# DISTRIBUTINONS OF PULSE DURATION IN THE POISSON NOISE AND ATMOSPHERIC NOISE

Taketosi NAKAI

**Abstract**

A simple model is taken for the time series of pulses of rectangular form resulted from amplifying, and limiting the atmospheric noise envelopes sliced at a given voltage level. This model is the Poisson noise process in which the original pulses with a constant duration are arriving at the time spacings determined by the Poisson distribution law. The probability density function and distribution function of duration are derived by the method of the characteristic function for the $K$-multiple pulses and all the observed pulses in the Poisson noise process. These functions are characterized by the product of the average number per second and the constant duration of the original pulses. For the case of the time series of pulses of rectangular form, when the atmospheric noise envelopes exceed a given voltage level, the duration of the original pulses is a random variable, and an accurate derivation of the distribution function of duration has not been obtained. In this case, the effects of overlappings between the original pulses on the distribution of duration of the original pulses can be approximately estimated with respect to the values of the product of the average duration and the average number of the original pulses per second.

## 1. Introduction

The detailed structure of the atmospheric radio noise can be most completely described by such amplitude functions and the time functions as the amplitude probability distribution and the crossing rate distribution of the atmospheric noise envelopes, and also by the distribution of pulse duration and the distribution of spacing between pulses when the atmospheric noise envelopes exceed a given voltage level. Now, it should be noted that the practical measurements of all the required parameters related to various distributions described as above are made for the same time series of pulses of rectangular form resulted from amplifying and limiting the atmospheric noise envelopes sliced at a given voltage level.

The author investigated the noise processes as a simple model for such time series of pulses of rectangular form, when the original pulses with a constant duration are arriving at the times spaced by the Poisson distribution law, and for it were deduced the average number of the observed pulses per second, the probability of time length which is occupied by the existence of the observed pulses and the average duration of the observed pulses per second. Thereafter proceeding from the above, we deduced the distribution of pulse duration for the Poisson noise process. The method is as follows. We deduced the probability density function of duration of the $K$-multiple pulses by the method of the characteristic function, and also the distribution function of duration of the $K$-multiple pulses. And by summing the distributions of duration of the $K$-multiple pulses multiplied by a weighting function for each value of $K$, over all the integral numbers of $K$, we obtained the distribution function of duration of all the observed pulses for the noise process under consideration, and by the differentiation of the function, we obtained the probability density function of duration of all the observed pulses. Due to the theoretical expression derived, the theoretical curves of the distribution of duration of all the observed pulses have been calculated for various values of the average number of the original pulses per second times the constant duration of the original pulses.

For the practical time series of pulses of rectangular form when the atmospheric noise envelopes are sliced at a given voltage level, the duration of the original pulses should not be considered to be constant, but a random variable. In this case, an accurate deduction of the distribution function is very difficult. And for the present, the only reasonable approximation of the observed distribution of duration for the atmospheric noise envelopes may be given by the distribution of duration of the observed pulses for the Poisson noise process with the average number of the original pulses per second times the average duration of the original pulses when the atmospheric noise envelopes exceed a given voltage level.

The details will be described in the paper.

## 2. Properties of $K$-multiple pulses

### 2. 1 Average number per second

The author has already reported on some properties of the Poisson noise processes elsewhere.[1] And a brief explanation of them will be given here. The Poisson noise process is defined as the noise process in which the original pulses of constant duration $\tau$ and constant amplitude $a$ are arriving at times determined by the Poisson distribution law. Now let it be that $\nu$ indicates the average number of the arriving pulses per second, and $x$ is a random variable that obeys the Poisson distribution law, and so its probability density function $\mathrm{P}(x)$ is given as follows,

$$\mathrm{P}(x)=\nu \exp(-\nu x) \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

The observed time series of pulses of rectangular form for the noise process under consideration are such as shown in Fig. 1, which are resulted by limiting the observed time series of pulses at the original amplitude of the pulses. Now we take a

