor over the sheaf of K-theory spaces *to a $\mathcal{K}$-torsor over the final object* $\operatorname{Spec} R$*, where $\mathcal{K}$ is regarded by Proposition 1.8 as a group object in the ∞-topos* $\mathrm{Shv}_{(\mathrm{Spaces})}(N\operatorname{Spec} R_{\mathrm{Nis}})$.

In general, for every group object G of an ∞-topos $\mathfrak{X}$ there is an object BG that classifies G-torsors, in the sense that for each object X of $\mathfrak{X}$ there is an equivalence between the ∞-groupoid of G-torsors over X and the mapping space from X to BG; the object BG is just given by the connected delooping of the group object G. (Theorem 3.19 of [15], recalled in section A.5 below as Theorem A.16.) We call the object BG the *classifying space object* of the group object G, or the *moduli space* of G-torsors.

The following is the geometric incarnation of Theorem 1.2, which describes the moduli space of torsors over the sheaf of K-theory spaces. The key technical ingredient of the proof is the fact that, in an ∞-topos, the classifying space object of a group object is just given by its connected delooping (Theorems A.16, A.13). We also remark that Drinfeld's theorem on the Nisnevich local vanishing of the first negative K-group ([9], Theorem 3.7), which is used in the form of Lemma 5.1 in section 5 below, plays a crucial role in applying this fact to our setting.

Theorem 1.10 *The moduli space of torsors over the sheaf of K-theory spaces is given by the K-theory sheaf of Tate vector bundles. I.e., in the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(N\operatorname{Spec} R_{\mathrm{Nis}})$ there is an equivalence between $B\mathcal{K}$ and $\mathcal{K}_{\mathrm{Tate}}$.*

Corollary 1.11 *A Tate vector bundle $M \in \operatorname{ob}(\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(R))^{\natural}$ defines a torsor $\mathfrak{D}_M$ over the sheaf of K-theory spaces.*

Let $\operatorname{Aut} M$ denote the sheaf of groups on $\operatorname{Spec} R_{\mathrm{Nis}}$ given by

$$R' \mapsto \operatorname{Aut}_{(\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(R'))^{\natural}} M \otimes_R R'.$$

Remark 1.12 *To see that* $\operatorname{Aut} M$ *is indeed a sheaf, we have to invoke Theorem 3.3 of Drinfeld [9], which says that his notion of Tate R-module is local for the fpqc topology, meaning that every faithfully flat morphism* $R' \to R$ *induces an equivalence of the category of Tate R-modules to the category of Tate* R'*-modules equipped with descent data. This implies that the presheaf* $\operatorname{Aut}_{\mathrm{Tate}^{\mathrm{Dr}}_{(-)}}(-) \otimes_R M$ *is a sheaf on the site of affine R-schemes with the fpqc topology, and in particular on the small Nisnevich site of* $\operatorname{Spec} R$*. Using the fully faithful embedding* $(\varprojlim\!\!\!\!\longleftrightarrow \mathcal{P}(R))^{\natural} \hookrightarrow \mathrm{Tate}^{\mathrm{Dr}}_R$ *(Theorem 5.22 of [6]), this shows that* $\operatorname{Aut} M$ *is a sheaf on* $\operatorname{Spec} R_{\mathrm{Nis}}$.

The sheaf $\operatorname{Aut} M$ is a group object in the ordinary topos $\mathrm{Shv}_{(\mathrm{Sets})}(\operatorname{Spec} R_{\mathrm{Nis}})$, which is regarded as the full subcategory of discrete objects of the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(N \operatorname{Spec} R_{\mathrm{Nis}})$.

Theorem 1.13 *There is a canonical action of the group object* $\operatorname{Aut} M$ *on the torsor* $\mathfrak{D}_M$ *over the sheaf of K-theory spaces.*

Comparison with Drinfeld's $\mathrm{Pic}^{\mathbb{Z}}$-torsor

It is not clear for the moment what is the precise relation between our torsor $\mathfrak{D}_M$ over the sheaf of K-theory spaces described above and Drinfeld's $\mathrm{Pic}^{\mathbb{Z}}$-torsor Det_M introduced in section 5 of [9], which classifies the categorical Tate central extension.

For the ∞-topos $\mathfrak{X} = \mathrm{Shv}_{(\mathrm{Spaces})}(N \operatorname{Spec} R_{\mathrm{Nis}})$ and for an integer $k \geq -2$, by Proposition 5.5.6.18 of [13], the inclusion of the subcategory $\tau_{\leq k}\mathfrak{X}$ of k-truncated objects in the sense of Definition 5.5.6.1 of [13] into the whole $\mathfrak{X}$ admits a left adjoint functor $\tau_{\leq k}$. Composing the map $[M] : \operatorname{Spec} R \to B\mathcal{K}$ classifying our torsor $\mathfrak{D}_M$ with the truncation map $B\mathcal{K} \to B\tau_{\leq 1}\mathcal{K}$, we get a map $\operatorname{Spec} R \to B\tau_{\leq 1}\mathcal{K}$ in the 2-topos (in the sense of [13], 6.4) $\tau_{\leq 1}\mathfrak{X}$, which classifies a $\tau_{\leq 1}\mathcal{K}$-torsor, which we denote by $\mathfrak{D}_{M,\leq 1}$. The 2-topos $\tau_{\leq 1}\mathfrak{X}$ should be equivalent, in some appropriate sense, to the 2-category of stacks of 1-groupoids on the small Nisnevich site $\operatorname{Spec} R_{\mathrm{Nis}}$, and under this equivalence the sheaf $\tau_{\leq 1}\mathcal{K}$ should correspond to the stack $\mathrm{Pic}^{\mathbb{Z}}$ of graded line bundles.

What is the relation between our $\tau_{\leq 1}\mathcal{K}$-torsor $\mathfrak{D}_{M,\leq 1}$ and Drinfeld's $\mathrm{Pic}^{\mathbb{Z}}$-torsor Det_M? The simplest guess is the following: Under this equivalence, $\mathfrak{D}_{M,\leq 1}$ coincides with Det_M. In fact, this is a part of the assertions of Beilinson-Drinfeld's "vague picture" described in section 5.5 of [9]. We leave the following problem.

Problem 1.14 *Establish a precise comparison statement between our* $\tau_{\leq 1}\mathcal{K}$*-torsor* $\mathcal{D}_{M,\leq 1}$ *and Drinfeld's* $\mathrm{Pic}_{\mathbb{Z}}$*-torsor* Det_M.

Organization of the thesis

The content of this thesis can be largely devided into two parts. Sections 2 through 4 are devoted for the proof of Previdi's delooping conjecture, Theorem 1.2, and section 5 and the Appendix for its geometric incarnation as a description of the moduli space, Theorem 1.10, together with the construction of the canonical action, Theorem 1.13. The main sections for the former and latter parts are sections 4 and 5, respectively, where the proofs of of Theorems 1.2, 1.10, and 1.13 are given. The other sections and the Appendix, based on the existing literature, serve for preliminary purposes. Section 2 collects the necessary materials on the connective and non-connective K-theory, with particular emphasis on Schlichting's machinery [21], [22], which we employ to prove Theorem 1.2. Section 3 recalls, following [3] and [17] the definition and description of Beilinson's category

$\underset{\longleftrightarrow}{\lim} \mathcal{A}$. We make an essential use of the theory of ∞-topoi in section 5, and the reader who is unfamiliar with it is referred to the Appendix where we collect the necessary materials on the theory of ∞-categories and ∞-topoi, with main references being [13] and [15].

Conventions of the terminology

There is an important difference in the terminology between the former part (sections 2–4) and the latter part (section 5 and the Appendix): In the former part we work in the framework of ordinary 1-categories, whereas in the latter part we use the higher category theory of (∞,1)-categories, i.e. of categories equipped with notions of *k-morphisms* for all $k \in \mathbb{Z}_{\geq 0}$, which are invertible if $k > 1$. Also, we adopt the formulation of $(\infty, 1)$-categories in terms of simplicial sets following Lurie [13] et al., and, accordingly, refer to an $(\infty, 1)$-category simply as an ∞-*category*. For instance, in the latter part, basic categorical terms such as *object*, *morphism*, and *functor*, should be understood in the sense as recalled in section A.1. The terms *limit* and *colimit* should be understood in the sense of Definition A.4, which correspond to the notions of homotopy limit and homotopy colimit in simplicial categories ([13], Theorem 4.2.4.1).

Other remarks on the conventions of the terminology adopted in this thesis:

1. By a *simplicial category* we mean a category enriched in simplicial sets.
2. The term *space* refers to an object of the ∞-category (Spaces) defined in section A.2, the simplicial nerve of the simplicial category of Kan complexes. Note that (Spaces) is not a 1-category.
3. The term *spectrum* is used in the sense of Bousfield-Friedlander [5], Definition 2.1. I.e. a spectrum is a sequence of pointed compactly generated weakly Hausdorff topological spaces $X = (X_0, X_1, X_2, \ldots)$ equipped with continuous maps $X_i \to \Omega X_{i+1}$ for $i \geq 0$. For the spectrum $X = (X_0, X_1, X_2, \ldots)$ we write $\Omega^\infty X$ for the topological space $\varinjlim_i \Omega^i X_i$, where the colimit is taken in the category of compactly generated weakly Hausdorff topological spaces. We write (Spectra) for the category of spectra, which has a simplicial model structure as given in Theorem 2.3 of [5].

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2 Connective and non-connective algebraic K-theory spectra

This section recalls necessary preliminary materials on algebraic K-theory. In subsections 2.1 and 2.2 we recall, with main references being [18], [21], [17], and [16], basic definitions and constructions on exact categories, and give motivational discussions for the definition of connective algebraic K-theory in terms of dimension and determinant theories. In subsections 2.3 and 2.4 we introduce the connective K-theory spectrum following [25] and the non-connective K-theory spectrum following [22], respectively.

2.1 Dimension theories on an exact category and the group K_0

Let k be a field. A vector space V over k has a finite *dimension* $\dim V = n$ if there exists a basis consisting of n vectors, and isomorphism classes of finite dimensional vector spaces are classified by dimensions. Recall the following theorem from undergraduate linear algebra.

Theorem 2.1 *Let*

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence of finite dimensional k-vector spaces. Then we have $\dim V = \dim V' + \dim V''$.

An assignment $\chi : \operatorname{ob} \operatorname{Vect}_0 k \to A$ that assigns to a finite dimensional k-vector space V an element $\chi(V)$ of an abelian group A satisfying $\chi(V) = \chi(V') + \chi(V'')$ for every short exact sequence

$$0 \to V' \to V \to V'' \to 0,$$

is called a *dimension theory*. The theorem above says that $\dim : \operatorname{ob} \operatorname{Vect}_0 k \to \mathbb{Z}$ is a dimension theory. Note that such a χ takes the 0 vector space to 0 and isomorphic vector spaces to an identical element of A. The fact that the isomorphism class of a finite dimensional vector space is determined by its dimension then implies that there is a unique map of groups $\phi : \mathbb{Z} \to A$ such that $\chi(V) = \phi(\dim V)$ for every $V \in \operatorname{ob} \operatorname{Vect}_0 k$. In this sense, the dimension theory $\dim : \operatorname{ob} \operatorname{Vect}_0 k \to \mathbb{Z}$ is universal among dimension theories, and we express this by saying that the 0-th K-group $K_0(k)$ of the field k is the integers $\mathbb{Z}$.

One can more generally consider the category $\operatorname{Vect}_0(X)$ of finite dimensional vector bundles over a scheme X and the assignment $\operatorname{rank} : \operatorname{ob} \operatorname{Vect}_0(X) \to \mathrm{H}^0(X, \mathbb{Z})$, which satisfies $\operatorname{rank} E = \operatorname{rank} E' + \operatorname{rank} E''$ for a short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of vector bundles. This assignment rank, however, needs not be universal among dimension theories on the scheme X. Nevertheless, a universal dimension theory exists, whose target abelian group is formally constructed by considering the free abelian group generated by the set of isomorphism classes $[E]$ of finite dimensional vector bundles E and quotienting it out by the relation $[E] = [E'] + [E'']$ for short exact sequences

$$0 \to E' \to E \to E'' \to 0.$$

The abelian group thus constructed is called the 0-th K-group of X and the assignment $E \mapsto [E]$ is by construction universal among dimension theories on X.

There is a general notion of a category equipped with an appropriate class of short exact sequences, to which the category $\operatorname{Vect}_0(X)$ of vector bundles provides a typical example, and the notion of a universal dimension theory and the group K_0 can also be defined for them.

Definition 2.2 *1. An* exact category $\mathcal{A}$ *is a full additive subcategory of an abelian category* $\mathcal{F}$*, which is closed under extensions, meaning that if*

$$0 \to a' \to a \to a'' \to 0$$

is a short exact sequence in $\mathcal{F}$ *and if the left and right terms* a' *and* a'' *lie in the subcategory* $\mathcal{A}$*, then the middle term* a *also lies in* $\mathcal{A}$*. A short sequence in* $\mathcal{A}$ *is called an* (admissible) short exact sequence *if it is a short exact sequence in* $\mathcal{F}$*. Morphisms in* $\mathcal{A}$ *which appear as the second (resp. third) maps of admissible short exact sequences are called* admissible monomorphisms*, and depicted* $\hookrightarrow$ *(resp.* admissible epimorphisms*, depicted* $\twoheadrightarrow$*).*

2. *A* dimension theory *on the exact category* $\mathcal{A}$ *taking values in an abelian group* A *is an assignment* $\chi : \operatorname{ob} \mathcal{A} \to A$ *that satisfies* $\chi(a) = \chi(a') + \chi(a'')$ *for every short exact sequence of* $\mathcal{A}$

$$0 \to a' \to a \to a'' \to 0.$$

3. *The dimension theory* $\chi : \operatorname{ob} \mathcal{A} \to A$ *is* universal *if for every other dimension theory* $\psi : \operatorname{ob} \mathcal{A} \to B$ *there is a unique map of groups* $\phi : A \to B$ *such that* $\psi(a) = \phi(\chi(a))$ *for every* $a \in \operatorname{ob} \mathcal{A}$*. The target abelian group* A *equipped with such a map* χ *is unique up to unique isomorphism, and is called the* 0-th K-group *and denoted by* $K_0(\mathcal{A})$*.*

We remark that the universal dimension theory and the group $K_0(\mathcal{A})$ admits a similar formal construction whenever the set of isomorphism classes of $\mathcal{A}$ is small. Giving a dimension theory on $\mathcal{A}$ valuing in B is equivalent to giving a map of abelian groups from $K_0(\mathcal{A})$ to B.

Before going on to the definition of the K_1 group, let us fix the terminology and recall basic properties of exact categories here.

Definition 2.3 *1. An* exact functor *from an exact category* $\mathcal{A}$ *to another exact category* $\mathcal{B}$ *is an additive functor* $\mathcal{A} \to \mathcal{B}$ *that preserves admissible short exact sequences.*

2. *An exact functor* $\mathcal{A} \to \mathcal{B}$ *is* cofinal *if every object in* $\mathcal{B}$ *is isomorphic to a direct summand of the image of an object of* $\mathcal{A}$*.*

3. *A* fully exact embedding *of exact categories is a fully faithful exact functor* $\mathcal{A} \hookrightarrow \mathcal{B}$ *whose essential image in* $\mathcal{B}$ *is closed under extensions and a short sequence in* $\mathcal{A}$ *is an admissible short exact sequence if and only if its image in* $\mathcal{B}$ *is an admissible short exact sequence.*

4. *An exact category* $\mathcal{A}$ *is* idempotent complete *if every idempotent (that is, a map* p *in* $\mathcal{A}$ *such that* $p \circ p = p$*) has a kernel.*

Proposition 2.4 (See [23], Theorem A.9.1-(a)) *For every exact category* $\mathcal{A}$ *there is an idempotent complete exact category* $\mathcal{A}^\natural$ *which contains* $\mathcal{A}$ *as a fully exact cofinal subcategory. Such a* $\mathcal{A}^\natural$ *is essentially unique (that is, unique up to equivalence of exact categories respecting the inclusions from* $\mathcal{A}$*) and called the idempotent completion of* $\mathcal{A}$*.*

Proposition 2.5 (Quillen [18]) *Every exact category* $\mathcal{A}$ *satisfies the following properties:*

1. *If*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & a' & \longrightarrow & a & \longrightarrow & a'' & \longrightarrow & 0 \\ & & \sim\downarrow & & \sim\downarrow & & \sim\downarrow & & \\ 0 & \longrightarrow & b' & \longrightarrow & b & \longrightarrow & b'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram in $\mathcal{A}$ where the upper horizontal sequence is admissible exact and where the vertical maps are isomorphisms, then the lower horizontal sequence is also an admissible exact sequence.

