Unified understanding of different quantization methods via resurgence

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Resurgence theory is a method to investigate the non-trivial relationship between perturbation and non-perturbation contributions by analysis based on the Stokes phenomenon of Borel summation and Picard Lefschetz theory. There are two different approaches in the resurgence of quantum mechanical systems: one based on perturbative expansions and semi-classical approximations, and the other is the exact-WKB analysis based on the Schrödinger equation. In this thesis, we clarify how the two resurgence are connected each other and determine the relation among the different quantization methods: Schrödinger equation, Bohr-Sommerfeld, Gutzwiller trace formula and path integral without any approximation. Furthermore, we show the resurgent structure of each quantization method, i.e., how the perturbative and non-perturbative contributions are related exactly. We also provide a new method to calculate the intersection number of Lefschetz thimbles in the path integral and the hidden relation between the Maslov index in Gutzwiller trace formula.

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CONTENTS

I. INTRODUCTION

There are two approaches in the study of non-perturbative aspects of quantum mechanical systems based on perturbative series. The first one, called the resurgence analysis [2–9], we utilize a nontrivial relation between the perturbative and non-perturbative contributions via the Borel resummation of the series expansion of physical quantities [13–27, 29–51, 56–58]. In the approach, we express physical quantities such as ground state energy or the partition function as series containing perturbative and non-perturbative contributions,

$$Z(\hbar) = \sum_n a_n \hbar^n + e^{-\frac{S_1}{\hbar}} \sum_n b_n \hbar^n + e^{-\frac{S_2}{\hbar}} \sum_n c_n \hbar^n + ..., \tag{1}$$

This type of series is called as *trans-series.* The perturbative and non-perturbative parts are usually calculated independently. However, they are connected through imaginary ambiguities called the Borel ambiguity, which can arise in the Borel summations of the series of each sector. The Borel ambiguity including the non-perturbative factor $e^{-\frac{S_1}{\hbar}}$ (S_1 is sometimes called the instanton action) is expressed as

$$(\mathcal{S}_+ - \mathcal{S}_-) \left[\sum_n a_n \hbar^n \right] \propto \, i e^{-\frac{S_1}{\hbar}} \,, \tag{2}$$

Here $\mathcal{S}_\pm$ stands for the operation of lateral Borel summation, in which the index $\pm$ means how the Laplace integral contour in the Borel summation avoids the Borel singularity, which is determined by the sign of $\mathrm{Im}\,\hbar$. This phenomenon is regarded as the Stokes phenomena in the Picard-Lefschetz theory, where the topological structure of the Lefschetz thimble decomposition discontinuously changes [119–131]. From a physical point of view, it means that the perturbative part must have information of the non-perturbative part since the physical quantity cannot be ambiguous and such ambiguities in the Borel summation should be cancelled out. This structure has been intensively studied not only in quantum mechanics but also in quantum field theories [84–118] and matrix models [59–83]. .

The second one is called the exact-WKB analysis [132, 133, 135–145], which has been studied mainly by mathematicians, one investigates the monodromy of solutions to certain differential equations using the Borel summation. Our main interest is in its application to the Schrödinger equation,

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+V(x)\right)\psi(x)=E\psi(x)\,. \tag{3}$$

Here $\psi(x)$ correspond to the wave function and E stands for the energy eigenvalue. We now consider the analytic continuation of the WKB-solution of $\psi(x)$ from one region to another region in complex plane of x, then its asymptotic behavior sometimes changes discontinuously.

$$\psi^+_{\mathrm{I}}(x)\to\psi^+_{\mathrm{II}}(x)+\psi^-_{\mathrm{II}}(x), \tag{4}$$

where $\psi^\pm(x)$ stands for the two independent solutions of the Schrödinger equation. This is another type of the Stokes phenomenon, and this change can be examined by studying the Borel summation of the WKB solutions of wave functions. From this information, we can investigate the properties of solutions and eigenvalues. In this study, we reveal how these two different Stokes phenomena in the two "seemingly" different analyses are related.

The other point we focus on is the equivalence of the different quantization conditions. The recent intensive studies on the resurgent structure in quantum mechanics, matrix models and quantum field theories, draw attention of theorists to exact quantization conditions since the resurgent structure should be hidden there. However, the relation or the equivalence among different quantization conditions such as Bohr-Sommerfeld, Gutzwiller [150], path-integral and spectral-summation ones has not yet been clarified. The exact-WKB analysis based on Stokes curves can be a powerful tool to investigate the quantization conditions although few works have attempted to do it so far. In this work, we figure out the nontrivial equivalence of the quantization conditions.

Let us briefly summarize our findings:

1. **Unified understanding of the two Stokes phenomena:** As well-known, in the semi-classical analysis of the double-well quantum system, the Stokes phenomenon occurs in each of perturbative and non-perturbative sectors, where the imaginary ambiguities are canceled out between the perturbative Borel resummation and the non-perturbative bion contribution [155–164]. This is the resurgent structure in the path integral formulation. In the exact-WKB analysis, we find the same Stokes phenomenon takes place as the change of the "topology" of the Stoke curve. This correspondence is expressed as the Delabaere-Dillinger-Pham (DDP) formula, which determines the resurgent structure of each quantization method.

2. **Generalizing the Gutzwiller trace formula:** The Gutzwiller trace formula is a semi-

classical method that ties spectrum of quantum theory to classical mechanical concepts, to periodic orbits calculations, actions, geometric phases [150]. However, there was no unified way to determine which periodic solution should be added up as a unit (prime periodic orbit)and how to sum up the units. In this work, we discover the uniform way to identify the unit orbits and how to sum them up with including instanton effects. Furthermore, we find the Stokes phenomenon in Gutzwiller trace formula is expressed as the reversal of the direction of rotation of the perturbative cycles.

3. **Novel meaning of quasi-moduli integral:** The above findings give new physical meaning to the quasi-moduli integral (QMI) in the semiclassical analysis of path integral. Using the perspective of Gutzwiller trace formula, the non-perturbative contribution obtained from QMI is shown to have a nontrivial relation with the perturbative contribution around the classical vacuum.

4. **Discovering the relation between Maslov index and the intersection number of Lefschetz thimble:** The resolvent $G(E)$ obtained from the quantization condition $D(E)$ in the exact-WKB analysis can be rewritten in the Gutzwiller trace formula and it can be compared to $G(E)$ derived in the Gutzwiller's quantization condition. We then find that $(-1)^n$ appearing in the non-perturbative sector is interpreted as the Maslov index.

5. **Equivalence of the different quantization conditions:** we clarify the relation among the different quantization methods: Schrödinger equation, Bohr-Sommerfeld, Gutzwiller trace formula and path integral without any approximation.

6. **Generalization to symmetric multi-well potential:** We generalize the exact-WKB analysis and the DDP formula to the quantum systems with generic symmetric multi-well potentials. Again, we show that the Stokes phenomena occur as the topological change of Stokes curve in the exact-WKB analysis, and the resurgent structure in the semiclassical analysis is completely incorporated in the DDP formula.

The paper is constructed as follows. In Sec. II A we review path integral, Lefschetz thimble decompositions, resolvent methods and Gutzwiller's quantization in quantum mechanics. In Sec. III and Sec. III D we review the exact-WKB method, and explain Stokes curves for potential problems in simple examples. In Sec. IV we apply the exact-WKB analysis to double-well potential quantum mechanics by studying the associated Stoke curves, where we find the equivalence of the two Stokes phenomena and show the equivalence among several quantization conditions. In Sec. V we extend our investigation to the systems with generic multi-well potentials and discuss outcomes. In Sec. VI we focus on the ordinal QMI calculation and the divergence problem called $1/\epsilon$ *problem*. Sec. VII is devoted to summary and discussion.

The main result of this paper is summarized by the following flowchart Fig.1.

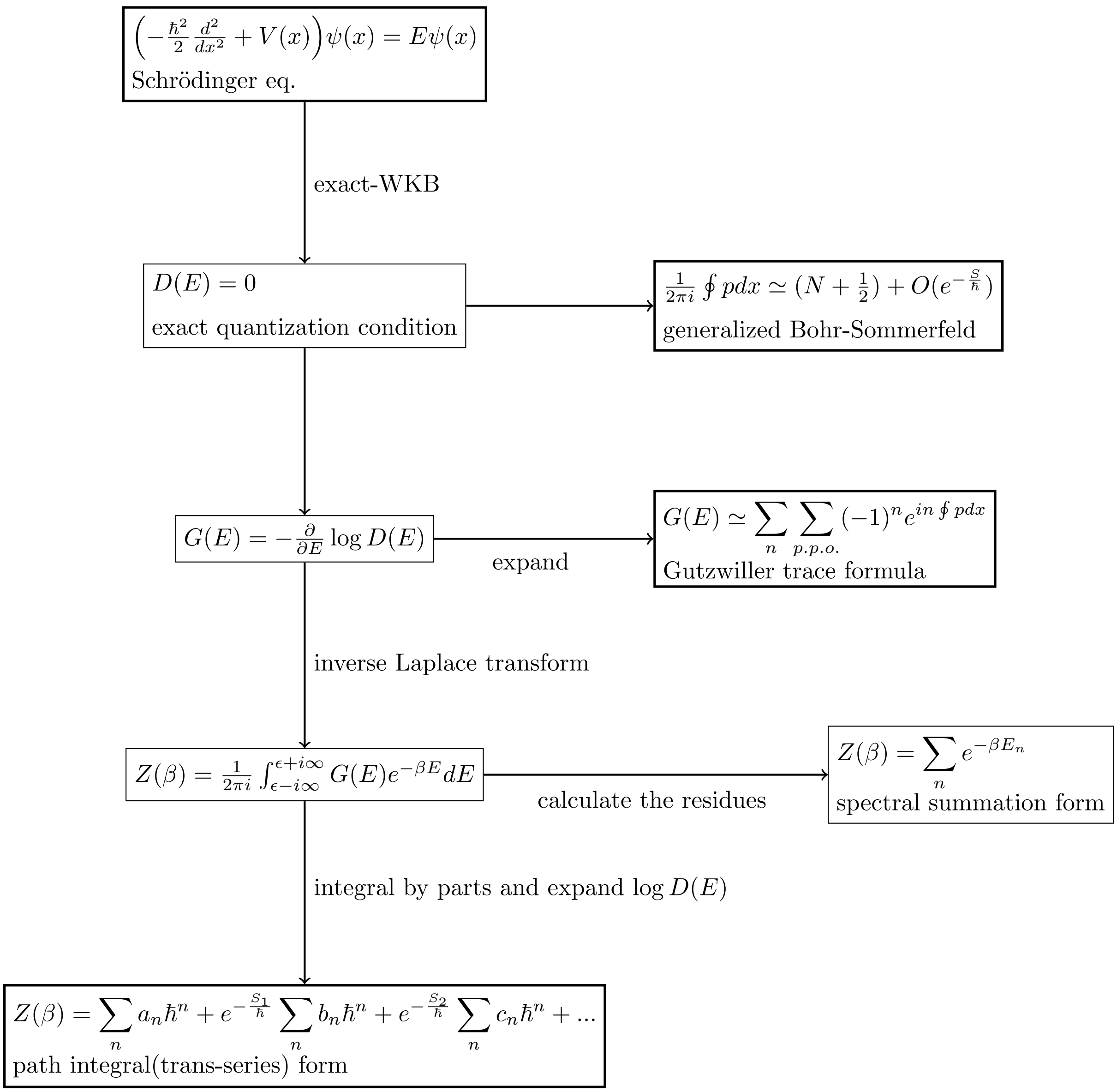

FIG. 1. The relation among several quantization methods. ($D(E)=\det\left(\hat{H}-E\right)$, $G(E)=\mathrm{tr}\,\frac{1}{\hat{H}-E}$, $Z(\beta)=\mathrm{tr}\,e^{-\beta\hat{H}}$) We can identify the resurgent structure of each case without approximation from the exact-WKB.

II. PREPARATION

In this section, we introduce the tools other than the exact-WKB analysis as prerequisite knowledge. These include saddle point decomposition of path integrals (Lefschetz thimbles), its relation to resolvent, Gutzwiller's quantization and Maslov index.

A. Borel summation and Lefschetz thimble decomposition

In quantum theory, perturbative expansion of the Euclidean partition function with field $x(\tau)$ is expressed as

$$Z(\beta,\hbar)=\int \mathcal{D}x\; e^{-\frac{S[x]}{\hbar}}=\sum_n a_n\hbar^n+e^{-\frac{S_1}{\hbar}}\sum_n b_n\hbar^n+e^{-\frac{S_2}{\hbar}}\sum_n c_n\hbar^n+\dots, \tag{5}$$

where β is Euclidean time period. This type of series is called trans-series and generally each of the series is asymptotic expansion, i.e. the radius of convergence is zero. This statement was first suggested by Dyson[1] in QED, and it was proved that in quantum systems, this comes from the factorial increase in the number of Feynman diagrams as the order of the perturbative series grows. This can be immediately understood using a simple integral (0−dim ϕ^4 theory), such as

$$Z=\int_{-\infty}^{\infty} dx\; e^{-\frac{S(x)}{\hbar}} \quad S(x)=\frac{1}{2}x^2+\frac{1}{4}x^4 \tag{6}$$

Using the rescale $x\to\sqrt{\hbar}x$ and expanding this into a perturbative expansion around $x=0$, i.e., expanding the interaction term $(-\frac{\hbar}{4}x^4)$,

$$\begin{aligned}Z(g)=\sqrt{\hbar}\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}x^2-\frac{\hbar}{4}x^4}&=\sqrt{\hbar}\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}x^2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\hbar}{4}\right)^n x^{4n}\\&=\sqrt{2\hbar}\sum_{n=0}^{\infty}\frac{\Gamma(2n+\frac{1}{2})}{\Gamma(n+1)}(-\hbar)^n\\&\cong\sqrt{2\hbar}\sum_n n!(-\hbar)^n\end{aligned} \tag{7}$$

and this is indeed an asymptotic series of factorial divergence with a zero convergence radius. On the other hand, the original integral (6) is convergent. The reason the well-defined function (6) became an divergent series is interchanging the sum and integration during the calculation. However, This procedure is nothing but ordinal pertubative expansion using Feynman diagram.

A method to make sense of factorially divergent series is the *Borel summation*, which makes the series convergent.

1. Borel summation

The Laplace transform of the Borel transform is called Borel summation, which has the same asymptotic expansion as the original series but is the convergent function: that is, for a series expansion such as

$$Z(\hbar) = e^{-\frac{A}{\hbar}} \sum_{n=0}^{\infty} a_n \hbar^{n+\alpha} \qquad \alpha \notin \{-1, -2, -3, ...\} \,. \tag{8}$$

The Borel transform of this series is defined as

$$\mathfrak{B}[Z](z) \equiv \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+\alpha)} (z-A)^{n+\alpha-1} \,, \tag{9}$$

The complex plane in which z is defined after the Borel transform is called Borel plane. The Borel summation $\mathcal{S}[Z]$ is defined as Laplace transform of the Borel transform:

$$\mathcal{S}[Z](\hbar) \equiv \int_A^{\infty e^{i\theta}} e^{-\frac{z}{\hbar}} \mathfrak{B}[Z](z)\, dz \,, \qquad \theta = \mathrm{Arg}(\hbar) \,. \tag{10}$$

If the series is absolutely convergent, this procedure just return the original series due to the identity

$$1 = \frac{1}{\Gamma(n+\alpha)} \int_0^{\infty} e^{-x} x^{n+\alpha-1} dx \,. \tag{11}$$

Because,

$$Z(\hbar) \cdot 1 = e^{-\frac{A}{\hbar}} \sum_{n=0} a_n \hbar^{n+\alpha} \frac{1}{\Gamma(n+\alpha)} \int_0^{\infty} e^{-x} x^{n+a-1}\, dx \tag{12}$$

$$\text{“}=\text{”} \int_0^{\infty} e^{-\frac{A}{\hbar} - x} \hbar \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+\alpha)} (x\hbar)^{n+\alpha-1}\, dx \tag{13}$$

$$\text{“}=\text{”} \int_A^{\infty e^{i\theta}} e^{-\frac{z}{\hbar}} \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+\alpha)} (z-A)^{n+\alpha-1}\, dz \,, \quad (z = A + \hbar x) \tag{14}$$

$$\text{“}=\text{”} \int_A^{\infty e^{i\theta}} e^{-\frac{z}{\hbar}} \mathfrak{B}[Z](z) = \mathcal{S}[Z](\hbar) \tag{15}$$

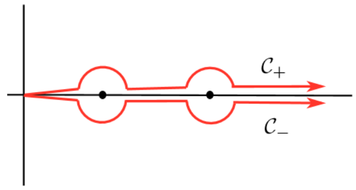

FIG. 2. The deformation of the integral path[53]. This procedure is same to $\hbar \to \hbar \pm i\epsilon$

Hence, if the original series is absolutely convergent, the interchange of integral and summation does nothing so this procedure just return the original series due to the identity (11). The Borel summation is a homomorphism, so that the following algebraic properties hold.

$$\begin{aligned} \mathcal{S}[A+B] &= \mathcal{S}[A] + \mathcal{S}[B] \\ \mathcal{S}[AB] &= \mathcal{S}[A]\mathcal{S}[B] \end{aligned} \tag{16}$$

Now, we consider applying this Borel summation to (7) with assuming $\hbar$ is real. The result is

$$\mathcal{S}[Z](\hbar) = \sqrt{\frac{\hbar}{\pi}}\Gamma\left(\frac{1}{4}\right)\int_0^\infty e^{-z}\frac{1}{z^{\frac{1}{4}}(1+4z\hbar)^{\frac{1}{4}}}dz \tag{17}$$

$$= \frac{1}{\sqrt{2}}e^{\frac{1}{8\hbar}}K_{\frac{1}{4}}\left(\frac{1}{8\hbar}\right), \tag{18}$$

where $K_{\frac{1}{4}}(x)$ is the modified Bessel function of the second kind. This is exactly same to the original integral (6).

When evaluating the Borel summation, singularities (or branch cut) can occur on the integral path. This is called Borel singularity, and in this case, in order to define the integral, we need to avoid singularities by deforming the integral path (2) or equivalently, introducing a small imaginary term to the parameter ($\hbar$). This gives ambiguity to the Borel summation. For example,

$$Z(\hbar) = \sum_{n=0}^{\infty} n!\hbar^{n+1} \quad \to \quad \mathcal{S}[Z](\hbar) = \int_0^\infty e^{-\frac{z}{\hbar}}\frac{1}{1-z}dz \tag{19}$$

Therefore

$$(\mathcal{S}_+ - \mathcal{S}_-)[Z](\hbar) = 2\pi i e^{\frac{1}{\hbar}}, \tag{20}$$

where $\pm$ in $\mathcal{S}_\pm$ stands for the sign of $\mathrm{Im}\,\hbar$.

This ambiguity is called *Borel ambiguity*, and it is actually the key to linking perturbative part and non-perturbative factor in resurgence. This can be understood by using Lefschetz thimble decomposition in Picard Lefschetz theory, which we explain in the next section.

