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Shape optimization problem of elastic bodies for controlling contact pressure

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Abstract

The present paper describes a numerical solution to shape optimization problems of contacting elastic bodies for controlling contact pressure. The contacting elastic problem is formulated as the minimization of potential energy with a constraint for penetration based on the large deformation theory. The contact pressure is defined as a Lagrange multiplier for the constraint of penetration in the minimization problem. An error norm of the contact pressure to a desired distribution is chosen as an objective functional. The shape derivative of the functional is theoretically evaluated. Numerical solutions are constructed by the traction method.

Keywords calculus of variations, shape optimization, elastic contact, shape derivative, traction method

Research Activity Group Mathematical Design

1. Introduction

The problem of determining the deformation and contact pressure in contacting elastic bodies appears in the design of products in which the role of contact is critical, such as tires, shoes, and implants. In order to improve the contact pressure and efficiency in the design process, the development of a numerical solution to shape optimization problems of contacting elastic bodies is in necessary.

The elastic contact problem can be formulated as a boundary-value problem involving a geometrically nonlinear elastic equation with constraint inequality for penetration [1]. An algorithm of the finite element method that passes the patch test for the contact problem was developed by Chen and Hisada [2].

Shape optimization problems for domains in which boundary value problems of partial differential equations are defined have been investigated extensively. General theories on shape derivatives are described in a number of studies [3–7]. A reshaping algorithm for shape optimization problems having a smoothing operation to compensate for a lack of regularity in the shape derivatives was previously presented by the authors [8,9]. This algorithm is referred to as the traction method because domain variations are obtained by solving the boundary value problem of an elliptic partial differential equation, such as the elastic problem, using the shape derivative for the Neumann condition [8], which is the traction condition in the elastic problem, or the Robin condition [9], which is the traction condition with a distributed spring. The mathematical considerations for the traction method are described in [10]. Another algorithm for the moving boundary using the Laplace operator on the boundary was proposed by Mohammadi and Pironneau [11].

Therefore, if a method by which to evaluate the shape derivatives for shape optimization problems could be determined, the shape optimization problems could be solved. In the present paper, we construct a shape optimization problem for controlling contact pressure as a minimization problem of the error norm between the contact pressure and a desired distribution. The goals of the present paper are to demonstrate how to evaluate the shape derivative of the error norm theoretically and to present the results obtained in a numerical example.

2. Elastic problem including contact

Throughout this paper, D denotes a fixed bounded domain (connected open set) in $\mathbb{R}^d$, $d = 2, 3$, as shown in Fig. 1. For $s = 1, 2, 3$, $r > 0$ and $M > 0$, we introduce a class of sub-domains of D, which is called the admissible set of domains $\mathcal{W}^{s,\infty}(r, M) = \mathcal{W}^{s,\infty}(D, r, M)$, in the following definition.

Definition 1 (Admissible set of domains) *A subdomain Ω of D, such that $\Omega \subseteq D$, belongs to $\mathcal{W}^{s,\infty}(r, M)$ if and only if the following conditions are satisfied. Ω is composed of disjoint subdomains Ω^{A}, Ω^{B}, and Ω^{C} such that $\Omega^{\mathrm{A}} \cap \Omega^{\mathrm{B}} = \Omega^{\mathrm{B}} \cap \Omega^{\mathrm{C}} = \Omega^{\mathrm{C}} \cap \Omega^{\mathrm{A}} = \emptyset$ and $\Omega = \Omega^{\mathrm{A}} \cup \Omega^{\mathrm{B}} \cup \Omega^{\mathrm{C}}$. The boundaries $\partial\Omega^{\mathrm{A}}$, $\partial\Omega^{\mathrm{B}}$, and $\partial\Omega^{\mathrm{C}}$ are of class*

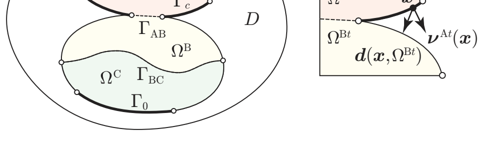

Fig. 1. Contacting elastic bodies and distance vector $\boldsymbol{d}(\boldsymbol{x}, \Omega^{\mathrm{B}t})$.

