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Error analysis of H1 gradient method for topology optimization problems of continua

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Abstract

The present paper describes the result of the error estimation of a numerical solution to topology optimization problems of domains in which boundary value problems are defined. In the previous paper, we formulated a problem by using density as a design variable, presented a regular solution, and called it the H1 gradient method. The main result in this paper is the proof of the first order convergence in the H1 norm of the solution in the H1 gradient method with respect to the size of the finite elements if first order elements are used for the design and state variables.

Keywords calculus of variations, boundary value problem, topology optimization, H1 gradient method, error analysis

Research Activity Group Mathematical Design

1. Introduction

The problem of finding the optimum layout of holes in a domain in which a boundary value problem is defined is called the topology optimization problem of continua [1]. One method for formulating this topology optimization problem uses density as a design variable; in this case the problem is called the SIMP problem. In the previous paper [2], we formulated the problem and presented a regular solution by using a gradient method in a function space, and called this method the H1 gradient method. The aim of the present paper is to show the error estimation of the H1 gradient method using standard finite element analyses.

2. SIMP problem

Let $D \in \mathbb{R}^d$, $d \in \{2, 3\}$, be a fixed bounded domain with boundary ∂D, $\Gamma_{\mathrm{D}} \subset \partial D$ be a fixed subboundary of $|\Gamma_{\mathrm{D}}| > 0$, and $\Gamma_{\mathrm{N}} = \partial D \setminus \bar{\Gamma}_{\mathrm{D}}$. Following [2], let $\phi \in C^\infty(\mathbb{R}; [0, 1])$ be the density given by a sigmoidal function of design variable $\theta \in S = \{W^{1,\infty}(D; \mathbb{R}) \mid \|\theta\|_{1,\infty} \leq M\}$ for a constant $M > 0$. Let u be the solution to the following problem.

Problem 1 *Let $f \in H^1(D; \mathbb{R})$, $p \in H^{3/2}(\Gamma_{\mathrm{N}}; \mathbb{R})$ and $u_{\mathrm{D}} \in H^3(D; \mathbb{R})$ be given functions, and $\alpha > 1$ be a constant. For a given $\theta \in S$, find $u \in H^1(D; \mathbb{R})$ such that*

$$-\boldsymbol{\nabla} \cdot (\phi^\alpha(\theta) \boldsymbol{\nabla} u) = f \quad \text{in } D,$$

$$\phi^\alpha(\theta) \partial_{\boldsymbol{\nu}} u = p \quad \text{on } \Gamma_{\mathrm{N}}, \quad u = u_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}}.$$

Here, $\partial_{\boldsymbol{\nu}} = \boldsymbol{\nu} \cdot \boldsymbol{\nabla}$ where $\boldsymbol{\nu}$ is the unit outward normal vector along ∂D. Moreover, we provide cost functions as

$$J^l(\theta, u) = \int_D g^l(\theta, u) \,\mathrm{d}x + \int_{\partial D} j^l(\theta, u) \,\mathrm{d}\gamma + c^l \tag{1}$$

for $l \in \{0, 1, \ldots, m\}$ with constants c^l and given functions g^l and j^l. By using J^l, we define the SIMP problem as follows [2].

Problem 2 *Find θ such that*

$$\min_{\theta \in S} \{J^0(\theta, u) \mid J^l(\theta, u) \leq 0, \ l \in \{1, \ldots, m\}\}.$$

3. θ derivative of J^l

The Fréchet derivative of J^l with respect to θ is obtained as

$$\begin{aligned} J^{l\prime}(\theta, u, v^l)[\rho] &= \int_D (g^l_\theta + G^l_a)\rho \,\mathrm{d}x + \int_{\partial D} j^l_\theta \rho \,\mathrm{d}\gamma \\ &= \langle G^l, \rho \rangle \end{aligned} \tag{2}$$

for all $\rho \in H^1(D; \mathbb{R})$ [2]. Here, $\langle \cdot, \cdot \rangle$ is the dual product, $G^l_a = -\alpha\phi^{\alpha-1}(\theta)\phi_\theta \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v^l$, and $(\cdot)_\theta$ denotes $\partial(\cdot)/\partial\theta$. The function v^l is the solution of the following problem.

