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Regular solution to topology optimization problems of continua

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Abstract

The present paper describes a numerical solution to topology optimization problems of domains in which boundary value problems of partial differential equations are defined. Density raised to a power is used instead of the characteristic function of the domain. A design variable is set by a function on a fixed domain which is converted to the density by a sigmoidal function. Evaluation of derivatives of cost functions with respect to the design variable appear as stationary conditions of the Lagrangians. A numerical solution is constructed by a gradient method in a design space for the design variable.

Keywords calculus of variations, boundary value problem, topology optimization, density method, H1 gradient method

Research Activity Group Mathematical Design

1. Introduction

A problem finding the optimum layout of holes in domain in which boundary value problem is defined is called the topology optimization problem of continua [1]. In the present paper, the Poisson problem is considered as a boundary value problem for the simplicity.

One of the most natural expressions of a topology optimization problem uses the characteristic function of the domain as a design variable. Let D be a fixed domain in $\mathbb{R}^d$, $d \in \{2,3\}$, $\Gamma_{\mathrm{D}} \subset \partial D$ be a fixed subboundary, define $\Gamma_{\mathrm{N}} = \partial D \setminus \bar{\Gamma}_{\mathrm{D}}$, and let functions f, p, and u_{D} be fixed functions on D. Denoting the characteristic function for $\Omega \subseteq D$ by $\chi_\Omega \in X = \{\chi \in L^\infty(D;\mathbb{R}) \mid 0 \le \chi \le 1 \text{ a.e. in } D\}$, the normal by $\boldsymbol{\nu}$, and $\partial_{\boldsymbol{\nu}} = \boldsymbol{\nu} \cdot \boldsymbol{\nabla}$, we can write the topology optimization problem as follows.

Problem 1 (Topology optimization problem)
For each $\chi_\Omega \in X$, let $u \in H^1(D;\mathbb{R})$ satisfy

$$-\boldsymbol{\nabla} \cdot (\chi_\Omega \boldsymbol{\nabla} u) = f \quad \text{in } D,$$

$$\chi_\Omega \partial_{\boldsymbol{\nu}} u = p \quad \text{on } \Gamma_{\mathrm{N}}, \quad u = u_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}}.$$

Find χ_Ω such that

$$\min_{\chi_\Omega \in X} \{J^0(\chi_\Omega, u) \mid \boldsymbol{J}(\chi_\Omega, u) \le \boldsymbol{0}\},$$

where J^0 and $\boldsymbol{J} = (J^1, \ldots, J^m)^\top$, $J^l \in C^0(X \times H^1(D;\mathbb{R});\mathbb{R})$ are cost functions.

However, it has been shown that Problem 1 does not always have a solution [2].

To avoid the non-existence of a solution, the idea of assuming that D consists of a micro-structure having rectangular holes was presented [3]. In this formulation, χ_Ω is substituted by a function evaluated by homogenization theory. A numerical scheme was demonstrated using the finite element method [4].

Moreover, it has been found that introducing a density $\phi : D \to [0,1]$ and a constant $\alpha > 1$, and replacing χ_Ω by ϕ^α obtains a similar result to that from the micro-structure model. This method is called the SIMP (solid isotropic material with penalization) method [1,5]. The meaning of the penalization is that the intermediate density is weakened by the nonlinear function ϕ^α.

However, numerical instabilities such as checkerboard patterns or mesh-dependencies are observed if the parameters of micro-structure or the density is constructed by a constant function in each finite element and they are varied using a gradient method [6,7]. If the design parameters are approximated by continuous functions [8], it is known that a numerical instability, such as the so-called island phenomena, is observed [9]. In addition, although many numerical schemes have been proposed to overcome such numerical instabilities [10,11], regularity in the sense of functional analysis has not been shown.

In the present paper, a regular solution which is free of numerical instability is presented, where the meaning of regular is as follows. First, the admissible set of a design variable is defined. Then, a solution is regular if any point obtained by the solution from a point in the admissible set also belongs to the admissible set.

