Nonlinear Rotordynamics Investigation on Self-Excited Vibration of Rotors Supported by Fixed Profile and Tilting Pad Journal Bearings

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Nonlinear Rotordynamics Investigation on Self-Excited Vibration of Rotors Supported by Fixed Profile and Tilting Pad Journal Bearings

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Abstract

As an increasing trend in a high-efficiency rotating machine, the rotor of the rotating machine has been designed to be more slender and operated at a higher speed. These design and operating parameters make the rotor supported by journal bearings (JBs) susceptible to self-excited vibration caused by JB's nonlinearity. When this self-excited vibration occurs, the rotor at the stable equilibrium starts to whirl at the Hopf bifurcation point. The rotor will whirl with subsynchronous frequency in either a bounded limit cycle (supercritical Hopf bifurcation) or unbounded limit cycles (subcritical Hopf bifurcation), which can cause damage from the rubbing between the rotor and stator. Furthermore, the rotor at speed below the Hopf bifurcation point can also become unstable if perturbed sufficiently in subcritical Hopf bifurcation. Hence, the identification of the Hopf bifurcation type is essential, particularly in the design stage. However, the linear analysis, which is still the industry's norm in rotordynamics analysis (RDA), cannot capture critical behavior such as limit cycles and Hopf bifurcation types. As a result, the nonlinear rotordynamics analysis is an essential tool for the designer to predict the nonlinear behavior, especially when the rotor becomes linearly unstable or perturbed by unforeseen excitation, such as earthquake and loss of rotating mass. Nevertheless, the nonlinear rotordynamics analysis requires considerably more computational time than the linear rotordynamics analysis. Hence, it is limited to the relatively simple rotor and bearing models, which have simplified mathematical expressions. However, these models might not correctly represent the real machine's behaviors. In this aspect, the numerically efficient nonlinear rotordynamics analysis of the complex rotor and bearing system that better represents the real machine is advantageous in understanding the rotor and bearing system's dynamic behavior. Even several research papers dealt with this kind of nonlinear rotordynamics analysis, the research on nonlinear self-excitation of a rotor supported by the rotordynamics analysis with complex rotor-bearing geometries and operating condition representing the real rotating machine is still limited.

In this thesis, a time-efficient nonlinear vibration analysis program for the autonomous, self-excitation problem of the rotor supported by the JB based on the shooting method with parallel computing and arclength continuation is developed. It is used to determine the solution branch originated from the Hopf bifurcation point and its stability, which is later used to identify the Hopf bifurcation type. The bearing forces in the program are numerically determined from the Reynolds equation in every time step, so the applicable range of the analysis is not restricted to a small orbit amplitude as in the analyses based on linear or nonlinear dynamics coefficients. The analysis program can accommodate the models of fixed profile JB and the JB with movable pads, such as tilting pad journal bearing (TPJB) and flexure pivot tilting pad journal bearing (FPTPJB). Also, the turbulence effect can be readily included in the Reynolds equation. The rotor model used in the analysis can be a rigid or flexible rotor model constructed from the beam's finite element (FE), which the latter model can generally represent the rotor of the real machine. The degrees of freedom (DOF) of the flexible rotor model can be reduced to those subjected to the nonlinear forces by model reduction schemes, such as modified Guyan reduction and component mode synthesis (CMS) to increase the calculation speed. The analysis program is employed to investigate the effects of the selected rotor and bearing parameters on the nonlinear vibration.

Firstly, the effects of turbulence on the nonlinear vibration of a rigid rotor supported by two identical bearings are examined. The investigated parameters are bearing length-to-diameter (L/D) ratios, profiles, and type of turbulence model (Constantinescu and Ng-Pan-Elrod). The results are compared to those obtained from the laminar bearing model. It is found that the bearing with preload always gives supercritical Hopf bifurcation regardless of the L/D ratio and turbulence model. In contrast, the Hopf bifurcation type of the bearing without preload shifts to the subcritical Hopf bifurcation at a longer L/D ratio. It is also observed that the L/D ratio that the Hopf bifurcation changed from supercritical to subcritical of both turbulence models is lower than that of the laminar model. The discrepancies between preloaded and non-preloaded bearings are caused by the downward force from the upper pad of the preloaded bearing, which becomes more pronounced as the preload is increased and further reinforced by the additional viscosity of the turbulence models. Lastly, the Hopf bifurcation point obtained from the Ng-Pan-Elrod turbulence model is always higher than that of the Constantinescu model. Hence, the findings suggest that the journal bearings operated in the turbulent regime should have a high pad preload and L/D ratio to obtain the higher onset speed of instability (OSI) and supercritical Hopf bifurcation with a lower limit cycle's growth rate. Furthermore, the Ng-Pan-Elrod model is considered more conservative than the Constantinescu model in predicting the OSI in all investigated bearing configurations.

Next, the importance of considering the journal's angular whirling motion to predict the bifurcation type correctly and the applicable range of the bearing models that consider and not consider this motion was investigated. Besides, the effect of rotor and bearing parameters on the bifurcation type were studied. The calculation results were compared with the experimental results, and it can be concluded that considering the journal's angular whirling motion was essential in the flexible rotor with long L/D ratio that has the main mass located close to the journal bearing. Hence, the model that does not consider the journal's angular whirling motion, which considerably uses less calculation time, could correctly predict the bifurcation type apart from these cases. Moreover, the supercritical bifurcation was likely when the bearing has short L/D ratio and preload, and the main mass was located near the journal bearing.

Lastly, the effect of load orientation: load-on-pad (LOP) and load-between-pad (LBP), on nonlinear vibration of a flexible rotor supported by a 4-lobe flexure pivot tilting pad journal bearing (FPTPJB) was numerically investigated along with the first critical speed and onset speed of instability (OSI). The investigation was extended to the tilting pad journal bearing (TPJB) and fixed profile journal bearing (JB) of the same pad geometry for comparison. The calculation results showed that FPTPJB gave considerably higher OSI than the fixed profile JB of the same orientation. In contrast, the rotor supported by TPJB remained stable until more than 10,000 rpm, so the limit cycles could not be obtained. Besides, FPTPJB in LOP orientation yielded considerably higher OSI than the LBP one, whereas the OSI of the fixed profile JB in LOP orientation was substantially higher than the LBP counterpart. The FPTPJB in LOP orientation produced the largest stable operating region, whereas the fixed profile JB in LBP configuration yield the smallest one. The calculation results were verified by the experiment, in which good agreement in shaft centerline, the first critical speed, OSI, and Hopf bifurcation type were observed in the most case except the cases of TPJB in both orientations and FPTPJB in LOP orientation, in which the OSI and Hopf bifurcation type could not be verified due to the test rig's speed limit.

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Nomenclature

$\mathbf{M}$	mass matrix, kg
$\mathbf{C}$	damping matrix, Ns/m
$\mathbf{G}$	gyroscopic matrix, Ns/m
$\mathbf{K}$	stiffness matrix, N/m
$\mathbf{f}$	force, moment vector, N, Nm
$\mathbf{q}$	lateral, angular displacement vector, m, rad
m	mass, kg
g	gravitational acceleration, m/s^2
ω, Ω	rotor's rotational, precession speed, rad/s
t	time, s
x, y	horizontal/vertical lateral displacement, m
θ_x, θ_y	horizontal/vertical angular displacement, rad
ϑ	inclination angle of the bearing housing, deg
L	length, m
D, R	diameter, radius, m
H	height, m
ρ	density, kg/m^3
E	Young's modulus, Pa
G	shear modulus, Pa
k	stiffness, N/m
I	area moment of inertia, m^4
C_{rp}, C_{rb}	pad/bearing assembly radial clearance, m
μ	dynamic viscosity, Pa-s
z	axial distance from bearing midplane, m
$\widehat{m}$	bearing preload
θ_z	angle measured from $+x$ to $+y$-axis, rad
ξ, η, δ	pad's radial, tangential, and tilt displacement, m, m, rad
t_p	pad's thickness at the pivot
α	over-relaxation factor
$\widehat{\alpha}$	pad's pivot offset
p	oil film pressure, Pa
$N\theta$, Nz	number of grid points in θ and z-directions
$\overline{Re}$	average Reynolds number
k_θ, k_z	turbulence correction parameters
a_θ, b_θ, a_z, b_z	constants of turbulence
$\hat{\varepsilon}$	convergence criteria of the SOR iteration
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	unit vector in x, y, z-axis
$\mathbf{r}_{PVCM}$	vector from the pivot to pad's center of mass, m
m_{pz}	fluid film's moment acting on the pad, Nm
κ	shear correction factor
$\mathbf{E}$	error vector

λ	mode eigenvalue
β, Ω	eigenvalue's real and imaginary part
$\boldsymbol{\varphi}$	eigenvector
$\boldsymbol{\eta}$	eigenvector matrix
$\mathbf{s}$	CMS's reduced displacement vector
c	modal generalized coordinate
$\mathbf{c}$	modal generalized coordinate vector
$\tilde{\mathbf{R}}$	dimensionless residual vector in shooting method
$\tilde{T}$	dimensionless period in shooting method
T	temperature
$\mathbf{\Phi}$	monodromy matrix
s, Δs	position along the arc, arclength
ε	eccentricity ratio of the journal, $\varepsilon = \sqrt{\tilde{x}_5^2 + \tilde{y}_5^2}$
Ω	limit cycle or torus

Subscript

g	gravitational force
id, od	disk's inner, outer
Td, Pd	disk's polar, transverse
is, os	shaft's inner, outer
Ts, Ps	shaft's polar, transverse
NE	number of shaft element
b, jb, bb	bearing, journal, ball bearing
arc	pad arc
r, s, d	rotor, shaft, disk
p, pi	pad, i^{th} pad
PV	pivot
CM	center of mass
TP, FP	tilting pad, flexure pivot
cp	the connecting point between the pad and the web
eq	equilibrium
x, y, z	components in x, y, z-directions
ξ, η, δ	components in ξ, η, δ-directions
i, j	index of grid in circumferential and axial direction
l, t	leading, trailing edge of bearing pad
st	static deflection
I, B	internal, boundary degrees of freedom
RD	reduced model
h	h^{th} retained lowest modes of internal coordinates
ref	reference value
0, T	start, end point of shooting method

Superscript

k	iteration step index
T	vector transpose

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Chapter 1

Introduction

1.1 Background

Rotating machinery, such as turbines, pumps, compressors, is essential in many crucial industrial sectors, such as power generation and refinery. Many modern rotating machines, particularly the high-performance ones, are operated at a high rotational speed. They usually have the rotor consisted of a flexible shaft and multiple disks, shown in Fig. 1.1. The journal part of the rotor is generally supported by journal bearings (JBs), either fixed profile JB, titling pad JB (TPJB) or the most recent flexure pivot TPJB (FPTPJB), which the pictures and schematics are depicted in Fig. 1.2a-c, respectively. To ensure the safe and smooth operation of the machine, accurate prediction of rotordynamics behavior is crucial.

Fig. 1.1 Typical power plant gas turbine rotor

Fig. 1.2 Commonly used JB: (a) fixed profile, (b) TPJB, (c) FPTPJB
(https://www.waukbearing.com/)

Typically, rotordynamics analysis (RDA) is performed to predict the critical speed, onset speed of instability (OSI), and unbalance response. The analysis is usually based on a linear rotor model and the equivalent stiffness and damping coefficients of the bearing. The rotor is modeled by a 2-D finite element model using beam elements for the shaft and lumped mass and moment of inertia for component attached to the shaft. The stiffness and damping coefficients of the bearing are obtained by linearizing the perturbed nonlinear bearing force about the equilibrium position. However, they are valid only when both the equilibrium position exists and the shaft vibration amplitude of the journal is small (<20% of bearing clearance). However, these assumptions can be easily violated since the unbalance response of the flexible rotor can be large, particularly at the critical speed. More importantly, in the stability aspect, linear analysis can only predict the OSI, or the speed at which the rotor becomes unstable. However, it cannot predict how the rotor becomes unstable. Above the OSI, the rotor's vibration amplitude from self-excited vibration can be bounded (supercritical Hopf bifurcation) without contact between the rotor's journal and the bearing, or unbounded (subcritical Hopf bifurcation). In the case of subcritical Hopf bifurcation, the rotor can become unstable even its rotational speed is lower than the OSI if the rotor-bearing system is substantially perturbed by an external force, such as blade loss, sudden loading, and earthquake. The linear analysis could mask this potentially dangerous bifurcation, which can only be described by the nonlinear bearing force, resulting in the machine unable to withstand excessive vibration before it is safely brought to stop and fails catastrophically as an example shown in Fig. 1.3a [1]. In this case, nonlinear behavior of partial-arc JB (PAJB), which is a kind of fixed profile JB, installed in the machine, shown in Fig. 1.3b, led to excessive journal vibration and transmitted bearing force when unbalance force, caused by blade loss in this case, exceeds a particular value. As a result, nonlinear rotordynamics analysis is essential.

(a) (b)

Fig. 1.3 (a) Catastrophic failure of a steam turbine,
(b) Calculated journal vibration & bearing force versus unbalance force of two JB types [1]

The nonlinear analysis, however, requires more computational time than the linear analysis. As the number of parametric studies in nonlinear rotordynamics increases due to the rotor-bearing system's complexity, the effective nonlinear model reduction and analysis method for the sizeable rotor-bearing system are necessary.

1.2 Literature Review

1.2.1 Nonlinearity Caused by Journal Bearing in Rotordynamics

Until now, the mainstream method for the rotor-bearing system's stability is based on eigenanalysis of the rotor combined with bearing's stiffness and damping coefficients proposed by Lund [2] in 1965. In this work, he analyzed the stability of a flexible rotor system and proposed a linear oil film force model expressed by four stiffness and four damping coefficients. However, this method could not match various dynamic behavior caused by nonlinear effects.

The early papers about the instability caused by fluid bearings were published by Newkirk in 1924 [3] and Newkirk and Taylor [4]. It was observed that the rotor whirled at a frequency of approximately half of the rotor speed when the instability occurred. In addition, the rotor behavior was unlike critical speed resonance, in which the vibration amplitude increases as the rotor's speed approaches its critical speed and then decreases as it passes the critical speed. Bently [5] observed the subharmonic whirling motion in a simple horizontal rigid rotor supported by two plain JBs. He concluded that the nonlinear response was from the bearing clearance. Muszynska [6, 7, 8] developed a nonlinear mathematical model of the symmetric rotor supported by one radially rigid bearing and one circular JB that considered the tangential destabilizing force that predicted the occurrence of oil whirl and oil whip when the rotational speed was slightly higher than 2 times of the first critical speed. This rotational speed was referred to as "Instability Threshold", which is the same as OSI used in this thesis. Akers [9] performed numerical integration of the unbalanced rotor supported by identical JBs of various length-to-diameter (L/D) ratios to observed the stability of the solution trajectory. They observed that the magnitude of the unbalance affects the OSI. Noah and Sundararajan [10] utilized the closed-form nonlinear bearing force of a finite length circular JB, proposed by Childs et al. [11] to calculate the Hopf bifurcation type of a balanced rigid rotor supported by two finite length circular JB. They found that at the rotational speed higher than the OSI, the rotor whirled in either bounded amplitude (supercritical Hopf bifurcation) or unbounded amplitude (subcritical Hopf bifurcation) increasing amplitude. They also pointed out that the subcritical Hopf bifurcation is more potentially dangerous due to a perturbation of the equilibrium point, which can be caused by blade loss [1], or earthquake [12], even for speeds below the OSI; may render uncontainable vibration amplitude that leads to catastrophic failure. Deepak and Noah [13] performed the experiment on the flexible unbalanced rotor supported by JBs to observed the Hopf bifurcation type. They observed both types of Hopf bifurcation, depending on the operating condition. They also perceived that JB became unstable below the rotor threshold speed of instability, following a small perturbation applied to the rotor in the subcritical Hopf bifurcation case. Furthermore, they remarked that the flexible unbalanced rotor had a lower threshold speed of instability.

For the machine with high journal surface speed or uses low viscosity lubricant, the operating regime of the fluid film may be shifted from the laminar to the turbulent one [14]. The turbulence affects the load-carrying capacity of the JB, and thus, the dynamic characteristics of the rotor system [15]. Hence, examining the effects of turbulence is crucial in predicting the machine's dynamic behaviors, which can be different from the prediction

using the conventional laminar model. There are several investigations about the effects of turbulence on nonlinear vibration: Wang and Khonsari [16] used Hopf bifurcation theory to investigate the Hopf bifurcation type of simple rigid rotor supported by identical short circular JBs in using both laminar and turbulent bearing model. They observed subcritical Hopf bifurcation in both bearing models. However, the turbulence effects reduced the OSI and made the stability envelope at speed below the OSI of the turbulence model much smaller. Hemmati et al. [17] expanded the investigation of the simple Jeffcott rotor model supported by identical short or long circular JBs using the Hopf bifurcation theory. They found that the shaft stiffness dictated the bifurcating regions and the stable envelopes. It was also observed that the subcritical Hopf bifurcation region could be alleviated with the high static load in the shaft supported by short JBs if the shaft stiffness is higher than a critical value.

The misalignment occurs when the axes of the journal and the JB are not parallel. It causes asymmetric thickness distribution in the oil film and resulting in different fluid film pressure from the aligned bearing, resulting in the difference in bearing's static and dynamic performances. The misalignment can be categorized into static and dynamic misalignment. The static misalignment could be caused by the deflection of the rotor under its weight, bearing support structure, and installation error. Several papers studied the effect of static misalignment and gave general conclusions that misalignment reduces the load-carrying capacity; however, it improves the OSI of all journal bearing systems. Experimental results indicated that journal bearings are relatively more stable under misalignment [18]. The most recent comprehensive review on the characteristic of misaligned JBs was given by Jang and Khonsari [19]. However, the study in dynamic misalignment, caused by the dynamic deflection of the rotor due to its mode shape, has not been sufficiently investigated. This type of misalignment happens when the instability in JB bearing occurs, and the amplitude could be substantial in the highly flexible rotor cases.

In 1946, Haggs [20] published a paper that included the experimental results of several TPJBs. They concluded that this bearing type could eliminate the self-excited vibration instability caused by the tangential destabilizing force encountered in the fixed profile JB due to the free-to-tilt nature of the pad. Furthermore, Adams [1] numerically observed that this bearing type, referred to as pivoted-pad JB (PPJB) in his paper, could eliminate the nonlinear jump phenomenon as the unbalance increased. However, TPJB has lower damping and stiffness than the fixed profile counterpart [21], and the moving pad could cause pivot of the loaded pad to wear [22] and pad fluttering [23, 24, 25] in the unloaded pad, resulting in subsynchronous self-excited vibration of the pad. More importantly, the manufacturing tolerance stack-up could result in variations of bearing clearance and preload [26]. The effect of these variations could be pronounced for relatively small-sized bearings commonly used in high-speed rotating machinery.

To alleviate the problems of pivot wear, pad fluttering, and tolerance stack-up of the TPJB, the FPTPJB was initially introduced by Armentrout and Paquette [27] for high-speed turbocompressor application. In their paper, the rotor-bearing's linear stability data was presented. The FPTPJB is machined from a solid blank by an electrical discharge machining (EDM). However, the flexure pivot does not allow the pad to tilt freely. Hence, even when the pad inertia is neglected, the destabilizing tangential oil film force always exists. As a result, the rotor-bearing stability must be carefully examined when the FPTPJB is used.

Since there is a broad range of rotor-bearing design and operating parameters, these parameters can affect the Hopf bifurcation type and are needed to be investigated.

1.2.2 Rotor System Modelling

Since introduced by Jeffcott [28] in 1919, The Jeffcott rotor model concluded that operation above the first critical speed was possible and stable. Even so, it was later discovered that the rotor system could still experience instability caused by oil whip and oil whirl, cross-coupling stiffness in bearings and seals, internal damping, and gyroscopic effects. This rotor model and 2 degrees of freedom (DOF) rigid rotor model are still used in the rotordynamics analysis due to their simplicity to analyze the various effects on the rotor system's response. However, for the rotordynamics analysis of more complex systems, such as large-scale turbines, compressors, and jet engines, these rotor models were not accurate enough to model those rotor systems. As a result, several methods of modelling the complex rotor system were developed. Due to the difficulties in deriving analytical solutions for complex rotor systems, the computational models were developed. Among them, the Transfer Matrix method and more recent Finite Element Method (FEM) are widely used.

The transfer matrix method, proposed by Myklestad [29] and Prohl [30], divides structural elements into linear arrangement [31]. This method needs a smaller memory compared to the FEM. However, it can be applied when the total system can be broken into a sequence of subsystems, and, more importantly, it has a limitation of processing nonlinear dynamic problems. An improved transfer matrix method was developed to analyze a complex rotor-bearing system, in which the nonlinear forces were given in closed-form [32, 33]. A comprehensive review of this method was given by Pestel and Leckie [34].

The FEM for the rotor systems was initially applied to the rotordynamics analysis by Ruhl and Booker [35] for a simple rotor based on the Euler beam theory. The method was further refined to cope with the effects of rotatory inertia, gyroscopic moments, and axial load is developed using the consistent matrix approach by Nelson and McVaugh [36]. The methods based on the Timoshenko beam theory, which considers shear deformation of the element, were published by Thomas et al. [37] and Nelson [38]. In this method, the attached parts are modeled as lumped masses and moments of inertia because their natural frequencies are generally much higher than the range of the rotor's operating speed and the first or second critical speeds. The support parts, such as bearings and flexible support, are modeled as either linear by calculating equivalent coefficients or nonlinear by solving the Reynolds equation for bearing force [39]. Modified beam elements have been applied to this method for more complex shaft geometries, such as hollow section [1], tapered beam section [40], and conical section [41]. The FEM requires a higher computational cost when compared to the transfer matrix method. However, it can represent the behavior of the flexible shaft, built on parts and the support system, making this approach widely used these days for rotordynamics analyses. More detail in the FEM can be found in [1, 42, 43, 44, 45].

1.2.3 Nonlinear Journal Bearing Modelling

The JB force can be calculated from pressure obtained by solving the Reynolds equation [46], which is the simplified version of the Navier-Stokes equation based on the following assumptions [47, 48, 49]:

1. The continuum description is valid.
2. The Navier-Stokes equation hold.
3. Incompressible fluid.
4. Constant viscosity fluid.
5. Thin fluid film. Hence, the flow is laminar, and fluid inertia is negligible.

However, the closed-form solution of the Reynolds equation that accounted for realistic boundary conditions and cavitation has not yet been obtained [50]. Except for the cases of plain circular bearing, which have the closed-form solution for infinitely short [51], infinitely long

[52], and finite length [11]. However, the plain circular bearing is most likely prone to fluid-induced instability [53, 54]. Hence, designers of modern rotating machinery tend to use other bearing designs, such as multilobe JB, pressure dam JB, TPJB, or FPTPJB, to increase the stability of the system. As a result, these closed-form solutions do not correctly reflect the commonly used bearing type in the real machine.

To obtain the noncircular bearing's pressure field, the Reynolds equation is solved numerically by discretizing the fluid domain in the gap between the journal and the bearing using the finite difference method (FDM), finite element method (FEM), or finite volume method (FVM). The FDM [48, 49, 55], which was originally applied to thrust bearing by Sternlicht and Sneck [56], has commonly been used, especially with the bearing that has no geometrical discontinuity on the pad, due to the simplicity of its application. The FEM [57, 58] has proven to be very effective and has been adopted when the FDM was proving difficult to use, such as, for example, in the case of discontinuous bearing geometry of the pressure dam JB. However, the FEM took a longer calculation time than the FDM for the same accuracy in the bearing without geometrical discontinuity [59]. More recently, the FVM [60, 61] has been developed for the problems that the equations express flux conservation. The flux conservation can be conservation of heat in thermodynamics or conservation of fluid flux in fluid dynamics.

Since the fluid used in the JB usually cannot withstand large and continuous negative pressure, the fluid will rupture at a certain location and causing cavitation. To account for this phenomenon, in 1904, Sommerfeld [52] proposed a half-Sommerfeld cavitation boundary condition by ignoring all of the negative pressure points. The calculation results using this boundary condition gave better agreement to the experimental results than not considering the cavitation, but the discrepancies still occurred. In 1932, McKee and McKee [62] experimentally determined the pressure distribution of the vertically oriented bearing in various clearances, viscosities, loads and speeds. They observed the pressure distribution was in bell shape in the circumferential direction and parabola shape in the axial direction. Besides, the positive pressure points were distributed in the circumferential extent less than 180 degrees. Later in 1949, Cameron and Wood [63] found that half-Sommerfeld cavitation boundary condition permitted flow discontinuity at the exit edge of the oil film. As a result, they used the Reynolds boundary condition, which the numerical recipe was previously proposed by Christopherson in 1941 [64], to achieve the flow continuity. The calculation results of finite length bearings using this boundary condition agreed well with previous experimental results.

As the operating regime of many machines has been shifted toward the turbulent regime, which could be determined by Reynolds number at which the turbulence comes into effect when the average Reynolds number exceeds 2000 [65], the original laminar flow assumption been relaxed by adding the turbulence correction terms [48]. The two most commonly used turbulence models were proposed by Ng-Pan-Elrod [66, 67], based on eddy viscosity; and Constantinescu [68], based on the Prandtl mixing length hypothesis. Hashimoto and Wada [69] used the closed-form short bearing model with Ng-Pan-Elrod correction terms to obtain shaft vibration of the rigid rotor and found that the rotor would go unstable at a lower speed when turbulence was considered. Hashimoto et al. [70] also utilized the simple correction terms, rotor and turbulence model to solve the turbulent lubrication equation of elliptical JBs by using the semi-analytical finite element method. They observed that the turbulence affected the rotor's static and dynamic characteristics, and the preload increased the OSI. Kumar and Rao [71] also used FDM to solve for stability region of the same rotor type supported by finite length circular

JB with Constantinescu turbulence model and found that decreasing L/D ratio improved the stability of the rotor.

For TPJB and FPTPJB, apart from the modelling of the fluid film, the modelling of the pad and pivot is equally essential. The simplest model of TPJB and FPTPJB is 1 DOF model that the pad can freely tilt [72] or can tilt under the rotational stiffness [27, 73], respectively. A more complex model of TPJB that considers the motion of the pivot was studied by San Andrés and Tao [74], and the model that considers both pivot motion and pad deflection was examined by Wilkes and Childs [75]. The model of FPTPJB that considers the flexure pivot as a cantilever beam was investigated by Chen [76].

After the fluid film force of the JB was obtained, there are two approaches to represent nonlinear bearing forces in the system's equation of motion (EOM). The first method is to represent the bearing force as it is. The second method is described by nonlinear bearing coefficients, which can be obtained from the bearing force of the perturbed displacement and velocity of the journal about the equilibrium position, similar to obtaining stiffness and damping coefficients [77, 78]. However, even the nonlinear coefficient representation provides faster calculation time than bearing force representation; it cannot give accurate force when the vibration amplitude is large.

To date, most of the research in nonlinear vibration of flexible rotors supported by journal bearings still uses bearing force models based on the assumption that the journal axis is parallel to the bearing axis [79, 80, 81]. The study of the dynamic misalignment effect caused by the journal's angular motion is mostly limited to a relatively small amplitude about the equilibrium position in the process to obtain linear [82] or nonlinear [83, 84] dynamic coefficients. Nevertheless, in the highly flexible rotor case, the journal's angular whirling amplitude can be larger than the applicable range of the dynamic coefficients, and the full nonlinear model is necessary for predicting the bifurcation type of the rotor-bearing system. Koondilogpiboon and Inoue [85] numerically and experimentally investigated the necessity of considering the journal angular motion. They found that, even at L/D ratio as low as 0.6, considering this motion is essential in predicting the correct bifurcation type in two out of three large disk position cases. Nonetheless, the unstable limit cycle in the case of subcritical bifurcation, which is the threshold that dictates whether the vibration amplitude jumps to a large amplitude or returns to the equilibrium position, was not calculated.

1.2.4 Model Reduction in Rotordynamics Analysis

By modelling the rotor using FEM, many degrees of freedom (DOF) are produced and resulting in calculation difficulty. However, typically, a few nodes are subjected to nonlinear forces. Hence, to reduce the computational cost, the total DOF of the rotor-bearing system can be reduced to the boundary DOF related to the nodes subjected to nonlinear forces. There are several methods to reduce the rotor model, such as Guyan reduction and Component mode synthesis (CMS). Guyan reduction [86, 87] assumes the dynamic relationship between the boundary DOF, and the internal DOF is the same as the static relationship. However, this approach is only applicable when the rotor spins significantly slower than its first critical speed because the inertial force is neglected [87]. Hence, it is not suitable for analyzing the oil whirl, which usually occurs above two times the first critical speed. As a result, Guyan reduction is modified to reduce the rotor-bearing model around the frequency of interest by incorporating the inertia force around that frequency. However, only one frequency can be evaluated, and it must be calculated beforehand from the eigenanalysis of the full rotor-bearing model [88]. Besides, the discrepancy between the OSI of the full and reduced model increases as the large

disk is moved away from the rotor midspan. This discrepancy is due to the gyroscopic term is not fully included in the reduction [85]. In contrast, CMS [89] can better handle this problem since it considers some of the substructure's modal coordinates constrained at the boundary as additional DOF, although it increases the DOF due to the additional modal coordinates [90]. Reviews of rotor model reduction in the rotordynamics analysis can be found in [91, 92].

Glasgow and Nelson [93] used CMS to perform a stability analysis of several rotor-bearing systems. They concluded that the first few forward and backward whirl modes could be predicted with high accuracy even with a high modal truncation level. Also, Nelson et al. [94] utilized CMS to determine the forced response of nonlinear rotor-bearing systems. They found that it was possible to achieve high simulation accuracy by retaining a small number of component modes. Sundararajan and Noah [95] developed a numerical algorithm to calculate the periodic response, stability, and bifurcation of a periodically excited multi-degrees of freedom (MDOF) rotor-bearing system with local nonlinearities using real mode CMS for model reduction. The shooting method and Floquet stability theory were applied to the reduced system to calculate the periodic response and its stability. The results indicated that accurate dynamic behavior could be obtained even with a few numbers of retained component modes. Kim and Palazzolo [96] applied CMS, shooting method, and continuation to an eight-stage compressor rotor supported by two identical five-pad tilting pad journal bearings (TPJB) to identify the solution curves of periodic responses under periodic excitation and determine their stability.

1.2.5 Nonlinear Rotordynamics Analysis Method

Due to the influence of nonlinear JB force, the rotor whirling motion could become asynchronous, especially when the fluid-induced instability occurred. To investigate the nonlinear behavior, several nonlinear analysis methods have been utilized.

The early nonlinear analysis attempts were based on numerical integration, such as applying short bearing's closed-form solution to the rigid rotor by Kirk and Gunter [97]. However, numerical integration needs very long-time integration to obtain the limit cycle and cannot obtain the unstable limit cycle, which indicates the nonlinear stable boundary. Khonsari and Chang [98] proposed the method to find this limit cycle. However, the method was computationally tedious. Most of the EOMs of the rotor support by JBs, especially the flexible rotor, exhibited the behavior of the stiff ordinary differential equations (ODEs) and required a suitable integration scheme to overcome numerical instability and be computationally efficient [99]. Among many integration methods, the ode15s, which is one of the ODE suites in MATLAB that is a variable-step, variable-order (VSVO) solver based on the numerical differentiation formulas (NDFs) [99], was successfully applied in the rotordynamics analysis [72, 96].

Hopf bifurcation theory was applied to identify the Hopf bifurcation type of the simple rotor model using the closed-form short bearing by Hollis and Taylor [100] and long bearing by Meyers [101]. Even though this method can efficiently obtain the Hopf bifurcation type via nonlinear dynamic coefficients, it cannot accurately obtain the limit cycle when the point of interest is away from the bifurcation point. Hence, it cannot be used to find the nonlinear stable boundary and saddle-node point. As a result, numerical continuation was implemented to obtain the nonlinear solutions at the point further away from the bifurcation point. However, its application seemed to be limited to the bearings with closed-form solutions [102].

The shooting method [103] was utilized to overcome the shortcoming of numerical integration, Hopf bifurcation theory, and numerical continuation. For determining the limit

cycle, it shoots out trajectories in different directions of the initial condition until the solution's trajectory becomes periodic using Newton-Raphson method. Hence, if a proper initial guess of initial condition is provided, it can converge to both stable and unstable limit cycles [104, 105, 106] with high computational efficiency, which can be further enhanced by using parallel computing [72]. In addition, the Monodromy matrix, which is a by-product of the shooting method, can be used to identify the stability of the obtained limit cycles to identify the bifurcation type. [107, 108]

1.3 Problem Statement

As stated in the previous section, the assumption of linear rotordynamics analysis can be easily violated in the real operating conditions, for instance, when journal vibration amplitude is large. In addition, it cannot predict the bifurcation type, which is the critical indicator that shows how the rotor behaves when it is disturbed by the external force or the OSI is exceeded. Hence, the nonlinear rotordynamics analysis is required to catch complete rotordynamics behavior, such as limit cycles and Hopf bifurcation types.

Since the journal's vibration amplitude is insignificant compared to the rotor's overall dimension, the rotor is considered a linear model in this thesis. However, the journal bearing forces are considered a nonlinear function of instantaneous displacements and velocities of the journal for fixed profile JB. Additionally, the forces of a tilting pad and flexure pivot tilting pad JB (TPJB and FPTPJB) are also functions of the pads' displacements and velocities.

The early nonlinear rotordynamics analyses were based on either the rigid rotor or the Jeffcott rotor [28] with the simplified bearing force models. However, the analyses did not represent the behaviors of the real machine, which has more complex geometries. The nonlinear rotordynamics analysis using the complex rotor model, based on finite elements of the beam, and bearing forces model, based on directly solving the Reynolds equation, can better represent the behaviors of the real machine. Nonetheless, the calculation time required for these rotor-bearing systems is considerably longer than that of the simplified rotor model, which hinders the investigation of the rotordynamics behavior, especially when various parametric studies are needed. As a result, the nonlinear rotordynamics analysis must be "efficient", which means it must obtain nonlinear solutions and their stability in a short time.

In this thesis, the nonlinear rotordynamics analysis is developed. The rotor model is based on the finite element method, and the journal bearing's forces are nonlinear. The forces are functions of the journal's instantaneous displacements and velocities for the fixed profile JB. Also, they are functions of pads' displacements and velocities in the cases of TPJB and FPTPJB.

To represent the lateral motion of the rotor, 4 degrees of freedom (DOF) are needed for each node of the rotor. Moreover, additional DOF are required for each pad of TPJB and FPTPJB, respectively. As a result, the number of DOF of the rotor-bearing system can be large. However, compared to the total number of DOF, there are a few numbers of DOF that interact with the nonlinear force. Hence, the model reduction can greatly improve the calculation efficiency.

Since the EOM of this rotor-bearing system shows the behavior of stiff ODEs. A suitable integration method is needed to obtain the convergence in solving the EOM.

1.4 Research Objective

This research aims to investigate the effects of the rotor and bearing parameters on the nonlinear self-excited vibration, such as limit cycle and Hopf bifurcation types, of the complex rotor and bearing models. The scopes of investigation are as follows:

1. The effect of turbulence on the rigid rotor supported by the journal bearings
2. The necessity of considering the angular whirling motion of the flexible rotor's journal to obtain the calculation results that give the same Hopf bifurcation type as the experimental results
3. The applicable range of the bearing model that considers and not consider the angular whirling motion of the flexible rotor's journal under various rotor and bearing parameters
4. The effect of load orientations on the bearing

For various parametric investigations, the efficient numerical method for nonlinear rotordynamics analysis must be created. This method needs to obtain the nonlinear periodic solutions and their stability of the complex rotor and bearing models in a relatively short time. To achieve the research objective, the following tasks must be performed:

1. Develop finite elements based modelling and assembling method for the general linear flexible rotor supported by the nonlinear bearing.
2. Apply proper and numerically efficient nonlinear bearing force calculation method.
3. Develop an efficient nonlinear calculation method that can determine both stable and unstable solutions of the rotor-bearing system's self-excited vibration to obtain bifurcation type.
4. Set up the test rig and experiment to verify the calculation results.

1.5 Dissertation Outline

The mathematical models of the rotor and bearing and numerical procedures are presented in Chapter 2, along with methods for linear and nonlinear rotordynamics analysis. Chapter 3 presents the test rig used for experimental verification, vibration signal measurement, and processing. Chapter 4 focuses on investigating the effect of turbulence model and bearing parameters on nonlinear self-excited vibration of the rigid rotor support by circular and 2-lobe journal bearings. Effect of considering journal angular whirling motion and rotor-bearing parameters on nonlinear self-excited vibration of the flexible rotor supported by 2-lobe journal bearings with experimental verification is presented in Chapter 5. The first part of this chapter focuses on the importance of considering the angular whirling motion of the journal in order to obtain the bifurcation type of the rotor system that conforms with the experimental results. In the second part, the effect of rotor-bearing parameters on nonlinear self-excited vibration is investigated along with the applicable range of the bearing model that considers or not considers the angular whirling motion of the journal. In Chapter 6, the effects of rotor-bearing parameters on nonlinear self-excited vibration of the flexible rotor supported by FPTPJB compared with TPJB and fixed profile JB are presented. Chapter 7 presents the conclusion of this work, contributions, and recommendations for future work.