Problem 3 *For the solution u to Problem 1 at $\theta \in S$, find $v^l \in H^1(D; \mathbb{R})$ such that*

$$-\boldsymbol{\nabla} \cdot (\phi^\alpha(\theta) \boldsymbol{\nabla} v^l) = g^l_u(\theta, u) \quad \text{in } D,$$

$$\phi^\alpha(\theta) \partial_{\boldsymbol{\nu}} v^l = j^l_u(\theta, u) \quad \text{on } \Gamma_{\mathrm{N}}, \quad v^l = 0 \quad \text{on } \Gamma_{\mathrm{D}}.$$

4. Solution to Problem 2

Following [2], we generate θ_i, $i \in \{1, 2, \ldots, n\}$, from θ_0 by the simplified steps as follows.

(i) Set a small constant $\varepsilon > 0$ for step size, and $i = 0$.
(ii) Compute $u_i = u$ by solving Problem 1 with $\theta = \theta_i$.
(iii) Compute $v^l_i = v^l$ by solving Problem 3 with $\theta = \theta_i$.
(iv) Compute $G^l_i = G^l$ by (2) using u_i, v^l_i and θ_i.
(v) Compute $\rho^l_{G,i} \in H^1(D; \mathbb{R})$ by solving

$$\int_D (\boldsymbol{\nabla} \rho^l_{G,i} \cdot \boldsymbol{\nabla} y + c\rho^l_{G,i} y) \,\mathrm{d}x = -\langle G^l_i, y \rangle \tag{3}$$

for a constant $c > 0$ and all $y \in H^1(D;\mathbb{R})$.

(vi) Solve $\boldsymbol{\lambda} = (\lambda_i^l)_l$ in $\boldsymbol{A\lambda} = -\boldsymbol{b}$ where $\boldsymbol{A} = (a_{jl})_{jl}$, $a_{jl} = \langle G_i^j, \rho_{G,i}^l\rangle$ and $\boldsymbol{b} = (J^j + a_{j0})_j$. Put $\lambda_i^0 = 1$ and construct

$$\rho_i = \frac{\rho_{G,i}}{\|\rho_{G,i}\|_{1,2}}, \quad \rho_{G,i} = \sum_{l=0}^{m} \lambda_i^l \rho_{G,i}^l. \tag{4}$$

(vii) Construct $\theta_{i+1} = \theta_i + \varepsilon\rho_i$ and return to (ii) with $i = i+1$.

5. Error analysis

We estimate the error of the numerical solution by the finite element method with respect to θ_n obtained in the solution in Section 4. Let $D_h = \cup\{K\}$ be a finite element approximation of D with elements $\{K\}$, $h = \max_{K\in\{K\}} \operatorname{diam}(K)$. For positive integer k and even number $q \geq d$, we restrict u_i, v_i^l and $\rho_{G,i}^l$ to $W^{k+1,q}(D_h;\mathbb{R})$, and θ_i on D_h. We denote $\theta_{h,i} = \theta_i + \delta\theta_i$ is the approximation of θ_i, $\hat{u}_i = u_i + \delta\hat{u}_i \in W^{k+1,q}(D_h;\mathbb{R})$ and $\hat{v}_i^l = v_i^l + \delta\hat{v}_i^l \in W^{k+1,q}(D_h;\mathbb{R})$ are the analytical solutions of Problems 1 and 3 replacing θ_i by $\theta_{h,i}$. Let $u_{h,i} = u_i + \delta u_i$, $v_{h,i}^l = v_i^l + \delta v_i^l$, $G_{h,i}^l = G_i^l + \delta G_i^l$, $\rho_{Gh,i}^l = \rho_{G,i}^l + \delta\rho_{G,i}^l$, $\rho_{Gh,i} = \rho_{G,i} + \delta\rho_{G,i}$, and $\rho_{h,i} = \rho_i + \delta\rho_i$ be the approximate functions of u_i, v_i^l, G_i^l, $\rho_{G,i}^l$, $\rho_{G,i}$, and ρ_i, respectively. $\hat{\rho}_{G,i}^l = \rho_{G,i}^l + \delta\hat{\rho}_{G,i}^l \in W^{k+1,q}(D_h;\mathbb{R})$ represents an analytical solution of (3) replacing G_i^l by $G_{h,i}^l$. Also, let $\boldsymbol{\lambda}_h = (\lambda_{h,i}^l)_l$ be the solution to $\boldsymbol{A}_h\boldsymbol{\lambda}_h = -\boldsymbol{b}_h$ with $\boldsymbol{A}_h = (a_{h,jl})_{jl}$, $a_{h,jl} = \langle G_{h,i}^j, \rho_{Gh,i}^l\rangle$, $\boldsymbol{b}_h = (J_h^j + a_{h,j0})_j$, $J_h^j = J^j(\theta_{h,i}, u_{h,i})$. We use

