DISTRIBUTION OF THE SPACING BETWEEN THE OCCURRENCE TIMES OF PULSES IN THE POISSON NOISE PROCESS

Taketoshi NAKAI

Abstract

The cumulative distribution of the spacing between the occurrence times of pulses, when the atmospheric noise envelopes exceed a given voltage level, contains an information about the time variations of the atmospheric noise field, and so it is available in determining the behaviors of the wireless communication system in existence of the atmospheric noise. Lately, the cumulative distributions of the time length of quiescent level between pulses have been measured[1],[2], when the atmospheric noise envelopes exceed various voltage levels. The measured distributions in such cases may be a good approximation to the distributions of the spacing between the occurrence times of pulses in the directions of the high voltage levels, but it is supposed that a departure arises between the two kinds of distributions in the directions of the low voltage levels. A theoretical analysis for this departure has not yet been published. In this paper, due to the analysis for the time series of pulses in the Poisson noise process,[3] the probability density and the cumulative distribution of the spacing between the occurrence times of pulses have been derived. And the calculated curves have been obtained due to the theoretically derived functions and it is shown that a remarkable departure arises between the cumulative distributions and the Poisson distribution, especially in the directions of low voltage levels.

1. Introduction

The author reported elsewhere[3] the cumulative distributions of pulse duration and some other behaviors of the pulses in the Poisson noise process. In this paper, by expanding the discussions there, we describe the theoretical derivations of the probability density function and the cumulative distribution function of the spacing between the occurrence times of pulses in the Poisson noise process, and the results of numerical calculations of the two functions.

2. Integral expression of the probability density function of the spacing between the occurrence times

Firstly, let us derive the integral representation of the probability density function of the spacing between the occurrence times of pulses. The time relations between various times associated with the two observed pulses successively arrived are such as shown in Fig. 1. Y indicates the pulse duration of the early arrived pulse of the two observed pulses, and y indicates the spacing between the beginning of the observed pulse and the beginning of the kth original pulse, counting from early original pulses to late ones, when the observed pulse is resulted from the overlappings between the k original pulses. And then it is found that

$$Y = y + \tau \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

where τ is the duration of the original pulse.
And s indicates the time length of the quiescent level between the two observed pulses, and so $(s+\tau)$ indicates the spacing between the occurrence time of the kthly arrived original pulse which is the component of the early arrived observed pulse and the beginning of the original pulse arrived soon after the original one specified above, that is, the beginning of the secondly arrived observed pulse of the two observed pulses.

Now, it is easily found that the $(s+\tau)$ also obeys the Poisson distribution law due to the assumption,[3] where its variable range is now modified as follows,

$$\tau < s + \tau < \infty \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$

Such change of the variable range is resulted from the overlappings between the original pulses in the noise process now considered. The probability density function of s is easily derived by making use of the equation (2) and is as follows,

$$q(s) = \nu e^{-\nu s} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3)$$

where $\quad 0 \leq s < \infty$

The equation (3) means that the time length of the quiescent level between the successively observed two pulses in the Poisson noise process is the random variable

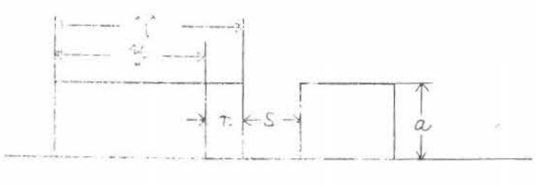


Fig. 1. Spacings between various times for two observed pulses.

obeying the Poisson distribution. Secondly, let it be that z indicates the spacing between the occurrence times of the two successively observed pulses. Then the following relation holds, that is,

$$z = Y + s \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (4)$$

where Y and s are independent random variables for each other and z is also random variable, because it is the sum of the two independent random variables. Now, by using the probability density functions of Y and s, the probability density function of z is expressed as follows,

$$P(z) = \int_{-\infty}^{+\infty} P(Y)\, q(z-Y)\, dY \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5)$$

and the following inequalities hold here

$$\tau \leqq Y < \infty, \qquad Y \leqq z < \infty \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (6)$$

where $P(z)$, $P(Y)$ and $q(z-Y)$ indicate the probability density functions of the spacing between the occurrence times of the observed pulses, the duration of the observed pulses, and the time-length of the quiescent level between the observed pulses, respectively.