2. Admissible set of design variable

To define a boundary value problem, the Lipschitz boundary is required for a domain. Accordingly, to determine a boundary from a level set of density ϕ, ϕ has to be an element of $W^{1,\infty}(D;\mathbb{R})$, where D also has a Lipschitz boundary. To avoid the restriction of the range of ϕ to $[0,1]$, we introduce a function θ belonging to

$$S = \{\theta \in H^1(D;\mathbb{R}) \mid \theta \in W^{1,\infty}(D;\mathbb{R}), \|\theta\|_{1,\infty} \le M\}$$

as a design variable and relate it to the density ϕ by a sigmoidal function, for which

$$\phi(\theta) = \frac{1}{\pi}\tan^{-1}\theta + \frac{1}{2} \qquad (1)$$

is used in the present paper. Because M is initially fixed, the set S is weakly compact in $H^1(D;\mathbb{R})$. If $\|\theta\|_{1,\infty} = \|\theta\|_{W^{1,\infty}(D;\mathbb{R})} \le M$ becomes active, let this condition be included among the constraints. In the present paper, let M be sufficiently large for simplicity.

To avoid loss of regularity on $\partial\Gamma_{\mathrm{D}}$ and a set $\Upsilon \subset \partial D$ on which $u \notin H^{k+2}(D;\mathbb{R})$ and $v^l \notin H^{3-k}(D;\mathbb{R})$, $l \in \{0,1,\dots,m\}$, $k \in \{0,1\}$, in Problems 2 and 6 respectively, we provide a fixed neighborhood $U_r = \{\boldsymbol{x} \in D \mid |\boldsymbol{x}-\boldsymbol{y}| < r,\ \boldsymbol{y} \in \partial\Gamma_{\mathrm{D}} \cup \Upsilon\}$ for a small positive constant r, and $D_r = D \setminus U_r$.

We call S the admissible set of the design variable. We call $H^1(D;\mathbb{R})$ the design space with respect to S because a Hilbert space is required for the gradient method.

3. SIMP problem

Let us consider a topology optimization problem of SIMP type by using $\theta \in S$. First, we define a boundary value problem as follows.

Problem 2 (Poisson problem) *For some $k \in \{0, 1\}$, let $f \in H^k(D,\mathbb{R})$, $p \in H^{k+1/2}(\Gamma_{\mathrm{N}};\mathbb{R})$ and $u_{\mathrm{D}} \in H^{k+2}(D;\mathbb{R})$ be fixed functions, and $\phi(\theta)$ as in (1). Find $u \in H^1(D;\mathbb{R})$ such that*

$$-\boldsymbol{\nabla}\cdot(\phi^\alpha(\theta)\boldsymbol{\nabla}u) = f \quad \text{in } D,$$

$$\phi^\alpha(\theta)\partial_{\boldsymbol{\nu}}u = p \quad \text{on } \Gamma_{\mathrm{N}}, \quad u = u_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}}.$$

From the assumptions for Problem 2, we have $u|_{D_r}$ belongs to $H^{k+2}(D_r;\mathbb{R})$. Moreover, Problem 2 gives the Lagrangian as

$$\begin{aligned}&\mathscr{L}^{\mathrm{BV}}(\theta,v,w)\\ &= \int_D \phi^\alpha(\theta)\boldsymbol{\nabla}v\cdot\boldsymbol{\nabla}w\,\mathrm{d}x - \int_D fw\,\mathrm{d}x\\ &\quad - \int_{\Gamma_{\mathrm{D}}} wp\,\mathrm{d}\gamma - \int_{\Gamma_{\mathrm{D}}} (v-u_{\mathrm{D}})\phi^\alpha(\theta)\partial_{\boldsymbol{\nu}}w\,d\gamma\\ &\quad - \int_{\Gamma_{\mathrm{D}}} w\phi^\alpha(\theta)\partial_{\boldsymbol{\nu}}v\,d\gamma \qquad (2)\end{aligned}$$

for all $v, w \in H^1(D;\mathbb{R})$ [12]. If u is a stationary point such that

$$\mathscr{L}^{\mathrm{BV}}(\theta,u,w) = 0$$

for all $w \in H^1(D;\mathbb{R})$, u is the solution to Problem 2.