$$\|u\|_{j,q} = \left(\sum_{k=0}^{j} |u|_{k,q}^q\right)^{1/q}, \quad |u|_{j,q} = \left[\int_{D_h} (\boldsymbol{\nabla}^j u)^q \mathrm{d}x\right]^{1/q}$$

as the $W^{j,q}$ norm $\|\cdot\|_{j,q}$ and seminorm $|\cdot|_{j,q}$ on D_h for $j \in \{0,1\}$, $q \in \{4,6,\ldots,\infty\}$ with $\boldsymbol{\nabla}^0 = 1$. We set the following necessary hypotheses to evaluate the error.

(H1) We take $\alpha \geq 2$ in Problems 1, 3 and (2).

(H2) There exist some positive constants C_1, C_2, C_3 independent of h such that

$$\|\hat{u}_i - u_{h,i}\|_{j,q} \leq C_1 h^{k+1-j} |\hat{u}_i|_{k+1,q}, \tag{5}$$

$$\|\hat{v}_i^l - v_{h,i}^l\|_{j,q} \leq C_2 h^{k+1-j} |\hat{v}_i^l|_{k+1,q}, \tag{6}$$

$$\|\hat{\rho}_{G,i}^l - \rho_{Gh,i}^l\|_{j,q} \leq C_3 h^{k+1-j} |\hat{\rho}_G^l|_{k+1,q}. \tag{7}$$

(H3) For $J^l(\theta,u)$, we restrict $j^l(\theta,u)$ to a function of u, i.e. $j^l(u)$, $j^l \in C^2(W^{1,q}(D;\mathbb{R}); L^1(D;\mathbb{R}))$, and $g^l \in C^2(Y; L^1(D;\mathbb{R}))$ for $Y = S \times W^{1,q}(D;\mathbb{R})$ such that $j_u^l \in C^1(W^{1,q}(D;\mathbb{R}); W^{1,\infty}(D;\mathbb{R}))$, g_θ^l, $g_u^l \in C^1(Y; L^\infty(D;\mathbb{R}))$, $j_{uu}^l \in C^0(W^{1,q}(D;\mathbb{R}); W^{1,\infty}(D;\mathbb{R}))$, and $g_{\theta\theta}^l$, $g_{\theta u}^l$, $g_{u\theta}^l$, $g_{uu}^l \in C^0(Y; L^\infty(D;\mathbb{R}))$, respectively.

(H4) There exists $C_4 > 0$ such that $\|\boldsymbol{A}^{-1}\|_\infty < C_4$, where $\|\cdot\|_\infty$ is the maximum norm on $\mathbb{R}^m$ and the corresponding operator norm for $m \times m$ matrices.

Then we have the following main theorem.

Theorem 4 (Error of θ_n) *Assume from* (H1) *to* (H4). *Then there exists a constant $C > 0$ independent of ε and h such that $\|\delta\theta_n\|_{1,q} \leq C\varepsilon n h^k$ holds for n.*

Here $\varepsilon n = T$ can be considered as the total amount of variation of θ. To prove this theorem, we introduce an induction hypothesis for $\theta_{h,i}$:

$$\|\delta\theta_i\|_{1,q} \leq C\varepsilon i h^k \tag{8}$$

for $i \in \{0,1,\ldots,n-1\}$ and the lemmas below.

Lemma 5 (Error of u_i) *Assume* (H1), (H2) *and* (8). *Then there exists a constant $C_1' > 0$ independent of ε and h such that $\|\delta u_i\|_{1,q} \leq C_1'(\varepsilon i + 1)h^k$ holds.*

Proof u_i and $\hat{u}_i$ satisfy

$$\int_{D_h} \phi^\alpha(\theta_i) \boldsymbol{\nabla}\delta\hat{u}_i \cdot \boldsymbol{\nabla} v^l \mathrm{d}x = \int_{D_h} (\phi^\alpha(\theta_i) - \phi^\alpha(\theta_{h,i})) \boldsymbol{\nabla}\hat{u}_i \cdot \boldsymbol{\nabla} v^l \mathrm{d}x. \tag{9}$$