Now, the integral appearing in the equation (5) is well known as the convolution integral.[4] By using the inequalities (6), showing each variable range of the duration of the observed pulses and the spacing between the occurrence times of the observed pulses, and the equation (4) showing the analized relation of the duration of the observed pulse, the integral on the right side of the equation (5) is transformed as follows,

$$P(z) = \int_{0}^{z-\tau} P(Y)\, q(z-y-\tau)\, dy \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (7)$$

The equation (7) expresses the probability-relation between the three random variables, that is, the duration of the observed pulse, the spacing between the occurrence times of the observed pulses and the time length of the quiescent level between the observed pulses. And the purpose of the following analysis is to derive an accurate and calculable expression for the cumulative distribution of the spacing between the occurrence times of the observed pulses, due to the equation (7).

3. Probability density and cumulative distribution functions of the spacing btween the occurrence times

3.1 Representation of the probability density function of the duration of the observed pulses

For the noise process now considered, it has been shown that the probability density function of durations of all the observed pulses, $P(Y)$ or $P(y)$ is expressed as follows,

$$P(Y) = P(y) = \sum_{k=1}^{\infty} (1-e^{-\nu\tau})^{k-1} \cdot e^{-\nu\tau} \cdot P_k(y) \cdots\cdots\cdots\cdots (8)$$

And the probability density function $P_k(y)$ of duration of the k-complex pulses has been derived for arbitrary values of positive integer except for $k=1$. Now, for the purpose of substituting into the integrand on the right side of the equation (7), let us derive a representation of the probability density function of all the observed pulses, which holds for the cases of positive real values of y and $y=0$, that is, the duration of the original pulse. The required expression is derived as follows. For the case $k=1$, the observed pulse is nothing but the arrived original pulse and its duration is always τ. Then the probability density function $P_1(y)$ is expressed as follows,

$$P_1(y) = \delta(y) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (9)$$

where $\delta(y)$ is so called the Dirac delta function and is defined as follows.

$$\delta(y) = \begin{cases} \infty, & y = 0 \\ 0, & y \neq 0 \end{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (10)$$

and

$$\int_0^{\infty} \delta(y)\, dy = 1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (11)$$

By using the equation (9) and the expression $P_k(y)$, the equation (8) is transformed into the followings,

$$\begin{aligned} P(Y) &= P(y) = e^{-\nu\tau} \cdot \delta(y) \\ &+ \sum_{k=2}^{\infty} \sum_{n=0}^{(s)<y/\tau} (-1)^n \cdot \nu^{k-1} {}_{k-1}C_n \frac{(y-n\tau)^{k-2}}{(k-2)!} e^{-\nu(\tau+y)} \cdots\cdots\cdots (12) \end{aligned}$$

where y is any real positive number containing zero.

3.2 Probability density function and cumulative distribution function of the spacing between the occurrence times

For the equation (7) representing the probability density function of the spacing between the occurrence times now considered, substituting the equations (3) and (12) into the two probability density functions, which are the integrand on the right side, the equation (7) are transformed as follows,

$$P(z) = \int_0^{z-\tau} \Big\{ \nu e^{-(z-y)}\, \delta(y)$$

$$+ \sum_{k=2}^{\infty} \sum_{n=0}^{(s)<y/\tau} (-1)^n \nu^k \underset{k-1\ \ n}{C} \frac{(y-n\tau)^{k-2}}{(k-2)!} e^{-\nu z} \Big\} dy \quad \cdots\cdots\cdots\cdots \quad (13)$$

Now, let us define m a positive real number, and equate $m\tau$ to the upper limit of the integral on the right side of the equation (13), that is

$$z - \tau = m\tau \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (14)$$

and define [m] zero or positive integer, satisfying the following inequalities,

$$[m+1] \geqq m > [m] \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (15)$$

Then, it is found that the maximum positive integer of n appearing in the second term of the right side of the equation (13), is nothing but [m] defined now.

Now, by taking

$$y - n\tau = t \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (16)$$

the equation (13) is transformed into the following

$$P(z) = \nu e^{-\nu\tau} + \sum_{n=1}^{[m]} I_n \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (17)$$

where

$$I_n = \int_0^{(m-n)\tau} \Big\{ \sum_{k=n+1}^{\infty} (-1)^n \underset{k-1\ \ n}{C} \frac{\nu(\nu t)^{k-2}}{(k-2)!} \Big\} \nu e^{-\nu(m+1)\tau} dt \quad \cdots\cdots \quad (18)$$