Chapter 2

Mathematical Model and Numerical Method

2.1 Introduction

In this chapter, the mathematical models of the rotor and bearing used throughout this thesis are explained. The rotor component modelling is described first. Then, the nonlinear bearing force modelling is portrayed followed by the equilibrium and linear dynamic coefficients calculation method. After that, the rotor-bearing system assembly method is explained. Next, the eigenanalysis is presented. It uses linear dynamic coefficients to determine the OSI, which is used as a starting point of nonlinear analysis. Then, model reduction methods, which are essential for DOF reduction in the subsequent nonlinear calculation, are discussed. Finally, the numerical procedure of nonlinear analysis is presented.

2.2 Rotor Component Modelling

In this section, the rigid rotor model and the flexible rotor model by finite element, which are in the opposite ends in terms of rotor modelling complexity, are described. The rigid rotor model is the simplest model that can encompass the journal's whirl motion, and the study of the nonlinear vibration of this rotor model will be presented in Chapter 4. The flexible rotor model by finite element is the model that can accommodate almost all of the complexity in the real machine's rotor. This model will be used for the investigation from Chapter 5 to Chapter 6.

2.2.1 Rigid Rotor Model

The model consists of a rigid symmetric rotor of mass $2m$ supported by two identical JBs, as shown in Fig. 2.1. The rotor is allowed to rigidly translate in a radial X-Y plane. It is connected to the ground through the identical nonlinear JBs. Due to symmetry, the EOM of the balanced rigid rotor can be depicted as [1, 72]:

Fig. 2.1 Schematic diagram of rotor and bearings

$$
\begin{aligned}
m\ddot{x} &= F_{jbx}\left(x, y, \dot{x}, \dot{y}\right) \\
m\ddot{y} &= F_{jby}\left(x, y, \dot{x}, \dot{y}\right) - mg
\end{aligned}
\tag{2.1}
$$

where F_{jbx} and F_{jby} are the horizontal and vertical nonlinear journal bearing forces, respectively.

2.2.2 Flexible Rotor Model by Finite Element

Since the simple rotor model cannot adequately represent the behavior of the rotor with complex geometry used in the real machines and no exact solution for an arbitrary rotor supported by JB exists, the finite element method (FEM) is usually employed to model the rotor. The FEM discretizes the rotor's continuous structure into fundamental elements

described by the simple governing equations obtained from the element's mechanics of material. For example, an arbitrary rotor model consisted of a shaft and rigid disks, as shown in Fig. 2.2, can be discretized into N-1 beam elements between the defined N nodes located at the endpoint of the beam elements. The nodes can contain the concentrated mass, such as disk, and each node has 4 DOFs of movement, as shown in Fig. 2.3. The EOM of the rotor in global XYZ coordinates can be represented in a matrix form as follows:

$$\mathbf{M}_r\ddot{\mathbf{q}} + (\mathbf{C}_r + \omega\mathbf{G}_r)\dot{\mathbf{q}} + \mathbf{K}_r\mathbf{q} = \mathbf{f} \tag{2.2}$$

where $\mathbf{q} = \left\{x_1, y_1, \theta_{x1}, \theta_{y1}, ..., x_N, y_N, \theta_{xN}, \theta_{yN}\right\}^T$, $\mathbf{M}_r, \mathbf{C}_r, \mathbf{G}_r, \mathbf{K}_r$ are mass, damping, gyroscopic and stiffness matrix of the rotor. $\mathbf{f}$ is force vector, consisted of forces and moments, acting on each DOF of the rotor. These forces can be a gravitational force, linear force, such as unbalance force, or nonlinear force, such as force from JB. These matrices and vectors are assembled from those of each rotor element. The finite element matrices and vectors in this section are excerpted from Adams [1], and Ishida and Yamamoto [109] and their derivations are presented in Appendix A.

Fig. 2.2 Nodes and elements of the rotor

Fig. 2.3 Degrees of freedom of each beam element

2.2.2.1 Equation of a Rigid Disk

Many concentrated masses of the rotor, such as thrust collar, coupling, impeller, gear, can be modeled as a rigid disk, shown in Fig. 2.4. In this thesis, it is assumed that the rigid disk does not contribute to the shaft element's flexibility. Mass properties of the disk are given as follows:

$$\begin{aligned}
&\text{Mass, } m_d = \frac{\rho_d \pi \left(D_{od}^2 - D_{id}^2\right) l}{4} \\
&\text{Transverse inertia, } I_{Td} = \frac{1}{12} m_d \left[\frac{3\left(D_{od}^2 + D_{id}^2\right)}{4} + l^2\right] \\
&\text{Polar inertia, } I_{Pd} = \frac{1}{2} m_d \frac{\left(D_{od}^2 + D_{id}^2\right)}{4}
\end{aligned} \tag{2.3}$$

where $\rho_d, D_{od}, D_{id}, l$ are disk's density, outer diameter, inner diameter and thickness, respectively. When the disk has lateral displacement x, y, inclination θ_x, θ_y, and rotational speed ω, the EOM of the disk are given as follows:

Fig. 2.4 Dimension of the rigid disk

$$\mathbf{M}_d\ddot{\mathbf{q}} + \mathbf{G}_d\dot{\mathbf{q}} = \mathbf{f}_d \tag{2.4}$$

where

$$\mathbf{M}_d = \begin{bmatrix} m_d & 0 & 0 & 0 \\ 0 & m_d & 0 & 0 \\ 0 & 0 & I_{Td} & 0 \\ 0 & 0 & 0 & I_{Td} \end{bmatrix}, \mathbf{G}_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{Pd} \\ 0 & 0 & -I_{Pd} & 0 \end{bmatrix}, \mathbf{q} = \begin{Bmatrix} x \\ y \\ \theta_x \\ \theta_y \end{Bmatrix}, \mathbf{f}_d = \begin{Bmatrix} f_x \\ f_y \\ f_{\theta x} \\ f_{\theta y} \end{Bmatrix} \tag{2.5}$$

$\mathbf{f}_d$ is force vector acting on a disk, such as unbalance forces and forces from the adjacent elements. The gravitational force acting on the disk of the horizontal rotor is:

$$\mathbf{f}_{gd} = \begin{Bmatrix} 0 & -m_d g & 0 & 0 \end{Bmatrix}^T \tag{2.6}$$

note that mass of the disk does not contribute to any moment on the rotor's node.

2.2.2.2 Equation of a Shaft Element

A shaft element obtained by dividing the shaft, shown in Fig. 2.2, is presented in Fig. 2.5. The global coordinates XYZ are the same as that of the rotor system, shown in Fig. 2.2.

Fig. 2.5 Dimension of the shaft element

Similar to those of the rigid disk, the mass properties of the shaft element are as follows:

$$\begin{aligned}
&\text{Mass, } m_s = \frac{\rho_s \pi \left(d_{os}^2 - d_{is}^2\right) L}{4} = \rho_s A_s L \\
&\text{Transverse inertia, } I_{Ts} = \frac{1}{12} m_s \left[\frac{3\left(d_{os}^2 + d_{is}^2\right)}{4} + L^2 \right] \\
&\text{Polar inertia, } I_{Ps} = \frac{1}{2} m_s \frac{\left(d_{os}^2 + d_{is}^2\right)}{4} \\
&\text{Area moment of inertia, } I_{As} = \frac{\pi\left(d_{os}^4 - d_{is}^4\right)}{64} \\
&\text{Young's modulus, } E
\end{aligned} \tag{2.7}$$

where $\rho_s, A_s, d_{os}, d_{is}, L$ are shaft element's density, cross-section area, outer diameter, inner diameter and thickness, respectively. After the derivation, the shaft element stiffness matrix $\mathbf{K}_s$ is given as follows:

$$\mathbf{K}_s = \frac{EI_{As}}{L^3}\begin{bmatrix} 12 & 0 & 0 & 6L & -12 & 0 & 0 & 6L \\ 0 & 12 & -6L & 0 & 0 & -12 & -6L & 0 \\ 0 & -6L & 4L^2 & 0 & 0 & 6L & 2L^2 & 0 \\ 6L & 0 & 0 & 4L^2 & -6L & 0 & 0 & 2L^2 \\ -12 & 0 & 0 & -6L & 12 & 0 & 0 & -6L^2 \\ 0 & -12 & 6L & 0 & 0 & 12 & 6L & 0 \\ 0 & -6L & 2L^2 & 0 & 0 & 6L & 4L^2 & 0 \\ 6L & 0 & 0 & 2L^2 & -6L^2 & 0 & 0 & 4L^2 \end{bmatrix} \tag{2.8}$$

Similarly, the shaft element mass matrix $\mathbf{M}_s$ can be obtained

$$\mathbf{M}_s = \mathbf{M}_{s,t} + \mathbf{M}_{s,r} \tag{2.9}$$

where $\mathbf{M}_{s,t}$ is translational mass matrix

$$\mathbf{M}_{s,t} = \frac{m_s}{420}\begin{bmatrix} 156 & 0 & 0 & 22L & 54 & 0 & 0 & -13L \\ 0 & 156 & -22L & 0 & 0 & 54 & 13L & 0 \\ 0 & -22L & 4L^2 & 0 & 0 & -13L & -3L^2 & 0 \\ 22L & 0 & 0 & 4L^2 & 13L & 0 & 0 & -3L^2 \\ 54 & 0 & 0 & 13L & 156 & 0 & 0 & -22L \\ 0 & 54 & -13L & 0 & 0 & 156 & 22L & 0 \\ 0 & 13L & -3L^2 & 0 & 0 & 22L & 4L^2 & 0 \\ -13L & 0 & 0 & -3L^2 & -22L & 0 & 0 & 4L^2 \end{bmatrix} \tag{2.10}$$

and $\mathbf{M}_{s,r}$ is rotational mass matrix

$$\mathbf{M}_{s,t} = \frac{I_{Ps}}{60L^2}\begin{bmatrix} 36 & 0 & 0 & 3L & -36 & 0 & 0 & 3L \\ 0 & 36 & -3L & 0 & 0 & -36 & -3L & 0 \\ 0 & -3L & 4L^2 & 0 & 0 & 3L & -L^2 & 0 \\ 3L & 0 & 0 & 4L^2 & -3L & 0 & 0 & -L^2 \\ -36 & 0 & 0 & -3L & 36 & 0 & 0 & -3L \\ 0 & -36 & 3L & 0 & 0 & 36 & 3L & 0 \\ 0 & -3L & -L^2 & 0 & 0 & 3L & 4L^2 & 0 \\ 3L & 0 & 0 & -L^2 & -3L & 0 & 0 & 4L^2 \end{bmatrix} \tag{2.11}$$

Also, the shaft element gyroscopic matrix $\mathbf{G}_s$ can be determined

$$\mathbf{G}_s = \frac{I_{Ps}}{30L^2}\begin{bmatrix} 0 & 36 & -3L & 0 & 0 & -36 & -3L & 0 \\ -36 & 0 & 0 & -3L & 36 & 0 & 0 & -3L \\ 3L & 0 & 0 & 4L^2 & -3L & 0 & 0 & -L^2 \\ 0 & 3L & -4L^2 & 0 & 0 & -3L & L^2 & 0 \\ 0 & -36 & 3L & 0 & 0 & 36 & 3L & 0 \\ 36 & 0 & 0 & 3L & -36 & 0 & 0 & 3L \\ 3L & 0 & 0 & -L^2 & -3L & 0 & 0 & 4L^2 \\ 0 & 3L & L^2 & 0 & 0 & -3L & -4L^2 & 0 \end{bmatrix} \tag{2.12}$$

Lastly, gravitational force and moment acting on the nodes of the element are derived. By considering Fig. 2.6, which represents only the shaft part of the rotor in Fig. 2.2:

Fig. 2.6 Nodes and elements of the rotor

From the shape functions used in the finite element derivation, the consistent force vector acting on the i-th and i+1-th nodes in YZ-plane can be obtained. The full gravitational force and moment acting on the i-th shaft element is as follows:

$$\mathbf{f}_{gs,i} = \left\{ 0 \quad -\frac{m_{s,i}g}{2} \quad -\frac{m_{s,i}gL_i}{12} \quad 0 \quad 0 \quad -\frac{m_{s,i}g}{2} \quad \frac{m_{s,i}gL_i}{12} \quad 0 \right\}^T \tag{2.13}$$

2.2.2.3 Rotor Finite Element Assembly

After the **M**, **G**, **K** matrices, and $\mathbf{f}_g$ vector of both rigid disk and shaft element were derived, they must be assembled to make the rotor system's EOM by linking all those matrices or vectors together. In this section, only the case of the free-free rotor is presented, and the rotor-bearing system assembly will be presented in Section 2.4.

According to the literature review in Section 1.2.1, the self-excited vibration caused by JB is observed in the first forward natural frequency. As this frequency is in the lower mode, it can be accurately modeled by the rotor model with few numbers of finite elements [109]. Hence, the rotor can be discretized in the minimum numbers of elements that can account for the shaft discontinuity, concentrated mass, and support bearing, as depicted by an example shown in Fig. 2.2. This rotor consists of a shaft divided into five shaft elements and three rigid disks located at Node 3, 4, and 5. Since this rotor has six nodes with 4 DOF for each node (x, y, θ_x, θ_x), the total DOF is 24.

Fig. 2.2 Nodes and elements of the rotor

The rotor's mass matrix $\mathbf{M}_r$ and gyroscopic matrix $\mathbf{G}_r$ are assembled by first overlaying each shaft element matrix $\mathbf{M}_s$ or $\mathbf{G}_s$ over the matrices of neighboring elements at the shared node.

Then, they are overlaid by the corresponding rigid disk matrix $\mathbf{M}_d$ or $\mathbf{G}_d$ at the node matching to the disk position, as shown in Fig. 2.7. Note that the rigid disk does not contribute to the shaft's stiffness, so only $\mathbf{K}_s$ matrices are used to assemble the rotor stiffness matrix $\mathbf{K}_r$. The gravitational force vector $\mathbf{f}_{gr}$ can be assembled from $\mathbf{f}_{gs}$ and $\mathbf{f}_{gd}$ in a similar manner as $\mathbf{M}_r$ and $\mathbf{G}_r$. Hence, the EOM of the free-free rotor with N_E shaft elements subjected to the gravitational force is as follows:

$$\mathbf{M}_r\ddot{\mathbf{q}}_r + \omega\mathbf{G}_r\dot{\mathbf{q}}_r + \mathbf{K}_r\mathbf{q}_r = \mathbf{f}_{gr} \tag{2.14}$$

where $\mathbf{q}_r = \begin{Bmatrix} x_1 & y_1 & \cdots & \theta_{xNE+1} & \theta_{yNE+1} \end{Bmatrix}^T$ is displacement of the rotor's all 4×(N_E+1) DOF.

Fig. 2.7 Free-free rotor assembly of **M, G, K** and $\mathbf{f}_g$

2.3 Bearing Force Modelling

In this section, the mathematical models and numerical methods for solving the nonlinear bearing forces are presented. These forces are obtained by integrating the fluid pressure profile in the gap between the journal and the bearing wall. This pressure profile is solved from the Reynolds equation [46], derived from the continuity equation and momentum equations. Since the fluid film thickness, which depends on journal position and bearing geometry, is needed for solving the Reynolds equation, the bearing geometry will be presented before fluid film force calculation.

2.3.1 Bearing Geometry

JBs used in rotating machines can be classified by the bearing wall's movability into two types: fixed profile JB and tilting pad JB (TPJB). This section will explain the bearing geometry and derive the film thickness of a plain circular journal bearing and a multilobe bearing, which represents the simplest and more complex geometry of the fixed profile JB, respectively; and TPJB and Flexure Pivot TPJB, which also represents the simplest and more complex geometry of the TPJB, respectively.

2.3.1.1 Plain Circular Journal Bearing

The bearing wall of a plain circular JB is continuous for 360 degrees. The gap between the journal and the bearing can be obtained by considering a misaligned journal in a bearing defined in a bearing $X_bY_bZ_b$-coordinates that have the axis Z_b coincides with the bearing axis,

as shown in Fig. 2.8. When the journal lateral displacement at the midplane of the bearing is x_b and y_b, and the inclination of the journal axis in Y_b- Z_b plane and X_b- Z_b plane is α_x and α_y, respectively; The unwrapped film thickness H can be derived as:

$$H(\theta_{zb}, z_b) = C_{rb} - (x_b + z_b \tan\alpha_y)\cos\theta_{zb} - (y_b - z_b \tan\hat{\alpha}_x)\sin\theta_{zb} \tag{2.15}$$

where θ_{zb} is the angle measured from X_b to Y_b. C_{rb} is bearing radial clearance obtained from:

$$C_{rb} = R_b - R_j \tag{2.16}$$

If the axes of the journal and the bearing are assumed to be parallel, α_x and α_y become zero, and Eq. (2.15) is reduced to [42, 49]:

$$H(\theta_{zb}) = C_{rb} - x_b \cos\theta_{zb} - y_b \sin\theta_{zb} \tag{2.17}$$

Fig. 2.8 Geometry and coordinates of plain circular JB

2.3.1.2 Multilobe Journal Bearing

The bearing wall of a multilobe JB is discontinuous because of the oil feed groove, shown in Fig. 2.9(a). In addition, many multilobe JBs have a non-circular profile, shown in Fig. 2.9(b), to increase the stability of the rotor [42, 54]. The degrees of non-circularity of the bearing can be expressed in terms of bearing preload $\hat{m}$:

Fig. 2.9 Multilobe JB: (a) photograph, (b) bearing geometry

$$\hat{m} = 1 - \frac{C_{rb}}{C_{rp}} \tag{2.18}$$

where $C_{rp} = R_p$- R_j is pad radial clearance. Similar to the full circular JB, the unwrapped i-th pad's film thickness H_{pi} of the multilobe JB can be derived as:

$$\left.\begin{aligned} H_{pi}(\theta_{zb}, z_b) = C_{rp} + (C_{rp} - C_{rb})\cos(\theta_{zb} - \theta_{Opi}) \\ -(x_b + z_b \tan\alpha_y)\cos\theta_{zb} - (y_b - z_b \tan\hat{\alpha}_x)\sin\theta_{zb} \end{aligned}\right\} \theta_{lpi} \le \theta_{zb} \le \theta_{tpi} \tag{2.19}$$

where α_x and α_y are inclination of the journal axis in Y_b- Z_b plane and X_b- Z_b plane, respectively. θ_{Opi}, θ_{lpi} and θ_{tpi} are the angle of the pad's center of curvature, leading and trailing edges of the

i-th pad measured from X_b axis in the counterclockwise direction, shown in Fig. 2.9, respectively. Similar to the plain circular JB, if the axes of the journal and the bearing are assumed to be parallel, Eq. (2.19) is reduced to [42]:

$$H_{pi}\left(\theta_{zb}\right)=C_{rp}+(C_{rp}-C_{rb})\cos(\theta_{zb}-\theta_{Opi})-x\cos\theta_{zb}-y_b\sin\theta_{zb}\Big\}\,\theta_{lpi}\le\theta_{zb}\le\theta_{tpi} \tag{2.20}$$

2.3.1.3 Tilting Pad Journal Bearing

Since the fixed profile JBs may exhibit instability under certain operating conditions due to the tangential, or cross-coupling, destabilizing force; The tilting pad journal bearing (TPJB), shown in Fig. 2.10(a), was invented to eliminate the destabilizing force by utilizing the pads that can tilt freely around the pivots. There are two types of TPJB: ball and socket type and rocker type. The pad of ball and socket type can tilt around all axes, whereas the pad of the rocker type can only tilt around the bearing's axis. Due to the difficulty in manufacturing ball and socket type TPJB for experimental verification, only rocker type TPJB is considered in this thesis. The TPJB can be categorized by load (usually the weight of the journal) direction relative to the orientation of the pivots as load-on-pad (LOP) and load-between-pad (LBP), as shown in Fig. 2.10(b) and (c), respectively.

Fig. 2.10 TPJB: (a) photograph (5 pads), (b) four-pad LOP, (c) four-pad LBP

The derivation of TPJB's film thickness is similar to those of multilobe JB but each pad can tilt around the pivot with angle δ, measured counterclockwise from the reference state shown in Fig. 2.10(b) and (c), the film thickness of the i-th pad's H_{pi} can be derived as:

$$\left.\begin{aligned}H_{pi}\left(\theta_{zb},z_b\right)=&\,C_{rp}-(C_{rp}-C_{rb})\cos(\theta_{zb}-\theta_{pvi})\\&-\delta_i\left(R_p+t_p\right)\sin(\theta_{zb}-\theta_{pvi})\\&-\left(x_b+z_b\tan\alpha_y\right)\cos\theta_{zb}-\left(y_b-z_b\tan\hat{\alpha}_x\right)\sin\theta_{zb}\end{aligned}\right\}\theta_{lpi}\le\theta_{zb}\le\theta_{tpi} \tag{2.21}$$

where R_p is pad's radius, t_p is pad's thickness at the pivot and θ_{pvi}, θ_{lpi} and θ_{tpi} are the pivot's angular location, leading and trailing edges of the i-th pad measured from X_b axis in the counterclockwise direction, respectively. If the axes of the journal and the bearing are assumed to be parallel, Eq. (2.21) is reduced to: [49, 72]

$$\left.\begin{aligned}H_{pi}\left(\theta_{zb},z_b\right)=&\,C_{rp}-(C_{rp}-C_{rb})\cos(\theta_{zb}-\theta_{pvi})\\&-\delta_i\left(R_p+t_p\right)\sin(\theta_{zb}-\theta_{pvi})\\&-x_b\cos\theta_{zb}-y_b\sin\theta_{zb}\end{aligned}\right\}\theta_{lpi}\le\theta_{zb}\le\theta_{tpi} \tag{2.22}$$

The pad offset, or pivot ratio ($\hat{\alpha}$), which is the fraction of the angle between the leading edge and the pad pivot angular location to the complete pad's angular extent, is introduced as follows [22]:

$$\hat{\alpha} = \frac{\theta_{pv} - \theta_{lp}}{\theta_{tp} - \theta_{lp}} = \frac{\phi_1}{\phi_1 + \phi_2} \tag{2.23}$$

where ϕ_1 and ϕ_2 are defined in Fig. 2.10. Most commonly used TPJBs have $\alpha = 0.5$ (centrally located pivot), or 0.6. The TPJB with pad offset provides a better oil film wedge, resulting in increased load capacity and bearing stiffness, and yields lower pad temperatures than the centrally located pivot counterpart [22].

2.3.1.4 Flexure Pivot Tilting Pad Journal Bearing

The FPTPJB is manufactured in one piece [27], as shown in Fig. 2.11(a), to solve the problems of pivot wear, pad fluttering, and tolerance stack-up encountered in the TPJB by using the flexure pivot. In this bearing type, the pad can move due to the flexibility of the flexure pivot. This flexure pivot can be modeled as a cantilever beam that the free end is rigidly connected with the pad. Hence, each pad can move in ξ, η, δ-direction, as shown in Fig. 2.11(c).

Fig. 2.11 FPTPJB: (a) photograph (4 pads), (b) four-pad LBP, (c) pad's coordinates

Due to the additional pad's motions in ξ and η directions, the FPTPJB's film thickness can be modified from Eq. (2.21) as follows:

$$\left.\begin{aligned} H_{pi}(\theta_{zb}, z_b) = C_{rp} &+ \left\{\xi - \left(C_{rp} - C_{rb}\right)\right\}\cos(\theta_{zb} - \theta_{pvi}) \\ &+ \left\{\eta - \delta_i\left(R_p + t_p\right)\right\}\sin(\theta_{zb} - \theta_{pvi}) \\ &- \left(x_b + z_b \tan\alpha_y\right)\cos\theta_{zb} - \left(y_b - z_b \tan\hat{\alpha}_x\right)\sin\theta_{zb} \end{aligned}\right\} \theta_{lpi} \le \theta_{zb} \le \theta_{tpi} \tag{2.24}$$

If the axes of the journal and the bearing are assumed to be parallel, Eq. (2.24) is reduced to [27]:

$$\left.\begin{aligned} H_{pi}(\theta_{zb}, z_b) = C_{rp} &+ \left\{\xi - \left(C_{rp} - C_{rb}\right)\right\}\cos(\theta_{zb} - \theta_{pvi}) \\ &+ \left\{\eta - \delta_i\left(R_p + t_p\right)\right\}\sin(\theta_{zb} - \theta_{pvi}) \\ &- x_b \cos\theta_{zb} - y_b \sin\theta_{zb} \end{aligned}\right\} \theta_{lpi} \le \theta_{zb} \le \theta_{tpi} \tag{2.25}$$

2.3.2 Fluid Film Force Calculation

2.3.2.1 Reynolds Equation

After the bearing's geometry was explained and film thickness was derived, the film thickness is substituted into the following Reynolds equation to solve the fluid film's pressure profile [42, 49].

$$\frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\frac{H^3}{12\mu}\frac{\partial p}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\frac{H^3}{12\mu}\frac{\partial p}{\partial z}\right)=\frac{\omega}{2}\frac{\partial H}{\partial\theta}+\frac{\partial H}{\partial t} \tag{2.26}$$

where R and θ are the journal radius and angular coordinate, this form of the Reynolds equation is readily compatible with film thickness H of various bearing types obtained in the previous section. The derivation of the Reynolds equation in Eq. (2.26) was depicted in Appendix B.

In some designs and applications, the rotor and the bearing may operate in the turbulence region. According to Diprima [65], the turbulence effect becomes notable when the JB's average Reynolds number

$$\overline{Re}=\frac{\rho\omega RC_{rb}}{\mu} \tag{2.27}$$

is higher than 2000. The turbulence effect is taken into account in the Reynolds equation by adding the correction parameters k_θ and k_z to the Reynolds equation

$$\frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\frac{H^3}{\mu k_\theta}\frac{\partial p}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\frac{H^3}{\mu k_z}\frac{\partial p}{\partial z}\right)=\frac{\omega}{2}\frac{\partial H}{\partial\theta}+\frac{\partial H}{\partial t} \tag{2.28}$$

where the turbulence parameters k_θ and k_z are functions of the local Reynolds number (Re) defined as follows:

$$\begin{aligned} k_\theta &= 12+a_\theta Re^{b_\theta} \\ k_z &= 12+a_z Re^{b_z} \\ Re &= \frac{\rho\omega RH}{\mu} \end{aligned} \tag{2.29}$$

where a and b in are constants of turbulence. The most commonly used values were proposed by Ng-Pan-Elrod [67], which is based on eddy viscosity, or by Constantinescu [68], which is based on the Prandtl mixing length hypothesis. The constants a and b of both models are presented in Table 2.1.

Table 2.1 Constants of turbulence a and b for Ng-Pan-Elrod and Constatinescu models

Constant	Ng-Pan-Elrod	Constatinescu
a_θ	0.0136	0.026
b_θ	0.9	0.8265
a_z	0.0044	0.0198
b_z	0.96	0.741

From Eq. (2.28) and (2.29), it can be seen that k_θ and k_z are always more than 12, and they can be considered as additional virtual viscosity when compared to Eq. (2.27), which is the laminar Reynolds equation. Hence, the effect of turbulence can increase the fluid film pressure and load-carrying capacity.

2.3.2.2 Numerical Method for Solving of Reynolds Equation

Since bearings with complex geometry, such as multilobe JB, TPJB, FPTPJB, or bearings operated in turbulence region, do not have analytical solutions [48, 49, 53], the numerical

method must be employed to solve the Reynolds equation. The finite difference method (FDM) is commonly used, especially in the bearings that contain no geometrical discontinuity, due to its implementation simplicity. The FDM method discretizes the bearing's surface into grids, as shown by an example of 2-lobe bearing in Fig. 2.12 to calculate pressure $p_{i,j}$ at node i, j, representing the i-th and j-th node in the circumferential and axial directions of the pad, respectively. According to Eq. (2.28) and (2.29), it can be seen that the pressure field can be obtained and continuously differentiable if the film thickness H is continuously differentiable, which is always true in the cases of the bearings without geometrical discontinuity. Thus, the number of grids in Fig. 2.12 should slightly affect the calculated bearing forces if the number of grids is not unreasonably low.

Fig. 2.12 Finite difference grid of bearing pads

Referring to Fig. 2.12, the appropriate finite difference operators for the partial derivatives using central difference are:

$$\begin{aligned}
\frac{\partial X}{\partial \theta} &= \frac{X_{i,j+1} - X_{i,j-1}}{2\Delta\theta} \\
\frac{\partial X}{\partial z} &= \frac{X_{i+1,j} - X_{i-1,j}}{2\Delta\theta} \\
\frac{\partial^2 X}{\partial \theta^2} &= \frac{X_{i,j+1} - 2X_{i,j} + X_{i,j-1}}{(\Delta\theta)^2} \\
\frac{\partial^2 X}{\partial z^2} &= \frac{X_{i+1,j} - 2X_{i,j} + X_{i-1,j}}{(\Delta z)^2}
\end{aligned} \tag{2.30}$$

where X denotes an arbitrary variable. Since the most general form of Reynolds equation is the turbulence Reynolds equation of misaligned journal (H=$H(\theta, z)$), Eq. (2.28) is expanded to:

$$\begin{aligned}
&\frac{1}{\mu R^2}\left[\frac{3H^2}{k_\theta}\frac{\partial H}{\partial \theta}\frac{\partial p}{\partial \theta} - \frac{H^3}{k_\theta^2}\frac{\partial k_\theta}{\partial \theta}\frac{\partial p}{\partial \theta} + \frac{H^3}{k_\theta}\frac{\partial^2 p}{\partial \theta^2}\right] \\
&+\frac{1}{\mu}\left[\frac{3H^2}{k_z}\frac{\partial H}{\partial z}\frac{\partial p}{\partial z} - \frac{H^3}{k_z^2}\frac{\partial k_z}{\partial z}\frac{\partial p}{\partial z} + \frac{H^3}{k_z}\frac{\partial^2 p}{\partial z^2}\right] = \frac{\omega}{2}\frac{\partial H}{\partial \theta} + \frac{\partial H}{\partial t}
\end{aligned} \tag{2.31}$$

As shown in Section 2.3.1, the terms H and its partial derivatives in Eq. (2.31) can be written in closed form at each node i, j. Hence, these terms are left as they are while other terms are rewritten in finite difference form using Eq. (2.30)

$$\frac{1}{\mu R^2}\left[\left(\frac{3H_{i,j}^2}{k_{\theta,i,j}}\frac{\partial H}{\partial\theta}\bigg|_{i,j}-\frac{H_{i,j}^3}{k_{\theta,i,j}^2}\frac{\partial k_\theta}{\partial\theta}\bigg|_{i,j}\right)\left(\frac{p_{i,j+1}-p_{i,j-1}}{2\Delta\theta}\right)+\frac{H_{i,j}^3}{k_{\theta,i,j}}\left(\frac{p_{i,j+1}-2p_{i,j}+p_{i,j-1}}{(\Delta\theta)^2}\right)\right]$$
$$+\frac{1}{\mu}\left[\left(\frac{3H_{i,j}^2}{k_{z,i,j}}\frac{\partial H}{\partial z}\bigg|_{i,j}-\frac{H_{i,j}^3}{k_{z,i,j}^2}\frac{\partial k_z}{\partial z}\bigg|_{i,j}\right)\left(\frac{p_{i+1,j}-p_{i-1,j}}{2\Delta z}\right)+\frac{H_{i,j}^3}{k_{z,i,j}}\left(\frac{p_{i+1,j}-2p_{i,j}+p_{i-1,j}}{(\Delta z)^2}\right)\right] \tag{2.32}$$
$$=\frac{\omega}{2}\frac{\partial H}{\partial\theta}\bigg|_{i,j}+\frac{\partial H}{\partial t}\bigg|_{i,j}$$

Rearranging Eq. (2.32) into groups of pressure at each node yields

$$\frac{1}{\mu}\left[\frac{H_{i,j}^3}{k_{z,i,j}}\frac{1}{(\Delta z)^2}-\left(\frac{3H_{i,j}^2}{k_{z,i,j}}\frac{\partial H}{\partial z}\bigg|_{i,j}-\frac{H_{i,j}^3}{k_{z,i,j}^2}\frac{\partial k_z}{\partial z}\bigg|_{i,j}\right)\left(\frac{1}{2\Delta z}\right)\right]p_{i-1,j}$$
$$+\frac{1}{\mu}\left[\frac{H_{i,j}^3}{k_{z,i,j}}\frac{1}{(\Delta z)^2}+\left(\frac{3H_{i,j}^2}{k_{z,i,j}}\frac{\partial H}{\partial z}\bigg|_{i,j}-\frac{H_{i,j}^3}{k_{z,i,j}^2}\frac{\partial k_z}{\partial z}\bigg|_{i,j}\right)\left(\frac{1}{2\Delta z}\right)\right]p_{i+1,j}$$
$$+\frac{1}{\mu R^2}\left[\frac{H_{i,j}^3}{k_{\theta,i,j}}\frac{1}{(\Delta\theta)^2}-\left(\frac{3H_{i,j}^2}{k_{\theta,i,j}}\frac{\partial H}{\partial\theta}\bigg|_{i,j}-\frac{H_{i,j}^3}{k_{\theta,i,j}^2}\frac{\partial k_\theta}{\partial\theta}\bigg|_{i,j}\right)\left(\frac{1}{2\Delta\theta}\right)\right]p_{i,j-1} \tag{2.33}$$
$$+\frac{1}{\mu R^2}\left[\frac{H_{i,j}^3}{k_{\theta,i,j}}\frac{1}{(\Delta\theta)^2}+\left(\frac{3H_{i,j}^2}{k_{\theta,i,j}}\frac{\partial H}{\partial\theta}\bigg|_{i,j}-\frac{H_{i,j}^3}{k_{\theta,i,j}^2}\frac{\partial k_\theta}{\partial\theta}\bigg|_{i,j}\right)\left(\frac{1}{2\Delta\theta}\right)\right]p_{i,j+1}$$
$$-2\left[\frac{1}{\mu}\frac{H_{i,j}^3}{k_{z,i,j}}\frac{1}{(\Delta z)^2}+\frac{1}{\mu R^2}\frac{H_{i,j}^3}{k_{\theta,i,j}}\frac{1}{(\Delta\theta)^2}\right]p_{i,j}=\frac{\omega}{2}\frac{\partial H}{\partial\theta}\bigg|_{i,j}+\frac{\partial H}{\partial t}\bigg|_{i,j}$$

By letting

$$A_{z,i,j}=\frac{1}{\mu}\frac{H_{i,j}^3}{k_{z,i,j}}\frac{1}{(\Delta z)^2} \qquad A_{\theta,i,j}=\frac{1}{\mu R^2}\frac{H_{i,j}^3}{k_{\theta,i,j}}\frac{1}{(\Delta\theta)^2}$$
$$B_{z,i,j}=\frac{1}{\mu}\frac{3H_{i,j}^2}{k_{z,i,j}}\frac{\partial H}{\partial z}\bigg|_{i,j}\frac{1}{2\Delta z} \qquad B_{\theta,i,j}=\frac{1}{\mu R^2}\frac{3H_{i,j}^2}{k_{\theta,i,j}}\frac{\partial H}{\partial\theta}\bigg|_{i,j}\frac{1}{2\Delta\theta} \tag{2.34}$$
$$C_{z,i,j}=\frac{1}{\mu}\frac{H_{i,j}^3}{k_{z,i,j}^2}\frac{\partial k_z}{\partial z}\bigg|_{i,j}\frac{1}{2\Delta z} \qquad C_{\theta,i,j}=\frac{1}{\mu R^2}\frac{H_{i,j}^3}{k_{\theta,i,j}^2}\frac{\partial k_\theta}{\partial\theta}\bigg|_{i,j}\frac{1}{2\Delta\theta}$$

pressure $p_{i,j}$ at node i, j can be obtained from those of the neighboring nodes as:

$$p_{i,j}=\frac{1}{2\left(A_{z,i,j}+A_{\theta,i,j}\right)}\Big[\left\{A_{z,i,j}-\left(B_{z,i,j}-C_{z,i,j}\right)\right\}p_{i-1,j}$$
$$+\left\{A_{z,i,j}+\left(B_{z,i,j}-C_{z,i,j}\right)\right\}p_{i+1,j}+\left\{A_{\theta,i,j}-\left(B_{\theta,i,j}-C_{\theta,i,j}\right)\right\}p_{i,j-1} \tag{2.35}$$
$$+\left\{A_{\theta,i,j}+\left(B_{\theta,i,j}-C_{\theta,i,j}\right)\right\}p_{i,j+1}-\left(\frac{\omega}{2}\frac{\partial H}{\partial\theta}\bigg|_{i,j}+\frac{\partial H}{\partial t}\bigg|_{i,j}\right)\Big]$$

which is the general form for pressure calculation. In the case of laminar flow, $k_\theta = k_z = 12$ [42, 49] and it makes the terms C_θ and C_z in Eq. (2.34) become zero.

If the journal's angular motion is not considered, H becomes a function of only θ and the terms B_z and C_z in Eq. (2.34) become zero. Additionally, the pressure profile is symmetric around the bearing midplane, making the calculation in half axial length sufficient and

requiring about half calculation time when compared to the model that considers this motion. The summary of the vanished terms and validity of calculation domain is shown in Table 2.2.