By taking $\boldsymbol{\nabla} v^l = (\boldsymbol{\nabla}\delta\hat{u}_i)^{q-1}$ and $m = \min_{\theta_i \in D_h} \phi^\alpha(\theta_i)$ in (9), we have

$$\begin{aligned} m|\delta\hat{u}_i|_{1,q}^q &\leq \alpha\|\delta\theta_i\|_{0,\infty} |\hat{u}_i|_{1,q} |\delta\hat{u}_i|_{1,q}^{q-1} \\ &\quad \times \max_{t\in[0,1]} \|\phi^{\alpha-1}(\theta_i + t\delta\theta_i)\phi_\theta(\theta_i + t\delta\theta_i)\|_{0,\infty}. \end{aligned} \tag{10}$$

By substituting (8) into (10) and dividing (10) by $|\delta\hat{u}_i|_{1,q}^{q-1}$, noticing (H1), we obtain $|\delta\hat{u}_i|_{1,q} \leq C_1'\varepsilon i h^k$. Using the Poincaré inequality, we get

$$\|\delta\hat{u}_i\|_{1,q} \leq C_1'\varepsilon i h^k \tag{11}$$

by rewriting $C_1' > 0$. By substituting (11) and (5) into $\|\delta u_i\|_{1,q} \leq \|\delta\hat{u}_i\|_{1,q} + \|\hat{u}_i - u_{h,i}\|_{1,q}$, the proof is complete. **(QED)**

Lemma 6 (Error of v_i^l) *Assume from* (H1) *to* (H3) *and* (8). *Then there exists a constant $C_2' > 0$ independent of ε and h such that $\left\|\delta v_i^l\right\|_{1,q} \leq C_2'(\varepsilon i + 1)h^k$ holds.*

Proof Noticing (H3), v_i^l and $\hat{v}_i^l$ satisfy

$$\begin{aligned} \int_{D_h} \phi^\alpha(\theta_i) \boldsymbol{\nabla}\delta\hat{v}_i^l \cdot \boldsymbol{\nabla} u' \mathrm{d}x &= \int_{D_h} (\phi^\alpha(\theta_i) - \phi^\alpha(\theta_{h,i})) \boldsymbol{\nabla}\hat{v}_i^l \cdot \boldsymbol{\nabla} u' \mathrm{d}x \\ &\quad + \int_{D_h} (g_u^l(\theta_{h,i}, u_{h,i}) - g_u^l(\theta_i, u_i)) u' \mathrm{d}x \\ &\quad + \int_{\partial D_h} (j_u^l(u_{h,i}) - j_u^l(u_i)) u' \mathrm{d}\gamma. \end{aligned} \tag{12}$$

By taking $\boldsymbol{\nabla} u' = (\boldsymbol{\nabla}\delta\hat{v}_i^l)^{q-1}$ and using the Poincaré inequality, we have

$$\begin{aligned} &\int_{D_h} (g_u^l(\theta_{h,i}, u_{h,i}) - g_u^l(\theta_i, u_i)) u' \mathrm{d}x \\ &\leq \|\delta\theta_i\|_{0,q} |\delta\hat{v}_i^l|_{1,q}^{q-1} \max_{t\in[0,1]} \|g_{u\theta}^l(\theta_i + t\delta\theta_i, u_i)\|_{0,\infty} \\ &\quad + \|\delta u_i\|_{0,q} |\delta\hat{v}_i^l|_{1,q}^{q-1} \max_{t\in[0,1]} \|g_{uu}^l(\theta_{h,i}, u_i + t\delta u_i)\|_{0,\infty} \end{aligned} \tag{13}$$

and

$$\int_{\partial D_h} (j_u^l(u_{h,i}) - j_u^l(u_i))u' \mathrm{d}\gamma$$
$$= \int_{D_h} \boldsymbol{\nabla}[(j_u^l(u_{h,i}) - j_u^l(u_i))u'] \,\mathrm{d}x$$
$$\leq |\delta u_i|_{1,q} |\delta \hat{v}_i^l|_{1,q}^{q-1} \max_{t\in[0,1]} |j_{uu}^l(u_i + t\delta u_i)|_{1,\infty}$$
$$+ \|\delta u_i\|_{0,q} |\delta \hat{v}_i^l|_{1,q}^{q-1} \max_{t\in[0,1]} \|j_{uu}^l(u_i + t\delta u_i)\|_{0,\infty}. \tag{14}$$