$W^{s,\infty}$ in the sense of [3]*. That is, a finite number of open balls of radius r covers the boundary, and, in each open ball, the boundary is identified with a level set of a $W^{s,\infty}$ function defined on the open ball. Moreover, all of the $W^{s,\infty}$ norms of these functions are bounded from above by M. With $\partial\Omega^{\mathrm{A}}$ and $\partial\Omega^{\mathrm{C}}$, we associate $\Gamma_{\boldsymbol{p}} \subset \partial\Omega^{\mathrm{A}}$, $\Gamma_c \subset \partial\Omega^{\mathrm{A}} \setminus \Gamma_{\boldsymbol{p}} \cup \Gamma_{\mathrm{AB}}$, and $\Gamma_0 \subset \partial\Omega^{\mathrm{C}} \setminus \Gamma_{\mathrm{BC}}$, where $\Gamma_{\mathrm{AB}} = \partial\Omega^{\mathrm{A}} \cap \partial\Omega^{\mathrm{B}}$ and $\Gamma_{\mathrm{BC}} = \partial\Omega^{\mathrm{B}} \cap \partial\Omega^{\mathrm{C}}$. Finally, suppose that* $\operatorname{meas}\Gamma_{\mathrm{BC}}$, $\operatorname{meas}\Gamma_0 > 0$.

Let $\boldsymbol{p} : \Gamma_p \times (0,T) \to \mathbb{R}^d$ be the nonzero traction. By $\boldsymbol{p}$, the elastic bodies are translated statically by $\boldsymbol{x}(\boldsymbol{X},t) : \Omega \times (0,T) \ni (\boldsymbol{X},t) \mapsto \boldsymbol{x} \in \mathbb{R}^d$ and are deform by $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ under $\boldsymbol{u} = \boldsymbol{0}$ on Γ_0. For $t \in (0,T)$, let $\Omega^t = \{\boldsymbol{X} + \boldsymbol{u} \mid \forall \boldsymbol{X} \in \Omega\}$. In Ω^t, penetration $g(\boldsymbol{u})$ can be defined as

$$g(\boldsymbol{u}) = -\boldsymbol{d}(\boldsymbol{x}, \Omega^{\mathrm{B}t}) \cdot \boldsymbol{\nu}^{\mathrm{A}t}(\boldsymbol{x}) \le 0 \quad \text{on } \Gamma_c, \tag{1}$$

where $\boldsymbol{d}(\boldsymbol{x}, \Omega^{\mathrm{B}t}) \in \mathbb{R}^d$ denotes the distance vector from $\boldsymbol{x}$ to $\Omega^{\mathrm{B}t}$, and $\boldsymbol{\nu}^{\mathrm{A}t}$ denotes the normal on $\Omega^{\mathrm{A}t}$ (Fig. 1).

Let us assume that the material is a Saint-Venant material, such that the second Piola-Kirchhoff stress $\boldsymbol{S}(\boldsymbol{u}) \in \mathbb{R}^{d\times d}$ is related to the Green-Lagrange strain $\boldsymbol{E}(\boldsymbol{u}) \in \mathbb{R}^{d\times d}$ by

$$\boldsymbol{S}(\boldsymbol{u}) = \boldsymbol{C}^m \boldsymbol{E}(\boldsymbol{u}),$$

where, using the notation $\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{\nabla}\boldsymbol{x} = (\partial x_i/\partial X_j)_{ij}$, we have

$$\boldsymbol{E}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{F}(\boldsymbol{u})\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u}) - \boldsymbol{I}\right) = \boldsymbol{E}^{\mathrm{L}}(\boldsymbol{u}) + \frac{1}{2}\boldsymbol{E}^{\mathrm{BL}}(\boldsymbol{u},\boldsymbol{u}),$$

$$\boldsymbol{E}^{\mathrm{L}}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{F}(\boldsymbol{u}) + \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\right),$$

$$\boldsymbol{E}^{\mathrm{BL}}(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{2}\left(\boldsymbol{F}(\boldsymbol{u})\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{v}) + \boldsymbol{F}(\boldsymbol{u})\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{v})\right),$$

and $\boldsymbol{C}^m \in L^\infty(D;\mathbb{R}^{d\times d\times d\times d})$, $m \in \{\mathrm{A},\mathrm{B},\mathrm{C}\}$, is the stiffness having ellipticity, that is, there exists $\alpha^m > 0$ such that $\boldsymbol{\xi}\cdot\boldsymbol{C}^m\boldsymbol{\xi} \ge \alpha^m|\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \{\boldsymbol{\xi} \in \mathbb{R}^{d\times d} \mid \boldsymbol{\xi} = \boldsymbol{\xi}^{\mathrm{T}}\}$.