2. Lefschetz thimble

Lefschetz thimble, also known as the steepest descent, refers to the integral path along which the Laplace-type integral converge the most. Each path is labeled by its saddle point, and it is known that the original integral path can be decomposed into the sum of the Lefschetz thimbles (Picard-Lefschetz theory). The discontinuous change of the topological structure of the Lefschetz thimbles and the existence of Borel ambiguity are closely related, and called as the Stokes phenomenon.

$$Z = \int_C e^{-\frac{S(z)}{\hbar}}\, dz = \int_C e^{F(z)}\, dz \tag{21}$$

Consider a path such that the imaginary part of $F(z)$ is constant and passes through a saddle point (z_σ) of $F(z)$. This is represented by the following flow equation:

$$\frac{dz}{dt} = a\left(\frac{dF}{dz}\right)^*, \quad a = \pm 1, \tag{22}$$

The one for $a = 1$ is called the Lefschetz thimble $(\mathcal{J}_\sigma)$ and the one for $a = -1$ is called the dual thimble $(\mathcal{K}_\sigma)$. According to Picard Lefshetz theory, the integral of (21) can be decomposed into sum of Lefshetz thimbles:

$$Z = \int_C e^{-\frac{S(z)}{\hbar}}\, dz = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} e^{-\frac{S(z)}{\hbar}}\, dz \tag{23}$$

$$n_\sigma =< C, \mathcal{K}_\sigma >, \tag{24}$$

where $n_\sigma =< C, \mathcal{K}_\sigma >= \pm 1$ or 0 is called the intersection number of Lefschetz thimble and it is determined by whether the dual thimble $\mathcal{K}_\sigma$ intersect on the original integral contour C or not[1].

Example:

As an example, we consider the following simple ordinal integral ($0-$dim double well)

$$Z = \int_{-\infty}^{\infty} e^{-\frac{S(z)}{\hbar}}\, dz \quad S(z) = -\frac{1}{2}z^2 + \frac{1}{4}z^4 \tag{25}$$

Then the Lefschetz thimble and dual thimble corresponding each the saddle point $z_\sigma = \pm 1, 0$ is

[1] Note that it is the number of intersections between $\mathcal{K}_\sigma$ and C, not the number of intersections between $\mathcal{J}_\sigma$ and C.

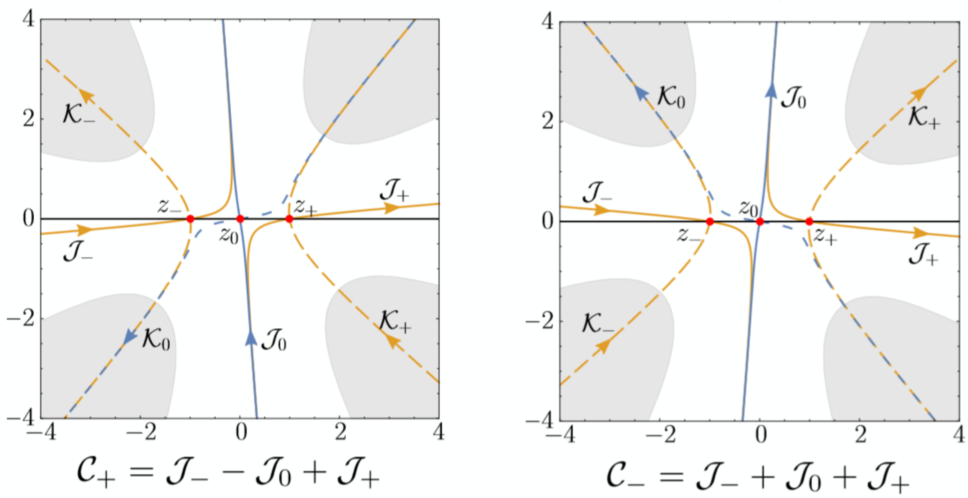

FIG. 3. The solid lines correspond to the Lefschetz thimbles, and the dotted line is the dual thimbles. Left: $\mathrm{Im}\,\hbar > 0$ Right: $\mathrm{Im}\,\hbar < 0$.

given by Fig.3 [10]

Therefore, the original integral contour $C = (-\infty, \infty)$ decomposed into

$$\int_C e^{-\frac{S(z)}{\hbar}}\, dz = \begin{cases} \left(\int_{\mathcal{J}_{-1}} - \int_{\mathcal{J}_0} + \int_{\mathcal{J}_{+1}}\right) e^{-\frac{S(z)}{\hbar}}\, dz & (\mathrm{Im}\,\hbar > 0) \\ \left(\int_{\mathcal{J}_{-1}} + \int_{\mathcal{J}_0} + \int_{\mathcal{J}_{+1}}\right) e^{-\frac{S(z)}{\hbar}}\, dz & (\mathrm{Im}\,\hbar < 0) \end{cases} \tag{26}$$

The topological structure of Lefschetz thimbles is changed discontinuously on $\mathrm{Im}\,\hbar = 0$. Actually, if the thimble decomposition is unambiguous, the Borel summation corresponding saddle is also unambiguous (Borel summable), and the position of Borel singularities determine the *Stokes phenomenon*, i.e, the change of topologies of the Lefschetz thimbles. The ambiguity in the thimble decomposition is exactly same to the Borel ambiguity. This relation is coming from the following property:

$$\int_{J_\sigma} e^{-\frac{S(z)}{\hbar}}\, dz = \mathcal{S}[Z_\sigma(\hbar)] \tag{27}$$

We can see this relation in this system. The perturbative expansion around the vacuum ($z = \pm 1$) is given by

$$Z_{\pm 1} = \sqrt{2\hbar} \sum_{n=0}^{\infty} \frac{\Gamma(2n + \frac{1}{2})}{n!} \hbar^n \tag{28}$$

Therefore its Borel summation is calculated as

$$\begin{aligned} \mathcal{S}[Z_1](\hbar) = \mathcal{S}[Z_{-1}](\hbar) &= \sqrt{\frac{\hbar}{\pi}}\Gamma\left(\frac{1}{4}\right)\int_0^{\infty}\frac{e^{-x}}{x^{\frac{1}{4}}(1-4x\hbar)^{\frac{1}{4}}}\,dx \\ &= \frac{\pi}{4}e^{\frac{1}{8\hbar}}\left[I_{\frac{1}{4}}\left(\frac{1}{8\hbar}\right)+I_{-\frac{1}{4}}\left(\frac{1}{8\hbar}\right)\pm i\frac{\sqrt{2}}{\pi}K_{\frac{1}{4}}\left(\frac{1}{8\hbar}\right)\right], \end{aligned} \tag{29}$$

where $I_{\pm\frac{1}{4}}(x)$ is the Bessel function of the first kind. The sign $\pm i$ corresponds to $\operatorname{Im}\hbar > 0, \operatorname{Im}\hbar < 0$. On the other hand, the contribution from the nonperturbative saddle $z = 0$ is

$$\mathcal{S}[Z_0](\hbar) = \mp i\sqrt{\frac{\hbar}{\pi}}\Gamma\left(\frac{1}{4}\right)\int_0^{\infty}e^{-x}\frac{1}{x^{\frac{1}{4}}(1+4x\hbar)^{\frac{1}{4}}}\,dx \tag{30}$$

$$= \mp i\frac{1}{\sqrt{2}}e^{\frac{1}{8\hbar}}K_{\frac{1}{4}}\left(\frac{1}{8\hbar}\right) \quad (- : \operatorname{Im}\hbar > 0, + : \operatorname{Im}\hbar > 0) \tag{31}$$

Therefore,

$$(\mathcal{S}_+ - \mathcal{S}_-)[Z_{\pm 1}] = \mathcal{S}[Z_0] \tag{32}$$

It implies the Borel ambiguity corresponding to the vacuum saddle point is exactly same to the sector of the other saddle point $z = 0$. This is indeed same to the thimble decomposition. Therefore,

$$\begin{aligned} Z = \int_C e^{-\frac{S(z)}{\hbar}}\,dz &= \sum_\sigma n_\sigma \int_{J_\sigma} e^{-\frac{S(z)}{\hbar}}\,dz \\ &= \mathcal{S}[Z_{+1}](\hbar) + \mathcal{S}[Z_0](\hbar) + \mathcal{S}[Z_{-1}](\hbar) \\ &= \frac{\pi}{2}e^{\frac{1}{8\hbar}}\left[I_{\frac{1}{4}}\left(\frac{1}{8\hbar}\right)+I_{-\frac{1}{4}}\left(\frac{1}{8\hbar}\right)\right] \end{aligned} \tag{33}$$

Then this is indeed same to (25).

In other words, perturbation expansions can have ambiguity arising from Borel singularity (thimble decomposition) in general, but if we can add up the contributions corresponding to all relevant saddle (thimble) and take their Borel sumamtion, the ambiguity is cancelled and gives the exact result. This is the claim of resurgence derived from the perspective of Lefschetz thimble.

Now, we consider the same argument for the path integral.

$$Z(\beta) = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathcal{D}x\, e^{-\frac{S[x]}{\hbar}}\,, \tag{34}$$

Here, we have to determine the intersection number, $n_\sigma(0 \text{ or } \pm 1)$, to obtain the correct result of the path integral. However, there is no efficient method for calculating this index except for calculating the thimble numerically and plotting it explicitly, which is a hard task (because the functional integral is essentially infinite dimensional integral). Therefore, we have no reliable way to determine *which are relevant saddles* in the generic cases from the integration itself. In this work, we propose a certain solution to this problem by using exact-WKB and simple Stokes graphs, and give a physical interpretation of the index in quantum mechanics.

B. Resolvent method

We now review the resolvent method [24] for quantum theories. In the latter part of this paper, this method will enable us to obtain the partition function of the system directly from the quantization conditions obtained from the exact-WKB analysis and to interpret the resurgent structure of the partition function in terms of the exact-WKB analysis.

First, we write down the partition function formally as a sum over saddle points

$$
\begin{aligned}
Z(\beta) = \operatorname{tr} e^{-\beta \hat{H}} &= \int \mathcal{D}x \; e^{-\frac{S[x]}{\hbar}} \\
&= n_0 \, \mathcal{S}\left[e^{-\frac{S[x_0]}{\hbar}} \sum_n a_n \hbar^n \right] + n_1 \, \mathcal{S}\left[e^{-\frac{S[x_1]}{\hbar}} \sum_n b_n \hbar^n \right] + \ldots \\
&= \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathcal{D}x \; e^{-\frac{S[x]}{\hbar}} = \sum_\sigma n_\sigma \, Z_\sigma(\beta)\,, \qquad (35)
\end{aligned}
$$

where $\mathcal{S}[\cdot]$ denotes the Borel summation of series expansions and x_σ stands for saddle points.

We then consider the Laplace transform of $Z(\beta)$, which gives the trace of resolvent $G(E)$. Since this transform is linear, we obtain the expression as

$$
\begin{aligned}
\operatorname{tr} \frac{1}{\hat{H} - E} = G(E) &= \int_0^\infty Z(\beta) e^{\beta E} \, d\beta \\
&= \sum_\sigma n_\sigma \int_0^\infty Z_\sigma(\beta) e^{\beta E} \, d\beta \\
&= \sum_\sigma n_\sigma G_\sigma(E)\,. \qquad (36)
\end{aligned}
$$

It is notable that the poles of $G(E)$ give the eigenvalues and $G_\sigma(E)$ stands for the trace of resolvent for each sector (each thimble). The trace of resolvent $G(E)$ can be connected to the Fredholm determinant $D(E) = \det\left(\hat{H} - E\right)$ via the relation $-\frac{\partial}{\partial E}\log D = G(E)$. Then, we have

$$D(E) = \prod_\sigma D_\sigma^{n_\sigma}(E)\,, \tag{37}$$

where $D_\sigma(E)$ stands for the Fredholm determinant for each thimble. We note that the zeros of $D(E)$ give the exact energy eigenvalues.[2]

The main point of our analysis is following. From the exact-WKB analysis, we will obtain an exact quantization condition, $D(E) = 0$. This formula will be expressed in terms of perturbative and non-perturbative cycles, which involve perturbative as well as non-perturbative instanton/bion data. By reexpressing the condition $D(E) = 0$ as a sum over P/NP cycles, we will be able extract the index n_σ and Maslov index from the exact-WKB analysis. Since we can go back to $Z(\beta)$ by inverse Laplace transform

$$G(E) = \int_0^\infty Z(\beta)e^{\beta E}d\beta \tag{38}$$

$$Z(\beta) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} G(E)e^{-\beta E}dE\,, \tag{39}$$

it will enable us to obtain the partition function from the exact-WKB method.

C. Gutzwiller's quantization

Gutzwiller's quantization, which is also known as *the Gutzwiller trace formula*[150], is a semi-classical construction that express the quantum mechanical density of states (the resolvent: $G(E)$), in terms of periodic orbits. The formalism uses path integral in Minkowski space formulation (in real time), hence one is dealing with amplitudes in real time $\mathcal{D}x\ e^{iS}$. In certain sense, Gutzwiller method can be interpreted as the intermediate quantization method between the path integral and the Bohr-Sommerfeld quantizations. Actually, the distribution of the poles of $G(E)$, which determines the energy eigenvalues, gives the Bohr-Sommerfeld quantization condition. Later, we will show how to derive this trace formula interms of exact-WKB method and resurgence.

We first express the "Lorentzian" partition function as

$$Z(T) = \mathrm{tr}\, e^{-i\hat{H}T} = \int_{periodic} \mathcal{D}x\ e^{iS}\,, \tag{40}$$

---

[2] Mathematically, the definition of Fredholm determinant (or resolvent) needs a regularization, e.g. $G_{reg.} \equiv G(E) - G(0)$ or $D_{reg.} \equiv \frac{D(E)}{D(0)}$ or zeta function regularization for $D(E)$.

The resolvent, which has quantum spectral data, is given by

$$G(E) = \int_0^\infty Z(t) e^{(iE-\epsilon)T} dT = \int_0^\infty \sum_n e^{(iE-iE_n-\epsilon)T} dT = -i \,\mathrm{tr}\, \frac{1}{\hat{H}-E} \,. \tag{41}$$

where $\lim_{\epsilon \searrow 0}$ is taken after integration. The resolvent can also be expressed as

$$G(E) = -i \,\mathrm{tr}\, \frac{1}{\hat{H}-E} = \int_0^\infty dT \int_{periodic} \mathcal{D}x \; e^{iS+iET} = \int_0^\infty dT \int_{periodic} \mathcal{D}x \; e^{i\Gamma} \,, \tag{42}$$

where $\Gamma = S + ET$. We also note that the action S is written as

$$S = \int^T p\dot{x} dt - \int^T H dt = \oint p dx - \int^T H dt \,. \tag{43}$$

We here evaluate the T integral in (42) by the stationary phase method with considering the T derivative of Γ

$$\frac{d\Gamma}{dT} = \frac{dS}{dT} + E \,, \tag{44}$$

Since $\oint pdx$ is the area of phase space, which depends on the trajectory but not on T (how long it takes to go around), i.e. $\frac{d}{dT} \oint pdx = 0$. Therefore, we obtain

$$\frac{d\Gamma}{dT} = \frac{dS}{dT} + E = -H + E \,. \tag{45}$$

It means the leading contributions of the T integral are periodic classical solutions whose energy is E. When a periodic orbit is a solution, the configuration rotating n times is also a solution. By taking this fact into account, we find that the contribution is obtained just by the replacement as $\oint pdx \to n \oint pdx$. Then, we obtain

$$\Gamma = S + ET = \left(n \oint pdx - ET \right) + ET = n \oint pdx \,, \qquad (n = 1, 2, 3 ...) \,. \tag{46}$$

After all, the contribution of the classical solutions to $G(E)$ is expressed as

$$G(E) \simeq \sum_{p.p.o.} \sum_{n=1}^{\infty} e^{in \oint_{p.p.o.} pdx} \,, \tag{47}$$

where *p.p.o.* stands for a *prime periodic orbit*, which is a topologically distinguishable orbit among the countless periodic orbits.

If we consider the sub-leading terms in stationary phase approximation, it gives

$$G(E) \simeq \sum_{p.p.o.} \sum_{n=1}^{\infty} \exp\left(i\, n \oint_{p.p.o.} p dx\right) \left(\det \frac{\delta^2 S}{\delta x \delta x}\right)^{-1/2}, \tag{48}$$

Here, $\det \frac{\delta^2 S}{\delta x \delta x}$ is a functional determinant taking into account the fluctuation operator around the saddle point. Evaluation of this part requires care, as described below, this operator has negative eigenvalues when considering the expansion around a periodic orbit in general. The number of negative eigenvalues is called *Maslov index*, which plays an important role in Gutzwiller's quantization.

1. Maslov index

Let x_{cl} denote the classical solution and δx denote the fluctuations around it. The integration over fluctuations at the quadratic level is determined by the functional determinant of the fluctuation operator:

$$M = \frac{\delta^2 S}{\delta x \delta x} = -\frac{d^2}{dt^2} - V''(x_{cl})\,. \tag{49}$$

The operator M has a zero eigenvalue if x_{cl} depends on t, and the operator M has 2n-1 negative eigenvalues for n-cycle orbit. We below give a brief proof of this fact:

Proof. Consider classical EoM:

$$-\frac{d^2 x_{cl}}{dt^2} - \frac{dV}{dx_{cl}} = 0\,. \tag{50}$$

Take t differential for this equation. Then we get

$$\left(-\frac{d^2}{dt^2} - V''(x_{cl})\right)\frac{dx_{cl}}{dt} = 0\,. \tag{51}$$

This expression is nothing but an eigenvalue equation for the zero eigenvalue of the fluctuation operator, $M\tilde{\psi}_0(t) = 0$, and the eigenfunction is proportional to $\tilde{\psi}_0(t) = \frac{dx_{cl}}{dt}$.

Next, let us consider a periodic classical solution x_{cl}. When it is a one-cycle solution, the derivative $\frac{dx_{cl}}{dt}$ typically has a behavior depicted in Fig. 4. The operator M is a Schrödinger-type operator, thus the level of the eigenfunction is determined by the number of zero points[3] (nodes).

[3] If the eigenfunction of M is on $\mathbb{R}^1 = (-\infty, \infty)$, the level is same to the number of nodes but when S^1, the level is

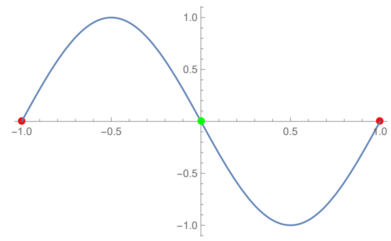

FIG. 4. The appearance of the derivative $\frac{dx_{cl}}{dt}$ for 1-cycle.

In the case of Fig. 4, M has two nodes since the periodic b.c. is imposed, and the endpoints are identical and regarded as a single node. The reason why it has one negative eigenvalues is that $\frac{dx_{cl}}{dt}$ is the first excited state, but at the same time, $\frac{dx_{cl}}{dt}$ is also an eigenfunction of the zero eigenvalue. Similarly, a n-cycle classical solution has $2n$ turning points, M has $2n-1$ negative eigenvalues because $\frac{dx_{cl}}{dt}$ is the $2n-1$-th excited state. $\square$

If we consider an analogy with the Morse theory, the operator M corresponds to (the diagonal part of) Hessian where the action is viewed as a Morse function, and its negative eigenvalue corresponds to the Morse index. Thus, the Maslov index is essentially regarded as the Morse index in the functional integral. To rephrase this, we now express the contribution of functional determinants as

$$\sqrt{\det M} = \sqrt{|\det M|}e^{i\alpha\pi}, \qquad \alpha = \frac{\nu}{2}\,. \tag{52}$$

Here, α is called the *Maslov index.* Here, ν is the number of negative eigenvalues of M. The determinant of the n-cycle is given by

$$\sqrt{\det M} = -i\sqrt{|\det M|}(-1)^n\,. \tag{53}$$

Therefore, the final form of $G(E)$ (up to higher order quantum corrections) is given as

$$G(E) = i\sum_{p.p.o.}\sum_{n=1}^{\infty} T(E)\, e^{in\oint_{p.p.o.} pdx}(-1)^n\left(\left|\det\frac{\delta^2 S}{\delta x\delta x}\right|\right)^{-1/2}, \tag{54}$$

where $T(E)$ is the period of each cycle, which comes from the zero eigenvalue of $M = \frac{\delta^2 S}{\delta x \delta x}$. Also, we call $(-1)^n$ as "Maslov index" instead of α in the latter calculation.