Table 2.2 Summary of the vanished terms and calculation domain

Journal's angular motion	Turbulence	B_z	C_θ	C_z	Calculation Domain
Not considered	Included	0	-	0	Half axial length
Not considered	Not included	0	0	0	Half axial length
Considered	Included	-	-	-	Full axial length
Considered	Not included	-	0	0	Full axial length

Note A_θ, A_z, and B_θ do not vanish in all cases

After the finite difference form of the Reynolds equation is obtained, Eq. (2.35) is written in the following standard form of successive over-relaxation (SOR) method [48, 53]:

$$\begin{aligned} p_{i,j}^{k+1} = p_{i,j}^{k} + \alpha \Bigg(& \frac{1}{2\left(A_{z,i,j}^{k} + A_{\theta,i,j}^{k}\right)} \Bigg[\left\{ A_{z,i,j}^{k} - \left(B_{z,i,j}^{k} - C_{z,i,j}^{k} \right) \right\} p_{i-1,j}^{k} \\ & + \left\{ A_{z,i,j}^{k} + \left(B_{z,i,j}^{k} - C_{z,i,j}^{k} \right) \right\} p_{i+1,j}^{k} + \left\{ A_{\theta,i,j}^{k} - \left(B_{\theta,i,j}^{k} - C_{\theta,i,j}^{k} \right) \right\} p_{i,j-1}^{k} \\ & + \left\{ A_{\theta,i,j}^{k} + \left(B_{\theta,i,j}^{k} - C_{\theta,i,j}^{k} \right) \right\} p_{i,j+1}^{k} - \left(\frac{\omega}{2} \frac{\partial H}{\partial \theta}\bigg|_{i,j}^{k} + \frac{\partial H}{\partial t}\bigg|_{i,j}^{k} \right) \Bigg] - p_{i,j}^{k} \Bigg) \end{aligned} \tag{2.36}$$

where k is an iteration step, and α is an over-relaxation factor. The procedure for pressure calculation using successive over-relaxation method can be initiated by specifying boundary conditions (presented in the next section) and assuming an initial pressure distribution of the interior nodes (in this thesis, the initial pressure of these nodes is set to zero). The key idea of successive over-relaxation method is to accelerate the convergence of residual terms (the terms in the parenthesis after α in Eq. (2.36)) to zero by the successive iterations. The over-relaxation factor can decrease the number of iterations required to make the following convergence criteria:

$$\hat{\varepsilon} = \frac{\sum_j \sum_i \left| p_{i,j}^{k} - p_{i,j}^{k-1} \right|}{\sum_j \sum_i \left| p_{i,j}^{k} \right|} \tag{2.37}$$

less than a specified value ($\hat{\varepsilon} = 10^{-14}$ in this thesis). The optimum value of α depends on the number of finite difference grids and can be determined beforehand by trial and error [110]. For the bearing shown in Fig. 2.12, the number of iterations required to reach the convergence criteria versus α of in various $N\theta \times Nz$ values is shown in Fig. 2.13.

Fig. 2.13 Convergence rate of SOR method in various $N\theta \times Nz$ values

2.3.2.3 Boundary Conditions

After the numerical method for solving the fluid film pressure is established, the boundary conditions, both fixed and cavitation boundary conditions, must be specified. Firstly, the fixed boundary conditions at pad edges for both plain circular and multilobe bearings are defined following Lund and Thomsen [55], as:

$$\begin{aligned} &\text{Plain Circular} \quad p(0,z) = p(2\pi,z), \quad p\left(\theta,-\frac{L}{2}\right) = p\left(\theta,\frac{L}{2}\right) = 0 \\ &\text{Multilobe} \quad p(\theta_{lpi},z) = p(\theta_{tpi},z) = 0, \quad p\left(\theta,-\frac{L}{2}\right) = p\left(\theta,\frac{L}{2}\right) = 0 \end{aligned} \tag{2.38}$$

If only fixed boundary conditions are used, the calculated fluid film pressure can be negative (lower than atmospheric pressure), as shown by the pressure profile of a plain circular bearing in Fig. 2.14. This boundary condition is called the full-Sommerfeld boundary condition. However, this boundary condition exists in exceptional circumstances, such as very high ambient fluid pressure. In typical bearings, fluid in the negative pressure zone boils or gases dissolved in the fluid escape. The streamer of flow interspersed with vapor creates a cavitation region that does not contribute to bearings' load-carrying capacity. This cavitation makes the fluid pressure in the cavitation region equals zero-gauge pressure.

Fig. 2.14 Non-cavitated pressure profile of a plain circular bearing

Initially, the cavitation was accounted for using half-Sommerfeld or Gumbel boundary condition, which substitutes the negative pressure with zero. However, this boundary condition violates flow continuity at the onset of the cavitation. As a result, the Swift-Stieber or Reynolds boundary condition which satisfies the following conditions

$$p = 0; \quad \frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0 \tag{2.39}$$

at the full film-cavitation boundary was proposed. This boundary condition gives good agreement with experimental data and easy to implement in iterative numerical method, such as successive over-relaxation method, by using Christopherson's algorithm [64]. It can be obtained by setting negative pressure in each iteration step to zero before calculating the pressure for the next iteration step. A comparison of pressure at the bearing midplane obtained from full-Sommerfeld, Gumbel, and Reynolds boundary conditions is shown in Fig. 2.15. It can be seen that the Reynolds boundary condition satisfies the zero-pressure gradient at the onset of cavitation, where the Gumbel boundary condition does not.

In this thesis, the Reynolds boundary condition is used. After the fluid film pressure is obtained, the fluid film force and moment acting on the journal $\mathbf{f}_{jb}$ are obtained by a numerical version of the following integrations:

$$\mathbf{f}_{\mathbf{jb}} = \begin{Bmatrix} f_{jx} \\ f_{jy} \\ m_{jx} \\ m_{jy} \end{Bmatrix} = -\sum_{pi} \begin{Bmatrix} f_{pxi} \\ f_{pyi} \\ m_{pxi} \\ m_{pyi} \end{Bmatrix} = -\sum_{pi} \int_{-L/2}^{L/2} \int_{\theta_{lpi}}^{\theta_{tpi}} p(\theta, z) \begin{Bmatrix} \cos\theta \\ \sin\theta \\ -z\sin\theta \\ z\cos\theta \end{Bmatrix} R d\theta dz \tag{2.40}$$

where the subscripts *pxi* and *pyi* denote the force and moment acting on the *i*-th pad in x and y-directions respectively, which are the reaction forces of the pad's contribution to the forces acting on the journal. Note that the moments vanish in case that journal and bearing axes are parallel. For TPJB and FPTPJB, this fluid film pressure also causes moment acting on the pad around its pivot as:

$$m_{pzi} = -\left(R_p + t_p\right) \int_{-L/2}^{L/2} \int_{\theta_{lpi}}^{\theta_{tpi}} p(\theta, z) \sin\left(\theta - \theta_{pvi}\right) R d\theta dz \tag{2.41}$$

where R_p and t_p are pad radius and thickness, shown in Fig. 2.10 and Fig. 2.11, respectively.

Fig. 2.15 Comparison of pressure at the bearing midplane

2.4 Rotor-Bearing System Assembly

After mathematical modelling of both rotor and journal bearing was described, the rotor and bearing's EOM are assembled to make the following rotor-bearing system EOM to be used in the dynamics analysis [81].

$$\mathbf{M}_{rb}\ddot{\mathbf{q}}_{rb} + (\mathbf{C}_{rb} + \omega\mathbf{G}_r)\dot{\mathbf{q}}_{rb} + \mathbf{K}_{rb}\mathbf{q}_{rb} = \mathbf{F}_{rb} \tag{2.42}$$

where subscript *rb* depicts matrices and vectors of the combined rotor and it support bearings. The assembly method is divided based on the movability of a bearing pad. To facilitate explanations, the rotor shown in Fig. 2.2 is used as an example. This rotor is support by a simple bearing and a JB at node 1 and 5, respectively.

Fig. 2.16 A rotor supported by a simple bearing and a JB

In this thesis, the simple bearing has only isotropic direct stiffness only in damping. The simple bearing forces $\mathbf{f}_{sb}$ acting on the rotor are as follows

$$\mathbf{f}_{sb} = -\mathbf{K}_{sb}\mathbf{x}_{sb} \tag{2.43}$$

where

$$\mathbf{K}_{sb} = \begin{bmatrix} k_{sb} & 0 & 0 & 0 \\ 0 & k_{sb} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{x}_{sb} = \begin{Bmatrix} x_2 \\ y_2 \\ \theta_{x2} \\ \theta_{y2} \end{Bmatrix}, \mathbf{f}_{sb} = \begin{Bmatrix} f_{sbx} \\ f_{sby} \\ m_{sbx} \\ m_{sby} \end{Bmatrix} \tag{2.44}$$

subscript *sb* depicts the values of the simple bearing. The EOM of a rotor supported at one end by a simple bearing can be written as the EOM a free-free rotor

$$\mathbf{M}_r\ddot{\mathbf{q}}_r + \omega\mathbf{G}_r\dot{\mathbf{q}}_r + \mathbf{K}_r\mathbf{q}_r = \mathbf{f}_{gr} \tag{2.14}$$

overlaid by $\mathbf{K}_{sb}$ from Eq. (2.43) at the DOF corresponding to that of rotor's node supported by the simple bearing. As a result, the EOM of the rotor supported at one end by the simple bearing of the example rotor in Fig. 2.16 is:

$$\mathbf{K}_{rs} = \mathbf{K}_r + \left[\begin{array}{c|cccc} \text{Node} & 1 & 2 & \cdots & 6 \\ \hline 1 & & & & \\ 2 & & \mathbf{K}_{sb} & & \\ \vdots & & & & \\ 6 & & & & \end{array}\right], \mathbf{M}_{rs} = \mathbf{M}_r, \mathbf{G}_{rs} = \mathbf{G}_s, \mathbf{F}_{rs} = \mathbf{f}_{gr}, \mathbf{q}_{rs} = \mathbf{q}_r \tag{2.45}$$

where subscript *rs* represents matrices and vectors of the rotor and simple bearing combined.

There are two ways to represent the assembled EOM in Eq. (2.42), depending on the representation of JB's force. The first one represents JB's force as the nonlinear force, which is useful in equilibrium position calculation and nonlinear analysis. The latter represents JB's force as linearized stiffness and damping, which is useful for OSI calculation by eigenanalysis. Hence, the assembly methods for both representations are presented in the following sections.

2.4.1 Fixed Profile Journal Bearing

Since the pad does not move in this case, the DOF of the rotor-bearing system equals the rotor's DOF.

Nonlinear Representation

For nonlinear representation, matrices and vectors can be assembled from those of Eq. (2.45) and the bearing force acting on the journal $\mathbf{f}_{jb}$ from Eq. (2.40) as follows

$$\mathbf{F}_{rb} = \mathbf{F}_{rs} + \underbrace{\left\{ \begin{array}{c|c} \text{Node} & \\ \hline 1 & \\ \vdots & \\ 5 & \mathbf{f}_{jb} \\ 6 & \end{array} \right\}}_{\mathbf{F}_j}, \mathbf{M}_{rb} = \mathbf{M}_{rs}, \mathbf{G}_{rb} = \mathbf{G}_{rs}, \mathbf{K}_{rb} = \mathbf{K}_{rs}, \mathbf{q}_{rb} = \mathbf{q}_{rs} = \mathbf{q}_r \tag{2.46}$$

Linearized Representation

For linearized representation, matrices and vectors can be assembled from those of Eq. (2.45) and parts that correspond to the journal's motion in Eq. (2.68)

$$\mathbf{K}_{rb} = \mathbf{K}_{rs} + \underbrace{\left[\begin{array}{c|ccc} \text{Node} & 1 & \cdots & 5 \quad 6 \\ \hline 1 & & & \\ \vdots & & & \\ 5 & & \mathbf{K}_{jb} & \\ 6 & & & \end{array} \right]}_{\mathbf{K}_j}; \mathbf{K}_{jb} = \begin{bmatrix} k_{xx} & k_{xy} & k_{x\theta x} & k_{x\theta y} \\ k_{x\theta y} & k_{yy} & k_{y\theta x} & k_{y\theta y} \\ k_{\theta xx} & k_{\theta xy} & k_{\theta x\theta x} & k_{\theta x\theta y} \\ k_{\theta yx} & k_{\theta yx} & k_{\theta y\theta x} & k_{\theta y\theta y} \end{bmatrix}$$

$$\mathbf{C}_{rb} = \underbrace{\left[\begin{array}{c|ccc} \text{Node} & 1 & \cdots & 5 \quad 6 \\ \hline 1 & & & \\ \vdots & & & \\ 5 & & \mathbf{C}_{jb} & \\ 6 & & & \end{array} \right]}_{\mathbf{C}_j}; \mathbf{C}_{jb} = \begin{bmatrix} c_{xx} & c_{xy} & c_{x\theta x} & c_{x\theta y} \\ c_{x\theta y} & c_{yy} & c_{y\theta x} & c_{y\theta y} \\ c_{\theta xx} & c_{\theta xy} & c_{\theta x\theta x} & c_{\theta x\theta y} \\ c_{\theta yx} & c_{\theta yx} & c_{\theta y\theta x} & c_{\theta y\theta y} \end{bmatrix} \tag{2.47}$$

$$\mathbf{M}_{rb} = \mathbf{M}_{rs}, \mathbf{G}_{rb} = \mathbf{G}_{rs}, \mathbf{F}_{rb} = \mathbf{F}_{rs} = \mathbf{f}_{gr}, \mathbf{q}_{rb} = \mathbf{q}_{rs} = \mathbf{q}_r$$

2.4.2 Tilting Pad Journal Bearing

Since the pad can move in this case, there are additional DOF added to the EOM of the rotor-bearing system. Before the assembly method is presented, the EOM of a pad of both TPJB and FPTPJB are explained. The free body diagrams (FBDs) of both bearing types are shown in Fig. 2.17.

Fig. 2.17 Reference coordinate, geometry, and FBD of TPJB's pad

TPJB

Since the pad of TPJB can tilt freely around the pivot, the EOM of the i-th pad becomes:

$$\begin{aligned} m_{pzi}\mathbf{k} - \left(\mathbf{r}_{PVCMi}\left(\theta_{pvi},\delta_i\right)\times m_{pi}g\mathbf{j}\right) &= I_{ppvi}\ddot{\delta}_i\mathbf{k} \\ \Rightarrow m_{pzi} - m_{pi}gx_{PVCMi}\left(\theta_{pvi},\delta_i\right) &= I_{ppvi}\ddot{\delta}_i \end{aligned} \tag{2.48}$$

where m_{pzi} is fluid film's moment acting on the pad obtained from Eq.(2.41), m_{pi} is pad mass, $\mathbf{r}_{PVCMi}$ is vector from the pivot to the pad's center of gravity (CG), and I_{ppv} is pad's polar moment of inertia about the pivot. For the pad with constant thickness, I_{ppv} can be defined by [49]:

$$\begin{gathered} I_{ppv} = 2R_p^2 m_{pi}\left[1+f_1-\left(\frac{\sin\phi_1+\sin\phi_2}{\phi_1+\phi_2}\right)f_2\right] \\ f_1 = \frac{3}{2}\left(\frac{t_p}{R_p}\right)+\frac{3}{4}\left(\frac{t_p}{R_p}\right)^2,\ f_2 = \left(1+\frac{t_p}{R_p}\right)\left(1+\frac{t_p}{R_p}+\frac{1}{2}\left(\frac{t_p}{R_p}\right)^2\right)\Big/\left(1+\frac{1}{2}\left(\frac{t_p}{R_p}\right)\right) \end{gathered} \tag{2.49}$$

where R_p, t_p, ϕ_1 and ϕ_2 are pad's radius, thickness, and angular extents defined in Fig. 2.17. I_{ppv} can be obtained by CAD software for the pad with a more complex shape. From Eq. (2.48), the moment from the pad's weight is a function of δ_{pi}. Hence, the EOMs of all pads can be written as:

$$\mathbf{M}_{TP}\ddot{\mathbf{q}}_{TP} + \mathbf{K}_{gTP}\mathbf{q}_{TP} = \mathbf{f}_{bTP} \tag{2.50}$$

where

$$\begin{gathered} \mathbf{M}_{TP} = \begin{bmatrix} I_{p1} & & \\ & \ddots & \\ & & I_{pN} \end{bmatrix},\ \mathbf{K}_{gTP} = \begin{bmatrix} m_{p1}g\dfrac{\partial}{\partial\delta}x_{PVCM1} & & \\ & \ddots & \\ & & m_{pN}g\dfrac{\partial}{\partial\delta}x_{PVCMN} \end{bmatrix}, \\ \mathbf{q}_{TP} = \begin{Bmatrix} \delta_1 \\ \vdots \\ \delta_N \end{Bmatrix},\ \mathbf{f}_{bTP} = \begin{Bmatrix} m_{pz1} \\ \vdots \\ m_{pzN} \end{Bmatrix} \end{gathered} \tag{2.51}$$

and the subscript TP denotes matrices and vectors of TPJB's pads.

FPTPJB

Each pad of FPTPJB can move in ξ, η, and δ-direction, as shown in Fig. 2.18. If the pad is rigidly connected to the top of the pivot, the EOM of the i-th pad's CG in each direction becomes:

Fig. 2.18 Reference coordinate and FBD of FPTPJB's pad

$$
\begin{gathered}
f_{\xi i} - k_{\xi\xi}\xi_i - m_{pi} g \sin\theta_{pvi} = m_{pi}\ddot{\xi}_i \\
f_{\eta i} - \left(k_{\eta\eta}\eta_i + k_{\eta\delta}\delta_i\right) - m_{pi} g \cos\theta_{pvi} = m_{pi}\ddot{\eta}_i \\
m_{pzi} - \left(k_{\delta\eta}\eta_i + k_{\delta\delta}\delta_i\right) - \left(k_{\eta\eta}\eta_i + k_{\eta\delta}\delta_i\right)\Delta\xi_{PVCG} + \left(k_{\xi\xi}\xi_i\right)\Delta\eta_{PVCG} = I_{CGpi}\ddot{\delta}_i
\end{gathered}
\tag{2.52}
$$

where the $k_{\xi\xi}$, $k_{\eta\eta}$, $k_{\delta\eta}$ and $k_{\eta\delta}$ are stiffnesses of the flexure pivot obtained from those of the Timoshenko cantilever beam as follows [44]:

$$
\begin{gathered}
k_{\xi\xi} = \frac{AE}{L},\ k_{\eta\eta} = \frac{3EI}{L^3\left(1+\Omega\right)},\ k_{\delta\eta} = k_{\eta\delta} = \frac{-2EI}{L^2\left(1+\Omega\right)},\ k_{\delta\delta} = \frac{EI\left(4+\Omega\right)}{4L\left(1+\Omega\right)}; \\
\Omega = \frac{12EI}{L^2\kappa AG};\ A = bw,\ I = \frac{bw^3}{12}
\end{gathered}
\tag{2.53}
$$

where E and G are Young and shear modulus of the pivot material. Ω is a bending-shear stiffness factor due to the shear deformation. κ is shear correction factor, which is 5/6 for rectangular cross-section pivot. For more complex pivot geometry, such as the pivot with fillets, the stiffnesses can be obtained from a CAD software's finite element function. The fluid film forces $f_{\xi i}$ and $f_{\eta i}$ acting on pad's CG in ξ and η-directions can be obtained by transforming forces acting on the pad in x and y-directions in Eq. (2.40) as follows:

$$
\begin{Bmatrix} f_{p\xi i} \\ f_{p\eta i} \end{Bmatrix} = \begin{bmatrix} \cos\theta_{CGi} & \sin\theta_{CGi} \\ -\sin\theta_{CGi} & \cos\theta_{CGi} \end{bmatrix} \begin{Bmatrix} f_{pxi} \\ f_{pyi} \end{Bmatrix}
\tag{2.54}
$$

where θ_{CGi} is the angle measured from X-axis to CG in the counterclockwise direction. Similar to the TPJB, the EOMs of all pads can be written from Eq.(2.52) as

$$
\mathbf{M}_{FP}\ddot{\mathbf{q}}_{FP} + \mathbf{K}_{pvFP}\mathbf{q}_{FP} + \mathbf{K}_{mpvFP}\mathbf{q}_{FP} = \mathbf{f}_{bFP} + \mathbf{f}_{gp}
\tag{2.55}
$$

where

$$
\mathbf{M}_{FP} = \begin{bmatrix}
m_{p1} & & & & & & \\
& m_{p1} & & & & & \\
& & I_{p1} & & & & \\
& & & \ddots & & & \\
& & & & m_{pN} & & \\
& & & & & m_{pN} & \\
& & & & & & I_{pN}
\end{bmatrix},\ \mathbf{K}_{pvFP} = \begin{bmatrix}
k_{\xi\xi} & & & & & & \\
& k_{\eta\eta} & k_{\eta\delta} & & & & \\
& k_{\delta\eta} & k_{\delta\delta} & & & & \\
& & & \ddots & & & \\
& & & & k_{\xi\xi} & & \\
& & & & & k_{\eta\eta} & k_{\eta\delta} \\
& & & & & k_{\delta\eta} & k_{\delta\delta}
\end{bmatrix},
$$

$$
\mathbf{K}_{mpvFP} = \begin{bmatrix}
-k_{\xi\xi}\Delta\eta_{CGPV} & k_{\eta\eta}\Delta\xi_{CGPV} & k_{\eta\delta}\Delta\xi_{CGPV} & & & & \\
& & & \ddots & & & \\
& & & & -k_{\xi\xi}\Delta\eta_{CGPV} & k_{\eta\eta}\Delta\xi_{CGPV} & k_{\eta\delta}\Delta\xi_{CGPV}
\end{bmatrix},
\tag{2.56}
$$

$$
\mathbf{q}_{FP} = \begin{Bmatrix} \xi_1 \\ \eta_1 \\ \delta_1 \\ \vdots \\ \xi_N \\ \eta_N \\ \delta_N \end{Bmatrix},\ \mathbf{f}_{bFP} = \begin{Bmatrix} f_{\xi 1} \\ f_{\eta 1} \\ m_{pz1} \\ \vdots \\ f_{\xi N} \\ f_{\eta N} \\ m_{pzN} \end{Bmatrix},\ \mathbf{f}_{gp} = \begin{Bmatrix} -m_{pi} g \sin\theta_{pv1} \\ -m_{pi} g \cos\theta_{pv1} \\ 0 \\ \vdots \\ -m_{pi} g \sin\theta_{pvN} \\ -m_{pi} g \cos\theta_{pvN} \\ 0 \end{Bmatrix}
$$

the subscript *FP* denotes matrices and vectors of FPTPJB's pads. After the additional EOMs of pads are obtained, the rotor-bearing EOM can be assembled as follows:

Nonlinear Representation

For nonlinear representation, matrices and vectors can be assembled from those of Eq. (2.45), $\mathbf{f}_{jb}$ from Eq. (2.40), and EOM of pads, either from Eq. (2.50) for TPJB or Eq. (2.55) for FPTPJB. The assembled matrices and vectors for TPJB are as follows:

$$\mathbf{M}_{rb} = \left[\begin{array}{c|c} \mathbf{M}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{M}_{TP} \end{array}\right], \mathbf{G}_{rb} = \left[\begin{array}{c|c} \mathbf{G}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right], \mathbf{K}_{rb} = \left[\begin{array}{c|c} \mathbf{K}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{K}_{gTP} \end{array}\right]$$
$$\mathbf{q}_{rb} = \left\{\begin{array}{c} \mathbf{q}_{rs} \\ \hline \mathbf{q}_{TP} \end{array}\right\}, \mathbf{F}_{rb} = \left\{\begin{array}{c} \mathbf{F}_{rs} + \mathbf{F}_{j} \\ \hline \mathbf{f}_{bTP} \end{array}\right\} \tag{2.57}$$

Similarly, for FPTPJB

$$\mathbf{M}_{rb} = \left[\begin{array}{c|c} \mathbf{M}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{M}_{FP} \end{array}\right], \mathbf{G}_{rb} = \left[\begin{array}{c|c} \mathbf{G}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \mathbf{G}_{rs}, \mathbf{K}_{rb} = \left[\begin{array}{c|c} \mathbf{K}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{K}_{pvFP} + \mathbf{K}_{mpvFP} \end{array}\right]$$
$$\mathbf{q}_{rb} = \left\{\begin{array}{c} \mathbf{q}_{rs} \\ \hline \mathbf{q}_{FP} \end{array}\right\}, \mathbf{F}_{rb} = \left\{\begin{array}{c} \mathbf{F}_{rs} + \mathbf{F}_{j} \\ \hline \mathbf{f}_{bFP} + \mathbf{f}_{gp} \end{array}\right\} \tag{2.58}$$

Linearized Representation

Firstly, JB's full stiffness $\mathbf{K}_{jb}$, and damping $\mathbf{C}_{jb}$ matrices are split into four matrices based on perturbation and force exertion of the journal or pads, as shown in Eq. (2.68). The linearized representation of TPJB can be written as:

$$\mathbf{K}_{rb} = \left[\begin{array}{c|c} \mathbf{K}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{K}_{gTP} \end{array}\right] + \left[\begin{array}{c|ccccc} \text{Node} & 1 & \cdots & 5 & 6 & \text{Pads} \\ \hline 1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & & \vdots & \vdots \\ 5 & \vdots & & \mathbf{K}_{jj} & \vdots & \mathbf{K}_{jp} \\ 6 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \hline \text{Pads} & \mathbf{0} & \cdots & \mathbf{K}_{pj} & \mathbf{0} & \mathbf{K}_{pp} \end{array}\right],$$
$$\mathbf{C}_{rb} = \left[\begin{array}{c|ccccc} \text{Node} & 1 & \cdots & 5 & 6 & \text{Pads} \\ \hline 1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & & \vdots & \vdots \\ 5 & \vdots & & \mathbf{C}_{jj} & \vdots & \mathbf{C}_{jp} \\ 6 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \hline \text{Pads} & \mathbf{0} & \cdots & \mathbf{C}_{pj} & \mathbf{0} & \mathbf{C}_{pp} \end{array}\right], \mathbf{M}_{rb} = \left[\begin{array}{c|c} \mathbf{M}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{M}_{TP} \end{array}\right],$$
$$\mathbf{G}_{rb} = \left[\begin{array}{c|c} \mathbf{G}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right], \mathbf{q}_{rb} = \left\{\begin{array}{c} \mathbf{q}_{rs} \\ \hline \mathbf{q}_{TP} \end{array}\right\}, \mathbf{F}_{rb} = \left\{\begin{array}{c} \mathbf{F}_{rs} \\ \hline \mathbf{0} \end{array}\right\} \tag{2.59}$$

Similarly, for FPTPJB

$$
\mathbf{K}_{rb} = \left[\begin{array}{c|c} \mathbf{K}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{K}_{pvFP} + \mathbf{K}_{mpvFP} \end{array}\right] + \left[\begin{array}{c|cccc|c} \text{Node} & 1 & \cdots & 5 & 6 & \text{Pads} \\ \hline 1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & & \vdots & \vdots \\ 5 & \vdots & & \mathbf{K}_{jj} & \vdots & \mathbf{K}_{jp} \\ 6 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \hline \text{Pads} & \mathbf{0} & \cdots & \mathbf{K}_{pj} & \mathbf{0} & \mathbf{K}_{pp} \end{array}\right],
$$

$$
\mathbf{C}_{rb} = \left[\begin{array}{c|cccc|c} \text{Node} & 1 & \cdots & 5 & 6 & \text{Pads} \\ \hline 1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & & \vdots & \vdots \\ 5 & \vdots & & \mathbf{C}_{jj} & \vdots & \mathbf{C}_{jp} \\ 6 & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \hline \text{Pads} & \mathbf{0} & \cdots & \mathbf{C}_{pj} & \mathbf{0} & \mathbf{C}_{pp} \end{array}\right], \mathbf{M}_{\mathbf{rb}} = \left[\begin{array}{c|c} \mathbf{M}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{M}_{FP} \end{array}\right], \tag{2.60}
$$

$$
\mathbf{G}_{rb} = \left[\begin{array}{c|c} \mathbf{G}_{rs} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right], \mathbf{q}_{rb} = \left\{\begin{array}{c} \mathbf{q}_{rs} \\ \hline \mathbf{q}_{FP} \end{array}\right\}, \mathbf{F}_{rb} = \left\{\begin{array}{c} \mathbf{F}_{rs} \\ \hline \mathbf{f}_{gp} \end{array}\right\}
$$

The EOM of combined rotor-bearing system Eq. (2.42) consists of N second-order ODEs, where N is the number of DOF. However, it is more convenient to convert this equation into an equivalent set of $2N$ first-order ODEs when using numerical calculation program, such as MATLAB. The equivalent first-order ODEs of Eq. (2.42) can be written as

$$
\dot{\mathbf{u}} = \mathbf{A}\mathbf{u} + \mathbf{F}_1 \tag{2.61}
$$

where

$$
\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_{rb}^{-1}\mathbf{K}_{rb} & \mathbf{M}_{rb}^{-1}(\mathbf{C}_{rb} + \omega\mathbf{G}_r) \end{bmatrix}, \mathbf{u} = \begin{Bmatrix} \mathbf{q}_{rb} \\ \dot{\mathbf{q}}_{rb} \end{Bmatrix}, \mathbf{F}_1 = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}_{rb}^{-1}\mathbf{F}_{rb} \end{Bmatrix} \tag{2.62}
$$

2.5 Equilibrium Position and Dynamics Coefficients

After the EOMs of the rotor-bearing system were established in the previous section, the equilibrium position of the rotor and the bearing pads (TPJB and FPTPJB) can be calculated from the static version of the nonlinear representation of the rotor-bearing system's EOM in Eq. (2.42) with Eq. (2.46), (2.57), and (2.58) for JB, TPJB, and FPTPJB, respectively. After that, the journal and pads are perturbed about the obtained equilibrium position to obtain bearing's stiffness and damping coefficients. The obtained coefficients are substituted in the linear representation of the rotor-bearing system's EOM in Eq. (2.42) using Eq. (2.47), (2.59), or (2.60) for JB, TPJB, and FPTPJB, respectively. The critical speed and onset speed of instability (OSI) of the rotor-bearing system can be obtained from the eigenanalysis of the linearized rotor-bearing system's EOM.

2.5.1 Equilibrium Position of The Rotor and Pads

The equilibrium position of the rotor and the bearing pads (TPJB and FPTPJB) can be calculated from the static version of the nonlinear representation of the rotor-bearing system's EOM in Eq. (2.42), which is:

$$
\mathbf{K}_{rb}\mathbf{q}_{rb,eq} = \mathbf{F}_{rb,eq}\left(\mathbf{q}_{rb,eq}\right) \tag{2.63}
$$

where the subscript eq depicts the equilibrium of rotor and pads and note that $\mathbf{F}_{rb}$ is a function of the members of $\mathbf{q}_{rb}$ that corresponding to displacements of journal and pads. $\mathbf{K}$, $\mathbf{q}$, and $\mathbf{F}$ for

fixed profile JB, TPJB and FPTPJB are obtained from Eq. (2.46), (2.57), and (2.58), respectively. The residual force is defined as:

$$\mathbf{E}\left(\mathbf{q}_{rb}\right)=\mathbf{K}_{rb}\mathbf{q}_{rb}-\mathbf{F}_{rb}\left(\mathbf{q}_{rb}\right) \tag{2.64}$$

which must be zero at the equilibrium position. The Newton-Raphson iteration of Eq. (2.64) can be written as:

$$\mathbf{q}_{rb}^{k+1}=\mathbf{q}_{rb}^{k}-\left(\left.\frac{\partial\mathbf{E}}{\partial\mathbf{q}_{rb}}\right|^{k}\right)^{-1}\mathbf{E}^{k} \tag{2.65}$$

where k is an iteration index. Eq. (2.65) is rearranged to find the equilibrium position of the next iteration as:

$$\mathbf{q}_{rb}^{k+1}=\mathbf{q}_{rb}^{k}-\left(\mathbf{K}_{rb}-\left.\frac{\partial\mathbf{F}_{rb}\left(\mathbf{q}_{rb}\right)}{\partial\mathbf{q}_{rb}}\right|^{k}\right)^{-1}\mathbf{E}^{k} \tag{2.66}$$

Note that only the nonzero members of the Jacobean matrix in Eq. (2.66) are those associated with the DOF consider in each bearing model. Jacobean matrix is numerically calculated by the central difference method. Eq. (2.66) is iteratively calculated until the Euclidean norm of $\mathbf{E}^k$ is less than 10^{-4}.