2. *The composition of two admissible monomorphisms is an admissible monomorphism, and the composition of two admissible epimorphisms is an admissible epimorphism.*

3. *If $a \hookrightarrow b$ is an admissible monomorphism and $a \to c$ any map, there is a pushout*

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

and the map $c \hookrightarrow d$ is an admissible monomorphism.

4. *If $b \twoheadrightarrow d$ is an admissible epimorphism and $c \to d$ any map, there is a pullback*

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

and the map $a \twoheadrightarrow c$ is an admissible epimorphism.

5. *For every pair of objects a and b in $\mathcal{A}$, the sequence*

$$0 \to a \overset{{}^t(1,0)}{\to} a \oplus b \overset{(0,1)}{\to} b \to 0$$

is an admissible short exact sequence.

Conversely, every additive category equipped with a distinguished class of short exact sequences satisfying the above properties is an exact category, i.e. it can be considered as an extension-closed full additive subcategory of an abelian category in a way that the distinguished class of short exact sequence precisely coincides with the class of short sequences whose image in that abelian category is exact.

2.2 Determinant theories and the group K_1

The notion of a determinant theory and the first K-group K_1 results from considering a 1-categorical analogue of dimension theories. The motivational example is provided by the usual notion of determinants in the linear algebra of finite dimensional vector spaces over a field. Recall that for a finite dimensional vector space V its *determinant* is the 1-dimensional space $\det V = \bigwedge^{\dim V} V$. The Theorem 2.1 has a determinantal analogue in the following form.

Theorem 2.6 *For all short exact sequences*

$$0 \to V' \to V \to V'' \to 0$$

of finite dimensional k-vector spaces, there are canonical isomorphisms $\alpha : \det V' \otimes_k \det V'' \xrightarrow{\sim} \det V$. *The isomorphisms* α *are such that for every nested triple* $V_1 \subset V_2 \subset V_3$ *with chosen subquotients* V_i/V_j *the diagram*

$$\begin{array}{ccc} \det V_1 \otimes_k \det(V_2/V_1) \otimes_k \det(V_3/V_2) & \xrightarrow{\mathrm{id} \otimes_k \alpha} & \det V_1 \otimes_k \det(V_3/V_1) \\ {\scriptstyle \alpha \otimes_k \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \alpha} \\ \det V_2 \otimes_k \det(V_3/V_2) & \xrightarrow{\alpha} & \det V_3 \end{array}$$

commutes (where the associativity constraints for the tensor products are omitted for simplicity).

We interpret this in terms of the following notion of Picard groupoids.

Definition 2.7 *A Picard groupoid is a (small) symmetric monoidal groupoid* $\mathcal{P}$, *whose tensor product structure* $\otimes$ *induces a group structure on the set* $\pi_0\mathcal{P}$ *of isomorphism classes of objects of* $\mathcal{P}$.

Example 2.8 1. *A typical example of a Picard groupoid is the symmetric monoidal groupoid* Pic_X *of line bundles over a scheme X and their isomorphisms, with the tensor product given by* $\otimes_{\mathcal{O}_X}$ *and the unit being the trivial line bundle. This is a strict Picard groupoid, meaning that the symmetry constraint* $\sigma_{L,L'} : L \otimes L' \xrightarrow{\sim} L' \otimes L, l \otimes l' \mapsto l' \otimes l$, *is such that* $\sigma_{L,L} = \mathrm{id}_{L\otimes L}$ *for* $L = L'$.

2. *An example of non-strict Picard groupoid is provided by a slight modification* $\mathrm{Pic}^{\mathbb{Z}}_X$ *of* Pic_X. *Its objects are* $\mathbb{Z}$*-graded line bundles on X, i.e. pairs* (n, L) *of locally constant functions* $n \in \mathrm{H}^0(X, \mathbb{Z})$ *and line bundles L. A morphism* $(n, L) \to (n', L')$ *in* $\mathrm{Pic}^{\mathbb{Z}}_X$ *is a pair of an equality* $n = n'$ *and an isomorphism* $L \xrightarrow{\sim} L'$ *of line bundles. The tensor product is the pair of the addition of integers and the tensor product of line bundles:* $(n, L) \otimes (n', L') = (n + n', L \otimes L')$. *The symmetry constraint* $\sigma^{\mathbb{Z}}$ *is defined by taking the grading into account, i.e.* $\sigma^{\mathbb{Z}} : (n, L) \otimes (n', L') \xrightarrow{\sim} (n', L') \otimes (n, L)$ *is the pair of the equality* $n + n' = n' + n$ *and the isomorphism* $L \otimes L' \xrightarrow{\sim} L' \otimes L, l \otimes l' \mapsto (-1)^{nn'} l' \otimes l$.

Consider the Picard groupoid $\mathrm{Pic}^{\mathbb{Z}}_{\mathrm{Spec}\,k} = \mathrm{Pic}^{\mathbb{Z}}_k$ of graded lines over k (Example 2.8-2). By a slight abuse of notation, we also write $\det V$ for the graded k-line $(\dim V, \det V)$. Then Theorem 2.6 is still true with this notation and the graded determinant functor $\det : \mathrm{Vect}_0\, k \to \mathrm{Pic}^{\mathbb{Z}}_k$ admits a universal description analogous to dimensions, as follows.

Definition 2.9 1. *A* determinant theory *on an exact category* $\mathcal{A}$ *taking values in a Picard groupoid* $\mathcal{P}$ *is a pair* (δ, α) *of a functor* $\delta : \mathcal{A} \to \mathcal{P}$ *and a collection of isomorphisms* $\alpha : \delta(a') \otimes \delta(a'') \xrightarrow{\sim} \delta(a)$, *one for all short exact sequences in* $\mathcal{A}$

$$0 \to a' \to a \to a'' \to 0,$$

such that for every nested triple $a_1 \hookrightarrow a_2 \hookrightarrow a_3$ *with chosen subquotients* a_i/a_j, *the diagram*

$$\begin{array}{ccc} \delta(a_1) \otimes \delta(a_2/a_1) \otimes \delta(a_3/_2) & \xrightarrow{\mathrm{id} \otimes \alpha} & \delta(a_1) \otimes \delta(a_3/a_1) \\ {\scriptstyle \alpha \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \alpha} \\ \delta(a_2) \otimes \delta(a_3/a_2) & \xrightarrow{\alpha} & \delta(a_3) \end{array}$$

commutes (where the associativity constraints for the tensor products are omitted for simplicity).

2. *The determinant theory $\delta : \mathcal{A} \to \mathcal{P}$ is* universal *if for every determinant theory $\delta' : \mathcal{A} \to \mathcal{P}'$ there is a symmetric monoidal functor $\phi : \mathcal{P} \to \mathcal{P}'$ such that $\phi \circ \delta'$ is naturally isomorphic to δ, respecting the isomorphism α, i.e. for every short exact sequence*

$$0 \to a' \to a \to a'' \to 0$$

the diagram

$$\begin{array}{ccc} \phi(\delta'(a')) \otimes \phi(\delta'(a'')) & \xrightarrow{\alpha} & \phi(\delta'(a)) \\ \downarrow & & \downarrow \\ \delta(a') \otimes \delta(a'') & \xrightarrow{\alpha} & \delta(a) \end{array}$$

commutes. The group $\pi_1 \mathcal{P}$ of the target Picard groupoid $\mathcal{P}$ is called the first K-group *of $\mathcal{A}$ and denoted by $K_1(\mathcal{A})$.*

It can be shown that the graded determinant functor $\det : \mathrm{Vect}_0\, k \to \mathrm{Pic}_k^{\mathbb{Z}}$ is universal among determinant theories on $\mathrm{Vect}_0\, k$, and hence the first K-group $K_1(\mathrm{Vect}_0\, k) = K_1(k)$ of the field k is the group $\pi_1 \mathrm{Pic}_k^{\mathbb{Z}} = k^* = k \setminus \{0\}$. We remark that for any scheme X we have the graded determinant functor $\det : \mathrm{Vect}_0\, X \to \mathrm{Pic}_X^{\mathbb{Z}}$, $E \mapsto (\operatorname{rank} E, \bigwedge^{\operatorname{rank} E} E)$, together with the canonical isomorphisms $\det E' \otimes \det E'' \xrightarrow{\sim} \det E$ for

$$0 \to E' \to E \to E'' \to 0,$$

and it forms a determinant theory on the exact category $\mathrm{Vect}_0\, X$. However, it is not a universal determinant theory on $\mathrm{Vect}_0\, X$ in general.

One reason for working with graded determinants instead of plain ones is that it is easier to give a universal description of the *pair* of the K-groups (K_0, K_1), rather than the sole group K_1. In the next subsection we will define the K-theory space as the universal target of ∞-determinants, and the above universal description for the pair (K_0, K_1) is simply obtained by truncating it. It is not always possible to truncating out the π_1-part only while keeping some universal property.

2.3 Higher categorical analogues of determinant theories and the connective algebraic K-theory spectrum

The discussions above motivate the idea that there should be a way of associating to each $n \geq 0$ and to an exact category $\mathcal{A}$ an abelian group $K_n(\mathcal{A})$ defined in terms of an n-categorical analogue of a determinant theory. Namely, the tuple $(K_0(\mathcal{A}), K_1(\mathcal{A}), K_2(\mathcal{A}), \ldots, K_n(\mathcal{A}))$ should be obtained as the sequence of groups $(\pi_0\mathcal{P}, \pi_1\mathcal{P}, \pi_2\mathcal{P}, \ldots, \pi_n\mathcal{P})$ encoded in a Picard n-groupoid $\mathcal{P}$ which is the target of a universal n-determinant theory. Moreover, there should be a notion of an ∞-determinant theory valuing in an ∞-Picard groupoid containing the notions of n-determinant theories.

This idea is elaborated and made precise by a recent paper [2] of Barwick, where he offers a universal characterization of connective K-theory in an ∞-categorical manner. Waldhausen's $S_\bullet$-construction [25] we recall below is considered as giving a concrete construction of the connective K-theory spectrum thus defined in terms of universality.

Definition 2.10 ([25]) *A Waldhausen category is a category $\mathcal{C}$ with a distinguished zero object 0, together with two subcategories $\mathrm{co}\,\mathcal{C} \subset \mathcal{C}$, whose morphisms are called cofibrations and depicted as $\hookrightarrow$, and $w\mathcal{C} \subset \mathcal{C}$, whose morphisms are called weak equivalences and depicted as $\xrightarrow{\sim}$, that satisfy the following conditions.*

1. *Both* $\mathrm{co}\,\mathcal{C}$ *and* $w\mathcal{C}$ *contain all objects of* $\mathcal{C}$ *and all isomorphisms in* $\mathcal{C}$*.*

2. *The unique map* $0 \to A$ *is contained in* $\mathrm{co}\,\mathcal{C}$ *for every object* A *of* $\mathcal{C}$*.*

3. *Given a pair of maps* $A \hookrightarrow B$ *and* $A \to C$ *in* $\mathcal{C}$*, where the first one is contained in* $\mathrm{co}\,\mathcal{C}$*, there is a pushout*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \cup_A C \end{array}$$

and the map $C \hookrightarrow B \cup_A C$ *is in* $\mathrm{co}\,\mathcal{C}$*.*

4. *Given a commutative diagram in* $\mathcal{C}$

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longleftarrow & A' & \longrightarrow & C', \end{array}$$

if $A \hookrightarrow B$ *and* $A' \hookrightarrow B'$ *are contained in* $\mathrm{co}\,\mathcal{C}$ *and all the three vertical maps are in* $w\mathcal{C}$*, then the induced map* $B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$ *is contained in* $w\mathcal{C}$*.*

Example 2.11 1. *Every exact category, e.g. the category* $\mathrm{Vect}_0(X)$ *of vector bundles on the scheme X, is a Waldhausen category, together with a chosen zero object and the subcategories of cofibrations and weak equivalences given respectively by admissible monomorphisms and isomorphisms. This is the standard Waldhausen structure on the exact category.*

2. *The category* $\mathrm{Perf}(X)$ *of perfect complexes of* $\mathcal{O}_X$*-modules on a scheme X with globally finite* Tor-*amplitude ([23], section 2) is a Waldhausen category, together with a chosen zero complex and the subcategories of cofibrations and weak equivalences given respectively by degree-wise split monomorphisms and quasi-isomorphisms.*

For each $n \geq 0$, write $\mathrm{Ar}[n]$ for the partially ordered set of pairs (i,j), $0 \leq i \leq j \leq n$, where $(i,j) \leq (i',j')$ if $i \leq i'$ and $j \leq j'$. Consider $\mathrm{Ar}[n]$ as a category whose objects are the pairs (i,j) and and whose morphisms are the relations $(i,j) \leq (i',j')$. For the Waldhausen category $\mathcal{C}$ define $S_n(\mathcal{C})$ to be the full subcategory of the category of functors from $\mathrm{Ar}[n]$ to $\mathcal{C}$, consisting of those functors A such that:

1. For every $0 \leq i \leq n$, the object $A_{(i,i)}$ is the chosen 0-object of $\mathcal{C}$.

2. For every $0 \leq i \leq j \leq k \leq n$, the sequence

$$0 \to A_{(i,j)} \to A_{(i,k)} \to A_{(j,k)} \to 0$$

is a cofibration sequence in $\mathcal{C}$, i.e. $A_{(i,j)} \hookrightarrow A_{(i,k)}$ is a cofibration in $\mathcal{C}$ and

$$\begin{array}{ccc} A_{(i,j)} & \longrightarrow & A_{(i,k)} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{(j,k)} \end{array}$$

is a pushout square.

Write $wS_n(\mathcal{C}) \subset S_n(\mathcal{C})$ for the subcategory of those morphisms $A \to A'$ in $S_n(\mathcal{C})$ for which $A_{(i,j)} \xrightarrow{\sim} A'_{(i,j)}$ is a weak equivalence in $\mathcal{C}$ for every $0 \leq i \leq j \leq n$. Taking the nerves of $wS_n(\mathcal{C})$ we get a bisimplicial set $([m],[n]) \mapsto N_m wS_n(\mathcal{C})$.

Definition 2.12 *We write $S(\mathcal{C})$ for the geometric realization of this bisimplicial set, and call it the* Waldhausen space *of the Waldhausen category $\mathcal{C}$.*

This construction can be iterated, i.e. for every n the category $S_n(\mathcal{C})$ inherits a structure of a Waldhausen category from that of $\mathcal{C}$, and we get a trisimplicial set $([l],[m],[n]) \mapsto N_l wS_m(S_n(\mathcal{C}))$, whose geometric realization we denote by $SS(\mathcal{C})$. For every $k \geq 0$ there result spaces $S^k(\mathcal{C})$, the geometric realization of the $(k+1)$-simplicial set $([n_0],\ldots,[n_k]) \mapsto N_{n_0} wS_{n_1} \cdots S_{n_k}(\mathcal{C})$, and Waldhausen [25] constructs a map $S^k(\mathcal{C}) \to \Omega S^{k+1}(\mathcal{C})$, to get a spectrum $K(\mathcal{C})$, in the sense of [5], Definition 2.1. This is an Ω-spectrum beyond the first term, so that the canonical map $\Omega S(\mathcal{C}) \xrightarrow{\sim} \Omega^\infty K(\mathcal{C})$ is a weak equivalence. See [25] for the proofs of these statements.

Definition 2.13 Connective K-theory of Waldhausen categories. *The connective spectrum $K(\mathcal{C})$ we call the* connective algebraic K-theory spectrum *of the Waldhausen category $\mathcal{C}$. The homotopy group $K_n(\mathcal{C}) = \pi_n K(\mathcal{C})$, $n \geq 0$, is called the* n-th K-group *of $\mathcal{C}$.*

Of exact categories. *The connective K-theory $K(\mathcal{A})$ of the exact category $\mathcal{A}$ is defined by the standard Waldhausen structure on $\mathcal{A}$. The 0-th and first homotopy groups of $K(\mathcal{A})$ recovers the groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ described above. ([16], Theorem 4.2.13.)*

Of schemes. *The connective K-theory $K(X)$ of the scheme X is the connective K-theory of the Waldhausen category* $\mathrm{Perf}(X)$ *of perfect complexes of $\mathcal{O}_X$-modules on X with globally finite* Tor*-amplitude.*

Remark 2.14 *The space $\Omega^\infty K(\mathcal{A})$ is homotopy equivalent to the K-theory space defined by Quillen [18]. (See [25], 1.9.) For a scheme X the K-theory $K(\mathrm{Vect}_0\, X)$ of the exact category $\mathrm{Vect}_0\, X$ of vector bundles on X agrees with Quillen's K-theory of the scheme. If X has an ample family of line bundles, $K(\mathrm{Vect}_0\, X)$ agrees with the K-theory of perfect complexes defined above, but not always. For the explanation of why the correct K-theory of schemes should be defined by perfect complexes, see [23], sections 8.5 and 8.6.*

2.4 Non-connective algebraic K-theory spectrum

The spectrum $K(\mathcal{A})$ is a *connective* one, i.e. it has trivial negative homotopy groups. There is a natural way of extending the K-theory spectrum to a spectrum $\mathbb{K}(\mathcal{A})$ with possibly non-trivial negative homotopy groups, which generalize the definition of K-groups to negative degrees. The first negative K-group measures how far the map $K_0(X) \to K_0(U)$ induced from an open embedding of schemes $U \hookrightarrow X$ is from being surjective, and the second negative K-group does this for the map $K_{-1}(X) \to K_{-1}(U)$, and so on. For regular schemes the negative K-groups vanish, so that negative K-theory can be considered as containing information on singular points of schemes.

