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Decidability of Innermost Termination and Context-Sensitive Termination for Semi-Constructor Term Rewriting Systems

Keita Uchiyama [1] Masahiko Sakai [2] Toshiki Sakabe [3]

Graduate School of Information Science
Nagoya University
Furo-cho, Chikusa-ku, Nagoya, 464-8603, Japan

Abstract

Yi and Sakai [13] showed that the termination problem is a decidable property for the class of semi-constructor term rewriting systems, which is a superclass of the class of right-ground term rewriting systems. Decidability was shown by the fact that every non-terminating TRS in the class has a loop. In this paper we modify the proof of [13] to show that both innermost termination and μ-termination are decidable properties for the class of semi-constructor TRSs.

Keywords: Context-Sensitive Termination, Dependency Pair, Innermost Termination

1 Introduction

Termination is one of the central properties of term rewriting systems (TRSs for short), where we say a TRS terminates if it does not admit any infinite reduction sequence. Since termination is undecidable in general, several decidable classes have been studied [6,8,9,12,13]. The class of semi-constructor TRSs is one of them [13], where a TRS is in this class if for every right-hand side of rules all its subterms having a defined symbol at root position are ground.

Innermost reduction, the strategy which rewrites innermost redexes, is used for call-by-value computation. Context-sensitive reduction is a strategy in which rewritable positions are indicated by specifying arguments of function symbols. Some non-terminating TRSs are terminating by context-sensitive reduction without loss of computational ability. The termination property with respect to innermost

[1] Email: uchiyama@sakabe.i.is.nagoya-u.ac.jp
[2] Email: sakai@is.nagoya-u.ac.jp
[3] Email: sakabe@is.nagoya-u.ac.jp

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(resp. context-sensitive) reduction is called innermost (resp. context-sensitive) termination. Since innermost termination and context-sensitive termination are also undecidable in general, methods for proving these terminations have been studied [2,4].

In this paper, we prove that innermost termination and context-sensitive termination for semi-constructor TRSs are decidable properties. We show that context-sensitive termination for μ-semi-constructor TRSs having no infinite variable dependency chain is a decidable property. We also extend the classes by using dependency graphs.

2 Preliminaries

We assume the reader is familiar with the standard definitions of term rewriting systems [5], dependency pairs [4], and context-sensitive rewriting [2]. Here we just review the main notations used in this paper.

A *signature* $\mathcal{F}$ is a set of function symbols, where every $f \in \mathcal{F}$ is associated with a non-negative integer by an arity function: arity: $\mathcal{F} \to \mathbb{N}$. The set of all *terms* built from a signature $\mathcal{F}$ and a countably infinite set $\mathcal{V}$ of *variables* such that $\mathcal{F} \cap \mathcal{V} = \emptyset$, is represented by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The set of *ground terms* is $\mathcal{T}(\mathcal{F}, \emptyset)$. The set of variables occurring in a term t is denoted by $\mathrm{Var}(t)$.

The set of all *positions* in a term t is denoted by $\mathcal{P}\mathrm{os}(t)$ and ε represents the root position. $\mathcal{P}\mathrm{os}(t)$ is: $\mathcal{P}\mathrm{os}(t) = \{\varepsilon\}$ if $t \in \mathcal{V}$, and $\mathcal{P}\mathrm{os}(t) = \{\varepsilon\} \cup \{iu \mid 1 \leq i \leq n, u \in \mathcal{P}\mathrm{os}(t_i)\}$ if $t = f(t_1, \ldots, t_n)$. Let C be a *context* with a hole $\Box$. We write $C[t]_p$ for the term obtained from C by replacing $\Box$ at position p with a term t. We sometimes write $C[t]$ for $C[t]p$ by omitting the position p. We say t is a subterm of s if $s = C[t]$ for some context C. We denote the *subterm relation* by $\trianglelefteq$, that is, $t \trianglelefteq s$ if t is a subterm of s, and $t \triangleleft s$ if $t \trianglelefteq s$ and $t \neq s$. The *root symbol* of a term t is denoted by $\mathrm{root}(t)$.

A *substitution* θ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that the set $\mathrm{Dom}(\theta) = \{x \in \mathcal{V} \mid \theta(x) \neq x\}$ is finite. We usually identify a substitution θ with the set $\{x \mapsto \theta(x) \mid x \in \mathrm{Dom}(\theta)\}$ of variable bindings. In the following, we write $t\theta$ instead of $\theta(t)$.

A *rewrite rule* $l \to r$ is a directed equation which satisfies $l \notin \mathcal{V}$ and $\mathrm{Var}(r) \subseteq \mathrm{Var}(l)$. A *term rewriting system* TRS is a finite set of rewrite rules. A *redex* is a term $l\theta$ for a rule $l \to r$ and a substitution θ. A term containing no redex is called a *normal form*. A substitution θ is *normal* if $x\theta$ is in normal forms for every x. The *reduction relation* $\underset{R}{\to} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ associated with a TRS R is defined as follows: $s \underset{R}{\to} t$ if there exist a rewrite rule $l \to r \in R$, a substitution θ, and a context $C[\,]_p$ such that $s = C[l\theta]_p$ and $t = C[r\theta]_p$, we say that s is reduced to t by contracting redex $l\theta$. We sometimes write $\overset{p}{\underset{R}{\to}}$ for $\underset{R}{\to}$ by displaying the position p.

A redex is *innermost* if all its proper subterms are in normal forms. If s is reduced to t by contracting an innermost redex, then $s \to_R t$ is said to be an *innermost reduction* denoted by $s \underset{in,R}{\longrightarrow} t$.

Proposition 2.1 *For a TRS R, if there is a reduction $s \xrightarrow[in,R]{} t$, then $C[s] \xrightarrow[in,R]{} C[t]$ for any context C.*

A mapping $\mu : \mathcal{F} \rightarrow \mathcal{P}(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if $\mu(f) \subseteq \{1, \dots, \text{arity}(f)\}$. The set of *$\mu$-replacing positions* $\mathcal{P}os_\mu(t)$ of a term t is: $\mathcal{P}os_\mu(t) = \{\varepsilon\}$, if $t \in \mathcal{V}$ and $\mathcal{P}os_\mu(t) = \{\varepsilon\} \cup \{iu \mid i \in \mu(f), u \in \mathcal{P}os_\mu(t_i)\}$, if $t = f(t_1, \dots, t_n)$. A context $C[\ ]_p$ is *μ-replacing* denoted by $C_\mu[\ ]_p$ if $p \in \mathcal{P}os_\mu(C)$. The set of all *$\mu$-replacing variables* of t is $\text{Var}_\mu(t) = \{x \in \text{Var}(t) \mid \exists C, C_\mu[x]_p = t\}$. The *$\mu$-replacing subterm relation* $\trianglelefteq_\mu$ is given by $s \trianglelefteq_\mu t$ if there is $p \in \mathcal{P}os_\mu(t)$ such that $t = C[s]_p$. A *context-sensitive rewriting system* is a TRS with an $\mathcal{F}$-map. If $s \xrightarrow{p} t$ and $p \in \mathcal{P}os_\mu(s)$, then $s \xrightarrow{p} t$ is said to be a *μ-reduction* denoted by $s \xrightarrow[\mu,R]{} t$.

