JSIAM Letters Vol.2 (2010) pp.37–40 JSIAM Letters

Numerical solution to shape optimization problems for non-stationary Navier-Stokes problems

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Received November 2, 2009, Accepted February 6, 2010

Abstract

The present paper describes a numerical solution of shape optimization problems for non-stationary Navier-Stokes problems. As a concrete example, we consider the problem of finding the shape of an obstacle in a flow field in order to minimize the energy loss integral for an assigned time interval. The primary goal of the present paper is to demonstrate the evaluation of the shape derivative of the energy loss. The traction method is used for the reshaping algorithm. Numerical results show that the shapes of the circle obstacle converge to wedge shapes for the cases of Reynolds numbers of 100 and 250.

Keywords calculus of variations, shape optimization, Navier-Stokes problem, shape derivative, traction method

Research Activity Group Mathematical Design

1. Introduction

Shape optimization problems for flow fields arise in the design of fluid machines. In the present paper, we consider flow fields of incompressible viscous fluids obtained as the solutions to non-stationary Navier-Stokes problems. As a concrete example, we consider a flow field in which an obstacle exists and minimize the energy loss integral in the flow field for an assigned time interval under the domain measure constraint.

A theoretical frame work on shape derivatives for stationary Stokes and Navier-Stokes problems has been investigated since the 1970's [1–7] based on general theories on shape derivatives [8]. Numerous numerical analyses of stationary problems have been conducted.

In these theoretical and numerical studies, a difficulty arose concerning the lack of regularity of the shape derivatives, which causes oscillation of boundaries in numerical analyses. This means that shape derivatives cannot be used directly as reshaping vectors. To compensate for the lack of regularity, the authors proposed the use of solutions to boundary value problems of elliptic partial equations using the shape derivatives for the Neumann condition [9] or the Robin condition [10]. This reshaping method is referred to as the traction method. Applications of the traction method to the shape optimization problems for stationary Navier-Stokes problems were presented in previous studies [11,12]. Another method by which to overcome the irregularity of the shape derivatives for a moving boundary was proposed using the Laplace operator on the boundary [5].

As described above, in the case of stationary Navier-Stokes problems, shape optimization problems have been investigated extensively, as compared to the case of non-stationary Navier-Stokes problems, and there seems to be no references to shape derivatives for non-stationary Navier-Stokes problems as a distributed parameter system, although numerical results obtained using the definition of derivatives of discrete systems have been reported. In general, we cannot obtain solutions to stationary Navier-Stokes problems for high Reynolds numbers. Therefore, the main objective of the present paper is to derive a theoretical result for computing the shape derivatives in non-stationary Navier-Stokes problems. Then, based on these results, we present a numerical scheme and numerical results for the obstacle problem using a newly developed program based on the algorithm of the traction method.

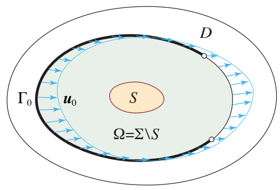


Fig. 1. Flow field Ω with obstacle S.

2. Non-stationary Navier-Stokes problem

Let us define the domains depicted in Fig. 1. D denotes a fixed bounded domain in $\mathbb{R}^d$, $d = 2, 3$. For $s = 1, 2, 3$, $r > 0$ and $M > 0$, we introduce a class of sub-domains of D, as denoted by $\mathcal{W}^{s,\infty}(r, M)$, in the following definition.

Definition 1 (Admissible set of domains) *A sub-domain Ω of D, such that $\Omega \subseteq D$, belongs to $\mathcal{W}^{s,\infty}(r, M)$*

if and only if the following conditions are satisfied. Ω *is composed of* $\Sigma \setminus S$ *such that* Σ *and* $S \subset \Sigma$ *are open subdomains of* D*. The boundaries* $\partial\Sigma$ *and* ∂S *are of class* $W^{s,\infty}$ *in the sense of* [8]*. That is, a finite number of open balls of radius* r *covers the boundary, and the boundary is identified with a level set of a* $W^{s,\infty}$ *function defined on each open ball. Moreover, the set of these functions is bounded with respect to the* $W^{s,\infty}$ *norm by* M*.*

The weak formulation of the non-stationary Navier-Stokes problem is defined as follows by referring to [13] with respect to the Dirichlet conditions.