Now, manipulating the equation (18) by using the mathematical formula

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} x^n = (1+x)e^x \qquad |x| < \infty \quad \cdots\cdots\cdots\cdots \quad (19)$$

and

$$\int_0^a \nu(\nu t)^n e^{-\nu t} dt = n! \Big\{ 1 - e^{-\nu a} \sum_{r=0}^{n} \frac{(\nu a)^r}{r!} \Big\} \quad \cdots\cdots\cdots\cdots \quad (20)$$

the following equation holds.

$$I_n = \frac{(-1)^n}{n!} \Big\{ (m-n)\nu\tau \Big\}^n \nu e^{-(n+1)\nu\tau} \quad \cdots\cdots\cdots\cdots\cdots \quad (21)$$

Then, the mathematical expression of the probability density function of the spacing between the occurrence times of the observed pulses is, due to the equations (17) and (21), derived as follows

$$P(z) = \sum_{n=0}^{[m]} \frac{(-1)^n \{(m-n)\nu\tau\}^n}{n!} \nu e^{-(n+1)\nu\tau} \quad \cdots\cdots\cdots\cdots \quad (22)$$

where

$$z = (m+1)\,\tau$$

And the mean $\bar{z}$ of the spacings between the occurrence times of the observed pulses per second is

$$\bar{z} = \frac{e^{\nu\tau}}{\nu} \qquad \cdots\cdots\cdots\cdots \qquad (23)$$

because it is the reciprocal of the average number of all the observed pulses per second.(3)

Furthermore, by integrating the expression on the right side of the equation (22), the probability P is derived that the spacing between the occurrence times of the observed pulses are larger than a given time length $(l+1)\tau$, that is

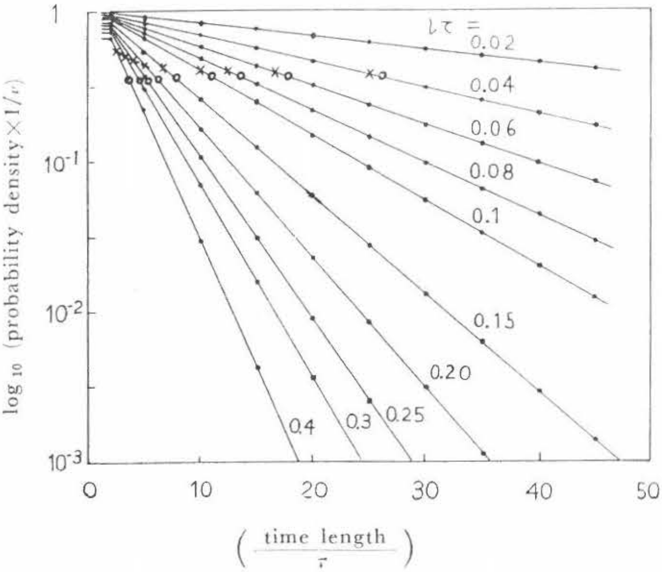
$$P\,(z \geqq (l+1)\,\tau) = 1 - \int_{\tau}^{l\tau} \sum_{n=0}^{m} \frac{(-1)^{n}\,\{(m-n)\,\nu\tau\}^{n}}{n\,!}\,\nu e^{-(n+1)\nu\tau}\,d\,(m\tau) \quad \cdots\cdots \quad (24)$$

where l is a positive real number.

4. Results of numerical calculation

Now, it is found from the equations (22) and (23) derived as above, that $1/\nu$ times the probability density function and the cumulative distribution of the spacing between the occurrence times of the observed pulses are the functions of $m\tau$ and $l\tau$ respectively. And the calculated curves of these functions are shown in Fig. 2 and Fig. 3 (a) and (b) for various values of $\nu\tau$, due to the numerical calculation of the equations (22) and (24). The signs of plots ∘ and × indicate the mean length $\bar{z}$ of the spacing between the occurrence times of the observed pulses and the mean spacing of the arrival times of the original pulses, respectively. Furthermore, the coordinate-systems in Figs. 2 and 3 (a) and 3 (b) are taken, so as to have the slopes of the corresponding straight lines $(-0.4343\nu\tau)$ and (-1), when the spacings between the occurrence times obey the Poisson distribution.

Now, it is found from the curves in Figs. 2 and the equation (22) that the cumulative distributions display the most remarkable departures from the Poisson distribution over the range of time length τ to 2τ for various values of $\nu\tau$, because the probability density of the spacing between the occurrence times of the observed pulses are uniform over the range specified above, and the probability density is $\nu \exp(\nu\tau)$ and the contribution to the probability due to the range of the time length now considered is $\nu\tau \exp(\nu\tau)$. The probability $\nu\tau \exp(\nu\tau)$ increases with $\nu\tau$ increasing, and after it attains the maximum value $\exp(-1)$ at $\nu\tau = 1$, it decreases with



○ average spacing between the arrival of the original pulses.