Working of Maslov index in simple harmonic oscillator: As an example, we now consider

the number of nodes -1.

the harmonic oscillator system. In this case, there is only one type of p.p.o. with constant $T(E)$ and $|\det \frac{\delta^2 S}{\delta x_i \delta x_j}|$, We then obtain

$$G(E) \propto \sum_{n=1}^{\infty} e^{in \oint pdx}(-1)^n = \frac{e^{i\oint pdx}}{1+e^{i\oint pdx}}\,. \tag{55}$$

Therefore the poles of $G(E)$ are given by

$$\oint pdx = 2\pi\left(n+\frac{1}{2}\right). \tag{56}$$

This is the Bohr-Sommerfeld quantization of harmonic oscillator. It should be emphasized that the contribution of the Maslov index is important in order to obtain the correct energy eigenvalues including the vacuum energy.

However, the way to determine p.p.o. in this method is not well understood in the most of cases and almost exclusively used in systems without tunneling phenomena (instantons) [150]. In retrospect, this is not surprising because usual instantons in real time correspond to imaginary singular configurations [153], and it is not so obvious how to deal with it. As we will show later, our finding gives the systematic method to determine p.p.o. including instanton-like configurations without any approximation. Furthermore, it also shows the relation between Maslov index and the intersection number of Lefschetz thimble.

III. EXACT WKB

In this section, we review the exact WKB method [52, 132, 133, 135–147] and the related techniques, including Borel resummation, Stokes curves and monodromy matrices. [4] For simplicity, we focus on the one-dimensional Schrödinger equation, and assume that the potential $V(x)$ doesn't include $\hbar$, i.e. it is a purely classical potential.

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+V(x)\right)\psi(x) = E\psi(x)\,. \tag{57}$$

We set $Q(x) = 2(V(x)-E)$ then rewrite the equation as

$$\left(-\frac{d^2}{dx^2}+\hbar^{-2}Q(x)\right)\psi(x) = 0\,, \tag{58}$$

[4] Although in standard quantum mechanics books WKB is presented as an approximation which applies to high-energy states, this perspective is not correct. It is an exact method, and applies every where in the spectrum.

In the WKB analysis, we consider the ansatz given by

$$\psi(x,\hbar) = e^{\int^x S(x,\hbar)dx}\,, \tag{59}$$

$$S(x,\hbar) = \hbar^{-1}S_{-1}(x) + S_0(x) + \hbar S_1(x) + \hbar^2 S_2(x) + \dots\,, \tag{60}$$

where $S(x,\hbar)$ is a formal power series expansion in expansion parameter $\hbar$, and $S_n(x)$ are functions of x. Substituting Eq. (59) into Eq. (58), leads to the non-linear Riccati equation

$$S(x)^2 + \frac{\partial S}{\partial x} = \hbar^{-2}Q(x)\,. \tag{61}$$

By substituting Eq. (60) into Eq. (61), we obtain the recursive relation

$$S_{-1}^2 = Q(x)\,, \qquad 2S_{-1}S_n + \sum_{j=0}^{n-1} S_j S_{n-j} + \frac{\partial S_{n-1}}{\partial x} = 0 \quad (n \geq 0)\,. \tag{62}$$

We note $S_{-1} = \pm\sqrt{Q(x)}$. Since S_n is recursively determined from S_{-1}, S_n has two independent solutions:

$$S^{\pm}(x,\hbar) = \hbar^{-1}S^{\pm}_{-1}(x) + S^{\pm}_0(x) + \hbar S^{\pm}_1(x) + \hbar^2 S^{\pm}_2(x) + \dots\,, \tag{63}$$

The first several terms are given by

$$S^{\pm}_{-1}(x) = \pm\sqrt{Q(x)}\,, \tag{64}$$

$$S^{\pm}_0(x) = -\frac{\frac{\partial Q}{\partial x}}{4Q}\,, \tag{65}$$

$$S^{\pm}_1(x) = \pm\left(-\frac{5}{32}\frac{\left(\frac{\partial Q}{\partial x}\right)^2}{Q^{5/2}} + \frac{\frac{\partial^2 Q}{\partial x^2}}{8Q^{3/2}}\right)\,. \tag{66}$$

From Eq. (62), one finds the relation $S^-_n = (-1)^n S^+_n$. Therefore, we reach the simple expression

$$S^{\pm}(x,\hbar) = \hbar^{-1}S^{\pm}_{-1}(x) + S^{\pm}_0(x) + \hbar S^{\pm}_1(x) + \hbar^2 S^{\pm}_2(x) + \dots \tag{67}$$

$$= \pm\hbar S^{+}_{-1} + S^{+}_0 \pm \hbar S^{+}_1 + \hbar^2 S^{+}_2 + \dots \tag{68}$$

$$= \pm S_{\mathrm{odd}} + S_{\mathrm{even}}\,. \tag{69}$$

Based on Eq. (69), Eq. (61) is rewritten as

$$(S_{\text{odd}}+S_{\text{even}})^2+\frac{\partial}{\partial x}(S_{\text{odd}}+S_{\text{even}})=\hbar^{-2}Q\,, \tag{70}$$

$$(-S_{\text{odd}}+S_{\text{even}})^2+\frac{\partial}{\partial x}(-S_{\text{odd}}+S_{\text{even}})=\hbar^{-2}Q\,. \tag{71}$$

These two equations give

$$\therefore S_{\text{even}}=-\frac{1}{2}\frac{\partial}{\partial x}\log S_{\text{odd}}\,. \tag{72}$$

Therefore the WKB wave function can be expressed as

$$\psi_a^{\pm}(x)=e^{\int^x S^{\pm}dx}=\frac{1}{\sqrt{S_{\text{odd}}}}e^{\pm\int_a^x S_{\text{odd}}dx}\,, \tag{73}$$

with a being an integral constant. For later calculations, we choose it as a turning point, which is a solution of $Q(x)=0$. At the leading order, WKB wave function is evaluated as

$$\psi_a^{\pm}(x)=\frac{1}{Q(x)^{1/4}}e^{\pm\frac{1}{\hbar}\int_a^x\sqrt{Q(x)}dx}\,, \tag{74}$$

which is nothing but the solution in the text-book level WKB approximation.

Since we have derived the WKB wave function recursively, it is regarded as a formal series in $\hbar$

$$\psi_a^{\pm}(x)=e^{\pm\frac{1}{\hbar}\int_a^x\sqrt{Q(x)}dx}\sum_{n=0}^{\infty}\psi_{a,n}^{\pm}(x)\hbar^{n+\frac{1}{2}}\,, \tag{75}$$

$$S_{\text{odd}}=\sum_{n=0}^{\infty}S_{2n-1}\hbar^{2n-1}\,. \tag{76}$$

Note that the factor $\frac{1}{2}$ in $\hbar^{n+\frac{1}{2}}$ comes from $\frac{1}{\sqrt{S_{\text{odd}}}}$. Here, both of these series turn out to be asymptotic expansions with respect to $\hbar$. In other words, the all orders WKB wave function is an divergent asymptotic expansion with respect to $\hbar$. In order to give it a precise meaning, we need another technology, the Borel resummation, applied to series for which the divergent coefficients $\psi_{a,n}^{\pm}(x)$ are x-dependent.

A. Borel summation

Let us consider the following formal series (not necessarily asymptotic) with respect to $\hbar$.

$$Z(\hbar) = e^{-\frac{A}{\hbar}} \sum_{n=0}^{\infty} a_n \hbar^{n+\alpha} \qquad \alpha \notin \{-1,-2,-3,...\}\,. \tag{77}$$

The Borel transform of this series is defined as

$$\mathfrak{B}[Z](z) \equiv \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+\alpha)} (z-A)^{n+\alpha-1}\,, \tag{78}$$

The directional Borel resummation $\mathcal{S}[Z]$ is defined as

$$\mathcal{S}[Z](\hbar) \equiv \int_A^{\infty e^{i\theta}} e^{-\frac{z}{\hbar}} \mathfrak{B}[Z](z)\,dz \qquad \theta = \mathrm{Arg}(\hbar)\,. \tag{79}$$

where θ denotes the direction of integration.

The resurgence property tells us that the Borel transform admits an analytic continuation in z plane, which allows via Borel resummation the reconstruction of the exact value of the result. If the series is convergent, this procedure just return the original series due to the identity

$$1 = \frac{1}{\Gamma(n+\alpha)} \int_0^{\infty} e^{-x} x^{n+\alpha-1} dx\,. \tag{80}$$

For an asymptotic divergent series, however, it gives one analytic functions which have the series as its asymptotic series.

The Borel summation is a homomorphism, so that the following algebraic properties hold.

$$\mathcal{S}[A+B] = \mathcal{S}[A] + \mathcal{S}[B] \tag{81}$$

$$\mathcal{S}[AB] = \mathcal{S}[A]\mathcal{S}[B] \tag{82}$$

Now, we apply the Borel summation procedure to the WKB wave function, then we obtain

$$\mathcal{S}[\psi_a^{\pm}](\hbar) = \int_{\mp z_0}^{\infty e^{i\theta}} e^{-\frac{z}{\hbar}} \mathfrak{B}[\psi_a^{\pm}(x)](z) dz, \qquad \theta = \mathrm{Arg}(\hbar)\,, \tag{83}$$

$$\mathfrak{B}[\psi_a^{\pm}(x)](z) = \sum_{n=0}^{\infty} \frac{\psi_{a,n}^{\pm}(x)}{\Gamma\left(n+\frac{1}{2}\right)} (z \pm z_0)^{n-\frac{1}{2}}, \qquad z_0 = \int_a^x \sqrt{Q(x)}\,. \tag{84}$$

Because the coefficient $\psi_n^{\pm}(x)$ depends on x, the position of Borel singularity also depends on

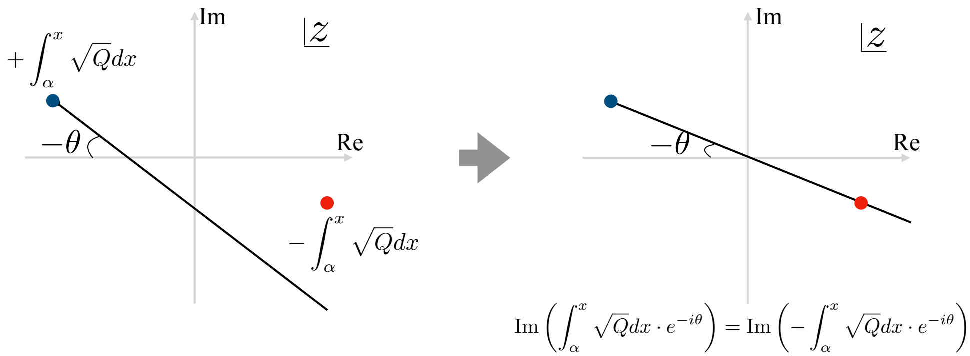

FIG. 5. The black line indicates the integration path on the Borel plane, the blue circle is the endpoint of the integral path and the red circle indicates the singularity of $\mathfrak{B}[\psi_a^\pm(x)](z)$.

x. The position of singularities of integrand is $z = \pm\int_a^x \sqrt{Q(x)}$ when the Stokes curve is not degenerate. From now on, we will express the Borel-summed wave function $\mathcal{S}[\psi_a^\pm](\hbar)$ as just $\psi_a^\pm(\hbar)$ unless otherwise noted.

We now look into details of the Borel singularities. One of them is always the endpoint of the integration path, and $\int \frac{1}{\sqrt{x}}$ is regular at $x = 0$. Therefore the other one contributes to Stokes phenomena. These two singularities are point-symmetric. Hence, Stokes phenomena occur due to the situations that one of the singularities is on the integration path as shown in Fig. 5. This condition can be expressed as

$$\begin{aligned} &\mathrm{Im}\, e^{-i\theta} \int_a^x \sqrt{Q(x)}dx = 0\,, \\ &\therefore \mathrm{Im}\,\frac{1}{\hbar}\int_a^x \sqrt{Q(x)}dx = 0\,. \end{aligned} \tag{85}$$

Note that $\frac{1}{\hbar}\int_a^x \sqrt{Q(x)} = \frac{1}{\hbar}S_{-1}$ is the leading term of WKB expansion. Therefore, in order to understand when Stokes phenomena occur, we do not have to calculate Borel summation of $\psi(x)$ explicitly, but just evaluate this term. The path derived from Eq. (85) is called a Stokes curve, which is part of a structure that is called Stokes graph. In principle, the exact energy spectrum of the theory can be calculated just from the Stokes curve data. We denote by $\mathcal{S}_\pm$ the lateral Borel resummation with a positive/negative (small) angle θ.

B. Stokes curves and Stokes phenomena

Let a be a turning point (a solution of $Q(x) = 0$). In this case, the Stokes curve associated with a is defined as

$$\mathrm{Im}\,\frac{1}{\hbar}\int_a^x \sqrt{Q(x)}dx = 0\,. \tag{86}$$

Also, each segment of the Stokes curve has an index, $\pm$. This index indicates which one of the ψ^+ and ψ^- pair increases exponentially when moving from the point a to infinity (more precisely, a to $\infty e^{i\theta^*}$ where θ^* is the phase of corresponding segment) along the Stokes curve. The parts between the Stokes lines is called Stokes regions or just regions.

When the index of the corresponding Stokes curve is $+$, ψ^+ increases exponentially [5] and

$$\mathrm{Re}\,\frac{1}{\hbar}\int_a^x \sqrt{Q(x)}dx > 0\,. \tag{87}$$

When the index is $-$, then ψ^- increases exponentially in the case

$$\mathrm{Re}\,\frac{1}{\hbar}\int_a^x \sqrt{Q(x)}dx < 0\,. \tag{88}$$

The Stokes curve indicates where the Stokes phenomena occur, when $\psi(x)$ is analytically connected between adjacent Stokes regions.

C. Connection formula and monodromy matrix

For a generic potential and at a typical value of the energy, the turning points are non-degenerate. To each such turning point, one attaches an Airy-type Stokes graph. Therefore, a Stokes graph for a general potential is a composite of elementary building blocks of Airy-type Stokes graphs.

We now give a connection formula for the Airy-type Stokes graph. When one considers the Borel-resummed wave functions for a given potential and its analytic continuation in terms of a complex x, one has to take into account the effect of the Borel singularity on the Borel plane, i.e., the Stoke phenomenon.

Roughly speaking, in order to compute the effect, we decompose a global Stokes graph into

[5] Note that, because of square-root, if ψ^+ increases in first Riemann sheet, it decreases in the second Riemann sheet. This point requires some care at various points.

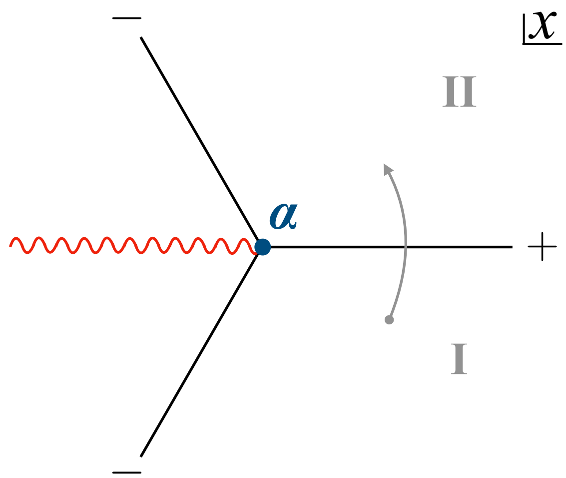

FIG. 6. The Airy-type Stokes graph emerging from a turning point a. The sign $+(-)$ labeling each lines means increasing (decreasing) $\mathrm{Re}\, \frac{1}{\hbar}\sqrt{Q(x)}dx$ as going out from the turning point along the line. The wavy line denotes a branch cut. By crossing the curve labeled by $+$ in anti-clockwise manner, the wave funtions in the I and II domains are related to each other as $\psi_{a,\mathrm{I}} = M_{+}\psi_{a,\mathrm{II}}$.

the Airy-type Stokes graphs locally and then consider the effect of crossing the Stokes line by using a connection formulas. As we emphasize, the Airy-type Stokes graph is a building block of any given graph, hence, it is important to understand it fully in simple examples. Apart from the connection formula, one also pay attention to the change of normalization point of the wave function corresponding to the change of turning points. We describe both below.

We suppose the wave function is normalized at a simple turning point a and consider analytic continuation from the region I to II as shown in Fig. 6. When x crosses a Stokes line, the relation between wave function can be expressed by

$$\begin{pmatrix} \psi^{+}_{a,\mathrm{I}} \\ \psi^{-}_{a,\mathrm{I}} \end{pmatrix} = M \begin{pmatrix} \psi^{+}_{a,\mathrm{II}} \\ \psi^{-}_{a,\mathrm{II}} \end{pmatrix}, \tag{89}$$

The monodromy matrix M multiplies the wave function according to following rules[143]:

$$M = \begin{cases} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} =: M_+ & \text{for anti-clockwise crossing of a curve labeled by } + \\ \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} =: M_+^{-1} & \text{for clockwise crossing of a curve labeled by } + \\ \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} =: M_- & \text{for anti-clockwise crossing of a curve labeled by } - \\ \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} =: M_-^{-1} & \text{for clockwise crossing of a curve labeled by } - \end{cases} \tag{90}$$

Furthermore, if it crosses the branch cut emerging from the simple turning point a, it moves between the first and second Riemann sheets as $S_{\text{odd}}(x) \to -S_{\text{odd}}(x)$. Thus, we have

$$M = \begin{cases} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} =: M_b & \text{for anti-clockwise crossing of a cut} \\ \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} =: M_b^{-1} & \text{for clockwise crossing of a cut} \end{cases} \tag{91}$$

A complete cycle around a turning point gives identity map:

$$M_- M_b M_- M_+ = M_+ M_b M_+ M_- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{92}$$

In order to consider the analytic continuation globally beyond the Airy-type Stokes curve, one has to incorporate the change of normalization point and then employ the connection formula due to curves or cuts emerging from other simple turning points.