Using θ and u, we define cost functions. Let us use the following notation: $(\,\cdot\,)_\theta = \partial(\,\cdot\,)/\partial\theta$ and $(\,\cdot\,)_u = \partial(\,\cdot\,)/\partial u$.

Definition 3 (Cost functions) *For $(\theta,u) \in S \times H^1(D;\mathbb{R}) = Y$ and $S \times H^1(D_r;\mathbb{R}) = Y_r$, let $g^l \in C^1(Y;L^1(D;\mathbb{R}))$ and $j^l \in C^1(Y;L^1(\partial D;\mathbb{R}))$, $l \in \{0,1,\dots,m\}$, are given functions such that $g^l_\theta \in C^0(Y_r;H^1(D_r;\mathbb{R}))$, $g^l_u \in C^0(Y;H^{1-k}(D;\mathbb{R}))$, $k \in \{0,1\}$ used in Problem 2, $j^l_\theta \in C^0(Y_r;H^{3/2}(\partial D_r;\mathbb{R}))$ and $j^l_u \in C^0(Y;H^{3/2-k}(\partial D;\mathbb{R}))$. We call J^0 and $\boldsymbol{J} = (J^1,\dots,J^m)^\top$,*

$$J^l(\theta,u) = \int_D g^l(\theta,u)\,\mathrm{d}x + \int_{\partial D} j^l(\theta,u)\,\mathrm{d}\gamma + c^l,$$

the cost functions, where J^0 is the objective function and $\boldsymbol{J}$ are the constraint functions.

We assume that constants c^l, $l \in \{0,1,\dots,m\}$, are set such that some $\theta \in S$ satisfies $\boldsymbol{J} \le \boldsymbol{0}$.

Based on the definitions above, we consider a SIMP problem as follows.

Problem 4 (SIMP problem) *Let u be the solution to Problem 2 for $\theta \in S$. Find θ such that*

$$\min_{\theta\in S}\{J^0(\theta,u) \mid \boldsymbol{J}(\theta,u) \le \boldsymbol{0}\}.$$

4. $\boldsymbol{\theta}$ derivatives of $\boldsymbol{J^l}$

To solve Problem 4 by a gradient method, the Fréchet derivatives of J^l with respect to θ are required. Let $\rho \in H^1(D;\mathbb{R})$ be a variation of θ and denote

$$\theta^\rho = \theta + \rho$$

as an updated function of θ. Also, let u^ρ be the solution to Problem 2 for θ^ρ.

Definition 5 (θ derivative of J^l) *For $J^l(\theta,u(\theta)) : H^1(D;\mathbb{R}) \supset S \ni \theta \mapsto J^l \in \mathbb{R}$, if $J^{l\prime}(\theta,u)[\rho]$ such that*

$$J^l(\theta^\rho,u^\rho) = J^l(\theta,u) + J^{l\prime}(\theta,u)[\rho] + o(\|\rho\|_{1,2})$$

is a bounded linear functional for all $\rho \in H^1(D;\mathbb{R})$, we call $J^{l\prime}(\theta,u) \in H^{1\prime}(D;\mathbb{R})$ the θ derivative of J^l at θ, and denoting as $J^{l\prime}(\theta,u)[\rho] = \langle G^l(\theta,u),\rho\rangle$ with the notation of dual product, $G^l(\theta,u) \in H^{1\prime}(D;\mathbb{R})$ the θ gradient.

Let us evaluate $G^l(\theta,u)$. The Lagrangian for $J^l(\theta,u)$ subject to Problem 2 is defined by

$$\begin{aligned}\mathscr{L}^l(\theta,u,v^l) &= \int_D g^l(\theta,u)\mathrm{d}x + \int_{\partial D} j^l(\theta,u)\mathrm{d}\gamma + c^l\\ &\quad - \mathscr{L}^{\mathrm{BV}}(\theta,u,v^l),\end{aligned}$$

where $v^l \in H^1(D;\mathbb{R})$ is used as the Lagrange multiplier for Problem 2, and $\mathscr{L}^{\mathrm{BV}}(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is as in (2).