× average spacing between the occurrence times of the observed pulses.

Fig. 2. Calculated curves of the probability density × $1/\nu$.

increasing $\nu\tau$.

The behavior of the change of the probability around $\nu\tau = 1$ relates to the characteristics of the calculated curves in Figs. 3(a) and (b). Otherwisely speaking, cumulative distributions for various values of $\nu\tau$ on each side around $\nu\tau = 1$, are found to display roughly similar characteristics. Furthermore, it is found that $1/\nu$ times the probability density and the cumulative distributions approach the Poisson distribution in the directions of large time length $l\tau$ and $m\tau$, respectively. Especially in the directions of very large or very small time length $\nu\tau$, the cumulative distributions almost obey the Poisson distribution.

5. Conclusion

It has been proved that the time length of quiescent level between the observed pulses also obeys the Poisson distribution, but the distributions of the spacing between the occurrence times of the observed pulses generally depart from the law, when the original pulses arrive at time spacings determined by the Poisson distribution law in the Poisson noise process.

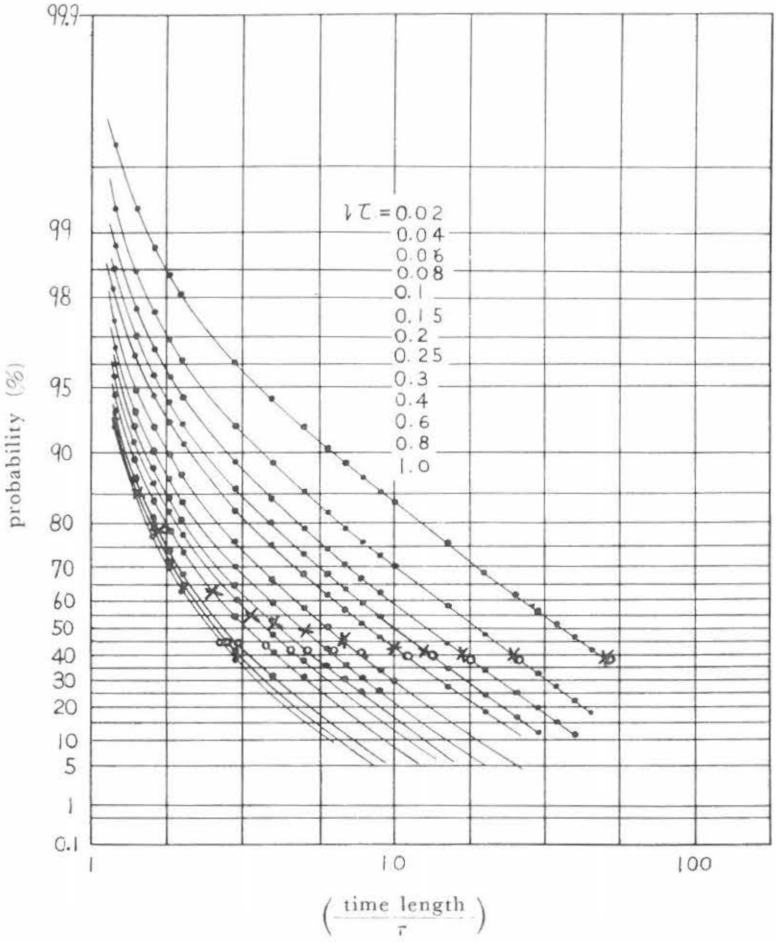


Fig. 3(a). Calculated curves of the distributions

Now, in measurements of the time functions of pulses in the past when the atmospheric noise envelopes exceed a series of voltage levels, the spacing between the occurrence times of the observed pulses were not measured, but the time length of quiescent level between the observed pulses, that is, the time spacing between the end of early arrived pulse and the beginning of successively arrived pulse, were measured. The measured distributions in such case give a good approximation for the distribution of the spacing between the occurrence times of pulses, because the duration of the observed pulses are generally small in the directions of high voltage levels. But in the directions of low voltage levels, such approximation as above does not hold by any means, because the duration of the observed pulses are not always small in this case. In general case, we can not know the accurate behavior of the distribution of the spacing between the occurrence times of the observed pulses, by such measuring method as utilized in practice. But it is very interesting to note that the time length