Problem 2 (Navier-Stokes problem) *Let* $\Omega \in \mathcal{W}^{s,\infty}(r,M)$*, and let* $\Gamma_0 \subset \partial\Sigma$*. In addition,* ρ *and* μ *are supposed to be positive constants, and*

$$\boldsymbol{f} \in L^2\left((0,T); W^{1,\infty}\left(D;\mathbb{R}^d\right)\right),$$
$$\boldsymbol{p}_0 \in L^2\left((0,T); W^{2,\infty}\left(D;\mathbb{R}^d\right)\right),$$
$$\boldsymbol{u}_0 \in \left\{\boldsymbol{v} \in H^1\left((0,T); W^{3,\infty}\left(D;\mathbb{R}^d\right)\right) \;\middle|\; \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \quad \boldsymbol{v}|_{\partial S} = \boldsymbol{0}\right\}.$$

Find velocity and pressure $(\boldsymbol{u},p) \in U \times Q$ *such that*

$$\int_0^T \int_\Omega \left[\rho\dot{\boldsymbol{u}}\cdot\boldsymbol{v} + \rho\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\boldsymbol{\nabla}\left(\boldsymbol{u}-\boldsymbol{u}_0\right)\cdot\boldsymbol{\nabla}\boldsymbol{v} - p\boldsymbol{\nabla}\cdot\boldsymbol{v}\right]\mathrm{d}x\mathrm{d}t = \int_0^T \left[\int_\Omega \left(\boldsymbol{f}\cdot\boldsymbol{v} - \mu\boldsymbol{\nabla}\boldsymbol{u}_0\cdot\boldsymbol{\nabla}\boldsymbol{v}\right)\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_0\cup\partial S} \boldsymbol{p}_0\cdot\boldsymbol{v}\,\mathrm{d}\gamma\right]\mathrm{d}t,$$

$$\int_0^T \int_{\Gamma_0\cup\partial S} \left(\boldsymbol{u}-\boldsymbol{u}_0\right)\cdot\left(\mu\nabla_{\boldsymbol{\nu}}\boldsymbol{v}\right)\mathrm{d}\gamma\mathrm{d}t = 0,$$

$$\int_0^T \int_{\Gamma_0\cup\partial S} \mu\nabla_{\boldsymbol{\nu}}\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\gamma\mathrm{d}t = 0,$$

$$\int_0^T \int_\Omega q\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = 0,$$

for all $(\boldsymbol{v},q) \in V \times Q$ *where*

$$U = \left\{\boldsymbol{v} \in H^1\left((0,T); H^1\left(\Omega;\mathbb{R}^d\right)\right) \;\middle|\; \boldsymbol{v} = \boldsymbol{0} \text{ at } t=0\right\},$$
$$V = \left\{\boldsymbol{v} \in H^1\left((0,T); H^1\left(\Omega;\mathbb{R}^d\right)\right) \;\middle|\; \boldsymbol{v} = \boldsymbol{0} \text{ at } t=T\right\},$$
$$Q = \left\{q \in H^1\left((0,T); L^2\left(\Omega;\mathbb{R}\right)\right) \;\middle|\; \int_\Omega q\,\mathrm{d}x = 0\right\},$$

$\dot{(\,\cdot\,)} = \partial(\,\cdot\,)/\partial t$ *for time* $t \in \mathbb{R}$*, and* $\boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\nabla}(\partial u_i/\partial x_j)_{ij}$ *for* $\boldsymbol{x} \in \mathbb{R}^d$*. The vector* $\boldsymbol{\nu} \in \mathbb{R}^d$ *denotes outer unit normal on* $\partial\Omega$*,* $\nabla_{\boldsymbol{\nu}} = \boldsymbol{\nu}\cdot\boldsymbol{\nabla}$*, and* $\nabla_{\boldsymbol{\nu}}\boldsymbol{u} = (\sum_{j=1}^d (\partial u_i/\partial x_j)\nu_j)_i$*.*

3. Shape optimization problem

Let us consider the following concrete problem.