If u is the solution to Problem 2, the stationary condition such that $\mathscr{L}^l_{v^l}(\theta,u,v^l)[w] = \mathscr{L}^{\mathrm{BV}}(\theta,u,w) = 0$ for all $w \in H^1(D;\mathbb{R})$ is satisfied.

The stationary condition such that

$$\begin{aligned}&\mathscr{L}^l_u(\theta,u,v^l)[w]\\ &= \langle \mathscr{L}^l_u(\theta,u,v^l),w\rangle\\ &= \int_D g^l_u w\,\mathrm{d}x + \int_{\partial D} j^l_u w\,\mathrm{d}\gamma - \int_D \phi^\alpha(\theta)\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}v^l\,\mathrm{d}x\\ &\quad + \int_{\Gamma_{\mathrm{D}}} \phi^\alpha(\theta)w\partial_{\boldsymbol{\nu}}v^l\,\mathrm{d}\gamma + \int_{\Gamma_{\mathrm{D}}} \phi^\alpha(\theta)v^l\partial_{\boldsymbol{\nu}}w\,\mathrm{d}\gamma\\ &= 0\end{aligned}$$

for all $w \in H^1(D;\mathbb{R})$ is satisfied if $v^l \in H^1(D;\mathbb{R})$ is the solution of the following adjoint problem.

Problem 6 (Adjoint problem for J^l) *For the solution u to Problem 2 at $\theta \in S$, find $v^l \in H^1(D;\mathbb{R})$ such* *that*

$$-\nabla\cdot\left(\phi^{\alpha}(\theta)\nabla v^{l}\right)=g_u^l(\theta,u)\quad in\ D,$$

$$\phi^{\alpha}(\theta)\partial_{\nu}v^{l}=j_u^l(\theta,u)\quad on\ \Gamma_{\mathrm{N}},\quad v^l=0\quad on\ \Gamma_{\mathrm{D}}.$$

Since $g_u^l\in H^{1-k}(D;\mathbb{R})$ and $j_u^l\in H^{3/2-k}(\partial D;\mathbb{R})$, $k\in\{0,1\}$ as in Problem 2, we have $v^l|_{D_r}\in H^{3-k}(D_r;\mathbb{R})$.

If u and v^l are the solutions of Problems 2 and 6, respectively, for $\theta\in S$, the θ derivative of $\mathscr{L}^l$ with respect to $\rho\in H^1(D;\mathbb{R})$ is given by

$$\begin{aligned}\mathscr{L}^{l\prime}(\theta,u,v^l)[\rho]&=\mathscr{L}^l_\theta(\theta,u,v^l)[\rho]\\&=\langle G^l,\rho\rangle\\&=\int_D(G_g^l+G_a^l)\rho\,\mathrm{d}x+\int_{\partial D}G_j^l\rho\,\mathrm{d}\gamma\end{aligned}\tag{3}$$

and agrees with $J^{l\prime}(\theta,u)[\rho]$, where

$$G_g^l(\theta,u)=g_\theta^l,\quad G_j^l(\theta,u)=j_\theta^l,$$

$$G_a^l(\theta,u,v^l)=-\alpha\phi^{\alpha-1}\phi_\theta\nabla u\cdot\nabla v^l.$$

Therefore, we have the following result.

Theorem 7 (θ derivative of J^l) *For the solutions u and v^l of Problems 2 and 6, respectively, for $\theta\in S$,*

$$J^{l\prime}(\theta,u)[\rho]=\langle G^l,\rho\rangle$$

holds for all $\rho\in H^1(D;\mathbb{R})$, where $G^l\big|_{D_r}$, $G_g^l\big|_{D_r}$, $G_a^l\big|_{D_r}$ and $G_j^l\big|_{\partial D_r}$ of (3) belong to $H^{1\prime}(D_r;\mathbb{R})$, $H^1(D_r;\mathbb{R})$s and $H^{3/2}(\partial D_r;\mathbb{R})$, respectively.