Let $\rightarrow$ be a binary relation on terms, the transitive closure of $\rightarrow$ is denoted by $\rightarrow^+$. The transitive and reflexive closure of $\rightarrow$ is denoted by $\rightarrow^*$. If $s \rightarrow^* t$, then we say that there is a *$\rightarrow$-sequence* starting from s to t or t is *$\rightarrow$-reachable from s*. We write $s \rightarrow^k t$ if t is $\rightarrow$-reachable from s with k steps. A term t *terminates* with respect to $\rightarrow$ if there exists no infinite $\rightarrow$-sequence starting from t.

Example 2.2 Let $R_1 = \{g(x) \rightarrow h(x),\ h(d) \rightarrow g(c),\ c \rightarrow d\}$ and $\mu_1(g) = \mu_1(h) = \emptyset$. A μ_1-reduction sequence starting from $g(d)$ is $g(d) \xrightarrow[\mu_1,R_1]{} h(d) \xrightarrow[\mu_1,R_1]{} g(c)$. We can not reduce $g(c)$ to $g(d)$ because c is not a μ_1-replacing subterm of $g(c)$.

Proposition 2.3 *For a TRS R and $\mathcal{F}$-map μ, if there is a reduction $s \xrightarrow[\mu,R]{} t$, then $C_\mu[s] \xrightarrow[\mu,R]{} C_\mu[t]$ for any μ-replacing context C_μ.*

For a TRS R (and $\mathcal{F}$-map μ), we say that R terminates (resp. innermost terminates, μ-terminates) if every term terminates with respect to $\rightarrow_R$ (resp. $\xrightarrow[in,R]{}$, $\xrightarrow[\mu,R]{}$).

For a TRS R, a function symbol $f \in \mathcal{F}$ is *defined* if $f = \text{root}(l)$ for some rule $l \rightarrow r \in R$. The set of all defined symbols of R is denoted by $D_R = \{\text{root}(l) \mid l \rightarrow r \in R\}$. A term t has a *defined root symbol* if $\text{root}(t) \in D_R$.

Let R be a TRS over a signature $\mathcal{F}$. The signature $\mathcal{F}^\sharp$ denotes the union of $\mathcal{F}$ and $D_R^\sharp = \{f^\sharp \mid f \in D_R\}$ where $\mathcal{F} \cap D_R^\sharp = \emptyset$ and $f^\sharp$ has the same arity as f. We call these fresh symbols *dependency pair symbols*. We define a notation $t^\sharp$ by $t^\sharp = f^\sharp(t_1, \dots, t_n)$ if $t = f(t_1, \dots, t_n)$ and $f \in D_R$, $t^\sharp = t$ if $t \in \mathcal{V}$. If $l \rightarrow r \in R$ and u is a subterm of r with a defined root symbol and $u \ntrianglelefteq l$, then the rewrite rule $l^\sharp \rightarrow u^\sharp$ is called a *dependency pair* of R. The set of all dependency pairs of R is denoted by $\text{DP}(R)$.

Example 2.4 Let $R_2 = \{a \rightarrow g(f(a)),\ f(f(x)) \rightarrow h(f(a), f(x))\}$. We have $\text{DP}(R_2) = \{a^\sharp \rightarrow a^\sharp,\ a^\sharp \rightarrow f^\sharp(a),\ f^\sharp(g(x)) \rightarrow a^\sharp,\ f^\sharp(g(x)) \rightarrow f^\sharp(a)\}$.

A rule $l \rightarrow r$ is said to be right ground if r is ground. Right-ground TRSs are TRSs that consist of right-ground rules.

Definition 2.5 [Semi-Constructor TRS] A TRS R is a *semi-constructor* system if every rule in $\text{DP}(R)$ is right ground.

Remark 2.6 The class of semi-constructor TRSs in this paper is a larger class of semi-constructor TRSs by the original definition because a rule $l^\sharp \to u^\sharp$ is not dependency pair if $u \lhd l$. The original definition of semi-constructor TRS is as follows [11]. A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is a *semi-constructor* term if every term s such that $s \unlhd t$ and $\mathrm{root}(s) \in D_R$ is ground. A TRS R is a semi-constructor system if r is a semi-constructor term for every rule $l \to r \in R$.

Example 2.7 The TRS R_2 (in Example 2.4) is a semi-constructor TRS but not in the original definition.

3 Decidability of Innermost Termination for Semi-Constructor TRSs

Decidability of termination for semi-constructor TRSs is proved based on the observation that there exists an infinite reduction sequence having a loop if it is not terminating [13]. In this section, we prove the decidability of innermost termination in a similar way.

Definition 3.1 [loop] Let $\to$ be a relation on terms. A reduction sequence *loops* if it contains $t \to^+ C[t]$ for some context C, and *head-loops* if containing $t \to^+ t$.

Proposition 3.2 *If there exists an innermost sequence that loops, then there exists an infinite innermost sequence.*

Definition 3.3 [Innermost DP-chain] For a TRS R, a sequence of the elements of $\mathrm{DP}(R)$ $s_1^\sharp \to t_1^\sharp, s_2^\sharp \to t_2^\sharp, \ldots$ is an *innermost dependency chain* if there exist substitutions $\tau_1, \tau_2, \ldots$ such that $s_i^\sharp \tau_i$ is in normal forms and $t_i^\sharp \tau_i \xrightarrow[in,R]{}^* s_{i+1}^\sharp \tau_{i+1}$ holds for every i.

Theorem 3.4 ([4]) *For a TRS R, R does not innermost terminate if and only if there exists an infinite innermost dependency chain.*

Let $\mathcal{M}_{\geq}^{\to}$ denote the set of all *minimal non-terminating terms* for a relation on terms $\to$ and an order on terms $\geq$.