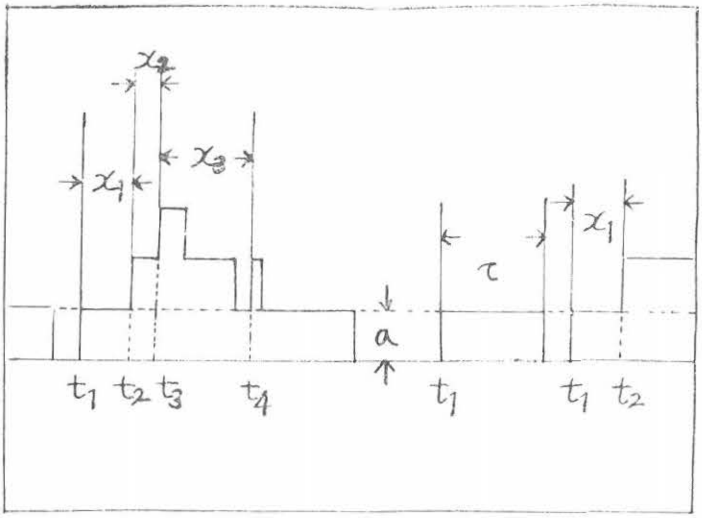


Fig. 1. Poisson noise process

reference time at the instant of the beginning of an arbitrary pulse observed and take the beginnings of successively arrived pulses, from early pulses to late ones as

$$t_1,\quad t_2,\quad t_3,\ \ldots\ldots\ldots$$

and $x_1$, $x_2$, ...... and so on in Fig. 1 are the random variables satisfying the equation (1).

For the time series of pulses as described above, the duration of a $K$-multiple pulse $\tau_K$ is expressed by the following expression

$$\tau_K = x_1 + x_2 + \cdots\cdots\cdots + x_{K-1} + \tau \tag{2}$$

where the following conditions must hold,

$$0 \leqq x_s \leqq \tau, \quad s = 1, 2, \cdots\cdots\cdots, \quad K-1 \tag{3}$$

and each of the $x_s$'s is an independent random variable for each other and satisfies the equation (1), and of course $\tau_K$ is a random variable. And the average number of the $K$-multiple pulses per second $R_K$ is given as follows,[1]

$$R_K = R\,(1 - e^{-\nu\tau})^{K-1} \cdot e^{-\nu\tau} \tag{4}$$

where R is the average number of all the observed pulses per second for the noise process under consideration.

### 2. 2 Probability density function of duration of $K$-multiple pulses

Let us now obtain the probability density function of duration of the $K$-multiple pulses observed in a very long time interval T by the method of the characteristic function. The duration of the $K$-multiple pulse is expressed by the right side of the equation (2) and a random variable. Each of $x_s$'s which appear in the equation (2) satisfies the inequalities (3) respectively and is an independent random variable satisfying the equation (1). And so its probability density function is given by

$$P(x_s) = \frac{\nu e^{-\nu x_s}}{1-e^{-\nu\tau}} \tag{5}$$

which holds for each of the $x_s$'s and its characteristic function is given by

$$Mx_s(jv) = \int_{-\infty}^{+\infty} P(x_s) \cdot e^{jvx_s} dx_s \tag{6}$$

When we transform the equation (2) as follows,

$$y = \tau_K - \tau = x_1 + x_2 + \cdots\cdots\cdots + x_{K-1} \tag{7}$$

y or $\tau_K - \tau$ is the sum of $(K-1)$ independent random variables and its characteristic function is given by

$$M_y(jv) = \prod_{S=1}^{K-1} Mx_s(jv) \tag{8}$$

which is the product of the $(K-1)$ characteristic functions of $x_s$. Also the probability density function of duration of the $K$-multiple pulses $P_K(y)$ is the Fourier transform of the characteristic function, that is,

$$P_K(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} M_y(jv) e^{-jvy} dv \tag{9}$$

By making use of the equations (5) and (6), the equation (8) is transformed as follows, that is,

$$M_y(jv) = \frac{\nu^{K-1}}{(1-e^{-\nu\tau})^{K-1}} \cdot \frac{1}{(\nu - jv)^{K-1}} \left\{1-e^{-(\nu-jv)\tau}\right\}^{K-1} \tag{10}$$