5. H^1 gradient method

Since $G^l\big|_{D_r}$ belongs to the dual space $H^{1\prime}(D_r;\mathbb{R})$ of $H^1(D_r;\mathbb{R})$, $\langle G^l,\rho\rangle$ is well defined in D_r. However, $\theta^{\epsilon G^l}=\theta+\epsilon G^l$ for a small $\epsilon>0$ does not belong to the admissible set S. This is considered to be the cause of the numerical instabilities discussed in the Introduction.

To avoid irregularity, we propose using an H^1 gradient method, which is an application of the traction method [13–15] to the SIMP problem, to determine a variation $\rho_G^l\in H^1(D;\mathbb{R})$ from $\theta\in S$ with $\bar{G}^l$ which is an extension of $G^l\big|_{D_r}$ to $H^1(D;\mathbb{R})$.

Problem 8 (H^1 gradient method) *Let $a: H^1(D;\mathbb{R})\times H^1(D;\mathbb{R})\to\mathbb{R}$ be a coercive bilinear form such that there exists $\beta>0$ that satisfies*

$$a(y,y)\geq\beta\|y\|_{1,2}^2$$

for all $y\in H^1(D;\mathbb{R})$. For G^l as in (3), find $\rho_G^l\in H^1(D;\mathbb{R})$ such that

$$a(\rho_G^l,y)=-\langle\bar{G}^l,y\rangle$$

for all $y\in H^1(D;\mathbb{R})$.

By the Lax-Milgram theorem, there exists a unique solution ρ_G^l to Problem 8. From Theorem 7, it is guaranteed that $\rho_G^l\big|_{D_r}$ belongs to $H^3(D_r;\mathbb{R})\subset W^{1,\infty}(D_r;\mathbb{R})$ and an extension $\bar{\rho}_G^l$ of $\rho_G^l\big|_{D_r}$ belongs to $W^{1,\infty}(D;\mathbb{R})$. Moreover, since

$$\begin{aligned}&J^l(\theta^{\epsilon\bar{\rho}_G^l},u^{\epsilon\bar{\rho}_G^l})-J^l(\theta,u)\\&=\langle G^l,\epsilon\bar{\rho}_G^l\rangle+o(\epsilon\|\bar{\rho}_G^l\|_{1,2})\\&\leq-\epsilon a(\bar{\rho}_G^l,\bar{\rho}_G^l)+o(\epsilon\|\epsilon\bar{\rho}_G^l\|_{1,2})\\&\leq-\epsilon\beta\|\bar{\rho}_G^l\|_{1,2}^2+o(\epsilon\|\bar{\rho}_G^l\|_{1,2})\\&<0\end{aligned}$$

for a sufficiently small positive number ϵ, $\bar{\rho}_G^l$ is a regular vector toward to a descent direction of J^l.

In the present paper, we use

$$a(y,z)=\int_D(\nabla y\cdot\nabla z+cyz)\,\mathrm{d}x\tag{4}$$

as a coercive bilinear form in Problem 8, where c is a positive constant.

6. Solution to SIMP problem

Let us consider a solution to Problem 4 by using a sequential quadratic approximation problem.

Problem 9 (SQ approximation) *Let G^0 and $\boldsymbol{G}=(G^1,\dots,G^m)^\top$ be θ derivatives of J^0 and $\boldsymbol{J}$, respectively, for a $\theta\in S$, $a(\cdot,\cdot)$ be given as in (4), and ϵ be a small positive constant. Find $\epsilon\rho$ such that*

$$\min_{\rho\in B}\{Q(\epsilon\rho)\mid\boldsymbol{J}(\theta,u)+\langle\boldsymbol{G},\epsilon\rho\rangle\leq\boldsymbol{0}\},$$

where $B=\{\rho\in H^1(D;\mathbb{R})\mid\|\rho\|_{1,2}=1\}$, and

$$Q(\epsilon\rho)=\frac{1}{2\epsilon}a(\epsilon\rho,\epsilon\rho)+\langle G^0,\epsilon\rho\rangle.$$