Two wave functions normalized at different turning points a_1, a_2 are related by the equation

$$\psi_{a_1}^{\pm}(x) = e^{\pm \int_{a_1}^{a_2} S_{\text{odd}}} \psi_{a_2}^{\pm}(x) . \tag{93}$$

The quantity, $\int_{a_1}^{a_2} S_{\text{odd}}$ is called the Voros multiplier. One may be tempted to think that the Voros multiplier is an asymptotic function, because S_{odd} is defined by the recursive relation and

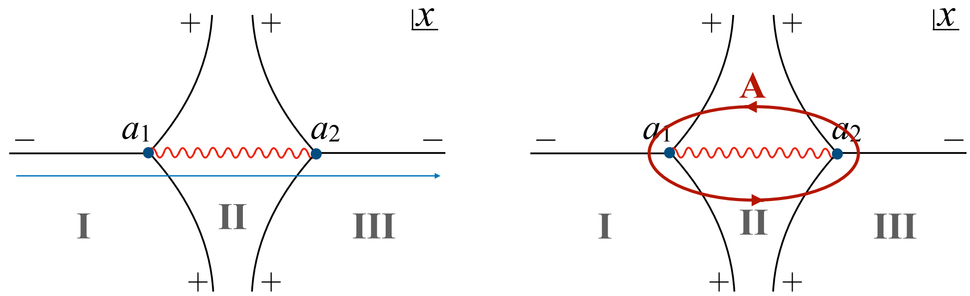

FIG. 7. The Stokes graph for harmonic oscillator. In order to obtain the quantization condition, we take the orbit from the left to the right below the real axis by taking into account the Stokes phenomena (Left panel). The cycle A is defined as an oriented cycle enclosing two turning points a_1 and a_2 (Right panel).

asymptotic itself. However, the Voros multiplier appearing here is Borel resummed because the $\psi^{\pm}_{a_1}(x)$ are already Borel resummed wave functions. We then write down the normalization matrix as

$$\begin{pmatrix} \psi^{+}_{a_1}(x) \\ \psi^{-}_{a_1}(x) \end{pmatrix} = N_{a_1 a_2} \begin{pmatrix} \psi^{+}_{a_2}(x) \\ \psi^{-}_{a_2}(x) \end{pmatrix}, \qquad N_{a_1 a_2} = \begin{pmatrix} e^{+\int_{a_1}^{a_2} S_{\mathrm{odd}}} & 0 \\ 0 & e^{-\int_{a_1}^{a_2} S_{\mathrm{odd}}} \end{pmatrix}. \tag{94}$$

The orientation of $N_{a_1 a_2}$ is flipped before/after crossing a branch cut.

$$M_b N_{a_1 a_2} = N^{-1}_{a_1 a_2} M_b \,. \tag{95}$$

D. Warm-up: Harmonic oscillator with Airy-type Stokes graph

For a general potential with multiple degenerate harmonic minima, we associate a Stokes graph which is a combination of the Stokes graph of harmonic oscillator. The Stokes graph for harmonic oscillator is a combination of two Airy-type graphs as shown in Fig.7. Therefore, one can quickly learn how the formalism works in practice in this simple, but essential example. Therefore, we first review this example, and then move to more interesting examples of double-well, triple-well and N-ple well examples.

The harmonic potential is given by $V(x) = \frac{1}{2}\omega^2 x^2$. Its Stokes curve is depicted in Fig. 7 assuming $E > 0$. There are two turning points, $a_1 = -\frac{\sqrt{2E}}{\omega}$ are $a_2 = \frac{\sqrt{2E}}{\omega}$, which satisfy $Q(x) = 2(V(x) - E) = 0$. The blue arrow is a trajectory of the analytic continuation. If we start with a decaying solution at the beginning of blue line, we will demand a decaying solution at the end of journey, for the full WKB solution to be normalizable. However, the Stokes phenomena will induce terms that will be exponentially growing at the end. In order to have a physical answer, we will demand that the pre-factor of the exponentially growing part to vanish. That will give us the quantization condition that will determine the spectrum of the theory.[6]

First, let us take wave function normalized at a_1 and consider analytic continuation from the region I to II. Then the wave function changes as

$$\begin{pmatrix} \psi^{+}_{a_1,\mathrm{I}}(x) \\ \psi^{-}_{a_1,\mathrm{I}}(x) \end{pmatrix} = M_{+} \begin{pmatrix} \psi^{+}_{a_1,\mathrm{II}}(x) \\ \psi^{-}_{a_1,\mathrm{II}}(x) \end{pmatrix}. \tag{96}$$

Second, consider analytic continuation from the region II to III. In this case, the Stokes curve to be crossed is starting from the other turning point, a_2. Therefore we have to change the normalization as

$$\begin{pmatrix} \psi^{+}_{a_1,\mathrm{II}}(x) \\ \psi^{-}_{a_1,\mathrm{II}}(x) \end{pmatrix} = N_{a_1 a_2} \begin{pmatrix} \psi^{+}_{a_2,\mathrm{II}}(x) \\ \psi^{-}_{a_2,\mathrm{II}}(x) \end{pmatrix}. \tag{97}$$

Then we can multiply the monodromy matrix as

$$\begin{pmatrix} \psi^{+}_{a_2,\mathrm{II}}(x) \\ \psi^{-}_{a_2,\mathrm{II}}(x) \end{pmatrix} = M_{+} \begin{pmatrix} \psi^{+}_{a_2,\mathrm{III}}(x) \\ \psi^{-}_{a_2,\mathrm{III}}(x) \end{pmatrix}. \tag{98}$$

As a result,[7], we obtain the connection formula

$$\begin{aligned} \begin{pmatrix} \psi^{+}_{a_1,\mathrm{I}}(x) \\ \psi^{-}_{a_1,\mathrm{I}}(x) \end{pmatrix} &= M_{+} N_{a_1 a_2} M_{+} N_{a_2 a_1} \begin{pmatrix} \psi^{+}_{a_1,\mathrm{III}}(x) \\ \psi^{-}_{a_1,\mathrm{III}}(x) \end{pmatrix} \qquad (99) \\ &= \begin{pmatrix} \psi^{+}_{a_1,\mathrm{III}}(x) + i(1+A)\psi^{-}_{a_1,\mathrm{III}}(x) \\ \psi^{-}_{a_1,\mathrm{III}}(x) \end{pmatrix}, \qquad (100) \end{aligned}$$

where the cycle $A = e^{\oint_A S_{\mathrm{odd}}} = e^{2\int_{a_1}^{a_2} S_{\mathrm{odd}}}$ is depicted in Fig. 7.

As $x \to -\infty$ in the region I, ψ^+ is normalizable, it decays as $x \to -\infty$. (This is true on first Riemann sheet which we stick through this argument.) Therefore we take ψ^+ in the region I and we find that it changes to $\psi^{+}_{a_1,\mathrm{III}}(x) + i(1+A)\psi^{-}_{a_1,\mathrm{III}}(x)$ in the region III. In region III, $\psi^{+}_{a_1,\mathrm{III}}(x)$ is decaying as $x \to -\infty$ while $\psi^{-}_{a_1,\mathrm{III}}(x)$ is blowing up. Therefore, in order to satisfy the normalization

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[6] There are several other ways to obtain the quantization condition using exact-WKB method. For example, the Wronskian constraint for each Stokes region is used in [52].

[7] One can omit the last $N_{a_2 a_1}$ because it just changes overall factors but this makes D simpler.

condition, the coefficient of ψ^- **must** be zero. This is the quantization condition:

$$D = 1 + A = 1 + e^{\oint_A S_{\rm odd}} = 0\,. \tag{101}$$

where $e^{\int_{a_1}^{a_2} S_{\rm odd}}$ is the Voros multiplier connecting two turning points. This is equivalent to

$$\oint_A S_{\rm odd} = -2\pi i\left(n+\frac{1}{2}\right) \quad \text{with } n\in\mathbb{Z}\,. \tag{102}$$

For the harmonic oscillator,

$$\oint_A S_{\rm odd} = \oint_A S_{-1} = \frac{1}{\hbar}\oint_A \sqrt{2(V(x)-E)}dx = -2\pi i\frac{E}{\hbar\omega}\,, \tag{103}$$

where $\omega = \frac{2\pi}{T}$ and T is the classical period, $T = \left|\oint_A (2(V(x)-E))^{-1/2}dx\right|$. Therefore, the quantization condition $D(E)=0$ obtained from the exact-WKB analysis gives

$$E = \hbar\omega\left(n+\frac{1}{2}\right)\,. \tag{104}$$

The Stokes curve in Fig.7 corresponds to $E>0$ and the turning points are real. This puts a restriction that $n=0,1,2...$, which is just the spectrum of simple harmonic oscillator.

E. Resolvent and Spectral form

We derive the partition function starting with the quantization condition $D(E)$ and resolvent $G(E)$. The reason we are presenting this is because we will follow verbatim the same procedure in the theories with instantons and we will reach to fairly non-trivial results. It is therefore useful to recall this tool in a simple example.

The quantization condition D is written as [8]

$$\begin{aligned} D &= 1 + e^{-2\pi i\frac{E}{\hbar\omega}} = e^{-\pi i\frac{E}{\hbar\omega}}2\sin\left(\pi\left(\frac{E}{\hbar\omega}+\frac{1}{2}\right)\right) \\ &= e^{-\pi i\frac{E}{\hbar\omega}}\frac{2\pi}{\Gamma(\frac{1}{2}+\frac{E}{\hbar\omega})\Gamma(\frac{1}{2}-\frac{E}{\hbar\omega})}\,, \end{aligned} \tag{106}$$

where we have used reflection formula: $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$. We then obtain the resolvent $G(E)$

[8] The zeta function regularization for Fredholm determinant $D(E) = \det(H-E)$ gives:

$$e^{-\zeta'(0,\frac{1}{2}-\frac{E}{\hbar\omega})} = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}-\frac{E}{\hbar\omega})} \tag{105}$$

It removes the irrelevant Gamma function, which does not contribute to the partition function defined through contour C.

from the quantization condition as:

$$\begin{aligned} G(E) &= -\frac{\partial}{\partial E}\log D \\ &= \frac{\pi i}{\hbar\omega} + \frac{\partial}{\partial E}\log\Gamma\left(\frac{1}{2}+\frac{E}{\hbar\omega}\right) + \frac{\partial}{\partial E}\log\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega}\right). \end{aligned} \tag{107}$$

The partition function is the inverse Laplace transform of resolvent:

$$Z = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} G(E)e^{-\beta E}dE\,. \tag{108}$$

To calculate this quantity, we consider the contour C depicted in Fig. 8, where C is determined by the condition $E>0$. It leads $-\frac{1}{2}\hbar\omega < \epsilon < \frac{1}{2}\hbar\omega$ and C is closing in the positive real region.

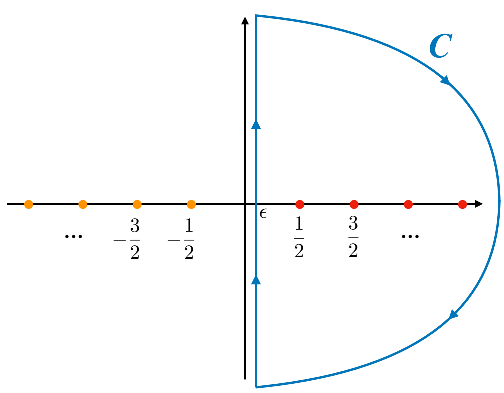

FIG. 8. C is the integration contour in the determination of the partition function as the inverse Laplace transform of resolvent.

Inside the contour C, $\frac{\pi i}{\hbar\omega}$ and $\frac{\partial}{\partial E}\log\Gamma\left(\frac{1}{2}+\frac{E}{\hbar\omega}\right)$ are holomorphic, hence do not contribute to integration. Furthermore, $\frac{\partial}{\partial E}\log\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega}\right)$ has infinitely many poles in C and all the residues[9] are 1. Therefore we find

$$Z(\beta) = \frac{1}{2\pi i}\int_C \left[\frac{\partial}{\partial E}\log\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega}\right)\right]e^{-\beta E}dE = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)}\,. \tag{109}$$

the partition function of harmonic oscillator. We will use the same strategy in more general cases involving instantons to describe the partition functions of the systems from the exact quantization condition.

[9] $\frac{\partial}{\partial x}\log\Gamma(x) \sim \frac{1}{x}$ in $x\to 0$

IV. SYMMETRIC DOUBLE-WELL POTENTIAL

We consider the exact-WKB analysis for the symmetric double-well potential. It is known that (1) the leading non-perturbative contribution to its ground state energy comes from the instanton configuration, and (2) the Borel ambiguity of the perturbation theory for the ground state is cancelled by that of the bion (correlated instanton–anti-instanton configuration) contribution. This pattern continue to higher states under the barrier. The exact form of the bion contribution can be obtained from the quasi-zero mode integration (quasi-moduli integral). We first review the resurgent structure of the partition function in this system. Then, we find an explicit mapping between this construction and Gutzwiller's quantization. In doing so, we figure out the relation between the phase ambiguity of quasi-moduli integral, the topological properties of the Stokes curve in terms of Gutzwiller's quantization.

For the symmetric double-well potential, $Q(x) = 2(V(x) - E) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$ where a_i are turning points. Then the Stokes curve of this systems[10] is schematically depicted as shown in Fig. 9.

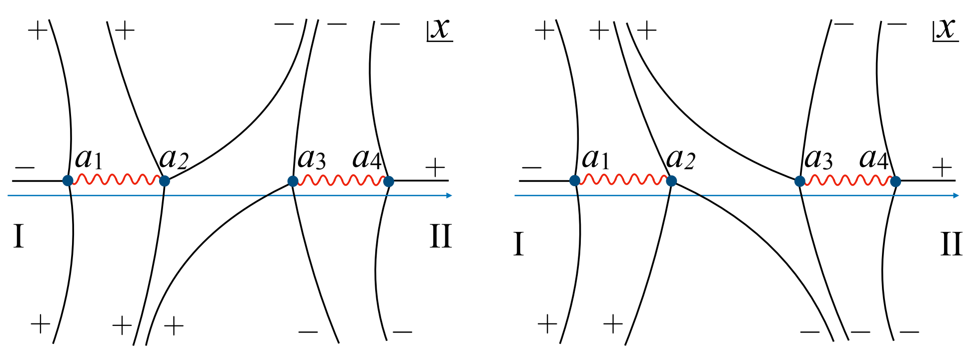

FIG. 9. The Stokes graph for the double-well potential with $\mathrm{Im}\,\hbar > 0$ (Left panel) and $\mathrm{Im}\,\hbar < 0$ (Right panel). We took the two branch cuts such that their end-points are turning points (a_1, a_2) and (a_3, a_4). We take the orbit for obtaining the quantization condition from the left to the right below the real contour.

[10] This Stokes curve corresponds to the low energy region, to energies below the barrier height. Above the barrier height, two of the real turning points, a_2 and a_3 turns into complex conjugate turning points. Even in high energy region, we can show the topological structure of Stokes curve is corresponding to phase ambiguity.

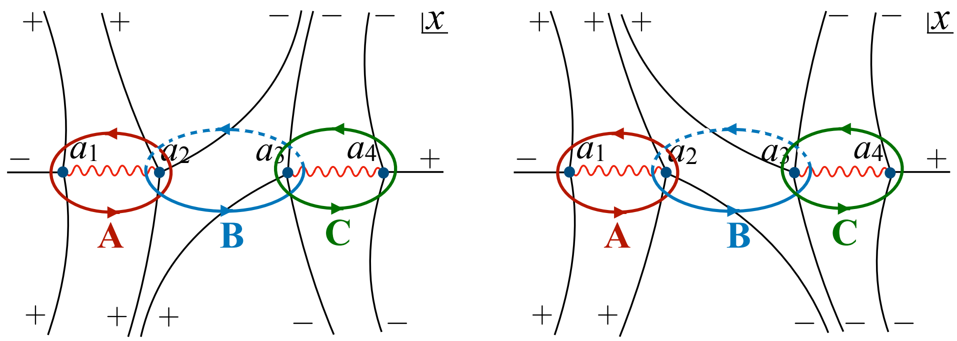

FIG. 10. Perturbative cycles (A, C) and a non-perturbative cycle (B) for the symmetric double-well potential. The perturbative cycles, A and C, are defined as oriented cycles enclosing (a_1, a_2) and (a_3, a_4), respectively, and the non-perturbative cycle, B, is an oriented cycle enclosing (a_2, a_3) and intersecting with the two branch cuts.

As we did for the harmonic oscillator in Sec. III D, we obtain the quantization condition from the normalized condition by performing the analytical continuation of the wave function from $-\infty$ to ∞ using the connection formulas.

$$\begin{cases} \begin{pmatrix} \psi^+_{a_1,\mathrm{I}}(x) \\ \psi^-_{a_1,\mathrm{I}}(x) \end{pmatrix} = M_+ N_{a_1a_2} M_+ N_{a_2a_3} M_+ M_- N_{a_3a_4} M_- N_{a_4a_3} N_{a_3a_2} N_{a_2a_1} \begin{pmatrix} \psi^+_{a_1,\mathrm{II}}(x) \\ \psi^-_{a_1,\mathrm{II}}(x) \end{pmatrix} & \text{for} \quad \mathrm{Im}\,\hbar > 0 \\ \\ \begin{pmatrix} \psi^+_{a_1,\mathrm{I}}(x) \\ \psi^-_{a_1,\mathrm{I}}(x) \end{pmatrix} = M_+ N_{a_1a_2} M_+ M_- N_{a_2a_3} M_- N_{a_3a_4} M_- N_{a_4a_3} N_{a_3a_2} N_{a_2a_1} \begin{pmatrix} \psi^+_{a_1,\mathrm{II}}(x) \\ \psi^-_{a_1,\mathrm{II}}(x) \end{pmatrix} & \text{for} \quad \mathrm{Im}\,\hbar < 0 \end{cases} \tag{110}$$

The quantization condition for this case is given by

$$D \propto \begin{cases} (1 + A^+)(1 + C^+) + A^+ B^+ = 0 & \text{for} \quad \mathrm{Im}\,\hbar > 0 \\ (1 + A^-)(1 + C^-) + C^- B^- = 0 & \text{for} \quad \mathrm{Im}\,\hbar < 0 \end{cases}, \tag{111}$$

where the cycles are defined as[11]

$$A = e^{\oint_A S_{\mathrm{odd}}}, \qquad B = e^{\oint_B S_{\mathrm{odd}}}, \qquad C = e^{\oint_C S_{\mathrm{odd}}} = 1/A \tag{112}$$

and $\mathfrak{C}^\pm := \mathcal{S}_\pm[\mathfrak{C}]$ for $\mathfrak{C} \in \{A, B, C\}$ as shown in Fig. 10. The A and C cycles are perturbative and

[11] C is the same as A on the other Riemann sheet (See the index of Stokes curve) since the potential is symmetric.

B cycle is non-perturbative, $B \propto e^{-\frac{S}{\hbar}}$, where S corresponds to the single bion contribution.

We here defined the notation, where $\mathfrak{C}$ is used as series forms and $\mathfrak{C}^{\pm}$ is used as Borel-summed forms. However, from now on, we would use a simplified notation where we use $\mathfrak{C}$ instead of $\mathfrak{C}^{\pm}$ for simplicity unless it causes a confusion.[12]

To evaluate the non-perturbative contribution to the ground state energy and the phase ambiguity term, let us consider the asymptotic form of A, which does not include non-perturbative contribution, before being Borel-resummed. It is:

$$A \to e^{-2\pi i \frac{E}{\hbar\omega_A(E,\hbar)}} \,. \tag{113}$$

This $\omega_A(E,\hbar)$ is an asymptotic expansion in $\hbar$. In the low energy limit, it can be regarded as a harmonic frequency of the classical (harmonic) vacuum:

$$\omega_A(E,\hbar)^2 = \sum_{n=0}^{\infty} c_n(E)\hbar^n \tag{114}$$

$$\lim_{E\to 0} c_0(E) = V''(x_{\rm vac}) \,, \tag{115}$$

where $x_{\rm vac}$ is a minimum of the potential. We emphasize that writing down this expression corresponds to taking the Borel-resummed A back to its asymptotic expansion form. This procedure helps us to see that the quantization condition D has the phase ambiguity. However, of course, this ambiguity disappears when we consider the Borel-resummed form. We will show it in the next subsection.

