Two-term silting complexes over algebras with small Loewy length and complete special biserial algebras
(短い Loewy 列を持つ多元環と完備特殊双列多元環上の２項準傾複体)

Toshitaka AOKI

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Abstract

Tilting theory plays an important role in the study of many areas of mathematics. Central notions of tilting theory are tilting complexes and silting complexes, which are generalizations of a progenerator in Morita theory. In fact, the endomorphism algebra of a tilting complex is derived equivalent to the original algebra. In this thesis, we mainly study two-term silting complexes. In representation theory of finite dimensional algebras, the g-vectors of two-term silting complexes are important numerical invariant. In the first part of this thesis, we classify all two-term silting complexes and their g-vectors over algebras with radical square zero. Using this result, we also study symmetric algebras with radical cube zero and determine the number of two-term tilting complexes over them. In the second part of this thesis, we study two-term silting theory for (complete) gentle algebras. A central role is played by their geometric realization on marked surfaces. Using our surface model, we show that a union of g-vector cones is dense in the real Grothendieck group. A main ingredient of our proof is the asymptotic behavior of g-vectors under Dehn twists. We also study a special class of special biserial algebras, called Brauer tree algebras, and show that any Brauer tree algebra has $\binom{2n}{n}$ two-term tilting complexes, where n is the number of edges of the corresponding Brauer tree.

Contents

Introduction

Tilting theory plays an important role in the study of many areas of mathematics. A central notion of tilting theory is a tilting module, which originates in a reflection functor by Bernstein–Gelfand–Ponomarev [BGP73] and is extensively studied by many authors such as Auslander–Platzeck–Reiten [APR79], Brenner–Butler [BB80] and Happel–Unger [HU05]. This viewpoint is adopted by Rickard [Ric89] for the notion of a tilting complex as a generalization of a progenerator in Morita theory. In fact, its endomorphism algebra is derived equivalent to the original algebra (that is, their derived categories are equivalent as triangulated categories).

From a viewpoint of mutation theory, the notion of silting complexes is introduced by Aihara–Iyama [AI12] as a generalization of that of tilting complexes. They are in bijection with various important concepts in representation theory of finite dimensional algebras, such as simple minded collections and bounded t-structures with length heart [KY14], and are closely related to cluster theory [BMR$^+$06, FT18, FZ07] and the study of Bridgeland's stability spaces for t-structures [Bri07, BS15, Qiu16]. In addition, if we focus on two-term complexes (i.e., complexes concentrated in only degree 0 and -1), then two-term silting complexes correspond bijectively with functorially finite torsion classes, support τ-tilting modules, two-term simple minded collections and intermediate t-structures (see e.g., [AIR14, HKM02, IJY14]). The bijections among them give partial orders on these objects respectively, which commute with the respective operation of mutations.

We say that a finite dimensional algebra A is *g-finite* if $2\text{-silt}\,A$ is finite (It was called τ-tilting finite in [DIJ19]), where $2\text{-silt}\,A$ is the set of isomorphism classes of basic two-term silting complexes for A. From the bijections, this means that there are only finitely many torsion classes of A, only finitely many isomorphism classes of basic support τ-tilting A-modules, and so on. Such an algebra A plays a fundamental role in mutation theory. In fact, it is known that the Hasse quiver of $2\text{-silt}\,A$ is connected and the g-vector fan of A has a nice property (cf. Theorem 0.0.4).

In this thesis, we study two-term silting complexes over a certain class of algebras, including algebras with radical square zero and gentle algebras. In particular, we give a classification of all two-term silting complexes for each class of algebras. As an application, we also determine the cardinality of the set of two-term silting complexes for these algebras.

Algebras with small Loewy length

In ring theory, the notion of Jacobson radical is basic, by definition, this is the intersection of all maximal (right) ideals. A local ring has the unique maximal ideal which coincides with its Jacobson radical. For a given finite dimensional algebra A, its Jacobson radical J tells us how far away the algebra A is from being semisimple. In fact, it is well-known that A is semisimple if and only if $J = 0$, if and only if $\ell\ell(A) = 1$. Here, $\ell\ell(A)$ is the *Loewy length* of A defined by

$$\ell\ell(A) := \min_{t \in \mathbb{Z}_{>0}} \{t \mid J^t = 0\}.$$

In this case, by Artin-Wedderburn theorem, A is isomorphic to a finite product of matrix algebras over division algebras. Thereby, its module category is completely understood up to Morita equivalence.

In the first part of this thesis, we mainly study algebras with Loewy length 2. Such algebras are called *algebras with radical square zero* (or RSZ algebras for short). A typical example of RSZ algebras is a factor algebra of a path algebra kQ of a finite quiver Q modulo the ideal generated by all path of length 2. In fact, if k is algebraically closed, then all RSZ algebras are obtained in this way up to Morita equivalence. In representation theory of RSZ algebras, a central role is played by a stable equivalence $\underline{\mathrm{mod}} A \cong \underline{\mathrm{mod}} \begin{pmatrix} A/J & J \\ 0 & A/J \end{pmatrix}$, see [Gab72]. Here, the algebra in the right-hand side is a RSZ algebra which is hereditary (that is, its global dimension is at most 1). Using this equivalence, one can characterize representation finite RSZ algebras in terms of Dynkin diagrams [DR74, Gab72].

In Chapter 1, we study silting theory of RSZ algebras by using hereditary RSZ algebras. A main result is the following. Now, we denote by $\mathrm{add}\, M$ a full subcategory consisting of all direct summands of finite direct sums of copies of an A-module M, and by $|M|$ the number of non-isomorphic indecomposable direct summands of M. Let $D := \mathrm{Hom}_k(-, k)$ be the k-dual.

Theorem 0.0.1. *(Theorem 1.3.3) Let A be a finite dimensional RSZ algebra over a field k. For each idempotent e of A with $f := 1 - e$, there is an isomorphism of partially ordered sets between*

(1) the set of isomorphism classes of basic two-term silting complexes $T = (T^{-1} \to T^0)$ for A such that $T^0 \in \mathrm{add}\, eA$ and $T^{-1} \in \mathrm{add}\, fA$, and

(2) the set of isomorphism classes of basic tilting modules over a hereditary RSZ algebra $\begin{pmatrix} f(A/J)f & D(eJf) \\ 0 & e(A/J)e \end{pmatrix}$.

Let $\mathcal{S}(A) := \{e_1, \ldots, e_{|A|}\}$ be a complete set of primitive orthogonal idempotents of A. For each subset $I \subseteq \{1, \ldots, |A|\}$, let e_I be a sum $\sum_{i \in I} e_i$ and $f_I := 1 - e_I$. Then the above maps give a bijection

$$\text{2-silt}\, A \overset{1-1}{\longleftrightarrow} \bigsqcup_{I \subseteq \{1, \ldots, |A|\}} \mathrm{tilt} \begin{pmatrix} f_I(A/J)f_I & D(e_I J f_I) \\ 0 & e_I(A/J)e_I \end{pmatrix},$$

where tilt A *is the set of isomorphism classes of basic tilting A-modules.*

In particular, the set 2-silt A is decomposed into $2^{|A|}$ subsets, which we call sign-decomposition in Section 1.1.7. As a corollary of Theorem 0.0.1, we get a characterization of g-finite RSZ algebras in terms of Dynkin diagrams (Corollary 1.4.3), which generalizes a result of Adachi [Ada16a, Theorem 3.1] (see also [Zha13]) to non-algebraically closed fields.

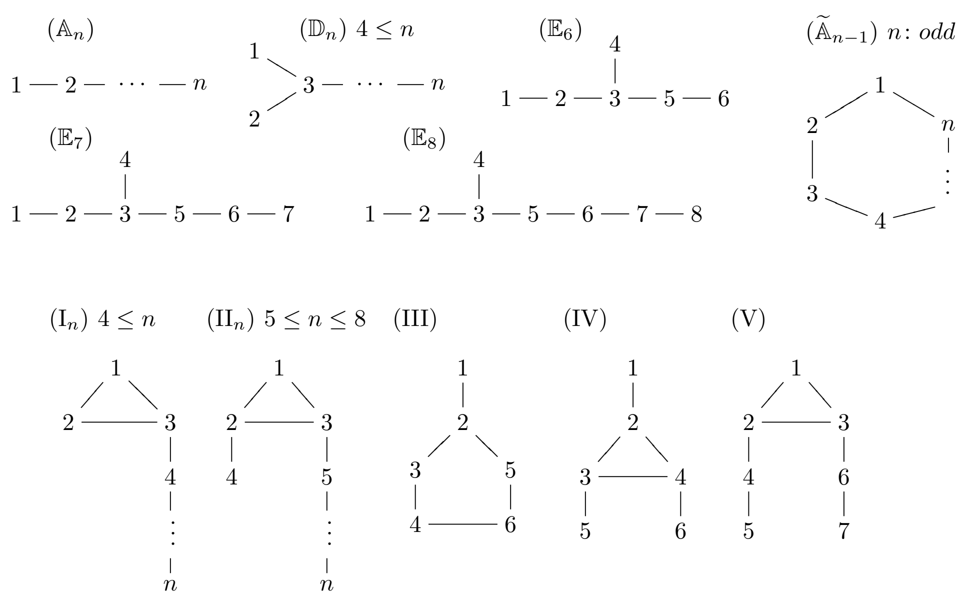
In Chapter 2, we study a certain class of symmetric algebras. For a symmetric algebra A, it is shown in [AI12, Example 2.8] that any silting complex for A is tilting. We denote by 2-tilt A the set of isomorphism classes of basic two-term tilting complexes for A. We especially consider symmetric algebras A with radical cube zero, that is, $\ell\ell(A) = 3$. This class of algebras has been studied by Okuyama [Oku86], Benson [Ben08] and Erdmann–Solberg [ES11], and also appears in several area of mathematics such as [CL12, HK01, Sei08]. Applying τ-tilting theory (Proposition 2.1.2), we have a natural bijection between 2-tilt A and 2-silt$(A/\operatorname{soc} A)$ for the corresponding RSZ algebra $A/\operatorname{soc} A$, where $\operatorname{soc} A$ is the socle of A.

Using results in Chapter 1, we show the following.

Theorem 0.0.2. *(Theorem 2.0.1) Let A be a finite dimensional connected symmetric algebra with radical cube zero over an algebraically closed field k. Then the following conditions are equivalent.*

(1) A is g-finite (i.e., 2-tilt A *is finite).*

(2) The Gabriel quiver of A is obtained by adding finite number of loops to the double quiver Q_G of a graph G in the following list.



Theorem 0.0.3. *(Theorem 2.0.2) In Theorem 0.0.2, the number* $\#\,\text{2-tilt}\,A$ *is given as follows.*

G	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\tilde{\mathbb{A}}_{n-1}$	I_n	II_5	II_6	II_7	II_8	III	IV	V
$\#\,\text{2-tilt}\,A$	$\binom{2n}{n}$	a_n	1700	8872	54066	2^{2n-1}	b_n	632	2936	11306	75240	3108	4056	17328

Here, for any $n \geq 4$, *let* $a_n := 6 \cdot 4^{n-2} - 2\binom{2(n-2)}{n-2}$ *and* $b_n := 6 \cdot 4^{n-2} + 2\binom{2n}{n} - 4\binom{2(n-1)}{n-1} - 4\binom{2(n-2)}{n-2}$.

We remark that the numbers for Dynkin graphs $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$ in the list are precisely biCatalan numbers introduced by [BR18] in the context of Coxeter-Catalan combinatorics. Our results for Dynkin graphs are independently obtained by [DIR+18] in the study of biCambrian lattices for preprojective algebras.

Complete gentle algebras

Gentle algebras, introduced in 1980's, form an important class of special biserial algebras and their representation theory has been studied by many authors (e.g. [AH81, AS87, BR87, CB89]). Moreover, the derived categories of gentle algebras are related to various subjects, such as discrete derived categories [BGS04, BPP16, Voß01], numerical derived invariants [APS19, AAG08, Nak18], and Fukaya categories of surfaces [HKK17, LP20, ST01].

In the second part of this thesis, we study two-term silting theory for gentle algebras. In this thesis, we don't assume that gentle algebras are finite dimensional. For our purpose, we consider the *complete gentle algebras* (see Definition 3.6.10). They are module-finite over $k[[t]]$ (i.e., finitely generated as an $k[[t]]$-module), where $k[[t]]$ is the formal power series ring in one valuable over an algebraically closed field k. In particular, finite dimensional gentle algebras are complete gentle algebras.

For a module finite $k[[t]]$-algebra A, one can discuss two-term silting theory similarly to that for a finite dimensional algebra, see Section 3.6 (cf. [AIR14, DIJ19, Kim20]). Each two-term presilting complex T has a numerical invariant $g_A(T) \in \mathbb{Z}^{|A|}$, called the g-vector of T. Then, one can define a cone in $\mathbb{R}^{|A|}$, called the g-vector cone of T, by

$$C_A(T) := \left\{ \sum_X a_X g_A(X) \mid a_X \in \mathbb{R}_{\geq 0} \right\} \subseteq \mathbb{R}^{|A|},$$

where X runs over all indecomposable direct summands of T. We denote by $\mathcal{F}(A)$ a collection of g-vector cones of all basic two-term presilting complexes for A, by $|\mathcal{F}(A)|$ its geometric realization in $\mathbb{R}^{|A|}$. It is shown in [DIJ19] that $\mathcal{F}(A)$ is a simplicial fan (i.e., every cone is a simplex) and its maximal faces correspond to basic two-term silting complexes for A. Namely,

$$|\mathcal{F}(A)| = \bigcup_{C \in \mathcal{F}(A)} C = \bigcup_{T \in \text{2-silt}\,A} C_A(T).$$

Such a fan plays an important role in the study of stability scattering diagrams and their wall-chamber structures (see e.g. [Asa19, Bri17, BST19, DF15, Yur18]).

The following result is well-known.

Theorem 0.0.4. *[Asa19, DIJ19] Let A be a finite dimensional algebra. Then the following conditions are equivalent.*

(1) A is g-finite.

(2) $|\mathcal{F}(A)| = \mathbb{R}^{|A|}$.

This result naturally leads to the following definition in a general setting.

Definition 0.0.5. Let A be a module finite $k[[t]]$-algebra. We say that A is *g-tame* if it satisfies

$$\overline{|\mathcal{F}(A)|} = \mathbb{R}^{|A|},$$

where $\overline{(-)}$ is the closure with respect to the natural topology on $\mathbb{R}^{|A|}$.

Note that a similar notion, called τ-tilting tame, was given in [BST19]. The g-tameness is known for path algebras of extended Dynkin quivers [Hil06], for complete preprojective algebras of extended Dynkin graphs [KM19], and for Jacobian algebras associated with triangulated surfaces [Yur20].

The aim of Chapter 3 is to prove the g-tameness of a new class.

Theorem 0.0.6. *(Section 3.6) Any complete special biserial algebra is g-tame.*

To prove Theorem 0.0.6, it suffices to prove that complete gentle algebras are g-tame. In fact, any complete special biserial algebra is a factor algebra of a complete gentle algebra (Proposition 3.6.9), and g-tameness is preserved under taking factor algebras (Proposition 3.6.6).

To study complete gentle algebras, their geometric realization plays a central role in this thesis. A similar construction has been developed in several area, such as [AAC18, CD20, KS02, Opp19, OPS18, PPP19]. For each dissection D of a marked surface (S, M), one can define a complete gentle algebra $A(D)$. Conversely, any complete gentle algebra arises in this way (see Sections 3.1 and 3.6.2 for the details). Note that the cardinality $n := |D|$ of D, which is equal to $|A(D)|$, is completely determined by (S, M) (Remark 3.1.3).

In Chapter 3, we study a geometric model of two-term silting theory for complete gentle algebras. For a dissection D of (S, M), we observe a certain class of non-self-intersecting curves of S, called *D-laminates*, and finite multi-set of pairwise non-intersecting D-laminates, called *D-laminations*. We construct a simplicial fan $\mathcal{F}(D)$ such that

$$\mathcal{F}(A(D)) = \mathcal{F}(D) \quad \text{and} \quad |\mathcal{F}(A(D))| = |\mathcal{F}(D)| \ \text{ in } \ \mathbb{R}^n,$$

where j-th dimensional faces of $\mathcal{F}(D)$ consist of reduced D-laminations having $j + 1$ elements. In particular, each maximal face of $\mathcal{F}(D)$, called complete D-lamination, corresponds to a two-term silting complex for $A(D)$.

We prove that the fan $\mathcal{F}(D)$ is dense in $\mathbb{R}^n$. Namely,

Theorem 0.0.7. *(Section 3.5) For a given dissection D of a marked surface (S, M), we have*

$$\overline{|\mathcal{F}(D)|} = \mathbb{R}^n.$$

Consequently, we show the g-tameness of complete gentle algebras. Thereby, we show the g-tameness of complete special biserial algebras.

A main ingredient of our proof of Theorem 0.0.7 is the asymptotic behavior of g-vectors under Dehn twists. This proof is inspired from the proof of [Yur20, Theorem 1.5].

In Chapter 4, we study dissections of marked surfaces (S, M) with no boundary. In this case, every dissection of (S, M) can be regarded as a ribbon graph (Example 3.6.28). We say that a dissection of (S, M) is a *tree* if its underlying graph has no cycles.

The aim of Chapter 4 is to prove the following result.

Theorem 0.0.8. *(Theorem 4.0.2 and 4.3.1) Let D be a dissection of (S, M) whose dual dissection is a tree with $n := |D|$. Let $A(D)$ be the complete gentle algebra associated with D and $B(D)$ the Brauer tree algebra (Definition 3.6.28) associated with D. Then all the following equations hold.*

(1) The number of complete D-laminations is $\binom{2n}{n}$.

(2) $\# \text{2-silt}\, A(D) = \binom{2n}{n}$.

(3) $\# \text{2-tilt}\, B(D) = \binom{2n}{n}$.

In particular, they are invariant on n.

Notice that the Brauer tree algebra $B(D)$ is a finite dimensional symmetric algebra, so silting complexes coincide with tilting complexes. Recently, the claim (3) of Theorem 0.0.8 is shown by [AMN20]. However, their proof is completely different from ours. In fact, they do not use any surface model of Brauer tree algebras.

This thesis is organized as follows. In the first part (Part I) of this thesis, we mainly study algebras with radical square zero. In Chapter 1, we first recall a relationship between torsion classes, tilting modules, two-term silting complexes and intermediate t-structures. After that, we study torsion classes for algebras with radical square zero to prove Theorem 0.0.1. In Chapter 2, we prove Theorem 0.0.2 by using results in the previous chapter. Chapter 1 is based on the author's work [Aok18], and Chapter 2 is based on the joint work with Takahide Adachi [AA18]. In the second part (Part II) of this thesis, we study complete gentle algebras via their geometric realization. In Chapter 3, a surface model of two-term silting theory for complete gentle algebras is established. A proof of Theorem 0.0.7 is given in Section 3.5. In Section 3.6, we define a complete gentle algebra associated with a dissection and prove Theorem 0.0.6. This chapter (Chapter 3) is based on the joint work with Toshiya Yurikusa [AY]. In Chapter 4, we study a class of algebras associated with marked surfaces with no boundary. We give a proof of Theorem 0.0.8 in Section 4.3. Chapter 4 is based on the author's work [Aok].

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Part I

Algebras with small Loewy length

Chapter 1

Classifying torsion classes for algebras with radical square zero

Throughout Part I, by an algebra we mean a finite dimensional basic algebra over a field k, by a module we mean a finitely generated right module. For a given algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $\operatorname{proj} A$ the category of finitely generated projective right A-modules. We denote by $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ the bounded derived category of $\operatorname{mod} A$, by $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ the homotopy category of bounded complexes of projective modules in $\operatorname{proj} A$. Let $D := \operatorname{Hom}_k(-, k)$ be the k-dual. We refer to [ASS06, ARS95] for the definitions and basic results on representation theory over finite dimensional algebras and [Hap88] for the basic definitions on triangulated categories.

This chapter (Chapter 1) is based on the author's work [Aok18].

1.1. Preliminaries

Let A be a finite dimensional algebra over a field k. In this section, we recall a relationship between several important objects in representation theory of finite dimensional algebras, such as torsion classes, tilting modules, silting complexes and t-structures, together with their partial order. The bijections among them will be explained in Section 1.1.6. In Section 1.1.7, we introduce the notion of sign-decomposition of these partially ordered sets.

Let P be a partially ordered set. For $x, y \in P$ with $x \leq y$, an interval $[x, y]$ is defined to be a subset of P consisting of all elements $z \in P$ such that $x \leq z \leq y$. We denote by $\operatorname{Hasse}(P)$ the Hasse quiver of P, by definition, its vertex set is P and there is an arrow $y \to x$ if and only if $x < y$ and there are no elements $z \in P$ satisfying $x < z < y$.

1.1.1 Torsion pairs

A *torsion* pair of $\operatorname{mod} A$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that

- $\operatorname{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$.
- For any $E \in \operatorname{mod} A$, there exists a short exact sequence

$$0 \to M \to E \to N \to 0$$

  with $M \in \mathcal{T}$ and $N \in \mathcal{F}$.

The full subcategories $\mathcal{T}$ and $\mathcal{F}$ are called *torsion class* and *torsion free-class* respectively. We have $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. Here, for a class $\mathcal{C}$ of A-modules, we define its orthogonal categories by

$$\mathcal{C}^{\perp} := \{X \in \operatorname{mod} A \mid \operatorname{Hom}_A(\mathcal{C}, X) = 0\} \text{ and } {}^{\perp}\mathcal{C} := \{X \in \operatorname{mod} A \mid \operatorname{Hom}_A(X, \mathcal{C}) = 0\}.$$

Thus, a torsion pair is determined by its torsion class (respectively, its torsion free-class).

For a class $\mathcal{C}$ of A-modules, we denote by $\operatorname{Filt} \mathcal{C}$ (resp., $\operatorname{Fac} \mathcal{C}$ and $\operatorname{Sub} \mathcal{C}$) the full subcategory consisting of A-modules which are filtered by (factor modules of, submodules of) finite direct sum of copies of A-modules in $\mathcal{C}$. We denote by $\mathsf{T}(\mathcal{C})$ the smallest torsion class containing $\mathcal{C}$. More precisely, it can be written as $\mathsf{T}(\mathcal{C}) = \operatorname{Filt} \operatorname{Fac} \mathcal{C}$.

Now, we denote by $\operatorname{tors} A$ the set of all torsion classes of $\operatorname{mod} A$. On this set, there is a natural partial order given by

$$\mathcal{U} \leq \mathcal{T} :\iff \mathcal{U} \subseteq \mathcal{T} \iff \mathcal{U}^{\perp} \supseteq \mathcal{T}^{\perp}$$

Clearly, $\operatorname{mod} A$ is the unique maximal element and $\{0\}$ is the unique minimal element with respect to this order.

Definition 1.1.1. A torsion class $\mathcal{T}$ is said to be *functorially finite* (resp., *faithful*) if $\mathcal{T} = \operatorname{Fac} M$ for some $M \in \operatorname{mod} A$ (resp., $\mathcal{T}$ contains DA).

We denote by $\operatorname{f-tors} A$ (resp., $\operatorname{fa-tors} A$) a subset of $\operatorname{tors} A$ consisting of all functorially finite (resp., faithful) torsion classes. With respect to the partial order, $\operatorname{f-tors} A$ (resp., $\operatorname{fa-tors} A$) has the unique maximal element $\operatorname{mod} A$ (resp., $\operatorname{mod} A$) and the unique minimal element 0 (resp., $\mathsf{T}(DA)$).

1.1.2 Tilting modules

In this section, we recall some results on tilting modules. An A-module M is said to be *pretilting* if $\operatorname{Ext}^1_A(M, M) = 0$ and its projective dimension is at most 1. It is said to be *tilting* if it is pretilting and there is a short exact sequence

$$0 \to A \to M' \to M'' \to 0$$

with M' and M'' in $\operatorname{add} M$, where $\operatorname{add} M$ is the full subcategory consisting of all direct summands of finite direct sums of copies of M.

The following result is first shown by Bongartz [Bon81]. Now, we say that an A-module M is basic if all indecomposable direct summands of M are pairwise non-isomorphic, and denote by $|M|$ the number of non-isomorphic indecomposable direct summands of M.

Proposition 1.1.2. *[Bon81] Any pretilting A-module M is a direct summand of a tilting module. In particular, M is tilting if and only if $|M| = |A|$.*

We denote by $\operatorname{tilt} A$ the set of isomorphism classes of basic tilting A-modules. On this set, we can define a partial order by

$$N \leq M :\iff N^{\perp_1} \subseteq M^{\perp_1}$$

where $M^{\perp_1}$ is the full subcategory consisting of A-modules X satisfying $\operatorname{Ext}^1_A(M, X) = 0$. In Section 1.1.3, we describe the Hasse quiver of this partially ordered set by using tilting mutation.

Every tilting module provides a torsion class. Now, we say that an A-module M is *faithful* if $DA \in \operatorname{Fac} M$.

Proposition 1.1.3. *[ASS06] Any tilting A-module M is faithful, and $\operatorname{Fac} M$ is a faithful torsion class of $\operatorname{mod} A$.*

1.1.3 Support τ-tilting modules

The notion of support τ-tilting modules is introduced in [AIR14]. We begin with the definition of τ-rigid modules. Here, τ is the Auslander-Reiten translation of $\operatorname{mod} A$.

Definition 1.1.4. An A-module M is said to be *τ-rigid* if $\operatorname{Hom}_A(M, \tau M) = 0$. It is said to be *$\tau$-tilting* if it is τ-rigid and $|M| = |A|$. An A-module M is said to be *support τ-tilting* if it is τ-titling $(A/\langle e \rangle)$-module for some idempotent e.

The notion of τ-tilting modules generalizes that of tilting modules.