We use Schlichting's non-connective K-theory of Frobenius pairs ([21]) as the general framework, and as particular cases introduce the non-connective K-theory of exact categories and of schemes.

Definition 2.15 ([22], section 3) *1. A* Frobenius category *is an exact category $\mathcal{E}$ which has enough projective objects and injective objects, and in which projective objects precisely coincide with injective objects.*

2. *The* stable category $\underline{\mathcal{E}}$ *of the Frobenius category* $\mathcal{E}$ *is the triangulated category with the same objects as* $\mathcal{E}$ *and with morphisms being equivalence classes of morphisms in* $\mathcal{E}$ *with respect to the equivalence relation that identifies morphisms factoring through projective-injective objects with zero. (See [10] for that* $\underline{\mathcal{E}}$ *has a triangulated category structure.)*

3. *A* Frobenius pair *is a pair* $(\mathcal{E}, \mathcal{E}_0)$ *of Frobenius categories* $\mathcal{E}$ *and* $\mathcal{E}_0$ *together with a fully exact embedding (in the sense of Definition 2.3-3)* $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ *which preserves projective-injective objects.*

4. *A* map of Frobenius pairs $(\mathcal{E}, \mathcal{E}_0) \to (\mathcal{E}', \mathcal{E}'_0)$ *is an exact functor* $\mathcal{E} \to \mathcal{E}'$ *that preserves projective-injective objects and that maps* $\mathcal{E}_0$ *into* $\mathcal{E}'_0$*.*

5. *The* derived category $D(\mathcal{E}, \mathcal{E}_0)$ *of the Frobenius pair* $(\mathcal{E}, \mathcal{E}_0)$ *is the triangulated category obtained as the Verdier quotient of the fully faithful triangulated functor* $\underline{\mathcal{E}_0} \hookrightarrow \underline{\mathcal{E}}$*.*

6. *To the Frobenius pair* $(\mathcal{E}, \mathcal{E}_0)$ *we associate a Waldhausen category* $W(\mathcal{E}, \mathcal{E}_0)$ *whose underlying category is* $\mathcal{E}$ *and whose subcategory of cofibrations is given by the subcategory of admissible monomorphisms and whose subcategory of weak equivalences is given by the subcategory of maps of* $\mathcal{E}$ *that become isomorphisms in the derived category* $D(\mathcal{E}, \mathcal{E}_0)$*. We write* $K(\mathcal{E}, \mathcal{E}_0) = K(W(\mathcal{E}, \mathcal{E}_0))$ *for the resulting connective K-theory spectrum.*

Example 2.16 1. *Let* $\mathcal{A}$ *be an exact category and consider the category* $\mathrm{Ch}^b \mathcal{A}$ *of bounded complexes in* $\mathcal{A}$ *and its full subcategory* $\mathrm{Ac}^b \mathcal{A}$ *of those complexes* $E^\bullet$*, whose differentials* $E^i \to E^{i+1}$ *admit factorizations* $E^i \to Z^{i+1} \to E^{i+1}$ *in a way that the sequence*

$$0 \to Z^i \to E^i \to Z^{i+1} \to 0$$

is an exact sequence in $\mathcal{A}$ *for every* $i \in \mathbb{Z}$*. There is an exact structure on* $\mathrm{Ch}^b \mathcal{A}$ *where a short sequence is exact if it is a degree-wise split short exact sequence, and since the inclusion* $\mathrm{Ac}^b \mathcal{A} \hookrightarrow \mathrm{Ch}^b \mathcal{A}$ *is closed under extensions with respect to this notion of exact sequences,* $\mathrm{Ac}^b \mathcal{A}$ *is also endowed with an exact structure. Both* $\mathrm{Ch}^b \mathcal{A}$ *and* $\mathrm{Ac}^b \mathcal{A}$ *are Frobenius categories, and since the inclusion* $\mathrm{Ac}^b \mathcal{A} \hookrightarrow \mathrm{Ch}^b \mathcal{A}$ *preserves projective-injective objects, we get the Frobenius pair* $(\mathrm{Ch}^b \mathcal{A}, \mathrm{Ac}^b \mathcal{A})$ *associated to the exact category* $\mathcal{A}$*. The derived category* $D(\mathrm{Ch}^b \mathcal{A}, \mathrm{Ac}^b \mathcal{A})$ *is the bounded derived category* $D^b\mathcal{A}$ *of the exact category* $\mathcal{A}$*. (For the agreement between the connective K-theory spectra* $K(\mathcal{A})$ *of the exact category* $\mathcal{A}$ *and* $K(\mathrm{Ch}^b \mathcal{A}, \mathrm{Ac}^b \mathcal{A})$ *of the Frobenius pair* $(\mathrm{Ch}^b \mathcal{A}, \mathrm{Ac}^b \mathcal{A})$*, when* $\mathcal{A}$ *is idempotent complete, see Theorem 2.17 below.)*

2. *Let* X *be a scheme and consider the category* $\mathrm{Perf}(X)$ *of perfect complexes of* $\mathcal{O}_X$*-modules with globally finite* Tor*-amplitude as an exact category where a short sequence is defined to be exact if it is a degree-wise split short exact sequence. Then it is a Frobenius category, and together with the full Frobenius subcategory* $\mathrm{Perf}(X)_0 \subset \mathrm{Perf}(X)$ *of those complexes* $E^\bullet$ *for which the map* $E^\bullet \to 0$ *is a quasi-isomorphism it forms a Frobenius pair* $(\mathrm{Perf}(X), \mathrm{Perf}(X)_0)$*. The category* $\mathrm{Perf}(X)$ *considered as a Waldhausen category whose cofibrations are degree-wise split monomorphisms and whose weak equivalences are quasi-isomorphisms, is the same as the Waldhausen category* $W(\mathrm{Perf}(X), \mathrm{Perf}(X)_0)$ *associated to the Frobenius pair* $(\mathrm{Perf}(X), \mathrm{Perf}(X)_0)$*. In particular, the connective K-theory spectra* $K(X)$ *and* $K(\mathrm{Perf}(X), \mathrm{Perf}(X)_0)$ *are identical.*

Theorem 2.17 (Gillet-Waldhausen Theorem, [23], 1.11.7.) *For every idempotent complete exact category* $\mathcal{A}$*, the embedding* $\mathcal{A} \hookrightarrow \mathrm{Ch}^b \mathcal{A}$*, given by considering an object of* $\mathcal{A}$ *as a complex concentrated in* *degree* 0*, induces a weak equivalence of spectra* $K(\mathcal{A}) \xrightarrow{\sim} K(W(\mathrm{Ch}^b \mathcal{A}, \mathrm{Ac}^b \mathcal{A}))$*, where the domain is the connective K-theory spectrum of* $\mathcal{A}$ *with respect to the standard Waldhausen structure, and the target is the connective K-theory spectrum of the Waldhausen category associated to the Frobenius pair* $(\mathrm{Ch}^b \mathcal{A}, \mathrm{Ac}^b \mathcal{A})$.

Given a Frobenius pair $(\mathcal{E}, \mathcal{E}_0)$, Schlichting constructs ([22], 4.7), in a functorial way, a new Frobenius pair $\mathcal{S}(\mathcal{E}, \mathcal{E}_0)$ (which is, in the notation of section 3 below, the pair $(\mathrm{Ind}^a_{\mathbb{N}} \mathcal{E}, \mathrm{S}_0(\mathcal{E}, \mathcal{E}_0))$, where $\mathrm{S}_0(\mathcal{E}, \mathcal{E}_0)$ is the full Frobenius subcategory of $\mathrm{Ind}^a_{\mathbb{N}} \mathcal{E}$ of objects that become trivial in the Verdier quotient $D\mathcal{F}(\mathcal{E}, \mathcal{E}_0)/D(\mathcal{E}, \mathcal{E}_0)$), together with a canonical map of spaces $\Omega^\infty K(\mathcal{E}, \mathcal{E}_0) \to \Omega\Omega^\infty K(\mathcal{S}(\mathcal{E}, \mathcal{E}_0))$. Iterated application of the construction $\mathcal{S}$ defines a map $\Omega^\infty K(\mathcal{S}^n(\mathcal{E}, \mathcal{E}_0)) \to \Omega\Omega^\infty K(\mathcal{S}^{n+1}(\mathcal{E}, \mathcal{E}_0))$ for every $n \geq 0$. See [22], 11.4 for details.

Definition 2.18 Non-connective K-theory of Frobenius pairs. *The spectrum* $\mathbb{K}(\mathcal{E}, \mathcal{E}_0)$ *whose n-th space is the space* $\Omega^\infty K(\mathcal{S}^n(\mathcal{E}, \mathcal{E}_0))$, $n \geq 0$*, and whose structure maps are given by the maps* $\Omega^\infty K(\mathcal{S}^n(\mathcal{E}, \mathcal{E}_0)) \to \Omega\Omega^\infty K(\mathcal{S}^{n+1}(\mathcal{E}, \mathcal{E}_0))$ *above, is called the* non-connective K-theory spectrum *of the Frobenius pair* $(\mathcal{E}, \mathcal{E}_0)$.

Of exact categories. *The* non-connective K-theory spectrum $\mathbb{K}(\mathcal{A})$ *of an exact category* $\mathcal{A}$ *is the non-connective K-theory spectrum* $\mathbb{K}(\mathrm{Ch}^b \mathcal{A}, \mathrm{Ac}^b \mathcal{A})$ *of the Frobenius pair* $(\mathrm{Ch}^b \mathcal{A}, \mathrm{Ac}^b \mathcal{A})$.

Of schemes. *The* non-connective K-theory spectrum $\mathbb{K}(X)$ *of a scheme* X *is the non-connective K-theory spectrum* $\mathbb{K}(\mathrm{Perf}(X), \mathrm{Perf}(X)_0)$ *of the Frobenius pair* $(\mathrm{Perf}(X), \mathrm{Perf}(X)_0)$.

Theorem 2.19 ([22], 11.7) *For every Frobenius pair* $(\mathcal{E}, \mathcal{E}_0)$ *there is a map of spectra* $K(\mathcal{E}, \mathcal{E}_0) \to \mathbb{K}(\mathcal{E}, \mathcal{E}_0)$ *which induces isomorphisms on non-negative homotopy groups.*

Theorems 2.19 plus 2.17 shows that for every idempotent complete exact category $\mathcal{A}$ the connective and non-connective K-theory spectra $K(\mathcal{A})$ and $\mathbb{K}(\mathcal{A})$ have isomorphic non-negative homotopy groups. The following proposition in addition tells that the positive homotopy groups of $K(\mathcal{A})$ and $\mathbb{K}(\mathcal{A})$ are always isomorphic without assuming the idempotent completeness of $\mathcal{A}$.

Proposition 2.20 (Cofinality, [22], 11.17) *If* $\mathcal{A} \hookrightarrow \mathcal{B}$ *is a fully exact, cofinal embedding of exact categories (in the sense of Definition 2.3), then the induced map of non-connective K-theory spectra* $\mathbb{K}(\mathcal{A}) \xrightarrow{\sim} \mathbb{K}(\mathcal{B})$ *is a weak equivalence. In particular, the inclusion into the idempotent completion* $\mathcal{A} \hookrightarrow \mathcal{A}^\natural$ *induces a weak equivalence* $\mathbb{K}(\mathcal{A}) \xrightarrow{\sim} \mathbb{K}(\mathcal{A}^\natural)$.

From Theorem 2.19 we also see that for a scheme X there is a map $K(X) \to \mathbb{K}(X)$ which induces isomorphisms on non-negative homotopy groups.

Proposition 2.21 *If* X *is a regular noetherian scheme then the map* $K(X) \xrightarrow{\sim} \mathbb{K}(X)$ *is a weak equivalence of spectra. In particular, the negative K-groups of* X *are trivial in this case.*

One of the most important properties of non-connective K-theory is the localization theorem stated as follows.

Definition 2.22 *An* exact sequence of Frobenius pairs *is a composable pair of maps of Frobenius pairs* $(\mathcal{E}', \mathcal{E}'_0) \xrightarrow{f} (\mathcal{E}, \mathcal{E}_0) \xrightarrow{g} (\mathcal{E}'', \mathcal{E}''_0)$*, together with a natural transformation* η *from* $g \circ f$ *to the constant functor* $0 : \mathcal{E}' \to \mathcal{E}''$ *onto the chosen zero object* 0*, such that for each object* E' *of* $\mathcal{E}'$ *the map* $\eta_{E'} : g(f(E')) \xrightarrow{\sim} 0$ *in* $\mathcal{E}''$ *is a weak equivalence in the Waldhausen category* $W(\mathcal{E}'', \mathcal{E}''_0)$*, and the functor* $D(\mathcal{E}', \mathcal{E}'_0) \to D(\mathcal{E}, \mathcal{E}_0)$ *is fully faithful, and the induced functor from the Verdier quotient* $D(\mathcal{E}, \mathcal{E}_0)/D(\mathcal{E}', \mathcal{E}'_0)$ *to* $D(\mathcal{E}'', \mathcal{E}''_0)$ *is cofinal (meaning that it is fully faithful and every object in the target is a direct summand of the image of an object of the domain).*

Theorem 2.23 (Localization, [22], 11.10) *Let $(\mathcal{E}', \mathcal{E}'_0) \xrightarrow{f} (\mathcal{E}, \mathcal{E}_0) \xrightarrow{g} (\mathcal{E}'', \mathcal{E}''_0)$ be an exact sequence of Frobenius pairs, together with a natural weak equivalence $\eta : g \circ f \xrightarrow{\sim} 0$. Then the sequence of spectra*

$$\mathbb{K}(\mathcal{E}', \mathcal{E}'_0) \to \mathbb{K}(\mathcal{E}, \mathcal{E}_0) \to \mathbb{K}(\mathcal{E}'', \mathcal{E}''_0),$$

together with the induced null-homotopy of the composed map $\mathbb{K}(\eta) : \mathbb{K}(g) \circ \mathbb{K}(f) \xrightarrow{\sim} *$, forms a homotopy fibration sequence, meaning that the induced map from $\mathbb{K}(\mathcal{E}', \mathcal{E}'_0)$ to the homotopy fiber of $\mathbb{K}(g) : \mathbb{K}(\mathcal{E}, \mathcal{E}_0) \to \mathbb{K}(\mathcal{E}'', \mathcal{E}''_0)$ is a weak equivalence of spectra.