We now set $E = \hbar\omega_A\left(\frac{1}{2}+\delta\right)$, where δ roughly stands for the energy deviation from that of the harmonic oscillator. The quantization condition $D = 0$ then becomes

$$\begin{aligned} 4\sin^2(\pi\delta) &= e^{-2\pi i\delta}B \qquad \operatorname{Im}\hbar > 0\,, \\ 4\sin^2(\pi\delta) &= e^{2\pi i\delta}B \qquad \operatorname{Im}\hbar < 0\,. \end{aligned} \tag{116}$$

Or equivalently,

$$\begin{aligned} \frac{1}{\Gamma(-\delta)} &= \pm\frac{\sqrt{B}}{2\pi}e^{-\pi i\delta}\Gamma(1+\delta) \qquad \operatorname{Im}\hbar > 0\,, \\ \frac{1}{\Gamma(-\delta)} &= \pm\frac{\sqrt{B}}{2\pi}e^{\pi i\delta}\Gamma(1+\delta) \qquad \operatorname{Im}\hbar < 0\,. \end{aligned} \tag{117}$$

[12] $\mathfrak{C}^{\pm}$ can be identified when we look only at its exponentially dominant sector asymptotically.

Here $\pm$ in the latter form stands for parity. We emphasize that this result is obtained without any approximation.

In [16][24],[51], The quantization condition was calculated using path integral(QMI) method. The result is

$$\frac{1}{\Gamma(-x)} = \pm\frac{e^{-S_{inst}}}{2\pi}e^{-\pi i x}\left(\frac{\hbar}{2}\right)^{-x-\frac{1}{2}}\sqrt{2\pi} \qquad \mathrm{Im}\,\hbar < 0\,.$$
$$\frac{1}{\Gamma(-x)} = \pm\frac{e^{-S_{inst}}}{2\pi}e^{\pi i x}\left(\frac{\hbar}{2}\right)^{-x-\frac{1}{2}}\sqrt{2\pi} \qquad \mathrm{Im}\,\hbar < 0\,., \tag{118}$$

where $x = E - \frac{1}{2}$. Considering that $\left(\frac{\hbar}{2}\right)^{-\delta-\frac{1}{2}}\sqrt{2\pi}$ in (118) is the contribution from quantum fluctuations, this part is included in B and ω_A in (117). The extra Gamma function $\Gamma(1+\delta)$ is coming from the negative energy part when we consider the argument in (107), so it can be ignored under the condition $E > 0$. Therefore, this result is regarded as the complete quantization condition with full quantum fluctuations.

A. Gutzwiller's quantization

Gutzwiller's quantization is based on prime-periodic orbit (p.p.o.) as a fundamental unit, but the way how to add up this p.p.o. has not been clearly known except for simple systems. We will see that one can exactly obtain the Gutzwiller's form from the quantization conditions in the exact-WKB analysis and it reveals a new physical meaning of the quasi-moduli integral in the path integral method.

First, let us rewrite the quantization condition Eq. (111), using $C = 1/A$, in terms of only A and B cycles

$$D(E) = (1+A)(1+A^{-1})\left(1+\frac{B}{D_A^2}\right) \tag{119}$$

where D_A=$1 + A^{-1}$ ($\mathrm{Im}\,\hbar > 0$) or $1 + A$ ($\mathrm{Im}\,\hbar < 0$). This rewriting allows us to write the trace of resolvent $G(E) = -\frac{\partial}{\partial E}\log D(E)$, derived from the quantization condition Eq. (111), in a useful form:

$$
\begin{aligned}
G(E) &= G_{\mathrm{p}}(E) + G_{\mathrm{np}}(E) \\
&= \left[-\frac{\partial}{\partial E}\log(1+A) - \frac{\partial}{\partial E}\log\bigl(1+A^{-1}\bigr)\right] + \left[-\frac{\partial}{\partial E}\log\left(1+\frac{B}{D_A^2}\right)\right] \\
&= \left[-\frac{\frac{\partial}{\partial E}A}{1+A} - \frac{\frac{\partial}{\partial E}A^{-1}}{1+A^{-1}}\right] + \left[-\frac{\frac{\partial}{\partial E}\bigl(D_A^{-2}B\bigr)}{1+(D_A^{-2}B)}\right],
\end{aligned} \tag{120}
$$

The derivative term $\frac{\partial}{\partial E}A$ produces the "period"

$$
\begin{aligned}
\frac{\partial}{\partial E}A = \frac{\partial}{\partial E}e^{\oint_A S_{\mathrm{odd}}} &= \left(\frac{\partial}{\partial E}\oint_A S_{\mathrm{odd}}\right)e^{\oint_A S_{\mathrm{odd}}} \\
&= \left(\oint_A \frac{1}{\hbar}\frac{-1}{\sqrt{2(V-E)}} + O(\hbar)\right)e^{\oint_A S_{\mathrm{odd}}} \equiv -\frac{1}{\hbar}iT_A A\,.
\end{aligned} \tag{121}
$$

and similarly,

$$
\frac{\partial}{\partial E}B = -\frac{1}{\hbar}iT_B B\,. \tag{122}
$$

$$
\frac{\partial}{\partial E}\bigl(D_A^{-2}B\bigr) = -i\frac{1}{\hbar}\sum_{n,m=1}^{\infty}(-1)^{(n+m)}\bigl(T_B \mp (n+m)T_A\bigr)B(A^{\mp})^{n+m}, \tag{123}
$$

where $\pm$ corresponds to $\operatorname{Im}\hbar > 0$ and $\operatorname{Im}\hbar < 0$ respectively. Since the classical solutions in the lower part of the potential are doubly-periodic, and our definition of $S_{-1} = \sqrt{Q}$ where $Q = 2(V-E)$, T_A is real ($E > V$) and T_B is purely imaginary ($E < V$). Our construction instructs us that *complex* periodic paths are part of Gutzwiller formula. This seems to be the mechanism through which Gutzwiller formula is able to capture the tunneling (instanton) effects. The magnitudes of the quantities T_A, T_B can be called *quantum periods* and its leading term corresponds exactly to

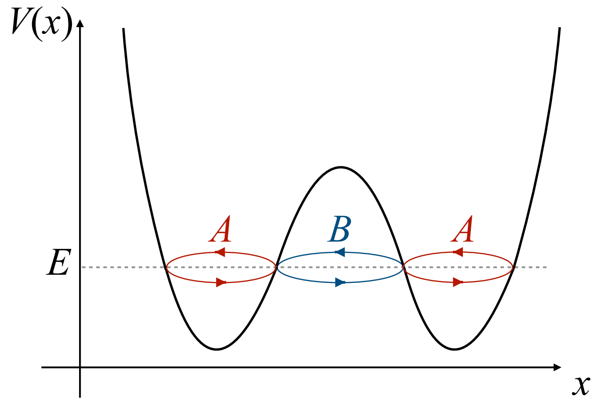

FIG. 11. Relationship between a periodic orbit and the Maslov index for the symmetric double-well potential. The index $(-1)^n$ is determined by counting $D_A^{-2}B$ in Eq.(126), as a unit. $D_A^{-2}B$ includes two infinite number of A-cycles ($D_A^{-1} = \frac{1}{1+A}$) and one B-cycle (B).

the period of the classical orbit. Using these quantities, $G(E)$ can be expressed as

$$G(E) = G_{\rm p} + G_{\rm np} \tag{124}$$

$$G_{\rm p}(E) = i\frac{1}{\hbar}T_A\sum_{n=1}^{\infty}(-1)^nA^n + i\frac{1}{\hbar}T_A\sum_{n=1}^{\infty}(-1)^nA^{-n}, \tag{125}$$

$$G_{\rm np}(E) = -\frac{\partial}{\partial E}\big(D_A^{-2}B\big)\sum_{n=0}^{\infty}(-1)^n(D_A^{-2}B)^n, \tag{126}$$

$$D_A^{-2}B = \begin{cases} B\big(\sum_{k=1}^{\infty}(-1)^kA^{-k}\big)\big(\sum_{l=1}^{\infty}(-1)^lA^{-l}\big) & (\mathrm{Im}\,\hbar > 0) \\ B\big(\sum_{k=1}^{\infty}(-1)^kA^{k}\big)\big(\sum_{l=1}^{\infty}(-1)^lA^{l}\big) & (\mathrm{Im}\,\hbar < 0) \end{cases} \tag{127}$$

This is exactly the form of Gutzwiller's quantization in Eq. (54) including the quantum corrections. Note that the quantum period and each cycle contain the quantum corrections (e.g. $T_A = T_{A,cl} + O(\hbar)$, $A = e^{\frac{i}{\hbar}\oint_A p} + O(\hbar)$). It is important to remind ourselves that the (-1) associated with each cycle can be interpreted as the factor coming from Maslov index (See Sec. II C 1).

Our result shows what p.p.o. are and how to add them up explicitly, and it is by no means obvious. Perhaps, we should take exact quantization condition and the corresponding resolvent Eq.(120) as the precise meaning of the Gutzwiller's sum. The perturbative part consists of the infinite number of A cycles and the non-perturbative part is made up of the infinite number of A cycles and B cycle. The change of topology of the Stokes curves corresponds to the reversal of the direction of the A cycle of the non-perturbative term. As we show later, this transition can give the new perspective of the quasi-moduli integration.

B. Partition function

In this subsection, we calculate the partition function based on the resolvent method in Sec.III E. In particular, when evaluating the partition function using path integral, it is important to evaluate the contribution of the integral called quasi-moduli integral(QMI). It is shown that this can be evaluated explicitly by the calculation using exact WKB. We also show that the partition function itself is invariant under the Borel sum.

1. *Comparison to quasi-moduli integral*

Using the decomposition of resolvent given in (120), we can write the partition function as

$$Z = Z_{\rm p}(\beta) + Z_{\rm np}(\beta) \tag{128}$$

where

$$Z_{\rm p}(\beta) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log(1+A)\right]e^{-\beta E}dE + \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log\big(1+A^{-1}\big)\right]e^{-\beta E}dE, \tag{129}$$

and

$$\begin{aligned} Z_{\rm np}(\beta) &= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log\left(1+\frac{B}{D_A^2}\right)\right]e^{-\beta E}dE \\ &= -\beta\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\log\left(1+\frac{B}{D_A^2}\right)e^{-\beta E}dE \\ &= \beta\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{B}{D_A^2}\right)^n e^{-\beta E}dE\,, \end{aligned} \tag{130}$$

where we have used integration by parts moving to the second line. We now clarify the relation between the above quasi-moduli integral and our result on the non-perturbative contribution

$$Z_{\rm np}(\beta) = \beta\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{B}{D_A^2}\right)^n e^{-\beta E}dE\,, \tag{131}$$

Using this asymptotic expansion in Eq. (113) $A \sim e^{-2\pi i \frac{E}{\hbar\omega_A(E,\hbar)}}$ again. Then D_A is given by

$$D_A = \begin{cases} e^{\pi i \frac{E}{\hbar\omega_A}} 2\sin\left(\pi\left(\frac{E}{\hbar\omega_A}+\frac{1}{2}\right)\right) = e^{\pi i \frac{E}{\hbar\omega_A}} \frac{2\pi}{\Gamma(\frac{1}{2}+\frac{E}{\hbar\omega_A})\Gamma(\frac{1}{2}-\frac{E}{\hbar\omega_A})} & (\mathrm{Im}\,\hbar > 0) \\ e^{-\pi i \frac{E}{\hbar\omega_A}} 2\sin\left(\pi\left(\frac{E}{\hbar\omega_A}+\frac{1}{2}\right)\right) = e^{-\pi i \frac{E}{\hbar\omega_A}} \frac{2\pi}{\Gamma(\frac{1}{2}+\frac{E}{\hbar\omega_A})\Gamma(\frac{1}{2}-\frac{E}{\hbar\omega_A})} & (\mathrm{Im}\,\hbar < 0)\,. \end{cases} \tag{132}$$

For our purpose, we drop the irrelevant Gamma function factor $\frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}+\frac{E}{\hbar\omega_A})}$, which corresponds to the negative eigenvalue and does not contributes to the integral in the case of harmonic oscillator.

Then, we rewrite Z_{np} as

$$Z_{\mathrm{np}}(\beta) = \beta \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \sum_{n=1}^{\infty} \frac{1}{n}(-1)^n \left(B \frac{e^{\mp 2\pi i \frac{E}{\hbar\omega_A}}}{4\sin^2\left(\pi\left(\frac{E}{\hbar\omega_A}+\frac{1}{2}\right)\right)}\right)^n e^{-\beta E} dE\,. \tag{133}$$

By defining $s \equiv E/(\hbar\omega_A) - 1/2$, it is expressed as

$$Z_{\mathrm{np}}(\beta) = \beta \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \sum_{n=1}^{\infty} \frac{1}{n}(-1)^n \left(B \frac{e^{\mp 2\pi i(\frac{1}{2}+s)}}{4\sin^2(\pi s)}\right)^n e^{-\beta(\hbar\omega_A(\frac{1}{2}+s))}\hbar\omega_A ds \tag{134}$$

Essentially $\frac{\sqrt{2\pi}}{\Gamma(1+s)}$ in $2\sin(\pi s) = \frac{2\pi}{\Gamma(-s)\Gamma(1+s)}$ corresponds to the negative eigenvalues, so if we define the integral path to take only positive eigenvalues, this integral can be approximated[13] as

$$\begin{aligned} Z_{\mathrm{np}}(\beta) &\simeq \beta \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \sum_{n=1}^{\infty} \frac{1}{n}(-1)^n \left(\Gamma(-s)^2 \frac{B}{2\pi} e^{\mp 2\pi i(1/2+s)}\right)^n e^{-\beta(\hbar\omega_A(1/2+s))}\hbar\omega_A ds & (135) \\ &= \beta \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \sum_{n=1}^{\infty} \frac{1}{n} \left(B\Gamma(-s)^2 \frac{1}{2\pi} e^{\mp 2\pi i s}\right)^n e^{-\beta\frac{\hbar\omega_A}{2}} e^{-s\beta}\hbar\omega_A ds\,. & (136) \end{aligned}$$

Here, the partition function obtained by calculating the path integral is as follows[51][24], (Appendix ??):

$$\frac{Z_{\mathrm{np}}}{Z_0} = \beta \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \sum_{n=1}^{\infty} \frac{1}{n} \left(e^{-S_{\mathrm{bion}}} \left(\frac{\det M_I}{\det M_0}\right)^{-1} \frac{S_{\mathrm{inst}}}{2\pi}\Gamma(-s)^2 \left(\frac{\hbar}{2}\right)^{-s} e^{\mp 2\pi i s}\right)^n e^{-s\beta} ds\,. \tag{137}$$

Comparing this with our results, we can see that we obtain the path integral representation and indeed each sector in Eq. (136) has physical meaning as follows: β corresponds to the zero-mode integral (translation symmetry of time-dependent solution), $\frac{1}{n}$ is cyclic permutation of multi-bions, $\Gamma(-s)^2 e^{\mp 2\pi i s}$ are quasi-moduli integrals(QMI) (See Appendix ??) with Stokes phenomena[14], $B =$

[13] The residues of $\frac{1}{\sin(\pi s)}$ and $\Gamma(-s)$ are different, so just removing $\Gamma(1+s)$, even though it essentially corresponds to the negative eigenvalues, would change the result. However if we only consider the residue around $s = 0$, which corresponds to the ground state energy, the factor $\Gamma(1+s)$ can be ignored.

[14] There are two QMI for one bion and it gives two Gamma functions.

$e^{-\frac{1}{\hbar}\oint_B p} + O(\hbar)$ is the bion contribution with quantum correction, the integral from $-i\infty$ to $i\infty$ corresponds to the delta function constraint in the quasi-moduli integral, and $e^{-\frac{1}{2}\beta\hbar\omega_A}$ is regarded as the partition function of perturbative part in the large β limit, Z_0. The missing part in Eq. (136) is $S_{\rm inst}\left(\frac{\det M_I}{\det M_0}\right)^{-1}$ and $\left(\frac{\hbar}{2}\right)^{-s}$. However, both are coming from the quantum fluctuation. But since the quantum fluctuation is included in B and ω_A, it is considered to be required by doing a higher-order expansion of them. $(-1)^n$ in Eq. (135) is regarded as the *Maslov index*. The origin of this index is easily understood by using Gutzwiller's quantization as shown in Sec. IV A. This factor is cancelled by $e^{\mp\pi i n}$ in Eq. (135), which looks like the hidden topological angle (HTA) [36]) of n-bion configuration though, the index has very important role: this quantity can be regarded as the intersection number of Lefschetz thimble[15].

Furthermore, comparing the QMI calculation and Gutzwiller's perspective, we can see the new physical meaning of QMI. The QMI calculation is based on the approximation that the cycle is sufficiently large, but from Gutzwiller's point of view, the B cycle is so short that it requires the A cycle to rotate infinite times in order to earn the sufficiently long cycle, and therefore it is considered to be represented in the form of $D_A^{-2}B$. This perspective explains the puzzle in the calculation of [51][16]: The structure of the Γ function derived from the vacuum contribution and the structure of the Γ function derived from QMI matched despite both were calculated entirely separately.

$$D(E) = \frac{1}{\Gamma\left(\frac{1}{2}-E\right)\Gamma\left(\frac{1}{2}-E\right)}\left(1 - Be^{\pm i\pi(1-2E)}\left(\frac{\hbar}{2}\right)^{(1-2E)}\Gamma\left(\frac{1}{2}-E\right)\Gamma\left(\frac{1}{2}-E\right)\right) = 0 \tag{138}$$

The first $\frac{1}{\Gamma\left(\frac{1}{2}-E\right)\Gamma\left(\frac{1}{2}-E\right)}$ are from two vacua and the latter ones are from QMI. This miracle is easily explained by this Gutzwiller's representation. Both have essentially the same origin, the infinite number of A cycles, $D_A^{-1} = \frac{1}{1+A} = \sum_n^{\infty}(-1)^n A^n$.

2. The intersection number of Lefschetz thimble

It is notable that the of quantization condition in Eqs. (111) determines the "relevant saddles" in the path integral and the intersection number of Lefschetz thimble (n_σ). Firstly, as we mentioned in Sec. II A, the Fredholm determinant can be expressed as

$$D(E) = \prod_\sigma D_\sigma^{n_\sigma}(E)\,. \tag{139}$$

[15] The relation between the Maslov index and Lefschetz thimbles is also discussed in [47]

Now, the quantization condition given by Eq. (111) can be rewritten as

$$
\begin{aligned}
D &= (1+A)(1+A^{-1})\left(1+\frac{B}{D_A^2}\right) \\
&= (1+A)(1+A^{-1})\left[e^{-\frac{B}{D_A^2}}\right]^{-1}\left[e^{-\frac{1}{2}\left(\frac{B}{D_A^2}\right)^2}\right]\left[e^{-\frac{1}{3}\left(\frac{B}{D_A^2}\right)^3}\right]^{-1}\left[e^{-\frac{1}{4}\left(\frac{B}{D_A^2}\right)^4}\right]\ldots \\
&= (1+A)(1+A^{-1})\prod_{n=1}^{\infty} D_n^{(-1)^n} \qquad (140)
\end{aligned}
$$

The first $(1+A)$ and $1+A^{-1}$ are regarded as the Fredholm determinant coming from the vacuum saddle points and the latter ones are ones from n-bion saddle points.

$$
D_n = e^{-\frac{1}{n}\left(\frac{B}{D_A^2}\right)^n} \qquad (141)
$$

$$
\begin{aligned}
Z_n &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\frac{\partial}{\partial E}\frac{1}{n}\left(\frac{B}{D_A^2}\right)^n\right]e^{-\beta E}dE \\
&= \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{1}{n}\left(\Gamma(-s)^2\frac{B}{2\pi}e^{\mp 2\pi i(1/2+s)}\right)^n e^{-\beta(\hbar\omega_A(1/2+s))}\hbar\omega_A ds\,. \qquad (142)
\end{aligned}
$$

This representation is a factorized form for each bion, and as explained in the previous section, the Z_n obtained from this D_n is indeed the partition function of n-bions. The power of each bracket [...] is nothing other than the Maslov index. Therefore, we can see the intersection number of Lefschetz thimble of non-perturbative contributions is exactly corresponding to the Maslov index.