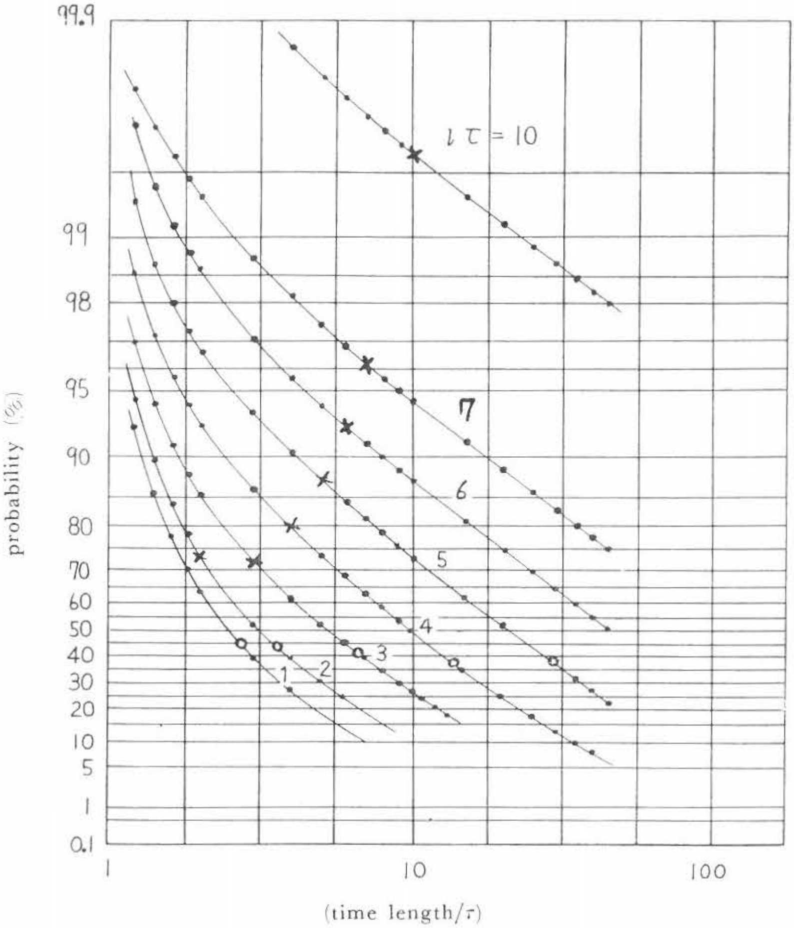


Fig. 3(b). Calculated curves of the distributions

of quiescent level between the observed pulses for the atmospheric radio noise has been measured to be likely to obey the Poisson distribution, when the atmospheric noise envelopes at VLF exceed a given voltage level in the directions of fairly low voltage levels.

The observational results agree with the analytical result as described above. And it may lead to the following conclusion, that is, it is true in practice that the atmospheric impulses arriving at the antenna obey the Poisson distribution. At the same time, it is clear from the analysis as described above that the spacing between the occurrence times between the observed pulses remarkably depart from the Poisson distribtion when the atmospheric noise envelopes exceed given especially low voltage levels.

Now, let us imagine that the amplitude distribution of the atmospheric impulses at the antenna, the characteristics of band-width of the receiver used are known.

Then, by assuming the approximate time series of pulses for the true time series of the parts of the noise envelopes, when the atmospheric noise envelopes exceed a given voltage level, we can calculate the average number per second and the duration of the original pulses from the knowledge of the impulse response, and so we can calculate the cumulative distribution of the spacing between the occurrence times of the observed pulses as a function of the voltage level.

6. Acknowledgement

The author expresses his deep gratitude to prof. A. Kimpara, director of the Research Institute of Atmospherics, for his encouragement and guidance in this study. The author expresses his deep appreciation to Mr. M. Nagatani and Miss. Y. Ito, for their helps in preparing this paper. And the calculation in this paper was done by making use of the electronic computer NEAC-2203 at Nagoya University.

References

(1) Watt, A. D. and Maxwell, E. L.: Proc. IRE, **45**, 1, p. 55 (1957)
(2) Nakai, T and Suzuki, Y: Proc. Res. Inst. Atmospherics, Nagoya Univ. **11** (1961)
(3) Nakai, T: Proc. Res. Inst. Atmospherics, Nagoya Univ. **11** (1964)
(4) Bennet, W. R.: Proc. IRE, **44**, 5 (1956)