Since Saint-Venant materials have potential energy, the elastic problem including contact is given by the equilibrium equation at $t = T$. Hereinafter, let $\boldsymbol{p}$ and $\boldsymbol{u}$ denote the $\boldsymbol{p}$ and $\boldsymbol{u}$ at $t = T$. The equilibrium equation at $t = T$ is originally given by the Cauchy stress. Using $\boldsymbol{S}(\boldsymbol{u})$ and $\boldsymbol{E}(\boldsymbol{u})$, we can convert the equilibrium equation to the weak form of the equilibrium equation written in the total Lagrange description by multiplying the equilibrium equation by a variational displacement $\boldsymbol{v}$ and integrating over Ω.

Problem 2 (Elastic problem including contact) *Let $\Omega \in \mathcal{W}^{s,\infty}(r,M)$, and let λ be the Lagrange multiplier, having the meaning of the contact pressure, for the constraint of penetration g in* (1)*. Then, find $(\boldsymbol{u},\lambda) \in H^1(\Omega;\mathbb{R}^{d+1})$ with respect to $\boldsymbol{p} \in H^1(D;\mathbb{R}^d)$ such that*

$$\int_\Omega \boldsymbol{S}(\boldsymbol{u})\cdot\delta\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v})\,dX = \int_{\Gamma_{\boldsymbol{p}}} \boldsymbol{p}\cdot\boldsymbol{v}\,d\gamma + \int_{\Gamma_c} \left(\lambda g_{\boldsymbol{u}}(\boldsymbol{u})\cdot\boldsymbol{v} + \mu g(\boldsymbol{u})\right)d\gamma + \int_{\Gamma_0} \left(\boldsymbol{u}\cdot\delta\boldsymbol{S}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{\nu} + \boldsymbol{v}\cdot\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nu}\right)d\gamma,$$

$$g(\boldsymbol{u}) \le 0, \quad g(\boldsymbol{v}) \le 0, \quad \lambda \ge 0, \quad \mu \ge 0 \quad \text{on } \Gamma_c$$

for all $(\boldsymbol{v},\mu) \in H^1(\Omega;\mathbb{R}^{d+1})$, where $\boldsymbol{\nu}$ denotes the normal, $g_{\boldsymbol{u}} = \partial g/\partial\boldsymbol{u}$, $\delta\boldsymbol{F}(\boldsymbol{v}) = (\partial v_i/\partial X_j)_{ij}$, and

$$\delta\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{2}\left(\delta\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{v})\boldsymbol{F}(\boldsymbol{u}) + \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\delta\boldsymbol{F}(\boldsymbol{v})\right) = \boldsymbol{E}^{\mathrm{L}}(\boldsymbol{v}) + \boldsymbol{E}^{\mathrm{BL}}(\boldsymbol{u},\boldsymbol{v}),$$

$$\delta\boldsymbol{S}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{C}^m\delta\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v}).$$

Problem 2 can be converted into velocity representation as follows.

Problem 3 (Velocity form of Problem 2) *Let $(\boldsymbol{u},\lambda)$ be the solution of Problem 2. Find $(\dot{\boldsymbol{u}},\dot{\lambda}) \in H^1(\Omega;\mathbb{R}^{d+1})$ with respect to $\dot{\boldsymbol{p}} \in H^1(D;\mathbb{R}^d)$ such that*