Example 2.24

1. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ be a sequence of exact categories, together with a natural isomorphism $\eta : g \circ f \xrightarrow{\sim} 0$, such that in the sequence of bouded derived categories*

   $$D^b\mathcal{A} \to D^b\mathcal{B} \to D^b\mathcal{C},$$

   the functor $D^b\mathcal{A} \to D^b\mathcal{B}$ is fully faithful and the induced map from the Verdier quotient $D^b\mathcal{B}/D^b\mathcal{A}$ to $D^b\mathcal{C}$ is cofinal. Then the associated sequence

   $$(\mathrm{Ch}^b\,\mathcal{A}, \mathrm{Ac}^b\,\mathcal{A}) \xrightarrow{\mathrm{Ch}^b(f)} (\mathrm{Ch}^b\,\mathcal{B}, \mathrm{Ac}^b\,\mathcal{B}) \xrightarrow{\mathrm{Ch}^b(g)} (\mathrm{Ch}^b\,\mathcal{C}, \mathrm{Ac}^b\,\mathcal{C})$$

   of Frobenius pairs, together with the natural isomorphism $\mathrm{Ch}^b(\eta)$ from the composed map to 0, forms an exact sequence of Frobenius pairs, since the derived category of the Frobenius pair of bounded chain complexes associated to an exact category is the bounded derived category of that exact category. Hence in this case there results a homotopy fibration sequence

   $$\mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{B}) \to \mathbb{K}(\mathcal{C}),$$

   together with the null-homotopy of the composed map given by $\mathbb{K}(\mathrm{Ch}^b(\eta))$

2. *(**Thomason's localization theorem, [23], 7.4**.) Let X be a quasi-compact, quasi-separated scheme and $U \subset X$ a quasi-compact open subscheme with closed complement $Z = X \setminus U$. We write $\mathrm{Perf}_Z(X) \subset \mathrm{Perf}(X)$ for the fully exact subcategory of perfect complexes which are acyclic over U. It forms a Frobenius pair $(\mathrm{Perf}_Z(X), \mathrm{Perf}_Z(X)_0)$ with the subcategory $\mathrm{Perf}_Z(X)_0 \subset \mathrm{Perf}_Z(X)$ of complexes $E^\bullet$ such that $E^\bullet \to 0$ is a quasi-isomorphism. Write $\mathbb{K}(X\ on\ Z)$ for the non-connective K-theory spectrum of the Frobenius pair $(\mathrm{Perf}_Z(X), \mathrm{Perf}_Z(X)_0)$. We note that for all complexes $E^\bullet$ in $\mathrm{Perf}_Z(X)$, the maps $\eta_{E^\bullet} : E^\bullet_{|U} \to 0$ are quasi-isomorphisms in $\mathrm{Perf}(U)$ by definition, constituting a natural quasi-isomorphism η from the composed functor $\mathrm{Perf}_Z(X) \to \mathrm{Perf}(X) \to \mathrm{Perf}(U)$ to the constant functor to 0. Then the sequence of Frobenius pairs*

   $$(\mathrm{Perf}_Z(X), \mathrm{Perf}_Z(X)_0) \to (\mathrm{Perf}(X), \mathrm{Perf}(X)_0) \to (\mathrm{Perf}(U), \mathrm{Perf}(U)_0),$$

   together with the natural quasi-isomorphism η, forms an exact sequence of Frobenius pairs, and hence there results a homotopy fibration sequence

   $$\mathbb{K}(X\ on\ Z) \to \mathbb{K}(X) \to \mathbb{K}(U),$$

   together with the null-homotopy of the composed map given by $\mathbb{K}(\eta)$.

Thomason's localization theorem has the following important geometric consequence.

Theorem 2.25 (Nisnevich descent, [23], 10.8) *Let X be a quasi-compact, quasi-separated, noetherian scheme of finite Krull dimension. Then the presheaf of spectra on the small Nisnevich site X_{Nis} of X given by*

$$\mathbb{K} : X_{\mathrm{Nis}}^{\mathrm{op}} \to (\mathrm{Spectra}), Y \mapsto \mathbb{K}(Y),$$

satisfies Nisnevich descent. I.e., it is a fibrant object with respect to Jardine's local model structure [11] on the category of presheaves of spectra on X_{Nis}.

3 Beilinson's category $\varprojlim\!\!\!\!\varinjlim \mathcal{A}$

We first recall in subsection 3.1 the notions of ind- and pro-objects in a category and constructions on exact categories involving them. We then define Beilinson's category $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ and its exact category structure following [3] and [17], in subsection 3.2.

3.1 Ind- and pro-objects in an exact category

If $\mathcal{C}$ is any category, the category $\mathrm{Ind}\,\mathcal{C}$ (resp. $\mathrm{Pro}\,\mathcal{C}$) of *ind-objects* (resp. *pro-objects*) in $\mathcal{C}$ is defined to have as objects functors $\mathcal{X} : J \to \mathcal{C}$ with domain J small and filtering (resp. $\mathcal{X} : I^{\mathrm{op}} \to \mathcal{C}$ with I small and filtering). The ind-object $\mathcal{X} : J \to \mathcal{C}$ (resp. pro-object $\mathcal{X} : I^{\mathrm{op}} \to \mathcal{C}$) defines a functor $\mathcal{C}^{\mathrm{op}} \to (\mathrm{sets})$, $C \mapsto \varinjlim_{j\in J} \mathrm{Hom}_{\mathcal{C}}(C, \mathcal{X}_j)$ (resp. $\mathcal{C} \to (\mathrm{sets})$, $C \mapsto \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(\mathcal{X}_i, C)$). A morphism $\mathcal{X} \to \mathcal{Y}$ of ind-objects (resp. pro-objects) is a natural transformation between the functors $\mathcal{C}^{\mathrm{op}} \to (\mathrm{sets})$ (resp. $\mathcal{C} \to (\mathrm{sets})$) associated to $\mathcal{X}$ and $\mathcal{Y}$. Equivalently, the sets of morphisms of ind- and pro-objects can be defined to be the projective-inductive limits $\mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(\mathcal{X}, \mathcal{Y}) = \varprojlim_i \varinjlim_k \mathrm{Hom}_{\mathcal{C}}(\mathcal{X}_i, \mathcal{Y}_k)$ and $\mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}}(\mathcal{X}, \mathcal{Y}) = \varprojlim_l \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(\mathcal{X}_j, \mathcal{Y}_l)$, respectively.

If $\mathcal{X}$ and $\mathcal{Y}$ have a common index category, a natural transformation $\mathcal{X} \to \mathcal{Y}$ between the functors $\mathcal{X}$ and $\mathcal{Y}$ defines a map between the ind- or pro-objects $\mathcal{X}$ and $\mathcal{Y}$. Conversely, every map of ind- or pro-objects $\mathcal{X} \to \mathcal{Y}$ can be "straightened" to a natural transformation, in the sense that there is a commutative diagram in $\mathrm{Ind}\,\mathcal{C}$ or $\mathrm{Pro}\,\mathcal{C}$

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \sim\downarrow & & \sim\downarrow \\ \widetilde{\mathcal{X}} & \longrightarrow & \widetilde{\mathcal{Y}} \end{array}$$

with the vertical maps isomorphisms, $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ having a common index category, and $\widetilde{\mathcal{X}} \to \widetilde{\mathcal{Y}}$ coming from a natural transformation. (See [1], Appendix, for details.)

If $\mathcal{C}$ is an exact category, the categories $\mathrm{Ind}\,\mathcal{C}$ and $\mathrm{Pro}\,\mathcal{C}$ possess exact structures. A pair of composable morphisms in $\mathrm{Ind}\,\mathcal{C}$ or $\mathrm{Pro}\,\mathcal{C}$ is a short exact sequence if it can be straightened to a sequence of natural transformations which is level-wise exact in $\mathcal{C}$ ([17], 4.15, 4.16). In this article we are mainly concerned with the full subcategories $\mathrm{Ind}^a\,\mathcal{C}$ and $\mathrm{Pro}^a\,\mathcal{C}$ of *admissible ind-* and *pro-objects* introduced by Previdi [17], 5.6: An ind-object $\mathcal{X} : J \to \mathcal{C}$ (resp. pro-object $\mathcal{X} : I^{\mathrm{op}} \to \mathcal{C}$) is *admissible* if for every map $j \to j'$ in J (resp. $i \to i'$ in I) the morphism $X_j \hookrightarrow X_{j'}$ is an admissible monomorphism in $\mathcal{C}$ (resp. $X_i \twoheadleftarrow X_{i'}$ an admissible epimorphism). These subcategories are extension-closed in the exact categories $\mathrm{Ind}\,\mathcal{C}$ and $\mathrm{Pro}\,\mathcal{C}$, respectively, so that they have induced exact structures. Since an object C of $\mathcal{C}$ can be considered as an admissible ind- or pro-object which is indexed by whatever small and filtering category and takes the constant value C, there are embeddings of exact categories $\mathcal{C} \hookrightarrow \operatorname{Ind}^a \mathcal{C}$ and $\mathcal{C} \hookrightarrow \operatorname{Pro}^a \mathcal{C}$.

We write $\operatorname{Ind}^a_{\mathbb{N}} \mathcal{C}$ and $\operatorname{Pro}^a_{\mathbb{N}} \mathcal{C}$ for the full, extension-closed subcategories of $\operatorname{Ind}^a \mathcal{C}$ and $\operatorname{Pro}^a \mathcal{C}$ consisting of admissible ind- and pro-objects, respectively, indexed by the filtering category of natural numbers. (There is precisely one morphism $j \to k$ if $j \leq k \in \mathbb{N}$.) The object C of $\mathcal{C}$ defines an object $C = C = C = \cdots$ in $\operatorname{Ind}^a_{\mathbb{N}} \mathcal{C}$ or $\operatorname{Pro}^a_{\mathbb{N}} \mathcal{C}$. Note that the resulting embedding $\mathcal{C} \hookrightarrow \operatorname{Ind}^a_{\mathbb{N}} \mathcal{C} \hookrightarrow \operatorname{Ind}^a \mathcal{C}$ (resp. $\mathcal{C} \hookrightarrow \operatorname{Pro}^a_{\mathbb{N}} \mathcal{C} \hookrightarrow \operatorname{Pro}^a \mathcal{C}$) is naturally isomorphic to the embedding $\mathcal{C} \hookrightarrow \operatorname{Ind}^a \mathcal{C}$ (resp. $\mathcal{C} \hookrightarrow \operatorname{Pro}^a \mathcal{C}$) mentioned above.

3.2 Definition of $\underset{\longleftrightarrow}{\lim} \mathcal{A}$

Let $\mathcal{A}$ be an exact category. We write Π for the ordered set $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i \leq j\}$, where $(i,j) \leq (i',j')$ if $i \leq i'$ and $j \leq j'$. A functor $X : \Pi \to \mathcal{A}$, where Π is viewed as a filtered category, is *admissible* if for every triple $i \leq j \leq k$, the sequence

$$0 \to X_{i,j} \to X_{i,k} \to X_{j,k} \to 0$$

is a short exact sequence in $\mathcal{A}$. We denote by $\operatorname{Fun}^a(\Pi, \mathcal{A})$ the exact category of admissible functors $X : \Pi \to \mathcal{A}$ and natural transformations, where a short sequence $X \to Y \to Z$ of natural transformations of admissible functors is a short exact sequence in $\operatorname{Fun}^a(\Pi, \mathcal{A})$ if

$$0 \to X_{i,j} \to Y_{i,j} \to Z_{i,j} \to 0$$

is a short exact sequence in $\mathcal{A}$ for every $i \leq j$. A bicofinal map $\phi : \mathbb{Z} \to \mathbb{Z}$ (ϕ is said to be *bicofinal* if it is nondecreasing and satisfies $\lim_{i \to \pm\infty} \phi(i) = \pm\infty$) induces a cofinal functor $\widetilde{\phi} : \Pi \to \Pi$, $(i,j) \mapsto (\phi(i), \phi(j))$. If ϕ and $\psi : \mathbb{Z} \to \mathbb{Z}$ are bicofinal maps such that $\phi(i) \leq \psi(i)$ for all i, and if $X : \Pi \to \mathcal{A}$ is an admissible functor, there is a natural transformation $u_{X,\phi,\psi} : X \circ \widetilde{\phi} \to X \circ \widetilde{\psi}$.

Definition 3.1 (Beilinson [3], A.3) *The category $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ is defined to be the localization of* $\operatorname{Fun}^a(\Pi, \mathcal{A})$ *by the morphisms $u_{X,\phi,\psi}$, where $X \in \operatorname{ob} \operatorname{Fun}^a(\Pi, \mathcal{A})$, and $\phi \leq \psi : \mathbb{Z} \to \mathbb{Z}$ are bicofinal.*

If $X : \Pi \to \mathcal{A}$ is an admissible functor, we have for each $j \in \mathbb{Z}$ an admissible pro-object in $\mathcal{A}$

$$X_{\bullet,j} : \{i \in \mathbb{Z} \mid i \leq j\} \to \mathcal{A}, i \mapsto X_{i,j}.$$

We get in turn an admissible ind-object in $\operatorname{Pro}^a \mathcal{A}$

$$X_{\bullet,\bullet} : \mathbb{Z} \to \operatorname{Pro}^a \mathcal{A}, j \mapsto X_{\bullet,j}.$$

Thus the admissible functor X can be viewed as an object of the iterated, admissible Ind-Pro category $\operatorname{Ind}^a \operatorname{Pro}^a \mathcal{A}$. If $\phi \leq \psi : \mathbb{Z} \to \mathbb{Z}$ are bicofinal, the map $u_{X,\phi,\psi}$ defines an isomorphism between the ind-pro-objects $X \circ \widetilde{\phi}$ and $X \circ \widetilde{\psi}$. We get a functor $\underset{\longleftrightarrow}{\lim} \mathcal{A} \to \operatorname{Ind}^a \operatorname{Pro}^a \mathcal{A}$. In view of the following theorem, we regard $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ as an exact subcategory of $\operatorname{Ind}^a \operatorname{Pro}^a \mathcal{A}$.

Theorem 3.2 (Previdi [17], 5.8, 6.1) *The functor $\underset{\longleftrightarrow}{\lim} \mathcal{A} \to \operatorname{Ind}^a \operatorname{Pro}^a \mathcal{A}$ is fully faithful. Moreover, the image is closed under extensions in $\operatorname{Ind}^a \operatorname{Pro}^a \mathcal{A}$. In particular, $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ has an exact structure where a sequence in $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ is exact if and only if its image in $\operatorname{Ind}^a \operatorname{Pro}^a \mathcal{A}$ is exact.*

By [17], 6.3, there is an embedding $\mathrm{Ind}^a_{\mathbb{N}} \mathcal{A} \hookrightarrow \underset{\longleftrightarrow}{\lim} \mathcal{A}$ (resp. $\mathrm{Pro}^a_{\mathbb{N}} \mathcal{A} \hookrightarrow \underset{\longleftrightarrow}{\lim} \mathcal{A}$) of exact categories that sends $X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots \in \mathrm{ob}\,\mathrm{Ind}^a_{\mathbb{N}} \mathcal{A}$ (resp. $X_1 \twoheadleftarrow X_2 \twoheadleftarrow X_3 \twoheadleftarrow \cdots \in \mathrm{ob}\,\mathrm{Pro}^a_{\mathbb{N}} \mathcal{A}$) to the object in $\underset{\longleftrightarrow}{\lim} \mathcal{A}$ determined by $X_{i,j} = X_{0,j} = X_j$ for $i \leq 0 < j$ (resp. $X_{i,j} = X_{i,1} = X_{-i+1}$ for $i \leq 0 < j$).

We refer to [17] for detailed discussions on ind/pro-objects in exact categories.

4 The delooping theorem

4.1 Schlichting's machinery

We prove Theorem 1.2 using the s-filtering localization sequence constructed by Schlichting [21].

Let $\mathcal{A} \hookrightarrow \mathcal{U}$ be a fully exact embedding of exact categories. Following Schlichting [21], we define a map in $\mathcal{U}$ to be a *weak isomorphism* with respect to $\mathcal{A} \hookrightarrow \mathcal{U}$ if it is either an admissible monomorphism that admits a cokernel in the essential image of $\mathcal{A} \hookrightarrow \mathcal{U}$ or an admissible epimorphism that admits a kernel in the essential image of $\mathcal{A} \hookrightarrow \mathcal{U}$. In particular, for every $A \in \mathrm{ob}\,\mathcal{A}$ the maps $0 \to A$ and $A \to 0$ are weak isomorphisms. The localization of $\mathcal{U}$ by weak isomorphisms with respect to $\mathcal{A}$ is denoted by $\mathcal{U}/\mathcal{A}$. Since the maps $\eta_A : A \to 0$ in $\mathcal{U}$ are weak isomorphisms for all objects A of $\mathcal{A}$, there results a natural isomorphism η from the composed functor $\mathcal{A} \hookrightarrow \mathcal{U} \to \mathcal{U}/\mathcal{A}$ to the constant functor onto the chosen zero object 0. Recall, from [21], that the fully exact embedding $\mathcal{A} \hookrightarrow \mathcal{U}$ is a *left s-filtering* if the following conditions are satisfied.

(1) If $A \twoheadrightarrow U$ is an admissible epimorphism in $\mathcal{U}$ with $A \in \mathrm{ob}\,\mathcal{A}$, then $U \in \mathrm{ob}\,\mathcal{A}$.

(2) If $U \hookrightarrow A$ is an admissible monomorphism in $\mathcal{U}$ with $A \in \mathrm{ob}\,\mathcal{A}$, then $U \in \mathrm{ob}\,\mathcal{A}$.

(3) Every map $A \to U$ in $\mathcal{U}$ with $A \in \mathrm{ob}\,\mathcal{A}$ factors through an object $B \in \mathrm{ob}\,\mathcal{A}$ such that $B \hookrightarrow U$ is an admissible monomorphism in $\mathcal{U}$.

(4) If $U \twoheadrightarrow A$ is an admissible epimorphism in $\mathcal{U}$ with $A \in \mathrm{ob}\,\mathcal{A}$, then there is an admissible monomorphism $B \hookrightarrow U$ with $B \in \mathrm{ob}\,\mathcal{A}$ such that the composition $B \twoheadrightarrow A$ is an admissible epimorphism in $\mathcal{A}$.