Proposition 1.1.5. *[ASS06, VIII.5.1] Let M be an A-module.*

(1) M is a tilting module if and only if it is a faithful τ-tilting module.

(2) If A is hereditary, then M is a tilting module if and only if it is a τ-tilting module.

We can regard a support τ-tilting module as a certain pair of modules.

Definition 1.1.6. [AIR14, Definition 0.3] Let M be an A-module and P a projective A-module. A pair (M, P) is said to be *τ-rigid* if M is τ-rigid and $\operatorname{Hom}_A(P, M) = 0$. It is said to be *$\tau$-tilting* if it is τ-rigid and $|M| + |P| = |A|$.

Every τ-rigid module M provides a τ-rigid pair $(M, 0)$. The next claim means that support τ-tilting modules correspond to τ-tilting pairs.

Proposition 1.1.7. *[AIR14, Proposition 2.3] If M is a τ-tilting $(A/\langle e\rangle)$-module for an idempotent e, then (M, eA) is a τ-tilting pair for A. On the other hand, any τ-tilting pair (M, P) provides a support τ-tilting module M.*

Moreover, every support τ-tilting module M determines a unique τ-tilting pair, in the sense that, if (M, P) and (M, Q) are τ-tilting pairs, then $\operatorname{add} P = \operatorname{add} Q$ holds. We denote by $\mathrm{s}\tau\text{-tilt}\, A$ the set of isomorphism classes of basic support τ-tilting A-modules. From [AIR14, Theorem 2.7], we can define a partial order on this set by

$$N \leq M :\iff \operatorname{Fac} N \subseteq \operatorname{Fac} M.$$

With respect to this partial order, $\mathrm{s}\tau\text{-tilt}\, A$ has the unique maximal element A and the unique minimal element 0. Furthermore, this order is compatible with one on $\operatorname{tilt} A$. Thus, we naturally regard $\operatorname{tilt} A$ as a subset of $\mathrm{s}\tau\text{-tilt}\, A$.

Now, we define the left mutation of support τ-tilting modules, which generalizes the mutation of tilting modules. Let $M = M_1 \oplus \cdots \oplus M_r$ be a basic support τ-tilting A-module with M_i indecomposable. For $i \in \{1, \ldots, r\}$, if $M_i \notin \operatorname{Fac} \bigoplus_{i \neq j} M_j$, then a left mutation of M with respect to M_i, denoted $\mu^-_{M_i}(M)$, is defined as $M'_i \oplus \bigoplus_{i \neq j} M_j$, where M'_i is the cokernel of a minimal left $(\operatorname{add} \bigoplus_{i \neq j} M_j)$-approximation of M_i

$$M_i \to M'.$$

By [AIR14, Proposition 2.30], $\mu^-_{M_i}(A)$ is a support τ-tilting A-module.

We define a support τ-tilting quiver $Q(\mathrm{s}\tau\text{-tilt}\, A)$ as follows:

- The set of vertices is $\mathrm{s}\tau\text{-tilt}\, A$.
- There is an arrow $M \to N$ if and only if N is a left mutation of M.

Proposition 1.1.8. *[AIR14, Corollary 2.34] The Hasse quiver of $\mathrm{s}\tau\text{-tilt}\, A$ coincides with $Q(\mathrm{s}\tau\text{-tilt}\, A)$.*

1.1.4 Two-term silting complexes

In this subsection, we recall basic properties of two-term silting complexes.

Definition 1.1.9. Let $T = (T^i, f^i)$ be a complex in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$.

(1) T is said to be *presilting* if $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, T[i]) = 0$ for all integers $i > 0$.

(2) T is said to be *silting* if it is presilting and $\operatorname{thick} T = \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$, where $\operatorname{thick} T$ is the smallest triangulated subcategory of $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ which contains T and is closed under taking direct summands.

(3) T is said to be *tilting* if it is a silting complex and satisfies $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, T[i]) = 0$ for all integers $i \neq 0$.

(4) T is said to be *two-term* if $T^i = 0$ for all $i \neq 0, -1$.

The following result is basic.

Proposition 1.1.10. *[AI12, Theorem 2.27] Let T be a basic silting complex for A. Then the set of isomorphism classes of indecomposable direct summands of T forms a basis of the Grothendieck group of the additive category $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$. In particular, $|T| = |A|$ holds.*

We first define the partial order on silting complexes.

Definition-Proposition 1.1.11. *[AI12, Theorem 2.11] For $T, U \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$, we write $U \leq T$ if $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, U[i]) = 0$ for all integers $i > 0$. The notation $\leq$ gives a partial order on the set of isomorphism classes of basic silting complexes.*

Furthermore, this partial order is described by using silting mutation as follows. Let T be a basic silting complex for A and X a direct summand of T. A *left mutation* of T with respect to X, denoted $\mu_X^-(T)$, is defined as $X' \oplus (T/X)$, where X' is a cone of a minimal left $\operatorname{add}(T/X)$-approximation f of X

$$X \xrightarrow{f} T' \to X' \to X[1].$$

By [AI12, Theorem 2.31], this is again a silting complex. It is called *irreducible left mutation* if X is indecomposable. Dually, we define a right mutation and an irreducible right mutation $\mu_X^+(T)$ of T.

It is shown in [AI12, Theorem 2.35] that the Hasse quiver of the partially ordered set of isomorphism classes of basic silting complexes coincides with the following quiver:

- The set of vertices is the set of isomorphism classes of basic silting complexes for A.
- We draw an arrow $U \to T$ if U is a irreducible left mutation of T.

Next, we observe two-term silting complexes. Recall that a given complex T is two-term if and only if $A[1] \leq T \leq A$ by definition. We denote by $\operatorname{2-silt} A$ the set of isomorphism classes of basic two-term silting complexes for A. By definition, the partial order on this set is written as

$$U \leq T \iff \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, U[1]) = 0$$

with $U, T \in \operatorname{2-silt} A$. With respect to this partial order, $\operatorname{2-silt} A$ has the unique maximal element A and the unique minimal element $A[1]$. Furthermore, the Hasse quiver of $\operatorname{2-silt} A$ is described by using silting mutation.

The following are basic properties of two-term presilting complexes.

Proposition 1.1.12. *Let $T = (T^{-1} \to T^0)$ be a two-term presilting complex for A. Then the following hold.*

(1) [Aih13, Proposition 2.16] T is a direct summand of a two-term silting complex. In particular, it is silting if and only if $|T| = |A|$.

(2) [AI12, Lemma 2.25] If T is basic, then $\operatorname{add} T^0 \cap \operatorname{add} T^{-1} = 0$.

(3) [AI12, Theorem 2.27] If T is silting, then $\operatorname{add}(T^0 \oplus T^{-1}) = \operatorname{add} A$.

Proposition 1.1.13. *[AIR14, Corollary 3.8] Let U be a basic two-term presilting complex for A. If $|U| = |A| - 1$, then it is a direct summand of precisely two basic two-term silting complexes $T, T' \in$ 2-silt A. In this case, T and T' are irreducible mutation of each other.*

Proposition 1.1.14. *For two basic two-term complexes $U = (U^{-1} \xrightarrow{f} U^0)$ and $T = (T^{-1} \xrightarrow{g} T^0)$ in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$, there is an exact sequence*

$$\operatorname{Hom}_A(T^{-1}, U^{-1}) \times \operatorname{Hom}_A(T^0, U^0) \xrightarrow{\eta} \operatorname{Hom}_A(T^{-1}, U^0) \xrightarrow{\pi} \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, U[1]) \to 0,$$

where π is a natural surjection and η is a map given by $(h^{-1}, h^0) \mapsto h^0 f - g h^{-1}$.

Proof. It is obvious from the definition. □

Let $\mathcal{S}(A) := \{e_1, \ldots, e_{|A|}\}$ be a complete set of primitive orthogonal idempotents of A. We have a decomposition $A = \bigoplus_{i=1}^{|A|} e_i A$, where $e_i A$ are indecomposable projective A-modules. There is a natural isomorphism of groups between $\mathbb{Z}^{|A|}$ and the Grothendieck group of the additive category $\operatorname{proj} A$ with canonical basis $e_1 A, \ldots, e_{|A|} A$.

Definition 1.1.15. The *g-vector* of a two-term complex $T = (T^{-1} \to T^0)$ is defined by

$$g_A(T) := (m_1 - n_1, \ldots, m_{|A|} - n_{|A|}) \in \mathbb{Z}^{|A|}$$

where m_i (resp., n_i) is the multiplicity of $e_i A$ as indecomposable direct summands of T^0 (resp., T^{-1}).

Proposition 1.1.16. *[AIR14, Theorem 5.5] Let $U, T \in$ 2-silt A. If $g_A(U) = g_A(T)$, then $T \cong U$.*

1.1.5 Intermediate t-structures

We recall the notion of t-structure which is first introduced by Beĭlinson–Bernstein–Deligne [BBD82]. A *t-structure* on $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ is a pair $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ of full subcategories of $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ such that

- $\mathcal{C}^{\leq -1} \subseteq \mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 1} \subseteq \mathcal{C}^{\geq 0}$.
- $\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)}(X, Y) = 0$ for all $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq 1}$.
- $\mathcal{C}^{\leq 0} * \mathcal{C}^{\geq 1} = \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$.

Here, $\mathcal{C}^{\leq -1} := \mathcal{C}^{\leq 0}[1]$, $\mathcal{C}^{\geq 1} := \mathcal{C}^{\geq 0}[-1]$, and $\mathcal{C}^{\leq 0} * \mathcal{C}^{\geq 1}$ is a full subcategory consisting of all objects E of $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ which admit triangles $X \to E \to Y \to X[1]$ with $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq 1}$. Two subcategories $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ are called *aisle* and *coaisle* respectively. It is known that the intersection $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$, called *heart* of a t-structure, of its aisle and coaisle is an abelian category.

The triangulated category $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ has a t-structure $(\mathcal{D}(A)^{\leq 0}, \mathcal{D}(A)^{\geq 0})$, called *standard t-structure*, defined by

$$\mathcal{D}(A)^{\leq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A) \mid H^i(X) = 0 \text{ for all } i > 0\}, \tag{1.1.1}$$
$$\mathcal{D}(A)^{\geq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A) \mid H^i(X) = 0 \text{ for all } i < 0\} \tag{1.1.2}$$

and its heart is $\operatorname{mod} A$.

An *intermediate t-structure* is a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ on $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$ such that

$$\mathcal{D}(A)^{\leq -1} \subseteq \mathcal{C}^{\leq 0} \subseteq \mathcal{D}(A)^{\leq 0}, \quad \text{or equivalently,} \quad \mathcal{D}(A)^{\geq 0} \subseteq \mathcal{C}^{\geq 0} \subseteq \mathcal{D}(A)^{\geq -1}.$$

We denote by int-t-str A the set of aisles of intermediate t-structures. Then it has a natural partial order given by

$$\mathcal{C}'^{\leq 0} \leq \mathcal{C}^{\leq 0} :\iff \mathcal{C}'^{\leq 0} \subseteq \mathcal{C}^{\leq 0} \iff \mathcal{C}'^{\geq 0} \supseteq \mathcal{C}^{\geq 0}$$

Clearly, int-t-str A has the unique maximal element $\mathcal{D}(A)^{\leq 0}$ and the unique minimal element $\mathcal{D}(A)^{\leq -1}$.

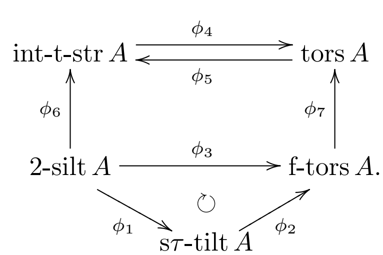
Next, we recall the Happel-Reiten-Smalø tilt on t-structures. For a torsion class $\mathcal{T}$ of $\operatorname{mod} A$, define two full subcategories of $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$:

$$\mathcal{C}_{\mathcal{T}}^{\leq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A) \mid H^i(X) = 0 \text{ for } i > 0 \text{ and } H^0(X) \in \mathcal{T}\},$$
$$\mathcal{C}_{\mathcal{T}}^{\geq 0} := \{X \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A) \mid H^i(X) = 0 \text{ for } i < -1 \text{ and } H^{-1}(X) \in \mathcal{F}\}.$$

By [HRS96, Proposition 2.10], a pair $(\mathcal{C}_{\mathcal{T}}^{\leq 0}, \mathcal{C}_{\mathcal{T}}^{\geq 0})$ forms an intermediate t-structure, which we call the *Happel-Reiten-Smalø tilt* with respect to $\mathcal{T}$. Furthermore, its heart is $\mathcal{F}[1] * \mathcal{T}$, where $\mathcal{F} = \mathcal{T}^{\perp}$.

1.1.6 The bijections

Here, we explain the bijections among objects we discussed. There is the following commutative diagram of order preserving maps (see [BY13] for the details):



Here, ϕ_1, ϕ_2, ϕ_3 are isomorphisms of partially ordered sets mapping

$$\phi_1 : T \mapsto H^0(T), \quad \phi_2 : M \mapsto \operatorname{Fac} M \quad \text{and} \quad \phi_3 : T \mapsto \operatorname{Fac} H^0(T),$$

ϕ_4 and ϕ_5 are mutually inverse isomorphisms given by the Happel-Reiten-Smalø tilt as

$$\phi_4\colon \mathcal{C}^{\leq 0} \mapsto \mathcal{D}(A)^{\geq 0} \cap \mathcal{C}^{\leq 0} \quad \text{and} \quad \phi_5\colon \mathcal{T} \mapsto \mathcal{C}^{\leq 0}_{\mathcal{T}},$$

ϕ_6 is an injective map given by

$$T \mapsto \mathcal{C}^{\leq 0}_{T} := \{X \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A) \mid \operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)}(T[i], X) = 0 \text{ for all } i > 0\},$$

and ϕ_7 is the inclusion map. In particular, all the above maps send the unique maximal (resp., minimal) element to the unique maximal (resp., minimal) element. In addition, by [AIR14, Corollary 2.8], the map ϕ_2 restricts to an isomorphism $\operatorname{tilt} A \xrightarrow{\sim} \text{f-tors}\, A \cap \text{fa-tors}\, A$ of partially ordered sets.

1.1.7 Sign-decompositions

In this subsection, we introduce the notion of sign-decomposition. Let $\mathcal{S}(A) := \{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of A, where $n := |A|$. Fix an element $\epsilon \in \{\pm 1\}^n$. We have a decomposition $A = e_1 A \oplus \cdots \oplus e_n A = e^+ A \oplus e^- A$, where e^+ (resp., e^-) is a sum of idempotents e_i with $\epsilon(i) = 1$ (resp., -1). Let $P^+ := e^+ A, P^- := e^- A$. In addition, let S^+, S^- be the semisimple modules corresponding to e^+, e^- respectively, namely, the projective cover of S^+ (resp., S^-) is P^+ (resp., P^-).

First, let

$$\operatorname{tors}_\epsilon A := \{\mathcal{T} \in \operatorname{tors} A \mid S^+ \in \mathcal{T}, S^- \notin \mathcal{T}\}.$$

Clearly, it forms an interval of $\operatorname{tors} A$ with the unique maximal element $\operatorname{Fac} P^+$ and the unique minimal element $\operatorname{Filt} S^+$, that is, $\operatorname{tors}_\epsilon A = [\operatorname{Filt} S^+, \operatorname{Fac} P^+]_{\operatorname{tors} A}$ as an interval. In addition, let $\text{f-tors}_\epsilon A := \operatorname{tors}_\epsilon A \cap \text{f-tors}\, A$.

Second, let $\mathsf{K}^2_\epsilon(\operatorname{proj} A)$ be the full subcategory of $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ consisting of all complexes isomorphic to a two-term complex $T = (T^{-1} \to T^0)$ such that $T^0 \in \operatorname{add} P^+$ and $T^{-1} \in \operatorname{add} P^-$, in other words, its g-vector $g_A(T)$ lies in an area

$$\{x \in \mathbb{Z}^n \mid x_i \in \epsilon(i) \cdot \mathbb{Z}_{\geq 0} \text{ for all } i \in \{1, \dots, n\}\}.$$

Let

$$\text{2-silt}_\epsilon A := \text{2-silt}\, A \cap \mathsf{K}^2_\epsilon(\operatorname{proj} A).$$

We find that the set $\text{2-silt}_\epsilon A$ forms an interval of $\text{2-silt}\, A$ with the unique maximal element $\mu^-_{P^-}(A)$ and the unique minimal element $\mu^+_{P^+}(A)[1]$. Similarly, we denote by $\text{2-tilt}\, A$ the set of isomorphism classes of basic two-term tilting complexes, by $\text{2-tilt}_\epsilon A$ the intersection of $\text{2-tilt}\, A$ and $\mathsf{K}^2_\epsilon(\operatorname{proj} A)$.

Finally, let $\operatorname{mod}_\epsilon A$ be the subcategory of $\operatorname{mod} A$ consisting of all A-modules M such that $(P_1 \to P_0) \in \mathsf{K}^2_\epsilon(\operatorname{proj} A)$, where $P_1 \to P_0 \to M \to 0$ is a minimal projective presentation of M in $\operatorname{mod} A$. Let

$$\text{s}\tau\text{-tilt}_\epsilon A := \text{s}\tau\text{-tilt}\, A \cap \operatorname{mod}_\epsilon A.$$

The following is basic but important.

Proposition 1.1.17. *The isomorphisms ϕ_1, ϕ_2, ϕ_3 given in Section 1.1.6 restrict to isomorphisms of partially ordered sets*

$$\begin{array}{ccc} \text{2-silt}_\epsilon A & \xrightarrow{\phi_3} & \text{f-tors}_\epsilon A \\ & {\scriptstyle \phi_1} \searrow \quad \circlearrowright \quad \nearrow {\scriptstyle \phi_2} & \\ & \text{s}\tau\text{-tilt}_\epsilon A & \end{array}$$

Proof. We show that $\operatorname{Fac} H^0(T) \in \operatorname{tors}_\epsilon A$ for any two-term silting complex $T \in$ 2-silt$_\epsilon A$. By Proposition 1.1.12(2) and (3), every indecomposable projective A-module lies in either $\operatorname{add} T^0$ or $\operatorname{add} T^{-1}$. Thus, $P^+ \in \operatorname{add} T^0$ and $P^- \in \operatorname{add} T^{-1}$ imply $\operatorname{add} P^+ = \operatorname{add} T^0$ and $\operatorname{add} P^- = \operatorname{add} T^{-1}$ respectively. It clearly implies $\operatorname{Filt} S^+ \subseteq \operatorname{Fac} H^0(T) \subseteq \operatorname{Fac} T^0 = \operatorname{Fac} P^+$ as desired. Therefore, the map ϕ_3 restricts to a bijection between 2-silt$_\epsilon A$ and f-tors$_\epsilon A$. On the other hand, ϕ_1 gives the desired bijection by the definition of g-vectors. It finishes a proof. □

On the other hand, our signature is compatible with taking factor algebras. Let $B = A/I$ be a factor algebra of A. Assume that none of e_i's are contained in I. In this case, the image of $\mathcal{S}(A)$ in B gives $\mathcal{S}(B)$. Let $\overline{\mathcal{T}} := \mathcal{T} \cap \operatorname{mod} B$ for a given full subcategory $\mathcal{T} \subseteq \operatorname{mod} A$.

Proposition 1.1.18. *Let $B = A/I$ be a factor algebra such that none of e_i's are contained in I. Then we have a commutative diagram*

$$\begin{array}{ccc} \operatorname{tors} A & \xrightarrow{\overline{(-)}} & \operatorname{tors} B \\ & {\scriptstyle \text{sign}} \searrow \quad \circlearrowright \quad \swarrow {\scriptstyle \text{sign}} & \\ & \{\pm 1\}^n & \end{array}$$

satisfying $\overline{\mathsf{T}(\mathcal{U})} = \mathcal{U}$ for any torsion class $\mathcal{U}$ in $\operatorname{mod} B$.

Proof. By [DIR+18, Proposition 5.7], we have the map $\overline{(-)}\colon \operatorname{tors} A \to \operatorname{tors} B$ such that $\overline{\mathsf{T}(\mathcal{U})} = \mathcal{U}$ for any torsion class $\mathcal{U}$ of $\operatorname{mod} B$. The assertion follows from the fact that $S \in \mathcal{T}$ if and only if $S \in \overline{\mathcal{T}}$ for any simple A-module S. □

1.2. Hereditary algebras with radical square zero

Let A be an (not necessarily connected) algebra and J its Jacobson radical. We say that A is an algebra with *radical square zero* (resp., *radical cube zero*) if $J^2 = 0$ but $J \neq 0$ (resp., $J^3 = 0$ but $J^2 \neq 0$). For simplicity, we abbreviate an algebra with radical square zero (resp., radical cube zero) by a RSZ (resp., RCZ) algebra. A typical example of this class of algebras is given by a factor algebra of a path algebra kQ of a finite quiver Q modulo the ideal generated by all path of length 2. In fact, if k is algebraically closed, then any RSZ algebra can be obtained in this way up to Morita equivalence. In particular, a path algebra kQ of Q is a RSZ algebra if and only if Q is *bipartite*, that is, every vertex of Q is either a source or a sink. In general, we need to use the notion of valued quivers defined in the next subsection.

1.2.1 Valued quivers of finite dimensional algebras

To each finite dimensional algebra A over a field k, one can associate the valued quiver Γ_A of A in the following way. Let $\mathcal{S}(A) := \{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A, where $n := |A|$. Now, we denote by $\operatorname{rad} M$ the radical of an A-module M.

Definition 1.2.1. [DR74] Suppose that $A/J = \prod_{i=1}^{n} F_i$, where F_i is a division algebra for each i. Each pair (i, j) determines an F_i-F_j-bimodule ${}_iN_j := e_i(J/J^2)e_j$. We define the *valued quiver* Γ_A of A as follows: The set of vertices of Γ_A is given by $\{1, \ldots, n\}$, and there is an arrow from i to j if ${}_iN_j \neq 0$ and assigning to this arrow the pair of non-negative integers (d'_{ij}, d''_{ij}), where d'_{ij} (resp., d''_{ij}) is the dimension of ${}_iN_j$ as a right F_j-module (resp., left F_i-module).

Note that the integer d'_{ij} is equal to j-th entry of the dimension vector of $\operatorname{rad} e_iA/\operatorname{rad}^2 e_iA$. Here, the *dimension vector* $\underline{\dim}_A M$ of an A-module M is given by

$$\underline{\dim}_A M = (m_1, \ldots, m_n) \in \mathbb{Z}^n,$$

where m_i is the multiplicity of the simple A-module S_i corresponding to e_i as composition factor of M.

Remark 1.2.2. A finite quiver can be regarded as a valued quiver by replacing m arrows from i to j by one arrow with valuation (m, m), vice versa. When a base field k is algebraically closed, the valued quiver of A is just a finite quiver and called the *Gabriel quiver* of A.

The following result is a generalization of [Gab72] to non-algebraically closed fields.

Proposition 1.2.3. *[DR74, Theorem(a)] Let A be a connected hereditary algebra and Γ_A the valued quiver of A. Then the following conditions are equivalent.*

(1) A is representation finite.

(2) $\operatorname{tilt} A$ *is finite.*

(3) The underlying graph of Γ_A is one of Dynkin diagrams $\mathbb{A}_n$, $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ and $\mathbb{G}_2$.

1.2.2 Torsion classes and two-term silting complexes

In this section, let A be a (not necessarily connected) hereditary RSZ algebra and $\mathcal{S}(A) := \{e_1, \ldots, e_n\}$ a complete set of primitive orthogonal idempotents of A, where $n := |A|$. Let Γ_A be the valued quiver of A.

- Γ_A is bipartite, that is, every vertex of Γ_A is either a source or a sink.

Therefore, there exists $\epsilon \in \{\pm 1\}^n$ such that i is a source (resp., sink) if $\epsilon(i) = 1$ (resp., -1). Notice that there are precisely 2^l elements satisfying our condition, where l is the number of isolated vertices, that is, the number of simple ring indecomposable summands of A. To such an element ϵ, we associate subalgebras e^+Ae^+ and e^-Ae^-, which are semisimple. Since $e^-Ae^+ = 0$, A is isomorphic to an upper triangular matrix algebra

$$A \cong \begin{pmatrix} e^+Ae^+ & J \\ 0 & e^-Ae^- \end{pmatrix}.$$

On the other hand,

$$A^! := \begin{pmatrix} e^-Ae^- & DJ \\ 0 & e^+Ae^+ \end{pmatrix}$$

is also a hereditary RSZ algebra. As in Section 1.1.7, let P^+, P^- (resp., S^+, S^-) be the projective (resp, semisimple) A-modules associated to the signature ϵ. From our description of A, we clearly have $P^- \cong S^-$ as an A-module.

Example 1.2.4. In the above, the valued quiver of $A^!$ is obtained from Γ_A of A by reversing all arrows. In particular, if k is algebraically closed, then A is isomorphic to a path algebra kQ of some bipartite quiver Q, and therefore $A^!$ is isomorphic to kQ^{op}, where Q^{op} is the opposite quiver of Q.

Now, let

$$T_A := \tau^- S^- \oplus P^+$$

be a complex of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)$, where $\tau := \nu[-1]$ and $\nu := D\,\mathrm{Hom}_A(-, A)$ is the Nakayama functor.