2.5.2 Dynamic Coefficients Calculation

After the equilibrium positions of the journal and pads (for TPJB and FPTPJB) are obtained, the stiffness and damping coefficients are calculated from the fluid film forces of perturbed displacement and velocity of each journal and pad about the equilibrium position using the pad perturbation method [111]. This method considers the pad's DOF and does not assume the pad's motion to a specific frequency. Hence, it is suitable for determining the OSI of the rotor supported by the TPJB and FPTPJB as the stiffness and damping coefficients of these bearing types are dependent upon the excitation force's frequency [49], which is unknown in the cases of self-excited vibration. The following example shows the numerical calculation method for stiffness and damping of a misaligned FPTPJB with one pad.

$$\begin{aligned}
f &= f\left(\underbrace{x,y,\theta_x,\theta_y}_{\text{Journal Disp.}},\underbrace{\xi_1,\eta_1,\delta_1}_{\text{Pad Disp.}},\underbrace{\dot{x},\dot{y},\dot{\theta}_x,\dot{\theta}_y}_{\text{Journal Vel.}},\underbrace{\dot{\xi}_1,\dot{\eta}_1,\dot{\delta}_1}_{\text{Pad Vel.}}\right)\\
k_{xy} &= -\frac{\partial f_{jx}}{\partial y}\cong-\frac{1}{2\Delta y}\begin{bmatrix} f_{jx}\left(x_{eq},y_{eq}+\Delta y,\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,0,0,0,0,0,0\right)\\ -f_{jx}\left(x_{eq},y_{eq}-\Delta y,\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,0,0,0,0,0,0\right)\end{bmatrix}\\
k_{\theta xy} &= -\frac{\partial f_{jx}}{\partial y}\cong-\frac{1}{2\Delta y}\begin{bmatrix} m_{jx}\left(x_{eq},y_{eq}+\Delta y,\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,0,0,0,0,0,0\right)\\ -m_{jx}\left(x_{eq},y_{eq}-\Delta y,\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,0,0,0,0,0,0\right)\end{bmatrix}\\
k_{\xi 1y} &= -\frac{\partial f_{\xi i}}{\partial y}\cong-\frac{1}{2\Delta y}\begin{bmatrix} f_{\xi 1}\left(x_{eq},y_{eq}+\Delta y,\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,0,0,0,0,0,0\right)\\ -f_{\xi 1}\left(x_{eq},y_{eq}-\Delta y,\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,0,0,0,0,0,0\right)\end{bmatrix}\\
k_{\delta 1y} &= -\frac{\partial f_{\xi i}}{\partial y}\cong-\frac{1}{2\Delta y}\begin{bmatrix} m_{pz1}\left(x_{eq},y_{eq}+\Delta y,\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,0,0,0,0,0,0\right)\\ -m_{pz1}\left(x_{eq},y_{eq}-\Delta y,\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,0,0,0,0,0,0\right)\end{bmatrix}\\
c_{xy} &= -\frac{\partial f_{jx}}{\partial \dot{y}}\cong-\frac{1}{2\Delta \dot{y}}\begin{bmatrix} f_{jx}\left(x_{eq},y_{eq},\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,+\Delta\dot{y},0,0,0,0,0\right)\\ -f_{jx}\left(x_{eq},y_{eq},\theta_{xeq},\theta_{yeq},\xi_{eq},\eta_{eq},\delta_{eq},0,-\Delta\dot{y},0,0,0,0,0\right)\end{bmatrix}
\end{aligned} \tag{2.67}$$

In this thesis, according to the recommendation from Someya [53], the perturbation sizes in Eq. (2.67) is set to $0.001C_{rb}$ for displacement and $0.001\omega C_{rb}$ for velocity, respectively. Adams [1] also suggested perturbation size selection by considering the significant digits of perturbed forces. The full stiffness matrix $\mathbf{K}_{jb}$ of FPTPJB with one pad that considers the journal angular motion from stiffnesses obtained by Eq. (2.67) can be written as:

$$\mathbf{K}_{jb} = \begin{array}{c|cccc:ccc} & \Delta x & \Delta y & \Delta\theta_x & \Delta\theta_y & \Delta\xi_1 & \Delta\eta_1 & \Delta\delta_1 \\ \hline -\Delta f_{jx} & k_{xx} & k_{xy} & k_{x\theta x} & k_{x\theta y} & k_{x\xi 1} & k_{x\eta 1} & k_{x\delta 1} \\ -\Delta f_{jy} & k_{yx} & k_{yy} & k_{y\theta x} & k_{y\theta y} & k_{y\xi 1} & k_{y\eta 1} & k_{y\delta 1} \\ -\Delta m_{jx} & k_{\theta xx} & k_{\theta xy} & k_{\theta x\theta x} & k_{\theta x\theta y} & k_{\theta x\xi 1} & k_{\theta x\eta 1} & k_{\theta x\delta 1} \\ -\Delta m_{jy} & k_{\theta yx} & k_{\theta yy} & k_{\theta y\theta x} & k_{\theta y\theta y} & k_{\theta y\xi 1} & k_{\theta y\eta 1} & k_{\theta y\delta 1} \\ \hdashline -\Delta f_{\xi 1} & k_{\xi 1x} & k_{\xi 1y} & k_{\xi 1\theta x} & k_{\xi 1\theta y} & k_{\xi 1\xi 1} & k_{\xi 1\eta 1} & k_{\xi 1\delta 1} \\ -\Delta f_{\eta 1} & k_{\eta 1x} & k_{\eta 1y} & k_{\eta 1\theta x} & k_{\eta 1\theta y} & k_{\eta 1\xi 1} & k_{\eta 1\eta 1} & k_{\eta 1\delta 1} \\ -\Delta m_{pz1} & k_{\delta 1x} & k_{\delta 1y} & k_{\delta 1\theta x} & k_{\delta 1\theta y} & k_{\delta 1\xi 1} & k_{\delta 1\eta 1} & k_{\delta 1\delta 1} \end{array} \quad \begin{array}{l} \text{Forces on Journal} \\ \\ \\ \\ \text{Forces on Pad} \\ \\ \\ \end{array} \tag{2.68}$$

$$\underbrace{\qquad\qquad\qquad}_{\text{Journal Pert.}} \qquad \underbrace{\qquad\qquad\qquad}_{\text{Pad Pert.}}$$

$$\mathbf{K}_{jb} = \left[\begin{array}{c:c} \mathbf{K}_{jj} & \mathbf{K}_{jp} \\ \hdashline \mathbf{K}_{pj} & \mathbf{K}_{pp} \end{array}\right]$$

For more pads, the dimension of $\mathbf{K}_{jp}$, $\mathbf{K}_{pj}$ and $\mathbf{K}_{pp}$ parts of $\mathbf{K}_{jb}$ in Eq. (2.68) are expanded according to the number of pads correspondingly. The full damping matrix $\mathbf{C}_{jb}$ can be obtained similarly from velocity perturbation. According to Eq. (2.67) and (2.68), The size of $\mathbf{K}_{jb}$ and $\mathbf{C}_{jb}$ is determined by the number of input variables of the bearing's fluid film force function, which can be summarized as the following table

Table 2.3 Summary of input variables of fluid film force function and size of **K** and **C**

Pad Type	Journal Angular Motion	θ_x	θ_y	ξ_i	η_i	δ_i	K, C Size
Fixed	Not Considered	No	No	No	No	No	2^2
Fixed	Considered	Yes	Yes	No	No	No	4^2
TPJB	Not Considered	No	No	No	No	Yes	$(2+N)^2$
TPJB	Considered	Yes	Yes	No	No	Yes	$(4+N)^2$
FPTPJB	Not Considered	No	No	Yes	Yes	Yes	$(2+3N)^2$
FPTPJB	Considered	Yes	Yes	Yes	Yes	Yes	$(4+3N)^2$

2.6 Eigenanalysis

The eigenanalysis is the common tool to calculate critical speed and OSI of the rotor-bearing system. The free vibration of Eq. (2.61) in linearized representation is

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) \tag{2.69}$$

where **u** is the concatenated displacement and velocity vector defined in Eq. (2.61). Eq. (2.69) has the solution that can be written in linear combination of modal coordinate and mode's product as:

$$\mathbf{u}(t) = \sum_{j=1}^{2N} X_j e^{\lambda_j t} \boldsymbol{\varphi}_j \tag{2.70}$$

where the subscript j represents the j-th mode. X_j is a real modal constant that controls the modal amplitude, λ_j is a complex scalar that indicates the modal stability and frequency, and $\boldsymbol{\varphi}_j$ are a complex vector that shows the mode shape. By inserting Eq. (2.70) into Eq. (2.69), each mode must satisfy:

$$\left(\mathbf{A}-\lambda_j\mathbf{I}\right)\boldsymbol{\varphi}_j=\mathbf{0} \tag{2.71}$$

where **I** is identity matrix. The solution of Eq. (2.71) consists of $2N$ sets of eigenvalues λ_j and its associated eigenvector $\boldsymbol{\varphi}_j$. Since λ_j and $\boldsymbol{\varphi}_j$ are complex in general but the matrix **A** is real and symmetric, λ_j and $\boldsymbol{\varphi}_j$ must have their complex conjugates $\overline{\lambda}_j$ and $\overline{\boldsymbol{\varphi}}_j$. As a result, the free vibration $\mathbf{u}(t)$ in Eq. (2.70) can be written as a linear combination of product of eigenvalue λ_i and its associated eigenvector $\boldsymbol{\varphi}_j$

$$\mathbf{u}(t)=\sum_{j=1}^{N}X_j\left(\boldsymbol{\varphi}_j e^{\lambda_j t}+\overline{\boldsymbol{\varphi}}_j e^{\overline{\lambda}_j t}\right);\ \lambda_j=\beta_j+i\Omega_j;\ \Omega_j\geq 0 \tag{2.72}$$

where β and Ω indicate the decay/growth and the angular velocity, respectively, of the free vibration solution of each mode. Only positive Ω is considered in the eigenanalysis.

To help explain the following parts, an example of the eigenanalysis result of the rotor in Fig. 2.16 supported by 2-lobe JB from shaft rotational speed (ω) 500 to 5000 rpm is presented in Fig. 2.19. Only four λ with lowest positive Ω are presented. In Fig. 2.19, λ_2 crosses zero at ω = 2259 RPM, which is the onset speed of instability (OSI), or 37.65 Hz with Ω = 16.66 Hz. Hence, the rotor becomes "linearly" unstable at 2259 RPM with whirl frequency ratio (Ω/ω: WFR) of 0.44, which agrees with the observations in literature reviews that the whirling frequency of the unstable vibration is about half of rotational speed.

Next, the whirling direction of the mode is classified as forward and backward whirl based on the co/counter-rotation of Ω and ω, as shown in Fig. 2.20. The whirling direction of each mode can be determined by using only sinusoidal parts of Eq. (2.72). The lateral coordinates, defined by x and y, are used to determine whirling direction at the particular axial location (node). The sinusoidal part of x and y of the j-th eigenvector in Eq. (2.72), which is a real number, can be written by summation of the product of positive-Ω eigenvalue and part of its associated eigenvector related to x and y with their conjugate counterpart.

$$\begin{Bmatrix} x_j(t) \\ y_j(t) \end{Bmatrix}=\begin{Bmatrix} a_{xj}+ib_{xj} \\ a_{yj}+ib_{yj} \end{Bmatrix}e^{i\Omega_j t}+\begin{Bmatrix} a_{xj}-ib_{xj} \\ a_{yj}-ib_{yj} \end{Bmatrix}e^{-i\Omega_j t} \tag{2.73}$$

Fig. 2.19 Eigenvalues of the rotor-bearing system in Fig. 2.16

Fig. 2.20 Definition of forward and backward whirl

From coordinate convention in Fig. 2.20, the journal position's vector ***r*** can be represented in complex form as

$$\mathbf{r}_j(t) = x_j(t) + iy_j(t) \tag{2.74}$$

Inserting Eq. (2.73) into Eq. (2.74) yields

$$\begin{aligned}\mathbf{r}_j(t) &= x_j(t) + iy_j(t) \\ &= \left[\left(a_{xj} + ib_{xj}\right)e^{i\Omega_j t} + \left(a_{xj} - ib_{xj}\right)e^{-i\Omega_j t}\right] \\ &\quad + i\left[\left(a_{yj} + ib_{yj}\right)e^{i\Omega_j t} + \left(a_{yj} - ib_{yj}\right)e^{-i\Omega_j t}\right] \\ &= \underbrace{\left[\left(a_{xj} - b_{yj}\right) + i\left(a_{yj} + b_{xj}\right)\right]}_{F} e^{i\Omega_j t} + \underbrace{\left[\left(a_{xj} + b_{yj}\right) + i\left(a_{yj} - b_{xj}\right)\right]}_{B} e^{-i\Omega_j t}\end{aligned} \tag{2.75}$$

which can be interpreted that an arbitrary (elliptical) journal's whirling motion or orbit can be represented by the summation of forward and backward circular whirls, as shown in Fig. 2.21. The criteria for whirling direction can be defined as:

$$\begin{aligned}|F| > |B| \quad &\Rightarrow \quad \text{Forward} \\ |F| < |B| \quad &\Rightarrow \quad \text{Backward}\end{aligned} \tag{2.76}$$

Fig. 2.21 Summation of forward and backward whirls

By using Eq. (2.75), (2.76) and $\boldsymbol{\varphi}_j$, λ_2 and λ_3 in Fig. 2.19 are forward eigenvalues, whereas λ_1 and λ_4 are backward eigenvalues. Thus, the rotor becomes linearly unstable due to the forward mode, which agrees with the literature review finding that positive cross-coupling stiffness can destabilize the rotor. The critical speed is the speed ω that the synchronous whirling ($\Omega=\omega$ or

"1X" line in Fig. 2.19) equals to Ω of forward eigenvalue. Hence, the first critical speed (ω_{cr1}) is 1000 rpm in this example. The mode shapes of λ_2 at the first critical speed and OSI are presented in Fig. 2.22. As the rotor becomes unstable, it whirls in the same mode shape similar to the one it whirls during passing through the first critical speed. Also, the displacements of nodes 4 and 6 are out of phase, indicating that the journal at node 5 has angular whirling motion. This angular whirling motion affects the thickness and forces of the fluid film, which may result in difference in rotordynamics behavior when compared to the bearing model that assumes the journal and bearing are parallel.

Fig. 2.22 Example of mode shapes of λ_2 at the first critical speed and OSI

2.7 Model Reduction

For nonlinear rotordynamics analysis that the rotor-bearing system's assembled EOM in nonlinear representation is integrated in the time domain, the integration of matrix **A** in Eq. (2.61), which has the size of $(2N)^2$ for of N-DOF system, can be time-consuming, especially for the rotor model with a large number of nodes. However, compared to the total DOF, only a few DOF are subjected to nonlinear forces, and a model reduction from the full model's DOF to these DOF can save a lot of calculation time. The EOM is prepared for the model reduction by rearranging the order of the original EOM's coordinates into groups of internal and boundary coordinates as follows:

$$\begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{IB} \\ \mathbf{M}_{BI} & \mathbf{M}_{BB} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_I \\ \ddot{\mathbf{q}}_B \end{Bmatrix} + \left(\begin{bmatrix} \mathbf{C}_{II} & \mathbf{C}_{IB} \\ \mathbf{C}_{BI} & \mathbf{C}_{BB} \end{bmatrix} + \omega \begin{bmatrix} \mathbf{G}_{II} & \mathbf{G}_{IB} \\ \mathbf{G}_{BI} & \mathbf{G}_{BB} \end{bmatrix} \right) \begin{Bmatrix} \dot{\mathbf{q}}_I \\ \dot{\mathbf{q}}_B \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_{BB} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_B \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_I(t) \\ \mathbf{f}_B(t) \end{Bmatrix} \tag{2.77}$$

where subscript $\mathbf{q}_I$ and $\mathbf{q}_B$ are internal and boundary coordinates, respectively. The boundary coordinates are those directly subjected to the nonlinear forces, such as $\mathbf{q}_5 = \{x_5, y_5, \theta_{x5}, \theta_{x5}\}^T$ in Fig. 2.16, and they are retained for the nonlinear rotordynamics analysis. The internal coordinates are those not directly subjected to the nonlinear forces. They are discarded but can be recovered later using the dynamic relationship with the boundary coordinates, depending on reduction methods.

There are two model reduction methods used in this thesis: modified Guyan reduction and real mode component-mode synthesis. The procedures of each method and comparison of reduction accuracy in both eigenvalue and nonlinear integration of selected cases are presented in the following sections.

2.7.1 Modified Guyan Reduction

Guyan reduction [86] assumes the dynamic relationship between $\mathbf{q}_I$ and $\mathbf{q}_B$ in Eq. (2.77) is the same as the static relationship.

$$\begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_{BB} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_B \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_B(t) \end{Bmatrix} \tag{2.78}$$

However, considering only the static relationship cannot accurately reduce the rotor system's model that spins faster than its first critical speed because the inertia force is neglected. From the eigenanalysis result of the full rotor-bearing model in Fig. 2.19, the eigenvalue contributing to the instability is λ_2. The reduction accuracy about the OSI can be increased if the inertia force of the associated whirling frequency Ω_2 is included in the static relationship [88]. Since Ω_2 is obtained from the eigenanalysis result of the full rotor-bearing model, it implicitly contains the effect of both damping and gyroscopic matrices in the EOM.

$$\left(\begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_{BB} \end{bmatrix} - \Omega(\omega)^2 \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{IB} \\ \mathbf{M}_{BI} & \mathbf{M}_{BB} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_B \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_B(t) \end{Bmatrix} \tag{2.79}$$

The top row of Eq. (2.79) gives

$$\boldsymbol{\chi}\mathbf{q}_B = \mathbf{q}_I \tag{2.80}$$

where

$$\boldsymbol{\chi} = -\left(\mathbf{K}_{II} - \Omega(\omega)^2 \mathbf{M}_{II}\right)^{-1} \left(\mathbf{K}_{IB} - \Omega(\omega)^2 \mathbf{M}_{IB}\right) \tag{2.81}$$

Inserting Eq. (2.80) into Eq. (2.77) yields

$$\begin{aligned} &\begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{IB} \\ \mathbf{M}_{BI} & \mathbf{M}_{BB} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix} \ddot{\mathbf{q}}_B + \left(\begin{bmatrix} \mathbf{C}_{II} & \mathbf{C}_{IB} \\ \mathbf{C}_{BI} & \mathbf{C}_{BB} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix} + \omega \begin{bmatrix} \mathbf{G}_{II} & \mathbf{G}_{IB} \\ \mathbf{G}_{BI} & \mathbf{G}_{BB} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix} \right) \dot{\mathbf{q}}_B \\ &+ \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_{BB} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix} \mathbf{q}_B = \begin{Bmatrix} \mathbf{F}_I(t) \\ \mathbf{F}_B(t) \end{Bmatrix} \end{aligned} \tag{2.82}$$

where $\mathbf{I}$ is identity matrix, multiplying by $\{\boldsymbol{\chi} \quad \mathbf{I}\}$ on both sides yields

$$\mathbf{M}_{RD}\ddot{\mathbf{q}}_B + \left(\mathbf{C}_{RD} + \omega\mathbf{G}_{RD}\right)\dot{\mathbf{q}}_B + \mathbf{K}_{RD}\mathbf{q}_B = \mathbf{F}_{RD}(t) \tag{2.83}$$

where

$$\begin{aligned} &\mathbf{M}_{RD} = \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix}^T \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{IB} \\ \mathbf{M}_{BI} & \mathbf{M}_{BB} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix}, \mathbf{C}_{RD} = \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix}^T \begin{bmatrix} \mathbf{C}_{II} & \mathbf{C}_{IB} \\ \mathbf{C}_{BI} & \mathbf{C}_{BB} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix} \\ &\mathbf{G}_{RD} = \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix}^T \begin{bmatrix} \mathbf{G}_{II} & \mathbf{G}_{IB} \\ \mathbf{G}_{BI} & \mathbf{G}_{BB} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix}, \mathbf{K}_{RD} = \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix}^T \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_{BB} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix}, \mathbf{F}_{RD}(t) = \begin{Bmatrix} \boldsymbol{\chi} \\ \mathbf{I} \end{Bmatrix}^T \begin{Bmatrix} \mathbf{F}_I(t) \\ \mathbf{F}_B(t) \end{Bmatrix} \end{aligned} \tag{2.84}$$

and the subscript *RD* refers to the reduced model. Hence, the total DOF are reduced to those of $\mathbf{q}_B$, which are physical coordinates. In this method, the eigenanalysis of the full model must be done in advance to obtain the whirl frequency of the eigenvalue that becomes unstable Ω_2.

2.7.2 Component Mode Synthesis (CMS)

The component mode synthesis for rotordynamics analysis was contributed by Craig and Bampton [89]. This reduction method replaces the internal coordinates, which are physical coordinates, with modal coordinates. The reduced model can accurately represent the vibration phenomena of interest if the modes related to the interested vibration phenomena are retained. Hence, this reduction method is useful for the nonlinear rotordynamics analysis related to the journal bearing instability because it occurs in lower modes.

The modal coordinates consist of the constrained normal modes, which are displacements relative to the fixed boundary coordinates, and static modal displacements, which are obtained by displacing only the boundary DOF. The constrained normal modes are obtained by setting the boundary coordinates in Eq. (2.77) to zero and solving a free-vibration problem of the internal coordinates.

$$\mathbf{M}_{II}\ddot{\mathbf{q}}_I + \left(\mathbf{C}_{II} + \omega\mathbf{G}_{II}\right)\dot{\mathbf{q}}_I + \mathbf{K}_{II}\mathbf{q}_I = \mathbf{0} \tag{2.85}$$

Generally, the eigenvalues and eigenvectors of Eq (2.85) are complex values, resulting in speed-dependent complex modal coordinate and requiring complex variable integration to solve the vibration of the reduced model. However, the eigenvalues and eigenvectors are real values if $\mathbf{C}_{II}$ and $\mathbf{G}_{II}$ are neglected; and Sundararajan and Noah [95], and Kobayashi and Aoyama [112] found that neglecting $\mathbf{C}_{II}$ and $\mathbf{G}_{II}$ in Eq. (2.85) causes only a small loss in model reduction accuracy. As a result, Eq. (2.85) is reduced to

$$\mathbf{M}_{II}\ddot{\mathbf{q}}_I + \mathbf{K}_{II}\mathbf{q}_I = \mathbf{0} \tag{2.86}$$

which yields NI real eigenvalues in ascending order λ_{I1}, … , λ_{INI}, and the associated eigenvectors $\boldsymbol{\varphi}_{I1}$, … , $\boldsymbol{\varphi}_{INI}$. The internal coordinates $\mathbf{q}_I$ can be represented as a linear combination of product of modal coordinate c and eigenvectors $\boldsymbol{\varphi}_I$

$$\mathbf{q}_I = \boldsymbol{\eta}_{NI}\mathbf{c}_{NI} \tag{2.87}$$

where

$$\boldsymbol{\eta}_{NI} = \begin{bmatrix}\boldsymbol{\varphi}_1 & \cdots & \boldsymbol{\varphi}_{NI}\end{bmatrix}, \mathbf{c}_{NI} = \begin{Bmatrix} c_1 \\ \vdots \\ c_{NI}\end{Bmatrix} \tag{2.88}$$

Note that in the real mode, the following orthogonality is satisfied:

$$\begin{aligned} \boldsymbol{\eta}_{NI}^T\mathbf{M}_{II}\boldsymbol{\eta}_{NI} &= \mathbf{I} \\ \boldsymbol{\eta}_{NI}^T\mathbf{K}_{II}\boldsymbol{\eta}_{NI} &= \begin{bmatrix}\lambda_{I1}^2 & & \\ & \ddots & \\ & & \lambda_{INI}^2\end{bmatrix} \end{aligned} \tag{2.89}$$

Model reduction by real mode CMS is done by truncating Eq. (2.87) to the retained h-th lowest modes. The matrix of the retained associated eigenvectors $\boldsymbol{\eta}_h$ is defined as:

$$\boldsymbol{\eta}_h = \begin{bmatrix}\boldsymbol{\varphi}_1 & \cdots & \boldsymbol{\varphi}_h\end{bmatrix} \tag{2.90}$$

The static relationship between $\mathbf{q}_B$ and $\mathbf{q}_I$ from Guyan reduction in Eq. (2.78) yields

$$\mathbf{q}_I = \boldsymbol{\kappa}\mathbf{q}_B \tag{2.91}$$

where

$$\boldsymbol{\kappa} = -\mathbf{K}_{II}^{-1}\mathbf{K}_{IB} \tag{2.92}$$

From Eq. (2.87), (2.90), and (2.91), the model reduction that the internal coordinates are truncated to the h-th lowest modes is obtained

$$\begin{Bmatrix}\mathbf{q}_I \\ \mathbf{q}_B\end{Bmatrix} = \boldsymbol{\Psi}_h\mathbf{s}_h \tag{2.93}$$

where

$$\boldsymbol{\Psi}_h = \begin{bmatrix}\boldsymbol{\kappa} & \boldsymbol{\eta}_h \\ \mathbf{I} & \mathbf{0}\end{bmatrix}, \mathbf{s}_h = \begin{Bmatrix}\mathbf{q}_B \\ \mathbf{c}_h\end{Bmatrix} \tag{2.94}$$

Hence, the total DOF are reduced to those of $\mathbf{q}_B$, which are physical coordinates, and the retained modal coordinates $\mathbf{c}_h$ in this reduction method. Inserting Eq. (2.93) into Eq. (2.77) with manipulation yields

$$\mathbf{M}_{RD,h}\ddot{\mathbf{s}}_h + \left(\mathbf{C}_{RD,h} + \omega\mathbf{G}_{RD,h}\right)\dot{\mathbf{s}}_h + \mathbf{K}_{RD,h}\mathbf{s}_h = \mathbf{F}_{RD,h}\left(t\right) \tag{2.95}$$

where

$$\begin{gathered}\mathbf{M}_{RD,h} = \mathbf{\Psi}_h^T \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{IB} \\ \mathbf{M}_{BI} & \mathbf{M}_{BB} \end{bmatrix} \mathbf{\Psi}_h, \mathbf{C}_{RD,h} = \mathbf{\Psi}_h^T \begin{bmatrix} \mathbf{C}_{II} & \mathbf{C}_{IB} \\ \mathbf{C}_{BI} & \mathbf{C}_{BB} \end{bmatrix} \mathbf{\Psi}_h, \mathbf{G}_{RD,h} = \mathbf{\Psi}_h^T \begin{bmatrix} \mathbf{G}_{II} & \mathbf{G}_{IB} \\ \mathbf{G}_{BI} & \mathbf{G}_{BB} \end{bmatrix} \mathbf{\Psi}_h, \\ \mathbf{K}_{RD,h} = \mathbf{\Psi}_h^T \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_{BB} \end{bmatrix} \mathbf{\Psi}_h, \mathbf{F}_{RD,h}\left(t\right) = \mathbf{\Psi}_h^T \begin{Bmatrix} \mathbf{F}_I\left(t\right) \\ \mathbf{F}_B\left(t\right) \end{Bmatrix}\end{gathered} \tag{2.96}$$

Compared to the modified Guyan reduction method, the number of reduced DOF obtained from this method is larger, depending on the retained modes. However, this method does not require eigenanalysis of the full rotor model, and it considers the mode shapes of the internal coordinates. The comparison of reduction accuracy of both reduction methods in both linear and nonlinear analyses are presented in the next section.

2.7.3 Comparison of Reduction Accuracy

In this section, the comparison of reduction accuracy of the modified Guyan reduction and real mode CMS of selected cases in both linear and nonlinear analyses are presented. Two rotor configurations are selected for comparison. The first one, or "Config. 1", is shown in Fig. 2.16 and has the large disk located that the rotor's midspan. Hence, the gyroscopic effect is minimal in this configuration. The second one, or "Config. 2", has the large disk moved toward the JB, as shown in Fig. 2.23, resulting in an increased gyroscopic effect. Both rotors are supported by one simple bearing and one 2-lobe JB. The full model of both configurations has 24 DOF.

Fig. 2.23 A rotor in "Config. 2"

2.7.3.1 Linear Analysis

The comparison plots of real and imaginary parts of the eigenvalue that becomes unstable obtained the full rotor model (24 DOF), modified Guyan reduction (4 DOF), and real mode CMS with h=2, 4, and 6 (6, 8, 10 DOF, respectively) of Config. 1 and 2 rotors are shown in Fig. 2.24 and Fig. 2.25, respectively.

It can be seen that all reduction methods give almost identical eigenvalues to that of the full model in "Config. 1", as shown in Fig. 2.24. However, in "Config. 2", the modified Guyan reduction gives the most deviated eigenvalue from that of the full rotor model while the real mode CMS yields the noticeably closer eigenvalue to that of the full model even h=2, and it becomes closer as h is increased. This is because the CMS considers the gyroscopic effect more than the modified Guyan reduction, and the degree of consideration increases with h. The comparison of the first critical speed and onset speed of instability (OSI) of "Config. 1" and "Config. 2" rotors are shown in Table 2.4 and Table 2.5, respectively. It can be seen in that the modified Guyan reduction noticeably overpredicts the OSI while the real mode CMS gives OSI close to that of the full rotor model even h=2.

Fig. 2.24 Comparison of eigenvalues that become unstable of "Config. 1" rotor

Fig. 2.25 Comparison of eigenvalues that become unstable of "Config. 2" rotor

Table 2.4 First critical speed and OSI of full and reduced rotor models of "Config. 1" rotor

(rpm)	Full (24 DOF)	Guyan (4 DOF)	CMS (h=2) (6 DOF)	CMS (h=4) (8 DOF)	CMS (h=6) (10 DOF)
First Critical	1000	1000	1000	1000	1000
OSI	2259	2259	2258	2259	2259

Table 2.5 First critical speed and OSI of full and reduced rotor models of "Config. 2" rotor

(rpm)	Full (24 DOF)	Guyan (4 DOF)	CMS (h=2) (6 DOF)	CMS (h=4) (8 DOF)	CMS (h=6) (10 DOF)
First Critical	1630	1641	1636	1634	1630
OSI	3446	3625	3462	3457	3446

2.7.3.2 Nonlinear Analysis

This section compares the steady-state solutions obtained from the numerical integration of the nonlinear rotor-bearing model in the full and reduced rotor models (the description of numerical integration is shown in Section 2.8.2). Since the rotor in "Config. 1" cannot give steady-state integration result before the journal contact the bearing at speed higher than the OSI, only the integration results obtained from full rotor model, Guyan reduction, and CMS of the rotor in "Config. 2" at ω=OSI+10 rpm, are presented in Fig. 2.26. This figure indicates that the result from Guyan reduction is the most deviated from that of the full rotor model. The models reduced by CMS give noticeably better conformity with the full model even *h*=2.

Fig. 2.26 Comparison of numerical integration of "Config. 2" rotor

2.8 Numerical Procedures for Nonlinear Analysis

In the previous section, the onset speed of instability (OSI), which is the point that the rotor becomes linearly unstable, was determined. The OSI is a good starting point of nonlinear analysis that can obtain the crucial information that the linear analysis cannot determine, such as limits cycle and its stability. In this section, numerical procedures for nonlinear analysis are introduced.

2.8.1 Nondimensionalization

The EOM should be nondimensionalized to improve numerical accuracy. For example, the reduced model in Eq. (2.95) can be made dimensionless (denoted by ~) as follows:

$$\tilde{\mathbf{s}}_h'' + \left(\tilde{\mathbf{C}}_{RD,h} + \tilde{\omega}\tilde{\mathbf{G}}_{RD,h}\right)\tilde{\mathbf{s}}_h' + \tilde{\mathbf{K}}_{RD,h}\tilde{\mathbf{s}}_h = \tilde{\mathbf{F}}_{RD,h}\left(\tilde{t}\right) \tag{2.97}$$

where the dimensionless variables are

$$\tilde{\mathbf{s}}_h = \frac{\mathbf{s}_h}{C_{rb}} \qquad \tilde{\mathbf{s}}_h' = \frac{\dot{\mathbf{s}}_h}{\omega_{OSI}C_{rb}} \qquad \tilde{\mathbf{s}}_h'' = \frac{\ddot{\mathbf{s}}_h}{\omega_{OSI}^2 C_{rb}}$$

$$\tilde{t} = \omega_{OSI}t \qquad \tilde{\omega} = \frac{\omega}{\omega_{OSI}} \qquad \tilde{\mathbf{C}}_{RD,h} = \frac{\mathbf{M}_{RD,h}^{-1}\mathbf{C}_{RD,h}}{\omega_{OSI}} \tag{2.98}$$

$$\tilde{\mathbf{G}}_{RD,h} = \frac{\mathbf{M}_{RD,h}^{-1}\mathbf{G}_{RD,h}}{\omega_{OSI}} \qquad \tilde{\mathbf{K}}_{RD,h} = \frac{\mathbf{M}_{RD,h}^{-1}\mathbf{K}_{RD,h}}{\omega_{OSI}^2} \qquad \tilde{\mathbf{F}}_{RD,h}\left(\tilde{t}\right) = \frac{\mathbf{M}_{RD,h}^{-1}\tilde{\mathbf{F}}_{RD,h}\left(t\right)}{\omega_{OSI}^2 C_{rb}}$$

2.8.2 Numerical Integration

The integration of the reduced rotor model is done by integrating the nondimensionalized version of the reduced EOM in Eq. or (2.95). Since the EOM of the rotor-bearing system is considered stiff, the implicit integration method is needed to ensure numerical stability. The

MATLAB's ode15s, which is the recommended solver for the stiff problem [113], is used in this thesis. It is a variable-step, variable-order (VSVO) solver based on the numerical differentiation formulas (NDFs) of orders 1 to 5. Optionally, it can use the backward differentiation formulas (BDFs, also known as Gear's method) that are usually less efficient.

The **initial condition (IC)** for numerical integration is crucial because the improper IC can lead to long integration time required. In this study, the displacement part of the IC is the equilibrium position of the rotor and pads discussed in Section 2.5.1. The small velocity perturbation is added to the journal to simulate the rotor at the equilibrium being disturbed. The integration result will return to or move away from that equilibrium if it is stable or unstable, equilibrium, respectively.

The numerical integration is terminated when the result becomes steady, or the journal touches bearing. The **Contact** between the journal and bearing can be checked by the film thickness at both axial ends of the bearing:

$$\min\left(H\left(\theta,\frac{L}{2}\right),H\left(\theta,-\frac{L}{2}\right)\right)\leq 0 \tag{2.99}$$

The initial condition for the numerical integration at the adjacent speed can be obtained from the last point in time expansion of the previous speed's integration result to reduce the calculation time for the integration to reach the solution.

2.8.3 Shooting Method

Since the numerical integration takes a long time before the solution becomes steady and cannot obtain the unstable solution that encircles the rotor's stability boundary, it is inefficient and cannot determine the entire bifurcation event. These shortcomings of the numerical integration are overcome by the shooting method. The shooting method changes the system of ODEs into a boundary value problem (BVP) and determines the initial conditions and periods that made the solution periodic. For the investigated rotor model, the effect of the unbalance combined with self-excited vibration makes the solution aperiodic, which makes the shooting method inapplicable. However, the unbalance does not noticeably change the OSI from that of the well-balanced system subjected to only self-excited vibration. Hence, the excitation force is assumed to come from the self-excited vibration caused by JB only. This assumption makes the system autonomous and shows the periodic motion of an unknown period depending on the system's parameters. For the pure forced vibration, such as unbalance response, the system is non-autonomous and has the solution's period as that of the excitation force.

2.8.3.1 Derivation

For the periodic solution, a dimensionless residual vector $\tilde{\mathbf{R}}$ can be defined as

$$\tilde{\mathbf{R}}\left(\tilde{\mathbf{U}}_0,\tilde{T}_0\right)\equiv\tilde{\mathbf{U}}_T\left(\tilde{\mathbf{U}}_0,\tilde{T}_0\right)-\tilde{\mathbf{U}}_0=\mathbf{0} \tag{2.100}$$

where $\tilde{\mathbf{U}}=\left\{\tilde{\mathbf{s}}^T,\tilde{\mathbf{s}}'^T\right\}^T$ and $\tilde{\mathbf{s}}$ are defined in Eq.(2.97). $\tilde{T}_0$ is dimensionless period. First-order Taylor series of Eq. (2.100) yields

$$\begin{aligned}&\tilde{\mathbf{U}}_T\left(\tilde{\mathbf{U}}_0^k,\tilde{T}_0^k\right)-\tilde{\mathbf{U}}_0^k+\left[\left.\frac{\partial\tilde{\mathbf{U}}_T\left(\tilde{\mathbf{U}}_0^k,\tilde{T}_0^k\right)}{\partial\tilde{\mathbf{U}}_0}\right|_k-\mathbf{I}\right]\left\{\tilde{\mathbf{U}}_0^{k+1}-\tilde{\mathbf{U}}_0^k\right\}\\&+\left.\frac{\partial\tilde{\mathbf{U}}_T\left(\tilde{\mathbf{U}}_0^k,\tilde{T}_0^k\right)}{\partial\tilde{T}_0}\right|_k\left(\tilde{T}_0^{k+1}-\tilde{T}_0^k\right)=\mathbf{0}\end{aligned} \tag{2.101}$$

where k is an index of iteration. For the self-excited vibration problem, the numbers of equations and unknowns are $2N$ and $2N+1$, respectively. However, the journal velocity in either x or y-direction is zero at a certain time within the period $\tilde{T}_0$. Hence x_0' is set to zero and the terms corresponding to partial derivative respected to x_0' in the Jacobean matrix of Eq. (2.101) are removed. So, both $\tilde{\mathbf{U}}_0$ and $\tilde{T}_0$ can be determined simultaneously by the Newton-Raphson method. For the forced vibration problem, the last term on the left-hand side of Eq. (2.101) vanishes, and the numbers of equations and unknowns are 2; hence $\tilde{\mathbf{U}}_0$ can be solved directly. The Jacobean in Eq. (2.101) is obtained by the central difference method. The numerical integrations for determining $\tilde{\mathbf{U}}_T$ of all perturbed initial conditions can be done parallelly using the "parfor" in MATLAB to reduce the calculation time. The shooting method is terminated when the norm of $\tilde{\mathbf{R}}$ is less than 10^{-5}.

2.8.3.2 Initial Guess of Shooting Method

There are two methods for finding an initial guess for $\tilde{\mathbf{U}}_0$ and $\tilde{T}_0$ (for the self-excited vibration) depending on the existence of the stable numerical integration solution at the rotational speed (ω) above the OSI.

The first method is used when this stable solution exists, the initial guess for $\tilde{\mathbf{U}}_0$ and $\tilde{T}_0$ is obtained from steady-state integration result. $\tilde{\mathbf{U}}_0$ is the last point in time expansion that x_0' crosses zero with a positive slope and $\tilde{T}_0$ is determined from the period between this point and the previous point that x_0' crosses zero with a positive slope. If the stable solution does not exist, the initial guess for $\tilde{\mathbf{U}}_0$ and $\tilde{T}_0$ is determined at ω below the OSI using the same procedure as the previous method but from several points in long time expansion that shows increasing amplitude of $\tilde{\mathbf{U}}_0$, which indicates an unstable solution. The selected initial guess for $\tilde{\mathbf{U}}_0$ and $\tilde{T}_0$ in this method is the one that makes the shooting method yields a periodic solution.