(Here $\mathrm{ob}\,\mathcal{A}$ denotes by slight abuse of notation the collection of objects of $\mathcal{U}$ contained in the essential image of $\mathcal{A} \hookrightarrow \mathcal{U}$.) A *right s-filtering* embedding is defined by dualizing the conditions above.

The following theorem due to Schlichting [21] we use as the main technical tool for the proof.

Theorem 4.1 (Schlichting [21], 1.16, 2.10) *If $\mathcal{A} \hookrightarrow \mathcal{U}$ is left or right s-filtering, then the localization $\mathcal{U}/\mathcal{A}$ has an exact structure where a short sequence is exact if and only if it is isomorphic to the image of a short exact sequence in $\mathcal{U}$. Moreover, if $\mathcal{A}$ is idempotent complete, the sequence of exact categories $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$, together with the above natural isomorphism η from the composed functor to the constant functor onto the chosen zero object, satisfies the conditions in Example 2.24-1. In particular, there results a homotopy fibration sequence of non-connective K-theory spectra*

$$\mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{U}/\mathcal{A}),$$

together with the null-homotopy of the composed map given by $\mathbb{K}(\eta)$.

Corollary 4.2 *If $\mathcal{A}$ is any (possibly not idempotent complete) exact category and $\mathcal{A} \hookrightarrow \mathcal{U}$ is left or right s-filtering, then the sequence*

$$\mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{U}/\mathcal{A}),$$

together with the null-homotopy $\mathbb{K}(\eta)$ of the composed map, forms a homotopy fibration sequence.

Proof. The single statement of Theorem 2.10 of [21] assumes the idempotent completeness of $\mathcal{A}$ for this K-theory sequence to be a homotopy fibration. But the theorem holds for general $\mathcal{A}$ in view of Lemma 1.20 of loc. cit., which assures whenever $\mathcal{A} \hookrightarrow \mathcal{U}$ is left or right s-filtering the existence of an extension-closed full subcategory $\widetilde{\mathcal{U}}^{\mathcal{A}}$ of $\mathcal{U}^\natural$ such that $\mathcal{U}$ is cofinally contained in $\widetilde{\mathcal{U}}^{\mathcal{A}}$, the induced embedding $\mathcal{A}^\natural \hookrightarrow \mathcal{U}^\natural$ factors through a left or right s-filtering embedding $\mathcal{A}^\natural \hookrightarrow \widetilde{\mathcal{U}}^{\mathcal{A}}$, and $\mathcal{U}/\mathcal{A} \xrightarrow{\sim} \widetilde{\mathcal{U}}^{\mathcal{A}}/\mathcal{A}^\natural$ is an equivalence of exact categories. We get a weak equivalence of short sequences of spectra

$$\begin{array}{ccccc}
\mathbb{K}(\mathcal{A}) & \longrightarrow & \mathbb{K}(\mathcal{U}) & \longrightarrow & \mathbb{K}(\mathcal{U}/\mathcal{A}) \\
\sim\downarrow & & \sim\downarrow & & \sim\downarrow \\
\mathbb{K}(\mathcal{A}^\natural) & \longrightarrow & \mathbb{K}(\widetilde{\mathcal{U}}^{\mathcal{A}}) & \longrightarrow & \mathbb{K}(\widetilde{\mathcal{U}}^{\mathcal{A}}/\mathcal{A}^\natural),
\end{array}$$

since a cofinal embedding of exact categories induces a weak equivalence of non-connective K-theory spectra (Proposition 2.20). Write η for the natural isomorphism from the composed functor $\mathcal{A} \hookrightarrow \mathcal{U} \to \mathcal{U}/\mathcal{A}$ to the constant functor onto the chosen zero object 0, given by the weak isomorphism $\eta_A : A \xrightarrow{\sim} 0$ for each $A \in \operatorname{ob} A$, and $\widetilde{\eta}$ for the similarly defined natural transformation for $\mathcal{A}^\natural \hookrightarrow \widetilde{\mathcal{U}}^{\mathcal{A}} \to \widetilde{\mathcal{U}}^{\mathcal{A}}/\mathcal{A}^\natural$. By Theorem 4.1 the null-homotopy $\mathbb{K}(\widetilde{\eta})$ induces a weak equivalence from $\mathbb{K}(\mathcal{A}^\natural)$ to the homotopy fiber of $\mathbb{K}(\widetilde{\mathcal{U}}^{\mathcal{A}}) \to \mathbb{K}(\widetilde{\mathcal{U}}^{\mathcal{A}}/\mathcal{A}^\natural)$, which is in turn wealy equivalent to the homotopy fiber of $\mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{U}/\mathcal{A})$ via the weak equivalence induced by the above weak equivalence of short sequences of spectra. It follows that the map from $\mathbb{K}(\mathcal{A})$ to the homotopy fiber of $\mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{U}/\mathcal{A})$ induced by the null-homotopy $\mathbb{K}(\eta)$ is a weak equivalence, as desired. ■

Lemma 4.3 ([21], 3.2) *For any exact category $\mathcal{A}$, the embedding $\mathcal{A} \hookrightarrow \operatorname{Ind}^a \mathcal{A}$ is left s-filtering.*

Proof. We start by checking condition (3) of left s-filtering. Let X be an object of $\mathcal{A}$ and Y an admissible ind-object in $\mathcal{A}$ indexed by a small filtering category J. A morphism $f : X \to Y$ in $\operatorname{Ind}^a \mathcal{A}$ is an element of $\varinjlim_{j\in J} \operatorname{Hom}_{\mathcal{A}}(X, Y_j)$, i.e. represented as the class of a map $f_j : X \to Y_j$ in $\mathcal{A}$ for some $j \in J$. The canonical map $Y_j \hookrightarrow Y$ is an admissible monomorphism because the diagram $j/J \to \mathcal{A}$, $i \mapsto Y_i/Y_j$ serves as its cockerel, where j/J is the under-category of j. We get a desired factorization $f : X \xrightarrow{f_j} Y_j \hookrightarrow Y$.

Condition (1) follows from (3). Indeed, an admissible epimorphism $X \twoheadrightarrow Y$ with X in $\mathcal{A}$ factors through some Z in $\mathcal{A}$ such that $Z \hookrightarrow Y$ is an admissible monomorphism. The composition $X \twoheadrightarrow Y \twoheadrightarrow Y/Z$ is 0, but since this composition is also an admissible epimorphism, Y/Z must be 0. This forces Y to be essentially constant.

To prove (4), let $Y \twoheadrightarrow X$ be an admissible epimorphism in $\operatorname{Ind}^a \mathcal{A}$ with X in $\mathcal{A}$, whose kernel we denote by Z. The short exact sequence

$$0 \to Z \hookrightarrow Y \twoheadrightarrow X \to 0$$

is isomorphic to a straight exact sequence

$$0 \to Z' \hookrightarrow Y' \twoheadrightarrow X' \to 0,$$

where Z', Y', and X' are all indexed by an identical small filtering category I and respectively isomorphic to Z, Y, and X. The isomorphism $X' \xrightarrow{\sim} X$ is a compatible collection of morphisms $g_i : X'_i \to X$ in $\mathcal{A}$, $i \in I$, such that there is a morphism $h : X \to X'_{i_0}$ for some $i_0 \in I$ such that $g_{i_0} \circ h = \mathrm{id}_X$ and $h \circ g_{i_0}$ is equivalent to $\mathrm{id}_{X'_{i_0}}$ in $\varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X'_{i_0}, X'_i)$. Since X' is an admissible ind-object this implies $h \circ g_{i_0} = \mathrm{id}_{X'_{i_0}}$, i.e. g_{i_0} is an isomorphism. (Note also that g_i are isomorphisms for all $i \in i_0/I$.) The map $Y'_{i_0} \hookrightarrow Y' \xrightarrow{\sim} Y$ is an admissible monomorphism as noted above, and its composition with $Y \twoheadrightarrow X$ equals the composition $Y'_{i_0} \twoheadrightarrow X'_{i_0} \xrightarrow[g_{i_0}]{\sim} X$, which is an admissible epimorphism in $\mathcal{A}$.

Finally, if $Y \hookrightarrow X$ is an admissible monomorphism with X in $\mathcal{A}$, its cokernel Z is in $\mathcal{A}$ by (1). Let

$$0 \to Y' \hookrightarrow X' \twoheadrightarrow Z' \to 0$$

be a straightening of the exact sequence

$$0 \to Y \hookrightarrow X \twoheadrightarrow Z \to 0,$$

whose common indices we denote by I. Then an argument similar to above shows that there is an $i_0 \in I$ such that for every $i \in i_0/I$, X'_i and Z'_i are isomorphic to X and Z, respectively. It follows that Y'_i is essentially constant above i_0, and we conclude that Y is contained in the essential image of $\mathcal{A}$, verifying condition (2). ∎

4.2 Proof of Theorem 1.2

We remark that, given a composable pair of embeddings of exact categories $\mathcal{A} \hookrightarrow \mathcal{V}$ and $\mathcal{V} \hookrightarrow \mathcal{U}$, if their composition is naturally isomorphic to a left s-filtering embedding $\mathcal{A} \hookrightarrow \mathcal{U}$ and if $\mathcal{V} \hookrightarrow \mathcal{U}$ detects admissible monomorphisms then $\mathcal{A} \hookrightarrow \mathcal{V}$ is also left s-filtering. This in particular implies that the embeddings $\mathcal{A} \hookrightarrow \mathrm{Ind}^a_{\mathbb{N}} \mathcal{A}$ and $\mathrm{Pro}^a_{\mathbb{N}} \mathcal{A} \hookrightarrow \underset{\longleftrightarrow}{\lim} \mathcal{A}$ are left s-filtering. For an object A of $\mathcal{A}$ (resp. of $\mathrm{Pro}^a_{\mathbb{N}} \mathcal{A}$) write η_A (resp. η'_A) for the image in $\mathrm{Ind}^a_{\mathbb{N}} \mathcal{A}/\mathcal{A}$ (resp. in $\underset{\longleftrightarrow}{\lim} \mathcal{A}/\, \mathrm{Pro}^a_{\mathbb{N}}$) of the unique map $A \to 0$ in $\mathcal{A}$ (resp. in $\mathrm{Pro}^a_{\mathbb{N}}$), so that we have a natural isomorphism η (resp. η') from the composed functor $\mathcal{A} \hookrightarrow \mathrm{Ind}^a_{\mathbb{N}} \mathcal{A} \to \mathrm{Ind}^a_{\mathbb{N}} \mathcal{A}/\mathcal{A}$ (resp. $\mathrm{Pro}^a_{\mathbb{N}} \mathcal{A} \hookrightarrow \underset{\longleftrightarrow}{\lim} \mathcal{A} \to \underset{\longleftrightarrow}{\lim} \mathcal{A}/\, \mathrm{Pro}^a_{\mathbb{N}}$) to the constant functor onto the chosen zero object. Then by Theorem 4.1 we get two homotopy fibration sequences of non-connective K-theory spectra

$$\mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathrm{Ind}^a_{\mathbb{N}} \mathcal{A}) \to \mathbb{K}(\mathrm{Ind}^a_{\mathbb{N}} \mathcal{A}/\mathcal{A}),$$

together with the null-homotopy $\mathbb{K}(\eta)$ of the composed map given by the natural isomorphism η and

$$\mathbb{K}(\mathrm{Pro}^a_{\mathbb{N}} \mathcal{A}) \to \mathbb{K}(\underset{\longleftrightarrow}{\lim} \mathcal{A}) \to \mathbb{K}(\underset{\longleftrightarrow}{\lim} \mathcal{A}/\, \mathrm{Pro}^a_{\mathbb{N}} \mathcal{A}),$$

together with the null-homotopy $\mathbb{K}(\eta')$ of the composed map given by the natural isomorphi η'.

Lemma 4.4 *There are canonical contractions for the non-connective K-theory spectra* $\mathbb{K}(\mathrm{Ind}^a_{\mathbb{N}} \mathcal{A})$ *and* $\mathbb{K}(\mathrm{Pro}^a_{\mathbb{N}} \mathcal{A})$.

Proof. The contraction for $\mathbb{K}(\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A})$ comes from the canonical flasque structure on $\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}$ (i.e. an endo-functor whose direct sum with the identity functor is naturally isomorphic to itself) given as follows. Let $X = (X_i)_{i\geq 1} \in \mathrm{ob}\,\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}$ be an $\mathbb{N}$-indexed admissible ind-object in $\mathcal{A}$ whose structure maps we denote by $\rho = \rho_{i,i'} : X_i \hookrightarrow X_{i'}$. Write $T(X) \in \mathrm{ob}\,\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}$ for the admissible ind-object $0 \to X_1 \xrightarrow{(\rho,0)} X_2 \oplus X_1 \xrightarrow{(\rho\oplus\rho,0)} X_3 \oplus X_2 \oplus X_1 \xrightarrow{(\rho\oplus\rho\oplus\rho,0)} \cdots$. A morphism $f \in \mathrm{Hom}_{\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}}(Y, X) = \underleftarrow{\lim}_i \underrightarrow{\lim}_j \mathrm{Hom}_{\mathcal{A}}(Y_i, X_j)$ with i-th component represented by $f_i : Y_i \to X_{j(i)}$ defines a morphism $T(f) : T(Y) \to T(X)$ whose i-th component is the class of the composition $Y_{i-1} \oplus \cdots \oplus Y_1 \xrightarrow{f_{i-1}\oplus\cdots\oplus f_1} X_{j(i-1)} \oplus \cdots \oplus X_{j(1)} \xrightarrow{\rho\oplus\cdots\oplus\rho} X_{k+i-1} \oplus \cdots \oplus X_{k+1} \hookrightarrow T(X)_{k+i}$, where k is chosen to be sufficiently large. The endo-functor T thus defined is a flasque structure on $\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}$ since $(X \oplus T(X))_i \xrightarrow{=} T(X)_{i+1}$ give a natural isomorphism of ind-objects.

The contraction for $\mathbb{K}(\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A})$ follows from the contraction for $\mathbb{K}(\mathrm{Ind}^a_{\mathbb{N}}(-))$ via the identification $\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A} = (\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}$ and the general equivalence $\mathbb{K}(\mathcal{B}^{\mathrm{op}}) \xrightarrow{\sim} \mathbb{K}(\mathcal{B})$. ∎

In view of the homotopy fibration sequences above, this gives us that the map $\mathbb{K}(\underleftrightarrow{\lim}\,\mathcal{A}) \to \mathbb{K}(\underleftrightarrow{\lim}\,\mathcal{A}/\,\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A})$ is a weak equivalence, and that $\mathbb{K}(\mathrm{Ind}^a_{\mathbb{N}}\,/\mathcal{A})$ deloops $\mathbb{K}(\mathcal{A})$ (which is Schlichting's delooping, Theorem 3.4 of [21]). We then note:

Lemma 4.5 *There is an equivalence* $\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}/\mathcal{A} \xrightarrow{\sim} \underleftrightarrow{\lim}\,\mathcal{A}/\,\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A}$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A} \\ \downarrow & & \downarrow \\ \mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A} & \longrightarrow & \underleftrightarrow{\lim}\,\mathcal{A}, \end{array}$$

whence there results a functor $F : \mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}/\mathcal{A} \to \underleftrightarrow{\lim}\,\mathcal{A}/\,\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A}$.

To construct a quasi inverse, we start by noticing that the functor $\mathrm{Fun}^a(\Pi, \mathcal{A}) \to \mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}$, $(X_{i,j})_{i\leq j} \mapsto X_{0,1} \hookrightarrow X_{0,2} \hookrightarrow \cdots$, induces a functor $\widetilde{G} : \underleftrightarrow{\lim}\,\mathcal{A} \to \mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}/\mathcal{A}$. Indeed, if $\phi \leq \psi : \mathbb{Z} \to \mathbb{Z}$ are bicofinal, the map $u_{X,\phi,\psi} : X \circ \widetilde{\phi} \to X \circ \widetilde{\psi}$ in $\mathrm{Fun}^a(\Pi, \mathcal{A})$ is sent to the map $X_{\phi(0),\phi(\bullet)} \to X_{\psi(0),\psi(\bullet)}$, which factors as $X_{\phi(0),\phi(\bullet)} \hookrightarrow X_{\phi(0),\psi(\bullet)} \twoheadrightarrow X_{\psi(0),\psi(\bullet)}$. The map $X_{\phi(0),\phi(\bullet)} \hookrightarrow X_{\phi(0),\psi(\bullet)}$ is an isomorphism in $\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}$ since it consists of natural isomorphisms $\underrightarrow{\lim}_j \mathrm{Hom}_{\mathcal{A}}(A, X_{\phi(0),\phi(j)}) \xrightarrow{\sim} \underrightarrow{\lim}_j \mathrm{Hom}_{\mathcal{A}}(A, X_{\phi(0),\psi(j)})$, $A \in \mathrm{ob}\,\mathcal{A}$, as ϕ and ψ are bicofinal. We also see that $X_{\phi(0),\psi(\bullet)} \twoheadrightarrow X_{\psi(0),\psi(\bullet)}$ is a weak isomorphism in $\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}$ with respect to $\mathcal{A}$, since it has the constant kernel $X_{\phi(0),\psi(0)} = X_{\phi(0),\psi(0)} = \cdots$. The functor $\widetilde{G}$ thus defined takes weak isomorphisms in $\underleftrightarrow{\lim}\,\mathcal{A}$ with respect to $\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A}$ to weak isomorphisms in $\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}$ with respect to $\mathcal{A}$, since if $X \in \mathrm{ob}\,\underleftrightarrow{\lim}\,\mathcal{A}$ is in the image of $\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A}$ then its 0-th row is constant $X_{0,1} = X_{0,1} = \cdots$, i.e. $\widetilde{G}(X)$ is in the image of $\mathcal{A}$. Hence $\widetilde{G}$ factors through a functor $G : \underleftrightarrow{\lim}\,\mathcal{A}/\,\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A} \to \mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}/\mathcal{A}$.