The Lagrangian of Problem 9 is defined as

$$\mathscr{L}^{\mathrm{SQ}}(\epsilon\rho,\boldsymbol{\lambda})=Q(\epsilon\rho)+\boldsymbol{\lambda}\cdot(\boldsymbol{J}(\theta,u)+\langle\boldsymbol{G},\epsilon\rho\rangle),$$

where $\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^m)^\top\in\mathbb{R}^m$ are the Lagrange multipliers for the constraints. The Karush-Kuhn-Tucker conditions for Problem 9 are given as

$$\frac{1}{\epsilon}a(\epsilon\rho,y)+\langle(G^0+\boldsymbol{\lambda}\cdot\boldsymbol{G}),y\rangle=0,\tag{5}$$

$$\boldsymbol{J}(\theta,u)+\langle\boldsymbol{G},\epsilon\rho\rangle\leq\boldsymbol{0},\tag{6}$$

$$\mathrm{diag}(\boldsymbol{\lambda})(\boldsymbol{J}(\theta,u)+\langle\boldsymbol{G},\epsilon\rho\rangle)=\boldsymbol{0},\tag{7}$$

$$\boldsymbol{\lambda}\geq\boldsymbol{0},\tag{8}$$

for all $y\in Y$.

Here, let ρ_G^0 and $\boldsymbol{\rho}_G=(\rho_G^1,\dots,\rho_G^m)^\top$ be the solutions to Problem 8 using $a(\cdot,\cdot)/\epsilon$ instead of $a(\cdot,\cdot)$, and $\rho_G=\rho_G^0+\boldsymbol{\lambda}\cdot\boldsymbol{\rho}_G$,

$$\rho=\rho^0+\boldsymbol{\lambda}\cdot\boldsymbol{\rho}=\frac{\rho_G}{\|\rho_G\|_{1,2}}.\tag{9}$$

Then, it is confirmed that $\rho_G=\epsilon\rho$ satisfies (5). If the all constraints in (6) are active, we have

$$\langle\boldsymbol{G},\epsilon\boldsymbol{\rho}^\top\rangle\boldsymbol{\lambda}=-\boldsymbol{J}(\theta,u)-\langle\boldsymbol{G},\epsilon\rho^0\rangle.\tag{10}$$

If the $G^1,\dots,G^m$ are linearly independent, (10) has a unique solution $\boldsymbol{\lambda}$. Using the $\boldsymbol{\lambda}$, if there are inactive constraints l such that $J^l(\theta,u)+\langle G^l,\epsilon\rho\rangle<0$ or $\lambda^l<0$, let us remove the constraints from (10), put $\lambda^l=0$, and resolving (10). Then, we can obtain $\boldsymbol{\lambda}$ which satisfies from (6) to (8). Since Problem 9 is a convex problem, $\boldsymbol{\lambda}$ is the unique solution to Problem 9.

To ensure the global convergence, we use the following criteria for ϵ in Problem 9. Let $\mathscr{L}(\theta, u, \boldsymbol{\lambda}) = J^0(\theta, u) + \boldsymbol{\lambda} \cdot \boldsymbol{J}(\theta, u)$ be the Lagrangian for Problem 4, and $\boldsymbol{\lambda}^{\epsilon\rho}$ be the $\boldsymbol{\lambda}$ for $(\theta^{\epsilon\rho}, u^{\epsilon\rho})$ that satisfies the Karush-Kuhn-Tucker conditions. For a constant $\xi \in (0,1)$, the Armijo criterion [16] gives the upper limit of ϵ as

$$\mathscr{L}(\theta^{\epsilon\rho}, u^{\epsilon\rho}, \boldsymbol{\lambda}^{\epsilon\rho}) - \mathscr{L}(\theta, u, \boldsymbol{\lambda}) \leq \xi \left\langle \left(G^0\left(u, v^0\right) + \boldsymbol{\lambda} \cdot \boldsymbol{G}(u, \boldsymbol{v})\right), \epsilon\rho \right\rangle. \quad (11)$$

For a constant $\mu \in (0,1)$ such that $0 < \xi < \mu < 1$, the Wolfe criterion [17] gives lower limit of ϵ as

$$\mu\langle (G^0(u, v^0) + \boldsymbol{\lambda} \cdot \boldsymbol{G}(u, \boldsymbol{v})), \epsilon\rho \rangle \leq \langle (G^0(u^{\epsilon\rho}, v^{0\,\epsilon\rho}) + \boldsymbol{\lambda}^{\epsilon\rho} \cdot \boldsymbol{G}(u^{\epsilon\rho}, \boldsymbol{v}^{\epsilon\rho})), \epsilon\rho \rangle. \quad (12)$$

We propose a numerical solution as follows. Let $\boldsymbol{J}(\theta^0, u^0) \leq \boldsymbol{0}$ is satisfied for θ^0 in the following.