Definition 3.5 [$\mathcal{C}$-min] For a TRS R, let $\mathcal{C} \subseteq \mathrm{DP}(R)$. An infinite reduction sequence in $R \cup \mathcal{C}$ in the form $t_1^\sharp \xrightarrow[in,R\cup\mathcal{C}]{} t_2^\sharp \xrightarrow[in,R\cup\mathcal{C}]{} t_3^\sharp \xrightarrow[in,R\cup\mathcal{C}]{} \cdots$ with $t_i \in \mathcal{M}_{\trianglerighteq}^{\overrightarrow{in,R}}$ for all $i \geq 1$ is called a *$\mathcal{C}$-min innermost reduction sequence*. We use $\mathcal{C}_{min}^{in}(t^\sharp)$ to denote the set of all $\mathcal{C}$-min innermost reduction sequences starting from $t^\sharp$.

Proposition 3.6 ([4]) *Given a TRS R, the following statements hold:*

(i) *If there exists an infinite innermost dependency chain, then $\mathcal{C}_{min}^{in}(t^\sharp) \neq \emptyset$ for some $\mathcal{C} \subseteq \mathrm{DP}(R)$ and $t \in \mathcal{M}_{\trianglerighteq}^{\overrightarrow{in,R}}$.*

(ii) *For any sequence in $\mathcal{C}_{min}^{in}(t^\sharp)$, reduction by rules of R takes place below the root while reduction by rules of $\mathcal{C}$ takes place at the root.*

(iii) *For any sequence in* $\mathcal{C}^{in}_{min}(t^\sharp)$*, there is at least one rule in* $\mathcal{C}$ *which is applied infinitely often.*

Lemma 3.7 ([4]) *For two terms* s *and* s'*,* $s^\sharp \xrightarrow[in,R\cup\mathcal{C}]{}^* s'^\sharp$ *implies* $s \xrightarrow[in,R]{}^* C[s']$ *for some context* C*.*

Proof. We use induction on the number n of reduction steps in $s^\sharp \xrightarrow[in,R\cup\mathcal{C}]{}^n s'^\sharp$. In the case that $n = 0$, $s \xrightarrow[in,R]{}^* C[s']$ holds where $C = \square$. Let $n \geq 1$. Then we have $s^\sharp \xrightarrow[in,R\cup\mathcal{C}]{}^{n-1} s''^\sharp \xrightarrow[in,R\cup\mathcal{C}]{} s'^\sharp$ for some $s''^\sharp$. By the induction hypothesis, $s \xrightarrow[in,R]{}^* C[s'']$.
• Consider the case that $s''^\sharp \xrightarrow[in,R]{} s'^\sharp$. Since $s'' \xrightarrow[in,R]{} s'$, we have $C[s''] \xrightarrow[in,R]{} C[s']$ by Proposition 2.1. Hence $s \xrightarrow[in,R]{}^* C[s']$.
• Consider the case that $s''^\sharp \xrightarrow[in,\mathcal{C}]{} s'^\sharp$. Since s'' is a normal form with respect to $\rightarrow_R$, we have $s'' \xrightarrow[in,R]{} C'[s']$ by the definition of dependency pairs. $C[s''] \xrightarrow[in,R]{} C[C'[s']]$, by Proposition 2.1. Hence $s \xrightarrow[in,R]{}^* C[C'[s']]$. □

Lemma 3.8 *For a semi-constructor TRS* R*, the following statements are equivalent:*

(i) R *does not innermost terminate.*

(ii) *There exists* $l^\sharp \rightarrow u^\sharp \in \mathrm{DP}(R)$ *such that* sq *head-loops for some* $\mathcal{C} \subseteq \mathrm{DP}(R)$ *and* $sq \in \mathcal{C}^{in}_{min}(u^\sharp)$*.*

Proof. ((ii) ⇒ (i)) : It is obvious from Lemma 3.7, and Proposition 3.2. ((i) ⇒ (ii)) : By Theorem 3.4 there exists an infinite innermost dependency chain. By Proposition 3.6(i), there exists a sequence $sq \in \mathcal{C}^{in}_{min}(t^\sharp)$. By Proposition 3.6(ii),(iii), there exists some rule $l^\sharp \rightarrow u^\sharp \in \mathcal{C}$, which is applied at root position in sq infinitely often. By Definition 2.5, $u^\sharp$ is ground. Thus sq contains a subsequence $u^\sharp \xrightarrow[in,R\cup\mathrm{DP}(R)]{}^* \cdot \rightarrow_{\{l^\sharp\rightarrow u^\sharp\}} u^\sharp$, which head-loops. □

Theorem 3.9 *Innermost termination of semi-constructor TRSs is decidable.*

Proof. The decision procedure for the innermost termination of a semi-constructor TRS R is as follows: consider all terms $u_1, u_2, \ldots, u_n$ corresponding to the right-hand sides of $\mathrm{DP}(R) = \{l^\sharp_i \rightarrow u^\sharp_i \mid 1 \leq i \leq n\}$, and simultaneously generate all innermost reduction sequences with respect to R starting from $u_1, u_2, \ldots, u_n$. The procedure halts if it enumerates all reachable terms exhaustively or it detects a looping reduction sequence $u_i \xrightarrow[in,R]{}^+ C[u_i]$ for some i.

Suppose R does not innermost-terminate. By Lemma 3.8 and 3.7, we have a looping reduction sequence $u_i \xrightarrow[in,R]{}^+ C[u_i]$ for some i and C, which we eventually detect. If R innermost terminates, then the execution of the reduction sequence generation eventually stops since the reduction relation is finitely branching. In the latter case, the procedure does not detect a looping sequence, otherwise it contradicts Proposition 3.2. Thus the procedure decides innermost termination of R in finitely many steps. □

4 Decidability of Context-Sensitive Termination for Semi-Constructor TRSs

The proof of decidability for innermost termination is straightforward. However, the proof for context-sensitive termination is not so straightforward because of the existence of a dependency pair whose right-hand side is variable.

Definition 4.1 [μ-Loop] Let $\rightarrow$ be a relation on terms and μ be an $\mathcal{F}$-map. A *reduction sequence μ-loops* if it contains $t \rightarrow^{+} C_\mu[t]$ for some context C_μ.

Example 4.2 Let $R_3 = \{a \rightarrow g(f(a)),\ f(g(x)) \rightarrow h(f(a), x)\}$, $\mu_2(f) = \{1\}$, $\mu_2(g) = \emptyset$ and $\mu_2(h) = \{1, 2\}$. The μ_2-reduction sequence with respect to R_3 $f(a) \xrightarrow[\mu_2,R_3]{} f(g(f(a))) \xrightarrow[\mu_2,R_3]{} h(f(a), f(a)) \xrightarrow[\mu_2,R_3]{} \cdots$ is μ_2-looping.