$$\int_\Omega \left(\dot{\boldsymbol{S}}(\boldsymbol{u})\cdot\delta\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v}) + \boldsymbol{S}(\boldsymbol{u})\cdot\delta\dot{\boldsymbol{E}}(\boldsymbol{u},\boldsymbol{v})\right)dX = \int_{\Gamma_{\boldsymbol{p}}} \dot{\boldsymbol{p}}\cdot\boldsymbol{v}\,d\gamma + \int_{\Gamma_c}\left(\dot{\lambda}g_{\boldsymbol{u}}(\boldsymbol{u})\cdot\boldsymbol{v} + \mu g_{\boldsymbol{u}}(\boldsymbol{u})\cdot\dot{\boldsymbol{u}}\right)d\gamma + \int_{\Gamma_0}\left(\dot{\boldsymbol{u}}\cdot\delta\boldsymbol{S}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{\nu} + \boldsymbol{v}\cdot\dot{\boldsymbol{S}}(\boldsymbol{u})\boldsymbol{\nu}\right)d\gamma$$

for all $(\boldsymbol{v},\mu) \in H^1(\Omega;\mathbb{R}^{d+1})$, where $\dot{(\,\cdot\,)} = \partial(\,\cdot\,)/\partial t$, and

$$\dot{\boldsymbol{S}}(\boldsymbol{u}) = \boldsymbol{C}^m\dot{\boldsymbol{E}}(\boldsymbol{u}), \quad \dot{\boldsymbol{E}}(\boldsymbol{u}) = \boldsymbol{E}^{\mathrm{L}}(\dot{\boldsymbol{u}}) + \boldsymbol{E}^{\mathrm{BL}}(\dot{\boldsymbol{u}},\boldsymbol{u}), \tag{2}$$

$$\delta\dot{\boldsymbol{E}}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{E}^{\mathrm{L}}(\dot{\boldsymbol{v}}) + \boldsymbol{E}^{\mathrm{BL}}(\dot{\boldsymbol{u}},\boldsymbol{v}) + \boldsymbol{E}^{\mathrm{BL}}(\boldsymbol{u},\dot{\boldsymbol{v}}) = \boldsymbol{E}^{\mathrm{BL}}(\dot{\boldsymbol{u}},\boldsymbol{v}). \tag{3}$$

Since Problem 3 is formulated in bilinear form for $(\dot{\boldsymbol{u}},\dot{\lambda})$ and $(\boldsymbol{v},\mu)$, its Galerkin approximation is readily considered. Then, we can apply the Newton-Raphson method to solve Problem 2 using the Galerkin approximation.

3. Shape optimization problem

As described in the introduction, let us define cost functionals and a shape optimization problem.

Definition 4 (Cost functionals J^0 and J^1) *Let $(\boldsymbol{u},\lambda)$ be a solution to Problem 2 for $\Omega \in \mathcal{W}^{s,\infty}(r,M)$. Let J^0 be the functional for the error norm between the contact pressure λ and $\alpha\lambda_0$ using a fixed element $\lambda_0 \in W^{2,\infty}(D;\mathbb{R})$ having a shape of distribution of desired contact pressure and a variable $\alpha \in \mathbb{R}$ controlling the magnitude, and let J^1 be the functional for a domain measure constraint:*

$$J^0(\Omega,\lambda,\alpha) = \int_{\Gamma_c} |\lambda - \alpha\lambda_0|^2\,d\gamma,$$

$$J^1(\Omega) = m_0 - \int_{\Omega^{\mathrm{B}}} dX,$$

where $m_0 > 0$ is a constant such that $J^1(\Omega^0) \le 0$ for some $\Omega^0 \in \mathcal{W}^{s,\infty}(r,M)$.

Problem 5 (Shape optimization) *Let $(\boldsymbol{u},\lambda)$ be a solution to Problem 2 for $\Omega \in \mathcal{W}^{s,\infty}(r,M)$ with respect to fixed $\boldsymbol{p} \in W^{2,\infty}(D;\mathbb{R}^d)$. Find Ω, for J^0 and J^1 as Definitions 4 with $\lambda_0 \in W^{2,\infty}(D;\mathbb{R})$, such that*

$$\min_{\Omega\in\mathcal{W}^{s,\infty}(r,M),\ \alpha\in\mathbb{R}} \{J^0(\Omega,\lambda,\alpha) \mid J^1(\Omega) \le 0\}.$$

Since, as a result of the correspondence with the characteristic functions for domains, $\mathcal{W}^{s,\infty}(r,M)$ is compact with respect to the $L^2(D)$ topology [3], we can approach a local solution by constructing a series of domains from some Ω^0 such that $J^1(\Omega^0) \leq 0$ by looking for descent domain variations under $J^1(\Omega) \leq 0$. Therefore, we define the following set of domain variations.