Definition 3 (Cost functionals: J^0 and J^1) *Let* $(\boldsymbol{u},p)$ *be the solution to Problem 2 for* $\Omega \in \mathcal{W}^{s,\infty}(r,M)$*. Let* $J^0(\Omega,\boldsymbol{u},p)$ *and* $J^1(\Omega)$ *be the energy loss and the functional for the domain measure constraint as*

$$J^0(\Omega,\boldsymbol{u},p) = \int_{t_0}^T \left(-\int_{\Gamma_0\cup\partial S} \mu\nabla_{\boldsymbol{\nu}}\boldsymbol{u}\cdot\boldsymbol{u}_0\,\mathrm{d}\gamma + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_0\cup\partial S} \boldsymbol{p}_0\cdot\boldsymbol{u}\,\mathrm{d}\gamma\right)\mathrm{d}t + \frac{1}{2}\int_\Omega \left|\boldsymbol{u}\left(\boldsymbol{x},T\right)\right|^2\,\mathrm{d}x,$$

$$J^1\left(\Omega\right) = m_0 - \int_\Omega \mathrm{d}\Omega,$$

respectively, where $\boldsymbol{u}_0$ *and* $\boldsymbol{p}_0$ *are defined in Problem 2. Here* $t_0 \in (0,T)$ *is a constant given by the designer, and* $m_0 > 0$ *is a constant such that* $J^1(\Omega^0) \le 0$ *for some* $\Omega^0 \in \mathcal{W}^{s,\infty}(r,M)$*.*

Problem 4 (Shape optimization problem) *Let* $(\boldsymbol{u},p)$ *be the solution to Problem 2 for* $\Omega \in \mathcal{W}^{s,\infty}(r,M)$*. For* J^0 *and* J^1 *as given in Definition 3, find* Ω *such that*

$$\min_{\Omega\in\mathcal{W}^{s,\infty}(r,M)} \left\{J^0\left(\Omega,\boldsymbol{u},p\right) \;\middle|\; J^1\left(\Omega\right) \le 0\right\}.$$

Since $\mathcal{W}^{s,\infty}(r,M)$ is compact with respect to the $L^2(D)$ topology [8], we can approach a local solution by constructing a series of domains from some Ω^0 such that $J^1(\Omega^0) \le 0$ by looking for descent domain variations among the admissible set $\mathcal{U}^{s,\infty}$, which is defined as follows.

Definition 5 (Domain variations) *Let*

$$\mathcal{U}^{s,\infty} = \left\{\boldsymbol{\rho} \in W_0^{s,\infty}\left(D;\mathbb{R}^d\right) \;\middle|\; \|\boldsymbol{\rho}\| \le 1\right\}$$

be a set of domain variations, and let the new domain $\Omega^{\epsilon\boldsymbol{\rho}}$ *from* $\Omega \in \mathcal{W}^{s,\infty}(r,M)$ *be constructed with domain variation* $\boldsymbol{\rho} \in \mathcal{U}^{s,\infty}$ *and a small constant* $\epsilon > 0$*, as follows:*

$$\Omega^{\epsilon\boldsymbol{\rho}} = \left\{\boldsymbol{x} + \epsilon\boldsymbol{\rho} \mid \forall\boldsymbol{x} \in \Omega\right\}.$$

Problem 6 (Optimum domain variation) *Let* $(\boldsymbol{u}, p)$ *and* $(\boldsymbol{u}^{\epsilon\boldsymbol{\rho}}, p^{\epsilon\boldsymbol{\rho}})$ *be solutions to Problem 2 for* Ω *and* $\Omega^{\epsilon\boldsymbol{\rho}} \in \mathcal{W}^{s,\infty}(r,M)$*, respectively, with a small fixed constant* $\epsilon > 0$ *and* $\boldsymbol{\rho} \in \mathcal{U}^{s,\infty}$*. For* J^0 *and* J^1*, as given in Definition 3, find* $\boldsymbol{\rho}$ *such that*

$$\min_{\boldsymbol{\rho}\in\mathcal{U}^{s,\infty}} \left\{J^0\left(\Omega^{\epsilon\boldsymbol{\rho}},\boldsymbol{u}^{\epsilon\boldsymbol{\rho}},p^{\epsilon\boldsymbol{\rho}}\right) \;\middle|\; J^1\left(\Omega^{\epsilon\boldsymbol{\rho}}\right) \le 0\right\}.$$

4. Shape derivatives

Let us define the shape derivatives as the Gâteaux derivatives with respect to domain variation. To solve Problem 6 using a gradient-based method, we need to evaluate the shape derivatives of the cost functionals.