By the same argument as in the proof of Lemma 5, substituting (13) and (14) into (12), we have

$$m\|\delta \hat{v}_i^l\|_{1,q} \leq C_1'' m |\delta \hat{v}_i^l|_{1,q}$$
$$\leq C_1'' \alpha \|\delta\theta_i\|_{0,\infty} \|\hat{v}_i^l\|_{1,q}$$
$$\times \max_{t\in[0,1]} \|\phi^{\alpha-1}(\theta_i + t\delta\theta_i)\phi_\theta(\theta_i + t\delta\theta_i)\|_{0,\infty}$$
$$+ C_1'' \|\delta\theta_i\|_{0,q} \max_{t\in[0,1]} \|g_{u\theta}^l(\theta_i + t\delta\theta_i, u_i)\|_{0,\infty}$$
$$+ C_1'' \|\delta u_i\|_{0,q} \max_{t\in[0,1]} \|g_{uu}^l(\theta_{h,i}, u_i + t\delta u_i)\|_{0,\infty}$$
$$+ C_1'' |\delta u_i|_{1,q} \max_{t\in[0,1]} |j_{uu}^l(u_i + t\delta u_i)|_{1,\infty}$$
$$+ C_1'' \|\delta u_i\|_{0,q} \max_{t\in[0,1]} \|j_{uu}^l(u_i + t\delta u_i)\|_{0,\infty} \tag{15}$$

for some constant $C_1'' > 0$. From (H3), substituting (8) and (11) into (15) and substituting (15) and (6) into $\|\delta v_i^l\|_{1,q} \leq \|\delta \hat{v}_i^l\|_{1,q} + \|\hat{v}_i^l - v_{h,i}^l\|_{1,q}$, the proof is complete. **(QED)**

Lemma 7 (Error of G_i) *Assume from* (H1) *to* (H3) *and* (8). *Then there exists a constant $C_3' > 0$ independent of ε and h, such that $\|\delta G_i^l\|_{0,q} \leq C_3'(\varepsilon i+1)h^k$ holds.*

Proof By (H3), G_i^l and $G_{h,i}^l$ satisfy

$$\delta G_i^l = g_\theta^l(\theta_i, u_i) - g_\theta^l(\theta_{h,i}, u_{h,i})$$
$$+ \alpha\phi^{\alpha-1}(\theta_{h,i})\phi_\theta(\theta_{h,i}) \boldsymbol{\nabla} u_{h,i} \cdot \boldsymbol{\nabla} v_{h,i}^l$$
$$- \alpha\phi^{\alpha-1}(\theta_i)\phi_\theta(\theta_i) \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} v_i^l. \tag{16}$$

We estimate the bound on the first and the second terms in the right-hand side of (16) as

$$\|g_\theta^l(\theta_i, u_i) - g_\theta^l(\theta_{h,i}, u_{h,i})\|_{0,q}$$
$$\leq \|\delta u_i\|_{0,\infty} \max_{t\in[0,1]} \|g_{\theta u}^l(\theta_{i,h}, u_i + t\delta u_i)\|_{0,\infty}$$
$$+ \|\delta\theta_i\|_{0,\infty} \max_{t\in[0,1]} \|g_{\theta\theta}^l(\theta_i + t\delta\theta_i, u_i)\|_{0,\infty}. \tag{17}$$

By using triangle inequality, we can estimate the remaining terms as

$$\alpha\|\phi^{\alpha-1}(\theta_{h,i}) - \phi^{\alpha-1}(\theta_i)\|_{0,\infty} \|\phi_\theta(\theta_i)\boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} v_i^l\|_{0,q}$$
$$+ \alpha\|\phi_\theta(\theta_{h,i}) - \phi_\theta(\theta_i)\|_{0,\infty} \|\phi^{\alpha-1}(\theta_{h,i})\boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} v_i^l\|_{0,q}$$
$$+ \alpha\|\phi^{\alpha-1}(\theta_{h,i})\phi_\theta(\theta_{h,i})\boldsymbol{\nabla} v_i^l\|_{0,\infty} |u_{h,i} - u_i|_{1,q}$$
$$+ \alpha\|\phi^{\alpha-1}(\theta_{h,i})\phi_\theta(\theta_{h,i})\boldsymbol{\nabla} u_{h,i}\|_{0,\infty} |v_{h,i}^l - v_i^l|_{1,q} \tag{18}$$

and

$$\|\phi^{\alpha-1}(\theta_{h,i}) - \phi^{\alpha-1}(\theta_i)\|_{0,\infty}$$
$$\leq (\alpha-1)\|\delta\theta_i\|_{0,\infty}$$
$$\times \max_{t\in[0,1]} \|\phi^{\alpha-2}(\theta_i + t\delta\theta_i)\phi_\theta(\theta_i + t\delta\theta_i)\|_{0,\infty},$$
$$\|\phi_\theta(\theta_{h,i}) - \phi_\theta(\theta_i)\|_{0,\infty}$$
$$\leq \|\delta\theta_i\|_{0,\infty} \max_{t\in[0,1]} \|\phi_{\theta\theta}(\theta_i + t\delta\theta_i)\|_{0,\infty}.$$