2.8.3.3 Arclength Continuation

Once the set of $\tilde{\mathbf{U}}_0\left(\tilde{\omega}_j\right)$ and $\tilde{T}_0\left(\tilde{\omega}_j\right)$ that yields periodic solution is obtained at the speed ω_j, another set of $\tilde{\mathbf{U}}_0\left(\tilde{\omega}_{j+1}\right)$ and $\tilde{T}_0\left(\tilde{\omega}_{j+1}\right)$ at nearby speed $\tilde{\omega}_{j+1}$ can be obtained by using $\tilde{\mathbf{U}}_0\left(\tilde{\omega}_j\right)$ and $\tilde{T}_0\left(\tilde{\omega}_j\right)$ as the initial guess for shooting method at speed $\tilde{\omega}_{j+1}$. After two sets of $\tilde{\mathbf{U}}_0$ and $\tilde{T}_0$ are obtained, the differences in $\tilde{\mathbf{U}}_0, \tilde{T}_0$ and $\tilde{\omega}$ can be obtained as follows

$$\begin{aligned} \tilde{\mathbf{U}}_0\left(\tilde{\omega}_{j+1}\right)-\tilde{\mathbf{U}}_0\left(\tilde{\omega}_j\right) &= \left\{\Delta\tilde{u}_1 \quad \cdots \quad \Delta\tilde{u}_{2N-1}\right\}^T \\ \tilde{T}_0\left(\tilde{\omega}_{j+1}\right)-\tilde{T}_0\left(\tilde{\omega}_j\right) &= \Delta\tilde{T}_0 \\ \tilde{\omega}_{j+1}-\tilde{\omega}_j &= \Delta\tilde{\omega} \end{aligned} \tag{2.102}$$

In Eq.(2.102), both $\tilde{\mathbf{U}}_0$ and $\tilde{T}_0$ are written as functions of $\tilde{\omega}$. However, there are cases that multiple sets of $\tilde{\mathbf{U}}_0$ and $\tilde{T}_0$ occur at the same $\tilde{\omega}$ as shown in Fig. 2.27(a). Hence, a continuation by tracing a branch of solutions by varying $\tilde{\omega}$ will fail at the saddle-node points A and B. The arclength continuation [106], which parametrizes the continuation of the shooting method's

initial conditions with the position s along the arc instead of the speed $\tilde{\omega}$, is used to avoid this problem. The step size Δs of the arc is defined as:

$$\left(\Delta s\right)^2 = \left(\Delta\tilde{u}_1\right)^2 + \cdots \left(\Delta\tilde{u}_{2N-1}\right)^2 + \left(\Delta\tilde{T}_0\right)^2 + \left(\Delta\tilde{\omega}\right)^2 \tag{2.103}$$

A branch of solutions can be determined by continuing $\tilde{\mathbf{U}}_0, \tilde{T}_0$ and $\tilde{\omega}$ as functions of the position s along the arclength as follows:

$$\begin{aligned} s_{j+1} &= s_j + \Delta s \\ \tilde{u}_{i,j+1} &= \tilde{u}_{i,j} + \left(\frac{\Delta\tilde{u}_i}{\Delta s}\right)\bigg|_j \Delta s \\ \tilde{T}_{0,j+1} &= \tilde{T}_{0,j+1} + \left(\frac{\Delta\tilde{T}_0}{\Delta s}\right)\bigg|_j \Delta s \\ \tilde{\omega}_{j+1} &= \tilde{\omega}_j + \left(\frac{\Delta\tilde{\omega}}{\Delta s}\right)\bigg|_j \Delta s \end{aligned} \tag{2.104}$$

where i and j are indices of state variables and points along the arclength, respectively. For the shooting method, the s-parameterized form of Eq. (2.100) can be written as

$$\tilde{\mathbf{R}}\left(\tilde{\mathbf{U}}_0\left(s\right), \tilde{T}_0\left(s\right), \tilde{\omega}\left(s\right)\right) = \mathbf{0} \tag{2.105}$$

From Eq. (2.104), $\tilde{\mathbf{U}}_0, \tilde{T}_0$ and $\tilde{\omega}$ at s_{j+1} can be thought as linear extrapolation from s_j using the rate of changes of theses variables respected to s. The rate of changes is calculated at every s_j step. the size of Δs is inversely proportional to the rate of change of theses variables respected to s. When the continuation reaches point A or B in Fig. 2.27, the linear extrapolation fails because $\Delta\tilde{\omega}/\Delta s$ changes sign at this point and the size of Δs is usually smaller than a certain value. When this situation occurs, the predicted $\tilde{\omega}$ is obtained from that of the s obtained before, while $\tilde{\mathbf{U}}_0$ and $\tilde{T}_0$ are determined from forward linear extrapolation as follows:

$$\begin{aligned} \tilde{u}_{i,j+1} &= \tilde{u}_{i,j} + \left(\frac{\Delta\tilde{u}_i}{\Delta s}\right)\bigg|_j \left(s_j - s_{j-k}\right) \\ \tilde{T}_{0,j+1} &= \tilde{T}_{0,j+1} + \left(\frac{\Delta\tilde{T}_0}{\Delta s}\right)\bigg|_j \left(s_j - s_{j-k}\right) \\ \tilde{\omega}_{j+1} &= \tilde{\omega}_{j-k} \end{aligned} \tag{2.106}$$

where k is the index of backward tracing along the arclength of the k-th previously obtained solution point, as shown in Fig. 2.27(b). The value of k starts from 1 ($\tilde{\omega}_{j+1} = \tilde{\omega}_{j-1}$, while still extrapolating $\tilde{\mathbf{U}}_0, \tilde{T}_0$ in the forward direction), and the value is increased (uses $\tilde{\omega}$ obtained from the further backward point) if the continuation fails. This approach usually makes the continuation able to transverse through point A or B. The extrapolation is switched back to Eq. (2.104) after the continuation transverses through point A or B.

Fig. 2.27 (a) Arc construction, (b) Continuation scheme at the saddle-node point

2.8.3.4 Floquet Multiplier Analysis

Once the periodic solution is obtained, its stability is needed to be determined to identify the bifurcation type. The following Floquet theory is excerpted from Nayfeh and Balachandran [106]. Firstly, consider an example of an autonomous system below:

$$\dot{\mathbf{v}} = \mathbf{f}(\mathbf{v}, \boldsymbol{\lambda}) \tag{2.107}$$

where $\mathbf{v}$ and $\boldsymbol{\lambda}$ represent N-dimensional state vector and M-dimensional parameter vector, respectively. Let Eq. (2.107) has a periodic solution $\mathbf{v_0}$ with a minimal period T when $\boldsymbol{\lambda} = \boldsymbol{\lambda_0}$. Imposing perturbation $\mathbf{w}$ on $\mathbf{v_0}$ yields

$$\mathbf{v}(t) = \mathbf{v}_0(t) + \mathbf{w}(t) \tag{2.108}$$

assuming that $\mathbf{f}$ is at least twice continuously differentiable (C^2), substituting Eq. (2.108) into Eq. (2.107) and expanding it in first-order Taylor series about $\mathbf{v_0}(t)$ yields

$$\begin{aligned} \dot{\mathbf{w}}(t) &= \left(\frac{\partial \mathbf{f}(\mathbf{v}_0, \boldsymbol{\lambda}_0)}{\partial \mathbf{v}} \right) \mathbf{w}(t) + O\left(\left\| \mathbf{w}(t) \right\|^2 \right) \\ \dot{\mathbf{w}}(t) &\simeq \mathbf{B}\left(t, \boldsymbol{\lambda}_0\right) \mathbf{w}(t) \end{aligned} \tag{2.109}$$

Note that the perturbation $\mathbf{w}$ is linearized, so the stability analysis is local. The matrix $\mathbf{B}$ has period T. The N-dimensional linear system in Eq. (2.109) has N linearly independent solutions $\mathbf{w}_1, \ldots, \mathbf{w}_N$ that can be written in a matrix form as

$$\mathbf{W}(t) = \left[\mathbf{w}_1(t) \quad \cdots \quad \mathbf{w}_N(t) \right] \tag{2.110}$$

Hence, Eq. (2.109) can be written as

$$\dot{\mathbf{W}}(t) = \mathbf{B}\left(t, \boldsymbol{\lambda}_0\right) \mathbf{W}(t) \tag{2.111}$$

Substituting $\tau = t + T$ into Eq. (2.111) yields

$$\frac{d\mathbf{W}}{d\tau} = \mathbf{B}\left(\tau - T, \boldsymbol{\lambda}_0\right) \mathbf{W}(t) = \mathbf{B}\left(\tau, \boldsymbol{\lambda}_0\right) \mathbf{W}(t) \tag{2.112}$$

From Eq. (2.112), if $\mathbf{W}(t)$ is solution of Eq. (2.109), $\mathbf{W}(t+T)$ is also solution. However, Eq. (2.109) has at most N linearly independent solutions, so $\mathbf{w}_i(t+T)$ must be linear combination of $\mathbf{w}_1(t), \ldots, \mathbf{w}_N(t)$, that is

$$\mathbf{W}(t+T) = \mathbf{W}(t) \boldsymbol{\Phi} \tag{2.113}$$

where $\boldsymbol{\Phi}$ is called the monodromy matrix. Note that $\boldsymbol{\Phi}$ depends on chosen $\mathbf{W}$ and is not unique. It can be thought as a transformation that maps an initial vector in R^N at $t = 0$ to another vector in R^N at $t = T$. From Eq. (2.113), specifying the initial condition $\mathbf{W} = \mathbf{I}$ and setting $t = 0$ yields

$$\mathbf{W}(T) = \boldsymbol{\Phi} \tag{2.114}$$

Let $\mathbf{P}$ be a nonsingular constant matrix and introducing the transformation $\mathbf{W}(t) = \mathbf{Q}(t)\mathbf{P}^{-1}$, Eq. (2.113) can be written as

$$\mathbf{Q}(t+T)\mathbf{P}^{-1} = \mathbf{Q}(t)\mathbf{P}^{-1}\mathbf{\Phi}$$
$$\mathbf{Q}(t+T) = \mathbf{Q}(t)\mathbf{P}^{-1}\mathbf{\Phi}\mathbf{P} = \mathbf{Q}(t)\mathbf{L} \tag{2.115}$$

When the eigenvalues $\rho_1, \ldots, \rho_N$ of the monodromy matrix $\mathbf{\Phi}$ are distinct, one can choose the matrix $\mathbf{P}$ that its columns $\mathbf{p}_1, \ldots, \mathbf{p}_N$ are the right eigenvectors associated with the eigenvalues of $\mathbf{\Phi}$. Hence

$$\mathbf{\Phi}\mathbf{p}_i = \rho_i \mathbf{p}_i \tag{2.116}$$

This selection yields

$$\begin{aligned}\mathbf{L} &= \mathbf{P}^{-1}\mathbf{\Phi}\mathbf{P} = \mathbf{P}^{-1}\mathbf{\Phi}\begin{bmatrix}\mathbf{p}_1 & \cdots & \mathbf{p}_N\end{bmatrix} \\ &= \mathbf{P}^{-1}\begin{bmatrix}\mathbf{\Phi}\mathbf{p}_1 & \cdots & \mathbf{\Phi}\mathbf{p}_N\end{bmatrix} \\ &= \mathbf{P}^{-1}\begin{bmatrix}\rho_1\mathbf{p}_1 & \cdots & \rho_N\mathbf{p}_N\end{bmatrix} \\ &= \mathbf{P}^{-1}\mathbf{P}\begin{bmatrix}\rho_1 & & \\ & \ddots & \\ & & \rho_N\end{bmatrix} = \mathbf{P}^{-1}\mathbf{P}\boldsymbol{\rho} = \boldsymbol{\rho}\end{aligned} \tag{2.117}$$

When the eigenvalues $\rho_1, \ldots, \rho_N$ of $\mathbf{\Phi}$ are distinct, the matrix $\mathbf{P}$ is formed by using the generalized eigenvectors of $\mathbf{\Phi}$ and the corresponding matrix L can be either diagonal or nondiagonal that has off-diagonal members of one. The eigenvalues ρ_i of $\mathbf{\Phi}$ are called Floquet multipliers and there is a unique set of ρ_i for the matrix $\mathbf{B}$ in Eq. (2.109). Each ρ_i indicates the local orbital convergence or divergence. From Eq. (2.115) and Eq. (2.117), it can be seen that the magnitude of perturbation $\mathbf{w}_i$ will diminish as time passes if

$$|\rho_i| < 1 \quad ; i = 1, \cdots, N \tag{2.118}$$

In short, the periodic solution is stable if the spectral radius of the monodromy matrix is less than 1. By considering Eq. (2.101) and Eq. (2.109)-(2.113), the monodromy matrix is already obtained as a by-product of the shooting method as a part of the third term in Eq. (2.101). Hence, the shooting method is considered efficient because separate stability calculation is not needed.

2.8.4 Comparison of Nonlinear Analysis Speed

In this section, the comparison of calculation time required to obtain solutions by using numerical integration and shooting method (with and without parallel computing) of the selected case is presented to show the efficiency of the method used in the thesis.

Since numerical integration can only obtain the stable solution, the rotor in "Config. 2", as shown in Fig. 2.23, is selected for the comparison as it gives stable solutions at speed above the OSI as shown by the numerical integration results in Fig. 2.26. In this comparison, the rotor model is reduced by real mode CMS with h=2. The calculation starts from the rotational speed ω=OSI+10 rpm to ω=OSI+1 rpm with 1 rpm decrement. The initial condition and convergence criteria of the limit cycle at each speed for both numerical integration and shooting method are kept identical to compare the calculation speed of each method solely. The calculation is performed on MATLAB using a laptop with a 4-Core 2.80 GHz CPU and 16 GB RAM. The total calculation time to obtain 10 stable limit cycles, shown in Fig. 2.28, is presented in Table 2.6.

Fig. 2.28 Limit cycles used for calculation speed comparison

Table 2.6 Calculation time for nonlinear analysis method to obtain the limit cycles

Method	**Integration**	**Shooting (without parallel computing)**	**Shooting (with parallel computing)**
Time (min)	1343	215	66

It can be seen from Table 2.6 that the shooting method with parallel computing uses about 31% calculation time of the shooting method without parallel computing and only about 5% of the numerical integration to obtain these 10 limit cycles. The significantly faster calculation speed is due to the fast convergence of the Newton-Raphson method utilized in the shooting method. The speed can be further increased by using parallel computing as the integration for the solution of each perturbed variable for the Jacobean matrix calculation can be done parallelly. Hence, the shooting method, especially with parallel computing, is considered numerically efficient for nonlinear rotordynamics analysis. It can also obtain the unstable limit cycle, which is essential in determining the Hopf bifurcation type but cannot be determined by numerical integration.

2.9 Hopf and Secondary Bifurcation Type Identification

The contents in this section are excerpted from Nguyen-Schäfer [114]. The bifurcation occurs when the vibration characteristic change from the steady-state equilibrium, also called the solution's trivial branch, to the periodic or quasi-periodic vibrations, or nontrivial branches, as the bifurcation parameter approaches the threshold value called the bifurcation point. Hopf bifurcation is a special kind of bifurcation that the equilibrium changes to the periodic motion at the bifurcation point. For the rotor supported by JB, this phenomenon is equivalent to the change from the rotor's equilibrium position to periodic whirling motion at $\omega = \omega_{OSI}$. For example, Hopf bifurcation starts when the real part β_2 of the forward eigenvalue λ_2 becomes zeros in Fig. 2.19. At this point, the rotor will whirl at frequency $\Omega_2(\omega_{OSI})$ in a trajectory that asymptotically moves to or away from a closed orbit called a limit cycle. The limit cycle generally has an ellipse shape and can be classified into two types based on its stability, which can be determined by the Floquet multiplier analysis explained in Section 2.8.3.4. The first one is the stable limit cycle that encircles an unstable focus. The unstable inner spiral moves away from the unstable focus and approaches the stable limit cycle, whereas the outer stable spiral moves towards the stable limit cycle, as shown in Fig. 2.29(a). The latter is the unstable limit cycle that encircles a stable focus. The inner stable spiral moves away from the unstable limit

cycle and approaches the stable focus, whereas the unstable outer spiral moves away from the unstable limit cycle, as shown in Fig. 2.29(b).

Fig. 2.29 Limit cycle: (a) stable, (b) unstable

Hopf bifurcation can be categorized into supercritical and subcritical Hopf bifurcation based on the stability of the branch of limit cycles, or nontrivial branches, connected to the bifurcation point. In the supercritical Hopf bifurcation, the stable limit cycles encircle the unstable equilibrium points, or trivial branch, starting from the bifurcation point, as shown in Fig. 2.30(a). When the rotor goes through the supercritical bifurcation, its whirling motion's amplitude gradually increases as ω is increased from the OSI. Hence, this bifurcation type is considered as the soft loss of stability because no sudden increase in vibration amplitude that can damage the journal and bearing occurs. In the subcritical Hopf bifurcation, the unstable nontrivial branches (limit cycles) encircle the stable trivial branch (equilibrium points) ending at the bifurcation point, as shown in Fig. 2.30(b). If the rotor goes through the subcritical bifurcation, its whirling motion's amplitude suddenly increases, or jumps, as ω is increased from the OSI. In addition, the rotor at equilibrium at $\omega <$ OSI can also become unstable if it is sufficiently perturbed. As a result, this bifurcation type is considered as the hard loss of stability and is more dangerous and harmful than the supercritical bifurcations. In this thesis, the bifurcation diagrams are presented in the eccentricity ratio diagram format, which shows the maximum dimensionless distance from the bearing center to the limit cycle's locus and equilibrium point at each rotational speed, as shown in Fig. 2.30(c) and (d).

The secondary bifurcation changes the stable limit cycle (Ω_1) to the quasi-periodic torus (Ω_2) at the Neimark-Sacker bifurcation point in case of the supercritical secondary bifurcation shown in Fig. 2.31(a). For the subcritical secondary bifurcation, the stable limit cycle (Ω_1) jumps to the larger stable limit cycle (Ω_2) at the Neimark-Sacker bifurcation point, and the larger limit cycle can jump down to the smaller one at another lower speed saddle-node point, as shown in Fig. 2.31(b).

Fig. 2.30 Hopf bifurcation type: (a) supercritical, (b) subcritical

Fig. 2.31 Saddle-node bifurcation type: (a) supercritical, (b) subcritical

Chapter 3

Test Rig Description

3.1 Test Rig

The test rig is shown in Fig. 3.1. It consists of a test rotor supported by a self-aligning ball bearing at the drive end (DE) and a changeable test JB at the non-drive end (NDE). The rotor is driven through a flexible coupling by a shaft connected to a variable speed motor via pulleys and a v-belt. Journal's displacements are measured by orthogonal proximity probes installed at the JB's housing at both inboard (IB) and outboard (OB) sides. Speed and phase are measured by a tachometer located near the coupling. The test rig can be operated from speed 90-4500 rpm.

Fig. 3.1 Test rig

3.2 Test Rotor

There were two rotors used for experimental verification of the flexible rotor's calculations in this thesis with their drawings are shown in Fig. 3.2. Both of them consist of a large disk and a small disk attached to the shaft through mechanical locks. A journal is shrink-fit to the shaft. The large disk's quantity and axial position (L_d) and are configurable. The parameters of the rotors are shown in Table 3.1.

Fig. 3.2 Drawings of test rotors

Table 3.1 Material properties of the rotors

Parameter	Value	Unit
Large disk quantity	1, 2	disk(s)
Disk diameter	0.18	m
Disk thickness	0.02	m
L_d of Rotor 1	0.34, 0.575	m
L_d of Rotor 2	0.41, 0.56, 0.65	m
Shaft diameter	14×10^{-3}	m
Shaft and disk density	7930	kg/m^3
Shaft's Young modulus	191.4×10^{9}	Pa

The rotor's sag occurs by the rotor's weight, which causes the angular misalignment between the journal and the bearing. The bearing housing was tilted corresponding to the calculated angular displacement of the journal in simple support condition by inserting shims under the outboard side of the bearing housing. The engineer's blue paste was applied to check the contact pattern between the journal and the bearing. The bearing housing position was fine-tuned until the contact pattern indicated good parallelism between the journal and the bearing. Balancing was performed whenever the rotor's configuration was changed. It was done by attaching weights to the large disk to make the synchronous vibration about the first critical minimum.

Fig. 3.3 Parallel rotor-bearing contact pattern

Shaft runouts of both rotors at the IB and OB planes are shown in Table 3.2. They were obtained from the AC part of the signal from proximity probes, as shown in Fig. 3.4 at the minimum speed of 90 rpm, where the effect of dynamics force is minute and negligible. Since shaft runouts are the geometrical quality of the shafts, they are invariance to configurations of rotor and bearing. The shaft runouts were used for subtracting the measured vibration at high speed to obtain the real shaft vibrations caused by dynamics forces. It can be seen that the shaft runouts of Rotor 2, which was later available, are noticeably less than Rotor 1. Thus, Rotor 1 was used in the early stage of the experiments, whereas Rotor 2 was used later. The shaft runout compensation method will be explained in Section 3.4.

Table 3.2 Measured shaft runout

Runout ($\mu m_{pk\text{-}pk}$)	IB	OB
Rotor 1	140	80
Rotor 2	30	60

Fig. 3.4 Measured shaft runout at 90 rpm: (a) Rotor 1, (b) Rotor 2

3.3 Test Bearing

All test bearings were manufactured from brass by electro-discharge machining (EDM), which can control the dimensional tolerance within 5 μm. The bearing's outer diameter is 5-10 μm larger than the housing's bore diameter, so the bearing will not rotate with the journal when clamped. The lubricant is VG-32, and its temperature was controlled by a heater in the supply tank to make the steady-state bearing's drain temperature, measured at the location shown in Fig. 3.5, at speed about 500 rpm below the calculated OSI yielded the corresponding dynamic viscosity (μ) about 40 mPa-s. According to Vogel's equation in the case of VG-32 lubricant, the viscosity-temperature relationship is given as follows [115]:

$$\mu[\text{Pa-s}] = 7.36317\times10^{-5}\exp\left(\frac{797.7122}{\left(T[°\text{C}]+273\right)-177.3562}\right) \tag{3.1}$$

Since it was difficult to control the viscosity at 40 mPa-s precisely, the viscosity was controlled within 38-42 mPa-s. According to Eq.(3.1), the tank temperature was controlled to make the bearing's drain temperature falls within 30-32 °C.

Fig. 3.5 Bearing drain temperature measurement location

3.3.1 Fixed Profile Journal Bearing

Assembly and detailed drawings of the bearing are shown in Fig. 3.6 (JB with length of 12mm (L/D=0.4) and preload of 0.25 is depicted). The lubricant is supplied to the bearing through axial inlets. Bearings' parameters are given in Table 3.3

Fig. 3.6 Assembly and detailed CAD drawings of a fixed profile journal bearing

Table 3.3 Parameters of fixed profile journal bearings

Parameter	Value	Unit
Nominal diameter	30	mm
Length	12, 18	mm
Bearing assembly radial clearance	60	μm
Preload	0, 0.25	-
Number of lobes	2	-
Pad's angular extent	140	deg

3.3.2 Tilting Pad Journal Bearing

Drawings of the bearing are shown in Fig. 3.7, and the bearing parameters are given in Table 3.4. The pads can tilt freely around their pivots and are retained to the bearing rings by the pad-pivot interfaces' interlocking shape. The pads can be locked by inserting pins in the locking pin cut-out, turning the bearing into a four-lobe fixed profile JB. The lubricant is supplied to the bearing axially through oil supply covers and bearing rings.

Fig. 3.7 CAD drawing of a TPJB (oil supply covers removed)

Table 3.4 Parameters of a TPJB

Parameter	Value	Unit
Nominal diameter	30	mm
Length	12	mm
Bearing assembly radial clearance	60	μm
Number of pads	4	-
Pad's angular extent	75	deg
Preload	0.5	-
Offset	0.5	-
Pad thickness from pivot center	6.96	mm
Distance from pad's CG to pivot center	4.58	mm
Pad mass	16.53	g
Pad mass moment of inertia around pivot	1140.76	g-mm^2

3.3.3 Flexure Pivot Tilting Pad Journal Bearing

Drawings of the bearing and pad's and reference axes are shown in Fig. 3.8, and the bearing parameters are given in Table 3.5. The pads can move due to the flexibility of pivots. The pivot's stiffnesses obtained from Solidworks are given in Table 3.6. Similar to the TPJB, the lubricant is supplied to the bearing in the same way, and the pads can be locked by inserting pins in the locking pin cut-out, making it becomes a four-lobe fixed profile JB.

Fig. 3.8 CAD drawing of a FPTPJB (oil supply covers removed) and reference axes

Table 3.5 Parameters of a FPTPJB

Parameter	Value	Unit
Nominal diameter	30	mm
Length	12	mm
Bearing assembly radial clearance	60	μm
Number of pads	4	-
Pad's angular extent	75	deg
Preload	0.5	-
Offset	0.5	-
Pad thickness from the top of pivot	7.44	mm
Distance from pad's CG to the top of pivot	4.95	mm
Pad mass	16.98	g
Pad mass moment of inertia around top of pivot	1214.29	g-mm^2
Pivot's web height	5.81	mm
Pivot's web thickness	1.5	mm
Pivot's Young modulus	100	GPa
Pivot's shear modulus	37	GPa

Table 3.6 Stiffnesses of the pivot web obtained from Solidworks

$k_{\xi\xi}$ (N/m)	$k_{\eta\eta}$ (N/m)	$k_{\delta\eta}$ (Nm/m)	$k_{\eta\delta}$ (N/rad)	$k_{\delta\delta}$ (Nm/rad)
3.714×10^{8}	6.219×10^{6}	-2.596×10^{4}	-2.596×10^{4}	73.26

3.4 Instrumentation and Signal Processing

3.4.1 Setup

Two proximity probes were rigidly attached to the bearing housing in orthogonal orientation in each of two measurement planes (IB and OB), as shown in Fig. 3.9. By assuming the rotor portion between IB and OB measurement planes be a rigid body, both lateral and angular whirling motions of the journal can be measured using two measurement planes. The signals from proximity probes and tachometer were recorded simultaneously by a multichannel data logger. Properties and settings of probes and data logger are shown in Table 3.7.

Fig. 3.9 Proximity probes locations and orientations

Table 3.7 Properties and settings of probes and data logger

Parameter	Value	Unit
Probe's scale factor	5×10^{-3}	mV/μm
Data logger's sampling rate	500	μs
Data length	4001	samples
Low-pass filter	500	Hz

3.4.2 Signal Processing

The signal (voltage) from the proximity probe consists of DC and AC parts, indicating the journal's static and dynamic position relative to the probe, respectively. The DC part was obtained from averaging the signal between the first and last tachometer trigger events. The full-length signal was then subtracted by the obtained DC part to acquire the AC part, which was then divided by the scale factor for dynamic displacement. The dynamic displacements in the same measurement plane were used to construct a complex signal. First, the lead and lag signals were identified using the order that the fixed spot on the shaft encountered the probes in 90 degrees phase difference in the shaft's rotational direction (IB-H > IB-V and OB-V > OB-H in Fig. 3.9). For each measurement plane, the lead signal (IB-H, OB-V) becomes the real part, and the lag signal (IB-V, OB-V) became the imaginary part of the complex signal, which is the same convention for forward and backward whirl direction identification

mentioned in Section 2.6. The fast Fourier transform (FFT) of this complex signal yielded the full spectrum, which was used to identify the whirling direction of the dynamic signal's frequency components. For example, Fig. 3.10 shows the full spectrum of shaft runout in Fig. 3.4 (b). It indicates that the signal mostly consisted of forward whirl, which means the whirling direction is the same as that of rotation.

Since the dimension tolerance of the bearing is within 5μm, and the contact between the journal and the bearing can be confirmed, as shown in Fig. 3.3, the signals' DC voltage of all four probes at the lowest running speed (90 rpm) were used as the bottom dead center reference values for the journal's static position. The difference in DC voltage value of this speed and that of the given speed divided by the scale factor yielded static displacement from the bottom dead center of the bearing measured by each probe. As a result, the journal's lateral displacements at both bearing ends can be obtained from the similar triangles of combined static and dynamic displacements measured from IB and OB measurement planes.

Fig. 3.10 Full spectrum of shaft runout in Fig. 3.4 (b)

The shaft runout compensation was done by curve fitting the measured shaft runout, shown in Fig. 3.4 from the first to the last tachometer trigger event using cosine equation, as shown in Fig. 3.11. Since the shaft runout is speed-independent, the vibration signal at any speed was subtracted by the runout obtained from the fitted equation of the corresponding measurement location of that speed to obtain the runout-compensated signal that shows only the pure dynamics part.

Fig. 3.11 Curve fitting of measured shaft runout in Fig. 3.4 (b)

Chapter 4

Effect of Turbulence on Bifurcation Type of Rigid Rotor Supported by Fixed Profile Journal Bearing [116]

4.1 Overview

This chapter investigated the effect of considering laminar and turbulence bearing models on bifurcation types of a rigid and axial-symmetric rotor supported by two identical fixed pad journal bearings under different bearing profiles and L/D ratios. The laminar model and two turbulence models (Ng-Pan-Elrod and Constantinescu) of the Reynolds equation were used. The OSI and the whirl frequency ratio (WFR) were calculated from the eigenanalysis to obtain the starting point for nonlinear analysis. The shooting method, arclength continuation, and Floquet multiplier analysis were then used to determine the limit cycles (LCs) and their stability, which implied the bifurcation type. The conclusions of the findings were compared to the published paper's results for verification.

The investigated parameters of rotor and bearing are presented in Section 4.2. The overview of calculation methods is given in Section 4.3. The comparison of LCs and their WFR obtained from the laminar/turbulence models of the Reynolds equation of the full circular bearings of various L/D ratios, and discussion are shown in Section 4.4. Similar comparison and discussion of the 2-lobe bearings of various L/D ratios are presented in Section 4.5. The conclusions of the results found are discussed in Section 4.6.

4.2 Investigated Parameters

Typically, the bearing profile and length are altered without changing the journal diameter to solve the rotor's instability problem in the industry application. This investigation follows the same approach by fixing the journal size and bearing load but changing the bearing profile and length. The investigation's parameters are mostly the same as those in the work of Wang and Khonsari [16]. Water is used as the lubricant as its low viscosity causes the rotor to operate in the turbulence region ($\overline{Re} > 2000$, according to Eq. (2.27)). The schematics of a rigid and axial-symmetric rotor and journal bearings are shown in Fig. 4.1. The rotor has a mass of $2m$ and is supported by two identical JBs. The investigated parameters are described in Table 4.1.

Table 4.1 Rotor and bearings parameters

Parameter	Variable	Value	Unit
Rotor mass	$2m$	4.54	kg
Bearing length	L	38-76	mm
Journal diameter	D	76	mm
Dynamic viscosity of the lubricant (water)	μ	1.38	mPa-s
Bearing radial clearance	C_{rb}	127	µm
Pad's angular extent (2-lobe bearing)	-	150	deg
Pad's preload	m	0, 0.25,0.5	-

Fig. 4.1 Schematics: (a) rotor, (b) full circular profile, (c) 2-lobe profile

4.3 Overview of Calculation Methods

Since the rotor is axial-symmetric and whirls in rigid body translation, the axes of journal and bearing are always parallel. Hence, the corresponding film thickness functions in Eq. (2.17) and (2.20) were used for the full circular bearing and 2-lobe bearing, respectively. Three models of the Reynolds equation (laminar, Ng-Pan-Elrod, and Constantinescu) were solved for the bearing force in Eq. (2.1) to determine the corresponding equilibrium position of the journal by using the procedure stated in Section 2.4. Then the stiffness and damping coefficients were calculated by the method mentioned in Section 2.5.2 and then used in the eigenanalysis, explained in Section 2.6, to calculate the onset speed of instability (OSI) and the linear whirl frequency ratio (WFR). The OSI was used as the initial point for nonlinear vibration analysis using the shooting method with arclength continuation and Floquet multiplier analysis, described in Section 2.8.3, to obtain a solution's branch and its members' stability. Then Hopf bifurcation type and secondary bifurcation type can be determined by the criteria defined in Section 2.9.

4.4 Effect of Turbulence on Bifurcation Type in Full Circular Bearing

To explain the plotting conventions used throughout this chapter, Fig. 4.2 shows plots of the LC's maximum eccentricity ratio, which is the maximum dimensionless displacement from the bearing center to the LC's locus, and the associated spectral radius of the monodromy matrix $\mathbf{\Phi}$ in the case of the circular bearing with L/D=0.5 using the laminar model. The maximum eccentricity ratio's plot indicates that two stable LCs and one unstable LC coexist in a particular speed range. The stable and unstable LCs are represented by thick solid and dashed lines, respectively. The eccentricity ratio of equilibrium position from the laminar, Ng-Pan-Elrod, and Constantinescu turbulence models, are depicted by thin solid dashed, and dashed-dotted lines, respectively. The Hopf bifurcation point is defined by the point at which the LC shrinks to the equilibrium position. Supercritical and subcritical Hopf bifurcation points are denoted by "■" and "□" symbols, respectively. The secondary (Neimark-Sacker) bifurcation point is defined by the point at which the stable LC originated from the Hopf bifurcation point start to branch off that LC. There are two points related to this bifurcation: jump-up (Neimark-Sacker), and jump-down (saddle-node) points. The jump-up point is where the smaller stable LC jumps up to the larger stable LC as the shaft rotational speed is increased; it is represented by a "△" symbol with a solid line arrow indicating the jump-up's direction. The jump-down point is where the larger stable LC jumps down to either smaller stable LC or equilibrium position as the shaft rotational speed is decreased; it is represented by the "▽" symbol with a dashed line arrow indicating the jump-down's direction.

The plots of maximum LC's eccentricity ratio of the full circular bearing with the L/D ratio of 0.5, 0.75, and 1 are presented in Fig. 4.3 to Fig. 4.5, respectively. The results indicate that the laminar model yielded a higher equilibrium position's eccentricity ratio than those of both

turbulence models for all L/D ratios. These results are due to the higher bearing load-carrying capacity of the turbulence models caused by k_θ and k_z in Eq. (2.28) act as an additional virtual viscosity. For this reason, the equilibrium position of the turbulence models is closer to the bearing center. This position is considered as a less stable point in terms of the lower OSI, as stated by Childs [42], and the tendency to give subcritical Hopf bifurcation, as stated by Wang and Khonsari [16].

Fig. 4.2(a) Eccentricity ratio of the limit cycles of laminar circular bearing with L/D=0.5, (b) Spectral radius of $\mathbf{\Phi}$

When L/D=0.5, the speed range between the Hopf bifurcation point and the jump-up point of the Ng-Pan-Elrod turbulence model was considerably shorter than those of the laminar model, as shown in Fig. 4.3(a). The bifurcation type was even changed to the subcritical for the Constantinescu turbulence model. For the longer L/D ratios of 0.75 and 1, all three models yielded subcritical Hopf bifurcation with the Hopf bifurcation points of both turbulence models that occurred at a notably lower speed, as shown in Fig. 4.4(a) and Fig. 4.5(a). The reason for the findings is due to the additional virtual viscosity mentioned earlier. These findings agreed with the conclusion that the rotor in the turbulent regime is more susceptible to a vibration jump stated by Hemmati et al. [117], which also used the same circular bearing profile.

The whirl frequency ratio (WFR: Ω/ω) for L/D=0.5, 0.75, and 1 are shown in Fig. 4.3(b) to Fig. 4.5(b), respectively. From these figures, all three models did not yield a notable difference in the value of WFR. The WFR obtained from the shooting method started to deviate from the WFR obtained from eigenanalysis, which confirms the convergence of the nonlinear and linear solutions at the bifurcation point. Both turbulence models gave the WFRs of the large stable LC slightly higher than that of the laminar model and changed slightly over the shaft rotational speed range.

Fig. 4.3(a) Limit cycle eccentricity, (b) WFR of circular bearing (L/D=0.5)

Fig. 4.4(a) Limit cycle eccentricity, (b) WFR of circular bearing (L/D=0.75)

Fig. 4.5(a) Limit cycle eccentricity, (b) WFR of circular bearing (L/D=1)

A summary of the OSI obtained from the eigenanalysis, shown by lines, Hopf bifurcation points, and jump-up/jump-down points from L/D=0.5 to 1 is presented in Fig. 4.6. Theoretically, OSI and Hopf bifurcation point should be the same point, but they are slightly different due to dissimilar numerical treatment and numerical precision. It shows that both OSI and Hopf bifurcation point, increased as the L/D ratio was increased in the range of 0.5-0.6 for the laminar model, and within the range of 0.5-0.55 for the Ng-Pan-Elrod and Constantinescu turbulence models. Nonetheless, both OSI and Hopf bifurcation point started decreasing if the L/D ratio was further increased. Also, both turbulence models yielded lower OSI and Hopf bifurcation points than those of the laminar model. In addition, the OSI from the Ng-Pan-Elrod model was lower than that of the Constantinescu model, which conforms to the findings of Hemmati et al. [117]. Furthermore, the L/D ratios that the bifurcation type changes from supercritical to subcritical of both turbulence models were lower than that of the laminar model. The decreases in both OSI and Hopf bifurcation point when the L/D ratio is higher than a certain value happened because the longer L/D ratio raises the load-carrying capacity, resulting in the lower rotor's operating eccentricity that causes more destabilization force. For all models, the speed range between the jump-up point (or Hopf bifurcation point in the case of subcritical bifurcation) and jump-down point increased as the L/D ratio increased from 0.5 to 0.8 and remained constant at higher L/D ratios. Additionally, the Ng-Pan-Elrod model gave the narrowest speed range.

From the results of the full circular bearing, it can be concluded that the Hopf bifurcation point of both turbulence models is lower than that of the laminar model and the Hopf bifurcation point of the Ng-Pan-Elrod model is lower than that of the Constantinescu model. Also, the L/D ratios that the bifurcation type changes from supercritical to subcritical of both

turbulence models are lower than that of the laminar model. Besides, in the subcritical bifurcation cases, the speed range between the bifurcation point and the jump-down point of the laminar and Constantinescu models are similar at the same L/D ratio, whereas that of the Ng-Pan-Elrod model is the narrowest.

Fig. 4.6 Bifurcation point and saddle-node point (jump-up/down) of circular bearing

4.5 Effect of Turbulence on Bifurcation Type in 2-Lobe Bearing

4.5.1 Preload = 0 (Axial Groove bearing)

The plots of maximum LC's eccentricity ratio of the 2-lobe bearing (preload=0) with the L/D ratio of 0.5, 0.75, and 1 are presented in Fig. 4.7(a) to Fig. 4.9(a), respectively. These plots indicate that the laminar model also yielded a higher equilibrium position's eccentricity ratio than that of each turbulence model in all L/D ratios, which was also observed in the full circular bearing. At the L/D ratio of 0.5, only supercritical Hopf bifurcation (■) without secondary bifurcation (jump-up/down point) was observed in all three models with the laminar model gave the lower LC's growth rate than both turbulence models. When L/D = 0.75, each turbulence model gave the subcritical bifurcation while that of the laminar model remained supercritical. Lastly, when L/D = 1, all models yielded the subcritical bifurcation, but the speed range between the jump-down point (▽) and bifurcation point (□) of each turbulence model was more expansive than that of the laminar model, indicating that the rotor can go unstable at speed notably lower than the OSI if it is sufficiently perturbed. The lower equilibrium position's eccentricity ratio and changed the bifurcation type in this bearing configuration were caused by the additional virtual viscosity in the turbulence models.