Let us introduce the Lagrangian $\mathscr{L}^0$ for J^0 using the Lagrange multipliers $(\boldsymbol{v}^0 - \boldsymbol{u}, q^0) \in V \times Q$ for Problem 2 as

$$\mathscr{L}^0\left(\Omega,\boldsymbol{u},p,\boldsymbol{v}^0,q^0\right) = \int_0^T \left\{\int_\Omega \left[-\rho\dot{\boldsymbol{u}}\cdot\boldsymbol{v}^0 - \rho\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}\cdot\boldsymbol{v}^0 - \mu\boldsymbol{\nabla}\left(\boldsymbol{u}-\boldsymbol{u}_0\right)\cdot\boldsymbol{\nabla}\boldsymbol{v}^0 + p\boldsymbol{\nabla}\cdot\boldsymbol{v}^0 + \boldsymbol{f}\cdot\left(\boldsymbol{v}^0+\chi_0\boldsymbol{u}\right) - \mu\boldsymbol{\nabla}\boldsymbol{u}_0\cdot\boldsymbol{\nabla}\boldsymbol{v}^0 + q^0\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\right]\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_0\cup\partial S} \boldsymbol{p}_0\cdot\left(\boldsymbol{v}^0+\chi_0\boldsymbol{u}\right)\mathrm{d}\gamma\right.$$

$$+\int_{\Gamma_0\cup\partial S}\left[(\boldsymbol{u}-\boldsymbol{u}_0)\cdot\left(\mu\nabla_{\boldsymbol{\nu}}\boldsymbol{v}^0\right)\right.$$
$$\left.\left.+\mu\nabla_{\boldsymbol{\nu}}\boldsymbol{u}\cdot\left(\boldsymbol{v}^0-\chi_0\boldsymbol{u}_0\right)\right]\mathrm{d}\gamma\right\}\mathrm{d}t$$
$$+\frac{1}{2}\int_\Omega|\boldsymbol{u}(\boldsymbol{x},T)|^2\,\mathrm{d}x$$

where $\chi_0=\chi_{(t_0,T)}$, and $\chi_{(\,\cdot\,)}$ denotes the characteristic function. If Ω is a local minimum point of Problem 6 and $(\boldsymbol{u},p)$ is the solution to Problem 2, then the Gâteaux derivatives of $\mathscr{L}^0$ with respect to $(\acute{\boldsymbol{u}},\acute{p})\in U\times Q$ are

$$\mathscr{L}^{0\prime}\left(\Omega,\boldsymbol{u},p,\boldsymbol{v}^0,q^0\right)(\acute{\boldsymbol{u}},\acute{p})$$
$$=\int_0^T\left\{\int_\Omega\left[\rho\acute{\boldsymbol{u}}\cdot\dot{\boldsymbol{v}}^0-\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\acute{\boldsymbol{u}}\cdot\boldsymbol{v}^0-\mu\boldsymbol{\nabla}\acute{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{v}^0\right.\right.$$
$$\left.+\acute{p}\boldsymbol{\nabla}\cdot\boldsymbol{v}^0+\chi_0\boldsymbol{f}\cdot\acute{\boldsymbol{u}}+q^0\boldsymbol{\nabla}\cdot\acute{\boldsymbol{u}}\right]\mathrm{d}x$$
$$+\int_{\partial\Omega\setminus\Gamma_0\cup\partial S}\chi_0\boldsymbol{p}_0\cdot\acute{\boldsymbol{u}}\,\mathrm{d}\gamma$$
$$+\int_{\Gamma_0\cup\partial S}\left[\acute{\boldsymbol{u}}\cdot\left(\mu\nabla_{\boldsymbol{\nu}}\boldsymbol{v}^0\right)+\mu\nabla_{\boldsymbol{\nu}}\acute{\boldsymbol{u}}\right.$$
$$\left.\left.\cdot\left(\boldsymbol{v}^0-\chi_0\boldsymbol{u}_0\right)\right]\mathrm{d}\gamma\right\}\mathrm{d}t=0.$$

Here, let us define an adjoint problem for J^0 as below.