Strictly speaking, the Maslov index is attached to both A and B cycles. However, the former is not regarded as the intersection number of Lefschetz thimble but the only latter's is. This difference is related to the following situation: In the case of harmonic oscillator, we often calculate the partition function around the vacuum, and indeed it gives the correct answer. However there are other classical solutions in this system s.t. time-dependent solutions like oscillating around the vacuum. If we choose such the solution as the saddle point, we still get the same partition function. The reason is such time-dependent solutions are included in quantum fluctuations around the vacuum. On the other hand, the non-perturbative saddle, bions should be summed up to obtain the correct partition function. It means n-bion (and the vacuum)is topologically separated in the functional space, or one can say 2-bion cannot be expressed as 1-bion with quantum fluctuation.

We would make some comments on the Maslov index and intersection number. We may also write the quantization conditions for the symmetric double well as follows:

$$\begin{aligned} D &= (1+A)(1+A^{-1})\left(1+\frac{B}{D_A^2}\right) \\ &= (1+A)(1+A^{-1})\sqrt{1+\frac{B}{D_A^2}}\sqrt{1+\frac{B}{D_A^2}} \\ &= (1+A)(1+A^{-1})\left(\prod_{n=1}^{\infty} D_n^{\frac{1}{2}(-1)^n}\right)\left(\prod_{n=1}^{\infty} D_n^{\frac{1}{2}(-1)^n}\right), \end{aligned} \tag{143}$$

and this form can be considered as giving a fractional intersection number, $\mp\frac{1}{2}$. In the similar way to the procedure for obtaining the partition function given by Eq.(142), the quantization condition (143) gives

$$Z \simeq Z_{1,\mathrm{pert}} + Z_{2,pert} + \frac{1}{2}\sum_{n=1}^{\infty} e^{-nS_{\mathrm{bion}}} + \frac{1}{2}\sum_{n=1}^{\infty} e^{-nS_{\mathrm{bion}}}. \tag{144}$$

This form corresponds to the representation of the vacuum with a different starting point as a separate term. However, while physically it is reasonable to write the contributions in this way, from the point of view of transseries, these term should be combined.

Also the following form is considerable.

$$\begin{aligned} D &= (1+A)(1+A^{-1})\left(1+\frac{B}{D_A^2}\right) \\ &= (1+A)(1+A^{-1})\left(1+i\frac{\sqrt{B}}{D_A}\right)\left(1-i\frac{\sqrt{B}}{D_A}\right) \end{aligned} \tag{145}$$

The latter parts corresponds to the instanton contributions. Because of $G_{inst.}(E) = -\frac{\partial}{\partial E}\log\left(1+i\frac{\sqrt{B}}{D_A}\right) \propto \sum_n (-i)^n \left(\frac{\sqrt{B}}{D_A}\right)^n$, the Maslov index of instanton ($\sqrt{B}$) is $-i$. Using parity operator, $\hat{P}\,|x\rangle = |-x\rangle$, we can consider the projected partition function $Z_{\pm} = \mathrm{tr}\left(\frac{1\pm\hat{P}}{2}e^{-\beta\hat{H}}\right)$. From [16], it corresponds to

$$\begin{aligned} Z_+ &\to (1+A)\left(1+i\frac{\sqrt{B}}{D_A}\right) \\ Z_- &\to (1+A^{-1})\left(1-i\frac{\sqrt{B}}{D_A}\right) \end{aligned} \tag{146}$$

Therefore, if we impose the non-periodic boundary condition on the path integral, such the non-integer intersection number can appear. However, if we consider only the periodic trajectory, the

Maslov index is always integer, which means the intersection number is also integer.

C. Delabaere-Dillinger-Pham (DDP) formula

In this subsection, we would like to briefly review the Delabaere-Dillinger-Pham (DDP) formula[135]. For the double-well potential, as we have seen the previous sections, the A, C-cycles are defined as asymptotic series. These series are non-Borel nonsummable if $\mathrm{Im}\,\hbar = 0$. When the an asymptotic expansion is Borel nonsummable for $\mathrm{Im}\,\hbar = 0$, the Borel transformed cycles have a singular point on the positive real axis of the Borel plane, in other words an imaginary ambiguity happens according to the choice of the sign of $\mathrm{Arg}(\hbar)$ for the Borel resummation.

By employing this imaginary ambiguity the information of A, C-cycles can be carried into the B-cycle via the Stokes automorphism. This relationship is so called *the resurgence relation*. This type of resurgence relation connects high orders of the asymptotic expansion of the A, C-cycles to low orders perturbative expansion of B-cycle. In the physical sense, the A, C-cycles and B-cycle are now interpreted as a perturbative expansion(fluctuation) in terms of $\hbar$ around (locally) bounded potential and the nonperturbative bion background, respectively, so that their asymptotic expansions can be related to each others by the resurgence relation. Instead of directly looking at the Borel plane, there exits a way to find the same relation from the Stokes graph of the exact WKB analysis, which is so called *the Delabaere-Dillinger-Pham (DDP) formula*. The DDP formula can be directly applied to any functions of the cycles, and it would also have the important role to see the cancellation of imaginary ambiguities for the partition function, discussed in the later section. From here, we would like to demonstrate some physical applications to (DDP) formula [135, 146] to potential-well problems.

For simplicity, let us start with the simple setup which is the double-well potential with low energy shown in Fig. 12. The Stokes graph has four turning points on the real axis labelled by $a_1 < \cdots < a_4$, and we consider three anti-clockwise cycles enclosing a pair of two turning points. These are (a_1, a_2), (a_2, a_3), (a_3, a_4). We named these oriented loops as $\gamma_{12}, \gamma_{23}, \gamma_{34}$ in Fig.12, and the quantities A, B, C-cycles are defined along the each loops. Since the branch-cut lay on the complex x-plane with the endpoints at turning points, the A and C-cycles are now defined on the first Riemann have nontrivial value. In contrast, the B-cycle twice crosses the independently defined branch-cut, and the lower(upper) half contour lies on the first(second) Riemann sheet. This means that the B-cycle has the intersection with each A- and C-cycles once on the first sheet but

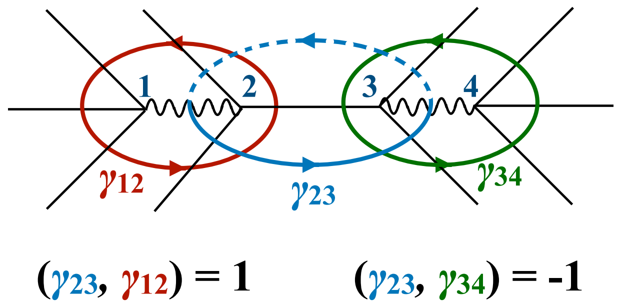

FIG. 12. Intersection number among cycles in the DDP formula. In this case, the intersection numbers of cycles are given by $(\gamma_{23}, \gamma_{12}) = +1$ and $(\gamma_{23}, \gamma_{34}) = -1$. (See fig.13 for the definition.)

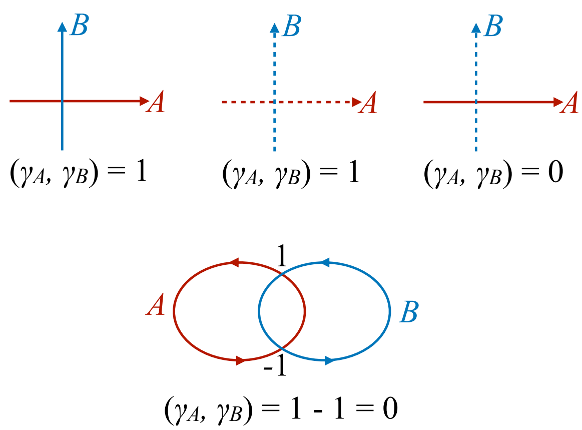

FIG. 13. The definition of intersection number. The solid and dashed lines denote a part of cycle on the first and second Riemann sheet, respectively. If two solid(dashed) lines given by γ_A and γ_B intersect with each other in respect to a right-handed coordinate, we say $(\gamma_A, \gamma_B) = +1$. If the intersection is given by lines on different sheets from each other, we say $(\gamma_A, \gamma_B) = 0$. For example, two cycles not crossing branch-cuts always give $(\gamma_A, \gamma_B) = 0$.

does not on the second sheet[16]. Under this setup, the DDP formula is obtained as follows. Here all symbols such as A, B used so far means Borel summed ones. In order to make this point clear, we will describe its asymptotic form as $\tilde{A}$, $\tilde{B}$ and Borel summed ones as $\mathcal{S}_{\pm}[\tilde{A}]$ and $\mathcal{S}_{\pm}[\tilde{B}]$ in this

---

[16] If the $A(C)$-cycle is defined on the second sheet, the relation with the cycle on the first sheet can be found as

$$A_{\text{1st.sheet}} = 1/A_{\text{2nd.sheet}}. \tag{147}$$

section. The DDP formula is given as

$$\mathcal{S}_+[\tilde{A}] = \mathcal{S}_-[\tilde{A}](1+\mathcal{S}[\tilde{B}])^{-1}, \tag{148}$$

$$\mathcal{S}_+[\tilde{B}] = \mathcal{S}_-[\tilde{B}] =: \mathcal{S}[\tilde{B}], \tag{149}$$

$$\mathcal{S}_+[\tilde{C}] = \mathcal{S}_-[\tilde{C}](1+\mathcal{S}[\tilde{B}])^{+1}, \tag{150}$$

where $\mathcal{S}_\pm$ is the Borel resummation for $\mathrm{sign}(\mathrm{Im}\,\hbar) = \pm 1$. The exponent of $(1+\mathcal{S}[\tilde{B}])$ is by intersection number $(\gamma_A, \gamma_B) = \pm 1$ which is defined as follows: If the intersection between perturbative and non-perturbative cycles, A and B, is right(left)-handed, we say that $(\gamma_A, \gamma_B) = +1(-1)$. If a perturbative cycle does not have intersection with non-perturbative cycles, then it gives $(\gamma_A, \gamma_B) = 0$. Fig. 13 shows how to determine the intersection number.

In the previous section we separately obtained the quantization condition D for $\mathrm{Im}\,\hbar > 0$ and $\mathrm{Im}\,\hbar < 0$, but one can see that these can be related to each other via the DDP formula:

$$\mathcal{S}_+[\tilde{D}^+] = \mathcal{S}_-[\tilde{D}^-], \tag{151}$$

where D^+ and D^- are given by eq.(111) for the positive and negative $\mathrm{Im}\hbar$, respectively, but replaced A, B, C with $A^\pm, B, C^\pm$. It is important to mention that from eq.(151) one can indeed show the imaginary ambiguity cancellation for *all order of bion sectors.*

V. GENERIC POTENTIALS (NPLE-WELL POTENTIAL)

The procedure to derive the quantization condition from the exact-WKB analysis and the construction of the path-integral explained in Sec. IV can be extended to the cases with more generic forms of potentials. Since the extension is straightforward, we would show only the quantization condition and make some comments.

We now focus on a parity-symmetric real-bounded potential, $V(x) \in \mathbb{R}[[x^2]]$ with $\lim_{|x|\to\infty} V(x) = +\infty$. There are N perturbative and $N-1$ non-perturbative cycles as shown in Fig. 14. By repeating the same argument as that in Sec. IV C, the DDP formula can be easily extended to these cases. When the N non-perturbative cycles ($\tilde{B}_i$) lays on the complex x-plane, the DDP formula

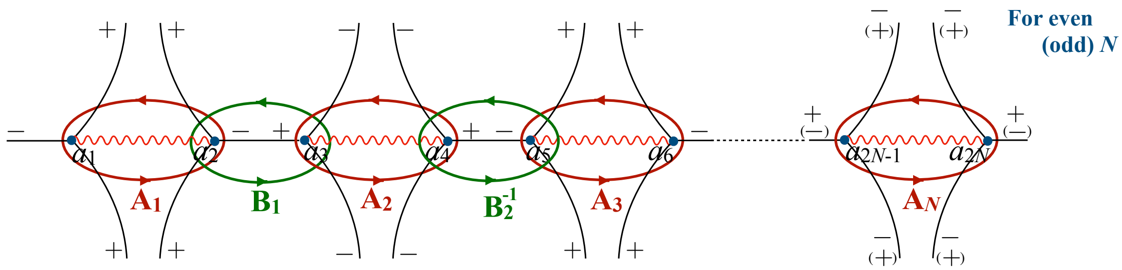

FIG. 14. The Stokes graph for the Nple-well potential for $\mathrm{Arg}(\hbar) = 0$. Perturbative cycles A_n and nonperturbative cycles B_n are defined as cycles enclosing a branch-cut and a stokes line having end points at turning points, respectively.

for a perturbative cycle $(\tilde{A}_j)$ can be expressed as [135]

$$\mathcal{S}_+[\tilde{A}_j] = \mathcal{S}_-[\tilde{A}_j] \prod_{i=1}^{N-1} (1 + \mathcal{S}[\tilde{B}_i])^{(-1)^i \cdot (\gamma_{B_i}, \gamma_{A_j})}, \tag{152}$$

$$\mathcal{S}_+[\tilde{B}_i] = \mathcal{S}_-[\tilde{B}_i] =: \mathcal{S}[\tilde{B}_i] \quad \text{for all} \quad 1 \leq i \leq N. \tag{153}$$

In this notation, we chose the orientation of $\tilde{B}_i$-cycles such that $\tilde{B}_i$ is exponentially small. Then, by considering the path-orbit of the wave function from the left to the right and using the connection formula, the quantization condition for the positive/negative $\mathrm{Arg}(\hbar)$ can be obtained as

$$\tilde{D}_N^+ = \Omega(\tilde{L}) \sum_{\mathbf{n}=\mathbf{0}}^{|\mathbf{n}|=N} \prod_{k=1}^{\lfloor (N+1)/2 \rfloor} \left[\left(1 + \tilde{B}_{2k-2}\right)^{1-n_{2k-2}} \left(1 + \tilde{B}_{2k-1}\right)^{1-n_{2k}} \right]^{n_{2k-1}} \cdot \Phi^{(\mathbf{n})}(\tilde{\mathbf{A}}), \tag{154}$$

$$\tilde{D}_N^- = \Omega(\tilde{L}) \sum_{\mathbf{n}=\mathbf{0}}^{|\mathbf{n}|=N} \prod_{k=1}^{\lfloor N/2 \rfloor} \left[\left(1 + \tilde{B}_{2k-1}\right)^{1-n_{2k-1}} \left(1 + \tilde{B}_{2k}\right)^{1-n_{2k+1}} \right]^{n_{2k}} \cdot \Phi^{(\mathbf{n})}(\tilde{\mathbf{A}}), \tag{155}$$

$$\Omega(\tilde{L}) = \frac{i^N}{\tilde{L}}, \qquad \tilde{L} := \left(\prod_{k=1}^{N} \tilde{A}_k \right) \left(\prod_{k=1}^{N-1} \tilde{B}_k \right), \tag{156}$$

where $\mathbf{n} = (n_0, n_1, \cdots, n_N, n_{N+1}) \in (\mathbb{Z}/2\mathbb{Z})^{\otimes N+2}$ with $n_0 = n_{N+1} = 0$, $|\mathbf{n}|$ denotes the 1-norm of $\mathbf{n}$, $\tilde{B}_0 = \tilde{B}_N = 0$, and $\Phi^{(\mathbf{n})}(\tilde{\mathbf{A}}) := \prod_{k=1}^{N} \tilde{A}_k^{n_k}$.[17] In the similar way to the double-well potential, one finds $\mathcal{S}_+[\tilde{D}^+] = \mathcal{S}_-[\tilde{D}^-]$ and shows that *the imaginary ambiguity is cancelled for all order of bion sectors.* After the cancellation, the quantization condition can be expressed in terms of

[17] For $\tilde{D}_{N=1}^-$, we assume that $\prod_{k=1}^{0} [\cdots] = 1$.

unambiguous parts as:

$$
\begin{aligned}
&\lim_{\mathrm{Arg}(\hbar)\to 0_\pm} \mathcal{S}_\pm[\tilde{D}_N^\pm/\Omega(\tilde{L})] \\
&= \sum_{\mathbf{n}=\mathbf{0}}^{|\mathbf{n}|=N} \frac{2\bar{\mathfrak{S}}_0^{(\mathbf{n})}(\hat{\mathbf{B}})}{\bar{\mathfrak{S}}_0^{(\mathbf{n})}(\hat{\mathbf{B}})+1} \cdot \prod_{k=1}^{\lfloor (N+1)/2 \rfloor} \left[\left(1+\hat{B}_{2k-2}\right)^{1-n_{2k-2}} \left(1+\hat{B}_{2k-1}\right)^{1-n_{2k}} \right]^{n_{2k-1}} \cdot \Phi^{(\mathbf{n})}(\hat{\mathbf{A}}), \\
&= \sum_{\mathbf{n}=\mathbf{0}}^{|\mathbf{n}|=N} \frac{2}{\bar{\mathfrak{S}}_0^{(\mathbf{n})}(\hat{\mathbf{B}})+1} \cdot \prod_{k=1}^{\lfloor N/2 \rfloor} \left[\left(1+\hat{B}_{2k-1}\right)^{1-n_{2k-1}} \left(1+\hat{B}_{2k}\right)^{1-n_{2k+1}} \right]^{n_{2k}} \cdot \Phi^{(\mathbf{n})}(\hat{\mathbf{A}}), \\
&= \sum_{\mathbf{n}=\mathbf{0}}^{|\mathbf{n}|=N} \frac{2\prod_{k=1}^N [(1+\hat{B}_{k-1})(1+\hat{B}_k)]^{n_k} \cdot \Phi^{(\mathbf{n})}(\hat{\mathbf{A}})}{\prod_{k=1}^{\lfloor (N+1)/2 \rfloor} [(1+\hat{B}_{2k-2})^{1+n_{2k-2}}(1+\hat{B}_{2k-1})^{1+n_{2k}}]^{n_{2k-1}} + \prod_{k=1}^{\lfloor N/2 \rfloor} [(1+\hat{B}_{2k-1})^{1+n_{2k-1}}(1+\hat{B}_{2k})^{1+n_{2k+1}}]^{n_{2k}}},
\end{aligned}
\tag{157}
$$

where

$$
\Phi^{(\mathbf{n})}(\hat{\mathbf{A}}) = \frac{\lim_{\mathrm{Arg}(\hbar)\to 0_+} \mathcal{S}_+[\Phi(\tilde{\mathbf{A}})] + \lim_{\mathrm{Arg}(\hbar)\to 0_-} \mathcal{S}_-[\Phi(\tilde{\mathbf{A}})]}{2}, \tag{158}
$$

$$
\hat{B}_n := \begin{cases} \lim_{\mathrm{Arg}(\hbar)\to 0_+} \mathcal{S}_+[\tilde{B}_n] = \lim_{\mathrm{Arg}(\hbar)\to 0_-} \mathcal{S}_-[\tilde{B}_n] & \text{for} \quad 1 \le n < N \\ 0 & \text{for} \quad n = 0, N \end{cases}, \tag{159}
$$

$\bar{\mathfrak{S}}_0^{(\mathbf{n})}(\hat{\mathbf{B}})$ is given by $\mathcal{S}_+[\Phi^{(\mathbf{n})}(\hat{\mathbf{A}})] = \mathcal{S}_- \circ \mathfrak{S}_0[\Phi^{(\mathbf{n})}(\hat{\mathbf{A}})] =: \bar{\mathfrak{S}}_0^{(\mathbf{n})}(\hat{\mathbf{B}})\Phi^{(\mathbf{n})}(\hat{\mathbf{A}})$ with the Stokes automorphism being expressed as $\mathfrak{S}_0$. Notice that $\Omega(L)$ does not contribute to the path-integral. Because of parity symmetry and the reality condition for the potential, the different cycles have nontrivial relation

$$
\hat{A}_n = \begin{cases} \hat{A}_{N-n+1}^{-1} = [\hat{A}_{N-n+1}]^* & \text{for even } N \\ \hat{A}_{N-n+1} = [\hat{A}_{N-n+1}^{-1}]^* & \text{for odd } N \end{cases}, \qquad \hat{B}_n = \hat{B}_{N-n} = [\hat{B}_{N-n}]^*. \tag{160}
$$

Thus, eq.(157) is real for an even N. For an odd N, the real $D_N^\pm$ can be obtained by dividing by $\hat{A}_{(N+1)/2}^{1/2} \cdot \prod_{n=1}^{(N-1)/2} \hat{A}_n$, which does not contribute to calculation of residue for reproducing the partition function due to the Cauchy's argument principle.