Proposition 1.2.5. *T_A is a tilting complex in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)$ satisfying $\mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)}(T_A) \cong A^!$.*

Proof. By definition, we have $T_A \cong \mu^-_{P^-}(A)$ and it is a tilting complex by [AI12, Theorem 2.50(c)]. The latter assertion is clear. □

By Rickard's theorem, we obtain a triangle equivalence

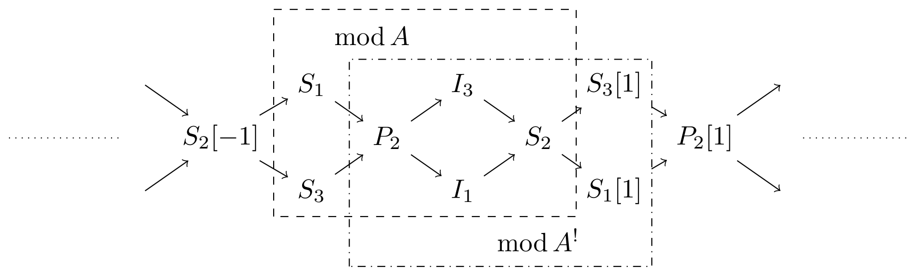
$$\mathbf{R} := \mathbf{R}\mathrm{Hom}_A(T_A, -) \colon \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A) \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A^!)$$

with quasi-inverse $\mathbf{L} := - \overset{\mathbf{L}}{\underset{A^!}{\otimes}} T_A$. In particular, we have $\mathbf{R}(T_A) \cong A^!$ and $\mathbf{R}(\nu T_A) \cong DA^!$.

The following is a main result in this subsection.

Theorem 1.2.6. *There is an isomorphism $\mathrm{R} \colon \mathrm{tors}_\epsilon\, A \xrightarrow{\sim} \text{fa-tors}\, A^!$ of partially ordered sets given by $\mathcal{T} \mapsto \mathbf{R}(\mathrm{add}(\mathcal{T}, S^-[1]))$. In addition, it preserves the property of being functorially finite.*

Example 1.2.7. Let $Q := (1 \leftarrow 2 \rightarrow 3)$ with $\epsilon := (-1, 1, -1) \in \{\pm 1\}^3$. Then $A = kQ$ is a RSZ hereditary algebra and $A^! \cong kQ^{\mathrm{op}}$. We describe two hearts mod A and mod $A^!$ in the Auslander-Reiten quiver of $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$



where S_i are simple A-modules corresponding to vertices i, and P_i (resp., I_i) are projective cover (resp., injective hull) of S_i. As $\epsilon = (-1, 1, -1)$, we have $S^+ = S_2$, $P^+ = P_2$ and $P^- = S^- = S_1 \oplus S_3$ of A-modules. There are 5 torsion classes in $\operatorname{tors}_\epsilon A$, all of which are functorially finite. In Figure 1.1, we describe the bijection R in Theorem 1.2.6.

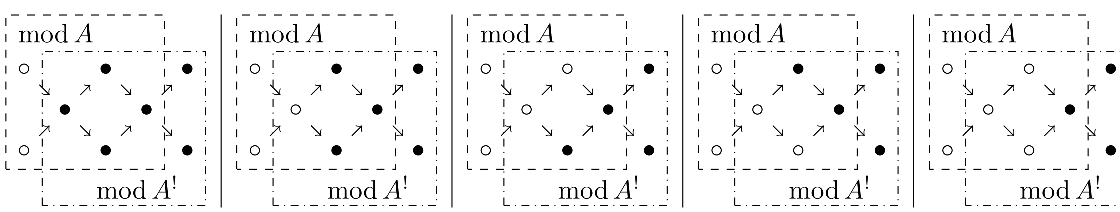


Figure 1.1: The bijection R between $\operatorname{tors}_\epsilon A$ and fa-tors $A^!$ given in Theorem 1.2.6. Each indecomposable object in a torsion class $\mathcal{T}$ (resp., $\mathrm{R}(\mathcal{T})$) corresponds to a black point in mod A (resp., mod $A^!$).

We begin with the following observations for two-term complexes. Now, let $\mathrm{B}_\epsilon := \operatorname{diag}(\epsilon(1), \ldots, \epsilon(n))$ be a diagonal matrix of size n.

Proposition 1.2.8. *The functor* $\mathbf{R}$ *restricts to*

$$\begin{array}{ccc} \mathbf{R}\colon\ \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A) & \xrightarrow{\sim} & \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A^!) \\ \cup & & \cup \\ \mathsf{K}^2_\epsilon(\operatorname{proj} A) & \xrightarrow{\sim} & \operatorname{mod} A^!. \end{array}$$

Moreover, it induces the following commutative diagram:

$$\begin{array}{ccc} K_0(\operatorname{proj} A) & \xrightarrow{\sim} & K_0(\operatorname{mod} A^!) \\ \downarrow g_A & \circlearrowleft & \downarrow \underline{\dim}_{A^!} \\ \mathbb{Z}^n & \xrightarrow[\mathrm{B}_\epsilon]{\sim} & \mathbb{Z}^n \end{array}$$

Proof. Since the functor $\mathbf{R}$ is fully faithful, it is enough to see the correspondence of objects. The tilting complex T_A determines a torsion pair $(\operatorname{Fac} P^+, \operatorname{add} S^-)$ in $\operatorname{mod} A$. By the Happel-Reiten-Smalø tilt, the heart $\operatorname{mod} A^!$ is given by the image of $\operatorname{add} S^-[1] * \operatorname{Fac} P^+$ by $\mathbf{R}$. Since A is hereditary, $\operatorname{add} S^-[1] * \operatorname{Fac} P^+ = \operatorname{add}(S^-[1], \operatorname{Fac} P^+)$. It coincides with the subcategory $\mathsf{K}^2_\epsilon(\operatorname{proj} A)$ in $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$. The latter assertion is obvious because the matrix B_ϵ presents the group homomorphism of Grothendieck groups induced by $\mathbf{R}$. □

Proposition 1.2.9. *(1)* $2\text{-silt}_\epsilon A = 2\text{-tilt}_\epsilon A$.

(2) The functor $\mathbf{R}$ *gives an isomorphism* $\mathbf{R}\colon 2\text{-tilt}_\epsilon A \xrightarrow{\sim} \operatorname{tilt} A^!$ *of partially ordered sets.*

Proof. (1) It is enough to show that any two-term complex $T = (T^{-1} \xrightarrow{f} T^0) \in \mathsf{K}^2_\epsilon(\operatorname{proj} A)$ satisfies $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, T[-1]) = 0$. Up to isomorphism, we can assume that f is in the radical of the category $\operatorname{proj} A$. Since A is a RSZ algebra, we have $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, T[-1]) \cong \operatorname{Hom}_A(T^0, T^{-1})$. On the other hand, $e^- A e^+ = 0$ implies $\operatorname{Hom}_A(T^0, T^{-1}) = 0$. Thus, we get the desired equality.

(2) It follows from Proposition 1.2.8 and (1) since $\mathbf{R}$ is a triangle equivalence. □

Next, we describe our partially ordered sets in terms of intervals.

Lemma 1.2.10. *As intervals, we have*

$$2\text{-tilt}_\epsilon A = [\nu T_A, T_A]_{2\text{-silt} A} \quad \textit{and} \quad \operatorname{tors}_\epsilon A = [\operatorname{Fac} H^0(\nu T_A), \operatorname{Fac} H^0(T_A)]_{\operatorname{tors} A}.$$

On the other hand, we have

$$\operatorname{tilt} A^! = [DA^!, A^!]_{\operatorname{tilt} A^!} \quad \textit{and} \quad \text{fa-tors}\, A^! = [\operatorname{Fac} DA^!, \operatorname{Fac} A^!]_{\operatorname{tors} A^!}$$

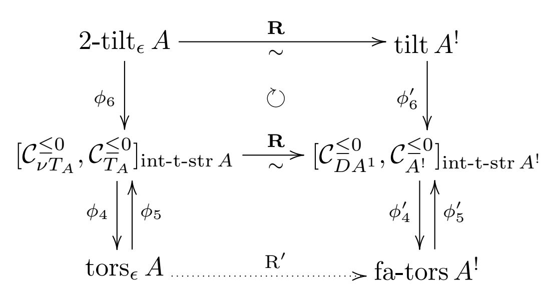
Proof. By Proposition 1.2.9(1), we have $2\text{-tilt}_\epsilon A = 2\text{-silt}_\epsilon A = [\mu^+_{P^+}(A)[1], \mu^-_{P^-}(A)]_{2\text{-silt} A}$. Then the former assertion follows from

$$T_A \cong \mu^-_{P^-}(A) \quad \text{and} \quad \nu T_A = \nu\tau^- S^- \oplus \nu P^+ = S^-[1] \oplus S^+ = \mu^+_{P^+}(A)[1].$$

The latter assertion is obvious because $DA^!$ is a tilting module in $\operatorname{mod} A^!$. □

Now, we are ready to prove Theorem 1.2.6.

Proof of Theorem 1.2.6. By Lemma 1.2.10, we get the following diagram:



where ϕ_i (resp., ϕ'_i) are maps given in Section 1.1.6 for A (resp., $A^!$) for $i \in \{4,5,6\}$. Now, let $\mathrm{R}' := \phi'_4 \circ \mathbf{R} \circ \phi_5$, that is,

$$\begin{aligned} \mathrm{R}'(\mathcal{T}) &:= \mathbf{R}(\mathcal{C}^{\leq 0}_{\mathcal{T}}) \cap \mathsf{D}(A^!)^{\geq 0} && (1.2.1) \\ &= \mathbf{R}(\mathcal{C}^{\leq 0}_{\mathcal{T}} \cap \mathcal{C}^{\geq 0}_{T_A}). && (1.2.2) \end{aligned}$$

Since A is hereditary, $\mathcal{C}^{\leq 0}_{\mathcal{T}} = \mathrm{add}(\mathsf{D}(A)^{\leq -1}, \mathcal{T})$ holds for any $\mathcal{T} \in \mathrm{tors}_\epsilon A$. On the other hand, we clearly have $\mathsf{D}(A)^{\leq -1} \cap \mathcal{C}^{\geq 0}_{T_A} = \mathrm{add}\, S^-[1]$. Consequently,

$$\mathcal{C}^{\leq 0}_{\mathcal{T}} \cap \mathcal{C}^{\geq 0}_{T_A} = \mathrm{add}(\mathsf{D}(A)^{\leq -1} \cap \mathcal{C}^{\geq 0}_{T_A}, \mathcal{T} \cap \mathcal{C}^{\geq 0}_{T_A}) = \mathrm{add}(\mathcal{T}, S^-[1]).$$

Since $\mathbf{R}$ is a triangle equivalence, $\mathrm{R} = \mathrm{R}'$ must hold. Finally, the map R preserves the property being functorially finite because of the commutativity of the diagram. □

1.3. Algebras with radical square zero

In this section, we study torsion classes of an arbitrary RSZ algebra. Throughout this section, let A be a RSZ algebra and $\mathcal{S}(A) := \{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents of A, where $n := |A|$.

1.3.1 Torsion classes

To each $\epsilon \in \{\pm 1\}^n$, we associate the following upper triangular matrix algebras:

$$A_\epsilon := \begin{pmatrix} e^+(A/J)e^+ & e^+Je^- \\ 0 & e^-(A/J)e^- \end{pmatrix} \quad \text{and} \quad A^!_\epsilon := \begin{pmatrix} e^-(A/J)e^- & D(e^+Je^-) \\ 0 & e^+(A/J)e^+ \end{pmatrix}.$$

Then, they are hereditary RSZ algebras such that $A^!_\epsilon = (A_\epsilon)^!$ each other. On the other hand, A_ϵ is a factor algebra of A modulo I_ϵ, where $I_\epsilon := e^+Je^+ + e^-Je^+ + e^-Je^-$ is a two-sided ideal in A by $J^2 = 0$.

We show the following result.

Theorem 1.3.1. *For any $\epsilon \in \{\pm 1\}^n$, there are isomorphisms between*

(1) $\mathrm{tors}_\epsilon A$,

(2) $\mathrm{tors}_\epsilon A_\epsilon$,

(3) $\text{fa-tors}\, A^!_\epsilon$,

where the map from (1) to (2) is given by $\mathcal{T} \mapsto \mathcal{T} \cap \mathrm{mod}\, A_\epsilon$, and the map from (2) to (3) is given in Theorem 1.2.6. In particular, we have a bijection

$$\mathrm{tors}\, A \overset{1-1}{\longleftrightarrow} \bigsqcup_{\epsilon \in \{\pm 1\}^n} \text{fa-tors}\, A^!_\epsilon$$

Proof. (1) and (2): Since A_ϵ is a factor algebra of A, we have a surjective map

$$\overline{(-)} := - \cap \operatorname{mod} A_\epsilon \colon \operatorname{tors}_\epsilon A \to \operatorname{tors}_\epsilon A_\epsilon$$

such that $\overline{\mathsf{T}(\mathcal{U})} = \mathcal{U}$ for any $\mathcal{U} \in \operatorname{tors}_\epsilon A_\epsilon$ by Proposition 1.1.18. In order to prove the injectivity, we check that, if $\mathcal{T} \in \operatorname{tors}_\epsilon A$ satisfies $\overline{\mathcal{T}} = \mathcal{U}$, then $\mathcal{T} = \mathsf{T}(\mathcal{U})$. By definition, we have $\mathsf{T}(\mathcal{U}) \subseteq \mathcal{T}$. Conversely, let $M \in \mathcal{T}$. There are short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\epsilon & \longrightarrow & A & \longrightarrow & A_\epsilon & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I_\epsilon & \longrightarrow & \operatorname{rad} A & \longrightarrow & \operatorname{rad} A_\epsilon & \longrightarrow & 0 \end{array}$$

where the bottom one splits since $\operatorname{rad} A$ is semisimple. Applying $M \otimes_A -$, we get $0 \to M \otimes_A I_\epsilon \to M \to M \otimes_A A_\epsilon \to 0$. Since $\mathcal{T}$ is closed under factor modules, we have $M \otimes_A A_\epsilon \in \overline{\mathcal{T}} = \mathcal{U}$. On the other hand, by $M \in \mathcal{T} \subset \operatorname{Fac} P^+$, we have $M \otimes_A I_\epsilon \cong (\operatorname{rad} M)e^+ \in \operatorname{add} S^+ \subseteq \mathcal{U}$. Since $\mathcal{T}$ is closed under extensions, we have $M \in \mathsf{T}(\mathcal{U})$ as desired.

(2) and (3): Applying Theorem 1.2.6 to $A := A_\epsilon$, we get the desired isomorphism. □

Let $\Gamma = \Gamma_A$ be the valued quiver of A. For a given $\epsilon \in \{\pm 1\}^n$, the valued quiver Γ_ϵ of A_ϵ is a bipartite subquiver of Γ given as follows:

- The set of vertices of Γ_ϵ is the same as Γ.
- The set of arrows of Γ_ϵ consists of all valued arrows $i \to j$ in Γ such that $(\epsilon(i), \epsilon(j)) = (1, -1)$.

In addition, that of $A^!_\epsilon$ is obtained by reversing all arrows of Γ_ϵ.

Example 1.3.2. We determine all torsion classes for a RSZ algebra $A = kQ/I$, where

$$Q = (\ 1 \underset{\xleftarrow{c}}{\overset{\xrightarrow{a}}{\xrightarrow{b}}} 2\ ) \quad \text{and} \quad I = \langle ac, bc, ca, cb \rangle.$$

Theorem 1.3.1 gives an isomorphism $\operatorname{Hasse}(\operatorname{tors}_\epsilon A) \cong \operatorname{Hasse}(\text{fa-tors}\, kQ_\epsilon^{\mathrm{op}})$ for each $\epsilon \in \{\pm 1\}^n$, where

$Q_{(1,1)}$	$Q_{(1,-1)}$	$Q_{(-1,1)}$	$Q_{(-1,-1)}$
$1 \qquad 2$	$1 \rightrightarrows 2$	$1 \longleftarrow 2$	$1 \qquad 2$

For $\epsilon \in \{(1,1), (-1,1), (-1,-1)\}$, we have

$$\operatorname{Hasse}(\text{fa-tors}\, k(1 \quad 2)) = (\bullet) \quad \text{and} \quad \operatorname{Hasse}(\text{fa-tors}\, k(1 \to 2)) = (\bullet \to \bullet).$$

On the other hand, our quiver $Q_{(1,-1)}$ is the Kronecker quiver. In this case, there are infinitely many faithful torsion classes and the corresponding Hasse quiver is given by

$$\operatorname{Hasse}(\text{fa-tors}\, k(\ 1 \leftleftarrows 2\ )) = (\bullet \to \bullet \to \bullet \to \cdots \operatorname{Hasse}(2^{\mathbb{P}^1(k)}) \cdots \to \bullet \to \bullet \to \bullet)$$

where $2^{\mathbb{P}^1(k)}$ is the power set of the projective line $\mathbb{P}^1(k)$ over k, and black points correspond to functorially finite torsion classes.

From the above argument, all torsion classes of $\operatorname{mod} A$ are described by using Theorem 1.3.1, see Figure 1.2. However, we find in Figure 1.2 that four arrows a_1, a_2, a_3 and a_4 in Hasse(f-tors A) are missing. In Section 1.3.3, we will reconstruct these arrows in terms of tilting mutation.

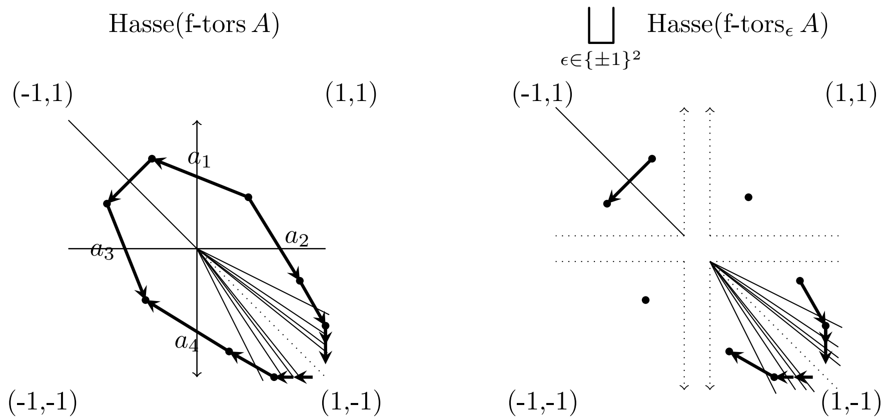


Figure 1.2: The Hasse quiver of f-tors A in the left and a disjoint union of the Hasse quivers of f-tors$_\epsilon$ A in the right.

1.3.2 Two-term silting complexes

Let A be a RSZ algebra. In this subsection, we prove the following result.

Theorem 1.3.3. *For any $\epsilon \in \{\pm 1\}^n$, there are isomorphisms between*

(1) 2-silt$_\epsilon$ A,

(2) 2-tilt$_\epsilon$ A_ϵ,

(3) tilt $A_\epsilon^!$,

where the map from (1) to (2) is given by $- \otimes_A A_\epsilon$, and the map from (2) to (3) is given in Proposition 1.2.9(2). In particular, we have a bijection

$$\text{2-silt}\, A \stackrel{1-1}{\longleftrightarrow} \bigsqcup_{\epsilon\in\{\pm 1\}^n} \operatorname{tilt} A_\epsilon^!.$$

Fix $\epsilon \in \{\pm 1\}^n$ and let

$$\mathbf{F} := - \otimes_A A_\epsilon \colon \operatorname{proj} A \to \operatorname{proj} A_\epsilon.$$

Proposition 1.3.4. *The following hold.*

(1) For any $X \in \operatorname{add} P^+$ *and any* $Y \in \operatorname{add} P^-$*, we have*

$$\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(Y, X) \cong \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A_\epsilon)}(\mathbf{F}(Y), \mathbf{F}(X)).$$

(2) The functor $\mathbf{F}$ *gives a bijection between isomorphism classes of indecomposable objects in* $\mathsf{K}^2_\epsilon(\operatorname{proj} A)$ *and in* $\mathsf{K}^2_\epsilon(\operatorname{proj} A_\epsilon)$ *that preserves the g-vector of complexes.*

(3) For any $T, U \in \mathsf{K}^2_\epsilon(\operatorname{proj} A)$*, we have*

$$\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, U[1]) \cong \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A_\epsilon)}(\mathbf{F}(T), \mathbf{F}(U)[1]).$$

Proof. (1) It follows from the equation

$$\operatorname{Hom}_A(P^-, P^+) = e^+ A e^- = e^+ J e^- = e^+ A_\epsilon e^- = \operatorname{Hom}_{A_\epsilon}(\mathbf{F}(P^-), \mathbf{F}(P^+)).$$

(2) It is clear by (1).

(3) By Lemma 1.1.14, we have the following diagram of exact sequences:

$$\begin{array}{ccccccc}
\operatorname{Hom}_A(T^0, U^0) \oplus \operatorname{Hom}_A(T^{-1}, U^{-1}) & \longrightarrow & \operatorname{Hom}_A(T^{-1}, U^0) & \longrightarrow & \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, U[1]) & \longrightarrow & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & \\
\operatorname{Hom}_{A_\epsilon}(\mathbf{F}(T^0), \mathbf{F}(U^0)) \oplus \operatorname{Hom}_{A_\epsilon}(\mathbf{F}(T^{-1}), \mathbf{F}(U^{-1})) & \longrightarrow & \operatorname{Hom}_{A_\epsilon}(\mathbf{F}(T^{-1}), \mathbf{F}(U^0)) & \longrightarrow & \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A_\epsilon)}(\mathbf{F}(T), \mathbf{F}(U)[1]) & \longrightarrow & 0
\end{array}$$

where f, g and h are surjective since $\mathbf{F}$ is a full functor. In addition, g is injective by (1). By the five lemma, it implies that h is injective, and hence an isomorphism as desired. □

Now, we give a proof of Theorem 1.3.3.

Proof of Theorem 1.3.3. (1) and (2): Recall that $2\text{-silt}_\epsilon A_\epsilon = 2\text{-tilt}_\epsilon A_\epsilon$ by Proposition 1.2.9(1). By Proposition 1.3.4(2), the functor $\mathbf{F}$ gives a bijection between isomorphism classes of indecomposable objects in $\mathsf{K}^2_\epsilon(\operatorname{proj} A)$ and in $\mathsf{K}^2_\epsilon(\operatorname{proj} A_\epsilon)$. Then it gives rise to an isomorphism $\mathbf{F}\colon 2\text{-silt}_\epsilon A \to 2\text{-tilt}_\epsilon A_\epsilon$ by Proposition 1.1.12(1) and 1.3.4(3).

(2) and (3): We just apply Proposition 1.2.9(2) to $A := A_\epsilon$ and $A^! := A^!_\epsilon$. □

Corollary 1.3.5. *The bijections in Theorem 1.3.1 preserve the property of being functorially finite.*

Proof. We have shown in Theorem 1.3.3 that there is an isomorphism $\mathbf{F}\colon 2\text{-silt}_\epsilon A \xrightarrow{\sim} 2\text{-tilt}_\epsilon A_\epsilon$. By [DIR+18, Proposition 5.6(b)], we have $\mathbf{F} \circ \operatorname{Fac} H^0 = \operatorname{Fac} H^0 \circ \overline{(-)}$, so the map $\overline{(-)}$ restricts to a bijection between functorially finite torsion classes. Then we get the assertion because the map in Theorem 1.2.6 preserves the property of being functorially finite. □

Corollary 1.3.6. *Let* $T \in 2\text{-silt}_\epsilon A$ *be a two-term silting complex and* $M \in \operatorname{tilt} A^!_\epsilon$ *the corresponding tilting module of* T *under the bijection in Theorem 1.3.3. Then we have* $g_A(T) = \mathrm{B}_\epsilon \cdot \underline{\dim}_{A^!_\epsilon}(M)$*, where* $\mathrm{B}_\epsilon := \operatorname{diag}(\epsilon(1), \ldots, \epsilon(n))$.

Proof. It follows from Propositions 1.2.8 and 1.3.4(2). □

On the other hand, it is important to know whether a given silting complex T is tilting. For the RSZ algebra A, we get the next criterion. This is a generalization of Proposition 1.2.9(1).

Proposition 1.3.7. *Let A be a RSZ algebra and Γ_A its valued quiver. For a given $\epsilon \in \{\pm 1\}^n$, the following conditions are equivalent.*

(1) $e^- A e^+ = 0$.

(2) $2\text{-silt}_\epsilon A = 2\text{-tilt}_\epsilon A$.

(3) There exists a tilting complex in $2\text{-silt}_\epsilon A$.

(4) There are no arrows $i \to j$ in Γ_A such that $(\epsilon(i), \epsilon(j)) = (-1, 1)$.

Proof. Let $T = (T^{-1} \xrightarrow{f} T^0) \in \mathsf{K}^2_\epsilon(\operatorname{proj} A)$. Up to isomorphism, we can assume that f is in the radical of the category $\operatorname{proj} A$. In this case, $T^0 \in \operatorname{add} P^+$ and $T^{-1} \in \operatorname{add} P^-$. Since $J^2 = 0$, we have $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, T[-1]) \cong \operatorname{Hom}_A(T^{-1}, T^0)$.

(1) ⇒ (2): If $e^- A e^+ = 0$, then we have $\operatorname{Hom}_A(T^{-1}, T^0) = 0$ as desired.

(2) ⇒ (3): It is clear since $\mu^-_{P^-}(A) \in 2\text{-silt}_\epsilon A = 2\text{-tilt}_\epsilon A$ is non-empty.

(3) ⇒ (1): By Proposition 1.1.12(2) and (3), $T \in 2\text{-silt}_\epsilon A$ satisfies $\operatorname{add} T^0 = \operatorname{add} P^+$ and $\operatorname{add} T^{-1} = \operatorname{add} P^-$. If T is tilting, then we have

$$\operatorname{Hom}_A(P^+, P^-) \subseteq \operatorname{Hom}_A(T^{-1}, T^0) \cong \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, T[-1]) = 0$$

as desired.