Problem 7 (Adjoint problem for J^0) *Let $(\boldsymbol{u},p)$ be the solution to Problem 2 for $\Omega\in\mathcal{W}^{s,\infty}(r,M)$. Find adjoint velocity and pressure $(\boldsymbol{v}^0-\boldsymbol{u},q^0)\in V\times Q$ such that*

$$\int_0^T\int_\Omega\left[-\rho\dot{\boldsymbol{v}}^0\cdot\acute{\boldsymbol{u}}+\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v}^0\cdot\acute{\boldsymbol{u}}\right.$$
$$\left.+\mu\boldsymbol{\nabla}\left(\boldsymbol{v}^0-\chi_0\boldsymbol{u}_0\right)\cdot\boldsymbol{\nabla}\acute{\boldsymbol{u}}-q^0\boldsymbol{\nabla}\cdot\acute{\boldsymbol{u}}\right]\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\left[\int_\Omega(\chi_0\boldsymbol{f}\cdot\acute{\boldsymbol{u}}-\mu\chi_0\boldsymbol{\nabla}\boldsymbol{u}_0\cdot\boldsymbol{\nabla}\acute{\boldsymbol{u}})\,\mathrm{d}x\right.$$
$$\left.+\int_{\partial\Omega\setminus\Gamma_0\cup\partial S}\chi_0\boldsymbol{p}_0\cdot\acute{\boldsymbol{u}}\,\mathrm{d}\gamma\right]\mathrm{d}t,$$
$$\int_0^T\int_{\Gamma_0\cup\partial S}\left(\boldsymbol{v}^0-\chi_0\boldsymbol{u}_0\right)\cdot(\mu\nabla_{\boldsymbol{\nu}}\acute{\boldsymbol{u}})\,\mathrm{d}\gamma\mathrm{d}t=0,$$
$$\int_0^T\int_{\Gamma_0\cup\partial S}\mu\nabla_{\boldsymbol{\nu}}\boldsymbol{v}^0\cdot\acute{\boldsymbol{u}}\,\mathrm{d}\gamma\mathrm{d}t=0,$$
$$\int_0^T\int_\Omega\acute{p}\boldsymbol{\nabla}\cdot\boldsymbol{v}^0\,\mathrm{d}x\mathrm{d}t=0,$$

for all $(\acute{\boldsymbol{u}},\acute{p})\in U\times Q$.

Note that this is a linear problem with respect to $(\boldsymbol{v}^0,q^0)$.

On the other hand, if $(\boldsymbol{u},p)$ and $(\boldsymbol{v}^0,q^0)$ are the solutions of Problems 2 and 7, by Lemmas 3 and 4 in [13], the shape derivatives of $\mathscr{L}^0$ with respect to $\boldsymbol{\rho}\in\mathcal{U}^{s,\infty}$ are obtained as

$$\mathscr{L}^{0\prime}\left(\Omega,\boldsymbol{u},p,\boldsymbol{v}^0,q^0\right)(\boldsymbol{\rho})$$
$$=\int_{\partial\Omega}G^0\left(\boldsymbol{u},p,\boldsymbol{v}^0,q^0\right)\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,\mathrm{d}\gamma$$
$$=\int_{\Gamma_0\cup\partial S}\left[-G_a^0\left(\boldsymbol{u}-\boldsymbol{u}_0,\boldsymbol{v}^0-\boldsymbol{u}_0\right)+G_b^0\left(\dot{\boldsymbol{u}},\boldsymbol{v}^0\right)\right.$$
$$+G_c^0(\boldsymbol{u})\left(\boldsymbol{u},\boldsymbol{v}^0\right)+G_{\boldsymbol{f}}^0\left(\boldsymbol{u}+\boldsymbol{v}^0\right)$$
$$\left.+G_T^0(\boldsymbol{u})\right]\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,\mathrm{d}\gamma$$
$$+\int_{\partial\Omega\setminus\Gamma_0\cup\partial S}\left[G_a^0\left(\boldsymbol{u},\boldsymbol{v}^0\right)+G_b^0\left(\dot{\boldsymbol{u}},\boldsymbol{v}^0\right)\right.$$
$$+G_c^0(\boldsymbol{u})\left(\boldsymbol{u},\boldsymbol{v}^0\right)+G_{\boldsymbol{f}}^0\left(\boldsymbol{u}+\boldsymbol{v}^0\right)$$
$$\left.+G_T^0(\boldsymbol{u})+G_{\boldsymbol{p}_0}^0\left(\boldsymbol{u}+\boldsymbol{v}^0\right)\right]$$
$$\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,\mathrm{d}\gamma, \tag{1}$$