Proposition 4.3 *If there exists a μ-looping μ-reduction sequence, then there exists an infinite μ-reduction sequence.*

Definition 4.4 [Context-Sensitive Dependency Pairs [2]] Let R be a TRS and μ be an $\mathcal{F}$-map. We define $\mathrm{DP}(R, \mu) = \mathrm{DP}_{\mathcal{F}}(R, \mu) \cup \mathrm{DP}_{\mathcal{V}}(R, \mu)$ to be the set of *context-sensitive* dependency pairs where:

$$\mathrm{DP}_{\mathcal{F}}(R, \mu) = \{l^\sharp \rightarrow u^\sharp \mid l \rightarrow r \in R, u \trianglelefteq_\mu r, \mathrm{root}(u) \in D_R, u \ntriangleleft_\mu l\}$$
$$\mathrm{DP}_{\mathcal{V}}(R, \mu) = \{l^\sharp \rightarrow x \mid l \rightarrow r \in R, x \in \mathrm{Var}_\mu(r) \setminus \mathrm{Var}_\mu(l)\}$$

Example 4.5 Consider TRS R_3 and $\mathcal{F}$-map μ_2 (in Example 4.2). $\mathrm{DP}_{\mathcal{F}}(R_3, \mu_2) = \{f^\sharp(g(x)) \rightarrow f^\sharp(a)\}$ and $\mathrm{DP}_{\mathcal{V}}(R_3, \mu_2) = \{f^\sharp(g(x)) \rightarrow x\}$.

For a given TRS R and an $\mathcal{F}$-map μ, we define $\mu^\sharp$ by $\mu^\sharp(f) = \mu(f)$ for $f \in \mathcal{F}$, and $\mu^\sharp(f^\sharp) = \mu(f)$ for $f \in D_R$. We write $s \trianglerighteq^\sharp_\mu t^\sharp$ for $s \trianglerighteq_\mu t$.

Definition 4.6 [Context-Sensitive Dependency Chain] For a TRS R and $\mathcal{F}$-map μ, a sequence of the elements of $\mathrm{DP}(R, \mu)$ $s^\sharp_1 \rightarrow t^\sharp_1,\ s^\sharp_2 \rightarrow t^\sharp_2,\ \ldots$ is a *context-sensitive dependency chain* if there exist substitutions $\tau_1, \tau_2, \ldots$ satisfying both:

- $t^\sharp_i \tau_i \xrightarrow[\mu^\sharp,R]{}{}^{*} s^\sharp_{i+1}\tau_{i+1}$, if $t^\sharp_i \notin \mathcal{V}$
- $x\tau_i \trianglerighteq^\sharp_\mu u^\sharp_i \xrightarrow[\mu^\sharp,R]{}{}^{*} s^\sharp_{i+1}\tau_{i+1}$ for some term u_i, if $t^\sharp_i = x$.

Example 4.7 Consider TRS R_3 and $\mathcal{F}$-map μ_2 (in Example 4.2). $f(a), f(g(f(a))) \in \mathcal{M}^{\xrightarrow[\mu_2,R_3]{}}_{\trianglerighteq_\mu}$ and $f(f(a)), h(f(a), f(a)) \notin \mathcal{M}^{\xrightarrow[\mu_2,R_3]{}}_{\trianglerighteq_\mu}$.

Theorem 4.8 **([2])** *For a TRS R and an $\mathcal{F}$-map μ, there exists an infinite context-sensitive dependency chain if and only if R does not μ-terminate.*

Let R be a TRS, μ be an $\mathcal{F}$-map and $\mathcal{C} \subseteq \mathrm{DP}(R, \mu)$. We define $\xhookrightarrow[\mu,R,\mathcal{C}]{}$ as $(\xrightarrow[\mu^\sharp,\mathcal{C}_\mathcal{F}]{} \cup (\xrightarrow[\mu^\sharp,\mathcal{C}_\mathcal{V}]{} \cdot \trianglerighteq^\sharp_\mu) \cup \xrightarrow[\mu^\sharp,R]{})$ where $\mathcal{C}_\mathcal{F} = \mathcal{C} \cap \mathrm{DP}_\mathcal{F}(R, \mu)$ and $\mathcal{C}_\mathcal{V} = \mathcal{C} \cap \mathrm{DP}_\mathcal{V}(R, \mu)$.

Definition 4.9 [μ-$\mathcal{C}$-min] Let R be a TRS, μ be an $\mathcal{F}$-map. An infinite sequence of terms in the form $t^\sharp_1 \xhookrightarrow[\mu,R,\mathcal{C}]{} t^\sharp_2 \xhookrightarrow[\mu,R,\mathcal{C}]{} t^\sharp_3 \xhookrightarrow[\mu,R,\mathcal{C}]{} \cdots$ is called a *$\mathcal{C}$-min μ-sequence* if

$t_i \in \mathcal{M}_{\trianglerighteq_\mu}^{\overrightarrow{\mu,R}}$ for all $i \geq 1$. We use $\mathcal{C}^{\mu}_{min}(t^\sharp)$ to denote the set of all $\mathcal{C}$-min μ-sequences starting from $t^\sharp$.

Note that $\mathcal{C}^{\mu}_{min}(t^\sharp) = \emptyset$ if $t \notin \mathcal{M}_{\trianglerighteq_\mu}^{\overrightarrow{\mu,R}}$.

Example 4.10 Let $\mathcal{C} = \mathrm{DP}(R_3, \mu_2)$, the sequence $f^\sharp(a) \hookrightarrow_{\mu_2,R_3,\mathcal{C}} f^\sharp(g(f(a))) \hookrightarrow_{\mu_2,R_3,\mathcal{C}} f^\sharp(a) \hookrightarrow_{\mu_2,R_3,\mathcal{C}} \cdots$ is a $\mathcal{C}$-min μ-sequence.