(i) Set $\theta^0 \in S$, $\epsilon > 0$, ξ and μ such that $0 < \xi < \mu < 1$, $\epsilon_0 > 0$ and $k = 0$.
(ii) Compute J^0, $\boldsymbol{J}$, G^0 and $\boldsymbol{G}$ at θ^0.
(iii) Solve $\rho_G^0 = \rho_G^{0\,k}$ and $\boldsymbol{\rho}_G = \boldsymbol{\rho}_G^k$ in Problem 8.
(iv) Solve $\boldsymbol{\lambda}$ in

$$\langle \boldsymbol{G}, \boldsymbol{\rho}_G^\top \rangle \boldsymbol{\lambda} = -\langle \boldsymbol{G}, \rho_G^0 \rangle. \quad (13)$$

- If (8) is satisfied, proceed to the next step.
- Otherwise, remove the constraints such that $\lambda^l < 0$, put $\lambda^l = 0$ and resolve (13) until (8) is satisfied.

(v) Using ρ defined by (9), compute J^0 and $\boldsymbol{J}$ at $\theta^{\epsilon\rho}$.
- Put $\lambda^l = 0$ for the inactive constraints such that $J^l(\theta^{\epsilon\rho}, u^{\epsilon\rho}) < 0$.
- If $\boldsymbol{J}(\theta^{\epsilon\rho}, u^{\epsilon\rho}) \leq \boldsymbol{0}$, proceed to the next step.
- Otherwise, set $\boldsymbol{\lambda} = \boldsymbol{\lambda}^0$ and $i = 0$, solve $\delta\boldsymbol{\lambda}$ in

$$\langle \boldsymbol{G}, \epsilon\boldsymbol{\rho}^\top \rangle \delta\boldsymbol{\lambda} = -\boldsymbol{J}(\theta^{\epsilon\rho(\boldsymbol{\lambda}^i)}, u^{\epsilon\rho(\boldsymbol{\lambda}^i)}) \quad (14)$$

for the active constraints such that $J^l(\theta^{\epsilon\rho}, u^{\epsilon\rho}) \geq 0$, replace $\boldsymbol{\lambda}^{i+1} = \boldsymbol{\lambda}^i + \delta\boldsymbol{\lambda}$ and $i+1$ with i, and resolve (14) until $\boldsymbol{J}(\theta^{\epsilon\rho}, u^{\epsilon\rho}) \leq \boldsymbol{0}$ is satisfied.

(vi) Compute G^0 and $\boldsymbol{G}$ at $\theta^{\epsilon\rho}$.
- If (11) and (12) hold, proceed to the next step.
- If (11) or (12) does not hold, update ϵ with a smaller or larger value. Return to (v).

(vii) Let $\theta^{k+1} = \theta^{\epsilon\rho}$, $\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}$, and judge terminal condition by $\|\theta^{k+1} - \theta^k\|_{1,\infty} \leq \epsilon_0$.
- If the condition holds, terminate the algorithm.
- Otherwise, replace $k+1$ with k and return to (iii).