by using $\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{u}_0)\cdot\boldsymbol{\nabla}\boldsymbol{u}=\nabla_{\boldsymbol{\nu}}(\boldsymbol{u}-\boldsymbol{u}_0)\cdot\nabla_{\boldsymbol{\nu}}\boldsymbol{u}$, where

$$G_a^0(\boldsymbol{u},\boldsymbol{v})=-\int_0^T\mu\boldsymbol{\nabla}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}t,$$
$$G_b^0(\boldsymbol{u},\boldsymbol{v})=-\int_0^T\rho\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}t,$$
$$G_c^0(\boldsymbol{u})(\boldsymbol{v},\boldsymbol{w})=-\int_0^T(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}t,$$
$$G_{\boldsymbol{f}}^0(\boldsymbol{u})=\int_0^T\boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}t,\quad G_T^0(\boldsymbol{u})=\frac{1}{2}|\boldsymbol{u}(\boldsymbol{x},T)|^2,$$
$$G_{\boldsymbol{p}_0}^0(\boldsymbol{u})=\int_0^T\left[\nabla_{\boldsymbol{\nu}}(\boldsymbol{p}_0\cdot\boldsymbol{u})+\kappa\boldsymbol{p}_0\cdot\boldsymbol{u}\right]\mathrm{d}t.$$

From the stationary conditions of $\mathscr{L}^0$, we have the following primary result.

Theorem 8 (Shape derivative of energy loss) *Let $\Omega\in\mathcal{W}^{2,\infty}(r,M)$, in which the subboundary of $\partial\Omega\setminus\Gamma_0\cup\partial S$, such that $\boldsymbol{p}\neq\boldsymbol{0}$ is of the $\mathcal{W}^{3,\infty}(r,M)$ class. Suppose that $(\boldsymbol{u},p)$ is the solution of Problem 2 and that $(\boldsymbol{v}^0,q^0)$ is the solution of Problem 7. Then, the shape derivative of J^0 with respect to $\boldsymbol{\rho}\in\mathcal{U}^{s,\infty}$ is given as*

$$J^{0\prime}(\Omega,\boldsymbol{u},p)(\boldsymbol{\rho})=\int_{\partial\Omega}G^0\left(\boldsymbol{u},p,\boldsymbol{v}^0,q^0\right)\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,\mathrm{d}\gamma,$$

where G^0 is defined in (1). Furthermore, shape gradient $G^0\boldsymbol{\nu}$ belongs to $W^{1,\infty}(D;\mathbb{R}^d)$.

Corollary 9 (In case of $D=\Sigma$) *If only S is variable in Theorem 8, then*

$$J^{0\prime}(\Omega,\boldsymbol{u},p)(\boldsymbol{\rho})=\int_{\partial S}G_0^0\left(\boldsymbol{u},\boldsymbol{v}^0\right)\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,\mathrm{d}\gamma,$$

where $G_0^0\boldsymbol{\nu}\in W^{1,\infty}(D;\mathbb{R}^d)$ is given as

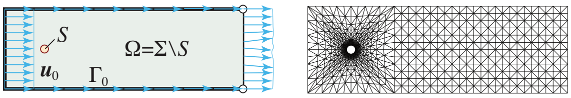
$$G_0^0(\boldsymbol{u},\boldsymbol{v})=\int_0^T\mu\nabla_{\boldsymbol{\nu}}\boldsymbol{u}\cdot\nabla_{\boldsymbol{\nu}}\boldsymbol{v}\,\mathrm{d}t.$$

For J^1, we have $J^{1\prime}(\Omega)(\boldsymbol{\rho})=\int_{\partial\Omega}G^1\boldsymbol{\nu}\cdot\boldsymbol{\rho}\,\mathrm{d}\gamma$, where $G^1=1$.

5. Numerical scheme

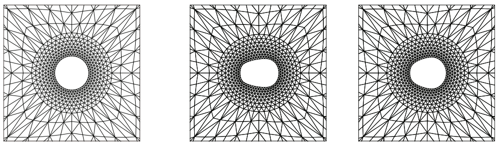
Based on the above results, we use the following scheme. Suppose that $(\,\cdot\,)_h$ denotes the Galerkin approximation.