Then, by substituting the above equation into the integrand on the right side of the equation (9), the probability density function becomes

$$P_K(y) = \frac{1}{2\pi} \cdot \frac{\nu^{K-1}}{(1-e^{-\nu\tau})^{K-1}} \int_{-\infty}^{+\infty} \frac{e^{-jvy}}{(\nu-jv)^{K-1}} \cdot \left\{1-e^{-(\nu-jv)\tau}\right\}^{K-1} dv \tag{11}$$

Here in the above equation by letting

$$\nu - jv = p \tag{12}$$

and multiplying $\exp(\nu y)$ each side of the equation (11), the following equation is obtained,

$$e^{\nu y} \cdot P_K(y) = \frac{-j}{2\pi} \cdot \frac{\nu^{K-1}}{(1-e^{-\nu\tau})^{K-1}} \int_{\nu-j\infty}^{\nu+j\infty} \frac{(1-e^{-p\tau})^{K-1}}{p^{K-1}} \cdot e^{py} \cdot dy \tag{13}$$

When we take p, a general complex number, the integral on the right side of the equation (13) is nothing but the Bromwich-Wagner integral in the plain of the complex number p. By developing the integrand of the complex integral on the right side of the equation (13) due to the binomial theorem the equation is transformed into

$$e^{\nu y} \cdot P_K(y) = \frac{-j}{2\pi} \cdot \frac{\nu^{K-1}}{(1-e^{-\nu\tau})^{K-1}} \cdot \sum_{n=0}^{K-1} (-1)^n \underset{K-1\ n}{C} \int_{\nu-j\infty}^{\nu+j\infty} \frac{e^{(y-n\tau)p}}{p^{K-1}} dp \tag{14}$$

where each term of the right side has the pole of degree $(K-1)$ at the origin $p=0$. And so the probability density function of the $K$-multiple pulses is expressed by

$$P_K(y)=\frac{\nu^{K-1}}{(1-e^{-\nu\tau})^{K-1}}\sum_{n=0}^{(s)<y/\tau}(-1)^n\cdot \underset{K-1\ n}{C}\cdot\frac{(y-n\tau)^{K-2}}{(K-2)\,!}\cdot e^{-\nu y} \tag{15}$$

where $\underset{K-1\ n}{C}$ indicates the binomial coefficient and (s) above the addition sign indicates the maximum positive integer of n and is smaller than $y/\tau$.

### 2. 3 Distribution function of duration of the $K$-multiple pulses

When we assign Y to the duration of the $K$-multiple pulses, it is of course the random variable determined by the equation (7). Now, let it be that

$$Y=y+\tau\leqq(m+1)\tau \tag{16}$$

where m is a given positive real number and satisfies the following inequalities,

$$(S-1)\tau\leqq m\tau\leqq S\tau\leqq(K-1)\tau \tag{17}$$

Then the probability that the pulse duration Y is less than $(m+1)\tau$ is obtained by the integration of the equation (15) as follows,

$$\int_{\tau}^{(m+1)\tau}P_K(Y)\cdot dY=\int_0^{m\tau}P_K(y)\cdot dy$$

$$=\frac{\nu^{K-1}}{(K-2)\,!\,(1-e^{-\nu\tau})^{K-1}}\left\{\underset{K-1\ 0}{C}\int_0^{m\tau}e^{-\nu y}\cdot y^{K-2}\,dy-\cdots\cdots\right.$$

$$-\underset{K-1\ 1}{C}\int_{\tau}^{m\tau}e^{-\nu y}(y-\tau)^{K-2}\,dy+\underset{K-2\ 2}{C}\int_{2\tau}^{m\tau}e^{-\nu y}(y-2\tau)^{K-2}\,dy-\cdots\cdots$$

$$+(-1)^{S-1}\underset{K-1\ S-1}{C}\int_{(S-1)\tau}^{m\tau}e^{-\nu y}\cdot(y-s\tau+\tau)^{K-2}\,dy+(-1)^S\underset{K-1\ S}{C}\int_{S\tau}^{m\tau}e^{-\nu y}\cdot(y-s\tau)^{K-2}dy \tag{18}$$