We can obtain the result in this lemma by substituting (17) and (18) into (16), using Lemmas 5, 6, (8), (H1) and (H3). **(QED)**

Lemma 8 (Error of ρ_i^l) *Assume from* (H1) *to* (H3) *and* (8). *Then there exists a constant $C_4' > 0$ independent of ε and h, such that $\|\delta\rho_{G,i}^l\|_{1,q} \leq C_4'(\varepsilon i+1)h^k$ holds.*

Proof $\rho_{G,i}^l$ and $\hat{\rho}_{G,i}^l$ satisfy

$$\int_{D_h} (\Delta\delta\hat{\rho}_{G,i}^l - c\delta\hat{\rho}_{G,i}^l) y \,\mathrm{d}x = \langle \delta G_i^l, y\rangle.$$

Taking $y = (q-1)\delta\hat{\rho}_{G,i}^l(\boldsymbol{\nabla}\delta\hat{\rho}_{G,i}^l)^{q-2}$ and considering $\boldsymbol{\nabla}\delta\hat{\rho}_{G,i}^l = 0$ on ∂D_h, we have

$$|\delta\hat{\rho}_{G,i}^l|_{1,q}^q + c(q-1)\int_{D_h} (\delta\hat{\rho}_{G,i}^l)^2(\boldsymbol{\nabla}\delta\hat{\rho}_{G,i}^l)^{q-2} \,\mathrm{d}x$$
$$\leq (q-1)\|\delta G_i^l\|_{0,q} |\delta\hat{\rho}_{G,i}^l|_{1,q}^{q-2} |\delta\hat{\rho}_{G,i}^l|_{0,q}. \tag{19}$$

Now we divide (19) by $|\delta\hat{\rho}_{G,i}^l|_{1,q}^{q-2}$. Then, since q $(> d)$ is even number and the Poincaré inequality, we get $\|\delta\hat{\rho}_{G,i}^l\|_{1,q} \leq C_4''(q-1)\|\delta G_i^l\|_{0,q}$ for some constant $C_4'' > 0$. By substituting (7) into

$$\|\delta\rho_{G,i}^l\|_{1,q} \leq \|\delta\hat{\rho}_{G,i}^l\|_{1,q} + \|\hat{\rho}_{G,i}^l - \rho_{Gh,i}^l\|_{1,q},$$

and using Lemma 7, the proof is complete. **(QED)**

Lemma 9 (Error of λ_i^l) *Assume from* (H1) *to* (H4) *and* (8). *Then there exists a constant $C_5' > 0$ independent of ε and h, such that $|\lambda_i^l - \lambda_{h,i}^l| \leq C_5'(\varepsilon i+1)h^k$ holds.*

Proof $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_h$ satisfy

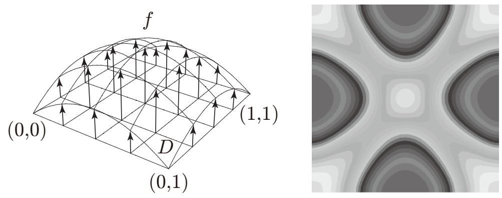
$$\boldsymbol{A}(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h) = \boldsymbol{b}_h - \boldsymbol{b} - (\boldsymbol{A} - \boldsymbol{A}_h)\boldsymbol{\lambda}_h$$

By (H4) and multiplying by $\boldsymbol{A}^{-1}$, we get

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_\infty$$
$$\leq \|\boldsymbol{A}^{-1}\|_\infty(\|\boldsymbol{b} - \boldsymbol{b}_h\|_\infty + \|\boldsymbol{A} - \boldsymbol{A}_h\|_\infty\|\boldsymbol{\lambda}_h\|_\infty)$$
$$\leq \|\boldsymbol{A}^{-1}\|_\infty(1 + m\|\boldsymbol{\lambda}_h\|_\infty) \max_{\substack{j\in\{1,\dots,m\},\\ l\in\{0,\dots,m\}}} |a_{jl} - a_{h,jl}|$$
$$+ \|\boldsymbol{A}^{-1}\|_\infty \max_{j\in\{1,\dots,m\}} |J^j(\theta_i, u_i) - J^j(\theta_{h,i}, u_{h,i})|.$$