(1) ⇔ (4): It is clear. □

1.3.3 Gluing the Hasse quivers

The bijection in Theorem 1.3.3 provides a partial data of the Hasse quiver of $2\text{-silt}\, A$ for a RSZ algebra A, in fact, some arrows are missing as in Example 1.3.2. In contrast, we can reconstruct it in terms of tilting mutation (Theorem 1.3.9). Recall that arrows of $\operatorname{Hasse}(2\text{-silt}\, A)$ correspond to irreducible left mutations of two-term siting complexes.

We define a partial order on $\{\pm 1\}^n$ by $\eta \leq \epsilon$ if $\eta(i) \leq \epsilon(i)$ for all $i \in \{1, \ldots, n\}$. With this partial order, there is an arrow $\alpha \colon \epsilon \to \eta$ in $\operatorname{Hasse}\{\pm 1\}^n$ if and only if there exists an element i_α such that $\epsilon(i_\alpha) = 1$, $\eta(i_\alpha) = -1$ and $\epsilon(j) = \eta(j)$ for all $j \neq i_\alpha$.

Let A be a RSZ algebra. For each arrow $\alpha \colon \epsilon_1 \to \epsilon_2$ in $\operatorname{Hasse}\{\pm 1\}^n$, there is a natural isomorphism

$$A^!_{\epsilon_1} / \langle e_{i_\alpha} \rangle \cong A^!_{\epsilon_2} / \langle e_{i_\alpha} \rangle.$$

We denote this factor algebra by $A^!_\alpha$. Notice that the valued quiver of $A^!_\alpha$ is obtained from that of $A^!_{\epsilon_i}$ by deleting the vertex i_α. There are two kinds of embedding of module categories:

$$\operatorname{mod} A^!_\alpha \hookrightarrow \operatorname{mod} A^!_{\epsilon_1} \quad \text{and} \quad \operatorname{mod} A^!_\alpha \hookrightarrow \operatorname{mod} A^!_{\epsilon_2}.$$

Proposition 1.3.8. *Let $\alpha\colon \epsilon_1 \to \epsilon_2$ be an arrow in* $\mathrm{Hasse}\{\pm 1\}^n$ *and $i \in \{1,2\}$. For each tilting module $M \in \operatorname{tilt} A^!_\alpha$, there exists a unique tilting module $M_i \in \operatorname{tilt} A^!_{\epsilon_i}$ having M as its direct summand.*

Proof. Let $i \in \{1,2\}$ and $Q := e_{i_\alpha} A^!_{\epsilon_i}$. Since $\operatorname{Hom}_{A^!_{\epsilon_i}}(Q, M) = 0$, a pair (M, Q) is a τ-rigid pair for $A^!_{\epsilon_i}$. Furthermore, it is τ-tilting because $|M|+|Q| = n$ holds. The corresponding two-term silting complex is $T \oplus Q[1]$, where $T = (P_1 \to P_0)$ is a two-term complex given by a projective presentation $P_1 \to P_0 \to M \to 0$ of M in $\operatorname{mod} A$. Consider a irreducible right mutation with respect to $Q[1]$ to get a two-term silting complex $T \oplus Y$ with $H^0(Y) \neq 0$. Since $A^!_{\epsilon_i}$ is hereditary, it is isomorphic to the 0-th cohomology, say M_i. This is the desired one because M_i has M as its direct summand and is a tilting module by Proposition 1.1.5(2). The uniqueness follows from Proposition 1.1.13. □

Consider a disjoint union of $\mathrm{Hasse}(\operatorname{tilt} A^!_\epsilon)$ for all $\epsilon \in \{\pm 1\}^n$. We add arrows $\alpha_M\colon M_1 \to M_2$ to this quiver for all pairs (α, M) of arrows $\alpha\colon \epsilon_1 \to \epsilon_2$ in $\mathrm{Hasse}\{\pm 1\}^n$ and $M \in \operatorname{tilt} A^!_\alpha$, where $M_1 \in \operatorname{tilt} A^!_{\epsilon_1}$ and $M_2 \in \operatorname{tilt} A^!_{\epsilon_2}$ are tilting modules obtained in Proposition 1.3.8. We denote the resulting quiver by $\mathrm{Q}(A)$.

Theorem 1.3.9. *There is an isomorphism* $\mathsf{Hasse}(2\text{-silt}\, A) \cong \mathrm{Q}(A)$ *of quivers which commutes with the bijection in Theorem 1.3.3.*

Proof. By Theorem 1.3.3, there are isomorphisms $\mathrm{Hasse}(2\text{-silt}_\epsilon A) \cong \mathrm{Hasse}(\operatorname{tilt} A^!_\epsilon)$ of subquivers for all $\epsilon \in \{\pm 1\}^n$. Thus, it is enough to show a bijection between arrows not lie in those subquivers.

Let $\alpha_M\colon M_1 \to M_2$ be an arrow in $\mathrm{Q}(A)$ given by $\alpha\colon \epsilon_1 \to \epsilon_2$ and $M \in \operatorname{tilt} A^!_\alpha$, where $M_i \in \operatorname{tilt} A^!_{\epsilon_i}$ are tilting modules obtained in Proposition 1.3.8. For $i \in \{1,2\}$, let T_i be the two-term silting complex in $2\text{-silt}_{\epsilon_i} A$ corresponding to M_i under the bijection in Theorem 1.3.3. Observe that T_1 and T_2 have a common direct summand T_M such that $|T_M| = n-1$ and satisfy $\operatorname{Fac} H^0(T_2) \subseteq \operatorname{Fac} H^0(T_1)$ by our choice of signatures $\epsilon_2 \leq \epsilon_1$. Thus, T_2 is a irreducible left mutation of T_1 by Proposition 1.1.13. Namely, there is an arrow $\overline{\alpha_M}\colon T_1 \to T_2$ in $\mathrm{Hasse}(2\text{-silt}\, A)$.

On the other hand, let $T_1 \to T_2$ be an arrow which does not lie in $\mathrm{Hasse}(2\text{-silt}_\epsilon A)$ for all $\epsilon \in \{\pm 1\}^n$. In this case, it is easy to see that $T_1 \in 2\text{-silt}_{\epsilon_1} A$ and $T_2 \in 2\text{-silt}_{\epsilon_2} A$ for some arrow $\alpha\colon \epsilon_1 \to \epsilon_2$ in $\mathrm{Hasse}\{\pm 1\}^n$. Following the previous paragraph in reverse, we find that the correspondence $\alpha_M \mapsto \overline{\alpha_M}$ is bijective. □

Example 1.3.10. In Example 1.3.2, we describe the Hasse quivers of $\text{f-tors}_\epsilon A \cong 2\text{-silt}_\epsilon A$ for all $\epsilon \in \{\pm 1\}^2$. Here, we recover arrows a_1, a_2, a_3 and a_4 in Figure 1.2.

For an arrow $\alpha_1\colon (1,1) \to (-1,1)$ in $\mathrm{Hasse}\{\pm 1\}^2$, we have

$$A^!_{(1,1)} = k(1 \quad 2), \quad A^!_{(-1,1)} = k(1 \to 2) \quad \text{and} \quad A^!_{\alpha_1} = k(2).$$

In this case, $M := A^!_{\alpha_1}$ is a unique tilting $A^!_{\alpha_1}$-module. Since M is simple projective, we have an arrow ${\alpha_1}_M\colon M_1 \to M_2$ in $Q(A)$, where $M_1 := A^!_{(1,1)}$ and

$M_2 := A^!_{(-1,1)}$. By Proposition 1.3.8, it gives rise to the arrow $a_1 := \overline{\alpha_{1M}}$ in Figure 1.2. See Figure 1.3.

Similarly, we get arrows a_2, a_3 and a_4 in Hasse(2-silt A) from arrows $\alpha_2 \colon (1,1) \to (1,-1)$, $\alpha_3 \colon (-1,1) \to (-1,-1)$ and $\alpha_4 \colon (1,-1) \to (-1,-1)$ respectively.

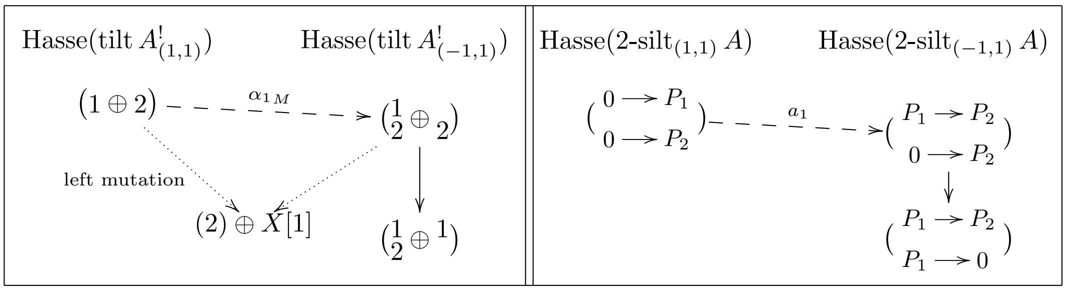


Figure 1.3: The correspondence $\alpha_{1M} \mapsto a_1$ of arrows, where we describe a module by its composition series as $M_1 = \big(1 \oplus 2\big), M_2 = \big(\substack{1\\2} \oplus 2\big)$ and $M = \big(2\big)$.

1.4. Characterizing g-finite algebras with radical square zero

Let A be a RSZ algebra and $\mathcal{S}(A) = \{e_1, \ldots, e_n\}$ a complete set of primitive orthogonal idempotents of A, where $n = |A|$. Let $\Gamma := \Gamma_A$ be the valued quiver of A.

Definition 1.4.1. We define a new valued quiver Γ^s, called *separated quiver* for Γ, as follows:

- The set of vertices is $\{i^\sigma \mid i \in \{1, \ldots, n\}, \sigma \in \{\pm 1\}\}$.
- For each valued arrow $i \to j$ in Γ, we draw a valued arrow $i^{+1} \to j^{-1}$ in Γ^s with the same valuation.

A full subquiver of Γ^s is called *single subquiver* if it contains at most one of i^{+1}, i^{-1} for each i.

Remark 1.4.2. For each $\epsilon \in \{\pm 1\}^n$, our quiver Γ_ϵ can be regarded as a maximal single subquiver of Γ^s by identifying vertices $i \in \epsilon^{-1}(\sigma)$ with i^σ.

As a consequence of Theorem 1.3.3, we obtain a slight generalization of the result of Adachi [Ada16a, Theorem 3.1] to non-algebraically closed fields.

Corollary 1.4.3. *Let A be a RSZ algebra and $\Gamma = \Gamma_A$ the valued quiver of A. The following conditions are equivalent.*

(1) A is g-finite (i.e., 2-silt A *is finite).*

(2) The underlying graph of any single subquiver of Γ^s *is a disjoint union of Dynkin diagrams* $\mathbb{A}_n$, $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ *and* $\mathbb{G}_2$.

(3) For every $\epsilon \in \{\pm 1\}^n$*, the underlying graph of* Γ_ϵ *is a disjoint union of Dynkin diagrams* $\mathbb{A}_n$, $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ *and* $\mathbb{G}_2$.

Proof. (1) $\Leftrightarrow$ (3): By Theorem 1.3.3, 2-silt A is finite if and only if tilt $A^!_\epsilon$ is finite for every $\epsilon \in \{\pm 1\}^n$. By Proposition 1.2.3, it is equivalent to the condition that the underlying graph of Γ_ϵ is a disjoint union of Dynkin diagrams for every ϵ.

(2) $\Leftrightarrow$ (3): By Remark 1.3.4, it is clear since any subquiver of a Dynkin diagram is Dynkin. $\square$

Chapter 2

The number of two-term tilting complexes over symmetric algebras with radical cube zero

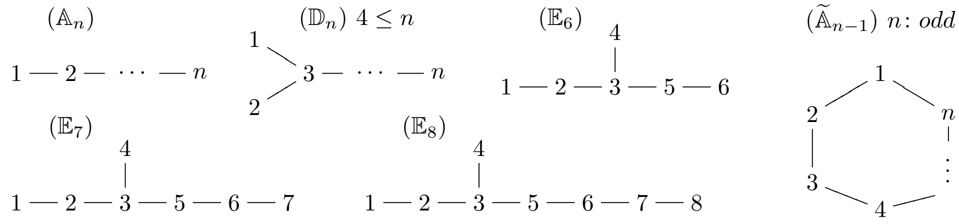
This chapter (Chapter 2) is based on the joint work with Adachi [AA18]. Throughout this chapter, we assume that a base field k is algebraically closed. In this setting, the valued quiver of a finite dimensional algebra A is a finite quiver, which we call the Gabriel quiver of A. Furthermore, the algebra A is assumed to be connected, by definition, its Gabriel quiver is connected.

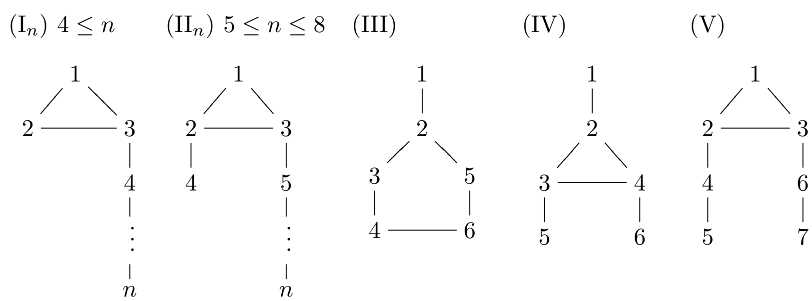
The aim of this section is to prove the following results. Note that silting complexes coincide with tilting complexes for a symmetric algebra [AI12, Example 2.8].

Theorem 2.0.1. *Let A be a symmetric RCZ algebra over an algebraically closed field k. Then the following conditions are equivalent.*

(1) A is g-finite (i.e., $\text{2-tilt}\, A$ *is finite).*

(2) The Gabriel quiver of A is obtained by adding finite number of loops to the double quiver Q_G of a graph G in the following list.





Theorem 2.0.2. *In Theorem 2.0.1, the number* $\#\,\text{2-tilt}\,A$ *is given as follows.*

G	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\tilde{\mathbb{A}}_{n-1}$	I_n	II_5	II_6	II_7	II_8	III	IV	V
$\#\,\text{2-tilt}\,A$	$\binom{2n}{n}$	a_n	1700	8872	54066	2^{2n-1}	b_n	632	2936	11306	75240	3108	4056	17328

Here, for any $n \geq 4$, *let* $a_n := 6 \cdot 4^{n-2} - 2\binom{2(n-2)}{n-2}$ *and* $b_n := 6 \cdot 4^{n-2} + 2\binom{2n}{n} - 4\binom{2(n-1)}{n-1} - 4\binom{2(n-2)}{n-2}$.

2.1. Symmetric algebras with radical cube zero via algebras with radical square zero

In this section, we study a connection between two-term tilting complexes for symmetric RCZ algebras and two-term silting complexes for the corresponding RSZ algebras. We refer to Section 1.1.4 for the definition and basic properties of silting complexes.

Let $Q = (Q_0, Q_1)$ be a connected finite quiver, where Q_0 is the vertex set and Q_1 is the arrow set. We denote by Q^{op} the opposite quiver of Q. For a map $\epsilon\colon Q_0 \to \{\pm 1\}$, we define a quiver $Q_\epsilon = (Q_0', Q_1')$, called a *single quiver* of Q, as

$$Q_0' := Q_0,\ Q_1' := \{(i \to j) \in Q_1 \mid \epsilon(i) = +1,\ \epsilon(j) = -1\}.$$

Note that Q_ϵ is bipartite (i.e., each vertex is either a sink or a source) but neither full nor connected as a subquiver of Q. Moreover, by definition, we have $Q_\epsilon = (Q^\circ)_\epsilon$, where Q° is the quiver obtained from Q by deleting all loops.

Proposition 2.1.1. *Let A be a RSZ algebra and Q its Gabriel quiver. Then the following statements hold.*

(1) There is a bijection

$$\text{2-silt}\,A \longrightarrow \bigsqcup_{\epsilon\colon Q_0 \to \{\pm 1\}} \operatorname{tilt} kQ_\epsilon^{\mathrm{op}}.$$

(2) ([Ada16a, Theorem 3.1]) *The following conditions are equivalent.*

(a) *A is g-finite.*

(b) *For every map $\epsilon\colon Q_0 \to \{\pm 1\}$, the single quiver Q_ϵ is a disjoint union of Dynkin quivers of type $\mathbb{A}, \mathbb{D}$ and $\mathbb{E}$.*

Proof. (1) Since k is algebraically closed, the algebra $A^!_\epsilon$ is isomorphic to a path algebra $kQ^{\mathrm{op}}_\epsilon$. Then the assertion follows from Theorem 1.3.3.

(2) It follows from Proposition 1.4.3 over an algebraically closed field k. $\square$

The classification of two-term tilting complexes for symmetric RCZ algebras is reduced to that of two-term silting complexes for RSZ algebras. Now, we denote by $\operatorname{soc} M$ the socle of an A-module M.

Proposition 2.1.2. *[Ada16b, DIR+18, EJR18] Let A be a symmetric RCZ algebra and Q its Gabriel quiver. Then $\overline{A} := A/\operatorname{soc} A$ is a RSZ algebra whose Gabriel quiver is also Q. Furthermore, the functor $- \otimes_A \overline{A}$ induces a bijection*

$$\text{2-tilt}\, A \to \text{2-silt}\, \overline{A}.$$

Next, we describe symmetric RCZ algebras in terms of finite graphs. For a connected finite graph G, we define the quiver Q_G, called the *double quiver* of G, as follows.

- The vertex set of Q_G is the vertex set of G.
- The arrow set of Q_G is as follows. For each edge e in G which has endpoints $i \neq j$ (i.e., it is not a loop), we draw two arrows $a\colon i \to j$ and $a'\colon j \to i$. For a loop l in G at a vertex i, we draw a loop b at i.

Proposition 2.1.3. *Let A be a symmetric RCZ algebra. Then the Gabriel quiver of A is the double quiver Q_G of a connected finite graph G with non-empty edge set.*

Proof. It is easy to see that every indecomposable projective A-module P has Loewy length 3, and P has simple top and simple socle isomorphic to S, where S is the simple A-module corresponding to P.

Since A is a finite dimensional algebra over an algebraically closed field k, there is a canonical surjection $kQ \to A$, where Q is the Gabriel quiver of A. We will write $\bar{x}$ for the image of $x \in kQ$ in A. Let $a\colon i \to j$ be an arrow in Q. If $\bar{a}^2$ is non-zero, then it lies in the socle soc$A$ of A, and $\overline{ab} = 0$ for all arrows $b \neq a$ by [GS16, Proposition 5.5]. Next, we assume that $\bar{a}^2 = 0$. Since every indecomposable projective module has Loewy length 3, there must be an arrow $a'\colon j \to i$ of Q such that $\overline{aa'}$ is a non-zero element of soc$A$, and we have $\overline{ac} = 0 = \overline{a'c}$ for all arrows $c \neq a, a'$ by [GS16, Proposition 5.5]. In this case, $\overline{a'a}$ is also a non-zero element of soc$A$ by [GS16, Lemma 5.4]. That is, the correspondence $a \mapsto a'$ gives an involution on the set of all arrows of Q which are not loops. Thus, we have $Q = Q_G$, where G is a finite graph obtained from Q by replacing each unordered pair $\{a, a'\}$ of arrows $a\colon i \to j$, $a'\colon j \to i$ which are not loops with an edge e between i and j, and each loop b in Q at i with a loop l at i. $\square$

We recall a construction of a symmetric RCZ algebra from a given finite graph due to [GS17]. Let G be a connected finite graph with non-empty edge set and Q_G the double quiver of G. Now, for an arrow a which is not a loop and a loop b in Q_G, we call the compositions aa', $a'a$ and b^2 *distinguished paths.* Let I_G be a two-sided ideal in kQ_G generated by all relations of the following form.

- $p - q$ for all distinguished paths p, q in Q_G such that p and q have the same starting point and ending point.
- aba for all arrows a, b in Q_G such that ab is a distinguished path.
- All non-distinguished paths of length 2.

We write $A_G := kQ_G/I_G$.

Lemma 2.1.4. *[GS17, Lemma 4.5] In the above, A_G is a symmetric RCZ algebra whose Gabriel quiver is Q_G.*

The next result implies that we may only consider the algebras of the form A_G for some connected finite graphs G when we prove Theorem 2.0.1.

Lemma 2.1.5. *Let A be a symmetric RCZ algebra and Q its Gabriel quiver. Suppose that $Q = Q_G$ for a connected finite graph G with non-empty edge set. Let A_G be the symmetric RCZ algebra associated to G. Then there is a natural bijection*

$$\text{2-tilt}\, A \to \text{2-tilt}\, A_G.$$

Proof. Let $\overline{A} := A/\operatorname{soc} A$ and $\overline{A}_G := A_G/\operatorname{soc} A_G$ be the corresponding RSZ algebras of A and A_G respectively. Since they have the same Gabriel quiver Q_G, we have $\text{2-silt}\,\overline{A} = \text{2-silt}\,\overline{A}_G$ by Proposition 2.1.1(1). Consequently, by Proposition 2.1.2, we have natural bijections

$$\text{2-tilt}\, A \cong \text{2-silt}\,\overline{A} = \text{2-silt}\,\overline{A}_G \cong \text{2-tilt}\, A_G$$

as desired. □

Combining Propositions 2.1.1 and 2.1.2, we have the following result, which plays a central role in our proof of main theorems.

Theorem 2.1.6. *Let G be a connected finite graph whose edge set is non-empty and $Q := Q_G$ the double quiver of G. Let A_G be the symmetric RCZ algebra associated to G and $\overline{A}_G := A_G/\text{soc}A_G$.*

(1) The following conditions are equivalent.

- *(a) $\text{2-tilt}\, A_G$ is finite.*
- *(b) $\text{2-silt}\,\overline{A}_G$ is finite.*
- *(c) For every map $\epsilon\colon Q_0 \to \{\pm 1\}$, the single quiver Q_ϵ is a disjoint union of Dynkin quivers of type $\mathbb{A}, \mathbb{D}$ and $\mathbb{E}$.*

(2) *Fix any vertex $v \in Q_0$. If one of the equivalent conditions in* (1) *is satisfied, then the following equalities hold.*

$$\#\,\text{2-tilt}\,A_G = \#\,\text{2-silt}\,\overline{A}_G = 2 \sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(v)=+1}} \#\,\text{tilt}\,kQ_\epsilon.$$

In particular, this number does not depend on the number of loops in G.

Proof. (1) It follows from Propositions 2.1.1(2) and 2.1.2.

(2) By Proposition 2.1.2, we have $\#\,\text{2-tilt}\,A_G = \#\,\text{2-silt}\,\overline{A}_G$. We show the second equality. Let v be a vertex in Q. By Proposition 2.1.1(1), we have

$$\#\,\text{2-silt}\,\overline{A}_G = \sum_{\epsilon\colon Q_0 \to \{\pm 1\}} \text{tilt}\,kQ_\epsilon^{\text{op}} = \sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(v)=+1}} \#\,\text{tilt}\,kQ_\epsilon^{\text{op}} + \sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(v)=-1}} \#\,\text{tilt}\,kQ_\epsilon^{\text{op}}.$$

For a map $\epsilon\colon Q_0 \to \{\pm 1\}$, we define a map $-\epsilon\colon Q_0 \to \{\pm 1\}$ by $(-\epsilon)(i) := -\epsilon(i)$ for all $i \in Q_0$. By the definition of the double quiver, we have $Q_\epsilon^{\text{op}} = Q_{-\epsilon}$. This implies that Q_ϵ and $Q_{-\epsilon}$ have the same underlying graph Δ. By our assumption, Δ is a disjoint union of Dynkin graphs. Thus we obtain $\#\,\text{tilt}\,kQ_\epsilon = \#\,\text{tilt}\,kQ_{-\epsilon}$ because it is well known (see Proposition 2.2.7) that the number of non-isomorphic basic tilting modules over a path algebra of Dynkin type does not depend on orientation. Hence we have

$$\sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(v)=+1}} \#\,\text{tilt}\,kQ_\epsilon = \sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(v)=+1}} \#\,\text{tilt}\,kQ_\epsilon^{\text{op}} = \sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(v)=-1}} \#\,\text{tilt}\,kQ_\epsilon^{\text{op}}.$$

This finishes the proof. □

Example 2.1.7. In Figure 2.1, we describe the double quiver of the graph $\mathbb{E}_6$ and its single quivers associated to maps ϵ with $\epsilon(6) = +1$, where the notation i^σ denotes the vertex i with $\sigma = \epsilon(i) \in \{\pm 1\}$. By Theorem 2.1.6, we find that there are 1700 isomorphism classes of two-term tilting complexes over $A_{\mathbb{E}_6}$ as in the list of Theorem 2.0.2.

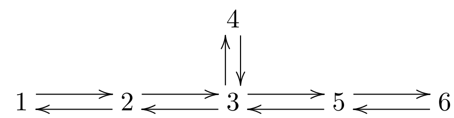
2.2. Proof of main theorems

In this section, we prove Theorems 2.0.1 and 2.0.2. Throughout this section, G is a connected finite graph with non-empty edge set, and $Q := Q_G$ is the double quiver of G. In order to show Theorems 2.0.1 and 2.0.2, we may assume that G has no loops because $Q_\epsilon = (Q^\circ)_\epsilon$ holds for each map $\epsilon\colon Q_0 \to \{\pm 1\}$.

2.2.1 Proof of Theorem 2.0.1

By Theorem 2.1.6(1), the proof is completed with the following proposition.