The LC's WFRs of L/D=0.5, 0.75, and 1 are shown in Fig. 4.7(b) to Fig. 4.9(b), respectively. There was no notable difference in the values of the WFRs obtained from all models.

The summary plot of the OSI, Hopf bifurcation point and jump-down point from L/D=0.5 to 1 is presented in Fig. 4.10. The OSI of this bearing type was lower than that of the full circular bearing, and it decreased almost linearly with increasing L/D ratio. These differences were caused by different boundary conditions in the film pressure calculation between both bearing types. However, similar to what observed in the full circular bearing, both turbulence models yielded the lower OSI and Hopf bifurcation points than the laminar model. Furthermore, the L/D ratio that Hopf bifurcation changed to subcritical of each turbulence model was lower than that of the laminar model. It was also observed that the speed range between the jump-down point (▽) and the subcritical bifurcation point (□) increased with the L/D ratio, but this speed range was much narrower than that of the full circular bearing as a result of different boundary conditions.

Fig. 4.7(a) Limit cycle eccentricity, (b) WFR of 2-lobe bearing (L/D=0.5, preload=0)

Fig. 4.8(a) Limit cycle eccentricity, (b) WFR of 2-lobe bearing (L/D=0.75, preload=0)

Fig. 4.9(a) Limit cycle eccentricity, (b) WFR of 2-lobe bearing (L/D=1, preload=0)

Fig. 4.10 Bifurcation point and jump-up/down points of 2-lobe bearing (preload=0)

4.5.2 Preload = 0.25

Since the pad's center and bearing center are not the same points, the converging fluid film of the upper pad exists even when the eccentricity ratio is zero, resulting in a downward force that tends to stabilize the rotor. The plots of maximum LC's eccentricity ratio of the 2-lobe bearing (preload = 0.25) with the L/D ratio of 0.5, 0.75, and 1 are presented in Fig. 4.11(a) to Fig. 4.13(a), respectively. It can be seen from these figures that only supercritical bifurcation was observed. The equilibrium position's eccentricity ratio of the laminar model was higher than that of both turbulence models due to the additional viscosity mentioned earlier. However, in contrast to the non-preloaded bearings mentioned earlier, the difference in eccentricity ratio between the laminar and turbulence models became smaller as the L/D ratio increased due to

the increasing downward fluid film force from the upper pad. In addition, each turbulence model gave higher OSI than that of the laminar model, which is a reversal of the trend noted in the results for the non-preloaded bearings. This reversal was caused by the downward force due to the combination of bearing preload and additional viscosity, which help to stabilize the rotor. When L/D = 0.5, each turbulence model gave steeper LC's growth rate than that of the laminar model. However, all models yielded similar LC's growth rates when L/D = 0.75. At L/D = 1, both turbulence models yielded considerably higher OSI and more gradual LC's growth rates when L/D = 1 due to the increasing downward force that suppressed the LC's amplitude.

The LC's WFRs of L/D = 0.5, 0.75, and 1 are shown in Fig. 4.11(b) to Fig. 4.13 (b), respectively. The characteristics of WFRs were similar to those observed in the supercritical bifurcation case of the 2-lobe bearing (preload=0). However, as the L/D ratio increased, WFRs from both shooting method and eigenanalysis of both turbulence models were higher than WFR of the laminar model.

The summary plot of the OSI and Hopf bifurcation point from L/D = 0.5 to 1 is presented in Fig. 4.14. Only supercritical bifurcation (■) was observed for all L/D ratios and laminar/turbulence models in this bearing type. In contrast to the non-preloaded bearings mentioned earlier, the OSI of turbulence models was higher than that of the laminar model in most of the L/D ratios. Besides, when L/D > 0.7, the OSI and bifurcation point of turbulence models increased sharply with the L/D ratio, whereas those of the laminar model were almost constant. The noted results were caused by the combination of bearing preload and additional viscosity that generated stabilizing force on the rotor.

Fig. 4.11(a) Limit cycle eccentricity, (b) WFR of 2-lobe bearing (L/D=0.5, preload=0.25)

Fig. 4.12(a) Limit cycle eccentricity, (b) WFR of 2-lobe bearing (L/D=0.75, preload=0.25)

Fig. 4.13(a) Limit cycle eccentricity, (b) WFR of 2-lobe bearing (L/D=1, preload=0.25)

Fig. 4.14 Bifurcation point and jump-up/down points of 2-lobe bearing (preload=0.25)

4.5.3 Preload = 0.5

In this bearing configuration, similar to preload = 0.25, the upper pad exerts a downward force due to the converging oil film even when the eccentricity ratio is zero. The plots of maximum LC's eccentricity ratio of the 2-lobe (preload=0.5) bearing with the L/D ratio of 0.5, 0.75, and 1 are presented in Fig. 4.15(a) to Fig. 4.17(a), respectively. Similar to preload = 0.25, The equilibrium position's eccentricity ratio of the laminar model was higher than that of both turbulence models, but the difference became smaller as the L/D ratio increased due to the increasing downward fluid film force from the upper pad. In the case of L/D=0.5, the OSI and bifurcation point of each laminar model were lower than those of the laminar model. The result became opposite in the case of L/D=0.75 with the laminar model's growth rate was slightly steeper. Lastly, the OSI and bifurcation point when L/D=1 of the turbulence models were much higher than those of the laminar model.

The LC's WFRs of L/D=0.5, 0.75, and 1 are shown in Fig. 4.15(b) to Fig. 4.17(b), respectively. The differences in WFRs of the turbulence models and the laminar model were higher than those of the bearing with preload=0.25 with the same L/D ratio. The higher preload causes the additional fluid force from the preload of all models to overcome the one from the additional viscosity of the turbulence model, resulting in the lower difference.

The summary plot of the OSI and Hopf bifurcation point from L/D=0.5 to 1 is presented in Fig. 4.18. Similar to the case of preload = 0.25, only supercritical bifurcation (■) was observed for all L/D ratios regardless of laminar/turbulence models in this bearing type. The OSI and Hopf bifurcation point of both turbulence models were higher than those of the laminar model when L/D > 0.75 and increased sharply as the L/D ratio was further increased while those of the laminar model decreased with the reducing rate as the L/D ratio was increased. This plot indicated that the effects of downward forced from both preload and additional viscosity helped to improve the rotor stability.

Fig. 4.15(a) Limit cycle eccentricity, (b) WFR of 2-lobe bearing (L/D=0.5, preload=0.5)

Fig. 4.16(a) Limit cycle eccentricity, (b) WFR of 2-lobe bearing (L/D=0.75, preload=0.5)

Fig. 4.17(a) Limit cycle eccentricity, (b) WFR of 2-lobe bearing (L/D=1, preload=0.5)

Fig. 4.18 Bifurcation point and jump-up/down points of 2-lobe bearing (preload=0.5)

From the 2-lobe bearing's results, it can be concluded that Hopf bifurcation point of each turbulence model is lower than those of the laminar model in all of L/D ratios for the bearings without preload. Also, L/D ratios that the bifurcation type changes from supercritical to subcritical of both turbulence models are lower than that of the laminar model. However, for the preloaded bearings, only supercritical bifurcation is observed, and Hopf bifurcation point of turbulence models becomes higher than that of the laminar model when L/D is higher than a specific value.

4.6 Conclusion

The effects of turbulence on nonlinear vibration (Hopf bifurcation type, jump-up/down point, and whirl frequency ratio (WFR)) of a rigid rotor supported by two identical bearings were studied by using a combination of shooting method, arclength continuation, and Floquet multiplier analysis. In this investigation, the bearing profile and length were altered without changing the journal diameter to reflect the practical approach in solving the rotor's instability problem. The bearings profiles used in the study were full circular and 2-lobe with preload of 0, 0.25, and 0.5. The bearing's length-to-diameter (L/D) ratios were varied from 0.5 to 1. Constantinescu and Ng-Pan-Elrod turbulence models were employed to compare the results with those obtained from the laminar model.

For the full circular bearing and 2-lobe bearing without preload, both the onset speed of instability (OSI) and Hopf bifurcation point of each turbulence model were lower than those of the laminar model regardless of L/D ratio. In addition, the L/D ratio that the Hopf bifurcation changed from supercritical to subcritical of each turbulence model was lower. However, if preload was increased to 0.25, all three models gave only supercritical Hopf bifurcation in all L/D ratios. Besides, the OSI and bifurcation point of the turbulence model became higher than those of the laminar model when L/D>0.7. Similarly, only supercritical Hopf bifurcation was observed regardless of laminar/turbulence models and L/D ratios as the preload was increased to 0.5. In this case, the OSI and bifurcation point increased sharply when L/D was increased from 0.75. For all bearing types, the WFRs of the limit cycle of both turbulence models were higher than that of the laminar one, especially for the bearings with high preload and L/D ratio. The discrepancies between the results of preloaded and non-preloaded bearings were caused by the downward force from the upper pad, which became more pronounced as the preload was increased and further reinforced by the additional viscosity of the turbulence models. Lastly, the OSI and bifurcation point obtained from the Ng-Pan-Elrod turbulence model was always than that of the Constantinescu model.

The findings suggested that the journal bearings operated in the turbulent regime should have a high pad preload and L/D ratio to obtain the higher OSI and supercritical Hopf bifurcation with a more gradual limit cycle's growth. The Ng-Pan-Elrod model was considered more conservative than the Constantinescu model in predicting the OSI in all investigated bearing configurations. Hence, this

Chapter 5

Nonlinear Vibration Analysis of a Flexible Rotor Supported by a Fixed Profile Journal Bearing [85, 118]

5.1 Overview

This chapter examined the nonlinear vibration of a flexible rotor supported by a fixed profile journal bearing under the various rotor and bearing parameters and the bearing film thickness model of the Reynolds equation that considers and does not consider the angular whirling motion of the journal, which can be notable in the case of the highly flexible rotor. This chapter was divided into two parts. The first part (Section 5.2) investigated the necessity of considering the journal's angular whirling motion to obtain the calculated Hopf bifurcation type that agreed with the experimental results [85]. The second part (Section 5.3) studied the applicable range of the bearing model that considers and does not consider the journal's angular whirling motion under the various rotor and bearing parameters, which were then verified by the experiments [118].

5.2 Effects of Journal Angular Motion Consideration on Hopf Bifurcation Type: Preliminary Investigation [85]

In this section, the bearing model that considers the journal's angular whirling motion (Model A) and the model that does not (Model B) were used to determine the nonlinear vibration (limit cycle's amplitude and bifurcation type) in the vicinity of the OSI. The obtained solutions of both models were then compared with the experimental results to investigate the importance of considering the journal's angular whirling motion.

5.2.1 Investigated Parameters

The flexible rotor, in which the drawing is shown in Fig. 5.1, was used for both calculation and experimental verification in this section. The axial position (L_d) of the large disk was configurable. The rotor parameters were the same as those of "Rotor 1" shown in Table 3.1 with additional bearing parameters shown in Table 5.1. The simple bearing was a self-aligning ball bearing and assumed to have isotropic radial stiffness, estimated using the formula stated by Harris [119]. Due to the oil's high viscosity, the rotor operated in the laminar region, and only the laminar model was considered.

Table 5.1 Rotor and bearings parameters

Parameter	Value	Unit
Large disk position from the simple bearing	0.41, 0.56, 0.65	m
Simple bearing's direct stiffness	1.34×10^{7}	N/m
JB's length	18×10^{-3}	m
Journal diameter	30×10^{-3}	m
JB radial clearance	60×10^{-6}	m
Pad preload	0	-
Pad number	2	pads
Pad angular extent	140	deg
Lubricant nominal viscosity	40×10^{-3}	Pa-s

Fig. 5.1 Drawings of test rotors

5.2.2 Investigation Method

The finite element model of the rotor was constructed using the method explained in Section 2.2.2. Then, the deflection caused by the rotor's weight was calculated, and the results are shown in Table 5.2. The bearing housing was tilted accordingly to make the axes of the bearing and journal parallel, which make the film thickness of Model A and B comparable.

Table 5.2 Journal's angular deflection and large disk's vertical deflection

Deflection / L_d (m)	**0.41**	**0.56**	**0.65**
Journal's angular (deg)	-0.30	-0.29	-0.24
Large disk's vertical (mm)	-1.43	-1.05	-0.60

Bearing Model A and B were used to determine equilibrium position, dynamic coefficients, and eigenvalue of the full rotor model. The modified Guyan Reduction, described in Section 2.7.1, was employed to reduce the rotor model, described in Section 2.4.1, for the nonlinear analysis. The eigenanalysis of the reduced model was done to check the reduction accuracy. The numerical integration of the reduced rotor model was initiated at speed 10 rpm above the OSI, or threshold speed (ω_{th}), of the reduced model. The shooting method was applied if the stable limit cycle was obtained without contact between the journal and the bearing to determine the solution at the successive speed. If not, the solution was determined using numerical integration that uses the integration result at 500 shaft revolutions before the contact occurred in the preceding speed as the initial condition. The integration was performed until either the journal touched the bearing or returned to the equilibrium position.

The experiments were conducted to verify the calculation results using the test rotor and bearing described in Chapter 3. The test in each case was done five times to ensure repeatability.

5.2.3 Calculation Results

The comparison plots of the journal's equilibrium position between bearing Model A and B of each L_d are shown in Fig. 5.2 to Fig. 5.4. Both models gave identical equilibrium positions.

The first forward and backward eigenvalues of bearing Model A and B using the full rotor model in each L_d are shown in Fig. 5.5 to Fig. 5.7. The real part of the forward mode, which indicates linear stability, of Model A was lower than that of Model B from low speed to OSI and then became higher at speed higher than OSI. The imaginary part, or the whirl frequency (Ω), of the forward mode of Model A was always higher than that of Model B in the entire speed range. The imaginary part of each model and L_d was then used for model reduction for the corresponding case. The summary of the OSI obtained from Model A and B in each L_d of full and reduced rotor model is presented in Table 5.3. The maximum difference in the OSI

obtained from Model A and B in either full or reduced rotor model was about 80 rpm. Moreover, the difference caused by model reduction increased with L_d with the maximum value of about 30 rpm. This difference is caused by the negligence of the gyroscopic effect in the reduction.

Fig. 5.2 Journal's equilibrium position of Model A and B: L_d = 0.41 m

Fig. 5.3 Journal's equilibrium position of Model A and B:

Fig. 5.4 Journal's equilibrium position of Model A and B: L_d = 0.65 m

Table 5.3 Summary of the first critical speed and OSI obtained from Model A and B

Parameter / L_d (m)	**Full order model**			**Reduced order model**		
	0.41	**0.56**	**0.65**	**0.41**	**0.56**	**0.65**
Model A's first critical speed (rpm)	800	928	1194	800	928	1197
Model B's first critical speed (rpm)	799	922	1188	799	923	1191
Model A's OSI (rpm)	1700	1871	2365	1700	1879	2395
Model B's OSI (rpm)	1722	1792	2335	1722	1800	2365

Fig. 5.5 The first forward and backward eigenvalues of Model A and B: L_d = 0.41 m

Fig. 5.6 The first forward and backward eigenvalues of Model A and B: L_d = 0.56 m

Fig. 5.7 The first forward and backward eigenvalues of Model A and B: L_d = 0.65 m

The results from numerical integration using Model A and B at speed 10 rpm above the OSI of the reduced model in each L_d are shown in Fig. 5.8 to Fig. 5.10. The results obtained from Model A indicated contact between the journal and the bearing in all L_d values. However, The results obtained from Model B indicated the contact only when L_d = 0.41 m, whereas the stable limit cycle without contact was observed in the other L_d values. Note that contact always occurred at the inboard (IB) side. Hence, only the eccentricity ratio of the IB side is presented to show the limit cycle's amplitude of the speed range about the OSI (ω_{th}) of each L_d and bearing model case, as depicted in Fig. 5.11 to Fig. 5.13 of each L_d value.

Fig. 5.8 Integration results at OSI+10 rpm: L_d = 0.41 m

Fig. 5.9 Integration results at OSI+10 rpm: L_d = 0.56 m

Fig. 5.10 Integration results at OSI+10 rpm: L_d = 0.65 m

Fig. 5.11 Limit cycle amplitude of model A and B vs speed: L_d = 0.41 m

Fig. 5.12 Limit cycle amplitude of model A and B vs speed: L_d = 0.56 m

Fig. 5.13 Limit cycle amplitude of model A and B vs speed: L_d = 0.65 m

In Fig. 5.11 to Fig. 5.13, the results obtained from Model A showed that the rotor at the equilibrium became unstable, and its whirling amplitude jumped to bearing clearance when the OSI (ω_{th}) was exceeded. In addition, the speed at which the large amplitude whirl returned to the equilibrium was lower than the OSI. These sudden jump and hysteresis implied that Model A yielded subcritical Hopf bifurcation in all L_d values. The result obtained from Model B at L_d = 0.41 m also indicated subcritical Hopf bifurcation. Nonetheless, the results at other L_d values showed a gradual increase in the whirling amplitude if the speed was higher than the OSI, and there was no hysteresis when the speed was decreased, which indicated supercritical Hopf

bifurcation. Hence, it can be seen that the consideration of the angular whirling motion in JB's force calculation can result in different Hopf bifurcation types.

5.2.4 Experimental Results

Comparison between the calculated shaft centerline plot and the measured one calculated from signals of orthogonal inboard and outboard probes, depicted in Fig. 3.9, of each L_d is presented in Fig. 5.14 to Fig. 5.16. The results indicated reasonably good agreement in the journal's equilibrium position obtained from the calculation and the experiment.

Fig. 5.14 Calculated and measured shaft centerline: L_d = 0.41 m

Fig. 5.15 Calculated and measured shaft centerline: L_d = 0.56 m

Fig. 5.16 Calculated and measured shaft centerline: L_d = 0.65 m

The measured signals were subtracted by the premeasured shaft runout to make runout-compensated signals by the method described in Section 3.4.2. The full cascade plot at the inboard side of each L_d is shown in Fig. 5.17 to Fig. 5.19. All figures indicate that when the rotor became unstable, the subsynchronous vibration locked up to the first critical speed with its amplitude suddenly increased. The measured first critical speed and OSI, both averaged from five tests, are depicted in Table 5.4. These values were agreeable with the calculation results shown in Table 5.3.

Fig. 5.17 Compensated full cascade plot at the inboard side: L_d = 0.41 m

Fig. 5.18 Compensated full cascade plot at the inboard side: L_d = 0.56 m

Fig. 5.19 Compensated full cascade plot at the inboard side: L_d = 0.65 m

Table 5.4 Experimented first critical speed and OSI

Parameter / L_d (m)	**0.41**	**0.56**	**0.65**
First critical speed (rpm)	810	940	1180
OSI (rpm)	1718	1998	2442

The compensated DC shaft orbit and full spectrum at the speeds before and after the rotor became unstable (the rotational speeds are shown in the legend) of each L_d are presented in Fig. 5.20 to Fig. 5.22. The sudden jump was caused by the subsynchronous vibration, and the contact was detected at the inboard side, which agreed with the calculation result of Model A.

Fig. 5.20 Compensated DC orbit and full spectrum: L_d = 0.41 m

Fig. 5.21 Compensated DC orbit and full spectrum: L_d = 0.56 m

Fig. 5.22 Compensated DC orbit and full spectrum: L_d = 0.65 m

5.2.5 Summary

This section investigated the necessity of considering the journal's angular whirling motion in fluid film force calculation to predict the Hopf bifurcation type that conformed with the experimental results. Three positions of the large disk were chosen, ranging from the midspan one to the one that was closed to the journal bearing. The rotor model was reduced by the modified Guyan reduction using the first forward mode's imaginary part. The calculation results showed no difference in the equilibrium position obtained from the model considered the journal's angular whirling motion (Model A) and the model that did not (Model B). Also, the first critical speed and the onset speed of instability (OSI) from each model were close. However, the Hopf bifurcation type from each model was different in two large disk positions. The experimental verifications showed that Model A was able to predict the Hopf bifurcation type in all cases correctly. Hence, considering the journal's angular whirling motion is crucial in nonlinear rotordynamics prediction of the highly flexible rotor in which the notable journal's angular whirling motion is likely. This section showed that modified Guyan reduction gave OSI close to that of the full model in the rotor with a small gyroscopic effect. However, the accuracy decreased with increased gyroscopic effect. In addition, the unstable limit cycle, which indicates the stable operation boundary of the rotor at speed below OSI, is not obtained. Hence, the combination of the reduction method that can better cope with the gyroscopic effect and the calculation method that can obtain the unstable limit cycle is advantageous.

5.3 Effects of Rotor-Bearing Parameters on Hopf Bifurcation Type: Essential Range of Considering Journal Angular Motion [118]

The previous section emphasized the importance of considering the flexible rotor journal's angular whirling motion to correctly determine the Hopf bifurcation type. However, several aspects could be further improved. Firstly, the rotor model's reduction accuracy could be improved by using real mode component mode synthesis (CMS). Secondly, the unstable limit cycle, which encircles the rotor's stable operating region, was needed to be determined. Lastly, a broader range of rotor and bearing parameters were needed to be investigated to see the applicable range of Model A and B. Model B is still preferred if both models yield the same bifurcation type because it requires about half of Model A's calculation time due to the axial symmetry property.

5.3.1 Investigated Parameters

The flexible rotor, shown in Fig. 5.23, which was slightly shorter than the rotor used in the previous section and not available during the investigation of the previous section, was used in this section because it has lower shaft runouts, as shown in Table 3.2. The large disk was configurable in both position and mass. The bearing's L/D ratio and preload were selectable. The parameters in this investigation that were different from those of the previous section are presented in Table 5.5.

Table 5.5 Rotor and bearings parameters

Parameter	Value	Unit
Large disk position from the simple bearing	0.340, 0.575	m
Large disk mass	4, 8	kg
JB's length	12×10^{-3}, 18×10^{-3}	m
JB's preload	0, 0.25	-

Fig. 5.23 Drawings of test rotors

5.3.2 Investigation Method

The investigation method was almost the same as the previous section. However, the Guyan reduction was replaced by CMS (h=2), explained in Section 2.7.2, to improve the rotor's model reduction accuracy. Also, the unstable limit cycle was searched by the method described in Section 2.8.3.2 to determine the stable operating region at speed below the OSI. The static deflections of the rotor, determined by using the same method as the previous section, are shown in Table 5.6, and the bearing housing was tilted accordingly to make the axes of the bearing and journal parallel. The experiments were conducted to verify the calculation results. The test in each case was done five times to ensure repeatability.

Table 5.6 Journal's angular deflection and large disk's vertical deflection

Large disk mass	**4**		**8**	
L_d (m)	**0.340**	**0.575**	**0.340**	**0.575**
Load on JB (N)	27.43	40.79	46.79	73.52
Journal's angular (deg)	-0.22	-0.16	-0.41	-0.29
Large disk's vertical (mm)	-0.89	-0.30	-1.65	-0.54

5.3.3 Calculation Results

In this section, the journal's equilibrium position plots were omitted because the main focus was the rotor's dynamic behavior, and the journal's equilibrium position of Model A and B was identical, as shown in the previous section. The comparison of the first forward eigenvalue that went unstable of the L/D ratio of 0.4 and 0.6 obtained from the full and reduced rotor model in each set of disk mass, position, and preload was presented in Fig. 5.24 to Fig. 5.31. In these figures, the full rotor model's values were depicted by the solid line, whereas the dashed line represented the reduced model's values. From these figures, the summary of the first critical speed and OSI obtained from Model A and B in each rotor-bearing configuration of the full and reduced rotor model are presented in Table 5.7 and Table 5.8, respectively. It can be seen from the figures that the real part of the full and reduced model was almost identical in most cases and the maximum discrepancy was observed when L_d = 0.575 m. In addition, the maximum discrepancy in the imaginary part also occurred in this L_d, where the reduced model gave a slightly higher imaginary part. Table 5.7 and Table 5.8 show that the maximum difference in the first critical speed and OSI was 7 and 28 rpm, respectively, indicating good reduction accuracy of the CMS even when h = 2. According to these tables, the preload raised the OSI in both L/D ratios when L_d = 0.34 m. Nonetheless, it gave the opposite results in most configurations when L_d = 0.575 m except when disk mass was 4 kg and L/D=0.6.

Fig. 5.24 Forward and backward eigenvalues: disk mass 4kg, preload 0, L_d 0.34m

Fig. 5.25 Forward and backward eigenvalues: disk mass 4kg, preload 0, L_d 0.575m

Fig. 5.26 Forward and backward eigenvalues: disk mass 4kg, preload 0.25, L_d 0.34m

Fig. 5.27 Forward and backward eigenvalues: disk mass 4kg, preload 0.25, L_d 0.575m

Fig. 5.28 Forward and backward eigenvalues: disk mass 8kg, preload 0, L_d 0.34m

Fig. 5.29 Forward and backward eigenvalues: disk mass 8kg, preload 0, L_d 0.575m

Fig. 5.30 Forward and backward eigenvalues: disk mass 8kg, preload 0.25, L_d 0.34m

Fig. 5.31 Forward and backward eigenvalues: disk mass 8kg, preload 0.25, L_d 0.575m

Table 5.7 First critical speed and OSI of the full rotor model

	Disk mass 4 kg, Preload 0							
Model	**A**				**B**			
L/D	**0.4**		**0.6**		**0.4**		**0.6**	
L_d **(m)**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	1000	1628	1005	1662	1000	1630	1004	1656
OSI (rpm)	2246	3374	2099	3478	2259	3366	2107	3469
	Disk mass 4 kg, Preload 0.25							
Model	**A**				**B**			
L/D	**0.4**		**0.6**		**0.4**		**0.6**	
L_d **(m)**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	995	1608	1001	1641	994	1604	1000	1636
OSI (rpm)	2418	3139	2271	3734	2420	3125	2280	3735
	Disk mass 8 kg, Preload 0							
	A				**B**			
	0.4		**0.6**		**0.4**		**0.6**	
$\hat{L}_d$ **(m)**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	734	1271	736	1281	733	1263	734	1270
OSI (rpm)	1985	2550	1535	2682	1989	2517	1561	2651
	Disk mass 8 kg, Preload 0.25							
Model	**A**				**B**			
L/D	**0.4**		**0.6**		**0.4**		**0.6**	
L_d **(m)**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	731	1261	733	1264	731	1252	732	1253
OSI (rpm)	2578	2315	1645	2564	2551	2265	1644	2515

Among 16 cases investigated, all cases indicated that the contact between the journal and the bearing occurred on the bearing's inboard side. Hence, only the comparison of the limit cycle obtained from Model A and B at the inboard side is shown in Fig. 5.32 to Fig. 5.39. The solid line, dashed line, and box in these figures represent the stable solution, unstable solution, and the reduced model's OSI, respectively. In all cases of L_d = 0.34m, both models gave subcritical Hopf bifurcation. On the other hand, both models gave supercritical bifurcation when L_d = 0.575m and L/D=0.4 except when the disk mass was 8 kg that subcritical bifurcation occurred, as shown in Fig. 5.38. There were three cases when L_d = 0.575m and L/D=0.6 that Model A and B gave different bifurcation types, as depicted in Fig. 5.34, Fig. 5.35, and Fig. 5.39. In these cases, Model A yielded subcritical bifurcation, whereas Model B yielded supercritical bifurcation. It can be summarized from these results that the discrepancy in bifurcation type occurred when the L/D ratio was long and the main mass of the rotor was located away from the rotor's midspan. Furthermore, decreasing the L/D ratio, increasing preload, and moving the main mass closer to the JB (increase JB's load) inclined to give supercritical bifurcation.

Table 5.8 First critical speed and OSI of the reduced rotor model

	Disk mass 4 kg, Preload 0							
Model	**A**				**B**			
L/D	**0.4**		**0.6**		**0.4**		**0.6**	
L_d (m)	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	1000	1634	1005	1669	1000	1636	1004	1662
OSI (rpm)	2245	3382	2098	3503	2258	3372	2107	3494
	Disk mass 4 kg, Preload 0.25							
Model	**A**				**B**			
L/D	**0.4**		**0.6**		**0.4**		**0.6**	
L_d (m)	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	995	1614	1001	1648	995	1610	1007	1642
OSI (rpm)	2414	3151	2270	3760	2415	3138	2279	3760
	Disk mass 8 kg, Preload 0							
	A				**B**			
	0.4		**0.6**		**0.4**		**0.6**	
$\hat{L}_d$ (m)	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	734	1277	736	1287	733	1269	734	1275
OSI (rpm)	1977	2560	1535	2708	1979	2527	1560	2675
	Disk mass 8 kg, Preload 0.25							
Model	**A**				**B**			
L/D	**0.4**		**0.6**		**0.4**		**0.6**	
L_d (m)	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	731	1266	733	1270	731	1258	732	1258
OSI (rpm)	2552	2323	1643	2591	2524	2272	1641	2542

Fig. 5.32 LC amplitude: disk mass 4kg, L_d 0.34m, preload 0

Fig. 5.33 LC amplitude: disk mass 4kg, L_d 0.34m, preload 0.25

Fig. 5.34 LC amplitude: disk mass 4kg, L_d 0.575m, preload 0

Fig. 5.35 LC amplitude: disk mass 4kg, L_d 0.575m, preload 0.25

Fig. 5.36 LC amplitude: disk mass 8kg, L_d 0.34m, preload 0

Fig. 5.37 LC amplitude: disk mass 8kg, L_d 0.34m, preload 0.25

Fig. 5.38 LC amplitude: disk mass 8kg, L_d 0.575m, preload 0

Fig. 5.39 LC amplitude: disk mass 8kg, L_d 0.575m, preload 0.25

5.3.4 Experimental Results

As observed in the previous section that the rotor whirled at the frequency coincided with its first critical speed when it became unstable. Hence, the amplitude at this frequency band, obtained by bandpass filtering the runout compensated signals, can indicate the first critical speed, bifurcation point (or OSI), and bifurcation type. The amplitudes at the inboard side are shown in Fig. 5.40 to Fig. 5.47. In these figures, the speed at which the brief peak occurred indicates the first critical speed, while the speed at which the amplitude started to increase monotonically indicates the bifurcation point or OSI. The bifurcation type could be determined from the slope of the increasing amplitude versus speed. If the slope was infinite (vertical line), the bifurcation type was subcritical and vice versa. The summary table of measured first critical speed, bifurcation point and type are presented in Table 5.9. The experimental results agreed

with the prediction in terms of the first critical speed and OSI. The observed bifurcation types conformed with those predicted by Model A in all cases, whereas those predicted by Model B agreed with the experimental results in most cases except the three cases of L_d = 0.575m and L/D=0.6 mentioned earlier.

Table 5.9 Measured first critical speed, OSI and bifurcation type

Disk mass 4 kg, Preload 0				
L/D	**0.4**		**0.6**	
L_d (m)	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	970	1610	1010	1630
OSI (rpm)	2283	3416	2090	3487
Bifurcation type	subcritical	supercritical	subcritical	subcritical
Disk mass 4 kg, Preload 0.25				
L/D	**0.4**		**0.6**	
L_d (m)	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	1004	1623	1008	1639
OSI (rpm)	2437	3157	2154	3612
Bifurcation type	subcritical	supercritical	subcritical	subcritical
Disk mass 8 kg, Preload 0				
L/D	**0.4**		**0.6**	
L_d (m)	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	741	1230	743	1256
OSI (rpm)	1900	2542	1580	2674
Bifurcation type	subcritical	subcritical	subcritical	subcritical
Disk mass 8 kg, Preload 0.25				
L/D	**0.4**		**0.6**	
L_d (m)	**0.340**	**0.575**	**0.340**	**0.575**
1st critical (rpm)	757	1217	744	1223
OSI (rpm)	2441	2250	1591	2468
Bifurcation type	subcritical	supercritical	subcritical	subcritical

Fig. 5.40 Amplitude of the frequency coincided with the first critical speed: disk mass 4kg, L_d 0.34m, preload 0

Fig. 5.41 Amplitude of the frequency coincided with the first critical speed: disk mass 4kg, L_d 0.575m, preload 0

Fig. 5.42 Amplitude of the frequency coincided with the first critical speed: disk mass 4kg, L_d 0.34m, preload 0.25

Fig. 5.43 Amplitude of the frequency coincided with the first critical speed: disk mass 4kg, L_d 0.575m, preload 0.25

Fig. 5.44 Amplitude of the frequency coincided with the first critical speed: disk mass 8kg, L_d 0.34m, preload 0

Fig. 5.45 Amplitude of the frequency coincided with the first critical speed: disk mass 8kg, L_d 0.575m, preload 0

Fig. 5.46 Amplitude of the frequency coincided with the first critical speed: disk mass 8kg, L_d 0.34m, preload 0.25

Fig. 5.47 Amplitude of the frequency coincided with the first critical speed: disk mass 8kg, L_d 0.575m, preload 0.25

5.3.5 Summary

The effects of the rotor and bearing parameter on the bifurcation type were investigated to determine the applicable range of the bearing models of the previous section. The rotor model was reduced by the real mode CMS, and shooting was used to determine stable and unstable limit cycles. Similar to the previous section, both bearing models gave close first critical speed and OSI. Also, there were cases that both models gave different bifurcation types. The test rig was set up with identical parameters as used in the calculation. The experimental results indicated that Model A was able to correctly predict the bifurcation type in all cases, whereas Model B, which requires only half the calculation time of Model A, could predict correctly in most cases except the three cases of L_d = 0.575m and L/D=0.6. Both numerical and experimental results indicated that the discrepancy in bifurcation type occurred when the L/D ratio was long and the rotor's main mass was located away from the rotor's midspan. Besides, decreasing the L/D ratio, increasing preload, and moving the main mass closer to the JB (increase JB's load) inclined to give supercritical bifurcation.

5.4 Conclusion

This chapter investigated the importance of considering the journal's angular whirling motion to predict the bifurcation type correctly and the applicable range of the bearing models that consider and not consider this motion. Also, the effect of rotor and bearing parameters on the bifurcation type were studied. The calculation results were compared with the experimental results, and it can be concluded that considering the journal's angular whirling motion was essential in the flexible rotor that has the main mass away from the midspan supported by the journal bearing with high L/D ratio. Hence, the model that does not consider the journal's angular whirling motion, which uses much less calculation time, could correctly predict the bifurcation type outside these cases. The supercritical bifurcation tended to occur when the bearing has low L/D ratio and preload, and the main mass was located near the journal bearing.

Chapter 6

Nonlinear Vibration Analysis of a Flexible Rotor Supported by a FPTPJB in Comparison to a Fixed Profile JB and a TPJB

6.1 Overview

This chapter examined the effect of load orientations on the nonlinear vibration of a flexible rotor supported by a flexure pivot tilting pad journal bearing (FPTPJB) and compared the results with a fixed profile journal bearing and a tilting pad journal bearing (TPJB) with the same geometry. The investigated load orientations were load-on-pad (LOP) and load-between-pad (LBP). Similar to Chapter 5, the calculation results were verified by the experiment.

6.2 Investigated Parameters

The rotor used in the investigation was identical to that of Section 5.3, as shown in Fig. 5.23. However, only one large disk position 0.34m was selected because it provided the lower OSI, and the minimum disk mass of 4 kg was used due to the safety reason in the experimental verification. The investigated load orientations were load-on-pad (LOP) and load-between-pad (LBP), as depicted in Fig. 6.1, which leads to the investigated parameters in Table 6.1. The rotor was supported by the investigated bearing at the NDE side. The FPTPJB's drawing, essential parameters, and pivot's stiffnesses are presented in Fig. 3.8 and Table 3.5, and Table 3.6, respectively. The FPTPJB could be converted to the fixed profile JB by inserting pad-locking pins, so the geometries of both bearings were considered identical. The drawing and crucial parameters of TPJB are shown in Fig. 3.7 and Table 3.4, respectively. Since the pad's EOM must be included in the calculation of FPTPJB and TPJB cases, the calculation time is longer than that of the fixed profile JB. However, due to the short L/D ratio (0.4) in the investigated cases and the conclusion from Section 5.3, only the model that considers the journal's lateral motion (Model B) was employed in this chapter.