One can find the origin of the intersection number (index) of the thimble decomposition, i.e. the Maslov index, in terms of the quantization condition. For example, the quantization condition (154), (155) for $N = 3$ can be expressed as

$$D^{+}_{N=3}/\Omega(L) = (1+A_1)(1+A_2)(1+A_1)\left[1+\frac{B}{D^{-2}_{A_1}D^{+}_{A_2}}\left(2D^{-}_{A_1}+B\right)\right], \tag{161}$$

$$D^{-}_{N=3}/\Omega(L) = (1+A_1)(1+A_2)(1+A_1)\left[1+\frac{B}{D^{+2}_{A_1}D^{-}_{A_2}}\left(2D^{+}_{A_1}+B\right)\right]. \tag{162}$$

where $\mathfrak{C} := \lim_{\mathrm{Arg}(\hbar)\to 0_{+(-)}} \mathcal{S}_{+(-)}[\tilde{\mathfrak{C}}]$ for cycles $\mathfrak{C} \in \{A_1, A_2, A_3, B_1, B_2\}$ in $D^{+(-)}_{N=3}$, and

$$D^{\pm}_{A_1} := 1 + A_1^{\pm 1}, \qquad D^{\pm}_{A_2} := 1 + A_2^{\pm 1}. \tag{163}$$

We also use the fact that $A_3 = A_1$ and $B_1 = B_2 =: B$. Therefore, we obtain

$$\begin{aligned} D^{\pm}_{N=3}/\Omega(L) =& (1+A_1)(1+A_2)(1+A_1) \\ & \cdot \prod_{n=1}^{\infty} \exp\left[-\frac{1}{n}\left\{\frac{2B}{D^{\mp}_{A_1}D^{\pm}_{A_2}} + \frac{B^2}{D^{\mp 2}_{A_1}D^{\pm}_{A_2}}\right\}^n\right]^{(-1)^n}. \end{aligned} \tag{164}$$

As shown in Eq. (140), the power $(-1)^n$ is the Maslov index of each nonperturbative cycle and is regarded as the intersection number of Lefschetz thimble. In the similar way to Eqs. (141)-(142), the partition function in the asymptotic limit can be expressed as

$$Z = Z_{\mathrm{p}} + Z_{\mathrm{np}}, \tag{165}$$

with

$$\begin{aligned} Z_{\mathrm{p}} =& \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log(1+A_1)\right]e^{-\beta E}dE \\ &+ \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log(1+A_2)\right]e^{-\beta E}dE + \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log(1+A_1)\right]e^{-\beta E}dE, \end{aligned} \tag{166}$$

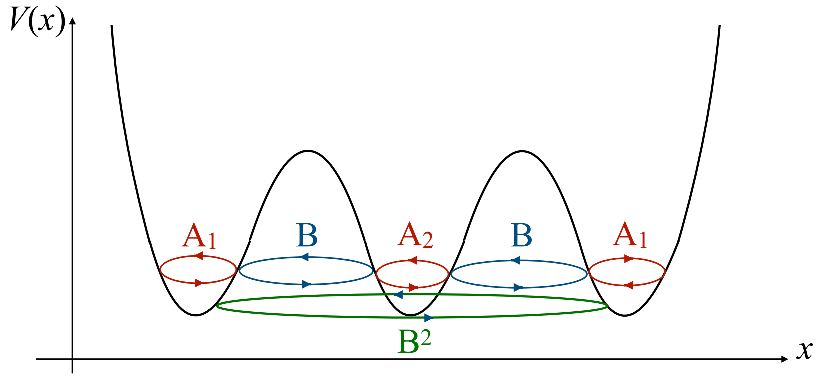

FIG. 15. Relationship between periodic orbits and the Maslov index for the symmetric triple-well potential. The index $(-1)^n$ in the n-the sector in Eq.(167) is determined by counting a cycle-unit which can be decomposed into the following parts; two $B/(D_{A_1}D_{A_2})$ and one $B^2/(D_{A_1}^2 D_{A_2})$. The former part including one B-cycle corresponds to an orbit running around two (locally) double-well potential, $[A_1$-B-$A_2]$ and $[A_2$-B-$A_1^{-1}]$, whereas the latter part including two B-cycle is related to an orbit which is globally running from the left to the right $[A_1$-B-A_2-B-$A_1^{-1}]$.

$$
\begin{aligned}
Z_{\rm np} &= \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\left[\frac{2B}{D_{A_1}^{\mp}D_{A_2}^{\pm}}+\frac{B^2}{D_{A_1}^{\mp 2}D_{A_2}^{\pm}}\right]^n \\
&\simeq \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\left[e^{\mp\pi i\left(\frac{E}{\hbar\omega_{A_1}}+\frac{E}{\hbar\omega_{A_2}}\right)}\frac{B}{2\pi}\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega_{A_1}}\right)\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega_{A_2}}\right)\right. \\
&\quad\left.+e^{\mp\pi i\left(2\frac{E}{\hbar\omega_{A_1}}+\frac{E}{\hbar\omega_{A_2}}\right)}\frac{B^2}{(2\pi)^{3/2}}\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega_{A_1}}\right)^2\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega_{A_2}}\right)\right]^n e^{-\beta E}dE\,,
\end{aligned}
\tag{167}
$$

where $\simeq$ means dropping the factor $\frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}+\frac{E}{\hbar\omega_A})}$, essentially the negative eigenvalue part. Now, the non-perturbative contribution of the partition function for the triple-well potential can be interpreted as follows. There are two types of bions as shown in Fig. 15: the one that reverses immediately at the turning point (blue) and the other passing through the barrier (green). The former gives $B\frac{1}{D_{A_1}D_{A_2}} \sim e^{-S_{bion}}\Gamma(\frac{1}{2}-\frac{E}{\hbar\omega_{A_1}})\Gamma(\frac{1}{2}-\frac{E}{\hbar\omega_{A_2}})$ and the latter gives $B^2\frac{1}{D_{A_1}D_{A_2}D_{A_1}} \sim e^{-2S_{bion}}\Gamma(\frac{1}{2}-\frac{E}{\hbar\omega_{A_1}})^2\Gamma(\frac{1}{2}-\frac{E}{\hbar\omega_{A_2}})$. We can identify how to sum up the non-perturbative contributions and this result shows the structure of partition function as a trans-series[18]. The phase ambiguity is interpreted as the reversal of the direction of the cycle A_2. We also note the Maslov index is not completely cancelled like the case of double-well potential because $e^{\mp\pi i\left(2\frac{E}{\hbar\omega_{A_1}}+\frac{E}{\hbar\omega_{A_2}}\right)}$ produces $(\mp i)^n$, not $(-1)^n$.

[18] The two blue bion and one green bion produce the same amplitude $B^2 \sim e^{-2S_{bion}}$ but these should be distinguished. This is quite natural from the path integral view because these are different classical solutions.

A. Exact resurgent structure

By using the DDP formula, we can say the following astonishing statement: *From only the perturbative information of partition function, we can recover the complete partition function including the non-perturbative contribution.*

As we have shown, the partition function $Z(\beta)$ obtained by the exact quantization condition $D(E)$ can be written as a function of the Voros multipliers A_i and B_j generally.

$$D = D(A_i, B_j) \leftrightarrow Z = Z(A_i, B_j), \tag{168}$$

where A_i, B_j: perturbative/nonperturbative Voros multiplier. We also know that the perturbative part of the partition function Z_p corresponds to

$$D_p = 1 + A_i \leftrightarrow Z_{p_i}, \tag{169}$$

where the index i corresponds to different vacua. Therefore, when the information of Z_{p_i} is completely known by means of the summation of Feynman diagram or other methods, it is possible to identify the functional form of A_i from it.

Then, by using the DDP formula153, the resurgent structure of A_i and B_j can be obtained rigorously, and we can find the functional form of B_j from the Borel ambiguity of A_i. Therefore, in a one-dimensional quantum mechanical system, it is possible to recover the complete partition function from only the information of the perturbative part of partition function.

VI. PATH INTEGRAL AND QMI CALCULATION

In this section, we explain resurgence analysis using the pertubative expansion and path integral considering quasi-moduli integral(QMI), which was done before the relationship between exact-WKB and path integral was clarified in this paper. We also discuss the divergence problem that arises when a deformation parameter is added to the symmetric double well potential, called the $1/\epsilon$ problem, and how to solve it.

A. $1/\epsilon$ problem

Now, we consider the Witten model, which is an supersymmetric extension of the one-dimensional quantum mechanical system.

$$S = \int_{-\infty}^{\infty} dt \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}(W'(x))^2 + i\eta^\dagger \dot{\eta} + W''(x)\eta^\dagger \eta \right) \tag{170}$$

where $W(x)$ is a superpotential. The Hamiltonian of this system can be written in terms of only bosonic variable after projecting to fermion number eigenstates:

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{171}$$

$$H_\pm = \frac{1}{2}p^2 + V_\pm(x) \tag{172}$$

$$= \frac{1}{2}p^2 + \frac{1}{2}(W'(x))^2 \mp \frac{1}{2}\hbar W''(x) \tag{173}$$

$$\tag{174}$$

The term $\mp\frac{1}{2}\hbar W''$ comes from the fermion terms. Now, we set the superpotential to $W(x) = \frac{1}{3}x^3 - a^2 x$. It gives

$$H_\pm = \frac{1}{2}p^2 + \frac{1}{2}(x^2 - a^2)^2 \mp \hbar x \tag{175}$$

This is tilted double well potential[19]. The zero energy eigenstate is

$$\langle x|0\rangle = e^{-\frac{W(x)}{\hbar}} = e^{-\frac{1}{\hbar}\left(\frac{1}{3}x^3 - a^2 x\right)} \tag{176}$$

This state is not normalizable in the real axis: $(-\infty, \infty)$. Therefore supersymmetry of this system is dynamically broken. the SUSY breaking is due to nonperturbative effects, because the perturbative contribution to the ground state energy vanishes. To examine the resurgent structure, we need to introduce deforming parameter ϵ here:

$$H_\pm = \frac{1}{2}p^2 + \frac{1}{2}(x^2 - a^2)^2 \mp \epsilon\hbar x \tag{177}$$

When $\epsilon = 1$, this system returns to the original SUSY Hamiltonian.

[19] More precisely, this is *quantum* tilted, which means the deformation term is proportional to $\hbar$. If we set the *classical* tilted potential, like $\mp x$, the complex bion solution does not contribute to the ground state energy. This difference is solved by degenerate-Weber type exact-WKB analysis.

The perturbative expansion of ground state energy[20] reads [36]

$$E_{0,pert} = \sum_{n=0}^{\infty} a_n \hbar^n \tag{178}$$

$$a_n = -\frac{6^{-\epsilon+1}}{2\pi}\frac{\Gamma(n-\epsilon+2)}{\Gamma(1-\epsilon)}\frac{1}{(2S_I)^n}, \tag{179}$$

where $S_I = \frac{3\hbar}{4a^3}$ is the one instanton action. The Borel summation of this series is

$$-\frac{6^{-\epsilon+1}}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\int_0^{\infty} e^{-z}\frac{z^{-\epsilon+1}}{1-\frac{z}{2S_I}}dz \tag{180}$$

It has this Borel ambiguity,

$$\operatorname{Im}\mathcal{S}[E_{0,pert}](\hbar) = \mp\frac{1}{2}6^{-\epsilon+1}\frac{1}{\Gamma(1-\epsilon)}2S_I^{-\epsilon}e^{-2S_I} \tag{181}$$

$\pm$ is corresponded to the sign of $\operatorname{Im}(\hbar)$.

Resurgence theory claims the Borel smbiguity from a perturbative expansion is cancelled by considering the contribution of other saddle points. In this system, there are classical solutions called the *complex bion* [36] :

$$x_{cb}(t) = x_1 - \frac{x_1 - x_T}{2}\coth\left(\frac{\omega_{cb}t_0}{2}\right)\left(\tanh\left(\omega_{cb}\frac{t+t_0}{2}\right) - \tanh\left(\omega_{cb}\frac{t-t_0}{2}\right)\right), \tag{182}$$

where x_1 is the position of potential minimum, $x_T = -x_1 + i\sqrt{\frac{\epsilon\hbar}{-x_1}}$ is the complex turning point, $\omega_{cb} = \sqrt{V''(x_1)}$, $t_0 = \frac{2}{\omega_{cb}}\text{arccosh}\left(\sqrt{\frac{3}{1-V''(x_T)/\omega_{cb}^2}}\right)$.

If we flip the parameter ϵ to $-\epsilon$, the complex bion becomes bounce, which is the other classical solution. However, naively the bounce is a high energy configuration and therefore is not expected to contribute to the ground state energy. In [36], the nonperturbative effect for the ground state energy is calculated by:

$$E_0 = -\frac{1}{\beta}\log Z = -\frac{1}{\beta}\log(Z_0 + Z_{cb} + Z_{2cb} + ...) \tag{183}$$

$$\simeq -\frac{1}{\beta}\log Z_0 - \frac{1}{\beta}\left(\frac{Z_{cb}}{Z_0}\right) \tag{184}$$

Z_0 and Z_{cb} are partition functions corresponding to the vauum and one complex bion solution,

[20] Of course when we set $\epsilon = 1$, the all coefficients are vanished by supersymmetry.

respectively. It suggests one complex bion is enough to obtain the leading nonperturbative contribution for ground state energy. Using this method, the nonperturbative contribution, which is from one complex bion is

$$I_{cb} = -\frac{1}{\beta}\frac{Z_{cb}}{Z_0} = \frac{1}{2\pi}\left(\frac{\hbar}{16a^3}\right)^{\epsilon-1}\left(-\cos(\epsilon\pi)\Gamma(\epsilon) \pm i\frac{\pi}{\Gamma(1-\epsilon)}\right)e^{-2S_I} \tag{185}$$

Again, $\pm$ is corresponded to the sign of $\mathrm{Im}(\hbar)$.[21] Combining the two results shows

$$\mathrm{Im}(S[E_{0,pert}] + I_{cb}) = 0 \tag{186}$$

Therefore the ground state energy is

$$E_0 = -\frac{1}{2\pi}\left(\frac{-\hbar}{16a^3}\right)^{\epsilon-1}\Gamma(\epsilon)e^{-\frac{8a^3}{3\hbar}}\cos(\epsilon\pi) \tag{187}$$

$$= -\frac{1}{2\pi}\left(\frac{\hbar}{16a^3}\right)^{-1}\left(\frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right)e^{-2S_I} \tag{188}$$

The Borel ambiguity from perturbative expansion is exactly cancelled by the other ambiguity from nonperturbative (complex bion) saddle. However, this expression has two strange facts: (i). This is singular as $\epsilon \to 0$, which is the case of symmetric double well potential. (ii). In the case of symmetric double well, the nonperturbative contribution of the ground state energy is coming from one-instanton.[165] So even if we remove the singularity by hand, the result is still incorrect. This is the $1/\epsilon$ problem in deformed SUSY quantum system. The problem does not only occur in the case of tilted double well, but is known to occur in CP^N and sine-Gordon model. [50]

B. The partition function

The method to calculate the ground state energy is based on Euclidean path integral with periodic boundary condition.

$$Z(\beta) = \int_{\text{periodic},\beta} \mathcal{D}x e^{-\frac{S_E[x]}{\hbar}} \tag{189}$$

For computing the partition function $Z(\beta)$ properly, all the classical solutions whose period are β should be taken into account. Also the partition function should be invariant under $\epsilon \to -\epsilon$

[21] This ambiguity is not Borel ambiguity but coming from Stokes phenomena of quasi-moduli integral. However, It does not conflict with the resurgence claim because of the equivalence of Borel summation and integral on the Lefschetz thimble.

because the spectrum doesn't change by this reflection. Therefore we have to *multi-complex bion and multi-bonce* with finite β to obtain the correct contribution. The calculation in [**?**], they considered only one complex bion and $\beta \to \infty$ limit first. These procedures do not treat the symmetry properly and lead $1/\epsilon$ problem as we show in the next section.

The partition function is shown to take the form:

$$\begin{aligned}\frac{Z}{Z_0} &= 1e^{\beta a\epsilon} + 1e^{-\beta a\epsilon} + \sum_{n=1}^{\infty}\left(e^{-2S_I}\frac{S_I}{2\pi}\left(\frac{\det M_I}{\det M_0}\right)^{-1}\right)^n \beta QMI^n(\epsilon) \\ &+ \sum_{n=1}^{\infty}\left(e^{-2S_I}\frac{S_I}{2\pi}\left(\frac{\det M_I}{\det M_0}\right)^{-1}\right)^n \beta QMI^n(-\epsilon)\end{aligned} \tag{190}$$

The first $1e^{\beta a\epsilon} + 1e^{-\beta a\epsilon}$ are from stationary classical solutions (vacuum and false vacuum). The factors $e^{\pm\beta a\epsilon}$ come from $e^{-S_{vac}/\hbar} = e^{-V(x_{vac})/\hbar} = e^{\beta a\epsilon}$. The latter terms come from the nonperturbative contributions, which are complex bions and bounces, respectively. The linear factor β is from the translation symmetry of (imaginary) time dependent solutions. $B = e^{-2S_I}\frac{S_I}{2\pi}\left(\frac{\det M_I}{\det M_0}\right)^{-1}$ is the square of one instanton contribution: bion contribution.

$$x_I(\tau) = a\tanh a(\tau - \tau_c) \tag{191}$$

$$S_I = \frac{S[x_I, \epsilon = 0]}{\hbar} = \frac{4a^3}{3\hbar} \tag{192}$$

$$\frac{\det M_I}{\det M_0} = \frac{1}{12} \tag{193}$$

The exact classical solution is not this instanton but complex bions and bounces (These solutions are interchanged by $\epsilon \to -\epsilon$). To calculate the contribution from these solutions for path integral, we have to consider quasi-moduli integral (QMI), which comes from a nearly flat direction in the configuration space.

QMI, comes from the nearly flat direction in complex bion solution, i.e. the separation between instanton and anti-instanton in a complex bion is

$$\tau = 2t_0 \simeq \frac{1}{2a}\left(\log\left(\frac{16a^3}{\epsilon\hbar} \pm i\pi\right)\right) \tag{194}$$

This can be infinite under $\hbar \to 0$, which leads to quasi-zero mode[22]. Therefore we have to consider the interaction potential $\mathcal{V}$ whose variable is τ. The complex bion itself is understood as the saddle point of $\mathcal{V}$.

[22] The existence of this direction comes from the $\hbar$ dependence of our potential.