Proposition 4.11 ([2]) *Given a TRS R and an $\mathcal{F}$-map μ, the following statements hold:*

(i) *If there exists an infinite context-sensitive dependency chain, then $\mathcal{C}^{\mu}_{min}(t^\sharp) \neq \emptyset$ for some $\mathcal{C} \subseteq \mathrm{DP}(R,\mu)$ and $t \in \mathcal{M}_{\trianglerighteq_\mu}^{\overrightarrow{\mu,R}}$.*

(ii) *For any sequence in $\mathcal{C}^{\mu}_{min}(t^\sharp)$, a reduction with $\xrightarrow[\mu^\sharp,R]{}$ takes place below the root while reductions with $\xrightarrow[\mu^\sharp,\mathcal{C}_\mathcal{F}]{}$ and $\xrightarrow[\mu^\sharp,\mathcal{C}_\mathcal{V}]{}$ take place at the root.*

(iii) *For any sequence in $\mathcal{C}^{\mu}_{min}(t^\sharp)$, there is at least one rule in $\mathcal{C}$ which is applied infinitely often.*

Lemma 4.12 *For two terms s and t, $s^\sharp \hookrightarrow^*_{\mu,R,\mathcal{C}} t^\sharp$ implies $s \xrightarrow[\mu,R]{}^* C_\mu[t]$ for some context C_μ.*

Proof. We use induction on the length n of the sequence. In the case that $n = 0$, it holds trivially. Let $n \geq 1$. Then we have $s^\sharp \hookrightarrow^*_{\mu,R,\mathcal{C}} u^\sharp \hookrightarrow_{\mu,R,\mathcal{C}} t^\sharp$ for some u.

• In the case that $u^\sharp \xrightarrow[\mu^\sharp,\mathcal{C}_\mathcal{F}]{} t^\sharp$, we have $u \xrightarrow[\mu,R]{} C'_\mu[t]$ by the definition of dependency pairs.

• In the case that $u^\sharp \xrightarrow[\mu^\sharp,\mathcal{C}_\mathcal{V}]{} v \trianglerighteq^\sharp_\mu t^\sharp$, we have $u \xrightarrow[\mu,R]{} C''_\mu[v]$ by the definition of dependency pairs and $v = C'''_\mu[t]$. Thus $u \xrightarrow[\mu,R]{} C''_\mu[C'''_\mu[t]] = C'_\mu[t]$.

• In the case that $u^\sharp \xrightarrow[\mu^\sharp,R]{} t^\sharp$, we have $u \xrightarrow[\mu,R]{} C'_\mu[t]$ for $C'_\mu[\ ] = \Box$.

Therefore $s \xrightarrow[\mu,R]{}^* C_\mu[u] \xrightarrow[\mu,R]{} C_\mu[C'_\mu[t]]$ by the induction hypothesis and Proposition 2.3. □

4.1 Context-Sensitive Semi-Constructor TRS

In this subsection, we discuss the decidability of μ-termination for context-sensitive semi-constructor TRSs.

Definition 4.13 [Context-Sensitive Semi-Constructor TRS] For an $\mathcal{F}$-map μ, a TRS R is a *context-sensitive semi-constructor (μ-semi-constructor) TRS* if all rules in $\mathrm{DP}_\mathcal{F}(R,\mu)$ are right ground.

For an $\mathcal{F}$-map μ, the class of μ-semi-constructor TRSs is a superclass of the class of semi-constructor TRSs from Definition 2.5 and 4.13.

For a TRS R and $\mathcal{F}$-map μ, we say R is free from the infinite variable dependency chain (FFIVDC) if and only if there exists no infinite context-sensitive dependency chain consisting of only elements in $\mathrm{DP}_{\mathcal{V}}(R,\mu)$. If R is FFIVDC, then $\mathcal{C}^{\mu}_{min}(t^\sharp)=\emptyset$ for any $\mathcal{C}\subseteq \mathrm{DP}_{\mathcal{V}}(R,\mu)$ and any term t.

Lemma 4.14 *Let μ be an $\mathcal{F}$-map. If a μ-semi-constructor TRS R is FFIVDC, then the following statements are equivalent:*

(i) *R does not μ-terminate.*

(ii) *There exists $l^\sharp \to u^\sharp \in \mathrm{DP}_{\mathcal{F}}(R,\mu)$ such that sq head-loops for $\mathcal{C}\subseteq \mathrm{DP}(R,\mu)$ and some $sq\in\mathcal{C}^{\mu}_{min}(u^\sharp)$.*

Proof. ((ii) ⇒ (i)) : It is obvious from Lemma 4.12, and Proposition 4.3. ((i) ⇒ (ii)) : By Theorem 4.8 there exists an infinite context-sensitive dependency chain. By Proposition 4.11(i), there exists a sequence $sq\in\mathcal{C}^{\mu}_{min}(t^\sharp)$. By Proposition 4.11(ii),(iii) and the fact that R is FFIVDC, there is some rule in $l^\sharp\to u^\sharp\in\mathcal{C}_{\mathcal{F}}$ which is applied at the root position in sq infinitely often.
By Definition 4.13, $u^\sharp$ is ground. Thus sq contains a subsequence $u^\sharp \xhookrightarrow[\mu,R,\mathcal{C}]{}{}^{+} u^\sharp$, which head-loops and is in $\mathcal{C}^{\mu}_{min}(u^\sharp)$. □

Theorem 4.15 *Let μ be an $\mathcal{F}$-map. If a μ-semi-constructor TRS R is FFIVDC, then μ-termination of R is decidable.*

Proof. The decision procedure for μ-termination of a μ-semi-constructor TRS R is as follows: consider all terms $u_1,u_2,\ldots,u_n$ corresponding to the right-hand sides of $\mathrm{DP}_{\mathcal{F}}(R,\mu)=\{l_i^\sharp\to u_i^\sharp \mid 1\le i\le n\}$, and simultaneously generate all μ-reduction sequences with respect to R starting from $u_1,u_2,\ldots,u_n$. The procedure halts if it enumerates all reachable terms exhaustively or it detects a μ-looping reduction sequence $u_i \xrightarrow[\mu,R]{}{}^{+} C_\mu[u_i]$ for some i.

Suppose R does not μ-terminate. By Lemma 4.14 and 4.12, we have a μ-looping reduction sequence $u_i \xrightarrow[\mu,R]{}{}^{+} C_\mu[u_i]$ for some i and C_μ, which we eventually detect. If R μ-terminates, then the execution of the reduction sequence generation eventually stops since the reduction relation is finitely branching. In the latter case, the procedure does not detect a μ-looping sequence, otherwise it contradicts to Proposition 4.3. Thus the procedure decides μ-termination of R in finitely many steps. □

We have to check the FFIVDC property in order to use Theorem 4.15. However, The FFIVDC property is not necessarily decidable. The following proposition provides a sufficient condition. The set $\mathrm{DP}^1_{\mathcal{V}}(R,\mu)$ is a subset of $\mathrm{DP}_{\mathcal{V}}(R,\mu)$ defined as follows:

$$\mathrm{DP}^1_{\mathcal{V}}(R,\mu)=\{f^\sharp(u_1,\ldots,u_k)\to x\in\mathrm{DP}_{\mathcal{V}}(R,\mu)\mid \exists i, 1\le i\le k, i\notin\mu(f), x\in Var(u_i)\}$$

Proposition 4.16 ([2]) *Let R be a TRS, μ be an $\mathcal{F}$-map and $\mathcal{C}\subseteq\mathrm{DP}^1_{\mathcal{V}}(R,\mu)$. $\mathcal{C}^{\mu}_{min}(t^\sharp)=\emptyset$ for any term t.*

If $\mathrm{DP}^1_{\mathcal{V}}(R,\mu)=\mathrm{DP}_{\mathcal{V}}(R,\mu)$ then R is FFIVDC by Proposition 4.16. Hence the following corollary directly follows from Theorem 4.15 and the fact that $\mathrm{DP}^1_{\mathcal{V}}(R,\mu)=\mathrm{DP}_{\mathcal{V}}(R,\mu)$ is decidable.