We have $G \circ F = \mathrm{id}_{\mathrm{Ind}^a_{\mathbb{N}}\,\mathcal{A}/\mathcal{A}}$ by definition. On the other hand, if $X = (X_{i,j})_{i\leq j} \in \mathrm{ob}\,\underleftrightarrow{\lim}\,\mathcal{A}$, then $F \circ G(X)$ is the object $\widetilde{X}$ of $\underleftrightarrow{\lim}\,\mathcal{A}$ determined by $\widetilde{X}_{i,j} = \widetilde{X}_{0,j} = X_{0,j}$, $(i \leq 0 < j)$. Define an admissible epimorphism $f_X : X \twoheadrightarrow \widetilde{X}$ in $\mathrm{Fun}^a(\Pi, \mathcal{A})$ (hence in $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$) by

$$(f_X)_{i,j} = \begin{cases} X_{i,j} = X_{i,j} & (0 \le i \le j) \\ X_{i,j} \twoheadrightarrow X_{0,j} & (i \le 0 < j) \\ X_{i,j} \twoheadrightarrow 0 & (i \le j \le 0). \end{cases}$$

The kernel coincides with the image of $0 \twoheadleftarrow X_{-1,0} \twoheadleftarrow X_{-2,0} \twoheadleftarrow X_{-3,0} \twoheadleftarrow \cdots \in \mathrm{ob}\,\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A}$ in $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$. Hence f_X is a weak isomorphism in $\underset{\longleftrightarrow}{\lim}\,\mathcal{A}$ with respect to $\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A}$. Thus we get an isomorphism $f : \mathrm{id}_{\underset{\longleftrightarrow}{\lim}\,\mathcal{A}/\,\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A}} \xrightarrow{\sim} F \circ G$, to conclude that G is a quasi inverse to F. ■

Proof of Theorem 1.2. Combining this with Schlichtings delooping and the weak equivalence $\mathbb{K}(\underset{\longleftrightarrow}{\lim}\,\mathcal{A}) \xrightarrow{\sim} \mathbb{K}(\underset{\longleftrightarrow}{\lim}\,\mathcal{A}/\,\mathrm{Pro}^a_{\mathbb{N}}\,\mathcal{A})$, we obtain that there is a natural zig-zag of weak equivalences between $\mathbb{K}(\mathcal{A})$ and $\Omega\mathbb{K}(\underset{\longleftrightarrow}{\lim}\,\mathcal{A})$. The statement of Theorem 1.2 follows since the inclusion of an exact category into its idempotent completion induces a weak equivalence of non-connective K-theory spectra (Proposition 2.20), and we are done.

4.3 Proof of the original statement of Conjecture 1.1

We can also prove the original statement of Conjecture 1.1, that the Waldhausen space $S(\underset{\longleftrightarrow}{\lim}\,\mathcal{A})$ is a delooping of $S(\mathcal{A})$, under the assumption that $\mathcal{A}$ is idempotent complete.

Theorem 4.6 *There is a weak equivalence of spaces between $S(\mathcal{A})$ and $\Omega S(\underset{\longleftrightarrow}{\lim}\,\mathcal{A})$ if $\mathcal{A}$ is idempotent complete.*

Remark 4.7 *(This remark is due to Marco Schlichting.) Theorem 4.6 implies that Conjecture 1.1 is true, because a partially abelian exact category $\mathcal{A}$ is always idempotent complete. To see this let A be an arbitrary object of $\mathcal{A}^\natural$. Since $\mathcal{A} \hookrightarrow \mathcal{A}^\natural$ is a cofinal embedding, there is an object B of $\mathcal{A}^\natural$ such that $A \oplus B$ is in $\mathcal{A}$. The two maps $f, g : A \oplus B \hookrightarrow A \oplus B \oplus A \oplus B$ given respectively by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ are admissible monomorphisms in $\mathcal{A}$ and*

$$\begin{array}{ccc} A & \xrightarrow{\;{}^t(1,0)\;} & A \oplus B \\ {\scriptstyle {}^t(1,0)}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A \oplus B & \xrightarrow[\;g\;]{} & A \oplus B \oplus A \oplus B \end{array}$$

is a pullback square. The partially abelian condition then tells $A \in \mathcal{A}$. Therefore the fully exact embedding $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^\natural$ is essentially surjective, hence an equivalence.

Proof. In [21] Schlichting shows that the a variant of Theorem 4.1 ([21], Theorem 2.1) stating that a left or right s-filtering embedding $\mathcal{A} \hookrightarrow \mathcal{U}$, together with a natural isomorphism η from the composed functor $\mathcal{A} \hookrightarrow \mathcal{U} \to \mathcal{U}/\mathcal{A}$ to the contant functor onto the chosen zero object, induces a homotopy fibration sequence of K-theory *spaces*

$$\Omega^\infty K(\mathcal{A}) \to \Omega^\infty K(\mathcal{U}) \to \Omega^\infty K(\mathcal{U}/\mathcal{A}),$$

with the null-homotopy of the composed map given by $\Omega^\infty K(\eta)$. The proof of this statement in [21] in fact shows that the sequence

$$S(\mathcal{A}) \to S(\mathcal{U}) \to S(\mathcal{U}/\mathcal{A})$$

is a homotoy fibration, with the null-homotopy of the composed map given by $S(\eta)$, so that by taking the Ω of this one gets the homotopy fibration sequence of for $\Omega^\infty K$.

The space $S(\operatorname{Ind}^a_{\mathbb{N}} \mathcal{A})$ is contractible since $\pi_0(S(\operatorname{Ind}^a_{\mathbb{N}} \mathcal{A})) = 0$, as the Waldhausen space of a Waldhausen category is always connected, and we also have $\pi_{i+1}(S(\operatorname{Ind}^a_{\mathbb{N}} \mathcal{A})) \cong \pi_i(K(\operatorname{Ind}^a_{\mathbb{N}} \mathcal{A}))$ are trivial for all $i \geq 0$ by Lemma 4.3. Hence the homotopy fibration sequence

$$S(\mathcal{A}) \to S(\operatorname{Ind}^a_{\mathbb{N}} \mathcal{A}) \to S(\operatorname{Ind}^a_{\mathbb{N}} \mathcal{A}/\mathcal{A}),$$

whose null-homotopy of the composed map is given by $S(\eta)$, gives a weak equivalence between $S(\mathcal{A})$ and $\Omega S(\operatorname{Ind}^a_{\mathbb{N}} \mathcal{A}/\mathcal{A})$. On the other hand, the space $S(\operatorname{Pro}^a_{\mathbb{N}} \mathcal{A})$ is also contractible by Lemma 4.4. The homotopy fibration sequence

$$S(\operatorname{Pro}^a_{\mathbb{N}} \mathcal{A}) \to S(\underset{\longleftrightarrow}{\lim} \mathcal{A}) \to S(\underset{\longleftrightarrow}{\lim} \mathcal{A}/ \operatorname{Pro}^a_{\mathbb{N}} \mathcal{A}),$$

whose null-homotopy of the coposed map is given by $S(\eta')$, shows $S(\underset{\longleftrightarrow}{\lim} \mathcal{A}) \to S(\underset{\longleftrightarrow}{\lim} \mathcal{A}/ \operatorname{Pro}^a_{\mathbb{N}} \mathcal{A})$ is a weak equivalence. Using the weak equivalence of $S(\operatorname{Ind}^a_{\mathbb{N}} \mathcal{A}/\mathcal{A})$ to $S(\underset{\longleftrightarrow}{\lim} \mathcal{A}/ \operatorname{Pro}^a_{\mathbb{N}} \mathcal{A})$ given Lemma 4.5 we get the desired conclusion. ■

5 The Classification theorem for torsors over the sheaf of K-theory spaces

We now turn to prove Theorems 1.10 and 1.13. Throughout this section, we work in the ∞-topos $\operatorname{Shv}_{(\text{Spaces})}(N \operatorname{Spec} R_{\text{Nis}})$ of sheaves of spaces (Example A.8) on the small Nisnevich site $\operatorname{Spec} R_{\text{Nis}}$ (Definition 1.5) of a fixed noetherian affine scheme $\operatorname{Spec} R$ of finite Krull dimension. See the Appendix for the necessary materials on the theory of ∞-topoi. We warn the reader that, in this section, categorical terms should be understood in the ∞-categorical sense. For example, limits and colimits are not usual limits and colimits in an ordinary category, but are used in the sense of Definition A.4.

5.1 Proof of Proposition 1.8, Theorem 1.10, and Corollary 1.11

We start with the following key lemma, which is based on Drinfeld's theorem on the Nisnevich local vanishing of the first negative K-group ([9], Theorem 3.7).

Lemma 5.1 *The object $\mathcal{K}_{\text{Tate}}$ is a connected pointed object of the ∞-topos $\operatorname{Shv}_{(\text{Spaces})}(N \operatorname{Spec} R_{\text{Nis}})$.*

Proof. The point $[0] \in \Omega^\infty \mathbb{K}((\varinjlim \mathcal{P}(R))^\natural)$ associated to the chosen zero object of the exact category $(\varinjlim \mathcal{P}(R))^\natural$ defines a map of simplicial presheaves $* \to \operatorname{Sing}\Omega^\infty\mathbb{K}((\varinjlim \mathcal{P}(-)^\natural)$ on $\operatorname{Spec} R_{\text{Nis}}$, which induces via θ the desired pointing $\operatorname{Spec} R \to \mathcal{K}_{\text{Tate}}$. The nontrivial part is the connectedness, which amounts to showing that the 0-th homotopy sheaf $\pi_0\mathcal{K}_{\text{Tate}}$ vanishes. The sheaf of sets $\pi_0\mathcal{K}_{\text{Tate}}$ is by Definition A.10 the sheafification of the presheaf $R' \mapsto \pi_0(\mathcal{K}_{\text{Tate}}(R')) = \pi_0(\operatorname{Sing}\Omega^\infty\mathbb{K}((\varinjlim \mathcal{P}(R'))^\natural))$, which is by Theorem 1.2 isomorphic to the presheaf given by $R' \mapsto \pi_0(\operatorname{Sing}\Omega^\infty\Sigma\mathbb{K}(\mathcal{P}(R'))) = K_{-1}(R')$. Now, it is a theorem of Drinfeld ([9], Theorem 3.7) that for every element $a \in K_{-1}(R')$ there exists a Nisnevich covering $\{\operatorname{Spec} R'_\alpha \to \operatorname{Spec} R'\}_\alpha$ such that the images of a in $K_{-1}(R'_\alpha)$ are zero. Therefore the Nisnevich sheafification of the presheaf K_{-1} vanishes and we get the desired triviality of the 0-th homotopy sheaf $\pi_0\mathcal{K}_{\text{Tate}}$. ∎

Lemma 5.2 *The loop space $\Omega\mathcal{K}_{\text{Tate}}$ of the pointed object $\mathcal{K}_{\text{Tate}}$ is equivalent to $\mathcal{K}$.*

Proof. Recall that the objects $\mathcal{K}$ and $\mathcal{K}_{\text{Tate}}$ of $\operatorname{Shv}_{(\text{Spaces})}(N\operatorname{Spec} R_{\text{Nis}})$ are the images by θ of the simplicial presheaves $\operatorname{Sing}\Omega^\infty\mathbb{K}(-)$ and $\operatorname{Sing}\Omega^\infty\mathbb{K}((\varinjlim \mathcal{P}(-))^\natural)$ (Definitions 1.6 and 1.7). In the simplicial category $(\text{Set}_\Delta^{\operatorname{Spec} R_{\text{Nis}}^{\text{op}}})^\circ$ we have that the object $\operatorname{Sing}\Omega^\infty\mathbb{K}(-)$ is equivalent to the homotopy limit $\operatorname{holim}(* \to \operatorname{Sing}\Omega^\infty\mathbb{K}((\varinjlim \mathcal{P}(-))^\natural) \leftarrow *)$ by Theorem 1.2. By Theorem 4.2.4.1 of [13] this translates into an equivalence in $N(\text{Set}_\Delta^{\operatorname{Spec} R_{\text{Nis}}^{\text{op}}})^\circ = \operatorname{Preshv}_{(\text{Spaces})}(N\operatorname{Spec} R_{\text{Nis}})$ between $\mathcal{K} = \theta(\operatorname{Sing}\Omega^\infty\mathbb{K}(-))$ and the limit $\varprojlim(\operatorname{Spec} R \to \theta(\operatorname{Sing}\Omega^\infty\mathbb{K}((\varinjlim \mathcal{P}(-))^\natural) \leftarrow \operatorname{Spec} R)$, which is by definition the loop space of the pointed object $\theta(\operatorname{Sing}\Omega^\infty\mathbb{K}((\varinjlim \mathcal{P}(-))^\natural) = \mathcal{K}_{\text{Tate}}$. ∎

Proof of Proposition 1.8. Recall Theorem A.13 which says that for an ∞-topos $\mathfrak{X}$ there is an equivalence

$$\Omega : \mathfrak{X}_{*,\text{conn}} \leftrightarrows \operatorname{Grp}(\mathfrak{X}) : B$$

between the ∞-categories $\mathfrak{X}_{*,\text{conn}}$ of connected pointed objects of $\mathfrak{X}$ and $\operatorname{Grp}(\mathfrak{X})$ of group objects of $\mathfrak{X}$. For $\mathfrak{X} = \operatorname{Shv}_{(\text{Spaces})}(N\operatorname{Spec} R_{\text{Nis}})$ we have by Lemma 5.1 that $\mathcal{K}_{\text{Tate}}$ is in $\mathfrak{X}_{*,\text{conn}}$, and by Lemma 5.2 that its loop space $\Omega\mathcal{K}_{\text{Tate}}$ is equivalent to $\mathcal{K}$. This provides with $\mathcal{K}$ the desired group structure, and the proof of Proposition 1.8 is complete.

Proof of Theorem 1.10. Applying the inverse functor B to the equivalence between $\Omega\mathcal{K}_{\text{Tate}}$ and $\mathcal{K}$ we obtain the desired equivalence between $\mathcal{K}_{\text{Tate}} \cong B\Omega\mathcal{K}_{\text{Tate}}$ and $B\mathcal{K}$, where $B\mathcal{K}$ serves as the classifying space object for $\mathcal{K}$-torsors in view of Theorem A.16, and the proof of Theorem 1.10 is complete.

Proof of Corollary 1.11. Thus we see that $\mathcal{K}$-torsors over $\operatorname{Spec} R$ are classified by maps from $\operatorname{Spec} R$ to $B\mathcal{K} \cong \mathcal{K}_{\text{Tate}}$. Let M be a Tate vector bundle over $\operatorname{Spec} R$. Then as an object of the exact category $(\varinjlim \mathcal{P}(R))^\natural$ it defines a point $[M]$ of the K-theory space $\Omega^\infty\mathbb{K}((\varinjlim \mathcal{P}(R))^\natural)$, inducing a map of simplicial presheaves $* \to \operatorname{Sing}\Omega^\infty\mathbb{K}((\varinjlim \mathcal{P}(-))^\natural)$ on $\operatorname{Spec} R_{\text{Nis}}$. For the resulting map $\operatorname{Spec} R \to \mathcal{K}_{\text{Tate}}$ in $\operatorname{Shv}_{(\text{Spaces})}(N\operatorname{Spec} R_{\text{Nis}})$ we also write $[M]$ by a slight abuse of notation. The desired torsor $\mathfrak{D}_M$ is the $\mathcal{K}$-torsor classified by this map $[M]$, i.e. it is the pullback $\mathfrak{D}_M = \varprojlim(\operatorname{Spec} R \xrightarrow{[M]} B\mathcal{K} \leftarrow \operatorname{Spec} R)$, where the map $B\mathcal{K} \leftarrow \operatorname{Spec} R$ denotes the base-point map for the classifying space object $B\mathcal{K}$. The proof of Corollary 1.11 is complete.