$Q := Q_{\mathbb{E}_6}$:



Q_ϵ	4^{+} $1^{+}\quad 2^{+}\quad 3^{+}\quad 5^{+}\quad 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1)$	4^{+} $1^{-}\leftarrow 2^{+}\quad 3^{+}\quad 5^{+}\quad 6^{+}$ $(\mathbb{A}_2,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1)$	4^{+} $1^{+}\to 2^{-}\leftarrow 3^{+}\quad 5^{+}\quad 6^{+}$ $(\mathbb{A}_3,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1)$	4^{+} $1^{-}\quad 2^{-}\leftarrow 3^{+}\quad 5^{+}\quad 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1)$
# tilt kQ_ϵ	1	2	5	2
	$4^{+}\downarrow$ $1^{+}\quad 2^{+}\to 3^{-}\leftarrow 5^{+}\quad 6^{+}$ $(\mathbb{A}_1,\mathbb{D}_4,\mathbb{A}_1)$	$4^{+}\downarrow$ $1^{-}\leftarrow 2^{+}\to 3^{-}\leftarrow 5^{+}\quad 6^{+}$ $(\mathbb{D}_5,\mathbb{A}_1)$	$4^{+}\downarrow$ $1^{+}\to 2^{-}\quad 3^{-}\leftarrow 5^{+}\quad 6^{+}$ $(\mathbb{A}_2,\mathbb{A}_3,\mathbb{A}_1)$	$4^{+}\downarrow$ $1^{-}\quad 2^{-}\quad 3^{-}\leftarrow 5^{+}\quad 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_3,\mathbb{A}_1)$
	20	77	10	5
	$4^{-}\uparrow$ $1^{+}\quad 2^{+}\quad 3^{+}\quad 5^{+}\quad 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_1,\mathbb{A}_1)$	$4^{-}\uparrow$ $1^{-}\leftarrow 2^{+}\quad 3^{+}\quad 5^{+}\quad 6^{+}$ $(\mathbb{A}_2,\mathbb{A}_2,\mathbb{A}_1,\mathbb{A}_1)$	$4^{-}\uparrow$ $1^{+}\to 2^{-}\leftarrow 3^{+}\quad 5^{+}\quad 6^{+}$ $(\mathbb{A}_4,\mathbb{A}_1,\mathbb{A}_1)$	$4^{-}\uparrow$ $1^{-}\quad 2^{-}\leftarrow 3^{+}\quad 5^{+}\quad 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_3,\mathbb{A}_1,\mathbb{A}_1)$
	2	4	14	5
	4^{-} $1^{+}\quad 2^{+}\to 3^{-}\leftarrow 5^{+}\quad 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_3,\mathbb{A}_1,\mathbb{A}_1)$	4^{-} $1^{-}\leftarrow 2^{+}\to 3^{-}\leftarrow 5^{+}\quad 6^{+}$ $(\mathbb{A}_4,\mathbb{A}_1,\mathbb{A}_1)$	4^{-} $1^{+}\to 2^{-}\quad 3^{-}\leftarrow 5^{+}\quad 6^{+}$ $(\mathbb{A}_2,\mathbb{A}_2,\mathbb{A}_1,\mathbb{A}_1)$	4^{-} $1^{-}\quad 2^{-}\quad 3^{-}\leftarrow 5^{+}\quad 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_1,\mathbb{A}_1)$
	5	14	4	2
	4^{+} $1^{+}\quad 2^{+}\quad 3^{+}\to 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_3)$	4^{+} $1^{-}\leftarrow 2^{+}\quad 3^{+}\to 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_2,\mathbb{A}_1,\mathbb{A}_3)$	4^{+} $1^{+}\to 2^{-}\leftarrow 3^{+}\to 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_5,\mathbb{A}_1)$	4^{+} $1^{-}\quad 2^{-}\leftarrow 3^{+}\to 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_4)$
	5	10	42	14
	$4^{+}\downarrow$ $1^{+}\quad 2^{+}\to 3^{-}\quad 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_1\mathbb{A}_3,\mathbb{A}_2)$	$4^{+}\downarrow$ $1^{-}\leftarrow 2^{+}\to 3^{-}\quad 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_4,\mathbb{A}_2)$	$4^{+}\downarrow$ $1^{+}\to 2^{-}\quad 3^{-}\quad 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_2,\mathbb{A}_2,\mathbb{A}_2)$	$4^{+}\downarrow$ $1^{-}\quad 2^{-}\quad 3^{-}\quad 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_2)$
	10	28	8	4
	$4^{-}\uparrow$ $1^{+}\quad 2^{+}\quad 3^{+}\to 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_4)$	$4^{-}\uparrow$ $1^{-}\leftarrow 2^{+}\quad 3^{+}\to 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_2,\mathbb{A}_4)$	$4^{-}\uparrow$ $1^{+}\to 2^{-}\leftarrow 3^{+}\to 5^{-}\leftarrow 6^{+}$ $(\mathbb{E}_6)$	$4^{-}\uparrow$ $1^{-}\quad 2^{-}\leftarrow 3^{+}\to 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_1,\mathbb{D}_5)$
	14	28	418	77
	4^{-} $1^{+}\quad 2^{+}\to 3^{-}\quad 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_1,\mathbb{A}_2)$	4^{-} $1^{-}\leftarrow 2^{+}\to 3^{-}\quad 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_3,\mathbb{A}_1,\mathbb{A}_2)$	4^{-} $1^{+}\to 2^{-}\quad 3^{-}\quad 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_2,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_2)$	4^{-} $1^{-}\quad 2^{-}\quad 3^{-}\quad 5^{-}\leftarrow 6^{+}$ $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_2)$
	4	10	4	2

Figure 2.1: A half of single quivers of the double quiver of $\mathbb{E}_6$.

Proposition 2.2.1. *Let G be a connected finite graph with no loops and $Q := Q_G$ the double quiver of G. Then the quiver Q_ϵ is a disjoint union of Dynkin quivers for every map $\epsilon\colon Q_0 \to \{\pm 1\}$ if and only if G is one of the list in Theorem 2.0.1.*

In the following, we give a proof of Proposition 2.2.1 by removing extended Dynkin quivers from the single quivers of Q. We start with removing extended Dynkin quivers of type $\widetilde{\mathbb{A}}$. A graph is called an *n-cycle* if it is a cycle with exactly n vertices. In particular, it is called an *odd-cycle* if n is odd, and an *even-cycle* if n even.

Lemma 2.2.2. *The following statements are equivalent:*

(1) There exists a map $\epsilon\colon Q_0 \to \{\pm 1\}$ such that Q_ϵ contains an extended

Dynkin quiver of type $\widetilde{\mathbb{A}}$ as a subquiver.*

(2) G contains an even-cycle as a subgraph.

Proof. (2)⇒(1): Let G' be a subgraph of G which is an even-cycle. Since an even-cycle is a bipartite graph, there exists a map $\epsilon\colon Q_0 \to \{\pm 1\}$ such that the underlying graph of Q_ϵ contains G' as a subgraph. Hence the assertion follows.

(1)⇒(2): Assume that for some map $\epsilon\colon Q_0 \to \{\pm 1\}$, the quiver Q_ϵ contains an extended Dynkin quiver Q' of type $\widetilde{\mathbb{A}}$. Since Q_ϵ is bipartite, so is Q'. Hence the underlying graph of Q' is an even-cycle and a subgraph of G. This finishes the proof. □

By Lemma 2.2.2, we may assume that G contains no even-cycle as a subgraph. In particular, Q has no multiple arrows. We give a connection between the single quivers of Q and subtrees of G. Recall that a *subtree* of G is a connected subgraph of G without cycles.

Proposition 2.2.3. *Assume that G contains no even-cycle as a subgraph. Let G' be a connected graph. Then the following statements are equivalent.*

(1) There exists a map $\epsilon\colon Q_0 \to \{\pm 1\}$ such that the underlying graph of Q_ϵ contains G' as a subgraph.

(2) G' is a subtree of G.

In particular, there exists a naturally two-to-one correspondence between the set of connected the single quivers of Q and the set of subtrees of G.

Proof. (2)⇒(1) is clear. We show (1)⇒(2). If G has no even-cycle as a subgraph, then the underlying graph Δ of Q_ϵ is tree by Lemma 2.2.2. Since Δ is a subgraph of G, any subgraph of Δ is a subtree of G. □

For a tree, we have the following result.

Corollary 2.2.4. *Assume G is a tree. Then the quiver Q_ϵ is a disjoint union of Dynkin quivers for each map $\epsilon\colon Q_0 \to \{\pm 1\}$ if and only if G is a Dynkin graph.*

Proof. It is well known that G is Dynkin if and only if all subtrees of G are Dynkin. The assertion follows from Proposition 2.2.3. □

We remove extended Dynkin quivers of type $\widetilde{\mathbb{D}}$ from single quivers of Q. If G contains at least two odd-cycles, then there exists a subtree G' of G such that G' is an extended Dynkin graph of type $\widetilde{\mathbb{D}}$. By Proposition 2.2.3, there exists a map $\epsilon\colon Q_0 \to \{\pm 1\}$ such that Q_ϵ contains an extended Dynkin quiver of type $\widetilde{\mathbb{D}}$ as a subquiver. Hence we may assume that G contains at most one odd-cycle. Moreover, by Corollary 2.2.4, it is enough to observe the case that G contains exactly one odd-cycle. Namely, G consists of an odd-cycle such that each vertex v in the odd-cycle is attached to a tree T_v (See Figure 2.2).

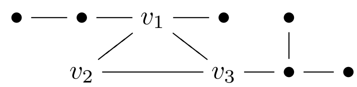


Figure 2.2: A tree T_v for a vertex v

Lemma 2.2.5. *Fix an integer $k \geq 1$ and $n := 2k + 1$. Assume that G consists of an n-cycle such that each vertex v in the n-cycle is attached to a tree T_v. Then the following statements are equivalent:*

(1) There exists a map $\epsilon \colon Q_0 \to \{\pm 1\}$ such that Q_ϵ contains an extended Dynkin quiver of type $\widetilde{\mathbb{D}}$ as a subquiver.

(2) G contains an extended Dynkin graph of type $\widetilde{\mathbb{D}}$ as a subgraph.

(3) G satisfies one of the following conditions.

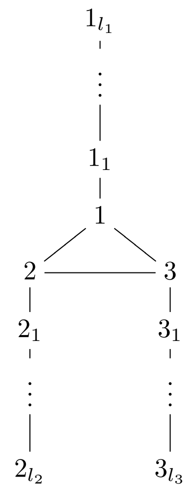
(a) There is a vertex v in the n-cycle such that the degree is at least four.

(b) There is a vertex v in the n-cycle such that the degree is exactly three and T_v is not Dynkin graph of type $\mathbb{A}$.

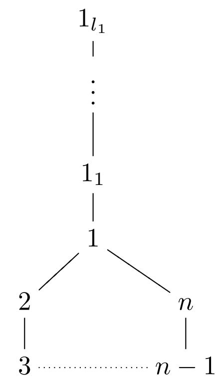
(c) $k \geq 2$ and there are at least two vertices in the n-cycle such that the degrees are at least three.

Proof. (1)⇔(2) follows from Proposition 2.2.3. Moreover, we can easily check (2)⇔(3) because $\widetilde{\mathbb{D}}_4$ has exactly one vertex whose degree is exactly four and $\widetilde{\mathbb{D}}_l$ ($l \geq 5$) has exactly two vertices whose degree is exactly three. □

Fix an integer $k \geq 1$ and $n := 2k+1$. By Lemma 2.2.5, we may assume that G is one of the following graphs:



$k = 1$



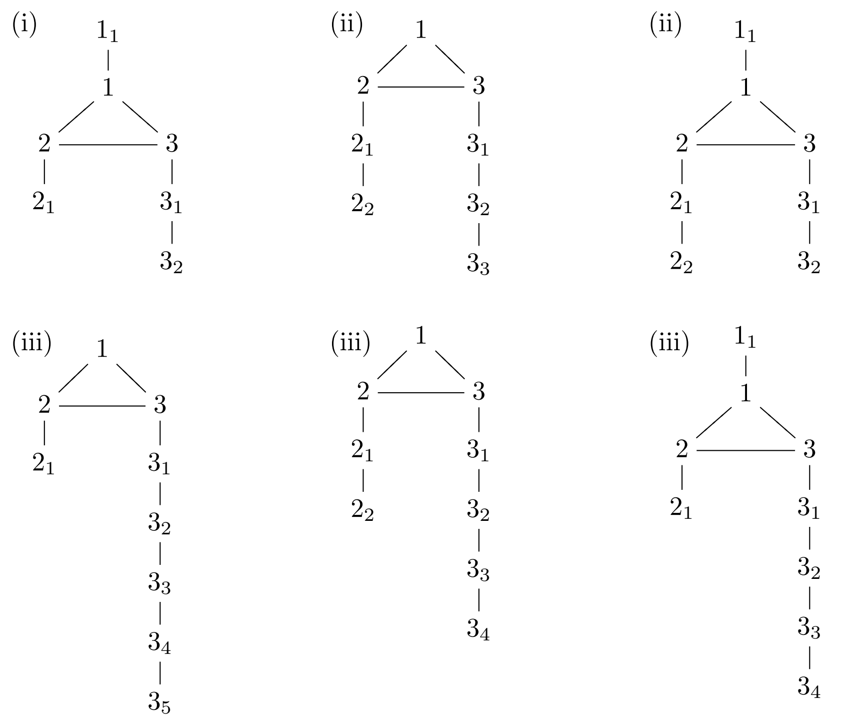
$k \geq 2$

Finally, we remove extended Dynkin graphs of type $\widetilde{\mathbb{E}}$ from the single quivers of Q.

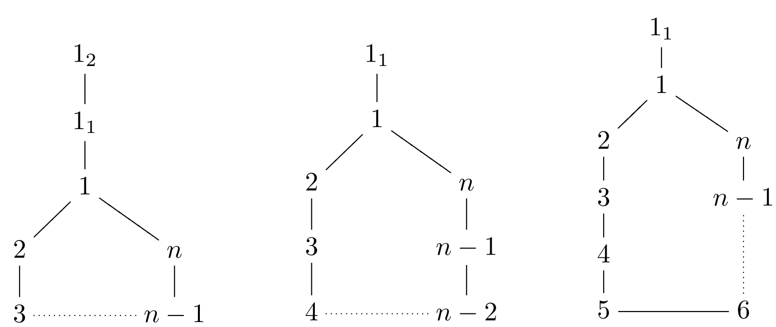
Lemma 2.2.6. *Fix an integer $k \geq 1$ and $n := 2k + 1$.*

(1) Assume that $k = 1$. The following graphs (i), (ii) *and* (iii) *are the minimal*

graphs containing extended Dynkin graphs $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ *and* $\widetilde{\mathbb{E}}_8$ *respectively.*



(2) Assume that $k \geq 2$. *The following graphs* (iv), (v) *and* (vi) *are the minimal graphs containing extended Dynkin graphs* $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ *and* $\widetilde{\mathbb{E}}_8$ *respectively.*



(iv) $k \geq 2$ (v) $k \geq 3$ (vi) $k \geq 4$

Proof. We can easily find extended Dynkin graphs $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$ in the graphs above. □

Now we are ready to prove Proposition 2.2.1.

Proof of Proposition 2.2.1. If G is a tree, then the assertion follows from Corollary 2.2.4. We assume that G is not a tree. All proper connected non-tree subgraphs of graphs appearing in Lemma 2.2.6(i–vi) give a complete list of graphs containing no extended Dynkin graphs as subgraphs. Then the complete list coincides with the graphs $(\tilde{\mathbb{A}}_{n-1})_{n:\mathrm{odd}}$, (I), (II), (III), (IV) and (V). By Proposition 2.2.3, G is one of the complete list if and only if for each map $\epsilon\colon Q_0 \to \{\pm 1\}$ the quiver Q_ϵ is a disjoint union of Dynkin quivers. This finishes the proof. □

We finish this subsection with proof of Theorem 2.0.1.

Proof of Theorem 2.0.1. Without loss of generality, we may assume that G has no loops by the definition of single quivers. This follows from Theorem 2.1.6(1) and Proposition 2.2.1. □

2.2.2 Proof of Theorem 2.0.2

We just compute the number of two-term tilting complexes for each graph in the list of Theorem 2.0.1. Since our calculation is based on Theorem 2.1.6(2), we need the number of tilting modules over a path algebra of Dynkin type.

Proposition 2.2.7 (see for example [ONFR15])**.** *Let Q be a quiver whose underlying graph Δ is one of a Dynkin graph of type $\mathbb{A}, \mathbb{D}$ and $\mathbb{E}$. Then the number of isomorphism classes of tilting module is given by the following table and does not depend on the orientation of Q.*

Δ	$\mathbb{A}_n$	$\mathbb{D}_n\ (n \geq 4)$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\#\,\mathrm{tilt}\, kQ$	$\frac{1}{n+1}\binom{2n}{n}$	$\frac{3n-4}{2n}\binom{2(n-1)}{n-1}$	418	2431	17342

Firstly, we enumerate the number of two-term tilting complexes for a Dynkin graph of type $\mathbb{A}$.

Proposition 2.2.8. *Let $G = \mathbb{A}_n$ with $n > 0$. We have*

$$\#\,2\text{-tilt}\, A_G = \binom{2n}{n}. \tag{2.2.1}$$

We need a few equations about Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Lemma 2.2.9. *For a positive integer m, the following equations hold.*

(1) $C_{m+1} = \sum_{k=0}^{m} C_k C_{n-k}$.

(2) $(m+2)C_{m+1} = 2(2m+1)C_n$.

(3) $\displaystyle\sum_{t=1}^{m} \binom{2(t-1)}{t-1}\binom{2(m-t)}{m-t} = 4^{m-1}$.

$$1 \qquad 2 \longrightarrow 3 \longleftarrow 4 \qquad 5 \longrightarrow 6$$

Figure 2.3: An example of the quiver Q_ϵ for $n = 6$ with $\epsilon = (1, 1, -1, 1, 1, -1)$ and $b = (1, 3, 2) \in Z_6^3$.

(4) $\displaystyle\sum_{t=1}^{m} \frac{1}{t}\binom{2(t-1)}{t-1}\binom{2(m-t)}{m-t} = \frac{1}{2}\binom{2m}{m}$.

Proof. The equality (1)-(3) is well-known. The equality (4) is obtained by

$$\sum_{t=1}^{m} \frac{1}{t}\binom{2(t-1)}{t-1}\binom{2(m-t)}{m-t} = \frac{m+1}{2}\sum_{t=1}^{m} C_{t-1}C_{m-t} = \frac{m+1}{2}C_m = \frac{1}{2}\binom{2m}{m}. \tag{2.2.2}$$

□

Now, we are ready to prove Theorem 2.2.8.

Proof of Theorem 2.2.8. Let $G = \mathbb{A}_n$ and Q its double quiver. We observe the quiver Q_ϵ for a given $\epsilon\colon Q_0 \to \{\pm 1\}$ with $\epsilon(1) = +1$. The tuple ϵ provides a sequence $(b_1, \dots, b_r)$ of positive integers by the equality

$$\{1 \leq i \leq n \mid \epsilon(i) = \epsilon(i+1)\} = \{b_1, b_1 + b_2, \dots, \sum_{s=1}^{r} b_s = n\}.$$

In this situation, Q_ϵ is a disjoint union of r quivers $Q^{(1)}, \dots, Q^{(r)}$ of type $\mathbb{A}$ having $b_1, \dots, b_r$ vertices respectively. By Proposition 2.2.7, we have

$$\#\operatorname{tilt} kQ_\epsilon = \prod_{s=1}^{r} \#\operatorname{tilt} kQ^{(s)} = \prod_{s=1}^{r} C_{b_s}. \tag{2.2.3}$$

On the other hand, we find that the above correspondence $\epsilon \mapsto (b_1, \dots, b_r)$ gives a bijection between $\{\epsilon\colon Q_0 \to \{\pm 1\} \mid \epsilon(1) = 1\}$ and $\bigcup_{r=1}^{n} Z_n^r$, where

$$Z_n^r := \{(b_1, \dots, b_r) \in \mathbb{Z}^r \mid \sum_{s=1}^{r} b_s = n,\ b_s > 0 \text{ for all } s = 1, \dots, r\}.$$

Consequently, we have

$$\frac{1}{2}\#\,2\text{-silt}\,A_G = \sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(1) = +1}} \#\operatorname{tilt} kQ_\epsilon = \sum_{r=1}^{n} \sum_{b \in Z_n^r} \prod_{s=1}^{r} C_{b_s}. \tag{2.2.4}$$

It remains to show the following claim.

Lemma 2.2.10.

$$\sum_{r=1}^{n} \sum_{b \in Z_n^r} \prod_{s=1}^{r} C_{b_s} = \frac{1}{2}\binom{2n}{n}.$$

Proof. Let $P_n^r := \sum_{b \in Z_n^r} \prod_{s=1}^r C_{b_s}$. We show that $\sum_{r=1}^n P_n^r = \frac{1}{2}\binom{2n}{n}$ by induction on n.

i) For $n = 1$, we have $P_1^1 = C_1 = 1$.

ii) Assume that $n > 1$ and $\sum_{r=1}^m P_m^r = \frac{1}{2}\binom{2m}{m}$ holds for $1 \leq m < n$.
Clearly, we have $P_n^1 = C_n$, and for $1 < r \leq n$,

$$P_n^r = \sum_{b \in Z_n^r} \prod_{s=1}^r C_{b_s} = C_1 P_{n-1}^{r-1} + C_2 P_{n-2}^{r-1} + \cdots + C_{n-r+1} P_{r-1}^{r-1}.$$

Therefore,

$$\begin{aligned}\sum_{r=1}^n P_n^r = C_n + \sum_{k=1}^{n-1} C_k \left\{ \sum_{r=1}^{n-k} P_{n-k}^r \right\} &\overset{\text{induction}}{=} C_n + \frac{1}{2}\sum_{k=1}^{n-1} C_k \binom{2(n-k)}{n-k} \\ &= C_n + \frac{1}{2}\sum_{k=1}^{n-1}(n-k+1)C_k C_{n-k} = C_n + \frac{1}{2}\sum_{k=1}^{n-1}(k+1)C_k C_{n-k}.\end{aligned}$$

Adding last two equations, we get

$$\begin{aligned}2\sum_{r=1}^n P_n^r = 2C_n + \frac{1}{2}(n+2)\sum_{k=1}^{n-1} C_k C_{n-k} &\overset{\text{Lem.2.2.9(1)}}{=} 2C_n + \frac{1}{2}(n+2)(C_{n+1} - 2C_n) \\ &\overset{\text{Lem.2.2.9(2)}}{=} \binom{2n}{n}\end{aligned}$$

as desired. □

We finish the proof of Theorem 2.2.8. □

Next, we consider the case when $G = \tilde{\mathbb{A}}_{n-1}$ for odd n. We give the concrete number of two-term tilting complexes.

Proposition 2.2.11. *If $G = \tilde{\mathbb{A}}_{n-1}$ for odd n, then we have*

$$\# \, \text{2-tilt}\, A_G = 2^{2n-1}. \tag{2.2.5}$$

We need the following equations coming from the proof of Lemma 2.2.10. Now, let $P_n^r := \sum_{b \in Z_n^r} \prod_{s=1}^r C_{b_s}$.

Lemma 2.2.12.

$$\sum_{r\,:\,odd} P_n^r = nC_{n-1} \quad \textit{and} \quad \sum_{r\,:\,even} P_n^r = (n-1)C_{n-1}. \tag{2.2.6}$$

Proof. Let $O_n := \sum_{r\,:\,\text{odd}} P_n^r$ and $E_n := \sum_{r\,:\,\text{even}} P_n^r$. We claim $O_n - E_n = C_{n-1}$ and show it by induction on n. After that, we obtain the desired equation (2.2.6) since we have already shown in Lemma 2.2.10 that $O_n + E_n = \sum_{r=1}^n P_n^r = \frac{1}{2}\binom{2n}{n}$.

i) For $n=1$, we have $O_1 - E_1 = 1 - 0 = 1$.

ii) Assume that $n > 1$ and $O_m - E_m = C_{m-1}$ holds for $1 \leq m < n$. Then we have

$$O_n := \sum_{r\colon \text{odd}} P_n^r = \sum_{k=1}^{n-1} C_k \left\{ \sum_{r\colon \text{odd}} P_{n-k}^{r-1} \right\} + C_n = \sum_{k=1}^{n-1} C_k E_{n-k} + C_n.$$

Similarly, we have

$$\begin{aligned} E_n := \sum_{r\colon \text{even}} P_n^r &= \sum_{k=1}^{n-1} C_k \left\{ \sum_{r\,:\,\text{even}} P_{n-k}^{r-1} \right\} = \sum_{k=1}^{n-1} C_k O_{n-k} \\ &= \sum_{k=1}^{n-1} C_k (E_{n-k} + C_{n-k-1}) = \sum_{k=1}^{n-1} C_k E_{n-k} + C_n - C_{n-1}. \end{aligned}$$

Therefore, we get the desired equation $O_n - E_n = C_{n-1}$ for n. □

Now, we are ready to prove Theorem 2.2.11.

Proof of Theorem 2.2.11. Let $G = \tilde{\mathbb{A}}_{n-1}$ for odd n. Let Q be the double quiver of G. We observe the quiver Q_ϵ for $\epsilon\colon Q_0 \to \{\pm 1\}$ such that $\epsilon(1) = +1$. Let Q' be a connected component of Q_ϵ containing the vertex 1. By definition, Q' is a Dynkin quiver of type $\mathbb{A}_t$, where t is the number of vertices of Q'.

(a) If $t = n$, then $Q' = Q_\epsilon$ and $\#\operatorname{tilt} kQ_\epsilon = \#\operatorname{tilt} kQ' = C_n$.

(b) We have $t \neq n-1$. It is because Q_ϵ is not a disjoint union of two connected subquivers by the definition of signatures.