Definition 6 (Domain variations) *Let*

$$\mathcal{U}^{s,\infty} = \{\boldsymbol{\rho} \in W_0^{s,\infty}(D;\mathbb{R}^d) \mid \|\boldsymbol{\rho}\| \leq 1\}$$

be a set of domain variations, and let the new domain $\Omega^{\epsilon\boldsymbol{\rho}}$ *from* $\Omega \in \mathcal{W}^{s,\infty}(r,M)$ *be constructed with domain variation* $\boldsymbol{\rho} \in \mathcal{U}^{s,\infty}$ *and a small constant* $\epsilon > 0$ *as*

$$\Omega^{\epsilon\boldsymbol{\rho}} = \{\boldsymbol{x} + \epsilon\boldsymbol{\rho} \mid \forall \boldsymbol{x} \in \Omega\}.$$

Note 7 (Domain variations) *To guarantee that* $\Omega^{\epsilon\boldsymbol{\rho}} \in \mathcal{W}^{s,\infty}(r,M)$, *we need more constraints using a formulation similar to the contact problem. For the sake of simplicity, in the present paper, we assume that the* $W^{s,\infty}$ *norm for* $\partial\Omega$ *is sufficiently smaller than* M *and that* ϵ *is sufficiently small such that* $\Omega^{\epsilon\boldsymbol{\rho}} \in \mathcal{W}^{s,\infty}(r,M)$.

To determine $\boldsymbol{\rho}$, let us construct the following problem.

Problem 8 (Optimum domain variation) *Let* $(\boldsymbol{u},\lambda)$ *and* $(\boldsymbol{u}^{\epsilon\boldsymbol{\rho}},\lambda^{\epsilon\boldsymbol{\rho}})$ *be solutions to Problem 2 for* Ω *and* $\Omega^{\epsilon\boldsymbol{\rho}} \in \mathcal{W}^{s,\infty}(r,M)$ *with a small fixed constant* $\epsilon > 0$ *and* $\boldsymbol{\rho} \in \mathcal{U}^{s,\infty}$ *by the fixed* $\boldsymbol{p} \in W^{2,\infty}(D;\mathbb{R}^d)$. *With* J^0 *and* J^1 *as Definitions 4 with* $\lambda_0 \in W^{2,\infty}(D;\mathbb{R})$, *find* $\boldsymbol{\rho}$ *such that*

$$\min_{\boldsymbol{\rho}\in\mathcal{U}^{s,\infty},\ \alpha\in\mathbb{R}} \{J^0(\Omega^{\epsilon\boldsymbol{\rho}},\lambda^{\epsilon\boldsymbol{\rho}},\alpha) \mid J^1(\Omega^{\epsilon\boldsymbol{\rho}}) \leq 0\}.$$

4. Solution to Problem 8

Next, we evaluate the shape derivatives in the same manner as [12] and present the solution.

4.1 Shape derivatives

We define the shape derivative of J^l, $l = 0, 1$, as follows.

Definition 9 (Shape derivatives) *For* J^0 *in Problem 8, we define the shape derivative of* J^0 *with respect to* $\boldsymbol{\rho} \in \mathcal{U}^{s,\infty}$ *by*

$$J^{0\prime}(\Omega,\lambda,\alpha)(\boldsymbol{\rho}) = \lim_{\epsilon\to+0} \frac{J^0(\Omega^{\epsilon\boldsymbol{\rho}},\lambda^{\epsilon\boldsymbol{\rho}},\alpha) - J^0(\Omega,\lambda,\alpha)}{\epsilon}.$$