5.2 Proof of Theorem 1.13

Let $M \in \mathrm{ob}(\varprojlim\!\!\!\!\varinjlim \mathcal{P}(R))^\natural$ be a Tate vector bundle over $\operatorname{Spec} R$, and consider the simplicial presheaf on $\operatorname{Spec} R_{\mathrm{Nis}}$ that assigns to R' the Kan complex $N\overline{\mathrm{Aut}_{(\overleftrightarrow{\lim}\mathcal{P}(R'))^\natural} M \otimes_R R'}$, the nerve of the groupoid $\overline{\mathrm{Aut}_{(\overleftrightarrow{\lim}\mathcal{P}(R'))^\natural} M \otimes_R R'}$ with a single object, the group of whose automorphisms is the group $\mathrm{Aut}_{(\overleftrightarrow{\lim}\mathcal{P}(R'))^\natural} M \otimes_R R'$. By taking the θ of the fibrant replacement of it we get an object $N\overline{\mathrm{Aut}\, M}$ of the ∞-category $\mathrm{Preshv}_{(\mathrm{Spaces})}(N \operatorname{Spec} R_{\mathrm{Nis}})$ of presheaves spaces on $N \operatorname{Spec} R_{\mathrm{Nis}}$, whose sheafification is denoted by $a(N\overline{\mathrm{Aut}\, M})$. We use the following lemma.

Lemma 5.3 *The classifying space object of the group object* $\mathrm{Aut}\, M$ *is given by* $a(N\overline{\mathrm{Aut}\, M})$.

Proof. The proof goes similarly to the proof of Theorem 1.10, once we notice that $a(N\overline{\mathrm{Aut}\, M})$ is a connected pointed object with its loop space object equivalent to $\mathrm{Aut}\, M$. With the obvious pointing $\operatorname{Spec} R \to N\overline{\mathrm{Aut}\, M}$ we have that $N\overline{\mathrm{Aut}\, M}$ is a pointed object, and so is its sheafification $a(N\overline{\mathrm{Aut}\, M})$. Recall the general fact that for every ordinary group G, the Kan complex $N\overline{G}$ is the Eilenberg-MacLane space $K(G,1)$, where $\overline{G}$ denotes the groupoid with a single object and morphisms given by elements of G. The 0-th homotopy sheaf $\pi_0 a(N\overline{\mathrm{Aut}\, M})$ is given by sheafifying the presheaf $R' \mapsto \pi_0(N\overline{\mathrm{Aut}\, M}(R'))$, and this vanishes since the Eilenberg-MacLane space $N\overline{G} = K(G,1)$ is always connected. Since the sheafification functor commutes with finite limits, the loop space $\Omega(a(N\overline{\mathrm{Aut}\, M}))$ is the sheafification of the loop space $\Omega(N\overline{\mathrm{Aut}\, M})$, which can be computed as the homotopy limit in the simplicial category $(\mathrm{Set}_\Delta^{\operatorname{Spec} R_{\mathrm{Nis}}^{\mathrm{op}}})^\circ$ by Theorem 4.2.4.1 of [13]. This in turn can be computed object-wise on the simplicial presheaf $R' \mapsto N\overline{\mathrm{Aut}_{(\overleftrightarrow{\lim}\mathcal{P}(R'))^\natural} M \otimes_R R'} = K(\mathrm{Aut}_{(\overleftrightarrow{\lim}\mathcal{P}(R'))^\natural} M \otimes_R R', 1)$, and using the general fact that $\Omega K(G,1) = G$ we get the desired conclusion. ■

Proof of Theorem 1.13. Now, recall that for a group object G of an ∞-topos $\mathfrak{X}$, giving a G-action on an object $P \in \mathrm{ob}\,\mathfrak{X}$ is equivalent to giving an object $\sigma \in \mathrm{ob}\,\mathfrak{X}_{/X \to BG \leftarrow *}$ for some $X \to BG$, such that σ is a limit of the diagram $X \to BG \leftarrow *$ and such that the cone point $\sigma|_{\Delta^0}(0)$ equals P. (See Definition A.14 and following discussions in section A.5.) Hence, constructing the desired $\mathrm{Aut}\, M$-action on the $\mathcal{K}$-torsor $\mathfrak{D}_M$ amounts to describing $\mathfrak{D}_M$ as a limit $\mathfrak{D}_M = \varprojlim(X \to B\,\mathrm{Aut}\, M \leftarrow \operatorname{Spec} R) \in \mathrm{ob}\,\mathrm{Shv}_{(\mathrm{Spaces})}(N \operatorname{Spec} R_{\mathrm{Nis}})_{/X \to B\,\mathrm{Aut}\, M \leftarrow \operatorname{Spec} R}$ for some X and some map $X \to B\,\mathrm{Aut}\, M$. It turns out that it suffices to have a map $[[M]] : B\,\mathrm{Aut}\, M \to B\mathcal{K}$ whose precomposition with the base-point map $\operatorname{Spec} R \to B\,\mathrm{Aut}\, M$ is equal to the map $[M] : \operatorname{Spec} R \to B\mathcal{K}$ classifying the $\mathcal{K}$-torsor $\mathfrak{D}_M$. Indeed, the successive pullback $\sigma = \varprojlim(X \to B\,\mathrm{Aut}\, M \leftarrow \operatorname{Spec} R)$, where $X = \varprojlim(B\,\mathrm{Aut}\, M \overset{[[M]]}{\to} B\mathcal{K} \leftarrow \operatorname{Spec} R)$, has the cone point $\sigma|_{\Delta^0}(0) = \mathfrak{D}_M$ if $[[M]] \circ (\text{base-point}) = [M]$, in view of Proposition 2.3 of [15].

To find such a map $[[M]]$, we notice that, in general, for any idempotent complete exact category $\mathcal{A}$ and an object a of $\mathcal{A}$ the space $|N\overline{\mathrm{Aut}_{\mathcal{A}}\, a}|$ admits a natural, canonical map to the space $\Omega|iS_\bullet(\mathcal{A})| = \Omega^\infty \mathbb{K}(\mathcal{A})$, where $S_\bullet$ denotes Waldhausen's $S_\bullet$-construction ([25], 1.3), $i(-)$ the subcategory of isomorphisms, and $|-|$ the geometric realization. This is the composition of the map $|N\overline{\mathrm{Aut}_{\mathcal{A}}\, a}| \to |Ni\mathcal{A}|$ (recall that we write $i\mathcal{A}$ for the subcategory of isomorphisms) with the first structure map $|Ni\mathcal{A}| \to \Omega|iS_\bullet(\mathcal{A})|$ of Waldhausen's connective algebraic K-theory spectrum ([25], 1.3). By adjunction there in particular results a map $N\overline{\mathrm{Aut}_{\mathcal{A}}\, a} \to \operatorname{Sing}\Omega^\infty\mathbb{K}(\mathcal{A})$. Applying this construction to $M \otimes_R R' \in \mathrm{ob}(\varprojlim\!\!\!\!\!\longleftrightarrow \mathcal{P}(R'))^\natural$ for étale R-algebras R', we get a map of simplicial presheaves $N\overline{\mathrm{Aut}_{(\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(-))^\natural}\, M \otimes_R (-)} \to \operatorname{Sing}\Omega^\infty\mathbb{K}((\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(-))^\natural)$. Via the equivalence θ this corresponds to a map $N\overline{\mathrm{Aut}\, M} \to \mathcal{K}_{\mathrm{Tate}}$ in $\mathrm{Preshv}_{(\mathrm{Spaces})}(N\operatorname{Spec} R_{\mathrm{Nis}})$, which in turn induces a map $[[M]] : B\,\mathrm{Aut}\, M \cong a(N\overline{\mathrm{Aut}\, M}) \to \mathcal{K}_{\mathrm{Tate}} \cong B\mathcal{K}$. Note that the precomposition of the map of simplicial presheaves $N\overline{\mathrm{Aut}_{(\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(-))^\natural}\, M \otimes_R (-)} \to \operatorname{Sing}\Omega^\infty\mathbb{K}((\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(-))^\natural)$ with the canonical pointing $\operatorname{Spec} R \to N\overline{\mathrm{Aut}_{(\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(-))^\natural}\, M \otimes_R (-)}$ equals the map of simplicial presheaves $\operatorname{Spec} R \to \operatorname{Sing}\Omega\mathbb{K}((\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(-))^\natural)$ defined by the point $[M] \in \Omega^\infty\mathbb{K}((\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(R))^\natural)$, so that the map $[[M]]$ satisfies the desired property $[[M]] \circ (\text{base-point}) = [M]$. The proof of Theorem 1.13 is complete.

5.3 Towards higher dimensions

We have thus shown that the moduli space of torsors over the sheaf of K-theory spaces on $\operatorname{Spec} R_{\mathrm{Nis}}$ is equivalent to the K-theory sheaf of Tate vector bundles, and in particular that each Tate vector bundle $M \in \mathrm{ob}(\underset{\longleftrightarrow}{\lim}\,\mathcal{P}(R))^\natural$ has a canonically associated torsor over the sheaf of K-theory spaces $\mathfrak{D}_M$ equipped with an action by the sheaf of automorphisms of M. It would then be natural to ask if this picture generalizes to higher Tate vector bundles. Here, for every $n \geq 0$, an *n-Tate vector bundle* we define to be an object of the iterated, idempotent-completed Beilinson category $(\underset{\longleftrightarrow}{\lim}{}^n\mathcal{P}(R))^\natural$. Unfortunately, our argument given above does not apply to higher Tate vector bundles. This is because the negative K-groups K_{-n} does not Nisnevich locally vanish if $n > 1$, that is, the object $\mathcal{K}_{\mathrm{Tate}_n}$ of the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(N\operatorname{Spec} R_{\mathrm{Nis}})$ given by $\theta(\operatorname{Sing}\Omega^\infty\mathbb{K}((\underset{\longleftrightarrow}{\lim}{}^n\mathcal{P}(-))^\natural)$ is not connected, and hence may differ from the n-times classifying space object $B^n\mathcal{K}$ of $\mathcal{K}$.

One possible way of adressing this difficulty is to use a stronger topology. For instance, assuming resolution of singularities in the sense of Hironaka, the negative K-groups locally vanish for the cdh topology by Proposition 2.21. A drawback of this approach is, however, that K-theory does not satisfy cdh descent.

Another possible way to treat the local non-vanishing of negative K-theory while respecting K-theory's descent property is to use Robalo's recently developed theory of *non-commutative Nisnevich topology* [19], [20]. He defines the opposite $\mathrm{NcS}(R)$ of the ∞-category of small dg-categories of finite type over R in the sense of Toën-Vaquié [24], localized by Dwyer-Kan equivalences, to be the *∞-category of non-commutative spaces over* $\operatorname{Spec} R$, and constructs an embedding $N\,\mathrm{AffSm}_R^{\mathrm{op}} \hookrightarrow \mathrm{NcS}(R)$ of the category of smooth affine schemes of finite type over $\operatorname{Spec} R$ into $\mathrm{NcS}(R)$, by sending $\operatorname{Spec} R'$ to the dg-category of perfect complexes on $\operatorname{Spec} R'$. There are connective and non-connective K-theory functors K and $\mathbb{K}$ on $\mathrm{NcS}(R)$, whose restrictions to $N\,\mathrm{AffSm}_R^{\mathrm{op}}$ recoverse the usual ones.

He introduces an analogue of the notion of Nisnevich descent for presheaves of spectra on non-commutative spaces, formulating it as sending certain distinguished squares of non-commutative spaces to pullback-pushout squares of spectra. He proves that the non-connective K-theory is the non-commutative Nisnevich sheafification of the connective K-theory, i.e. $\mathbb{K}$ is the image of K by the left adjoint of the inclusion $\mathrm{Fun}_{\mathrm{Nis}}(\mathrm{NcS}(R), (\mathrm{Spectra})) \hookrightarrow \mathrm{Fun}(\mathrm{NcS}(R), (\mathrm{Spectra}))$ of the ∞-category of non-commutative Nisnevich sheaves of spectra into the ∞-category of presheaves of spectra on $\mathrm{NcS}(R)$.

This, in particular, implies that all the negative K-groups are non-commutative Nisnevich locally trivial. Hence, if the ∞-category of non-commutative Nisnevich sheaves of spaces on $\mathrm{NcS}(R)$ is an ∞-topos, the group object $\mathcal{K} = \theta(\mathrm{Sing}\,\Omega^\infty \mathbb{K}(-))$ admits for every $n \geq 0$ an equivalence between the n-times classifying space $B^n\mathcal{K}$ and the K-theory of n-Tate vector bundles $\mathcal{K}_{\mathrm{Tate}_n} = \theta(\mathrm{Sing}\,\Omega^\infty \mathbb{K}((\underset{\longleftrightarrow}{\lim}{}^n \mathcal{P}(-))^\natural)$. Thus the construction described in the previous subsections applies to higher Tate vector bundles in the non-commutativce world. However, it is not clear for the moment that the non-commutative Nisnevich sheaves of spaces actually form an ∞-topos.

We leave it to a future paper to carry out the full details of this story and study its possible applications.

A Appendix: ∞-topoi

In this appendix we collect material on ∞-topos theory used in section 5. Our exposition is mainly based on [13], but the definitions and properties of group actions and torsors in ∞-topoi are also taken from [15]. We begin with a brief introduction to the language of ∞-categories, recalling basic definitions and constructions on general ∞-categories (subsections A.1 through A.4), and then go on to survey the theory of ∞-topoi (subsection A.5).

A.1 ∞-categories

Let Set_Δ denote the category of simplicial sets, i.e. contravariant functors from the category of the linearly ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$, $n \geq 0$, and order preserving maps to the category of sets and set maps. We write Δ^n for the simplicial set $[m] \mapsto \mathrm{Hom}_{\mathrm{Set}_\Delta}([m], [n])$ represented by $[n]$, called the *standard n-simplex*. For every $0 < i < n$, the *i-th horn* is the simplicial set $\Lambda^n_i \subset \Delta^n$ given by $(\Lambda^n_i)_m = \{p : [m] \to [n] \mid \text{order preserving}, \{i\} \cup p([m]) \neq [n]\}$.

Definition A.1 ([13], 1.1.2.1, 1.1.2.4) *A simplicial set $S \in \mathrm{ob}\,\mathrm{Set}_\Delta$ is an* ∞-category *(resp. a* Kan complex*) if for every $0 < i < n$ (resp. for every $0 \leq i \leq n$) and every map $f_0 : \Lambda^n_i \to S$ there is a map $f : \Delta^n \to S$ which restricts to f_0 on $\Lambda^n_i \subset \Delta^n$. We call a map of simplicial sets $S \to S'$ a* functor *if both S and S' are ∞-categories.*

There is a model category structure on Set_Δ, called the *Joyal model structure*, where fibrant objects are precisely ∞-categories. The category Cat_Δ of small categories enriched over Set_Δ also has a model structure whose fibrant objects are categories enriched over Kan complexes, and there is a Quillen equivalence

$$\mathfrak{C} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Cat}_\Delta : N.$$

See [13], section 1.1.5, for details. For a small category $\mathcal{C}$ enriched over Set_Δ we call the ∞-category $N\mathcal{C}$ the *simplicial nerve* of $\mathcal{C}$. For an ∞-category S, the Set_Δ-enriched category $\mathfrak{C}(S)$ has as the set of objects the set of 0-simplices of S. A functor between ∞-categories is called an *equivalence* if the induced functor of Set_Δ-enriched categories is an equivalence.

A 0-simplex $\Delta^0 \to S$ is called an *object* of S and a 1-simplex is called a *morphism*. We write $\mathrm{ob}\,S$ for the set of objects of S. An object x of S defines the *identity morphism* $\mathrm{id}_x = s_0(x) : \Delta^1 \to S$, where s_0 is the 0-the degeneracy map for the simplicial set S. A morphism ϕ of S defines a pair of objects of S: The *domain* $d_1(\phi) = \phi(0)$ and the *target* $d_0(\phi) = \phi(1)$, where d_i are the face maps for S. If ϕ and ϕ' are morphisms with the same domain x and target y, a *homotopy* from ϕ to ϕ' is a 2-simplex $\Delta^2 \to S$ with $d_2 = \phi$, $d_1 = \phi'$, and $d_0 = \mathrm{id}_y$. For every pair of objects x, y of S we have the *mapping space* $\mathrm{Map}_S(x,y) = |\mathrm{Hom}_{\mathfrak{C}(S)}(x,y)|$, where $|-|$ is the geometric realization of the simplicial set. A morphism from x to y defines a point in $\mathrm{Map}_S(x,y)$, and two morphisms from x to y are homotopic if and only if the associated points in the mapping space lie in the same path connected component.