Corollary 4.17 *For an $\mathcal{F}$-map μ and a μ-semi-constructor TRS R, μ-termination of R is decidable if* $\mathrm{DP}_{\mathcal{V}}(R,\mu) = \mathrm{DP}^1_{\mathcal{V}}(R,\mu)$.

4.2 Semi-Constructor TRS

In this subsection, we try to remove FFIVDC condition from the results of the previous subsection. As a result, it appears that μ-termination of semi-constructor TRSs (not μ-semi-constructor) is decidable. The arguments of following Lemma 4.18 and 4.19 are similar to those of Lemma 3.5 and Proposition 3.6 in [3].

Lemma 4.18 *Consider a reduction* $s^\sharp = C_{\mu^\sharp}[l\theta]_p \xrightarrow[\mu^\sharp,R]{} t^\sharp = C_{\mu^\sharp}[r\theta]_p = C'[u]_q$ *where* $s,u \in \mathcal{M}_{\trianglerighteq_\mu}^{\overrightarrow{\mu,R}}$ *and* $q \in \mathcal{P}\mathrm{os}(t) \backslash \mathcal{P}\mathrm{os}_\mu(t)$. *Then one of the following statements holds*

- (i) $s \rhd u$
- (ii) $v\theta = u$ *and* $r = C''[v]_{q'}$ *for some* θ, $v \notin \mathcal{V}$, C'', *and* $q' \in \mathcal{P}\mathrm{os}(r) \setminus \mathcal{P}\mathrm{os}_\mu(r)$

Proof. Since $q \in \mathcal{P}\mathrm{os}(t) \setminus \mathcal{P}\mathrm{os}_\mu(t)$, p is not below or equal to q. In the case that p and q are in parallel positions, $s \rhd u$ trivially holds. In the case that p is above q, it is obvious that $s \rhd u$ holds or, $v\theta = u$ and $r = C''[v]_{q'}$ for some θ, $v \notin \mathcal{V}$, C''. Here the fact that $q' \in \mathcal{P}\mathrm{os}(r) \setminus \mathcal{P}\mathrm{os}_\mu(r)$ follows from $p \in \mathcal{P}\mathrm{os}_\mu(t)$ and $q \notin \mathcal{P}\mathrm{os}_\mu(t)$. □

Lemma 4.19 *Let R be a semi-constructor TRS, μ be an $\mathcal{F}$-map. For a $\mathcal{C}$-min μ-sequence* $s_1^\sharp \xrightarrow[\mu^\sharp,R]{}^* t_1^\sharp \xrightarrow[\mu^\sharp,\mathcal{C}_\mathcal{V}]{} u_1 \trianglerighteq_\mu^\sharp s_2^\sharp \xrightarrow[\mu^\sharp,R]{}^* t_2^\sharp \xrightarrow[\mu^\sharp,\mathcal{C}_\mathcal{V}]{} u_2 \trianglerighteq_\mu^\sharp \cdots$ *with no reduction by rules in $\mathcal{C}_\mathcal{F}$, one of the following statements holds for each i:*

- (i) $s_i \rhd s_{i+1}$
- (ii) *There exists* $l^\sharp \to s_{i+1}^\sharp \in \mathrm{DP}(R)$ *for some* l

Proof. Since $t_i^\sharp \xrightarrow[\mu^\sharp,\mathcal{C}_\mathcal{V}]{} u_i \trianglerighteq_\mu^\sharp s_{i+1}^\sharp$, we have $t_i^\sharp = C[s_{i+1}]_q$ for some $q \in \mathcal{P}\mathrm{os}(t_i) \setminus \mathcal{P}\mathrm{os}_\mu(t_i)$. We show (i) or the following (ii') by induction on the number n of steps of $s_i^\sharp \xrightarrow[\mu^\sharp,R]{}^n t_i^\sharp = C[s_{i+1}]$.

(ii') There exists a reduction by $l \to r$ in $s_i^\sharp \xrightarrow[\mu^\sharp,R]{}^* t_i^\sharp$ and $l^\sharp \to s_{i+1}^\sharp \in \mathrm{DP}(R)$

- In the case that $n = 0$, trivially $s_i = t_i \rhd s_{i+1}$.
- In the case that $n > 0$, let $s_i^\sharp \xrightarrow[\mu^\sharp,R]{} s'^\sharp \xrightarrow[\mu^\sharp,R]{}^{n-1} t_i^\sharp = C[s_{i+1}]_q$. By the induction hypothesis, $s' \rhd s_{i+1}$ or the condition (ii') follows. In the former case, we have $s_i \rhd s_{i+1}$, or, we have $v\theta = s_{i+1}$ and $r = C'[v]_{q'}$ for some $l \to r \in R$, θ, $v \notin \mathcal{V}$, C' and $q' \in \mathcal{P}\mathrm{os}(r) \setminus \mathcal{P}\mathrm{os}_\mu(r)$ by Lemma 4.18. Hence $v\theta = v$ due to $\mathrm{root}(s_{i+1}) \in D_R$ and Definition 2.5. Therefore (ii') follows. □

One may think that the Lemma 4.19 would hold even if $\mathrm{DP}(R)$ were replaced with $\mathrm{DP}(R,\mu)$. However, it does not hold as shown by the following counter example.

Example 4.20 Consider the semi-constructor TRS $R_4 = \{f(g(x)) \to x,\ g(b) \to g(f(g(b)))\}$, $\mu_3(f) = \{1\}$ and $\mu_3(g) = \emptyset$. There exists a $\mathcal{C}$-min μ_3-sequence

$f^\sharp(g(b)) \xrightarrow[\mu_3^\sharp, R_4]{} f^\sharp(g(f(g(b))) \xrightarrow[\mu_3^\sharp, \mathcal{C}_\mathcal{V}]{} f(g(b)) \trianglerighteq^\sharp_{\mu_3} f^\sharp(g(b))$ where $\mathcal{C}_\mathcal{V} = \mathrm{DP}_\mathcal{V}(R_4, \mu_3)$. However there exists no dependency pair having $f^\sharp(g(b))$ in the right-hand side in $\mathrm{DP}(R, \mu)$.