Here,

$$|a_{jl} - a_{h,jl}| \leq |\langle \delta G_i^j, \rho_{G,i}^l\rangle| + |\langle G_{h,i}^j, \delta\rho_{G,i}^l\rangle|$$
$$\leq \|\delta G_i^j\|_{0,2}\|\rho_{G,i}^l\|_{0,2} + \|G_{h,i}^j\|_{0,2}\|\delta\rho_{G,i}^l\|_{0,2}.$$



(a) f in Problem 1 (b) $\phi(\theta_{1/20,100})$

Fig. 1. Setting for Problem 1 and converged ϕ.

Table 1. Results of $-\log_2 \|\delta\theta_n\|_{1,2}$ with $T = \varepsilon n = 10$ to Problem 1.

n	h=1/5	h=1/10	h=1/20	h=1/40	h=1/80
50	0.9012	1.8513	2.8614	3.8983	5.0598
incr.		0.9501	1.0101	1.0369	1.1615
100	0.9201	1.8655	2.8397	3.8761	5.0407
incr.		0.9454	0.9742	1.0364	1.1646
200	1.4861	2.4064	3.4106	4.4414	5.6005
incr.		0.9203	1.0042	1.0308	1.1591
400	1.0518	2.0617	3.0481	4.0759	5.2331
incr.		1.0099	0.9864	1.0278	1.1572
800	0.7343	1.6836	2.7203	3.7521	4.9121
incr.		0.9493	1.0367	1.0318	1.1600

and

$$
\begin{aligned}
|J^j - J_h^j| \\
&\le \|\delta\theta_i\|_{0,\infty} \max_{t\in[0,1]} \|g_\theta^j(\theta_i + t\delta\theta_i, u_i)\|_{0,\infty} \\
&\quad + \|\delta u_i\|_{0,\infty} \max_{t\in[0,1]} \|g_u^j(\theta_{h,i}, u_i + t\delta u_i)\|_{0,\infty} \\
&\quad + |\delta u_i|_{1,\infty} \max_{t\in[0,1]} |j_{uu}^j(u_i + t\delta u_i)|_{1,\infty} \\
&\quad + \|\delta u_i\|_{0,\infty} \max_{t\in[0,1]} \|j_{uu}^j(u_i + t\delta u_i)\|_{0,\infty}.
\end{aligned}
$$

From (8), Lemmas 5, 6, 7 and 8, the lemma is proven.

(QED)

Lemma 10 (Error of ρ_i) *Assume from* (H1) *to* (H4) *and* (8). *Then there exists a constant $C_6' > 0$ independent of ε and h, such that $\|\delta\rho_i\|_{1,q} \le C_6'(\varepsilon i + 1)h^k$ holds.*

Proof By (4), ρ_i and $\rho_{h,i}$ satisfy $\|\delta\rho_i\|_{1,q} \le 2\|\delta\rho_{G,i}\|_{1,q}/\|\rho_{G,i}\|_{1,2}$ and

$$
\begin{aligned}
\|\delta\rho_{G,i}\|_{1,q} \\
&\le (m+1) \max_{l\in\{0,\dots,m\}} |\lambda_{h,i}^l| \max_{l\in\{0,\dots,m\}} \|\delta\rho_{G,i}^l\|_{1,q} \\
&\quad + m \max_{l\in\{1,\dots,m\}} |\lambda_i^l - \lambda_{h,i}^l| \max_{l\in\{1,\dots,m\}} \|\rho_{G,i}^l\|_{1,q}.
\end{aligned}
$$

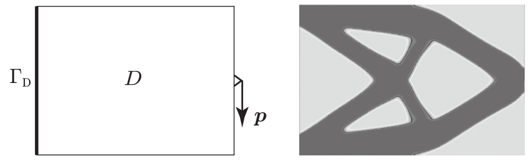
By using Lemmas 8 and 9, the theorem is proven.

(QED)

Proof of Theorem 4 If $n = 0$, we have Theorem 4 by $\theta_0 = \theta_{h,0}$.