$J^{1\prime}(\Omega)(\boldsymbol{\rho})$ *are also defined in the same manner.*

In order to evaluate $J^{0\prime}$, we introduce the Lagrangian $\mathscr{L}^0(\Omega,\boldsymbol{u},\lambda,\boldsymbol{v}^0,\mu^0,\alpha)$ for the minimization problem of J^0 subject to Problem 2, using the Lagrange multipliers $(\boldsymbol{v}^0,\mu^0) \in H^1(\Omega;\mathbb{R}^{d+1})$ for Problem 2, as

$$\begin{aligned}\mathscr{L}^0 = {} & J^0(\Omega,\lambda,\alpha) - \int_\Omega \boldsymbol{S}(\boldsymbol{u}) \cdot \delta\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v}^0)\,dX \\ & + \int_{\Gamma_{\boldsymbol{p}}} \boldsymbol{p}\cdot\boldsymbol{v}^0 d\gamma + \int_{\Gamma_c} \left(\lambda g_{\boldsymbol{u}}(\boldsymbol{u})\cdot\boldsymbol{v}^0 + \mu^0 g(\boldsymbol{u})\right) d\gamma \\ & + \int_{\Gamma_0} \left(\boldsymbol{u}\cdot\delta\boldsymbol{S}(\boldsymbol{u},\boldsymbol{v}^0)\boldsymbol{\nu} + \boldsymbol{v}^0\cdot\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nu}\right) d\gamma.\end{aligned}$$

The stationary condition of $\mathscr{L}^0$ can be determined as follows. If Ω is a local minimum point in Problem 8 and $(\boldsymbol{u},\lambda)$ is the solution of Problem 2, we have $\alpha \in \mathbb{R}$ from $\partial J^0/\partial\alpha = 0$ as

$$\alpha = \int_{\Gamma_c} \lambda\lambda_0\,d\gamma \Big/ \int_{\Gamma_c} \lambda^2\,d\gamma. \tag{4}$$

Moreover, if $(\acute{\boldsymbol{u}},\acute{\lambda}) \in H^1(\Omega;\mathbb{R}^{d+1})$ denotes the arbitrary variations of $(\boldsymbol{u},\lambda)$ at a fixed Ω, we have

$$\begin{aligned}& \mathscr{L}^{0\prime}(\Omega,\boldsymbol{u},\lambda,\boldsymbol{v}^0,\alpha)(\acute{\boldsymbol{u}},\acute{\lambda}) \\ & = J^{0\prime}(\Omega,\lambda,\alpha)(\acute{\lambda}) - \int_\Omega \acute{\boldsymbol{S}}(\boldsymbol{u})\cdot\delta\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v}^0)\,dX \\ & \quad - \int_\Omega \boldsymbol{S}(\boldsymbol{u})\cdot\delta\acute{\boldsymbol{E}}(\boldsymbol{u},\boldsymbol{v}^0)dX \\ & \quad + \int_{\Gamma_c} \left(\acute{\lambda} g_{\boldsymbol{u}}(\boldsymbol{u})\cdot\boldsymbol{v}^0 + \mu^0 g_{\boldsymbol{u}}(\boldsymbol{u})\cdot\acute{\boldsymbol{u}}\right) d\gamma \\ & \quad + \int_{\Gamma_0} \left(\acute{\boldsymbol{u}}\cdot\delta\boldsymbol{S}(\boldsymbol{u},\boldsymbol{v}^0)\boldsymbol{\nu} + \boldsymbol{v}^0\cdot\acute{\boldsymbol{S}}(\boldsymbol{u})\boldsymbol{\nu}\right) d\gamma = 0,\end{aligned} \tag{5}$$

where $(\acute{\cdot})$ are defined by replacing $(\dot{\cdot})$ with $(\acute{\cdot})$ in (2) and (3).

If we set the adjoint problem for J^0 as follows, this solution satisfies (5).

Problem 10 (Adjoint problem for J^0) *Let* $(\boldsymbol{u},\lambda)$ *be the solution of Problem 2. Find* $(\boldsymbol{v}^0,\mu^0) \in H^1(\Omega;\mathbb{R}^{d+1})$ *such that*

$$\begin{aligned}& \int_\Omega \acute{\boldsymbol{S}}(\boldsymbol{u})\cdot\delta\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v}^0)\,dX + \int_\Omega \boldsymbol{S}(\boldsymbol{u})\cdot\delta\acute{\boldsymbol{E}}(\boldsymbol{u},\boldsymbol{v}^0)\,dX \\ & = \int_{\Gamma_c} 2(\lambda - \alpha\lambda_0)\acute{\lambda} d\gamma \\ & \qquad + \int_{\Gamma_c} \left(\acute{\lambda} g_{\boldsymbol{u}}(\boldsymbol{u})\cdot\boldsymbol{v}^0 + \mu^0 g_{\boldsymbol{u}}(\boldsymbol{u})\cdot\acute{\boldsymbol{u}}\right) d\gamma \\ & \qquad + \int_{\Gamma_0} \left(\acute{\boldsymbol{u}}\cdot\delta\boldsymbol{S}(\boldsymbol{u},\boldsymbol{v}^0)\boldsymbol{\nu} + \boldsymbol{v}^0\cdot\acute{\boldsymbol{S}}(\boldsymbol{u})\boldsymbol{\nu}\right) d\gamma\end{aligned}$$