The form of quasi-moduli integral for n-complex bions is[23]

$$QMI^n(\epsilon) = e^{\beta a\epsilon}\frac{1}{2n}\left(\prod_{i=1}^{2n}\int_0^\infty d\tau_i e^{-\mathcal{V}_i(\tau_i)}\right)\delta\left(\sum_{k=1}^{2n}\tau_k - \beta\right)$$
$$\mathcal{V}_i(\tau) = \begin{cases} -\frac{16a^3}{\hbar}e^{-2a\tau} + 2a\epsilon\tau & (i = odd) \\ -\frac{16a^3}{\hbar}e^{-2a\tau} & (i = even) \end{cases} \tag{195}$$

The factor $\frac{1}{2n}$ in front of the integral arises because the configuration is invariant under cyclic permutation of the τ_i. The factor $e^{\beta a\epsilon}$ comes from changing the off-set because the interaction potential $\mathcal{V}_i$ is determined from the true vacuum (the minimum point) but the two vacua have the potential $e^{\beta a\epsilon}$. Actually this procedure is equivalent to setting the $\mathcal{V}_i(\tau)$ as

$$\mathcal{V}_i(\tau) = \begin{cases} -\frac{16a^3}{\hbar}e^{-2a\tau} + a\epsilon\tau & (i = odd) \\ -\frac{16a^3}{\hbar}e^{-2a\tau} - a\epsilon\tau & (i = even) \end{cases} \tag{196}$$

and omitting the factor $e^{\beta a\epsilon}$. When $i = even$, this integral is ill-defined, but if we do the analytic continuation after performing the integral, it coincides with (195). Hence, if we consider an infinite number of complex bions and bounces instead of one, we can say that the contributions of them enter on the same order of magnitude.

If we set $\mathcal{V}_i = 0$ and $\epsilon = 0$, this integral becomes

$$\left(\prod_{i=1}^{2n}\int_0^\infty d\tau_i\right)\delta\left(\sum_{k=1}^{2n}\tau_k - \beta\right) = \frac{1}{(2n-1)!}\beta^{2n-1} \tag{197}$$

Therefore the partition function is evaluated as

$$\frac{Z}{Z_0} = 2\sum_{n=0}^{\infty}\frac{B^n\beta^{2n}}{(2n)!} = 2\cosh\left(\sqrt{B}\beta\right) \tag{198}$$

This is the result that are obtained by employing the standard dilute instanton gas approximation of symmetric double well potential.

[23] There are two quasi-moduli integrals for one complex bion because we consider finite β now.

From here we set $a=\frac{1}{2}$ for simplicity. This integral can be evaluated as

$$QMI^n(\epsilon) = e^{\beta\frac{\epsilon}{2}}\frac{1}{2n}\prod_{i=1}^{2n}\left(\int_0^\infty d\tau_i e^{-\mathcal{V}_i(\tau_i)}\right)\delta\left(\sum_{k=1}^{2n}\tau_k-\beta\right) \tag{199}$$

$$= e^{\beta\frac{\epsilon}{2}}\frac{1}{2n}\prod_{i=1}^{2n}\left(\int_0^\infty d\tau_i e^{-\mathcal{V}_i(\tau_i)}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} dl e^{il\sum_{k=1}^{2n}(\tau_k-\beta)} \tag{200}$$

$$= e^{\beta\frac{\epsilon}{2}}\frac{1}{2n}\frac{1}{2\pi}\int_{-\infty}^{\infty} dl e^{-il\beta}\left(\left(\int_0^\infty d\tau e^{(il\tau+\frac{2}{\hbar}e^{-\tau})}\right)\left(\int_0^\infty d\tau e^{(il\tau-\epsilon\tau+\frac{2}{\hbar}e^{-\tau})}\right)\right)^n \tag{201}$$

$$= e^{\beta\frac{\epsilon}{2}}\frac{1}{2n}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds e^{-s\beta}\left(\left(\int_0^\infty d\tau e^{(s-\epsilon)\tau+\frac{2}{\hbar}e^{-\tau}}\right)\left(\int_0^\infty d\tau e^{s\tau+\frac{2}{\hbar}e^{-\tau}}\right)\right)^n \tag{202}$$

We can show the integral is evaluated as

$$\int_0^\infty d\tau e^{(s-\epsilon)\tau+\frac{2}{\hbar}e^{-\tau}} = e^{\pm i\pi(\epsilon-s)}\left(\frac{\hbar}{2}\right)^{\epsilon-s}\Gamma(\epsilon-s) \tag{203}$$

Here the $\pm$ is corresponded to the sign of $\mathrm{Im}(\hbar)$, which comes from Stokes phenomenon of this quasi-moduli integral (See Appendix F in [50]).

Therefore, the form of quasi-moduli integral for finite β becomes

$$QMI^n(\epsilon) = \frac{1}{4\pi i n}e^{\beta\frac{\epsilon}{2}}\int_{-i\infty}^{i\infty} ds e^{-s\beta}\left(\left(e^{\pm i\pi(\epsilon-s)}\left(\frac{\hbar}{2}\right)^{\epsilon-s}\Gamma(\epsilon-s)\right)\left(e^{\pm i\pi(-s)}\left(\frac{\hbar}{2}\right)^{-s}\Gamma(-s)\right)\right)^n \tag{204}$$

C. Calculating the resolvent

Using $Z_0=\sum_{k=0}^\infty e^{-\beta(k+1/2)}\sim e^{-\beta\frac{1}{2}}$(for large β), (190) can be written as

$$Z = \left\{Z_0 e^{\beta\frac{\epsilon}{2}}+\sum_{n=1}^\infty B^n e^{-\beta(1/2)}\beta e^{\beta\frac{\epsilon}{2}}\frac{1}{4\pi i n}\int_{-i\infty}^{i\infty} ds e^{-s\beta}(I(s,\epsilon)I(s,0))^n\right\}+\left\{(\epsilon\to-\epsilon)\right\} \tag{205}$$

$$B = e^{-2S_I}\frac{S_I}{2\pi}\left(\frac{\det M_I}{\det M_0}\right)^{-1} = \frac{e^{-\frac{1}{3\hbar}}}{2\pi}\frac{2}{\hbar} \tag{206}$$

$$I(s,\epsilon) = e^{\pm i\pi(\epsilon-s)}\left(\frac{\hbar}{2}\right)^{\epsilon-s}\Gamma(\epsilon-s) \tag{207}$$

The Laplace transform gives the trace of resolvent $G(E)$.

$$
\begin{aligned}
G(E) &= \Big\{G_0(E+\epsilon/2) + \sum_{n=1}^{\infty} B^n \frac{1}{4\pi i n}\int_0^{\beta} d\beta\, \beta \int_{-i\infty}^{i\infty} ds e^{(E-s-1/2+\epsilon/2)\beta}(I(s,\epsilon)I(s,0))^n\Big\} + \Big\{(\epsilon\to-\epsilon)\Big\}\\
&= \Big\{G_0(E+\epsilon/2) + \sum_{n=1}^{\infty} B^n \frac{1}{4\pi i n}\int_0^{\beta} d\beta\, \frac{\partial}{\partial E} \int_{-i\infty}^{i\infty} ds e^{(E-s-1/2+\epsilon/2)\beta}(I(s,\epsilon)I(s,0))^n\Big\} + \Big\{(\epsilon\to-\epsilon)\Big\}\\
&= \Big\{G_0(E+\epsilon/2) + \sum_{n=1}^{\infty} B^n \frac{1}{4\pi i n}\frac{\partial}{\partial E} \int_{-i\infty}^{i\infty} ds \frac{1}{s-E+1/2-\epsilon/2}(I(s,\epsilon)I(s,0))^n\Big\} + \Big\{(\epsilon\to-\epsilon)\Big\}\\
&= \Big\{G_0(E+\epsilon/2) + \frac{1}{2}\sum_{n=1}^{\infty} B^n \frac{1}{n}\frac{\partial}{\partial E}(I(s=E-1/2+\epsilon/2,\epsilon)I(s=E-1/2+\epsilon/2,0))^n\Big\} + \Big\{(\epsilon\to-\epsilon)\Big\},
\end{aligned}
\tag{208}
$$

where $G_0(E)=\frac{\partial}{\partial E}\log\Gamma(1/2-E)$ is the resolvent of harmonic oscillator[24].
Using $-\log(1+x)=\sum_{n=1}^{\infty}\frac{(-x)^n}{n}$ gives

$$
\begin{aligned}
G(E) =& -\frac{\partial}{\partial E}\Big\{-\log\Gamma\Big(\frac{1}{2}-E-\frac{\epsilon}{2}\Big) + \frac{1}{2}\log(1-BI(s=E-1/2+\epsilon/2,\epsilon)I(s=E-1/2+\epsilon/2,0))\Big\}\\
&-\frac{\partial}{\partial E}\Big\{(\epsilon\to-\epsilon)\Big\}
\end{aligned}
\tag{209}
$$

Using $G(E)=-\frac{\partial}{\partial E}\log D$, we obtain the Fredholm determinant as follows

$$
D(E)=\frac{1}{\Gamma\big(\frac{1}{2}-E-\frac{\epsilon}{2}\big)\Gamma\big(\frac{1}{2}-E+\frac{\epsilon}{2}\big)}\sqrt{1-BI(s_{+\epsilon},\epsilon)I(s_{+\epsilon},0)}\sqrt{1-BI(s_{-\epsilon},-\epsilon)I(s_{-\epsilon},0)}, \tag{210}
$$

where $s_{\pm\epsilon}=E-1/2\pm\epsilon/2$. Substituting $I(s,\epsilon)=e^{\pm i\pi(\epsilon-s)}\big(\frac{\hbar}{2}\big)^{\epsilon-s}\Gamma(\epsilon-s)$ into (210), $D(E)$ becomes

$$
D(E)=\frac{1}{\Gamma\big(\frac{1}{2}-E-\frac{\epsilon}{2}\big)\Gamma\big(\frac{1}{2}-E+\frac{\epsilon}{2}\big)}\left(1-Be^{\pm i\pi(1-2E)}\left(\frac{\hbar}{2}\right)^{1-2E}\Gamma\left(\frac{1}{2}-E-\frac{\epsilon}{2}\right)\Gamma\left(\frac{1}{2}-E+\frac{\epsilon}{2}\right)\right) \tag{211}
$$

Therefore, $D(E)=0$ gives

$$
\frac{1}{\Gamma\big(\frac{1}{2}-E-\frac{\epsilon}{2}\big)\Gamma\big(\frac{1}{2}-E+\frac{\epsilon}{2}\big)}-Be^{\pm i\pi(1-2E)}\left(\frac{\hbar}{2}\right)^{(1-2E)}=0, \tag{212}
$$

where $B=\frac{e^{-\frac{1}{3\hbar}}}{2\pi}\frac{2}{\hbar}$. If we only consider the finite number of bions, the partition function (205) is still singular like $\sim 1/\epsilon^n$. However, using the analytic continuation $(-\log(1+x)=\sum_{n=1}^{\infty}\frac{(-x)^n}{n})$

[24] This is obtained by zeta function regularization of $\prod_{n=0}^{\infty}\left(n+\frac{1}{2}-E\right)$

give analytic function around $\epsilon = 0$. Therefore the summation of all classical periodic solutions is necessary to solve the $1/\epsilon$ problem.

This result is exactly corresponds to the result (117). We can also see that this result is due to the reordering of the limits in evaluating the partition function. There are two limit operations, $\beta \to \infty$ and $\epsilon \to 0$. If we take the limit of $\beta \to \infty$ first, the bounce solutions disappear as a contribution because of the $e^{-\beta a\epsilon}$. This is not a problem in the case of $\epsilon \neq 0$, however when $\epsilon = 0$, because the contributions of both are equal, it causes problem. Therefore, if we want to get the analytic energy for ϵ, we need to consider finite β and add the infinite number of complex bions and bounces contributions.

D. Calculating the ground state energy

Now, we obtain the Fredholm determinant of this system, we consider the ground state energy in the physically important cases of $\epsilon = 0$ and $\epsilon = 1$.

1. For $\epsilon = 0$, symmetric double well

$$\frac{1}{\Gamma\left(\frac{1}{2} - E\right)\Gamma\left(\frac{1}{2} - E\right)} - Be^{\pm i\pi(1-2E)}\left(\frac{\hbar}{2}\right)^{(1-2E)} = 0 \tag{213}$$

We can write the ground state energy as $E = \frac{1}{2} + x$, where x is exponential small factor. Then (213) becomes

$$\frac{1}{\Gamma(-x)} = \sqrt{B}e^{\mp\pi ix}\left(\frac{\hbar}{2}\right)^{-x} \tag{214}$$

$$\frac{1}{\Gamma(-x)} = -\sqrt{B}e^{\mp\pi ix}\left(\frac{\hbar}{2}\right)^{-x} \tag{215}$$

The $\pm$ corresponds to the parity. Using the reflection formula: $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$,

$$\frac{\sin \pi x}{\pi} = \sqrt{B}e^{(\mp\pi i - \log\frac{\hbar}{2})x}\frac{1}{\Gamma(1+x)} \tag{216}$$

$$\frac{\sin \pi x}{\pi} = -\sqrt{B}e^{(\mp\pi i - \log\frac{\hbar}{2})x}\frac{1}{\Gamma(1+x)} \tag{217}$$

Finally, the nonperturbative contribution is

$$x = \sqrt{B}\Bigg(1 + \Bigg(\mp\pi i - \log\left(\frac{\hbar}{2}\right) + \gamma\Bigg)x + O(x^2)\Bigg) \tag{218}$$

$$= \sqrt{B} + \Bigg(\mp\pi i - \log\left(\frac{\hbar}{2}\right) + \gamma\Bigg)B + O(B^{3/2}) \tag{219}$$

and (220)

$$x = -\sqrt{B} - \Bigg(\mp\pi i - \log\left(\frac{\hbar}{2}\right) + \gamma\Bigg)B + O(B^{3/2}) \tag{221}$$

Therefore, the result gives indeed energy splitting by one instanton: $\sqrt{B}$, and the imaginary ambiguity is proportional to B (bion) for the symmetric double well. The ambiguity of ground state energy(221) is exactly cancelled by Borel ambiguity from the perturbative expansion around the vacuum. (set $\epsilon = 0$ in (181))

2. *For $\epsilon = 1$, SUSY case*

The Fredholm determinant (212) is valid for any ϵ and we can show the correct nonperturbative effect for $\epsilon = 1$.

$$\frac{1}{\Gamma(-E)\Gamma(1-E)} - Be^{\pm i\pi(1-2E)}\left(\frac{\hbar}{2}\right)^{(1-2E)} = 0 \tag{222}$$

We know the pertubative part of the ground state energy is 0. Setting $E = 0 + x$ gives

$$\frac{1}{\Gamma(-x)\Gamma(1-x)} - Be^{\pm i\pi(1-2x)}\left(\frac{\hbar}{2}\right)^{(1-2x)} = 0 \tag{223}$$

This is rewritten as

$$\frac{\sin \pi x}{\pi} = B\frac{\hbar}{2}e^{\mp 2\pi i x}\left(\frac{\hbar}{2}\right)^{-2x} \tag{224}$$

This is solved as

$$x = B\frac{\hbar}{2} + O(x) \tag{225}$$

$$= \frac{e^{-1/3\hbar}}{2\pi} + O(B^2) \tag{226}$$

The leading nonpertubative contribution is due to a bion and coincides to (187).

As you can see from the above calculation process, the calculation we have done so far to find the partition function using Exact-WKB is essentially the reverse of the method we used to solve the $1/\epsilon$ problem. In particular, It is quite surprising and beautiful that the calculation of QMI, which is a non-trivial contribution from the viewpoint of ordinary path integrals used here, is derived naturally by using the exact-WKB.

VII. SUMMARY

In this thesis, we focused on the relation between exact-WKB and saddle point analysis based on path integral, which are two resurgences in quantum theory, and through them we showed how various quantizations such as Schrödinger equation, Bohr Sommerfeld quantization, Gutzwiller trace formula, and path integral, are related. Furthermore, for each quantization method, we revealed the resurgent structure, the hidden relation between perturbative and non-perturbative contributions. Our main findings can be listed as follows:

1. Both exact WKB and saddle point method applied to path integrals take place in complex domain. In exact WKB, position x is complexified, and in path integrals, the space of paths is complexified. Stokes phenomena permeate through both constructions. We showed that the Stokes phenomena in exact-WKB expressed through exact quantization condition maps to the Stokes phenomena in the saddle-point method in path integration.

2. The Maslov index that appears in the non-perturbative contribution is identified with the intersection number of corresponding Lefschetz thimble.

3. We revealed the relation of Bohr-Sommerfeld, Gutzwiller, path-integral quantization, including the resurgent structure of them and showed that it is possible to recover the complete partition function including non-perturbative effects from only the information of the perturbative part of partition function.

4. The resurgent structures of partition functions naturally continues to hold for generic symmetric multi-well potential, and ambiguity cancellations holds to all orders in semi-classical expansion. We showed this by using exact-WKB.

Our results not only uncover the unknown facts on quantization conditions and resurgent structures in quantum mechanics but also exhibit that the exact-WKB method and the Stokes curves can be powerful tools to study physical problems. Below, we describe a couple of examples to which we hope to apply our methods

- In semi-classics approach to Euclidean path integral formulation, the action must be complexified at the beginning to determine the set of saddles that can possibly contribute. But it is not always easy to know which complex saddles contribute and which do not. Exact WKB method, via the use of basic Stokes graph, immediately answers this part of question. Furthermore, it can also be used to determine the Stokes multipliers of the corresponding saddles. There is clearly much to be learned in semi-classical approach to path integral by using the knowledge of exact (complex) WKB.

- A streamlined construction of the semi-classical expansion of Euclidean partition function at finite β, which is capable of addressing all the states in the Hilbert space, addressing both below and above the barrier in the spectrum, and not just restricted to few lowest lying states as usually done in literature.

- Analysis of graded partition functions in quantum mechanical systems coupled to Grassmann valued fields (e.g. supersymmetric or quasi-exactly solvable systems) or equivalently, Wess-Zumino terms, i.e, path integral of a particle $x(\tau)$ with intrinsic spin. We hope to discuss both path integral as well as the exact-WKB construction for these systems.

- Quantum mechanics on S^1 with topological theta angle. The resurgent structure of this system is considered to be closed in each topological charge, but we believe that a more rigorous discussion of this is possible.

- Although the resurgence structures of bound state spectrum and partition functions have been investigated, it is possible to apply the exact-WKB to scattering problems. We plan to use it to establish a non-perturbative method for calculating the S-matrix in scattering problems and to analyze its resurgent structure.

- Using exact-WKB, it is possible to identify *prime periodic orbit* in the Gutzwiller trace formula. This implies that complex classical solutions (even chaotic cases) can be classified topologically in phase space. From this point of view, it is possible to understand the relationship between the thimble decomposition and the complex classical solution in more detail.

- For the application to QFT, we can apply our method to simplified QFT with approximations or reductions giving 1D QM systems, such as S^1 compactification from 2D QFTs and integrable systems. (Schwinger model in two dimensions, $\mathbb{C}P^N$-model, Toda lattice, etc.) Furthermore, since the Gutzwiller trace formula itself can be applied to QFTs under semi-classical approximation [151], we can consider the hidden relation between the Maslov index and the intersection number of Lefschetz thimbles in the similar fashion.

Thus, by using exact WKB, we can rigorously investigate the relationship between various quantization methods. In addition, this method is applicable not only to quantum mechanical systems but also to general second-order linear differential equations, which makes it possible to approach a wider range of problems.

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