(c) Assume that $1 \leq t \leq n-2$. Observe that our quiver Q_ϵ is a disjoint union of Q' and quivers $Q^{(1)}, \ldots, Q^{(r)}$ of Dynkin type $\mathbb{A}$ having $b_1, \ldots, b_r$ vertices respectively, where r is even. In this case, we have $\#\operatorname{tilt} kQ_\epsilon = \#\operatorname{tilt} kQ' \cdot \prod_{s=1}^{r} \#\operatorname{tilt} kQ^{(s)} = C_t \prod_{s=1}^{r} C_{b_s}$.

Running over all elements $\epsilon\colon Q_0 \to \{\pm 1\}$ with $\epsilon(1) = +1$, we have

$$\sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(1)=+1}} \#\operatorname{tilt} kQ_\epsilon = nC_n + \sum_{t=1}^{n-2} tC_t \left\{ \sum_{r\colon \text{even}} P_{n-t}^r \right\} \overset{\text{Lem.2.2.12.}}{=} nC_n + \sum_{t=1}^{n-2} tC_t (n-t-1) C_{n-t-1}.$$

Here, the last term is equal to

$$nC_n + \sum_{t=0}^{n-1} (t+1) C_t (n-t) C_{n-t-1} - n \sum_{t=0}^{n-1} C_t C_{n-t-1} \overset{\text{Lem.2.2.9(1)}}{=} \sum_{t=0}^{n-1} \binom{2t}{t} \binom{2(n-t-1)}{n-t-1} \overset{\text{Lem.2.2.9(3)}}{=} 4^{n-1}.$$

Consequently, we get the desired equality

$$\#\,2\text{-silt}\, A_G = 2 \sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(1)=+1}} \#\operatorname{tilt} kQ_\epsilon = 2 \cdot 4^{n-1} = 2^{2n-1}.$$

It completes the proof. □

Thirdly, we consider the case when G is a Dynkin graph of type $\mathbb{D}$. For simplicity, let $c_0 = 1$, $c_l := \binom{2l}{l}$ for each $l \geq 1$ and $\#\,2\text{-tilt}\, A_{\mathbb{A}_0} := 2$.

Proposition 2.2.13. *Let $n \geq 4$ and $G = \mathbb{D}_n$. Then we have*

$$\#\,2\text{-tilt}\, A_G = 6 \cdot 4^{n-2} - 2c_{n-2}.$$

Proof. Let $G = \mathbb{D}_n$ in the list of Theorem 2.0.1 and Q its double quiver. By Theorem 2.1.6(2),

$$\#\,2\text{-tilt}\, A_G = 2 \sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(3) = +1}} \#\,\text{tilt}\, kQ_\epsilon.$$

Let M be the set of maps $\epsilon\colon Q_0 \to \{\pm 1\}$ such that $\epsilon(3) = +1$. Clearly, M is a disjoint union of the following subsets:

- $M_1 := \{\epsilon \in M \mid \epsilon(1) = \epsilon(2) = \epsilon(3)\}$.
- $M_2 := \{\epsilon \in M \mid \epsilon(1) = -\epsilon(2) = \epsilon(3)\}$.
- $M_3 := \{\epsilon \in M \mid -\epsilon(1) = \epsilon(2) = \epsilon(3)\}$.
- $M_4 := \{\epsilon \in M \mid -\epsilon(1) = -\epsilon(2) = \epsilon(3) = \epsilon(4)\}$.
- $M_5 := \{\epsilon \in M \mid -\epsilon(1) = -\epsilon(2) = \epsilon(3) = -\epsilon(4)\} = \bigsqcup_{t=4}^{n} M_5(t)$, where

$$M_5(t) := \{\epsilon \in M_5 \mid t = \min\{4 \leq j \leq n \mid \epsilon(j) = \epsilon(j+1)\}\}.$$

From now, we compute $\mathsf{n}(i) := \sum_{\epsilon \in M_i} \#\,\text{tilt}\, kQ_\epsilon$ for each $i \in \{1, \ldots, 5\}$. In the following, the notation $j \sim\!\sim\!\sim j+1$ is replaced by one of three types: $j \to j+1$ if $(\epsilon(j), \epsilon(j+1)) = (+1, -1)$, $j \leftarrow j+1$ if $(\epsilon(j), \epsilon(j+1)) = (-1, +1)$, otherwise nothing between them.

(i) Let $\epsilon \in M_1$. Then the quiver Q_ϵ is given by

$$\begin{array}{ll} 1 & \\ & 3 \sim\!\sim\!\sim 4 \sim\!\sim\!\sim \cdots \sim\!\sim\!\sim n-1 \sim\!\sim\!\sim n. \\ 2 & \end{array}$$

Let G' be a subgraph of G without the vertices $\{1, 2\}$ and Q' its double quiver. Then $\#\,\text{tilt}\, kQ_\epsilon = \#\,\text{tilt}\, kQ'_{\epsilon|_{\{3,\ldots,n\}}}$. Since G' is a Dynkin graph $\mathbb{A}_{n-2}$, we obtain $2\mathsf{n}(1) = \#\,2\text{-tilt}\, A_{\mathbb{A}_{n-2}} = c_{n-2}$ from Proposition 2.2.8.

By an argument similar to (1), we can calculate other cases.

(ii) For each $\epsilon \in M_2$, the quiver Q_ϵ is given by

$$\begin{array}{ll} 1 & \\ & 3 \sim\!\sim\!\sim 4 \sim\!\sim\!\sim \cdots \sim\!\sim\!\sim n-1 \sim\!\sim\!\sim n. \\ 2 \swarrow & \end{array}$$

Then we can check $2\mathsf{n}(2) = \#\,\text{2-tilt}\, A_{\mathbb{A}_{n-1}} - \#\,\text{2-tilt}\, A_{\mathbb{A}_{n-2}} = c_{n-1} - c_{n-2}$.

(iii) By the symmetry of G, we have $\mathsf{n}(3) = \mathsf{n}(2)$.

(iv) Let $\epsilon \in M_4$. Then the quiver Q_ϵ is described as

$$1 \leftarrow 3 \rightarrow 2 \qquad 4 \sim\sim \cdots \sim\sim n-1 \sim\sim n.$$

Thus we can show $2\mathsf{n}(4) = \#\,\text{tilt}\, kQ' \times \#\,\text{2-tilt}\, A_{\mathbb{A}_{n-3}} = 5c_{n-3}$, where $Q' = 1 \leftarrow 3 \rightarrow 2$.

(v) For $\epsilon \in M_5(t)$, the quiver Q_ϵ is given by

$$1 \leftarrow 3 \rightarrow 2, \quad 3 \longrightarrow 4 \longleftarrow \cdots \longrightarrow t \qquad t+1 \sim\sim \cdots \sim\sim n.$$

Then we obtain

$$2\mathsf{n}(5) = \sum_{t=4}^{n} \#\,\text{tilt}\, kQ' \times \#\,\text{2-tilt}\, A_{\mathbb{A}_{n-t}} = \frac{3n-4}{2n} c_{n-1} + \sum_{t=4}^{n} \frac{3t-4}{2t} c_{t-1} c_{n-t},$$

where Q' is the left connected component in Q_ϵ, that is, the underlying graph of Q' is $\mathbb{D}_t$.

By Lemma 2.2.9(3) and (4), we obtain the equality

$$\begin{aligned}
\sum_{t=4}^{n} \frac{3t-4}{2t} c_{t-1} c_{n-t} &= \frac{3}{2} \sum_{t=4}^{n} c_{t-1} c_{n-t} - 2 \sum_{t=4}^{n} \frac{1}{t} c_{t-1} c_{n-t} \\
&= \frac{3}{2} \{4^{n-1} - c_{n-1} - 2c_{n-2} - 6c_{n-3}\} - 2\{\frac{1}{2} c_n - c_{n-1} - c_{n-2} - 2c_{n-3}\} \\
&= 6 \cdot 4^{n-2} - c_n + \frac{1}{2} c_{n-1} - c_{n-2} - 5c_{n-3}.
\end{aligned}$$

By (i–v), we have

$$\begin{aligned}
\#\,\text{2-tilt}\, A_{\mathbb{D}_n} &= c_{n-2} + 2(c_{n-1} - c_{n-2}) + 5c_{n-3} + 6 \cdot 4^{n-2} - c_n + \frac{2n-2}{n} c_{n-1} - c_{n-2} - 5c_{n-3} \\
&= 6 \cdot 4^{n-2} - c_n + \frac{4n-2}{n} c_{n-1} - 2c_{n-2} \\
&= 6 \cdot 4^{n-2} - 2c_{n-2},
\end{aligned}$$

where the last equality follows from $c_n = \frac{2(2n-1)}{n} c_{n-1}$ by Proposition 2.2.9(2). □

Next, we consider the case when G is of type (I). The number is obtained by using the result on type $\mathbb{D}$.

Proposition 2.2.14. *If $G = \mathrm{I}_n$, then we have*

$$\#\,2\text{-tilt}\, A_G = 6 \cdot 4^{n-2} + 2c_n - 4c_{n-1} - 4c_{n-2}.$$

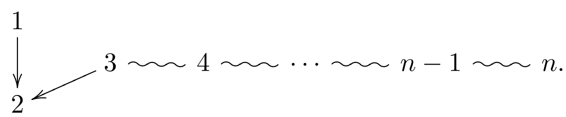
Proof. Let G be the graph I_n and Q its double quiver. By using a similar method of the proof of Proposition 2.2.13, we calculate the right-hand side of

$$\#\,2\text{-tilt}\, A_G = 2 \sum_{\substack{\epsilon\colon Q_0 \to \{\pm 1\} \\ \epsilon(3)=+1}} \#\,\text{tilt}\, kQ_\epsilon \tag{2.2.7}$$

Let M and M_i $(1 \le i \le 5)$ be sets of maps given in the proof of Proposition 2.2.13 and $\mathsf{m}(i) := \sum_{\epsilon \in M_i} \#\,\text{tilt}\, kQ_\epsilon$. Let Q' be the double quiver of the graph $\mathbb{D}_n$. For each map $\epsilon \in M_1 \sqcup M_4 \sqcup M_5$, we have $Q_\epsilon = Q'_\epsilon$. Hence for each $i \in \{1, 4, 5\}$, we have

$$\mathsf{m}(i) = \sum_{\epsilon \in M_i} \#\,\text{tilt}\, kQ_\epsilon = \sum_{\epsilon \in M_i} \#\,\text{tilt}\, kQ'_\epsilon = \mathsf{n}(i).$$

Since $\mathsf{m}(2) = \mathsf{m}(3)$ holds by the symmetry of G, we have only to calculate $\mathsf{m}(2)$. For each map $\epsilon \in M_2$, the quiver Q_ϵ is given by



Then the calculation of $\mathsf{m}(2)$ is reduced to graphs of type $\mathbb{A}$. Let Q'' be the double quiver of the graph $\mathbb{A}_n$. Indeed, we have

$$\begin{aligned}
\mathsf{m}(2) &= \sum_{\substack{\epsilon\colon Q''_0 \to \{\pm 1\} \\ \epsilon(3)=+1}} \text{tilt}\, kQ''_\epsilon - \sum_{\substack{\epsilon\colon Q''_0 \to \{\pm 1\} \\ \epsilon(2)=\epsilon(3)=+1}} \text{tilt}\, kQ''_\epsilon - \sum_{\substack{\epsilon\colon Q''_0 \to \{\pm 1\} \\ -\epsilon(1)=-\epsilon(2)=\epsilon(3)=+1}} \text{tilt}\, kQ''_\epsilon \\
&= \frac{1}{2}\#\,2\text{-tilt}\, A_{\mathbb{A}_n} - \frac{1}{4}\#\,2\text{-tilt}\, A_{\mathbb{A}_2} \cdot \#\,2\text{-tilt}\, A_{\mathbb{A}_{n-2}} - \mathsf{n}(2). \\
&= \frac{1}{2}c_n - \frac{3}{2}c_{n-2} - \mathsf{n}(2).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\#\,2\text{-tilt}\, A_G &= \{\#\,2\text{-tilt}\, A_{\mathbb{D}_n} - 4\mathsf{n}(2)\} + 4\mathsf{m}(2) \\
&= 6 \cdot 4^{n-2} - 2c_{n-2} - 4\mathsf{n}(2) + 2c_n - 6c_{n-2} - 4\mathsf{n}(2) \\
&= 6 \cdot 4^{n-2} - 2c_n - 4c_{n-1} - 4c_{n-2}.
\end{aligned}$$

This finishes the proof. □

For the remained finite series $\mathbb{E}$, (II), (III), (IV) and (V), we just compute the number by using Theorem 2.1.6(2) with Proposition 2.2.7.

Proposition 2.2.15. *For each case* $\mathbb{E}$, (II), (III), (IV) *and* (V)*, the number* 2-tilt A_G *is given by the list of Theorem 2.0.2.*

Proof. They are calculated by similar way. □

Part II

Complete gentle algebras via marked surfaces

Chapter 3

Complete special biserial algebras are g-tame

This chapter (Chapter 3) is based on the joint work with Yurikusa [AY].

3.1. Marked surfaces

In this section, we recall the notions and results of [APS19, PPP19] (see also [OPS18]). Our notations are slightly different from theirs for the convenience of our purpose.

3.1.1 Dissections of marked surfaces

Definition 3.1.1. A *marked surface* is the pair (S, M) consisting of the following data:

(a) S is a connected compact oriented Riemann surface with (possibly empty) boundary ∂S.

(b) $M = M_\circ \sqcup M_\bullet$ is a non-empty finite set of marked points on S such that

- both $M_\circ$ and $M_\bullet$ are not empty;
- each component of ∂S has at least one marked point;
- the points of $M_\circ$ and $M_\bullet$ alternate on each boundary component.

Any marked point in the interior of S is called a *puncture*.

Let (S, M) be a marked surface.

Definition 3.1.2. (1) A $\circ$-*arc* (resp., $\bullet$-arc) γ of (S, M) is a curve in S with endpoints in $M_\circ$ (resp., $M_\bullet$), considered up to isotopy, such that the following conditions are satisfied:

- γ does not intersect itself except at its endpoints;
- γ is disjoint from M and ∂S except at its endpoints;
- γ does not cut out a monogon without punctures.

(2) A $\circ$-*dissection* (resp., $\bullet$-*dissection*) is a maximal set of pairwise non-intersecting $\circ$-arcs (resp., $\bullet$-arcs) on (S, M) which does not cut out a subsurface without marked points in $M_\bullet$ (resp., $M_\circ$).

Remark 3.1.3. Let g be the genus of S, b be the number of boundary components and $p_\circ$ (resp., $p_\bullet$) be the number of punctures in $M_\circ$ (resp., $M_\bullet$). By [APS19, Proposition 1.11], a $\circ$-dissection (resp., $\bullet$-dissection) of (S, M) consists of $|M_\circ| + p_\bullet + b + 2g - 2 = |M_\bullet| + p_\circ + b + 2g - 2$ $\circ$-arcs (resp., $\bullet$-arcs).

By symmetry, the claims in this thesis hold if we permute the symbols $\circ$ and $\bullet$. Thus we state only one side of each claim. A dissection divides (S, M) into polygons with exactly one marked point.

Proposition 3.1.4. *[APS19, Proposition 1.12] For a $\bullet$-dissection D of (S, M), each connected component of $S \setminus D$ is homeomorphic to one of the following:*

- *an open disk with precisely one marked point in $M_\circ \cap \partial S$;*
- *an open disk with precisely one marked point in $M_\circ$, but not in ∂S.*

For a $\bullet$-dissection D of (S, M), the closure of a connected component of $S \setminus D$ is called a *polygon of D*. Proposition 3.1.4 implies that any polygon of D has exactly one marked point in $M_\circ$. We denote by $\triangle_v$ the polygon with marked point $v \in M_\circ$ (see Figure 3.1).

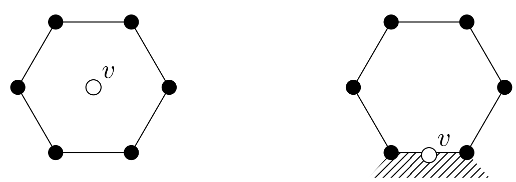


Figure 3.1: Polygon $\triangle_v$ for a marked point $v \in M_\circ$

Definition-Proposition 3.1.5. *[PPP19, Proposition 3.6] For a $\bullet$-dissection D of (S, M), there is a unique $\circ$-dissection D^* whose each $\circ$-arc intersects exactly one $\bullet$-arc of D. We have $D^{**} = D$. We call D^* the dual dissection of D. For $d \in D$, we write the corresponding $\circ$-arc by $d^* \in D^*$.*

3.1.2 g-vectors of D-laminates and D-laminations

We fix a $\bullet$-dissection D of (S, M).

Definition 3.1.6. (1) A $\circ$-*laminate* of (S, M) is a curve γ in S, considered up to isotopy relative to M, that is either

- a closed curve, or
- a curve whose ends are unmarked points on ∂S or spirals around punctures in $M_\circ$ either clockwise or counterclockwise (see Figure 3.2).

(2) A *D-laminate* is a non-self-intersecting $\circ$-laminate γ of (S, M) intersecting at least one $\bullet$-arc of D such that the following condition is satisfied:

($*$) Whenever γ intersects $d \in D$, the endpoints v and v' of d^* lie on opposite sides of γ in $\triangle_v \cup \triangle_{v'}$.

Here, we consider that the point v lies on the right (resp., left) to γ if γ circles clockwise (resp., counterclockwise) around v in $\triangle_v$.

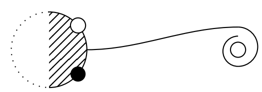


Figure 3.2: Example of a $\circ$-laminate

A D-laminate is called a *closed D-laminate* if it is a closed curve. Remark that non-closed D-laminates coincide with D-slaloms in [PPP19]. Now, we treat a certain collection of D-laminates, that is central in this chapter.

Definition 3.1.7. We say that two D-laminates are *compatible* if they don't intersect. A finite multi-set of pairwise compatible D-laminates is called a *D-lamination*. A D-lamination is said to be

- *reduced* if it consists of pairwise distinct non-closed D-laminates, and
- *complete* if it is reduced and is the maximal as a set.

Let γ be a D-laminate. Using the notations in the condition ($*$), let p be an intersection point of γ and d such that γ leaves $\triangle_v$ to enter $\triangle_{v'}$ via p. Then p is said to be *positive* (resp., *negative*) if v is to its right (resp., left), or equivalently, v' is to its left (resp., right). See Figure 4.3. For $d \in D$, we define an integer

$$g_d(\gamma) := \#\{\text{positive intersection points of } \gamma \text{ and } d\} \\ -\#\{\text{negative intersection points of } \gamma \text{ and } d\}. \tag{3.1.1}$$

The *g-vector* $g(\gamma)$ of γ is given by $\left(g_d(\gamma)\right)_{d \in D} \in \mathbb{Z}^{|D|}$, where $|D|$ is the number of $\bullet$-arcs of D. Remark that if $p_1 \dots, p_s$ are intersection points of γ and d, then all of them are positive or all of them are negative. Thus, the absolute value of $g_d(\gamma)$ just counts the number of intersection points of γ and d. For a D-lamination $\mathcal{X}$, the g-vector of $\mathcal{X}$ is defined to be $g(\mathcal{X}) = \sum_{\gamma \in \mathcal{X}} g(\gamma)$, where $g(\emptyset) := 0$. We denote by $C(\mathcal{X})$ a cone in $\mathbb{R}^{|D|}$ spanned by $g(\gamma)$ for all $\gamma \in \mathcal{X}$ and call it the *g-vector cone* of $\mathcal{X}$. We denote by $\mathcal{F}(D)$ a collection of all g-vector cones of reduced D-laminations.

The invariants, g-vectors and g-vector cones, have good properties.

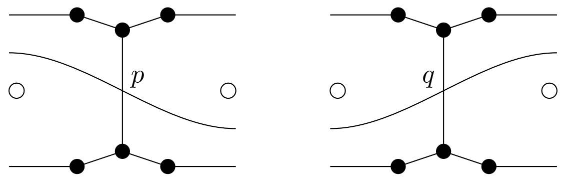


Figure 3.3: Positive intersection point p and negative intersection point q

Theorem 3.1.8. *[PPP19, Theorems 5.12 and 6.12]*

(1) $\mathcal{F}(D)$ is a simplicial fan whose maximal faces correspond to complete D-laminations.

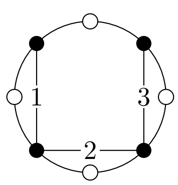
(2) A reduced D-lamination is complete if and only if it has precisely $|D|$ elements.

Theorem 3.1.9. *[PPP19, Theorem 6.14] If $\mathcal{F}(D)$ is finite, then all D-laminates are non-closed. In this case, we have $|\mathcal{F}(D)| = \mathbb{R}^{|D|}$.*

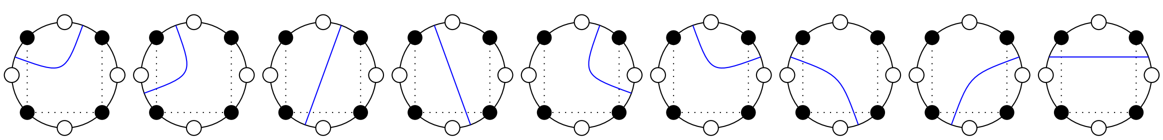
3.2. Examples

In this section, we examine our notions defined in the previous section.

(1) Let (S, M) be a disk with $|M| = 8$ such that all marked points lie on ∂S. For a $\bullet$-dissection of (S, M)

$D =$  ,

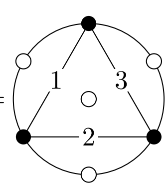
all D-laminates and the corresponding g-vectors are given as follows:

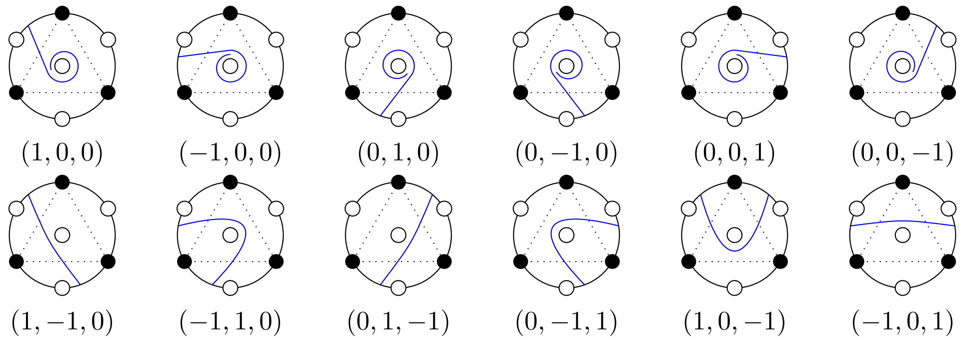


$(1,0,0)$ $(-1,0,0)$ $(0,1,0)$ $(0,-1,0)$ $(0,0,1)$ $(0,0,-1)$ $(1,-1,0)$ $(0,1,-1)$ $(1,0,-1)$

There are 14 complete D-laminations. The corresponding fan $\mathcal{F}(D)$ of g-vector cones for D is given as in the top left diagram of Figure 3.4.

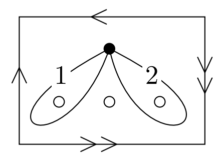
(2) Let (S, M) be a disk with $|M| = 7$ such that one marked point in $M_\circ$ is a puncture and the others lie on ∂S. For a $\bullet$-dissection of (S, M)

$D =$  , all D-laminates and the corresponding g-vectors are given as follows:



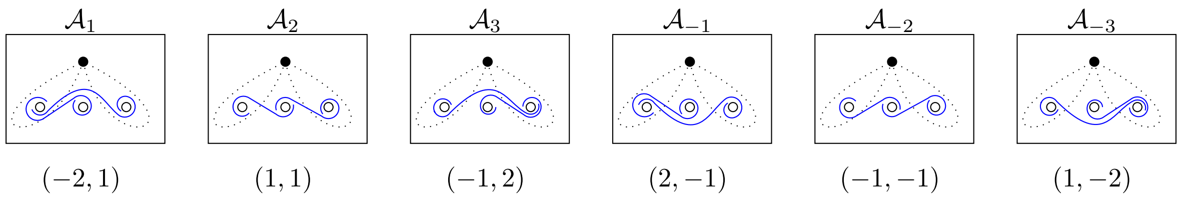
There are 20 complete D-laminations. The fan $\mathcal{F}(D)$ is given as in the top right diagram of Figure 3.4.

(3) Consider a sphere S with $\partial S = \emptyset$ and $|M_\bullet| = 1, |M_\circ| = 3$ (i.e., all marked points are punctures). In this case, every $\bullet$-dissection consists of $|M| - 2 = 2$ $\bullet$-arcs by Remark 3.1.3. Let D be a $\bullet$-dissection D of (S, M) given by



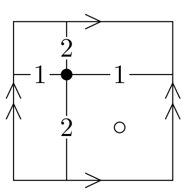
,

where we identify the sides of the square having a common symbol in the same direction. We find that there are no closed D-laminates. All complete D-laminations and the corresponding g-vectors are described as follows:



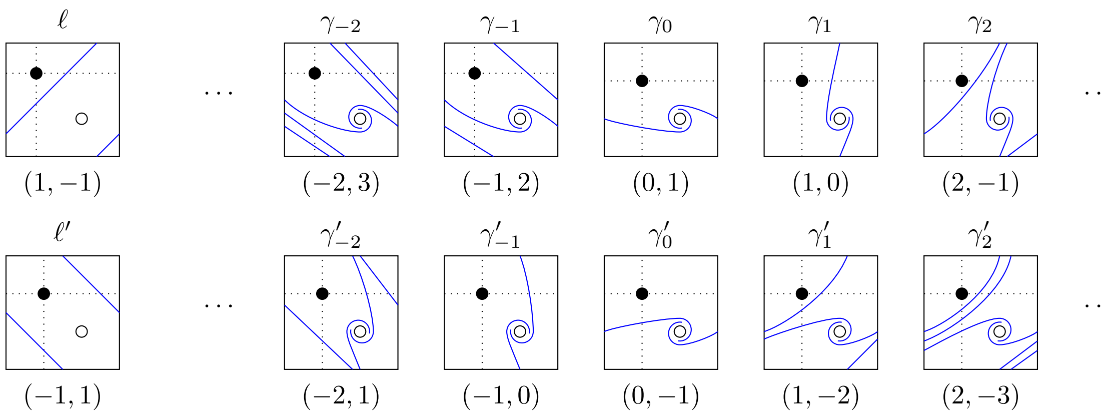
The fan $\mathcal{F}(D)$ is given as in the bottom left diagram in Figure 3.4.