for all $(\acute{\boldsymbol{u}},\acute{\lambda}) \in H^1(\Omega;\mathbb{R}^{d+1})$.

Comparing Problem 10 with Problem 3 reveals that $(\boldsymbol{v}^0,\mu^0)$ is computed with the coefficient matrix, which is constructed by the Galerkin method for Problem 3 and is transposed, and with the force term of $2(\lambda-\alpha\lambda_0)$ at the adjoint position to $\acute{\lambda}$.

In addition, let $\boldsymbol{\rho} \in \mathcal{U}^{s,\infty}$ be the arbitrary variation of Ω at a fixed $(\boldsymbol{u},\lambda,\boldsymbol{v}^0,\mu^0)$, which we can extend to $H^1(D;\mathbb{R}^{2(d+1)})$ [3]. Then, based on Lemmas 3 and 4 in [12], we have

$$\begin{aligned}& \mathscr{L}^{0\prime}(\Omega,\boldsymbol{u},\lambda,\boldsymbol{v}^0,\alpha)(\boldsymbol{\rho}) \\ & = \int_{\Gamma_c} G_J^0\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma + \int_{\partial\Omega} G_a^0\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma + \int_{\Gamma_{\boldsymbol{p}}} G_{\boldsymbol{p}}^0\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma \\ & \quad + \int_{\Gamma_c} G_c^0\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma + \int_{\Gamma_0} G_0^0\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma,\end{aligned}$$

where $\nabla_{\boldsymbol{\nu}} = \boldsymbol{\nabla}\cdot\boldsymbol{\nu}$, $\kappa = \Delta\boldsymbol{\nu}$, and

$$G_J^0 = \nabla_{\boldsymbol{\nu}} |\lambda - \alpha\lambda_0|^2 + \kappa |\lambda - \alpha\lambda_0|^2, \tag{6}$$

$$\begin{aligned}G_a^0 &= -\boldsymbol{S}(\boldsymbol{u})\cdot\delta\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v}^0) \\ &= \begin{cases} -\boldsymbol{S}(\boldsymbol{u})\cdot\delta\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v}^0) & \text{on } \partial\Omega\setminus\Gamma_0, \\ -\boldsymbol{E}^{\mathrm{L}}(\boldsymbol{u})\boldsymbol{\nu}\cdot\boldsymbol{\sigma}(\boldsymbol{v}^0)\boldsymbol{\nu} & \text{on } \Gamma_0, \end{cases}\end{aligned} \tag{7}$$

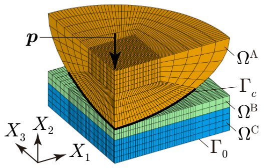

Fig. 2. Finite element model.

$$G^0_{\boldsymbol{p}} = \nabla_{\boldsymbol{\nu}}(\boldsymbol{p}\cdot\boldsymbol{v}^0) + \kappa\boldsymbol{p}\cdot\boldsymbol{v}^0, \tag{8}$$

$$G^0_c = (\nabla_{\boldsymbol{\nu}} + \kappa)\left(\lambda g_{\boldsymbol{u}}(\boldsymbol{u})\cdot\boldsymbol{v}^0 + \mu^0 g(\boldsymbol{u})\right), \tag{9}$$

$$G^0_0 = 2\boldsymbol{E}^{\mathrm{L}}(\boldsymbol{u})\boldsymbol{\nu}\cdot\boldsymbol{\sigma}(\boldsymbol{v}^0)\boldsymbol{\nu}.$$

Therefore, we have the following result.