7. Numerical example

A SIMP problem for a three-dimensional linear elastic continuum is solved by the method shown above. Let $\boldsymbol{p}$ be a traction force and $\boldsymbol{u}$ be a displacement. Set $\boldsymbol{u}_\mathrm{D} = \boldsymbol{0}$. A mean compliance $J^0(\theta, \boldsymbol{u}) = \int_{\Gamma_\mathrm{N}} \boldsymbol{p} \cdot \boldsymbol{u}\, \mathrm{d}\gamma$ and a mass $J^1(\theta) = \int_D (\phi(\theta) - 0.4)\, \mathrm{d}x$ are used as cost functions. We have $G_g^0 = G_j^0 = 0$ and $G_a^0 = -\alpha\phi^{\alpha-1}\phi_\theta \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u})$ for J^0, and $G_g^1 = \phi_\theta$ and zeros of the other terms for J^1, where $\boldsymbol{\sigma}(\boldsymbol{u})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denote the stress and the strain. We use $\alpha = 2$ and $c = 1/(10L)^2$ in (4) for the width L of D. Finite element model consists of eight-node brick elements with three nonconforming modes and a bubble mode of $120 \times 160 \times 1$. Fig. 1 shows the result of the density obtained by the present method. We did not encounter any numerical instability.

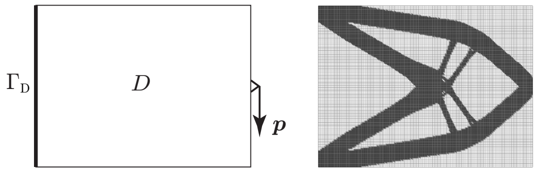

Fig. 1. Converged density (right) to the mean compliance minimization problem with mass constraint for a linear elastic problem as cantilever (left).

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References

[1] M. P. Bendsøe, Optimization of Structural Topology, Shape, and Material, Springer-Verlag, Berlin, 1995.
[2] F. Murat, Contre-exemples pour divers problémes ou le contrôle intervient dans les coefficients, Ann. Mat. Pura ed Appl., Serie 4, **112** (1977), 49–68.
[3] M. P. Bendsøe and N. Kikuchi, Generating optimal topologies in structural design using a homogenization method, Comput. Meths. Appl. Mech. Engrg., **71** (1988), 197–224.
[4] K. Suzuki and N. Kikuchi, A homogenization method for shape and topology optimization, Comput. Meths. Appl. Mech. Engrg., **93** (1991), 291–318.
[5] G. I. N. Rozvany, M. Zhou and T. Birker, Generalized shape optimization without homogenization, Struct. Optim., **4** (1992), 250–254.
[6] A. R. Diaz and O. Sigmund, Checkerboard patterns in layout optimization, Struct. Optim., **10** (1995), 40–45.
[7] O. Sigmund and J. Petersson, Numerical instabilities in topology optimization: a survey on procedures dealing with checkerboards, mesh-dependencies and local minima, Struct. Optim., **16** (1998), 68–75.
[8] K. Matsui and K. Terada, Continuous approximation of material distribution for topology optimization, Int. J. Numer. Meth. Engng., **59** (2004), 1925–1944.
[9] S. F. Rahmatalla and C. C. Swan, A Q4/Q4 continuum structural topology optimization implementation, Struct. Multidisc. Optim., **27** (2004), 130–135.
[10] J. Petersson and O. Sigmund, Slope constrained topology optimization, Int. J. Numer. Meth. Engng., **41** (1998), 1417–1434.
[11] G. -W. Jang, J. H. Jeong, Y. Y. Kim, D. Sheen, C. Park and M. -N. Kim, Checkerboard-free topology optimization using non-conforming finite elements, Int. J. Numer. Meth. Engng., **57** (2003), 1717–1735.
[12] G. Allaire, F. Jouve and A. M. Toader, Structural optimization using sensitivity analysis and a level-set method, J. Comput. Phys., **194** (2004), 363–393.
[13] H. Azegami, Solution to domain optimization problems (in Japanese), Trans. JSME, Ser. A, **60** (1994), 1479–1486.
[14] H. Azegami and K. Takeuchi, A smoothing method for shape optimization: traction method using the Robin condition, Int. J. Comput. Methods, **3** (2006), 21–33.
[15] S. Kaizu and H. Azegami, Optimal shape problems and traction method (in Japanese), Trans. JSIAM, **16** (2006), 277–290.
[16] L. Armijo, Minimization of functions having Lipschitz continuous first partial derivatives, Pacific J. Math., **16** (1966), 1–3.
[17] P. Wolfe, Convergence conditions for ascent methods, SIAM Review, **11** (1969), 226–235.