(4) Consider a torus $S = T^2$ with $\partial S = \emptyset$ and $|M| = 2$ (i.e., both marked points are punctures). Let D be a $\bullet$-dissection of (S, M) given by



,

where we identify the opposite sides of the square in the same direction. All D-laminates and the corresponding g-vectors are given as follows:



where ℓ, ℓ' are closed D-laminates and γ_m, γ'_m are non-closed D-laminates for all $m \in \mathbb{Z}$. We find that the set $\{\{\gamma_m, \gamma_{m+1}\}, \{\gamma'_m, \gamma'_{m+1}\} \mid m \in \mathbb{Z}\}$ provides all complete D-laminations. The fan $\mathcal{F}(D)$ is given as in the bottom right diagram of Figure 3.4.

For the closed D-laminate ℓ, its g-vector $g(\ell) = (1, -1) \in \mathbb{Z}^2$ does not contained in $|\mathcal{F}(D)|$. It will be approximated by using the Dehn twist T_ℓ along ℓ (we refer to Section 3.4 for the details). In fact, we have $\mathsf{T}_\ell(\gamma_i) = \gamma_{i+1}$ for any $i \in \mathbb{Z}_{>0}$ and hence

$$g(\ell) = (1, -1) \in \overline{\bigcup_{m \geq 0} C(\mathsf{T}_\ell^m(\{\gamma_1\}))}.$$

3.3. g-vectors and lattice points

The aim of this section is to prove the following result. This is an analog of [FT18, Theorems 12.3, 13.6] and a generalization of [PPP19, Proposition 6.14] to an arbitrary dissection.

Theorem 3.3.1. *Let D be a $\bullet$-dissection of (S, M). Then there is a bijection*

$$\{D\text{-laminations}\} \to \mathbb{Z}^{|D|}$$

given by the map $\mathcal{X} \mapsto g(\mathcal{X}) = \sum_{\gamma \in \mathcal{X}} g(\gamma)$.

To prove Theorem 3.3.1, we first consider the following two cases:

(a) Let (S_1, M_1) be a disk with $|M_1| = 2n + 2$ such that all marked points lie on ∂S_1. Let D_1 be a $\bullet$-dissection of (S_1, M_1) as in the left diagram of Figure 3.5.

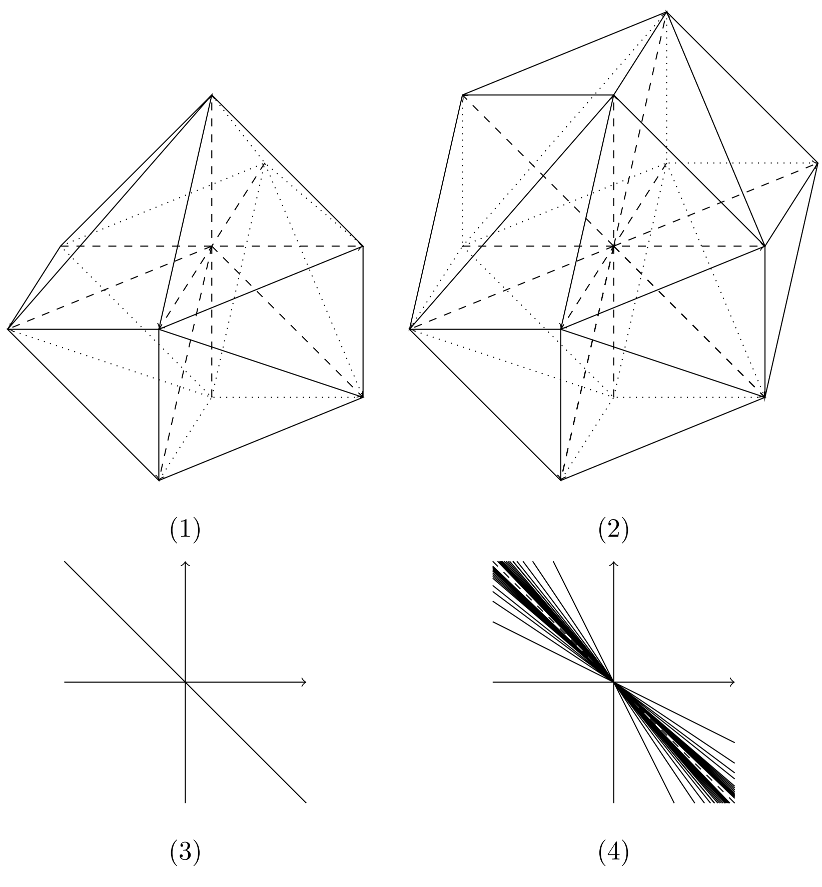


Figure 3.4: A fan $\mathcal{F}(D)$ of g-vector cones for Examples (1)-(4)

(b) Let (S_2, M_2) be a disk with $|M_2| = 2n+1$ such that one marked point in $(M_2)_\circ$ is a puncture and the others lie on ∂S_2. Let D_2 be a $\bullet$-dissection of (S_2, M_2) as in the right diagram of Figure 3.5.

In both cases, we have $|D_i| = n$.

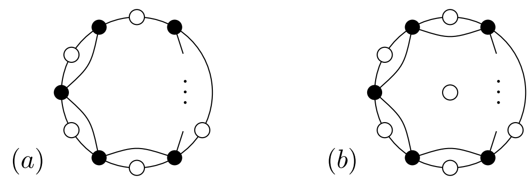


Figure 3.5: Special cases (a) and (b)

Proposition 3.3.2. *For $i \in \{1,2\}$, $\mathcal{F}(D_i)$ is finite. In particular, we have $|\mathcal{F}(D_i)| = \mathbb{R}^{|D_i|}$.*

Proof. In the same way as (1) and (2) in Section 3.2, one can check that the number of D_1-laminates is equal to $\frac{1}{2}n(n+3)$ and the number of D_2-laminates is equal to $n(n+1)$, in particular, they are finite. The latter assertion follows from Theorem 3.1.9. $\square$

Corollary 3.3.3. *Theorem 3.3.1 holds for $D = D_1$ or $D = D_2$.*

Proof. For $i \in \{1,2\}$, $\mathcal{F}(D_i)$ is a simplicial fan satisfying $|\mathcal{F}(D_i)| = \mathbb{R}^{|D_i|}$ by Proposition 3.3.2. This implies that the map $\mathcal{X} \mapsto g(\mathcal{X})$ provides a one-to-one correspondence between the set of D_i-laminations consisting only of non-closed D_i-laminates and $\mathbb{Z}^{|D_i|}$. More precisely, for any $g \in \mathbb{Z}^{|D_i|}$, there is exactly one reduced D_i-lamination $\mathcal{X}'$ such that g is contained in the interior of $C(\mathcal{X}')$. Since $C(\mathcal{X}')$ is simplicial, g is uniquely written by $g = \sum_{\gamma \in \mathcal{X}'} a_\gamma g(\gamma)$ for $a_\gamma \in \mathbb{Z}_{>0}$. Then a D_i-lamination $\mathcal{X}$ consisting of a_γ elements $\gamma \in \mathcal{X}'$ is a unique one such that $g(\mathcal{X}) = g$. $\square$

Now, we are ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Let D be a $\bullet$-dissection of (S, M) and $g = (g_d)_{d \in D}$ an arbitrary integer vector in $\mathbb{Z}^{|D|}$. In the following, we construct a D-lamination $\mathcal{X}$ such that $g = g(\mathcal{X})$.

Recall that (S, M) is divided into polygons $\triangle_v$ for all $v \in M_\circ$. For $v \in M_\circ$, we can naturally embed $\triangle_v$ into the above $\bullet$-dissection D_i of (S_i, M_i) for $i = 1$ or 2. More precisely, (S_i, M_i) is obtained from $\triangle_v$ by gluing a digon with one $\circ$-marked point on each $\bullet$-arc of $D \cap \triangle_v$, where $D \cap \triangle_v$ form D_i in (S_i, M_i). By Corollary 3.3.3, there is a unique D_i-lamination $\mathcal{X}_v$ such that $g(\mathcal{X}_v) = (g_d)_{d \in D \cap \triangle_v}$. We regard $\mathcal{X}_v \cap \triangle_v$ as a set of pairwise non-intersecting curves in (S, M) with $|g_d|$ endpoints on $d \in D \cap \triangle_v$. Then we can glue the curves of $\mathcal{X}_v \cap \triangle_v$ for all $v \in M_\circ$ at their endpoints on D. As a result, we obtain a set $\mathcal{X}$ of pairwise non-intersecting $\circ$-laminates of (S, M). From our construction, every ∘-laminate of $\mathcal{X}$ is a D-laminate, and hence $\mathcal{X}$ forms a D-lamination such that $g(\mathcal{X}) = g$ as desired.

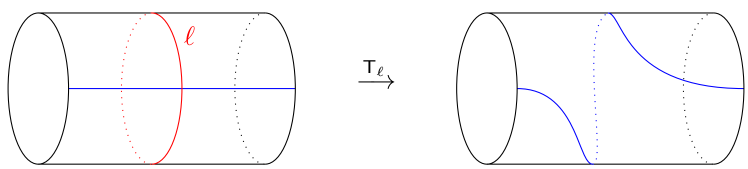
On the other hand, the uniqueness of $\mathcal{X}$ follows from that of $\mathcal{X}_v$ for any $v \in M_\circ$. □

3.4. Positive position and Dehn twists

In this section, we fix a •-dissection D of (S, M) and make preparations for proving Theorem 0.0.7. The proof of Theorem 0.0.7 appears in the next section.

3.4.1 Dehn twist along a closed D-laminate

We denote by T_ℓ the Dehn twist along a closed curve ℓ with the orientation defined as follows:



In general, $\mathsf{T}_\ell(\gamma)$ is not a D-laminate for a given D-laminate γ. We will give a condition that Dehn twists work well.

Let γ and δ be D-laminates. For each intersection point p of γ and δ, we can assume that p lies in $S \setminus D$, thus $p \in \triangle_v$ for some $v \in M_\circ$. We set orientations of the segments of γ and δ in $\triangle_v$ such that v lies on the right to them. We say that γ *is in positive position for* δ if γ and δ don't intersect or γ intersects δ from right to left at each intersection point (see Figure 3.6).

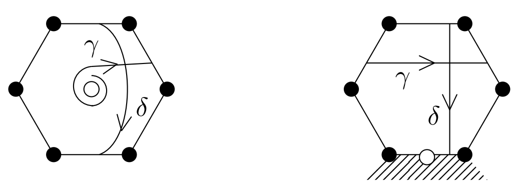


Figure 3.6: A D-laminate γ is in positive position for a D-laminate δ

Lemma 3.4.1. *Let ℓ be a closed D-laminate and γ a non-closed D-laminate which is in positive position for ℓ. Then*

(1) $\mathsf{T}_\ell(\gamma)$ is a non-closed D-laminate;

(2) $g(\mathsf{T}_\ell(\gamma)) = g(\gamma) + \#(\gamma \cap \ell) g(\ell)$;

(3) if a D-laminate γ' does not intersect ℓ, then

$$\#(\gamma' \cap \gamma) = \#(\gamma' \cap \mathsf{T}_\ell(\gamma)).$$

Proof. The assertions immediately follow from the assumption. □

In the situation of Lemma 3.4.1, we can repeat the Dehn twist T_ℓ. Moreover, Lemma 3.4.1 is generalized for D-laminations. For closed curves $\ell_1, \dots, \ell_k$ and $m_1, \dots, m_k \in \mathbb{Z}_{\geq 0}$, we write

$$\mathsf{T}_{(\ell_1,\dots,\ell_k)}^{(m_1,\dots,m_k)} := \mathsf{T}_{\ell_1}^{m_1} \cdots \mathsf{T}_{\ell_k}^{m_k}.$$

Note that if $\ell_1, \dots, \ell_k$ are pairwise non-intersecting, then all T_{ℓ_i} are commutative.

Proposition 3.4.2. *Let $\ell_1, \dots, \ell_k$ be a D-lamination consisting only of closed D-laminates and $\gamma_1, \dots, \gamma_h$ a D-lamination consisting only of non-closed D-laminates which are in positive position for any ℓ_i. Then for any $m_1, \dots, m_k \in \mathbb{Z}_{\geq 0}$ and $\mathsf{T} := \mathsf{T}_{(\ell_1,\dots,\ell_k)}^{(m_1,\dots,m_k)}$,*

(1) $\{\mathsf{T}(\gamma_1), \dots, \mathsf{T}(\gamma_h)\}$ is a D-lamination consisting only of non-closed D-laminates;

(2) we have the equality

$$\sum_{i=1}^{h} g(\mathsf{T}(\gamma_i)) = \sum_{i=1}^{h} g(\gamma_i) + \sum_{i=1}^{h}\sum_{j=1}^{k} m_j \#(\gamma_i \cap \ell_j) g(\ell_j).$$

Proof. (1) Let $\mathcal{X} := \{\gamma_1, \dots, \gamma_h\}$ and $\mathcal{Y} := \{\ell_1, \dots, \ell_k\}$. For any $\gamma \in \mathcal{X}$ and $\ell \in \mathcal{Y}$, by Lemma 3.4.1(1), $\mathsf{T}_\ell(\gamma)$ is a non-closed D-laminate. Lemma 3.4.1(3) says that $\mathsf{T}_\ell(\gamma) \cap \ell'$ is naturally identified with $\gamma \cap \ell'$ for any $\ell' \in \mathcal{Y}$. In particular, $\mathsf{T}_\ell(\gamma)$ is also in positive position for ℓ', thus $\mathsf{T}_{\ell'}\mathsf{T}_\ell(\gamma)$ is a non-closed D-laminate. Repeating this process, $\mathsf{T}(\gamma)$ is a non-closed D-laminate. Since $\mathsf{T}(\gamma)$ and $\mathsf{T}(\gamma')$ don't intersect for any $\gamma, \gamma' \in \mathcal{X}$, the assertion holds.

(2) The equality is calculated from Lemma 3.4.1(2) since Lemma 3.4.1(3) says that the number of all intersection points of $\mathcal{X}$ and $\mathcal{Y}$ is not changed by the Dehn twists. □

3.4.2 Non-closed D-laminate ℓ^d for a closed D-laminate ℓ

Let $\mathcal{X}$ be a D-lamination consisting only of non-closed D-laminates. We assume that there is a closed D-laminate ℓ such that $\mathcal{X} \sqcup \{\ell\}$ is a D-lamination. By the definition of D-laminates, there exists $d \in D$ such that $g_d(\ell) > 0$. From now, we construct a non-closed D-laminate ℓ^d such that

(a) ℓ^d is a non-closed D-laminate which is compatible with any D-laminate of $\mathcal{X}$;

(b) ℓ^d intersects with ℓ so that ℓ^d is in positive position for ℓ.

It plays an important role to prove Theorem 0.0.7 in the next section.

First, for $d \in D$, we define a D-laminate d_+^* (resp., d_-^*) as follows (see Figure 3.7):

- d^*_+ (resp., d^*_-) is a laminate running along d^* in a small neighborhood of it;
- If d^* has an endpoint $v \in M_\circ$ on a component C of ∂S, then the corresponding endpoint of d^*_+ (resp., d^*_-) is located near v on C in the counterclockwise (resp., clockwise) direction;
- If d^* has an endpoint at a puncture $p \in M_\circ$, then the corresponding end of d^*_+ (resp., d^*_-) is a spiral around p counterclockwise (resp., clockwise).

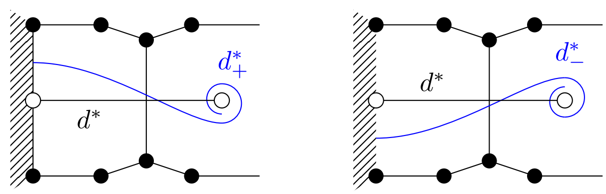


Figure 3.7: Two D-laminates d^*_+ and d^*_-

That is, $g_e(d^*_+) = \delta_{ed}$ (resp., $g_e(d^*_-) = -\delta_{ed}$) for $e \in D$, where δ is the Kronecker delta.

On this notation, $g_d(\ell) > 0$ implies $\ell \cap d^*_+ \neq \emptyset$ and d^*_+ is in positive position for ℓ. Without loss of generality, we can assume that $p \in \ell \cap d^*_+$ lie on d as in the left diagram of Figure 3.8 .

Second, for each endpoint v of d^*, we define a curve ℓ_v of S as follows: Consider the segment $\alpha := d^*_+ \cap \triangle_v$.

- If α intersects none of $\mathcal{X}$, then let $\ell_v := \alpha$ (see the center diagram of Figure 3.8);
- Otherwise, let p_v be the nearest intersection point of α and $\mathcal{X}$ from p, where $p_v \in \alpha \cap \gamma$ for $\gamma \in \mathcal{X}$. We denote by q an endpoint of a connected segment in $\gamma \cap \triangle_v$ containing p_v such that the intersection point $q \in \gamma \cap D$ is negative. Then ℓ_v is a curve obtained by gluing the following two curves at p_v (see the right diagram of Figure 3.8):
  - (i) a segment of α between p and p_v;
  - (ii) a segment of γ obtained by cutting γ at p_v, that contains q.

Finally, we define ℓ^d as a curve obtained by gluing ℓ_v and $\ell_{v'}$ at p for endpoints v and v' of d^*. A segment of ℓ^d between p_v and $p_{v'}$ is called its *center segment*, where p_v is a point on ℓ_v sufficiently close to v if $\ell_v = \alpha$. It follows from the construction that ℓ^d satisfies (a) and (b) above. Moreover, (b) is generalized as follows.

Lemma 3.4.3. *In the above situations, if a D-laminate γ is compatible with $\mathcal{X}$, then ℓ^d is in positive position for γ.*

Proof. If γ intersects ℓ^d, then all the intersection points lie on the center segment of ℓ^d, thus the assertion holds. □

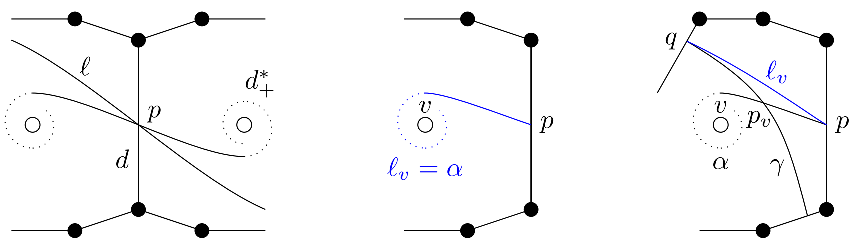


Figure 3.8: A closed D-laminate ℓ and $d \in D$ with $g_d(\ell) > 0$ (left), constructions of a curve ℓ_v (center, right)

3.5. Proof of Theorem 0.0.7

In this section, we prove Theorem 0.0.7. Namely, we show that

$$\overline{|\mathcal{F}(D)|} = \mathbb{R}^{|D|}$$

holds for any $\bullet$-dissection D of a marked surface (S, M). The idea of its proof comes from [Yur20]. Fix a $\bullet$-dissection D of (S, M). Let $g \in \mathbb{Z}^{|D|}$. By Theorem 3.3.1, there is a D-lamination $\mathcal{X}$ such that $g = g(\mathcal{X}) = \sum_{\gamma \in \mathcal{X}} g(\gamma)$. It is sufficient to find D-laminations $\{\mathcal{X}_m\}_{m \in \mathbb{Z}_{\geq 0}}$ consisting only of non-closed D-laminates such that

$$\mathcal{X}^{\mathrm{nc}} \subseteq \mathcal{X}_m \quad \text{and} \quad g \in \overline{\bigcup_{m \in \mathbb{Z}_{\geq 0}} C(\mathcal{X}_m)}.$$

where $\mathcal{X} = \mathcal{X}^{\mathrm{nc}} \sqcup \mathcal{X}^{\mathrm{cl}}$ is a decomposition such that $\mathcal{X}^{\mathrm{nc}}$ (resp., $\mathcal{X}^{\mathrm{cl}}$) consists of all non-closed D-laminates (resp., closed D-laminates) in $\mathcal{X}$.

If $\mathcal{X}^{\mathrm{cl}} = \emptyset$, then a family of $\mathcal{X}_m := \mathcal{X}$ for all $m \in \mathbb{Z}_{>0}$ is the desired one. Assume that $\mathcal{X}^{\mathrm{cl}}$ is non-empty. For $\ell_1 \in \mathcal{X}^{\mathrm{cl}}$ and $d_1 \in D$ with $g(\ell_1)_{d_1} > 0$, we obtain a non-closed D-laminate $\ell_1^{d_1}$ by the construction of Section 3.4.2 for $\mathcal{X}^{\mathrm{nc}}$. By Lemma 3.4.3, $\ell_1^{d_1}$ is in positive position for every $\ell \in \mathcal{X}^{\mathrm{cl}}$.

If the set $\{\ell \in \mathcal{X}^{\mathrm{cl}} \mid \ell \cap \ell_1^{d_1} = \emptyset\}$ is non-empty, then we take $\ell_2 \in \{\ell \in \mathcal{X}^{\mathrm{cl}} \mid \ell \cap \ell_1^{d_1} = \emptyset\}$ and $d_2 \in D$ with $g(\ell_2)_{d_2} > 0$. By the construction of Section 3.4.2 for $\mathcal{X}^{\mathrm{nc}} \sqcup \{\ell_1^{d_1}\}$, we obtain a non-closed D-laminate $\ell_2^{d_2}$. Notice that $\ell_2^{d_2}$ consists of some of segments of D-laminates in $\mathcal{X}^{\mathrm{nc}}$, the center segment of $\ell_2^{d_2}$ and that of $\ell_1^{d_1}$, where the third type may not appear. In the same way as Lemma 3.4.3, we can show that $\ell_2^{d_2}$ is in positive position for every $\ell \in \mathcal{X}^{\mathrm{cl}}$. Repeating this process, we finally get an integer $h \in \{1, \ldots, k = |\mathcal{X}^{\mathrm{cl}}|\}$ and non-closed D-laminates $\ell_1^{d_1}, \ldots, \ell_h^{d_h}$ such that

$$\{\ell \in \mathcal{X}^{\mathrm{cl}} \mid \ell \cap \ell_1^{d_1} = \cdots = \ell \cap \ell_h^{d_h} = \emptyset\} = \emptyset.$$

Moreover, our construction provides the following properties:

- $\mathcal{X}^{\mathrm{nc}} \cup \{\ell_1^{d_1}, \ldots, \ell_h^{d_h}\}$ is a D-lamination consisting only of non-closed D-laminates;

- For $i \in \{1, \ldots, h\}$, $\ell_i^{d_i}$ is in positive position for every $\ell \in \mathcal{X}^{\mathrm{cl}}$.

We set $\mathcal{X}^{\mathrm{cl}} = \{\ell_1, \ldots, \ell_h\} \sqcup \{\ell_{h+1}, \ldots, \ell_k\}$ and fix the notations $n_j^{(i)} := \#(\ell_i^{d_i} \cap \ell_j)$, $n_j := \sum_{i=1}^h n_j^{(i)}$, and $N := n_1 \cdots n_k$. Set

$$\mathsf{T} := \mathsf{T}_{(\ell_1, \ldots, \ell_k)}^{(\frac{N}{n_1}, \ldots, \frac{N}{n_k})}.$$

By Proposition 3.4.2, $\mathsf{T}^m(\ell_i^{d_i})$ are non-closed D-laminates for all $m \in \mathbb{Z}_{\geq 0}$ and $i \in \{1, \ldots, h\}$, and we get the equalities

$$\begin{aligned}
\sum_{i=1}^h g(\mathsf{T}^m(\ell_i^{d_i})) &= \sum_{i=1}^h g(\ell_i^{d_i}) + m \sum_{i=1}^h \sum_{j=1}^k \frac{N}{n_j} n_j^{(i)} g(\ell_j) \\
&= \sum_{i=1}^h g(\ell_i^{d_i}) + mN \sum_{j=1}^k g(\ell_j) \\
&= \sum_{i=1}^h g(\ell_i^{d_i}) + mNg(\mathcal{X}^{\mathrm{cl}}).
\end{aligned}$$

It gives

$$\lim_{m \to \infty} \frac{\sum_{i=1}^h g(\mathsf{T}^m(\ell_i^{d_i}))}{m} = Ng(\mathcal{X}^{\mathrm{cl}}).$$

Then the D-lamination

$$\mathcal{X}_m := \mathcal{X}^{\mathrm{nc}} \cup \{\mathsf{T}^m(\ell_i^{d_i})\}_{i=1}^h$$

is the desired one because

$$g = g(\mathcal{X}^{\mathrm{nc}}) + g(\mathcal{X}^{\mathrm{cl}}) \in C(\mathcal{X}^{\mathrm{nc}}) + \overline{\bigcup_{m \in \mathbb{Z}_{\geq 0}} C(\{\mathsf{T}^m(\ell_i^{d_i})\}_{i=1}^h)} \subseteq \overline{\bigcup_{m \in \mathbb{Z}_{\geq 0}} C(\mathcal{X}_m)}.$$

3.6. Representation theory

In this section, we study the algebraic aspects of our results. We can see their examples in the next section.