Theorem 11 (Shape derivative for J^0) *Let $\Omega \in \mathcal{W}^{2,\infty}(r,M)$ and assume that $\Gamma_c \cup \Gamma_{\boldsymbol{p}}$ is of class $W^{3,\infty}$. Suppose that $(\boldsymbol{u},\lambda)$ is the solution of Problem 2 with respect to $\boldsymbol{p} \in W^{2,\infty}(D;\mathbb{R}^d)$ and $\boldsymbol{C}^m \in W^{2,\infty}(D;\mathbb{R}^{d\times d\times d\times d})$, and that $(\boldsymbol{v}^0,\mu^0)$ is the solution of Problem 10. Recall that α is defined by (4). Then, the shape derivative $J^{0\prime}(\Omega,\lambda,\alpha)(\boldsymbol{\rho})$ is given as*

$$\begin{aligned} &J^{0\prime}(\Omega,\lambda,\alpha)(\boldsymbol{\rho}) \\ &= \int_{\partial\Omega} G^0\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma \\ &= \int_{\Gamma_c}(G^0_J + G^0_a + G^0_c)\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma + \int_{\Gamma_{\boldsymbol{p}}}(G^0_a + G^0_{\boldsymbol{p}})\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma \\ &\quad - \int_{\Gamma_0} G^0_a\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma + \int_{\partial\Omega\setminus\Gamma_c\cup\Gamma_{\boldsymbol{p}}\cup\Gamma_0} G^0_a\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,d\gamma, \end{aligned}$$

where G^0_J, G^0_a, $G^0_{\boldsymbol{p}}$, and G^0_c are defined as (6) through (9). Furthermore, shape gradient $G^0\boldsymbol{\nu}$ belongs to $W^{1,\infty}(D;\mathbb{R}^d)$.

The shape gradient for J^1 is obtained as $G^1\boldsymbol{\nu} = \boldsymbol{\nu}$.

4.2 Solution

Using the shape gradients above, we can solve Problem 8 by means of the algorithm using the sequential quadratic programming method [13], in which the traction method is used to obtain the descent domain variations for J^0 and J^1.

5. Numerical example

Following [2], we have developed a program based on an algorithm used in the finite-element method with eight-node hexahedral elements. Figure 2 shows a quarter of the target model in which Ω^{A} is a half ellipsoid with axes of 500 mm, 250 mm, and 500 mm, and Ω^{B} and Ω^{C} are rectangular bodies of 500 mm × 50 mm × 500 mm and 500 mm × 100 mm × 500 mm, respectively. We used Young's moduli of 5×10^9, 5×10^5, and 5×10^6 [Pa] for Ω^{A}, Ω^{B}, and Ω^{C}, respectively, and a Poisson's ratio of 0.3 for Ω. We assumed that $\boldsymbol{p}$ is a nodal force of $|\boldsymbol{p}| = 1$ kN and that Γ_{AB} is the point of contact in the initial state. We also assumed that, on Γ_0, $u_2 = 0$ and symmetry conditions. Finally, we assumed that only Γ_{BC} is variable, that Γ_c is a quadratic curved surface of approximately 350 mm × 350 mm, and that $\lambda_0 = 1$. Figure 3 shows that the distribution of contact pressure became uniform.

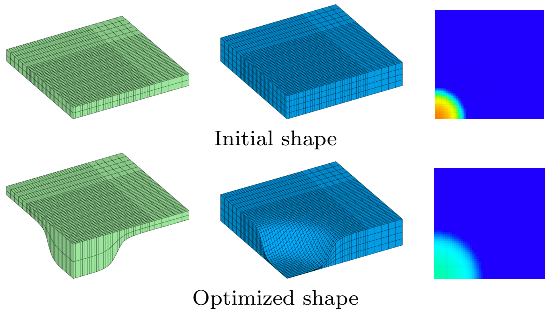

Fig. 3. Shapes of Ω^{B} and Ω^{C} and contact pressures on $\partial\Omega^{\mathrm{B}}$.

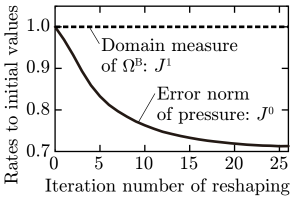

Fig. 4. Iteration history with respect to reshaping.

Furthermore, Fig. 4 shows that J^0 decayed monotonically and then converged, whereas J^1 remained unchanged.

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