Furthermore, by taking

$$y-t\tau=X\,;\qquad (t=0,\ 1,\ \cdots\cdots\cdots,\ s) \tag{19}$$

for each integral on the right side of the equation (18), and making use of the formula of indefinite integral

$$\int x^n\cdot e^{ax}\,dx=\frac{e^{ax}}{a}\sum_{r=0}^{n}(-1)^r\cdot\frac{n\,!\;x^{n-r}}{(n-r)\,!\;a^r} \tag{20}$$

the equation (18) is transformed into a more simplified form, that is

$$\int_{\tau}^{(m+1)\tau}P_K(Y)\,dY=\frac{\nu^{K-1}}{(1-e^{-\nu\tau})^{K-1}}\sum_{t=0}^{S}(-1)^t\cdot\underset{K-1\ t}{C}\;e^{-\nu t\tau}\cdot$$

$$\left\{\frac{1}{\nu^{K-1}}-\frac{e^{-\nu(m-t)\tau}}{\nu}\sum_{r=0}^{K-2}\frac{(m\tau-t\tau)^{K-2-r}}{(K-2-r)\,!\;\nu^r}\right\} \tag{21}$$

### 3. Distribution of duration of all the observed pulses

The equation (21) derived as above shows the probability that the durations of $K$-multiple pulses in the time interval under consideration are less than the time length $(m+1)\tau$. Accordingly, the probability that the durations of all the observed pulses are less than $(m+1)\tau$ in the same time interval, is clearly given by

$$\int_{\tau}^{(m+1)\tau} P(Y)\cdot dY = \sum_{K=1}^{\infty} \frac{R_K}{R}\cdot\int_{0}^{m\tau} P_K(y)\cdot dy \qquad (22)$$

where $R_K/R$ is the ratio of the average number of the $K$-multiple pulses to that of all the observed pulses per second. And for the case of $(m+1)\cdot\tau \geqq K\tau$, the following relation always holds

$$\int_{0}^{m\tau} P_K(y)\cdot dy = 1 \qquad (23)$$

And the probability density function of duration of all the observed pulses for the time interval under consideration, $P(Y)$ is obtained by differentiation of the equation (22) with respect to the variable $m\tau$, that is,

$$P(Y) = \sum_{K=1}^{\infty} \frac{R_K}{R}\cdot P_K(y) \qquad (24)$$

Now, for the purpose of deriving the accurate expression of calculable form for the distribution function and the probability density function of duration of all the observed pulses which are given by the equations (22) and (24), we will proceed to manipulate the equation (22) by making use of the equations (2), (15) and (24). After some manipulation of the equation (22), the distribution function of duration of all the observed pulses is transformed into the following form

$$\int_{\tau}^{(m+1)\tau} P(Y)\,dY = \sum_{K=1}^{[m]+1} (1-e^{-\nu\tau})\cdot e^{-\nu\tau}$$

$$+\sum_{K=[m]+2}^{\infty}\ \sum_{n=0}^{(S)<y/\tau} \int_{0}^{m\tau} (-1)^n\cdot\nu^{K-1}\cdot \underset{K-1\ n}{C}\cdot\frac{(y-n\tau)^{K-2}}{(K-2)!}\cdot e^{-\nu(\tau+y)}\,dy \qquad (25)$$

where $[m]$ indicates the positive integer which satisfies the following inequalities

$$m \geqq [m] \geqq m-1 \qquad (26)$$

At this stage, proceeding along the line of thought that we will change the order of the addition sign with respect to $K$ and $n$ in the second term on the right side of the equation (25), and divide it into two parts, each of which corresponds to $n=0$ and $n \geqq 1$ respectively, and simplify each part individually, making use of the mathematical formulas

$$\int_{0}^{P\tau} \nu(\nu y)^r\cdot e^{-\nu y}\cdot dy = r!\left\{1-e^{-p\nu\tau}\sum_{S=0}^{r}\frac{(p\nu\tau)^s}{s!}\right\} \qquad (27)$$

and

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} x^n = (1+x)\,e^x \quad (|x| < \infty) \tag{28}$$

we finally arrives at the following expression for the distribution function of duration of all the observed pulses.