An ∞-category S defines an ordinary category $\mathrm{h}\,S$ called the *homotopy category*, whose objects are the objects of S and whose morphisms are the homotopy classes of morphisms of S. The composition of the classes of two morphisms ϕ and ψ with $\phi(1) = \psi(0)$ is given as the classes of the first face of a chosen 2-simplex $\sigma : \Delta^2 \to S$ such that $d_0 = \psi$ and $d_2 = \phi$. By the defining condition for an ∞-category such a σ exists and it can be shown that the class of the first face of σ is independent of the choices involved. A morphism is a *homotopy equivalence* if it induces an isomorphism in the homotopy category.

If S and S' are ∞-categories, then the *∞-category* $\mathrm{Fun}(S, S')$ *of functors from S to S'* is the ∞-category $[n] \mapsto \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n \times S, S')$.

A.2 The ∞-category of ∞-categories and the ∞-category of spaces

The full subcategory $\mathrm{Cat}_\infty^\Delta \subset \mathrm{Set}_\Delta$ of ∞-categories has a canonical enrichment over Kan complexes. The mapping Kan complex from an ∞-category $\mathcal{C}$ to an ∞-category $\mathcal{D}$ is the largest Kan complex contained in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. This makes $\mathrm{Cat}_\infty^\Delta$ into a fibrant object of the model category Cat_Δ. Recall the Quillen equivalence

$$\mathfrak{C} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Cat}_\Delta : N.$$

The simplicial nerve $N(\mathrm{Cat}_\infty^\Delta)$ is the *∞-category of ∞-categories* and we denote it by Cat_∞ ([13], 3.0.0.1).

The full, Kan-complex-enriched subcategory $\mathrm{Kan} \subset \mathrm{Cat}_\infty^\Delta \subset \mathrm{Set}_\Delta$ of Kan complexes defines in the same way an ∞-category $N(\mathrm{Kan})$ which we call the *∞-category of spaces* and denote by (Spaces) ([13], 1.2.16.1).

Remark A.2 *Strictly speaking, we have to do some universe considerations to avoid set-theoretic inconsistencies in the above argument. If the Quillen equivalence between small simplicial sets and small simplicial-set enriched categories mentioned in the previous subsection is taken with respect to some Grothendieck universe $\mathfrak{U}$ (that is, being "small" means being an element of $\mathfrak{U}$), then the one recalled in this subsection should be considered in a larger universe $\mathfrak{U}'$. See [13], sections 1.2.15 and 1.2.16, for a more detailed account.*

A.3 Limits in an ∞-category

Let S and T be ∞-categories. The *join* is the simplicial set (in fact an ∞-category) $S \star T$ given by

$$(S \star T)_I = \left(\coprod_{I = J \coprod K, \emptyset \neq J < K \neq \emptyset} S(J) \times T(K) \right) \coprod S(I) \coprod T(I),$$

where I is a non-empty finite linearly ordered set and the first disjoint union runs over decompositions of I into two non-empty linearly ordered subsets J and K with empty intersection satisfying that every element of j in J lies over every element k of K. Note that the join $S \star T$ contains both S and T as simplicial subsets. For map of simplicial sets $p : D \to S$, called a *diagram*, the *undercategory* $S_{/p}$ and the *overcategory* $S_{p/}$ are the simplicial sets given by $(S_{/p})_n = \{f : \Delta^n \star D \to S \mid f|_D = p\}$ and $(S_{p/})_n = \{f : D \star \Delta^n \to S \mid f|_D = p\}$, respectively. It can be shown that $S_{/p}$ and $S_{p/}$ are ∞-categories whenever S is. For an object $\sigma \in S_{/p}$ or $S_{p/}$, we refer to $\sigma|_{\Delta^0}(0)$ as the *cone point* of σ.

Definition A.3 ([13]; 1.2.12.1) *Let S be an ∞-category. An object $x \in \operatorname{ob} S = S_0$ is a* terminal object *(resp. an* initial object*) of S if for every $y \in \operatorname{ob} S$ the mapping space $\operatorname{Map}_S(y, x)$ (resp. $\operatorname{Map}_S(x, y)$) is contractible.*

Definition A.4 ([13], 1.2.13.4) *A* colimit *of the diagram $p : D \to S$ is an initial object $\varinjlim p$ of $S_{p/}$. Dually, a* limit *of the diagram $p : D \to S$ is a terminal object $\varprojlim p$ of $S_{/p}$. By abuse of language we also refer to the objects of S given by the cone points $(\varinjlim p)|_{\Delta^0}(0)$ and $(\varprojlim p)|_{\Delta^0}(0)$ as the* colimit *and* limit *of p, respectively.*

The limit and colimit are not strictly unique but unique up to contractible ambiguity, in the sense that if σ and σ' are both limits (resp. colimits) for the diagram p, the mapping space $\operatorname{Map}_{S_{/p}}(\sigma, \sigma')$ (resp. $\operatorname{Map}_{S_{p/}}(\sigma, \sigma')$) is contractible by definition.

A.4 Adjoint functors between ∞-categories

Definition A.5 ([13], 5.2.2.1, 5.2.2.7, 5.2.2.8) *A pair of functors between ∞-categories*

$$f : \mathcal{C} \rightleftarrows \mathcal{D} : g$$

is called an adjunction *if there exists a morphism $u : \mathrm{id}_{\mathcal{C}} \to g \circ f$ in $\operatorname{Fun}(\mathcal{C}, \mathcal{C})$ such that for every pair of objects C of $\mathcal{C}$ and D of $\mathcal{D}$, the map*

$$\operatorname{Map}_{\mathcal{D}}(f(C), D) \xrightarrow{g} \operatorname{Map}_{\mathcal{C}}(g(f(C)), g(D)) \xrightarrow{u^*} \operatorname{Map}_{\mathcal{C}}(C, g(D))$$

is a weak equivalence of spaces. In this case f is called a left adjoint *to g and g is called a* right adjoint *to f.*

Proposition A.6 ([13], 5.2.3.5) *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories which has a right adjoint $g : \mathcal{D} \to \mathcal{C}$. Then f preserves all colimits and g preserves all limits.*

A.5 ∞-topoi

If $\mathcal{C}$ is an ∞-category and C' a full subcategory of the homotopy category $\mathrm{h}\mathcal{C}$, the pullback (taken in the category of simplicial sets)

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ NC' & \longrightarrow & N\mathrm{h}\mathcal{C} \end{array}$$

is again an ∞-category and called the *full ∞-subcategory of $\mathcal{C}$ spanned by the objects contained in C'*.

Definition A.7 ([13], 6.1.0.4) *An ∞-topos $\mathfrak{X}$ is a full, accessible (see [13], 5.4.2.1, for the definition) ∞-subcategory of the ∞-category $\mathrm{Preshv}_{(\mathrm{Spaces})}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Spaces}))$ of presheaves of spaces on some ∞-category $\mathcal{C}$, such that the inclusion $\mathfrak{X} \hookrightarrow \mathrm{Preshv}_{(\mathrm{Spaces})}(\mathcal{C})$ has a left adjoint that preserves finite limits. The left adjoint is called the sheafification functor.*

Example A.8 (∞-topos of sheaves of spaces ([13], 6.2.2)) *Let $\mathcal{C}$ be an ∞-category. A* sieve *on an object C of $\mathcal{C}$ is a full subcategory $\mathcal{C}^{(0)}_{/C}$ of the overcategory $\mathcal{C}_{/C}$ such that if a morphism in $\mathcal{C}_{/C}$ has its target in $\mathcal{C}^{(0)}_{/C}$ then it also has its source in $\mathcal{C}^{(0)}_{/C}$. By Proposition 6.2.2.5 of [13] there is a canonical bijection between sieves on the object C and monomorphisms in the ∞-category $\mathrm{Preshv}_{(\mathrm{Spaces})}(\mathcal{C})$ whose target is $j(C)$, where $j : \mathcal{C} \hookrightarrow \mathrm{Preshv}_{(\mathrm{Spaces})}(\mathcal{C})$ denotes the Yoneda embedding ([13], 5.1.3).*

A Grothendieck topology *on $\mathcal{C}$ is an assignment of a collection of sieves on C to each object C of $\mathcal{C}$. A sieve on C belonging to that assigned collection is called a* covering sieve *on C. A presheaf $F \in \mathrm{Preshv}_{(\mathrm{Spaces})}(\mathcal{C})$ on $\mathcal{C}$ is called a* sheaf of spaces *on $\mathcal{C}$ if for every object C of $\mathcal{C}$ and for every monomorphism $U \hookrightarrow j(C)$ corresponding to a covering sieve on C the induced map $\mathrm{Map}_{\mathrm{Preshv}_{(\mathrm{Spaces})}(\mathcal{C})}(j(C), F) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Preshv}_{(\mathrm{Spaces})}}(U, F)$ is a weak equivalence. The full subcategory $\mathrm{Shv}_{(\mathrm{Spaces})}(S) \subset \mathrm{Preshv}_{(\mathrm{Spaces})}(S)$ of sheaves of spaces on $\mathcal{C}$ is an ∞-topos ([13], 6.2.2.7).*

An ∞-category equipped with a Grothendieck topology is called an ∞-site*. An ordinary site can be seen as an ∞-site by taking the nerve.*

The following proposition is useful in that it allows to translate descent results stated in terms of Jardine's local model structure on simplicial presheaves ([11]) to statements on ∞-sheaves of spaces.

Proposition A.9 ([13], 6.5.2.14) *Let $\mathcal{C}$ be an ordinary site. Write $\mathbb{A}$ for the simplicial category $\mathrm{sSet}^{\mathcal{C}^{\mathrm{op}}}$ of simplicial presheaves on $\mathcal{C}$, considered with Jardine's local model structure [11], and $\mathbb{A}^\circ \subset \mathbb{A}$ for the full subcategory of fibrant-cofibrant objects. Then there is an equivalence*

$$\theta : N(\mathbb{A}^\circ) \xrightarrow{\sim} \mathrm{Shv}_{(\mathrm{Spaces})}(N\mathcal{C})^\wedge \subset \mathrm{Shv}_{(\mathrm{Spaces})}(N\mathcal{C})$$

of the simplicial nerve of $\mathbb{A}^\circ$ to the full ∞-subcategory of hypercomplete sheaves (see [13], 6.5.2) of the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(N\mathcal{C})$ of sheaves of spaces on the ∞-site $N\mathcal{C}$.

Definition A.10 (Homotopy sheaves; [13], 6.5.1.1) *Let $\mathfrak{X} \subset \mathrm{Preshv}_{(\mathrm{Spaces})}(\mathcal{C})$ be an ∞-topos and X a pointed object. For each non-negative integer $n \geq 0$, the n-th* homotopy sheaf *of X is the sheaf of sets on $\mathcal{C}$ given by sheafifying the presheaf of sets on $\mathcal{C}$ that assigns to each object C of $\mathcal{C}$ the n-th homotopy set $\pi_n(X(C))$ of the pointed space $X(C)$.*

Definition A.11 (Connected objects; [13], 6.5.1.11, 6.5.1.12) *An object X of an ∞-topos $\mathfrak{X}$ is* connected *if its 0-th homotopy sheaf $\pi_0 X$ is trivial.*

Write Δ_{big} for the category of non-empty finite linearly ordered sets. A *simplicial object* in an ∞-category $\mathcal{C}$ is a functor $N(\Delta^{\mathrm{op}}_{\mathrm{big}}) \to \mathcal{C}$. The notions of group objects and their actions are formulated in terms simplicial objects, as follows.

Definition A.12 (group objects; [13], 6.1.2.7, 7.2.2.1) *A* group object *of an ∞-topos $\mathfrak{X}$ is a simplicial object $G : N(\Delta^{\mathrm{op}}_{\mathrm{big}}) \to \mathfrak{X}$ in $\mathfrak{X}$ such that $G([0])$ is a terminal object of $\mathfrak{X}$ and for every $n \geq 0$ and for every partition $[n] = S \cup S'$ with $S \cap S' = \{s\}$, the maps $G([n]) \to G(S)$ and $G([n]) \to G(S')$ exhibit $G([n])$ as a product of $G(S)$ and $G(S')$.*

By a slight abuse of language we usually refer to the object $G([1]) \in \operatorname{ob}\mathfrak{X}$ as a group object and call the simplicial object G as the *group structure* on $G([1])$.

The following theorem says that in an ∞-topos the classifying space object of a group is obtained just by the connected delooping of the group.

Theorem A.13 ([13], 7.2.2.11-(1)) *There is an equivalence*

$$B : \mathrm{Grps}(\mathfrak{X}) \rightleftarrows \mathfrak{X}_{*,\mathrm{conn}} : \Omega$$

between the ∞-category of group objects in an ∞-topos $\mathfrak{X}$ and the ∞-category $\mathfrak{X}_{*,\mathrm{conn}} \subset \mathrm{Fun}(\Delta^1, \mathfrak{X})$ of pointed, connected objects of $\mathfrak{X}$ (i.e. morphisms $* \to X$ in $\mathfrak{X}$ whose source is a terminal object and whose target is a connected object). On the level of objects, the functor B takes a group object G to the colimit $BG = \varinjlim G$, where G is seen as a diagram in $\mathfrak{X}$ indexed by $N(\Delta^{\mathrm{op}}_{\mathrm{big}})$, with the pointing given by $* = G([0]) \to \varinjlim G$. The functor Ω takes a connected pointed object X to its loop space $\Omega X = \varprojlim(* \to X \leftarrow *)$.

Definition A.14 (Action of a group object; [15], 3.1) *Let G be a group object of the ∞-topos $\mathfrak{X}$. An* action *of G on an object $P \in \operatorname{ob}\mathfrak{X}$ is a map of simplicial objects $\rho \to G$ in $\mathfrak{X}$ such that $\rho([0]) = P$ and for every $n \geq 0$ and for every partition $[n] = S \cup S'$ with $S \cap S' = \{s\}$, the maps $\rho([n]) \to \rho(S)$ and $\rho([n]) \to G(S')$ exhibit $\rho([n])$ as a product of $\rho(S)$ and $G(S')$.*

An action $\rho \to G$ of G on $P = \rho([0])$ gives rise to a pullback $\sigma = \varprojlim(X \to BG \leftarrow *) \in \operatorname{ob}\mathfrak{X}_{/X\to BG\leftarrow *}$ for a certain map $X \to BG$, such that the cone point $\sigma|_{\Delta^0}(0)$ equals P, as shown in [15], Proposition 3.15. Conversely, given a pullback $\sigma = \varprojlim(X \to BG \leftarrow *) \in \operatorname{ob}\mathfrak{X}_{/X\to BG\leftarrow *}$ for some map $X \to BG$, there results an action $\rho \to G$ such that $\rho([0]) = \sigma|_{\Delta^0}(0)$, via the Cech nerve construction ([13], 6.1.2). These constructions are mutually inverses to each other, due to the Giraud axiom saying that in an ∞-topos every groupoid object is effective. See [15], section 3, for a details. Therefore, in an ∞-topos, giving an action of a group object G on an object P is equivalent to giving a pullback $\sigma = \varprojlim(X \to BG \leftarrow *)$ for some map $X \to BG$, such that the cone point $\sigma|_{\Delta^0}(0)$ equals P.

Definition A.15 (Torsors; [15], 3.4) *Let G be a group object in an ∞-topos $\mathfrak{X}$ and X an object. A G-*torsor *over X is a G-action $\rho \to G$ together with a map $\rho([0]) \to X$ such that the induced map to X from the colimit $(\varinjlim \rho)|_{\Delta^0}(0)$ is an equivalence.*

It is notable that this simple definition automatically implies, in the setting of ∞-topoi, the usual conditions for torsors, such as the principality condition and the local triviality. See [15], 3.7 and 3.13. Moreover, we have the following very simple classification theorem of torsors.

Theorem A.16 ([15], 3.19) *Let $\mathfrak{X}$ be an ∞-topos and G a group object. The ∞-category (which can be shown to be an ∞-groupoid; [15], 3.18) of G-torsors over a fixed object X is equivalent to the space (i.e. an ∞-groupoid) $\mathrm{Map}_{\mathfrak{X}}(X, BG)$ of maps from X to BG.*

In this sense we call BG the *classifying space object* of the group object G, or the *moduli space* of G-torsors, and say that a map $X \to BG$ *classifies* the G-torsor $\check{C}(X \times_{BG} * \to X) \to G$ over X, where $\check{C}$ denotes the Cech nerve ([13], 6.1.2). Theorem A.13 says that, in an ∞-topos, the classifying space for torsors is just given by the connected delooping of the group.

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