Lemma 4.21 *For a semi-constructor TRS R and an $\mathcal{F}$-map μ, the following statements are equivalent:*

(i) *R does not μ-terminate.*

(ii) *There exists $l^\sharp \to u^\sharp \in \mathrm{DP}(R)$ such that sq head-loops for $\mathcal{C} \subseteq \mathrm{DP}(R, \mu)$ and some $sq \in \mathcal{C}^\mu_{min}(u^\sharp)$.*

Proof. ((ii) $\Rightarrow$ (i)) : It is obvious from Lemma 4.12, and Proposition 4.3. ((i) $\Rightarrow$ (ii)) : By Theorem 4.8 there exists a context-sensitive dependency chain. By Proposition 4.11(i), there exists a sequence $sq \in \mathcal{C}^\mu_{min}(t^\sharp)$. By Proposition 4.11(ii),(iii), there exists a rule in $\mathcal{C}$ applied at root position in sq infinitely often.

• Consider the case that there exists a rule $l^\sharp \to r^\sharp \in \mathcal{C}_\mathcal{F}$ with infinite use in sq. Since u is ground by Proposition 4.11(ii) and $\mathcal{C}_\mathcal{F} \subseteq \mathrm{DP}(R)$, sq has a subsequence $u^\sharp \hookrightarrow^+_{\mu,R,\mathcal{C}} u^\sharp$.

• Otherwise, sq has an infinite subsequence without the use of the rules in $\mathcal{C}_\mathcal{F}$. The subsequence is in $\mathcal{C}^\mu_{min}(s^\sharp)$ for some $s^\sharp$. Then the condition (ii) of Lemma 4.19 holds for infinitely many i's; otherwise, we have an infinite sequence $s_k \triangleright s_{k+1} \triangleright \cdots$ for some k, which is a contradiction. Hence there exists a $l^\sharp \to u^\sharp \in \mathrm{DP}(R)$ such that $u^\sharp$ occurs more than once in sq. Thus the sequence $u^\sharp \hookrightarrow^+_{\mu,R,\mathcal{C}} u^\sharp$ appears in sq. □

Theorem 4.22 *The property μ-termination of semi-constructor TRSs is decidable.*

Proof. The decision procedure for μ-termination of a semi-constructor TRS R is as follows: consider all terms $u_1, u_2, \ldots, u_n$ corresponding to the right-hand sides of $\mathrm{DP}(R) = \{l_i^\sharp \to u_i^\sharp \mid 1 \le i \le n\}$, and simultaneously generate all μ-reduction sequences with respect to R starting from $u_1, u_2, \ldots, u_n$. The procedure halts if it enumerates all reachable terms exhaustively or it detects a μ-looping reduction sequence $u_i \xrightarrow[\mu,R]{}^+ C_\mu[u_i]$ for some i.

Suppose R does not μ-terminate. By Lemma 4.21 and 4.12, we have a μ-looping reduction sequence $u_i \xrightarrow[\mu,R]{}^+ C_\mu[u_i]$ for some i and C_μ, which we eventually detect. If R μ-terminates, then the execution of the reduction sequence generation eventually stops since the reduction relation is finitely branching. In the latter case, the procedure does not detect a μ-looping sequence, otherwise it contradicts to Proposition 4.3. Thus the procedure decides μ-termination of R in finitely many steps. □

5 Extending the Classes by DP-graphs

5.1 Innermost Termination

In this subsection, we extend the class for which innermost termination is decidable by using the dependency graph.

Lemma 5.1 *Let R be a TRS whose innermost termination is equivalent to the non-existence of an innermost dependency chain that contains infinite use of right-ground dependency pairs. Then innermost termination of R is decidable.*

Proof. We apply the procedure used in the proof of Lemma 3.9 starting with terms $u_1, u_2, \ldots, u_n$, where $u_i^\sharp$'s are all ground right-hand sides of dependency pairs. Suppose R is innermost non-terminating, then we have an innermost dependency chain with infinite use of a right-ground dependency pair. Similarly to the semi-constructor case, we have a looping sequence $u_i \xrightarrow[in,R]{}^+ C[u_i]$, which can be detected by the procedure. □

Definition 5.2 [Innermost DP-Graph [4]] The *innermost dependency graph* (innermost DP-graph for short) of a TRS R is a directed graph whose nodes are the dependency pairs and there is an arc from $s^\sharp \to t^\sharp$ to $u^\sharp \to v^\sharp$ if there exist normal substitutions σ and τ such that $t^\sharp\sigma \xrightarrow[in,R]{}^* u^\sharp\tau$ and $u^\sharp\tau$ is a normal form with respect to R.

An approximated innermost DP-graph is a graph that contains the innermost DP-graph as a subgraph. Such computable graphs are proposed in [4], for example.

Theorem 5.3 *Let R be a TRS and G be an approximated innermost DP-graph of R. If at least one node in the cycle is right-ground for every cycle of G, then innermost termination of R is decidable.*

Proof. From Lemma 5.1. □

Example 5.4 Let $R_5 = \{f(s(x)) \to g(x),\ g(s(x)) \to f(s(0))\}$. Then $\mathrm{DP}(R_5) = \{f^\sharp(s(x)) \to g^\sharp(x),\ g^\sharp(s(x)) \to f^\sharp(s(0))\}$. The innermost DP-graph of R_5 has one cycle, which contains a right-ground node [Fig. 1]. The innermost termination of R_5 is decidable by Theorem 5.3. Actually we know R_5 is innermost terminating from the procedure in the proof of Theorem 3.9 since all innermost reduction sequences from $f(s(0))$ terminate.

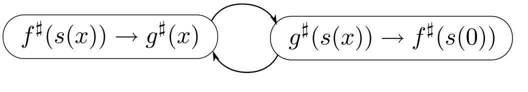

Fig. 1. The innermost DP-graph of R_5

Example 5.5 Let $R_6 = \{a \to b,\ f(a,x) \to x,\ f(x,b) \to g(x,x),\ g(b,x) \to h(f(a,a),x)\}$. Then $\mathrm{DP}(R_6) = \{f^\sharp(x,b) \to g^\sharp(x,x),\ g^\sharp(b,x) \to f^\sharp(a,a),\ g^\sharp(b,x) \to a^\sharp\}$. The innermost DP-graph of R_6 has one cycle, which contains a right-ground node [Fig. 2]. The innermost termination of R_6 is decidable by Theorem 5.3. Actually we know R_6 is not innermost terminating from the procedure in the proof of Theorem 3.9 by detecting the looping sequence $f(a,a) \xrightarrow[in,R_6]{} f(b,b) \xrightarrow[in,R_6]{} g(b,b) \xrightarrow[in,R_6]{} h(f(a,a),b)$.