$$\begin{aligned}
\int_{\tau}^{(m+1)\tau} P(Y)\,dY =& \sum_{K=1}^{[m]+1} (1-e^{-\nu\tau})^{K-1} \cdot e^{-\nu\tau} \\
&+ m\nu\tau \cdot e^{-\nu\tau} - \sum_{S=0}^{[m]-1} e^{-\nu\tau} \left\{ 1 - e^{-m\nu\tau} \sum_{t=0}^{S} \frac{(m\nu\tau)^t}{t!} \right\} \\
&+ \sum_{n=1}^{[m]} (-1)^n \cdot \frac{e^{-\nu(n+1)\tau}}{n!} \left[ \left\{ \nu\tau(m-n) \right\}^n + \frac{\{\nu\tau(m-n)\}^{n+1}}{n+1} \right] \\
&+ \sum_{n=1}^{[m]} \sum_{S=0}^{[m]-n} (-1)^{n+1} \cdot \frac{e^{-\nu(n+1)\tau}}{n!} \cdot \frac{s+n}{s!} \cdot (s+n-1)! \left[ 1 - \right. \\
&\left. e^{-(m-n)\nu\tau} \sum_{t=0}^{S+n-1} \frac{\{(m-n)\nu\tau\}^t}{t!} \right]
\end{aligned} \tag{29}$$

And the probability density function of duration of all the observed pulses is obtained by the differentiation of the equation (29), with respect to $m\tau$, that is

$$\begin{aligned}
P(Y) =& \nu e^{-\nu\tau} - \sum_{S=0}^{[m]-1} \nu \cdot e^{-(m+1)\nu\tau} \cdot \frac{(m\nu\tau)^s}{s!} \\
&+ \sum_{n=1}^{[m]} (-1)^n \cdot \frac{\nu \cdot e^{-\nu(n+1)\tau}}{n!} \left[ n \left\{ (m-n)\nu\tau \right\}^{n-1} \right. \\
&\left. + \left\{ (m-n)\nu\tau \right\}^n \right] + \sum_{n=1}^{[m]} \sum_{S=0}^{[m]-n} (-1)^{n+1} \cdot \frac{\nu e^{-\nu(n+1)\tau}}{n!} \\
&\cdot \frac{s+n}{s!} \cdot e^{-(m-n)\nu\tau} \left\{ (m-n)\nu\tau \right\}^{s+n-1}
\end{aligned} \tag{30}$$

which also corresponds to the equation (24).

Fig. 2 shows the calculated curves of the distributions of durations of all the observed pulses for the Poisson noise process due to the expression (29), where the abscissa indicates the time length in the unit of the duration of the original pulses $\tau$ and the ordinate indicates the log-log probability by which the durations of the observed pulses exceed the abscissa, and the parameter on the curves indicates various values of the average number of the original pulses per second times the constant duration of the original pulses.