Fig. 5.23 Drawings of test rotors

Fig. 6.1 4-pad FPTPJB in LOP, LBP orientations

Table 6.1 Rotor and bearings parameters

Parameter	Value	Unit
Large disk position from the simple bearing	0.340	m
Large disk mass	4	kg
Bearing orientation	LOP, LBP	-

Fig. 3.8 CAD drawing of a FPTPJB (oil supply covers removed) and reference axes

Table 3.5 Parameters of a FPTPJB

Parameter	Value	Unit
Nominal diameter	30	mm
Length	12	mm
Bearing assembly radial clearance	60	μm
Number of pads	4	-
Pad's angular extent	75	deg
Preload	0.5	-
Offset	0.5	-
Pad thickness from the top of pivot	7.44	mm
Distance from pad's CG to the top of pivot	4.95	mm
Pad mass	16.98	g
Pad mass moment of inertia around top of pivot	1214.29	g-mm^2
pivot's web height	5.81	mm
pivot's web thickness	1.5	mm
pivot's Young modulus	100	GPa
pivot's shear modulus	37	GPa

Table 3.6 Stiffnesses of the pivot web obtained from Solidworks

$k_{\xi\xi}$ (N/m)	$k_{\eta\eta}$ (N/m)	$k_{\delta\eta}$ (Nm/m)	$k_{\eta\delta}$ (N/rad)	$k_{\delta\delta}$ (Nm/rad)
3.714×10^8	6.219×10^6	-2.596×10^4	-2.596×10^4	73.26

Fig. 3.7 CAD drawing of a TPJB (oil supply covers removed)

Table 3.4 Parameters of a TPJB (only those differ from FPTPJB shown)

Parameter	Value	Unit
Pad thickness from pivot center	6.96	mm
Distance from pad's CG to pivot center	4.58	mm
Pad mass	16.53	g
Pad mass moment of inertia around pivot	1140.76	g-mm^2

6.3 Investigation Method

Since the rotor was the same as Section 5.3, the static deflection was the same, as shown in Table 5.6. Bearing housing was tilted accordingly to make the axes of the bearing and journal parallel, which is the assumption of Model B. The eigenanalysis of the rotor supported by TPJB and FPTPJB used the stiffness and damping coefficients determined by the pad perturbation method [111] that is suitable for calculating the onset speed of instability as it does not assume the pad motion frequency, which is unknown in the case of self-excited vibration. Hence, the size of bearing stiffness and bearing matrices of TPJB and FPTPJB in this investigation was 6×6 and 14×14, respectively, which corresponds to the total DOF of journal and pads. The investigation method of Section 5.3 was applied to the EOM of the rotor supported by the fixed profile JB and TPJB, depicted in Section 2.4.1 and 2.4.2, respectively. The total DOF of the reduced flexible rotor model using real mode component mode synthesis (h=2) supported by a fixed profile JB, TPJB, and FPTPJB are 6, 10, and 18, respectively.

The experiments were conducted to verify the calculation results. The load orientation was change by rotating the bearing to the desired load orientation then tightening the bearing housing to clamp the bearing in place. Pins were inserted into FPTPJB's locking pin cut-outs, shown in Fig. 3.8, to convert the FPTPJB to 4-lobe JB.

6.4 Calculation Results

The calculated shaft centerline plots in LOP and LBP orientations of the FPTPJB are shown in Fig. 6.2. The equilibrium deflections of pads' CG in each direction are depicted in Fig. 6.3 and Fig. 6.4 for LOP and LBP orientations, respectively. For TPJB, shaft centerline plots in LOP and LBP orientations are shown in Fig. 6.5, and the equilibrium pads' tilting angles are depicted in Fig. 6.6. Lastly, the calculated shaft centerline plots in LOP and LBP orientations of fixed profile JB are shown in Fig. 6.7.

The shaft centerline of the fixed profile JB will be discussed first as the bearing's load-bearing mechanism is the simplest because only the journal moves. From the centerline in Fig. 6.8, the journal position rose in a curved line, which means the vertical journal load results in journal horizontal displacement. In other words, the horizontal journal displacement is needed to balance the horizontal force acting on the rotor. This cross-coupling between the journal's vertical load and horizontal displacement explains why the centerline in LBP orientation rose more vertically as there was no load-carrying pad directly under the journal load line. Hence, the journal load must be countered by the vector summation of the two lower pads' forces, which increased the symmetry in the horizontal force. In the LBP orientation, the bottom pad carried most of the journal load with the trailing pad came next, which made the journal rose diagonally.

In TPJB, the moment on all pads of TPJB must be balanced by the pads' tilt. The centerline of TPJB rose almost vertically in both orientations, as presented in Fig. 6.5, which indicates the vertical journal load results in almost purely vertical journal movement. Note that the

journal position of the LBP orientation at low speed was lower because the journal load line and the pad's pivot were not in line. Hence, the journal load must be carried by vector summation of the forces generated mainly from the two lower pads as in the case of the fixed profile JB counterpart. As the speed increased, the shaft centerlines of both orientations approached one another, and the pads tilted toward their high-speed equilibrium, depicted in Fig. 6.6. The pad tilting angles of top and bottom pads in the LOP orientation remained relatively unchanged with speed because the journal load is in line with the pivots of top and bottom pads, making the journal vertical movement not considerably affects the moment on the pad that leads to a tilting angle. However, the left and right pads' pivots are perpendicular to the journal load, making the change in journal vertical position alters the moment on the pads. Hence, the pads must tilt in relatively larger angles as the speed is changed. In the LBP orientation, all pads' pivots are diagonal to the load line, which makes the changes in tilting angles of all pads comparable.

Due to the flexure pivots' stiffnesses, the maximum tilting angle (δ) of FPTPJB, shown in Fig. 6.3 and Fig. 6.4 for LOP and LBP orientations, respectively, was considerably less than that of the TPJB counterpart. This restricted tilting motion creates a degree of cross-coupling between the vertical journal load and journal horizontal displacement, as fixed-profile JB cases. Hence, the investigated FPTPJB's shaft centerline was similar to that of the fixed profile JB, as depicted in Fig. 6.2. Moreover, the flexure pivots allow the pads to move in the radial (ξ) and tangential (η) directions. It can be seen that the movement in the radial direction was negligible, but the movement in the tangential direction was significant as the movement can be as large as half of the bearing assembly clearance (C_{rb}) at low speed. Also, all pads deflected in the same direction of the shaft rotation' tangent at high speed. This tangential movement causes the difference in shaft centerline compared to the fixed profile JB and the journal's equilibrium position at low speed lies outside of the bearing assembly clearance (C_{rb}).

Fig. 6.2 Shaft centerline plot of FPTPJB

Fig. 6.3 Equilibrium pad's CG deflection of FPTPJB in LOP orientation

Fig. 6.4 Equilibrium pad's CG deflection of FPTPJB in LBP orientation

Fig. 6.5 Shaft centerline plot of TPJB

Fig. 6.6 Equilibrium pad's tilting angle of TPJB in LOP and LBP orientations

Fig. 6.7 Shaft centerline plot of fixed profile JB

The first forward eigenvalues in LOP and LBP orientations of FPTPJB, TPJB, and fixed profile JB are compared in Fig. 6.8 to Fig. 6.10, respectively. The full and reduced rotor models yielded similar eigenvalues, resulting in almost identical first critical speed and very close OSI, as shown in Table 6.2. The first critical speed of each case was close to each other. TPJB in both orientations remained stable until 10,000 rpm. However, the LOP case gave the less steep slope of the eigenvalue around this speed range, implying that the OSI of the LOP case can be higher. The higher OSI was caused by asymmetry in the direct stiffnesses, as pointed out by Zeidan and Paquette [120]. Since FPTPJB in this investigation had relatively stiff pivot stiffnesses as shown by the resemblance of shaft centerline to that of the fixed profile JB in the same orientation, the stability of FPTPJB is expected to be between that of TPJB and fixed profile JB. FPTPJB in both orientations gave higher OSI than that of the fixed profile JB due to lower cross-coupling stiffnesses. The FPTPJB in LOP orientation gave higher OSI than the LBP one due to asymmetry in the direct stiffnesses. However, for the fixed profile JB, the LBP orientation gave higher OSI than the LOP one due to the lower cross-coupling stiffness.

Fig. 6.8 First forward eigenvalues of FPTPJB

Fig. 6.9 First forward eigenvalues of TPJB

Fig. 6.10 First forward eigenvalues of fixed profile JB

Table 6.2 Calculated first critical speed and OSI

Bearing type		FPTPJB		TPJB		Fixed Profile	
Load orientation		LOP	LBP	LOP	LBP	LOP	LBP
First critical speed (rpm)	Full	1003	1003	1006	1005	1003	1002
	CMS	1003	1003	1006	1005	1003	1002
OSI (rpm)	Full	4749	3601	>10000	>10000	3040	2523
	CMS	4743	3595	>10000	>10000	3031	2521

Since TPJB remained stable until at least 10,000 rpm in both orientations, the limit cycles of this bearing type could not be obtained, so only those of FPTPJB and fixed profile JB are presented. The journal's eccentricity ratios of the equilibrium position and the farthest point of the limit cycle's locus in each orientation of FPTPJB and fixed profile JB are presented in Fig. 6.11 and Fig. 6.12, respectively. All cases showed subcritical Hopf bifurcation and no stable limit cycle with a smaller amplitude than the bearing clearance existed at speed above the OSI, which means the journal at equilibrium will whirl in increasing amplitude until it touches the bearing when rotor speed exceeds OSI. Besides, the unstable limit cycle surrounded the equilibrium in a speed range between the OSI and the saddle-node point. Hence, the journal at the equilibrium in this speed range can also whirl in the increasing amplitude until the journal touches the bearing or reaches the stable limit cycle if it is displaced outside the stable operating boundary encircled by the unstable limit cycle. It could be seen that this boundary was the largest and occupied the narrowest speed range in the case of FPTPJB in LOP orientation, whereas it was the smallest and occupied the widest speed range in the case of fixed profile JB in LBP orientation. As a result, the rotor supported by FPTPJB in LOP orientation is less susceptible to the instability caused by external perturbation.

Fig. 6.11 Limit cycles of FPTPJB in LOP and LBP orientations

Fig. 6.12 Limit cycles of fixed profile JB in LOP and LBP orientations

6.5 Experimental Results

The comparison of measured and calculated shaft centerlines in LOP and LBP orientations of FPTPJB, TPJB, and fixed profile JB is shown in Fig. 6.13 to Fig. 6.18. The measured results agreed well with the calculated results in the speed range 500 rpm, which is the speed the oil film can solely support the journal load, to the speed the rotor became unstable. When the rotor became unstable and vibration amplitude increased, the shaft centerline shifted toward the bearing center because the journal's orbit occupied the entire bearing clearance and the rubbing between the journal and the bearing occurred. This rubbing caused a hysteresis when the speed was lowered until a certain speed before the shaft centerline converges to the centerline during speed increase.

The full cascade plot of each case is shown in Fig. 6.19 to Fig. 6.24. Apart from the FPTPJB with LOP orientation and TPJB presented in Fig. 6.19, Fig. 6.21, and Fig. 6.22, the instability caused by the forward precession subsynchronous vibration was observed in the remaining cases. The subsynchronous vibration's frequency coincided with the first critical speed and its amplitude increased continuously until its amplitude was close to bearing clearance, which indicates subcritical Hopf bifurcation. When the speed was lowered from this point, the frequency components' hysteresis occurred until a certain speed before the frequency components became the same as the ones during speed increase. This speed range was much broader than the speed range between the OSI and the saddle-node point in the calculation. This discrepancy was caused by the rubbing excitation. However, the contact force from the rubbing is beyond the scope of this study, and the rotor-bearing system should be designed to avoid rubbing in its operating speed range. In the case of FPTPJB with LOP orientation, the subsynchronous vibration also existed at a certain speed range, but it was well-suppressed and diminished at a higher speed, indicating the stable operating circle is larger than the amplitude of the subsynchronous excitation, making the rotor remains stable. The measured first critical speed, OSI, and Hopf bifurcation type are presented in Table 6.3, which shows good agreement with the calculation in terms of the first critical speed in all cases and OSI, and Hopf bifurcation in the cases that the rotor became unstable at speed lower than the test rig's maximum speed.

Fig. 6.13 Measured shaft centerline of FPTPJB in LOP orientation

Fig. 6.14 Measured shaft centerline of FPTPJB in LBP orientation

Fig. 6.15 Measured shaft centerline of TPJB in LOP orientation

Fig. 6.16 Measured shaft centerline of TPJB in LBP orientation

Fig. 6.17 Measured shaft centerline of fixed profile JB in LOP orientation

Fig. 6.18 Measured shaft centerline of fixed profile JB in LBP orientation

Fig. 6.19 Full cascade plot of FPTPJB in LOP orientation

Fig. 6.20 Full cascade plot of FPTPJB in LBP orientation

Fig. 6.21 Full cascade plot of TPJB in LOP orientation

Fig. 6.22 Full cascade plot of TPJB in LBP orientation

Fig. 6.23 Full cascade plot of fixed profile JB in LOP orientation

Fig. 6.24 Full cascade plot of fixed profile JB in LBP orientation

Table 6.3 Measured first critical speed, OSI, and Hopf Bifurcation Type

Bearing type	**FPTPJB**		**TPJB**		**Fixed Profile**	
Load orientation	**LOP**	**LBP**	**LOP**	**LBP**	**LOP**	**LBP**
First critical speed (rpm)	987	984	986	985	986	984
OSI (rpm)	> 4500	3588	> 4500	> 4500	2494	2998
Hopf Bifurcation Type	-	sub	-	-	sub	sub

6.6 Summary

The effect of load orientation: load-on-pad (LOP) and load-between-pad (LBP), on nonlinear vibration (limit cycle and Hopf bifurcation type) of a flexible rotor supported by a 4-lobe flexure pivot tilting pad journal bearing (FPTPJB) was numerically investigated along with the first critical speed and onset speed of instability (OSI). The investigation was extended to the tilting pad journal bearing (TPJB) and fixed profile journal bearing (JB) of the same pad geometry for comparison. The rotor was discretized into finite elements, and the equations of motion (EOM) were reduced to those of the node subjected to the nonlinear bearing force, obtained from Reynolds equation, using the real mode component mode synthesis (CMS). The FPTPJB's pad was modeled to move in radial, tangential, and tilting directions. The eigenanalysis was done to obtain the first critical speed and OSI of both full and reduced rotor models to compare the reduction accuracy. The shooting method with parallel computing was applied to the reduced model to determine the limit cycle and Hopf bifurcation type.

The calculation results showed that CMS retained good accuracy of the eigenvalue. FPTPJB gave considerably higher OSI than the fixed profile JB of the same orientation. In comparison, the rotor supported by TPJB remained stable until more than 10,000 rpm. Besides, FPTPJB in LOP orientation yielded considerably higher OSI than the LBP one, whereas the OSI of the fixed profile JB in LOP orientation was substantially higher than the LBP counterpart. The FPTPJB in LOP orientation produced the largest stable operating region, whereas the fixed profile JB in LBP configuration yield the smallest one. The calculation results were verified by the experiment, in which good agreement in shaft centerline, the first critical speed, OSI, and Hopf bifurcation type were observed in the most case except the cases of TPJB in both orientations and FPTPJB in LOP orientation, in which the OSI and Hopf bifurcation type could not be verified due to the test rig's speed limit.

In conclusion, applying the shooting method with parallel computing to the rotor model reduced by real mode CMS provides a numerically efficient tool for nonlinear rotordynamics analysis. FPTPJB provides superior stability than the fixed profile JB with the same profile. However, due to the web's stiffnesses, it is inferior to TPJB in terms of stability. The load orientation has a noticeable effect on the stable operating region in both linear (OSI) and nonlinear (limit cycle) aspects. Hence, the effect of rotor and bearing parameters, especially those of FPTPJB, should be further investigated.

Chapter 7

Conclusions

7.1 Summary

This thesis aims to develop an efficient calculation method to determine the periodic self-excited vibration and its stability of a rotor supported by a journal bearing (JB). The calculation method must be able to cope with complex rotor geometry and nonlinear forces of the JB in various types, including the non-fixed profile bearing, such as tilting pad journal bearing (TPJB) and flexure pivot tilting pad journal bearing (FPTPJB). As a result, the calculation method based on the shooting method with parallel computing with arclength continuation was developed. The nonlinear bearing forces were numerically solved from Reynolds equation using the finite difference method (FDM). The calculation speed in finite element flexible rotor cases was improved while retaining the accuracy by utilizing model reduction.

The developed method was used to study the effects of the various rotor and bearing parameters, which can be categorized into three groups in the ascending order of the modelling complexity: a rigid symmetric rotor supported by two identical fixed profile JBs with consideration of the turbulence effect, a flexible rotor supported by a fixed profile JB, and a flexible rotor supported by a FPTPJB. Several crucial rotor and bearing parameters, such as rotor and bearing geometries, were varied to study their effects of nonlinear vibration throughout this research and several conclusions were made. The calculation results were mostly verified by the experiments on the test rig with identical parameters. Generally, good agreements were observed. The summary of the investigation in each aspect can be concluded as follows:

7.1.1 Effect of Turbulence and Bearing Geometry on Nonlinear Vibration of a Rigid Rotor

The limit cycle and bifurcation type of a rigid rotor supported by two identical journal bearings were investigated using the laminar model and two turbulent models (Ng-Pan-Elrod and Constantinescu) of Reynolds equation. The bearing L/D ratio was varied from 0.5 to 1, and its profiles were full circular and 2-lobe with preload value of 0, 0.25, and 0.5.

The results indicated that the onset speed of instability (OSI) and Hopf bifurcation point of both turbulence models were lower than those of the laminar model regardless of the L/D ratio in the bearings without preload. In addition, the L/D ratio that the Hopf bifurcation changed from supercritical to subcritical of each turbulence model was lower. Nonetheless, the laminar and turbulence models yielded only supercritical Hopf bifurcation in preloaded bearings. Moreover, both OSI and bifurcation point of the turbulence models became higher than those of the laminar model when L/D is higher than a certain value, depending on the preload. The limit cycles' whirl frequency ratio (WFR) of the turbulence models were higher than that of the laminar one, especially for the bearings with high preload and L/D ratio. The increased WFR was caused by the downward force from the upper pad, which became more pronounced as the preload was increased and further reinforced by the additional viscosity of the turbulence models. Lastly, the OSI and bifurcation point obtained from the Ng-Pan-Elrod turbulence model was always higher than that of the Constantinescu model, regardless of bearing profiles and L/D ratios.

Hence, the journal bearings operated in the turbulent regime should have a high pad preload and L/D ratio, which inclines to give a higher OSI and supercritical Hopf bifurcation with a

gradual limit cycle's growth. The Ng-Pan-Elrod model was considered more conservative than the Constantinescu model in predicting the OSI in all investigated bearing configurations.

7.1.2 Effects of Journal Angular Motion Consideration on Hopf Bifurcation

The combination of numerical integration and shooting method was utilized to identify the Hopf bifurcation type of the flexible rotor supported by a 2-lobe JB. The rotor's total DOF were reduced to those of the journal's node by the modified Guyan reduction using the first forward mode's imaginary part. Three positions of the large disk were chosen, ranging from the midspan one to the one that was closed to the journal bearing. Two bearing's film thickness models: considers the journal's angular whirling motion (Model A) and does not consider (Model B) were used in the investigation. Due to an axial symmetry property of Model B, it requires about half calculation time of Model A.

Numerical results indicated that even the equilibrium position, the first critical speed, and the OSI obtained from each film thickness model were similar, the Hopf bifurcation type obtained from each model can be different. The test rig was set up to verify the numerical results, and it confirmed the results obtained from Model A. Hence, using this model is essential for predicting the Hopf bifurcation type.

It was also observed in the numerical results that the discrepancy between the OSI obtained from the full and reduced models using the modified Guyan reduction increased with the gyroscopic effect. Hence, another reduction method that can retain good reduction accuracy in high gyroscopic effect cases was required. Also, a method for finding an initial condition for the shooting method to obtain the unstable limit cycle, which encircles the stable operating region, was needed.

7.1.3 Essential Range of Considering Journal Angular Motion

To improve the deficiencies of the previous investigation, the model reduction method was changed to real mode component mode synthesis (CMS). Also, the procedure for obtaining the initial condition for the shooting method that yielded the unstable limit cycle was established. Since the model that does not consider the journal angular motion (Model B) requires only about half the calculation time of the model that considers the journal angular motion (Model A), identifying the parameter range that using Model B is sufficient could improve the calculation speed. Hence, a broader range of rotor and bearing parameters was investigated to identify the cases that Model B was sufficient to predict the bifurcation type correctly. The verification of all cases was done by the experiment.

As observed in the previous section, the experimental results indicated that Model A was able to correctly predict the bifurcation type in all cases, whereas Model B, which requires only half the calculation time of Model A, could predict correctly in most cases except the three cases of L_d = 0.575m and L/D=0.6. These findings indicated that the discrepancy in bifurcation type tended to occur when the L/D ratio was increased and the rotor's main mass was located away from the rotor's midspan. Hence, the Model B is sufficient when the L/D ratio is short, and the main mass of the rotor is located near the midspan. Besides, decreasing the L/D ratio, increasing preload, and moving the main mass closer to the JB (increase JB's load) inclined to give supercritical bifurcation.

7.1.4 Effect of Load orientation on Nonlinear Vibration of a Flexible Rotor Supported by FPTPJB in Comparison to TPJB and a Fixed Profile JB

The previous section's calculation method was extended to a flexible rotor supported by FPTPJB to investigate the effect of load orientation: load-on-pad (LOP) and load-between-pad (LBP), on nonlinear vibration. The FPTPJB's pad was modeled to move in radial, tangential, and tilting directions. This calculation method was also used to solve the solutions of the rotor supported by TPJB and fixed profile JB of the same bearing profile for the comparison. The calculation results were then verified by the experiment.

FPTPJB gave considerably higher OSI of is than the JB of the same orientation. Besides, FPTPJB in LOP orientation yielded higher OSI than the LBP one, whereas the OSI of JB in LOP orientation was substantially higher than the LBP counterpart. The FPTPJB in LOP orientation produced the largest stable operating region, whereas JB in LBP configuration yield the smallest one. Good agreement between calculation and experiment was observed in most cases except TPJB in both orientations and FPTPJB in LOP orientation, in which the OSI and Hopf bifurcation type cannot be verified due to the test rig's speed limit.

7.2 Contributions

This work investigated the effects of several parameters on the nonlinear self-excited vibration (limit cycles and Hopf bifurcation type) of the rotor supported by several types of journal bearing (JB) by utilizing the efficient numerical method that substantially reduces the calculation time and can determine both stable and unstable limit cycle. The mathematical modelling and numerical method can be directly applied to the complex rotor-bearing system representing the actual machine.

The completed work of this dissertation has several contributions to the nonlinear rotordynamics analysis field as follows:

1. Development of the efficient numerical method to determine autonomous periodic solutions and their stability of rigid and flexible rotors supported by JB, including FPTPJB, using the shooting method with parallel computing and arclength continuation. The proposed method can obtain both stable and unstable limit cycles.
2. Recognition of the turbulence effect on nonlinear vibration of a rigid rotor supported by plain circular and 2-lobe JB under various values of L/D ratio and preload.
3. Identification of the necessity of considering the angular whirling motion for the nonlinear vibration prediction of flexible rotors supported by 2-lobe JB in the various rotor and bearing parameters with experimental verification.
4. Finding of the rotor-bearing parameters that considering the angular whirling motion is essential for the Hopf bifurcation type prediction in the flexible rotors supported by 2-lobe JB with experimental verification.
5. Identification of the effect of load orientation (LOP and LBP) on the nonlinear vibration of a flexible rotor supported by FPTPJB compared to the fixed profile JB with experimental verification.

7.3 Future Perspective and Recommendations for Future Work

Based on the contributions, more nonlinear rotordynamics analysis of the self-excited vibration in the complex rotor and bearing system can be performed in the same amount of time using the proposed efficient numerical method. Hence, the broad range of parametric study is possible for investigating the effect of the various rotor and bearing parameters on the rotor and bearing system's nonlinear behavior, which is essential in rotating machinery design and troubleshooting. Additionally, since the rotor model reduction and numerical method for obtaining the solution can work with any bearing force model, it is possible to extend the nonlinear analysis of the rotor supported by the bearing that its force cannot be determined by FDM, such as pressure dam bearing. The calculation speed can also be further increased by applying the novel bearing force calculation algorithm to the existing numerical method.

It is apparent that the works in this thesis cannot cover all aspects of the nonlinear analysis of self-excited vibration. There are several points that can be improved from the current work as follows:

1. Since the nonlinear forces of the bearing must be performed every integration's time step, the numerical method that provides faster pressure convergence is advantageous. However, the successive over-relaxation (SOR) used in this thesis is not the algorithm that yields the fastest convergence speed. A more novel algorithm that focuses on fast convergence rates, such as the multigrid method, can substantially improve the calculation speed.
2. The finite difference discretization of the oil film in this thesis is suitable for a bearing pad that does not contain any geometrical discontinuity. However, several bearings have this discontinuity, such as pressure groove and scallop cut. Using finite element discretization seems to be more suitable in pressure calculation for the bearing with the pad's geometrical discontinuity.
3. The Reynolds cavitation boundary condition used in this work satisfies mass conservation at the onset of the cavitation, but not at the onset of full film reformation. Theoretically, the mass-conserving cavitation algorithm, which conserves the mass at both the onset of cavitation and full film reformation, should give more accurate results. However, this cavitation algorithm requires a considerably longer calculation time. The author tried to apply this cavitation algorithm in the dynamic bearing forces calculation, but it seemed impractical to apply the existing cavitation algorithm. Hence, developing the mass-conserving cavitation algorithm that is practical for dynamic bearing forces calculation can be another interesting area.
4. In the experimental verification, the Hopf bifurcation type was confirmed. However, the amplitude of the unstable limit cycle that encircles the stable operating region was not measured. This measurement requires a non-contact actuator, such as active magnetic bearing (AMB), to perturb the rotor from the equilibrium position.
5. For the bearings with low cross-coupling stiffness, such as TPJB and FPTPJB, the OSI and bifurcation point can exceed the maximum speed of the test rig. The feedback-controlled AMB with a controller that can inject the known destabilization forces into the rotor system is a promising method for experimental verification.

Appendix A

Derivation of Finite Element of The Rotor

The following derivations for the rotor's finite element are taken from Adams [1], and Ishida and Yamamoto [109]. The rotor consists of a shaft and rigid disks and their EOM are derived by using Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)-\frac{\partial T}{\partial q_s}+\frac{\partial V}{\partial q_s}+\frac{\partial F}{\partial \dot{q}_s}=Q_s \tag{A.1}$$

where q_s is generalized coordinate, T is kinetic energy, V is potential energy, F is dissipation function, Q_s is non conservative generalized force.

Rigid Disk

It is assumed that the rigid disk, shown in Fig. 2.4, does not contribute to shaft element's flexibility. Mass properties of the disk are given as follows:

Fig. 2.4 Dimension of the rigid disk

$$\begin{aligned}
&\text{Mass, } m_d=\frac{\rho_d\pi\left(D_{od}^2-D_{id}^2\right)l}{4}\\
&\text{Transverse inertia, } I_{Td}=\frac{1}{12}m_d\left[\frac{3\left(D_{od}^2+D_{id}^2\right)}{4}+l^2\right]\\
&\text{Polar inertia, } I_{Pd}=\frac{1}{2}m_d\frac{\left(D_{od}^2+D_{id}^2\right)}{4}
\end{aligned} \tag{A.2}$$

where $\rho_d, D_{od}, D_{id}, l$ are disk's density, outer diameter, inner diameter and thickness, respectively. When the disk has lateral displacements x, y, inclinations θ_x, θ_y, and rotational speed ω, the kinetic energy of the disk T_d can be given by:

$$\begin{aligned}
T_d&=\frac{1}{2}m_d\left(\dot{x}^2+\dot{y}^2\right)+\frac{1}{2}I_{Pd}\left[\omega^2+\omega\left(\dot{\theta}_x\theta_y-\theta_x\dot{\theta}_y\right)\right]+\frac{1}{2}I_{Td}\left(\dot{\theta}_x^2+\dot{\theta}_y^2\right)\\
&=\frac{1}{2}\begin{Bmatrix}\dot{x}\\ \dot{y}\end{Bmatrix}^T\begin{bmatrix}m_d & 0\\ 0 & m_d\end{bmatrix}\begin{Bmatrix}\dot{x}\\ \dot{y}\end{Bmatrix}+\frac{1}{2}\begin{Bmatrix}\dot{\theta}_x\\ \dot{\theta}_y\end{Bmatrix}^T\begin{bmatrix}I_{Td} & 0\\ 0 & I_{Td}\end{bmatrix}\begin{Bmatrix}\dot{\theta}_x\\ \dot{\theta}_y\end{Bmatrix}\\
&\quad+\frac{1}{2}I_{Pd}\left[\omega^2+\omega\left(\dot{\theta}_x\theta_y-\theta_x\dot{\theta}_y\right)\right]
\end{aligned} \tag{A.3}$$

by substituting Eq. (A.3) into Lagrange's equation, Eq.(A.1), and by noting that only the terms that contain T_d in the left-hand-side are non-zero terms. The EOM of the disk are given as follows:

$$\begin{bmatrix} m_d & 0 & 0 & 0 \\ 0 & m_d & 0 & 0 \\ 0 & 0 & I_{Td} & 0 \\ 0 & 0 & 0 & I_{Td} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta}_x \\ \ddot{\theta}_y \end{Bmatrix} + \omega \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{Pd} \\ 0 & 0 & -I_{Pd} & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta}_x \\ \dot{\theta}_y \end{Bmatrix} = \begin{Bmatrix} f_x \\ f_y \\ f_{\theta x} \\ f_{\theta y} \end{Bmatrix} \tag{A.4}$$

$$\mathbf{M}_d \ddot{\mathbf{q}} + \mathbf{G}_d \dot{\mathbf{q}} = \mathbf{f}_d$$

where $\mathbf{f}_d$ represents forces acting on a disk, such as unbalance forces and forces from the adjacent elements. The gravitational force acting on the disk is as follows:

$$\mathbf{f}_{gd} = \{0 \quad -m_d g \quad 0 \quad 0\}^T \tag{A.5}$$

note that mass of the disk does not contribute to any moment on the rotor.

Flexible Shaft

A shaft element obtained by dividing the shaft is shown in Fig. A.1. The global coordinates *XYZ* is the same as that of the rotor system, shown in Fig. 2.2. the local coordinates *XYS*, originated from the right end of the element, is fixed to the shaft element.

Fig. A.1 Dimension of the shaft element

In the similar manner to those of the rigid disk, the mass properties of the shaft element are as follows:

$$\begin{aligned} &\text{Mass, } m_s = \frac{\rho_s \pi \left(d_{os}^2 - d_{is}^2\right) L}{4} = \rho_s A_s L \\ &\text{Transverse inertia, } I_{Ts} = \frac{1}{12} m_s \left[\frac{3\left(d_{os}^2 + d_{is}^2\right)}{4} + L^2 \right] \\ &\text{Polar inertia, } I_{Ps} = \frac{1}{2} m_s \frac{\left(d_{os}^2 + d_{is}^2\right)}{4} \\ &\text{Area moment of inertia, } I_{As} = \frac{\pi \left(d_{os}^4 - d_{is}^4\right)}{64} \\ &\text{Young's modulus, } E \end{aligned} \tag{A.6}$$

where $\rho_s, A_s, d_{os}, d_{is}, L$ are shaft element's density, cross-section area, outer diameter, inner diameter and thickness, respectively.

Consider the sliced disk of thickness *ds* located at position *s* in the local coordinates of Fig. A.1, the time-dependent displacements x, y, θ_x, θ_y of this sliced disk are functions of position *s* and have the relationship as follows:

$$\theta_x = -\frac{\partial y}{\partial s}, \theta_y = \frac{\partial x}{\partial s} \tag{A.7}$$

by using interpolation of the nodal displacement, the displacement of an arbitrary point on this shaft element can be obtained. Since the rotation of the sliced disk can be obtained from Eq.

(A.7), the lateral displacements x, y can be written in forms of linear combinations of the nodal displacements as follows:

$$\begin{aligned} x(s,t) &= \psi_{x,x1}x_1 + \psi_{x,y1}y_1 + \cdots + \psi_{x,\theta x2}\theta_{x2} + \psi_{x,\theta y2}\theta_{x2} \\ y(s,t) &= \psi_{y,x1}x_1 + \psi_{y,y1}y_1 + \cdots + \psi_{y,\theta x2}\theta_{x2} + \psi_{y,\theta y2}\theta_{x2} \end{aligned} \tag{A.8}$$

where ψ is shape function, Eq. (A.8) can be written in matrix form as:

$$\begin{Bmatrix} x(s,t) \\ y(s,t) \end{Bmatrix} = \mathbf{\Psi}(\mathbf{s})\mathbf{q}_s = \begin{Bmatrix} \mathbf{\Psi}_x \\ \mathbf{\Psi}_y \end{Bmatrix}\mathbf{q}_s \tag{A.9}$$

where

$$\begin{aligned} \mathbf{q}_s &= \begin{Bmatrix} x_1 & y_1 & \cdots & \theta_{x2} & \theta_{y2} \end{Bmatrix}^T \\ \mathbf{\Psi} &= \begin{bmatrix} \psi_{x,x1} & \psi_{x,y1} & \cdots & \psi_{x,\theta x2} & \psi_{x,\theta y2} \\ \psi_{y,x1} & \psi_{y,y1} & \cdots & \psi_{y,\theta x2} & \psi_{y,\theta y2} \end{bmatrix} \end{aligned} \tag{A.10}$$

Under Euler beam theory, the shaft element in Fig. 2.3 is postulated to have a uniform cross section in bending. Thus, its static deflection can be expressed as cubic functions in the *X-S* and *Y-S* planes as follows:

$$\psi(s) = as^3 + bs^2 + cs + d \tag{A.11}$$

A general state of shaft element deflection in the plane of local coordinates can be expressed as a linear superposition of four cases, each having a displacement of 1 for one of the four generalized coordinates in the plane with zero displacement for the remaining three coordinates in the plane. These four cases are specified by the sets of boundary conditions shown in Table A.1.

Table A.1 Correspondence between shape functions and rotor element coordinates

X-S	x_1	θ_{y1}	x_2	θ_{y2}
Y-S	y_1	$-\theta_{x1}$	y_2	$-\theta_{x2}$
Case 1	$\psi_1(0)=1$	$\psi_1'(0)=0$	$\psi_1(L)=0$	$\psi_1'(L)=0$
Case 2	$\psi_2(0)=0$	$\psi_2'(0)=1$	$\psi_2(L)=0$	$\psi_2'(L)=0$
Case 3	$\psi_3(0)=0$	$\psi_3'(0)=0$	$\psi_3(L)=1$	$\psi_3'(L)=0$
Case 4	$\psi_4(0)=0$	$\psi_4'(0)=0$	$\psi_4(L)=0$	$\psi_4'(L)=1$

Note: $'$ denotes the differentiation respected to s

By substituting these sets of boundary conditions into Eq. (A.11), the shape functions are:

$$\begin{aligned} \psi_1(s) &= 1-3\left(\frac{s}{L}\right)^2 + 2\left(\frac{s}{L}\right)^3 \qquad \psi_2(s) = s\left(1-2\left(\frac{s}{L}\right)+\left(\frac{s}{L}\right)^2\right) \\ \psi_3(s) &= 3\left(\frac{s}{L}\right)^2 - 2\left(\frac{s}{L}\right)^3 \qquad \psi_4(s) = \frac{s^2}{L}\left(\frac{s}{L}-1\right) \end{aligned} \tag{A.12}$$

From Eq. (A.8), (A.10), (A.12) and Fig. 2.3, the translation shape function matrix $\mathbf{\Psi}$ is as follows:

$$\mathbf{\Psi} = \begin{bmatrix} \psi_1 & 0 & 0 & \psi_2 & \psi_3 & 0 & 0 & \psi_4 \\ 0 & \psi_1 & -\psi_2 & 0 & 0 & \psi_3 & -\psi_4 & 0 \end{bmatrix} = \begin{Bmatrix} \mathbf{\Psi}_x \\ \mathbf{\Psi}_y \end{Bmatrix} \tag{A.13}$$

From Eq. (A.7), (A.9), (A.12) and (A.13), the rotation can be written in the similar manner as translation as follows:

$$\begin{Bmatrix} \theta_x(s,t) \\ \theta_y(s,t) \end{Bmatrix} = \mathbf{\Phi}(\mathbf{s})\mathbf{q}_s = \begin{Bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_y \end{Bmatrix} \mathbf{q}_s \tag{A.14}$$

where $\mathbf{\Phi}$ is rotation shape function matrix given by

$$\mathbf{\Phi} = \begin{bmatrix} 0 & -\psi_1' & \psi_2' & 0 & 0 & -\psi_3' & \psi_4' & 0 \\ \psi_1' & 0 & 0 & \psi_2' & \psi_3' & 0 & 0 & \psi_4' \end{bmatrix} = \begin{Bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_y \end{Bmatrix} \tag{A.15}$$

where $'$ denotes the differentiation respected to s.