If $n > 0$, for $i \in \{0,\dots,n-1\}$, we have $\|\delta\theta_{i+1}\|_{1,q} \le \varepsilon\|\delta\rho_i\|_{1,q} + \|\delta\theta_i\|_{1,q}$ with $\|\delta\theta_0\|_{1,q} = 0$. By applying Lemma 10 and (8) to the previous inequality, we have $\|\delta\theta_{i+1}\|_{1,q} \le \max\{C_6', C\}\varepsilon(i+1)h^k + C_6'\varepsilon^2 i h^k$. Since ε is a small constant, $C = \max\{C_6', C\}$, and $n = i+1$, we obtain Theorem 4.

(QED)



(a) Boundary condition (b) $\phi(\theta_{1/20,800})$

Fig. 2. Setting for linear elastic problem and converged ϕ.

Table 2. Results of $-\log_2 \|\delta\theta_n\|_{1,2}$ with $T = \varepsilon n = 80$ to a linear elastic problem.

n	h=1/5	h=1/10	h=1/20	h=1/40	h=1/80
400	−5.7374	−4.9172	−4.0073	−2.9443	−1.6998
incr.		0.8202	0.9099	1.0630	1.2445
800	−5.7580	−4.9492	−4.0596	−3.0060	−1.7628
incr.		0.8088	0.8896	1.0536	1.2432
1600	−5.7598	−4.9506	−4.0618	−3.0086	−1.7656
incr.		0.8092	0.8888	1.0532	1.2430
3200	−5.7607	−4.9514	−4.0629	−3.0099	−1.7669
incr.		0.8093	0.8885	1.0530	1.2430
6400	−5.7611	−4.9517	−4.0635	−3.0106	−1.7676
incr.		0.8094	0.8882	1.0529	1.2430

6. Numerical examples

For Problem 1, we use the setting $D = [0,1]^2$, $\Gamma_{\mathrm{D}} = \partial D$, $f = 2[x_1^2 + x_2^2 - (x_1 + x_2)]$, $u_{\mathrm{D}} = 0$, $\phi(\theta) = (\tanh\theta + 1)/2$, $\alpha = 2$. The cost functions are assumed as $J^0(\theta,u) = \int_D f u \,\mathrm{d}x$ and $J^1(\theta) = \int_D \phi(\theta)\,\mathrm{d}x - c^1$, where c^1 is taken as $J^1(\theta_0) = 0$ for $\theta_0 = 0$. We take $c = 1$ in (3). D is approximated as D_h using triangular element. We take $k = 1$ in (H2). Fig. 1 shows f and converged ϕ obtained by the present method. Table 1 shows the results of $-\log_2 \|\delta\theta_n\|_{1,2}$ with $T = n\varepsilon = 10$.

Another example is a SIMP problem for linear elastic continuum. Let $D = [0,3] \times [0,2]$, $\boldsymbol{p} \in H^{3/2}(\Gamma_{\mathrm{N}};\mathbb{R}^2)$ be a traction force, $\boldsymbol{u}_{\mathrm{D}} = \boldsymbol{0} \in H^2(\Gamma_{\mathrm{D}};\mathbb{R}^2)$, and $\boldsymbol{u} \in H^1(D;\mathbb{R}^2)$ be a displacement as a solution of the linear elastic problem for $\boldsymbol{p}$. A mean compliance $J^0(\theta,\boldsymbol{u}) = \int_{\Gamma_{\mathrm{N}}} \boldsymbol{p}\cdot\boldsymbol{u}\,\mathrm{d}\gamma$ and a mass $J^1(\theta) = \int_D \phi(\theta)\,\mathrm{d}x - c^1$ are used as cost functions. We have $G_a^0 = -\alpha\phi^{\alpha-1}\phi_\theta \boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})$ for J^0 where $\boldsymbol{\sigma}(\boldsymbol{u})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are denoted by the stress and the strain, respectively. The space approximation of D, c^1, α, c and k are the same as above. Fig. 2 shows the problem setting and the result ϕ obtained by the present method. Table 2 shows the results of $-\log_2 \|\delta\theta_n\|_{1,2}$ with $T = n\varepsilon = 80$.

From Tables 1 and 2, we can observe $\|\delta\theta_n\|_{1,2}$ achieves first order convergence in the H^1 norm with respect to h expected by Theorem 4 with $k = 1$. Also, these tables show $\|\delta\theta_n\|_{1,2}$ is independent of $T = \varepsilon n$.

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