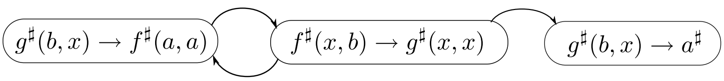

Fig. 2. The innermost DP-Graph of R_6

5.2 Context-Sensitive Termination

We extend the class for which μ-termination is decidable by using the dependency graph. The class extended in this subsection is the class that satisfies the condition of Corollary 4.17.

Lemma 5.6 *Let R be a TRS and μ be an $\mathcal{F}$-map. If μ-termination of R is equivalent to the non-existence of a context-sensitive dependency chain that contains infinite use of right-ground rules in $\mathrm{DP}_{\mathcal{F}}(R,\mu)$, then μ-termination of R is decidable.*

Proof. We apply the procedure used in the proof of Lemma 4.22 starting with terms $u_1, u_2, \ldots, u_n$, where $u_i^\sharp$'s are all ground right-hand sides of rules in $\mathrm{DP}_{\mathcal{F}}(R,\mu)$. Suppose R is non-μ-terminating, then we have a context-sensitive dependency chain with infinite use of right-ground rules in $\mathrm{DP}_{\mathcal{F}}(R,\mu)$. Similar to the μ-semi-constructor case, we have a looping sequence $u_i \xrightarrow[\mu,R]{}^{+} C_\mu[u_i]$, which can be detected by the procedure. □

Definition 5.7 [Context-Sensitive DP-Graph [2]] The *context-sensitive dependency graph* (context-sensitive DP-graph for short) of a TRS R and an $\mathcal{F}$-map μ is a directed graph whose nodes are elements of $\mathrm{DP}(R,\mu)$:

(i) There is an arc from $s \to t \in \mathrm{DP}_{\mathcal{F}}(R,\mu)$ to $u \to v \in \mathrm{DP}(R,\mu)$ if there exist substitutions σ and τ such that $t\sigma \xrightarrow[\mu^\sharp,R]{}^{*} u\tau$.

(ii) There is an arc from $s \to t \in \mathrm{DP}_{\mathcal{V}}(R,\mu)$ to each dependency pair $u \to v \in \mathrm{DP}(R,\mu)$.

Similar to the innermost case, a computable approximated context-sensitive DP-graph is proposed [2,3].

Theorem 5.8 *Let R be a TRS, μ be an $\mathcal{F}$-map and G be an approximated context-sensitive DP-graph of R. The property μ-termination of R is decidable if one of following holds for every cycle in G.*

(i) *The cycle contains at least one node that is right-ground.*

(ii) *All nodes in the cycle are elements in $\mathrm{DP}^1_{\mathcal{V}}(R,\mu)$.*

Proof. From Lemma 5.6 and Theorem 4.16. □

Example 5.9 Let $R_7 = \{h(x) \to g(x,x),\ g(a,x) \to f(b,x),\ f(x,x) \to h(a),\ a \to b\}$ and $\mu_4(f) = \mu_4(g) = \mu_4(h) = \{1\}$ [10]. Then $\mathrm{DP}(R_7,\mu_4) = \{h^\sharp(x) \to g^\sharp(x,x),\ g^\sharp(a,x) \to f^\sharp(b,x),\ f^\sharp(x,x) \to h^\sharp(a),\ f^\sharp(x,x) \to a^\sharp\}$. The context-sensitive DP-graph of R_7 and μ_4 has one cycle, which contains a right-ground node [Fig.3]. The μ_4-termination of R_7 is decidable by Theorem 5.8. Actually we know

R_7 is μ_4-terminating from the procedure in the proof of Theorem 4.15 since all μ_4-reduction sequences from $h(a)$ terminate.

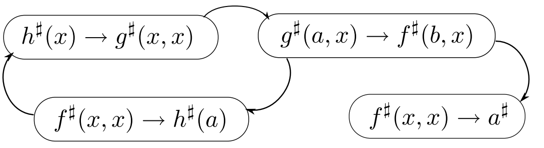

Fig. 3. The context-sensitive DP-Graph of R_7 and μ_4

Example 5.10 Let $\mu_5(g) = \{2\}$ and $\mu_5(f) = \mu_5(h) = \{1\}$. Consider the μ_5-termination of R_7. The context-sensitive DP-graph for R_7 and μ_5 is the same as the one for R_7 and μ_4 [Fig.3]. The μ_5-termination of R_7 is decidable by Theorem 5.8. By the decision procedure, we can detect the μ_5-looping sequence $h(a) \xrightarrow[\mu_5,R_7]{} g(a,a) \xrightarrow[\mu_5,R_7]{} g(a,b) \xrightarrow[\mu_5,R_7]{} f(b,b) \xrightarrow[\mu_5,R_7]{} h(a)$. Thus R_7 is non-μ_5-terminating.

The class of TRSs that satisfy the conditions of Theorem 5.8 is a superclass of the class of TRS that satisfy the conditions of Corollary 4.17. The class of semi-constructor TRSs and the class of TRSs that satisfy the conditions of Theorem 5.8 are not included in each other.

Example 5.11 The TRS R_7 with an $\mathcal{F}$-map μ_4 satisfies the condition of Theorem 5.8, but is not semi-constructor TRS. On the other hand, the TRS R_3 with an $\mathcal{F}$-map μ_2 is a semi-constructor TRS, but does not satisfy the second condition of Theorem 5.8.

6 Conclusion

We have shown that innermost termination for semi-constructor TRSs is a decidable property and μ-termination for semi-constructor TRSs and μ-semi-constructor TRSs are decidable properties.

It is not difficult to implement the procedures in proofs of Theorem 3.9, Theorem 4.15 and Theorem 4.22. The class of semi-constructor TRSs are a rather small class: approximately 3 % of the TRSs in the termination problem data base 4.0 [1] are in this class. We can extend the decidable classes if we succeed in developing a method for good approximated DP-graphs.

In the future we will study the decidability of innermost termination and μ-termination by applying known techniques for termination results [7,13]. Currently, innermost termination for shallow TRSs is known to be decidable [7]. There are several future works, studying whether the condition FFIVDC is removed from Theorem 4.15 or not, and extending the class of semi-constructor TRSs by using notions of context-sensitive DP-graph.

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