- (i) Solve Problems 2 and 7 using the finite element method, as shown below, and calculate G_h^0.
- (ii) Compute the domain variations $\boldsymbol{\rho}_G^0$ and $\boldsymbol{\rho}_G^1$ decreasing J^0 and J^1, respectively, using G_h^0 and $G_h^1=1$, respectively, using the traction method implemented by the finite element method.
- (iii) Solve Problem 6 using the algorithm based on the SQP method [14].



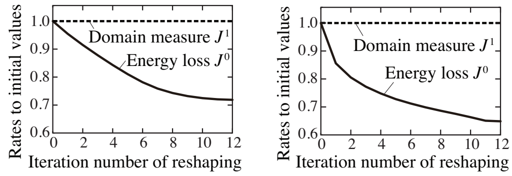
(a) Outline (b) Finite element mesh

Fig. 2. Example setting.



(a) Initial (circle) (b) Optimum Re = 100 (c) Optimum Re = 250

Fig. 3. Initial and optimum shapes.



(a) Re = 100 (b) Re = 250

Fig. 4. Iteration histories with respect to shape variation.

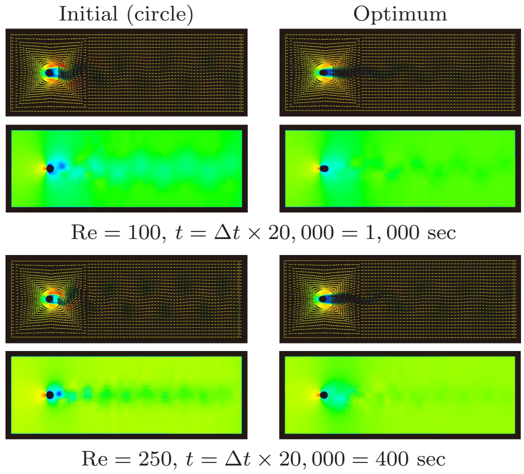


Fig. 5. Velocities $\boldsymbol{u}$ (upper) and pressures p (lower).

For simplicity, let $d = 2$ and $\boldsymbol{f} = \mathbf{0}$. The P2 + babble element and the P1 element are employed for $\boldsymbol{u}$ and p in Problem 2, respectively. In addition, we use a semi-implicit time-advancement scheme with the Adams-Bashforth method for the convection term and the Crank-Nicolson scheme for the other terms.

For Problem 7, we use the same elements for Problem 2, and the Crank-Nicolson scheme for all terms. We make use of the P1 element for the domain variation in the traction method.

6. Numerical examples

We developed a program based on the above scheme and obtained numerical results for the obstacle problem. Fig. 2 shows an outline of the example setting and a finite element mesh. We assumed that the boundary conditions are given as follows:

$$\boldsymbol{u}_0 = \begin{pmatrix} u_{01} \\ 0 \end{pmatrix} \theta(t), \quad u_{01} = \frac{\mu \mathrm{Re}}{\rho d_0},$$

$$\theta(t) = \begin{cases} ct & \left(t \in \left(0, \dfrac{1}{c}\right)\right) \\ 1 & \left(t \in \left(\dfrac{1}{c}, T\right)\right) \end{cases},$$

where $c = 1/(100\Delta t)$ on the left-hand side of $\partial\Sigma$, $p_{01} = 0$ and $u_{02} = 0$ on the upper and lower sides of $\partial\Sigma$, and $\boldsymbol{p}_0 = \mathbf{0}$ on right-hand side of $\partial\Sigma$. Here d_0 denotes the diameter of the initial shape of the obstacle in Fig. 3 (a), and Δt means the time step size. We analyzed the cases for Reynolds numbers of Re = 100 and 250. For Re = 100, $T = \Delta t N = 0.05 \times 20,000 = 1,000$ sec, and $t_0 = \Delta t \times 6,000 = 300$ sec. For Re=250, $T = \Delta t N = 0.02 \times 20,000 = 400$ sec, and $t_0 = \Delta t \times 6,000 = 120$ sec. The assumption of Corollary 9 is satisfied. The traction method of Neumann type was used for the domain variation.

Fig. 4 shows that the energy loss decreases monotonically while satisfying the domain measure constraint, and Fig. 5 shows that the size of the Kármán vortex decreased compared to the initial shapes.

Acknowledgments

The present study was supported by JSPS KAKENHI (20540113).

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