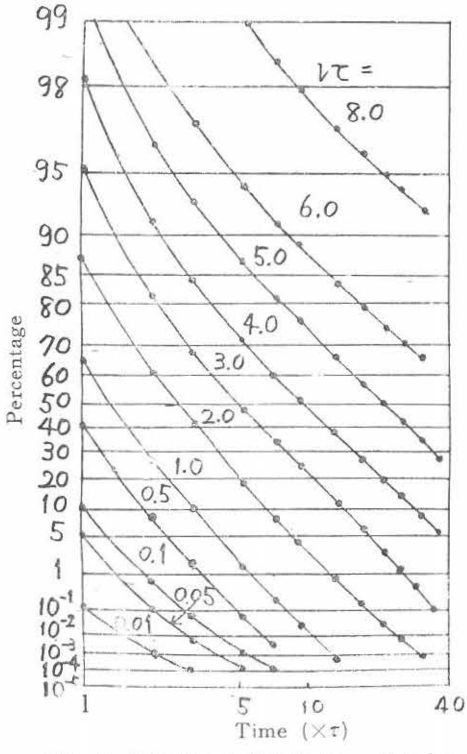


Fig. 2. Calculated distribution of pulse duration

## 4. Distribution of pulse duration for the atmospheric noise envelopes

Now, we will consider the behavior of the pulse durations when the atmospheric noise envelopes exceed a given voltage level. As the writer did elsewhere,[1] let us assume again that the peak amplitude distribution of the impulses in the atmospheric noise field at the input of the receiver is represented by a power function of the voltage level and the impulse response, i. e, the waveform of the envelope voltage generated by an impulse of a unit amplitude may be expressed at the output of the receiver as follows,

$$E(t) = 2\pi B G_0 \exp(-\pi B t) \qquad (31)$$

where B is the bandwidth between 3-db points of the receiver response curve and $G_0$ is the receiver gain at a reference frequency. And the probability density function of peak amplitude of the impulses[1] is given as follows,

$$P(v) = \frac{1}{n_m} \cdot \frac{r}{A} \left(\frac{v}{A}\right)^{-r-1} \qquad (32)$$

where A is the voltage level exceeded just by one impulse per second, r is the constant parameter and $n_m$ is an expected number per second that exceeds the least measurable amplitude of the impulses. Now, because the pulse duration $\tau$ when the impulse response of a given amplitude v exceeds a given voltage $v_0$ is given by the equation (31) as follows,

$$\tau = \frac{-1}{\pi B} \log_e \left(\frac{v_0}{2\pi B G_0 v}\right) \qquad (33)$$

the distribution function of duration of the pulses when the atmospheric noise envelopes exceed a given voltage level, $Q(\tau)$ may be expressed as follows,

$$Q(\tau) = \int_{\frac{v_0}{2\pi BG_0} \cdot \exp \tau/\pi B}^{\infty} \frac{1}{n_m} \cdot \frac{r}{A} \cdot \left(\frac{v}{A}\right)^{-r-1} \cdot dv \qquad (34)$$

Of course, we consider the above distribution of duration of the pulses to be original or imaginary, at the stage before the effects of interference between the successively arriving impulse responses result in the observed behavior of the pulse durations at the output of the receiver. Such an original distribution of pulse duration changes depending on the values of the parameter r which characterizes the peak amplitude ditsribution of the impulses. Generally, the durations of original pulses when the atmospheric noise envelopes exceed a given voltage level change randomly and the accurate distribution of pulse duration for this case has not yet been obtained. And this problem may be very difficult to solve.

On the basis of the analysis as described above, for the Poisson noise process with a constant duration of the original pulses, it may be stated as follows. The observed distribution of pulse duration of the atmospheric noise envelopes at a given voltage level may be a slightly transformed one from the original distribution of pulse duration, when the average number of pulses per second times the average duration of the original pulses is small. And with the increase of the product under consideration, the observed distribution of the pulse duration may be largely transformed from the original distribution. For the latter case, if the average number of the original pulses per second times the average duration of the original pulses is obtained, when the atmospheric noise envelopes exceed a given voltage level, the distribution of pulse duration calculated with this value by the equation (29) is expected to express a reasonable approximation for the observed distribution of duration of the pulses.

## 5. Conclusion

We have derived a physically satisfactory expression of the distribution of duration of pulses for the Poisson noise process as described in the paper. For the case of the atmospheric noise envelopes, the behaviors of pulse duration are very complex and the accurate expression of the distribution function has not been obtained. In future, we think it very interesting if a comparison is made between the observed distribution and the original distribution and the calculated distribution due to the expression (29) when the atmospheric noise envelopes exceed a given voltage level.

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## Reference

(1) Nakai, T.: Proc. Res. Inst. Atmospherics, Nagoya Univ. 10, 13 (1963)