After the shape function were obtained, the energy expressions can be derived. First, let's consider the potential energy stored in a beam of length L deflecting in the XZ-plane. According to mechanics of materials, this potential energy is given by:

$$U = \int_0^L dU,\, dU = \frac{1}{2} EI_{As} \left(\frac{\partial^2 x}{\partial s^2} \right)^2 ds \tag{A.16}$$

Similary, the potential energy dU_s stored in a sliced disk with length ds shown in Fig. 2.5 can be derived as follows:

$$dU_s = \frac{1}{2} \begin{Bmatrix} x'' \\ y'' \end{Bmatrix}^T \begin{bmatrix} EI_{As} & 0 \\ 0 & EI_{As} \end{bmatrix} \begin{Bmatrix} x'' \\ y'' \end{Bmatrix} ds \tag{A.17}$$

By substituting Eq. (A.9) into Eq. (A.17), the following is obtained:

$$dU_s = \frac{1}{2} EI_{As} \mathbf{q}_s^T \mathbf{\Psi}''^T \mathbf{\Psi}'' \mathbf{q}_s ds \tag{A.18}$$

The potential energy of the element is obtained by integrating this from 0 to L:

$$\begin{aligned} U_s &= \int_0^L dU_s = \frac{1}{2} \int_0^L \mathbf{q}_s^T \mathbf{\Psi}''^T \mathbf{\Psi}'' \mathbf{q}_s EI_{As} ds \\ U_s &= \frac{1}{2} \mathbf{q}_s^T \mathbf{K}_s \mathbf{q}_s \end{aligned} \tag{A.19}$$

where $\mathbf{K}_s$ is the shaft element stiffness matrix. Next, the kinetic energy is considered. The transverse inertia dI_{Ts} of the thin disk with infinitesimal thickness ds, shown in Fig. 2.5, can be obtained from Eq. (2.6) as follows:

$$dI_{Ts} = \frac{1}{12} m_s \left[\frac{3\left(d_o^2 + d_i^2\right)}{4} + \underbrace{ds^2}_{\approx 0} \right] \approx \frac{1}{2} dI_{Ps} = \frac{1}{2} \overline{I}_{Ps} ds \tag{A.20}$$

where $\overline{I}_{Ps}$ is polar moment of inertia per unit length. In the same manner as obtaining kinetic energy of the rigid disk, shown in Eq. (A.3), the kinetic energy of the thin disk is as follows:

$$\begin{aligned} dT_s &= \frac{1}{2} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix}^T \begin{bmatrix} \rho_s & 0 \\ 0 & \rho_s \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} A_s ds + \frac{1}{2} \omega^2 \overline{I}_{Ps} ds \\ &+ \frac{1}{4} \begin{Bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \end{Bmatrix}^T \begin{bmatrix} \overline{I}_{Ps} & 0 \\ 0 & \overline{I}_{Ps} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \end{Bmatrix} ds + \frac{1}{2} \omega \left(\dot{\theta}_x \theta_y - \theta_x \dot{\theta}_y \right) \overline{I}_{Ps} ds \end{aligned} \tag{A.21}$$

By substituting Eq. (A.9) and (A.14) into Eq. (A.21) to obtain:

$$\begin{aligned} dT_s &= \frac{1}{2} \dot{\mathbf{q}}_s^T \mathbf{\Psi}^T \mathbf{\Psi} \dot{\mathbf{q}}_s \rho_s A_s ds + \frac{1}{2} \omega^2 \overline{I}_{Ps} ds + \frac{1}{4} \dot{\mathbf{q}}_s^T \mathbf{\Phi}^T \mathbf{\Phi} \dot{\mathbf{q}}_s \overline{I}_{Ps} ds \\ &+ \frac{1}{2} \omega \left(\dot{\theta}_x \theta_y - \theta_x \dot{\theta}_y \right) \overline{I}_{Ps} ds \end{aligned} \tag{A.22}$$

From Eq. (A.22), only the last term on the right-hand is still not written in term of $\mathbf{q_s}$. However, this term can be manipulated by using Eq. (A.14) as follows:

$$\begin{aligned}\dot{\theta}_x\theta_y - \theta_x\dot{\theta}_y &= \mathbf{\Phi}_x\dot{\mathbf{q}}_s\mathbf{\Phi}_y\mathbf{q}_s - \mathbf{\Phi}_x\mathbf{q}_s\mathbf{\Phi}_y\dot{\mathbf{q}}_s \\ &= \dot{\mathbf{q}}_s^T\mathbf{\Phi}_x^T\mathbf{\Phi}_y\mathbf{q}_s - \dot{\mathbf{q}}_s^T\mathbf{\Phi}_y^T\mathbf{\Phi}_x\mathbf{q}_s\end{aligned} \tag{A.23}$$

Sutstituting Eq. (A.23) into (A.22) yields:

$$\begin{aligned}dT_s &= \frac{1}{2}\dot{\mathbf{q}}_s^T\mathbf{\Psi}^T\mathbf{\Psi}\dot{\mathbf{q}}_s\rho_s A_s ds + \frac{1}{2}\omega^2\overline{I}_{Ps}ds + \frac{1}{4}\dot{\mathbf{q}}_s^T\mathbf{\Phi}^T\mathbf{\Phi}\dot{\mathbf{q}}_s\overline{I}_{Ps}ds \\ &+ \frac{1}{2}\omega\dot{\mathbf{q}}_s^T\mathbf{\Phi}_x^T\mathbf{\Phi}_y\mathbf{q_s}\overline{I}_{Ps}ds - \frac{1}{2}\omega\dot{\mathbf{q}}_s^T\mathbf{\Phi}_y^T\mathbf{\Phi}_x\mathbf{q}_s\overline{I}_{Ps}ds\end{aligned} \tag{A.24}$$

The kinetic energy of the element is obtained by integrating Eq. (A.24) from 0 to L.

$$T_s = \frac{1}{2}\dot{\mathbf{q}}_s^T\mathbf{M}_s\dot{\mathbf{q}}_s + \frac{1}{2}\omega^2\overline{I}_{Ps}L + \frac{1}{2}\omega\dot{\mathbf{q}}_s^T\mathbf{N}_s\mathbf{q}_s \tag{A.25}$$

where

$$\begin{aligned}&\mathbf{M}_s = \mathbf{M}_{s,t} + \mathbf{M}_{s,r} \\ &\mathbf{M}_{s,t} = \int_0^L \mathbf{\Psi}^T\mathbf{\Psi}\rho_s A_s ds \quad \mathbf{M_{s,r}} = \frac{1}{2}\int_0^L \mathbf{\Phi}^T\mathbf{\Phi}\overline{I}_{Ps}ds \\ &\mathbf{N}_s = \mathbf{N}_{s,a} - \mathbf{N}_{s,b} \\ &\mathbf{N}_{s,a} = \int_0^L \mathbf{\Phi}_x^T\mathbf{\Phi}_y\overline{I}_{Ps}ds \quad \mathbf{N_{s,a}} = \int_0^L \mathbf{\Phi}_y^T\mathbf{\Phi}_x\overline{I}_{Ps}ds\end{aligned} \tag{A.26}$$

Next, the virtual work is considered. Let the distributed nonconservative forces in the X- and Y-directions and nonconservative moments about X- and Y-axes be $f_x(s, t)$, $f_y(s, t)$, $m_x(s, t)$ and $m_y(s, t)$, respectively. Also, let the forces and moments exerted by the adjacent elements on the right and left ends following the order of displacements in $\mathbf{q_s}$ as:

$$\mathbf{f}^* = \left\{ f_{x1}^* \quad f_{y1}^* \quad \cdots \quad f_{\theta x2}^* \quad f_{\theta y2}^* \right\} \tag{A.27}$$

The virtual work δW done by the distributed nonconservative forces that have the virtual displacements $\delta x(s, t)$, $\delta y(s, t)$, $\delta\theta_x(s, t)$, $\delta\theta_y(s, t)$ at position s, and the nodal forces and moments exerted by the adjacent elements that have virtual displacements δx_1, δy_1, …, $\delta\theta_{x2}$, $\delta\theta_{y2}$ at the ends of the element is given as follows:

$$\begin{aligned}\delta W &= \int_0^L f_x(s,t)\delta x(s,t)ds + \int_0^L f_y(s,t)\delta y(s,t)ds \\ &+ \int_0^L m_x(s,t)\delta\theta_x(s,t)ds + \int_0^L m_y(s,t)\delta\theta_y(s,t)ds \\ &+ f_{x1}^*(t)\delta x_1(t) + f_{y1}^*(t)\delta y_1(t) + \cdots + f_{\theta x2}^*(t)\delta\theta_{x2}(t) + f_{\theta y2}^*(t)\delta\theta_{y2}(t)\end{aligned} \tag{A.28}$$

In Eq. (A.28), the virtual displacements at position s can be expressed in terms of shape function interpolation using Eq. (A.9) and (A.14) as follows:

$$\begin{aligned}\delta x(s,t) &= \mathbf{\Psi}_x(s)\delta\mathbf{q}_s(t) \\ \delta y(s,t) &= \mathbf{\Psi}_y(s)\delta\mathbf{q}_s(t) \\ \delta\theta_x(s,t) &= \mathbf{\Phi}_x(s)\delta\mathbf{q}_s(t) \\ \delta\theta_y(s,t) &= \mathbf{\Phi}_y(s)\delta\mathbf{q}_s(t)\end{aligned} \tag{A.29}$$

By substituting Eq. (A.29) into Eq. (A.28), the following pattern can be obtained:

$$
\begin{aligned}
\delta W &= \sum_{i=1}^{8}\Big[\int_0^L f_x(s,t)\Psi_{x,i}(s)\delta q_{s,i}(t)ds + \int_0^L f_y(s,t)\Psi_{y,i}(s)\delta q_{s,i}(t)ds \\
&\quad + \int_0^L m_x(s,t)\Phi_{x,i}(s)\delta q_{s,i}(t)ds + \int_0^L m_y(s,t)\Phi_{y,i}(s)\delta q_{s,i}(t)ds \\
&\quad + f_i^*(t)\delta q_{s,i}(t)\Big] \\
&= \sum_{i=1}^{8}\delta q_{s,i}(t)\Big[\int_0^L f_x(s,t)\Psi_{x,i}(s)ds + \int_0^L f_y(s,t)\Psi_{y,i}(s)ds \\
&\quad + \int_0^L m_x(s,t)\Phi_{x,i}(s)ds + \int_0^L m_y(s,t)\Phi_{y,i}(s)ds + f_i^*(t)\Big] \\
&= \mathbf{f}_s(t)\delta\mathbf{q}_s
\end{aligned}
\tag{A.30}
$$

where the subscript i represents the i-th member of the corresponding vector and $\mathbf{f}_s$ is a row vector composed of the nodal forces $f_{s,i}$ (i=1-8).

$$
\begin{aligned}
f_{s,i}(t) &= \int_0^L f_x(s,t)\Psi_{x,i}(s)ds + \int_0^L f_y(s,t)\Psi_{y,i}(s)ds \\
&\quad + \int_0^L m_x(s,t)\Phi_{x,i}(s)ds + \int_0^L m_y(s,t)\Phi_{y,i}(s)ds + f_i^*(t)
\end{aligned}
\tag{A.31}
$$

After the potential energy, kinetic energy, and non-conservative force are derived. The EOM can be obtained by substituting Eq. (A.19), (A.25) and (A.31) into Lagrange's equation:

$$
\frac{d}{dt}\left(\frac{\partial T_s}{\partial \dot{q}_s}\right) - \frac{\partial T_s}{\partial q_s} + \frac{\partial U_s}{\partial q_s} = f_{s,i} \tag{A.32}
$$

From Eq. (A.19), the partial derivative respected to q_s is as follows:

$$
\frac{\partial U_s}{\partial q_s} = \mathbf{K}_s\mathbf{q}_s \tag{A.33}
$$

From Eq. (A.25), the partial derivatives respected to $\dot{q}_s$ and q_s are as follows:

$$
\begin{aligned}
\left\{\frac{d}{dt}\left(\frac{\partial T_s}{\partial \dot{q}_s}\right)\right\} &= \mathbf{M}_s\ddot{\mathbf{q}}_s + \frac{1}{2}\omega\mathbf{N}_s\dot{\mathbf{q}}_s \\
\frac{\partial T_s}{\partial q_s} &= \frac{1}{2}\omega\dot{\mathbf{q}}_s^T\mathbf{N}_s = \frac{1}{2}\omega\mathbf{N}_s^T\dot{\mathbf{q}}_s
\end{aligned}
\tag{A.34}
$$

By substituting Eq. (A.33) and (A.34) into Eq. (A.32), the following is obtained:

$$
\mathbf{M}_s\ddot{\mathbf{q}}_s + \frac{1}{2}\omega\left(\mathbf{N}_s - \mathbf{N}_s^T\right)\dot{\mathbf{q}}_s + \mathbf{K}_s\mathbf{q}_s = \mathbf{f}_s(t) \tag{A.35}
$$

or

$$
\mathbf{M}_s\ddot{\mathbf{q}}_s + \mathbf{G}_s\dot{\mathbf{q}}_s + \mathbf{K}_s\mathbf{q}_s = \mathbf{f}_s(t) \tag{A.36}
$$

where $\mathbf{G}_s$ is shaft element gyroscopic matrix:

$$
\mathbf{G}_s = \frac{1}{2}\omega\left(\mathbf{N}_s - \mathbf{N}_s^T\right) \tag{A.37}
$$

The term $\mathbf{K}_s$ in Eq. (A.36) can be obtained from Eq. (A.19)

$$
\mathbf{K}_s = \int_0^L \mathbf{\Psi}''^T\mathbf{\Psi}''EI_{As}ds \tag{A.38}
$$

By substituting $\psi_1,\cdots,\psi_4$ in Eq. (A.12) into $\mathbf{\Psi}$ in Eq. (A.13) and then substituting the obtained $\mathbf{\Psi}$ into Eq. (A.38), the shaft element stiffness matrix $\mathbf{K}_s$ is given as follows:

$$\mathbf{K}_s = \frac{EI_{As}}{L^3}\begin{bmatrix} 12 & 0 & 0 & 6L & -12 & 0 & 0 & 6L \\ 0 & 12 & -6L & 0 & 0 & -12 & -6L & 0 \\ 0 & -6L & 4L^2 & 0 & 0 & 6L & 2L^2 & 0 \\ 6L & 0 & 0 & 4L^2 & -6L & 0 & 0 & 2L^2 \\ -12 & 0 & 0 & -6L & 12 & 0 & 0 & -6L^2 \\ 0 & -12 & 6L & 0 & 0 & 12 & 6L & 0 \\ 0 & -6L & 2L^2 & 0 & 0 & 6L & 4L^2 & 0 \\ 6L & 0 & 0 & 2L^2 & -6L^2 & 0 & 0 & 4L^2 \end{bmatrix} \tag{A.39}$$

Similarly, by substituting the obtained $\mathbf{\Psi}$ into Eq. (A.26), $\mathbf{M}_{s,t}$ can be obtained as follows:

$$\mathbf{M}_{s,t} = \frac{m_s}{420}\begin{bmatrix} 156 & 0 & 0 & 22L & 54 & 0 & 0 & -13L \\ 0 & 156 & -22L & 0 & 0 & 54 & 13L & 0 \\ 0 & -22L & 4L^2 & 0 & 0 & -13L & -3L^2 & 0 \\ 22L & 0 & 0 & 4L^2 & 13L & 0 & 0 & -3L^2 \\ 54 & 0 & 0 & 13L & 156 & 0 & 0 & -22L \\ 0 & 54 & -13L & 0 & 0 & 156 & 22L & 0 \\ 0 & 13L & -3L^2 & 0 & 0 & 22L & 4L^2 & 0 \\ -13L & 0 & 0 & -3L^2 & -22L & 0 & 0 & 4L^2 \end{bmatrix} \tag{A.40}$$

Also, $\mathbf{M}_{s,r}$ can be obtained by substituting derivatives respected to s $\psi_1',\cdots,\psi_4'$ of Eq. (A.12) into Eq. (A.15) to get $\mathbf{\Phi}$ then substituting the resulted $\mathbf{\Phi}$ into Eq. (A.26):

$$\mathbf{M}_{s,r} = \frac{I_{Ps}}{60L^2}\begin{bmatrix} 36 & 0 & 0 & 3L & -36 & 0 & 0 & 3L \\ 0 & 36 & -3L & 0 & 0 & -36 & -3L & 0 \\ 0 & -3L & 4L^2 & 0 & 0 & 3L & -L^2 & 0 \\ 3L & 0 & 0 & 4L^2 & -3L & 0 & 0 & -L^2 \\ -36 & 0 & 0 & -3L & 36 & 0 & 0 & -3L \\ 0 & -36 & 3L & 0 & 0 & 36 & 3L & 0 \\ 0 & -3L & -L^2 & 0 & 0 & 3L & 4L^2 & 0 \\ 3L & 0 & 0 & -L^2 & -3L & 0 & 0 & 4L^2 \end{bmatrix} \tag{A.41}$$

Hence, the shaft element mass matrix $\mathbf{M}_s$ in Eq. (A.26) is obtained:

$$\mathbf{M}_s = \mathbf{M}_{s,t} + \mathbf{M}_{s,r} \tag{A.42}$$

Then, the shaft element gyroscopic matrix $\mathbf{G}_s$ can be determined from Eq. (A.37) using the already substituted $\mathbf{\Phi}$ used during $\mathbf{M}_{s,r}$ calculation:

$$\mathbf{G}_s = \frac{I_{Ps}}{30L^2}\begin{bmatrix} 0 & 36 & -3L & 0 & 0 & -36 & -3L & 0 \\ -36 & 0 & 0 & -3L & 36 & 0 & 0 & -3L \\ 3L & 0 & 0 & 4L^2 & -3L & 0 & 0 & -L^2 \\ 0 & 3L & -4L^2 & 0 & 0 & -3L & L^2 & 0 \\ 0 & -36 & 3L & 0 & 0 & 36 & 3L & 0 \\ 36 & 0 & 0 & 3L & -36 & 0 & 0 & 3L \\ 3L & 0 & 0 & -L^2 & -3L & 0 & 0 & 4L^2 \\ 0 & 3L & L^2 & 0 & 0 & -3L & -4L^2 & 0 \end{bmatrix} \tag{A.43}$$

Lastly, gravitational force and moment acting on the nodes of the element are derived. By considering Fig. 2.6, which represents only the shaft part of the rotor in Fig. 2.2:

Fig. 2.6 Nodes and elements of the rotor

Let the weight per length of the i-th shaft element be $-m_{s,i}g / L_i$. From the shape functions in Eq. (A.12), the consistent force vector acting on the i-th and i+1-th nodes in YZ-plane can be written as:

$$\begin{Bmatrix} f_{gy,i} \\ m_{gx,i} \\ f_{gy,i+1} \\ m_{gx,i+1} \end{Bmatrix} = -\frac{m_{s,i}g}{L_i}\int_0^{L_i}\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_3 \end{Bmatrix} ds = \begin{Bmatrix} -\dfrac{m_{s,i}g}{2} \\ -\dfrac{m_{s,i}gL_i}{12} \\ -\dfrac{m_{s,i}g}{2} \\ \dfrac{m_{s,i}gL_i}{12} \end{Bmatrix} \tag{A.44}$$

Hence, the full gravitational force and moment acting on the i-th shaft element is as follows:

$$\mathbf{f}_{gs,i} = \begin{Bmatrix} 0 & -\dfrac{m_{s,i}g}{2} & -\dfrac{m_{s,i}gL_i}{12} & 0 & 0 & -\dfrac{m_{s,i}g}{2} & \dfrac{m_{s,i}gL_i}{12} & 0 \end{Bmatrix}^T \tag{A.45}$$

Appendix B

Derivation of Reynolds Equation

The Reynolds equation was firstly derived by Reynolds [46] from the following continuity and Navier-Stokes equations in Cartesian coordinates. The following derivations are taken from Szeri [49].

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \tag{B.1}$$

$$\begin{aligned}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)&=\frac{\partial}{\partial x}\left[-p+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}\right]\\
&+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\rho f_x\\
\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)&=\frac{\partial}{\partial y}\left[-p+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial v}{\partial y}\right]\\
&+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\rho f_y\\
\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)&=\frac{\partial}{\partial z}\left[-p+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}\right]\\
&+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\rho f_z
\end{aligned} \tag{B.2}$$

where u, v, w are velocity component in X, Y, Z direction, respectively. p is pressure, ρ is density, f is body force, μ and λ is viscosity and dilation viscosity, respectively. For bearing application, the in-plane dimensions L_{xz} of the oil film are much larger than the film thickness L_y. For example, the plain circular bearing with R_j = 15 mm and C_r = 60 μm, shown in Fig. B.1 has the circumferential length of the fluid domain 94.6 mm whereas the maximum possible film thickness obtained from Eq. (2.16) is 120 μm. Hence the aspect ratio $\varepsilon = L_y / L_{xz}$ is in order of $O(10^{-3})$.

Fig. B.1 Dimensional comparison of the fluid film

To perform thin-film approximation of Eq. (B.1) and (B.2), the coordinates are nondimensionalized to:

$$\left(\bar{x},\bar{y},\bar{z}\right)=\frac{1}{L_{xz}}\left(x,\frac{1}{\varepsilon}y,z\right) \tag{B.3}$$

this will make $\bar{x},\bar{y},\bar{z}$ in the same order of magnitude. Similarly, the velocity is also nondimensionalized in the same manner:

$$\left(\bar{u},\bar{v},\bar{w}\right)=\frac{1}{U^*}\left(u,v,w\right) \tag{B.4}$$

where U^* is characteristic value of the velocity in *XZ* plane. In addition, nondimensional pressure, time, density, viscosity, and dilation viscosity are defined as follows:

$$\bar{p}=R_\varepsilon\frac{p}{\rho^* U^{*2}};\ \bar{t}=\frac{U^* t}{L_{xz}};\ \bar{\rho}=\frac{\rho}{\rho^*};\ \bar{\mu}=\frac{\mu}{\mu^*};\ \bar{\lambda}=\frac{\lambda}{\lambda^*} \tag{B.5}$$

where R_ε is reduced Reynolds number defined by:

$$R_\varepsilon=\mathrm{Re};\ \mathrm{Re}=\frac{\rho^* L_y U^*}{\mu^*} \tag{B.6}$$

By substitute Eq. (B.3) to (B.5) into Eq. (B.1), the nondimensional continuity equation is:

$$\frac{d\bar{\rho}}{d\bar{t}}+\bar{\rho}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)=0 \tag{B.7}$$

Similarly, the nondimensional and Navier-Stokes equation is

$$\begin{aligned}R_\varepsilon\bar{\rho}\left(\frac{\partial\bar{u}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}+\bar{w}\frac{\partial\bar{u}}{\partial\bar{z}}\right)&=-\frac{\partial\bar{p}}{\partial\bar{x}}+\frac{\partial}{\partial\bar{y}}\left(\bar{\mu}\frac{\partial\bar{u}}{\partial\bar{y}}\right)\\&+\varepsilon^2\left[\left(\frac{\lambda_0}{\mu_0}\right)\frac{\partial}{\partial\bar{x}}\left\{\bar{\lambda}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)\right\}+\frac{\partial}{\partial\bar{x}}\left(2\bar{\mu}\frac{\partial\bar{u}}{\partial\bar{x}}\right)\right.\\&\left.+\frac{\partial}{\partial\bar{y}}\left(\bar{\mu}\frac{\partial\bar{v}}{\partial\bar{x}}\right)+\frac{\partial}{\partial\bar{z}}\left\{\bar{\mu}\left(\frac{\partial\bar{u}}{\partial\bar{z}}+\frac{\partial\bar{w}}{\partial\bar{x}}\right)\right\}\right]\end{aligned} \tag{B.8a}$$

$$\begin{aligned}\varepsilon^2\left[R_\varepsilon\bar{\rho}\left(\frac{\partial\bar{v}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}+\bar{w}\frac{\partial\bar{v}}{\partial\bar{z}}\right)-\left(\frac{\lambda_0}{\mu_0}\right)\frac{\partial}{\partial\bar{y}}\left\{\bar{\lambda}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)\right\}\right.\\-\frac{\partial}{\partial\bar{y}}\left(2\bar{\mu}\frac{\partial\bar{v}}{\partial\bar{y}}\right)-\frac{\partial}{\partial\bar{x}}\left(\bar{\mu}\frac{\partial\bar{u}}{\partial\bar{y}}\right)-\frac{\partial}{\partial\bar{z}}\left(\bar{\mu}\frac{\partial\bar{w}}{\partial\bar{y}}\right)\\\left.-\varepsilon^2\left\{\frac{\partial}{\partial\bar{x}}\left(\bar{\mu}\frac{\partial\bar{v}}{\partial\bar{x}}\right)+\frac{\partial}{\partial\bar{z}}\left(\bar{\mu}\frac{\partial\bar{v}}{\partial\bar{z}}\right)\right\}\right]=-\frac{\partial\bar{p}}{\partial\bar{y}}\end{aligned} \tag{B.8b}$$

$$\begin{aligned}R_\varepsilon\bar{\rho}\left(\frac{\partial\bar{w}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{w}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{w}}{\partial\bar{y}}+\bar{w}\frac{\partial\bar{w}}{\partial\bar{z}}\right)&=-\frac{\partial\bar{p}}{\partial\bar{z}}+\frac{\partial}{\partial\bar{y}}\left(\bar{\mu}\frac{\partial\bar{w}}{\partial\bar{y}}\right)\\&+\varepsilon^2\left[\left(\frac{\lambda_0}{\mu_0}\right)\frac{\partial}{\partial\bar{z}}\left\{\bar{\lambda}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)\right\}+\frac{\partial}{\partial\bar{z}}\left(2\bar{\mu}\frac{\partial\bar{w}}{\partial\bar{z}}\right)\right.\\&\left.+\frac{\partial}{\partial\bar{y}}\left(\bar{\mu}\frac{\partial\bar{v}}{\partial\bar{z}}\right)+\frac{\partial}{\partial\bar{x}}\left\{\bar{\mu}\left(\frac{\partial\bar{u}}{\partial\bar{z}}+\frac{\partial\bar{w}}{\partial\bar{x}}\right)\right\}\right]\end{aligned} \tag{B.8c}$$

Since, ε is in order of $O(10^{-3})$, the ε^2 terms in Eq. (B.8) can be neglected:

$$R_\varepsilon \bar{\rho}\left(\frac{\partial \bar{u}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}\right)=-\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{\partial}{\partial \bar{y}}\left(\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{y}}\right) \tag{B.9a}$$

$$\frac{\partial \bar{p}}{\partial \bar{y}}=0 \tag{B.9b}$$

$$R_\varepsilon \bar{\rho}\left(\frac{\partial \bar{w}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{w}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{w}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{w}}{\partial \bar{z}}\right)=-\frac{\partial \bar{p}}{\partial \bar{z}}+\frac{\partial}{\partial \bar{y}}\left(\bar{\mu}\frac{\partial \bar{w}}{\partial \bar{y}}\right) \tag{B.9c}$$

Equation (B.9b) indicates that the pressure does not vary across the thickness of the film. So, $p=p(x, z, t)$. To further reduce Eq. (B.9), Reynolds used the following assumptions:

1. The continuum description is valid.
2. The Navier-Stokes equation hold.
3. Incompressible fluid.
4. Constant viscosity fluid.
5. Thin fluid film. Hence, the flow is laminar and fluid inertia is negligible.

Assumption (5) is valid, even the fluid film is bounded by curved surfaces of the journal and the bearing, because the ratio of film thickness and the radius of the pad is very small (also in the order of ε). Hence, segments of the journal and the bearing can be considered as planes as shown in Fig. B.2.

Fig. B.2 Planes of journal and bearing segments

Also, according to assumption (5), the temporal and convection acceleration terms on the left-hand side of Eq. (B.9a) and (B.9c) can be neglected. Hence the dimensional form of the simplified Eq. (B.9) is as follows:

$$\frac{\partial p}{\partial x}=\mu\frac{\partial^2 u}{\partial y^2} \tag{B.10a}$$

$$\frac{\partial p}{\partial y}=0 \tag{B.10b}$$

$$\frac{\partial p}{\partial z}=\mu\frac{\partial^2 w}{\partial y^2} \tag{B.10c}$$

Since $\partial p / \partial y = 0$, Eq. (B.10a) and (B.10c) are integrated twice with respect to y to obtain

$$\begin{aligned} u &= \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + Ay + B \\ w &= \frac{1}{2\mu}\frac{\partial p}{\partial z}y^2 + Cy + D \end{aligned} \tag{B.11}$$

From Fig. B.2, the boundary conditions for u and w are:

$$\begin{aligned} y=0: &\quad u=U_1 \quad v=V_1 \quad w=0 \\ y=H: &\quad u=U_2 \quad v=V_2 \quad w=0 \end{aligned} \tag{B.12}$$

Substituting Eq. (B.12) into Eq. (B.11) yields velocity distribution

$$\begin{aligned} u &= \frac{1}{2\mu}\frac{\partial p}{\partial x} y\left(y-H\right)+\left(1-\frac{y}{H}\right)U_1+\frac{y}{H}U_2 \\ w &= \frac{1}{2\mu}\frac{\partial p}{\partial z} y\left(y-H\right) \end{aligned} \tag{B.13}$$

In addition, the continuity equation (B.1) can be reduced by assumption (5) to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{B.14}$$

Substitute Eq. (B.13) into Eq. (B.14) yields

$$\begin{aligned} \frac{\partial v}{\partial y} &= -\frac{\partial}{\partial x}\left[\frac{1}{2\mu}\frac{\partial p}{\partial x} y\left(y-H\right)\right]-\frac{\partial}{\partial z}\left[\frac{1}{2\mu}\frac{\partial p}{\partial z} y\left(y-H\right)\right] \\ &\quad -\frac{\partial}{\partial x}\left[\left(1-\frac{y}{H}\right)U_1+\frac{y}{H}U_2\right] \end{aligned} \tag{B.15}$$

Integrating Eq. (B.15) with respect to y using the boundary conditions in Eq. (B.12) yields:

$$\begin{aligned} V_2-V_1 &= -\int_0^H \frac{1}{2\mu}\frac{\partial}{\partial x}\left[\frac{\partial p}{\partial x} y\left(y-H\right)\right]dy-\int_0^H \frac{1}{2\mu}\frac{\partial}{\partial z}\left[\frac{\partial p}{\partial z} y\left(y-H\right)\right]dy \\ &\quad -\int_0^H \frac{\partial}{\partial x}\left[\left(1-\frac{y}{H}\right)U_1+\frac{y}{H}U_2\right]dy \end{aligned} \tag{B.16}$$

By employing Leibnitz's rule for differentiating under the integral sign:

$$\frac{d}{dx}\int_A^B f\left(x,y\right)dy=\int_A^B \frac{\partial f\left(x,y\right)}{\partial x}dy+f\left(x,B\right)\frac{dB}{dx}-f\left(x,A\right)\frac{dA}{dx} \tag{B.17}$$

In this case, $A = 0$, $B = H$. Thus

$$\int_0^H \frac{\partial f\left(x,y\right)}{\partial x}dy=\frac{d}{dx}\int_0^H f\left(x,y\right)dy-f\left(x,H\right)\frac{dH}{dx} \tag{B.18}$$

the terms in square brackets in Eq. (B.16) correspond to $f\left(x,y\right)$ in Eq. (B.18). Substituting Eq. (B.18) into Eq. (B.17) gives

$$\begin{aligned} V_2-V_1 &= -\left[\frac{\partial}{\partial x}\int_0^H \frac{1}{2\mu}\frac{\partial p}{\partial x} y\left(y-H\right)dy-0\right]-\left[\frac{\partial}{\partial z}\int_0^H \frac{1}{2\mu}\frac{\partial p}{\partial z} y\left(y-H\right)dy-0\right] \\ &\quad -\left[\frac{\partial}{\partial x}\int_0^H \left\{\left(1-\frac{y}{H}\right)U_1+\frac{y}{H}U_2\right\}dy-U_2\frac{\partial H}{\partial x}\right] \\ &= \left[\frac{\partial}{\partial x}\left(\frac{H^3}{12\mu}\frac{\partial p}{\partial x}\right)\right]+\left[\frac{\partial}{\partial z}\left(\frac{H^3}{12\mu}\frac{\partial p}{\partial z}\right)\right] \\ &\quad -\left[\frac{\partial}{\partial x}\left\{U_1\left(\frac{H}{2}\right)+U_2\left(\frac{H}{2}\right)\right\}-U_2\frac{\partial H}{\partial x}\right] \end{aligned} \tag{B.19}$$

which yields Reynolds equation for lubrication between planes

$$\frac{\partial}{\partial x}\left(\frac{H^3}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{H^3}{\mu}\frac{\partial p}{\partial z}\right)=12\left(V_2-V_1\right)+6\left(U_1-U_2\right)\frac{\partial H}{\partial x}+6H\frac{\partial}{\partial x}\left(U_1+U_2\right) \tag{B.20}$$

The first, second, and third terms on the right-hand side of Eq. (B.20) are characterized as squeeze, wedge, and stretch terms, respectively. The squeeze and wedge terms are generated by relative radial and tangential velocities, respectively.

After Reynolds equation is obtained, the velocities on the right-hand side of Eq. (B.20) is derived from journal translation and rotation shown in the kinematics diagram of Fig. B.3.

Fig. B.3 Kinematics diagram of journal and bearing

Let the point Q is on the journal's surface that has velocity $\boldsymbol{V}_2$ consisted of velocities from journal's rotation $\boldsymbol{V}_{2,r}$ and translation $\boldsymbol{V}_{2,t}$, respectively. H is film thickness between point Q and P, which is on the bearing. From Fig. B.3 and thin film assumption

$$U_{2,r} = \left|\boldsymbol{V}_{2,r}\right|\cos\alpha = \left|\boldsymbol{V}_{2,r}\right|\left[1-\frac{1}{2}\left(\frac{\partial H}{\partial x}\right)^2+\cdots\right] \approx \left|\boldsymbol{V}_{2,r}\right|$$
$$\because \frac{\partial H}{\partial x} \ll 1 \Rightarrow \therefore U_{2,r} \approx \left|\boldsymbol{V}_{2,r}\right| \tag{B.21}$$

Similarly

$$V_{2,r} = \left|\boldsymbol{V}_{2,r}\right|\sin\alpha = U_{2,r}\tan\alpha \approx \left|\boldsymbol{V}_{2,r}\right|\frac{\partial H}{\partial x} \tag{B.22}$$

$$U_2 = U_{2,r} + U_{2,t} = U_{2,r}\left(1+\frac{U_{2,t}}{U_{2,r}}\right); \because \frac{U_{2,t}}{U_{2,r}} = O\left(10^{-3}\right) \Rightarrow \therefore U_2 \approx U_{2,r} \tag{B.23}$$

With Eq. (B.23), Eq. (B.22) takes the following form

$$V_{2,r} \approx U_2\frac{\partial H}{\partial x}$$
$$V_2 = V_{2,t} + V_{2,r} \approx V_{2,t} + U_2\frac{\partial H}{\partial x} \tag{B.24}$$

Consider film thickness H in Fig. B.3, it can be seen that H changes with the journal's translation and rotation, which the latter forces both Q and H to change with the speed of $U_{2,r}$. Hence, the time rate of change of H can be given as

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\frac{dx}{dt} = \frac{\partial H}{\partial t} + U_{2,r}\frac{\partial H}{\partial x} = \left(V_{2,t} - V_1\right) + U_{2,r}\frac{\partial H}{\partial x} \tag{B.25}$$

substitute Eq. (B.24) into Eq. (B.20) yields

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{H^3}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{H^3}{\mu}\frac{\partial p}{\partial z}\right) &= 12\left(V_{2,t}+U_2\frac{\partial H}{\partial x}-V_1\right)+6\left(U_1-U_2\right)\frac{\partial H}{\partial x} \\
&\quad +6H\frac{\partial}{\partial x}\left(U_1+U_2\right) \\
&= 12\left(V_{2,t}-V_1\right)+6\left(U_1+U_2\right)\frac{\partial H}{\partial x}+6H\frac{\partial}{\partial x}\left(U_1+U_2\right) \\
&= 12\frac{\partial H}{\partial t}+6\left(U_1+U_2\right)\frac{\partial H}{\partial x}+6H\frac{\partial}{\partial x}\left(U_1+U_2\right)
\end{aligned}
\tag{B.26}
$$

If the bearing is stationary ($U_1 = V_1 = 0$), Eq. (B.26) is reduced to

$$
\frac{\partial}{\partial x}\left(\frac{H^3}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{H^3}{\mu}\frac{\partial p}{\partial z}\right)=12\frac{\partial H}{\partial t}+6U_2\frac{\partial H}{\partial x}+6H\frac{\partial U_2}{\partial x} \tag{B.27}
$$

From Eq. (B.23), the last two terms of Eq. (B.27) can be written as

$$
\begin{aligned}
6U_2\frac{\partial H}{\partial x}+6H\frac{\partial U_2}{\partial x} &= 6\frac{\partial}{\partial x}\left(HU_2\right) \\
&= 6\frac{\partial}{\partial x}\left[H\frac{U_{2,r}}{R}\left(1+\frac{U_{2,t}}{U_{2,r}}\right)\right] \\
&= 6\frac{\partial}{\partial x}\left[H\frac{\omega R}{R}\left(1+\frac{U_{2,t}}{U_{2,r}}\right)\right] \\
&\approx 6\omega\frac{\partial H}{\partial x}
\end{aligned}
\tag{B.28}
$$

For JB, it is more convenient to replace the circumferential variable x by $R\theta$, where R and θ are the journal radius and angular coordinate shown in Fig. B.3, respectively. Changing coordinate and substituting, Eq. (B.28) into Eq. (B.27) yields

$$
\frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\frac{H^3}{12\mu}\frac{\partial p}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\frac{H^3}{12\mu}\frac{\partial p}{\partial z}\right)=\frac{\omega}{2}\frac{\partial H}{\partial\theta}+\frac{\partial H}{\partial t} \tag{B.29}
$$

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