3.6.1 Two-term silting complexes for module-finite algebras

Let $R := k[[t]]$ be the formal power series ring in one valuable over an algebraically closed field k. Let A be a basic R-algebra which is module-finite (i.e., A is finitely generated as an R-module). We denote by $\operatorname{proj} A$ the category of finitely generated projective right A-modules, by $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ the homotopy category of bounded complexes of projective modules in $\operatorname{proj} A$. In particular, $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ is an R-linear category and $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(X, Y)$ is a finitely generated R-module for any $X, Y \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$.

We begin with the following observation.

Proposition 3.6.1. *The category* $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ *is a Krull-Schmidt triangulated category.*

Proof. For any $X \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$, $E = \operatorname{End}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(X)$ is a module-finite algebra over the complete local noetherian ring R. Therefore, E is semiperfect by [CR62, p.132] and hence $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ is Krull-Schmidt. □

Now, we study two-term silting theory for a module-finite R-algebra A. We refer to [ADI, Kim20] for two-term silting theory of module-finite algebras.

Definition 3.6.2. Let $P = (P^i, f^i)$ be a complex in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$.

(1) We say that P is *two-term* if $P^i = 0$ for any integer $i \neq 0, -1$.

(2) We say that P is *presilting* if $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(P, P[m]) = 0$ for any positive integer m.

(3) We say that P is *silting* if it is presilting and $\operatorname{thick} P = \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$, where $\operatorname{thick} P$ is the smallest triangulated subcategory of $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ which contains P and is closed under taking direct summands.

We denote by $\operatorname{2-ips} A$ (resp., $\operatorname{2-presilt} A$, $\operatorname{2-silt} A$) the set of isomorphism classes of indecomposable two-term presilting (resp., basic two-term presilting, basic two-term silting) complexes for A. Here, we say that a complex P is *basic* if all indecomposable direct summands of P are pairwise non-isomorphic. We denote by $|P|$ the number of non-isomorphic indecomposable direct summands of P and by $\operatorname{add} P$ the category of all direct summands of finite direct sums of copies of P.

Let $A = \bigoplus_{i=1}^{|A|} P_i$ be a decomposition of A, where P_i is an indecomposable projective A-module.

Definition 3.6.3. Let $P = (P^{-1} \xrightarrow{f} P^0)$ be a two-term complex in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$.

(1) The *g-vector* of P is defined by

$$g_A(P) := (m_1 - n_1, \dots, m_{|A|} - n_{|A|}) \in \mathbb{Z}^{|A|}$$

where m_i (resp., n_i) is the multiplicity of P_i as indecomposable direct summands of P^0 (resp., P^{-1}).

(2) The *g-vector cone* $C_A(P)$ is defined to be a cone in $\mathbb{R}^{|A|}$ spanned by g-vectors of all indecomposable direct summands of P.

We denote by $\mathcal{F}(A)$ a collection of g-vector cones of all basic two-term presilting complexes for A.

The following are basic properties of two-term presilting complexes.

Proposition 3.6.4. *[Kim20] Let* $P = (P^{-1} \xrightarrow{f} P^0) \in \operatorname{2-presilt} A$*. Then the following hold:*

(1) *P is a direct summand of some basic two-term silting complex for A.*

(2) *P is silting if and only if $|P| = |A|$.*

(3) $\operatorname{add} P^0 \cap \operatorname{add} P^{-1} = 0$.

Proposition 3.6.5. *[Kim20] The collection $\mathcal{F}(A)$ is a simplicial fan whose maximal faces correspond to basic two-term silting complexes for A.*

Let I be an ideal in A and $B := A/I$. In particular, B is module-finite over R. The functor $- \otimes_A B\colon \operatorname{proj} A \to \operatorname{proj} B$ induces a triangle functor

$$\overline{(-)} := - \otimes_A B\colon \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A) \to \mathsf{K}^{\mathrm{b}}(\operatorname{proj} B).$$

Proposition 3.6.6. *[ADI] In the above, the following hold.*

(1) *If P is a two-term presilting complex for A, then $\overline{P}$ is a two-term presilting complex for B.*

(2) *If A is g-tame, then so is B.*

Proposition 3.6.7. *[Kim20, Theorem 1.4] If I is generated by central elements and contained in the radical, then the correspondence $\overline{(-)}$ induces bijections*

$$\text{2-presilt}\, A \to \text{2-presilt}\, B \quad \textit{and} \quad \text{2-silt}\, A \to \text{2-silt}\, B.$$

In particular, we have $\mathcal{F}(A) = \mathcal{F}(B)$.

3.6.2 Complete special biserial algebras

We define complete special biserial algebras and complete gentle algebras. For a given finite connected quiver Q, let $\widehat{kQ}$ be the complete path algebra, that is, the completion of a path algebra kQ of Q with respect to kQ_+-adic topology, where kQ_+ is the arrow ideal. For arrows α and β, we denote by $s(\alpha)$ and $t(\alpha)$ the starting point and the terminal point of α, respectively. Also we write $\alpha\beta$ for the path from $s(\alpha)$ to $t(\beta)$.

Definition 3.6.8. Let Q be a finite connected quiver and I an ideal in the path algebra kQ of Q. We say that $\widehat{kQ}/\overline{I}$ is a *complete special biserial algebra*, where $\overline{I}$ is the closure of I, if all the following conditions are satisfied:

(SB1) For each vertex i of Q, there are at most two arrows starting at i and there are at most two arrows ending at i.

(SB2) For every arrow α in Q there exists at most one arrow β such that $t(\alpha) = s(\beta)$ and $\alpha\beta \notin I$.

(SB3) For every arrow α in Q, there exists at most one arrow γ such that $s(\alpha) = t(\gamma)$ and $\gamma\alpha \notin I$.

It is called *complete gentle algebra* if in addition:

(SB4) For every arrow α in Q, there exists at most one arrow β such that $t(\alpha) = s(\beta)$ and $\alpha\beta \in I$.

(SB5) For every arrow α in Q, there exists at most one arrow γ such that $s(\alpha) = t(\gamma)$ and $\gamma\alpha \in I$.

(SB6) The ideal I is generated by paths of length 2.

Here, we don't assume that complete special biserial algebras are finite dimensional. Notice that finite dimensional special biserial (resp., gentle) algebras are complete special biserial (resp., gentle) algebras. The following observation is basic.

Proposition 3.6.9. *Complete special biserial algebras are precisely factor algebras of complete gentle algebras.*

Proof. It is immediate from the definition. □

Therefore, to prove Theorem 0.0.6, it suffices to show the g-tameness of complete gentle algebras by Proposition 3.6.6(2).

3.6.3 Gentle algebras from dissections

It is known in [PPP19, Theorem 4.10] that complete gentle algebras are precisely obtained by the following construction.

Definition 3.6.10. For a $\bullet$-dissection D of (S, M), we define a quiver $Q(D)$ and an ideal $I(D)$ in $kQ(D)$ as follows:

- The set of vertices of $Q(D)$ corresponds bijectively with D;
- The set of arrows of $Q(D)$ is a disjoint union of sets of arrows in C_v for all $v \in M_\circ$ defined as follows (see Figure 3.9):
  - If v is a puncture and $d_1, \dots, d_m \in D$ are sides of $\triangle_v$ in counterclockwise order, then there is a cycle $C_v \colon d_1 \overset{a_1}{\to} d_2 \overset{a_2}{\to} \cdots \overset{a_{m-1}}{\to} d_m \overset{a_m}{\to} d_1$ in $Q(D)$, that is uniquely determined up to cyclic permutation.
  - If v lies on a boundary segment b, and $d_1, \dots, d_m \in D$ are sides of $\triangle_v$ in counterclockwise order, then there is a path $C_v \colon d_1 \overset{a_1}{\to} d_2 \cdots \overset{a_{m-1}}{\to} d_m$ in $Q(D)$.
- $I(D)$ is generated by all paths of length 2 which is not a sub-path of any C_v.

Let $A(D) := \widehat{kQ(D)}/\overline{I(D)}$.

Proposition 3.6.11. *[PPP19, Theorem 4.10] For a $\bullet$-dissection D of (S, M), the algebra $A(D)$ is a complete gentle algebra, and any complete gentle algebra arises in this way.*

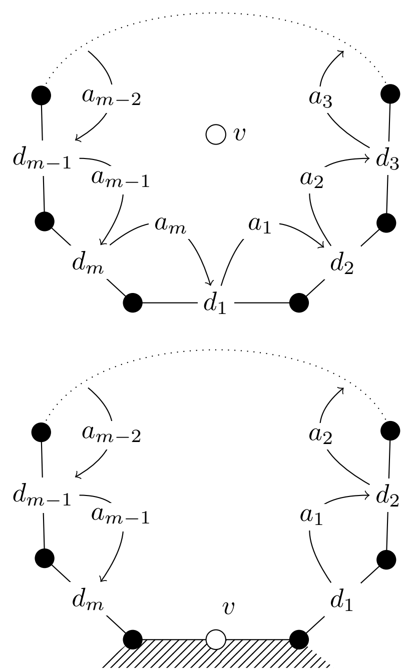


Figure 3.9: Sub-quiver of $Q(D)$ in $\triangle_v$

We prepare a few terminology.

Definition 3.6.12. Let $Q(D)$ be the quiver in Definition 3.6.10.

- For a puncture $v \in M_\circ$, a cycle C_v is called a *special cycle* at v. If it is a representative of its cyclic permutation class starting and ending at $d \in D$, then we call it a *special d-cycle* at v.
- For $v \in M_\circ$, every non-constant sub-path of C_v is called a *short path*.

3.6.4 Two-term silting complexes for $A(D)$ via D-laminates

Let D be a $\bullet$-dissection of (S, M) and $A(D) := \widehat{kQ(D)}/\overline{I(D)}$ the complete gentle algebra associated with D. In this subsection, we establish a geometric model of two-term silting theory for $A(D)$ and prove Theorem 0.0.6.

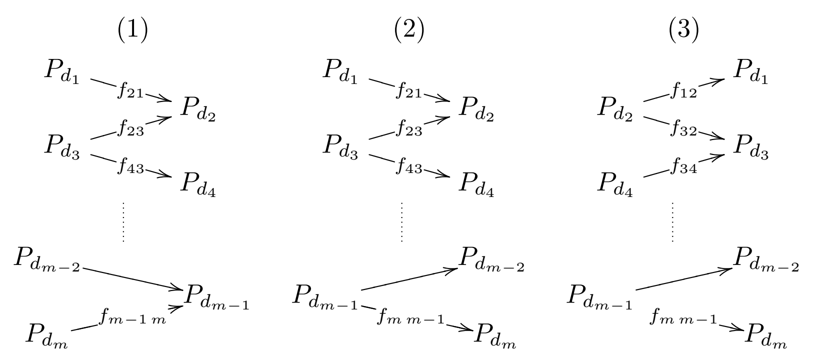
Let P_d be an indecomposable projective $A(D)$-module corresponding to $d \in D$. For $d, e \in D$, every short path in $Q(D)$ from d to e provides a non-vanishing homomorphism $P_e \to P_d$ in $\operatorname{proj} A(D)$, that we call *short map*. We observe that a sum t of all special cycles of $Q(D)$ is in the center of $A(D)$.

Proposition 3.6.13. *The complete gentle algebra $A(D)$ is a module-finite $k[[t]]$-algebra.*

Proof. It follows from the fact that $A(D)$ is generated by all short paths and constant paths as an $k[[t]]$-module. □

We would discuss a class of complexes in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A(D))$ obtained from D-laminates.

Definition 3.6.14. An indecomposable two-term complex in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A(D))$ is called a *two-term string complex* if it can be written as one of the following forms:



where each f_{ij} is of the form $f_{ij} = t^h f'$ for a short map $f' \colon P_j \to P_i$ and a non-negative integer h. It is called *two-term short string complex* if all f_{ij} are short maps.

We denote by 2-scx $A(D)$ the set of isomorphism classes of two-term short string complexes P such that $\operatorname{add} P^0 \cap \operatorname{add} P^{-1} = 0$. To prove the following proposition, we use τ-tilting theory.

Proposition 3.6.15. *Any indecomposable two-term presilting complex in* $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ *is a two-term short string complex, that is,* 2-ips $A(D) \subseteq$ 2-scx $A(D)$.

Proof. Since $B := A(D)/(t)$ is a finite dimensional special biserial algebra, every indecomposable non-projective Λ-module is either a string module or a band module (see [BR87, WW85]). A band module M satisfies $M = \tau M$, in particular, $\operatorname{Hom}_B(M, \tau M) \neq 0$, where τ is the Auslander-Reiten translation for B. By [AIR14, Lemma 3.4], if $P \in$ 2-ips B is a non-stalk complex, then it is a minimal projective presentation of a string module and hence a two-term string complex. In addition, P must be short since $t = 0$ on B. Therefore, Proposition 3.6.4(3) gives $P \in$ 2-scx B. By Proposition 3.6.7, so is any complex in 2-ips $A(D)$ because the functor $- \otimes_{A(D)} B$ preserves the property being short. □

First, we give a geometric model of short maps in $\operatorname{proj} A(D)$.

Definition 3.6.16. A *D-segment* is a non-self-intersecting curve, considered up to isotopy relative to M, in a polygon $\triangle_v$ of D for some $v \in M_\circ$ whose ends are unmarked points on sides of $\triangle_v$ or spirals around v.

Let η be a D-segment in $\triangle_v$ whose endpoints lie on $d, e \in \triangle_v \cap D$. We orient it to satisfy that v is to its right and its starting point lies on d. Then it corresponds to a short path $d_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{s-1}} d_s$ in $Q(D)$, where $d_1, \ldots, d_s \in D$ are sides of $\triangle_v$ in counterclockwise order with $d_1 = e$ and $d_s = d$. It induces a short map $\sigma(\eta)\colon P_d \to P_e$ in $\operatorname{proj} A(D)$.

Proposition 3.6.17. *The map σ induces a bijection*

$$\sigma : \{D\text{-}segments\ whose\ endpoints\ lie\ on\ D\} \to \{short\ maps\ in\ \operatorname{proj} A(D)\}.$$

Proof. The assertion immediately follows from the definition of σ. □

Next, we give a geometric model of two-term short string complexes in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A(D))$.

Definition 3.6.18. A *generalized D-laminate* is a $\circ$-laminate γ intersecting at least one $\bullet$-arc of D such that the condition $(*)$ in Definition 3.1.6(2) and the following conditions are satisfied:

- Each connected component of γ in $\triangle_v$ does not intersect itself for any $v \in M_\circ$;
- For any $d \in D$, all intersection points of γ and d are either positive or negative simultaneously.

Note that a D-laminate is precisely a non-self-intersecting generalized D-laminate.

A non-closed generalized (NCG, for short) D-laminate γ is decomposed into D-segments $\gamma_0, \dots, \gamma_m$ in polygons such that γ_{i-1} and γ_i have a common endpoint p_i on $d_i \in D$ for every $i \in \{1, \dots, m\}$. In particular, an end of γ_0 (resp., γ_m) does not lie on D and both endpoints of γ_i lie on D for $i \in \{1, \dots, m-1\}$. By Proposition 3.6.17, each γ_i provides a short map $\sigma(\gamma_i)$ between $T_\gamma^{(i)}$ and $T_\gamma^{(i+1)}$ for $i \in \{1, \dots, m-1\}$, where $T_\gamma^{(i)} := P_{d_i}$. It yields a complex T_γ in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A(D))$. More precisely, T_γ is a two-term complex $(T_\gamma^{-1} \xrightarrow{f} T_\gamma^0)$ given by

$$T_\gamma^{-1} := \bigoplus_{p_j:\text{negative}} T_\gamma^{(j)}, \quad T_\gamma^0 := \bigoplus_{p_i:\text{positive}} T_\gamma^{(i)},$$

$$f = (f_{ij})_{i,j \in \{1,\dots,m-1\}}, \quad \text{where} \quad f_{ij} := \begin{cases} \sigma(\gamma_{j-1}) & \text{if } i = j-1, \\ \sigma(\gamma_j) & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

From our construction, we have the equality $g(T_\gamma) = g(\gamma)$ under the identification of $\mathbb{Z}^{|D|}$ and $\mathbb{Z}^{|A|}$ via the map $d \mapsto P_d$, where the vector $g(\gamma) \in \mathbb{Z}^{|D|}$ is defined by the equality (3.1.1).

Lemma 3.6.19. *Suppose that two NCG D-laminates γ and γ' are decomposed into D-segments $\gamma_0, \dots, \gamma_m$ and $\gamma'_0, \dots, \gamma'_{m'}$ as above, respectively. If $m = m' > 1$ and $\gamma_i = \gamma'_i$ for $i \in \{1, \dots, m-1\}$, then $\gamma' = \gamma$.*

Proof. The D-segment γ_0 (resp., γ_m) only depends on the sign of p_1 (resp., p_m), that is uniquely determined by γ_1 (resp., γ_{m-1}). Thus we have $\gamma_0 = \gamma'_0$ and $\gamma_m = \gamma'_m$. □

Proposition 3.6.20. *The map $T_{(-)} \colon \gamma \mapsto T_\gamma$ induces a bijection*

$$T_{(-)} \colon \{\textit{NCG } D\textit{-laminates}\} \to \text{2-scx}\, A(D)$$

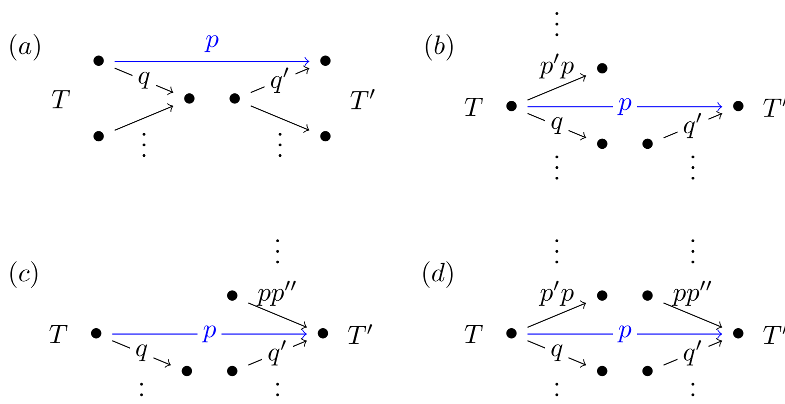
such that $g(\gamma) = g(T_\gamma)$ for any NCG D-laminate γ.

Proof. By our construction, T_γ is a stalk complex or a two-term short string complex such that $\operatorname{add} T_\gamma^0 \cap \operatorname{add} T_\gamma^{-1} = 0$, that is, $T_\gamma \in \text{2-scx}\, A(D)$. By Lemma 3.6.19, this map is injective.

In order to prove surjectivity of the map, we give the inverse map. If $P \in \text{2-scx}\, A(D)$ is a stalk complex P_d with $d \in D$ concentrated in degree 0 (resp., -1), then we just take $\gamma = d_+^*$ (resp., $\gamma = d_-^*$). Next, let $P \in \text{2-scx}\, A(D)$ be a short string complex which is one of (1)-(3) in Definition 3.6.14. We only consider the form (1) since the others can be proved similarly. By Proposition 3.6.17, $\gamma_1 := \sigma^{-1}(f_{21}), \gamma_2 := \sigma^{-1}(f_{23}), \dots, \gamma_{m-1} := \sigma^{-1}(f_{m-1\,m})$ are D-segments, and γ_{i-1} and γ_i have a common endpoint on d_i for $i \in \{2, \dots, m-1\}$. Then, by Lemma 3.6.19, there are two D-segments γ_0 and γ_m such that the curve γ obtained by gluing $\gamma_0, \dots, \gamma_m$ one by one is an NCG D-laminate. From our construction, we have $P = T_\gamma$. □

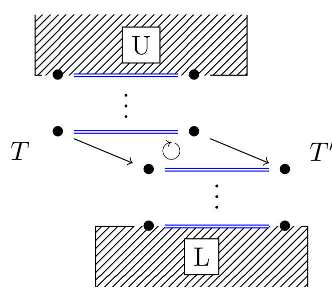
Finally, we give a geometric model of two-term presilting/silting complexes in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A(D))$. To do it, we need some preparations. First of all, for $T, T' \in$ 2-scx $A(D)$, we consider two kinds of morphisms from T to $T'[1]$ in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A(D))$.

Definition 3.6.21. For $T, T' \in$ 2-scx $A(D)$, a morphism $f \in \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A(D))}(T, T'[1])$ is called a *singleton single map* if it is induced by a short map p as one of the following forms:

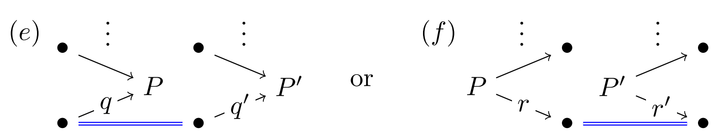


where p and q (resp., q') have no common arrows as paths, and p' and p'' are not constant.

Definition 3.6.22. For $T, T' \in$ 2-scx $A(D)$, a morphism $f \in \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A(D))}(T, T')$ is called a *quasi-graph map* if it is induced by the following form:



where $\boxed{\mathrm{U}}$ and $\boxed{\Gamma}$ are

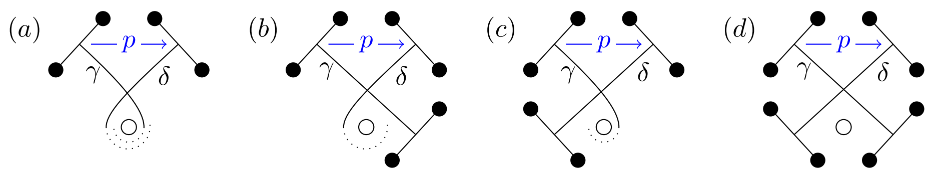


and there is no $p \in \operatorname{Hom}_{A(D)}(P, P')$ such that $pq = q'$ or $r = r'p$. Note that it implies $q' \neq 0$ and $r \neq 0$. A quasi-graph map in $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A(D))}(T, T')$ naturally induces a unique morphism in $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A(D))}(T, T'[1])$, called a *quasi-graph map representative*.

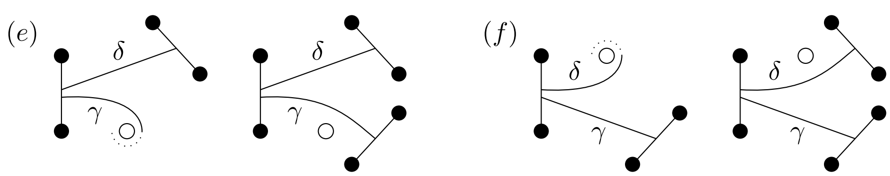
Regarding two-term short string complexes as homotopy strings defined as in [ALP16] (see also [BM03]), we can obtain the following result from [ALP16].

Proposition 3.6.23. *[ALP16, Propositions 4.1 and 4.8] For $T, T' \in$ 2-scx $A(D)$, singleton single maps and quasi-graph map representatives between T and T' give a basis of $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A(D))}(T, T'[1])$.*

Let γ and δ be NCG D-laminates. By Propositions 3.6.20 and 3.6.23, $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A(D))}(T_\gamma, T_\delta[1])$ has a basis consisting of singleton single maps and quasi-graph map representatives. It follows from the definition that a singleton single map in $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A(D))}(T_\gamma, T_\delta[1])$ is given by one of the following local figures:



where p is the associated short map. For a quasi-graph map representative in $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A(D))}(T_\gamma, T_\delta[1])$, each of $\boxed{\mathrm{U}}$ and $\boxed{\mathrm{L}}$ as in Definition 3.6.22 is given by one of the following local figures:



where the left figure of (e) (resp., (f)) is in the case of $q = 0$ (resp., $r' = 0$).

Proposition 3.6.24. *The following conditions are equivalent for two NCG D-laminates γ and δ:*

(1) $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A(D))}(T_\gamma, T_\delta[1]) = 0$*;*

(2) γ is in positive position for δ.

Proof. The assertion follows from Proposition 3.6.23 and the above observations. □

We are ready to state our results.

Proposition 3.6.25. *The following hold.*

(1) The map $T_{(-)}$ in Proposition 3.6.20 restricts to a bijection

$$\{\textit{non-closed } D\textit{-laminates}\} \to \text{2-ips}\, A(D).$$

(2) *Two non-closed D-laminates γ and η are compatible if and only if $T_\gamma \oplus T_\eta$ is presilting.*

Proof. In general, two NCG D-laminates γ and η are compatible if and only if they are in positive position each other. By Proposition 3.6.24, this is equivalent to the condition that $\mathrm{Hom}(T_\gamma, T_\delta[1]) = \mathrm{Hom}(T_\delta, T_\gamma[1]) = 0$. Since a non-closed D-laminate is precisely a non-self-intersecting NCG D-laminate, we get (1) and (2). □

Theorem 3.6.26. *The map $\mathcal{X} \mapsto T_\mathcal{X} := \bigoplus_{\gamma\in\mathcal{X}} T_\gamma$ gives bijections*

$$\{\textit{reduced } D\textit{-laminations}\} \to \text{2-presilt}\, A(D) \quad \textit{and} \quad \{\textit{complete } D\textit{-laminations}\} \to \text{2-silt}\, A(D)$$

such that $C(\mathcal{X}) = C_A(T_\mathcal{X})$ for all reduced D-laminations $\mathcal{X}$. In particular, we have

$$\mathcal{F}(D) = \mathcal{F}(A(D)) \quad \textit{and} \quad |\mathcal{F}(D)| = |\mathcal{F}(A(D))|.$$

Proof. It follows from Proposition 3.6.25. □

Now, we prove Theorem 0.0.6.

Proof of Theorem 0.0.6. By Proposition 3.6.11, any complete gentle algebra is given as $A(D)$ for some $\bullet$-dissection D of marked surfaces, and it follows from Theorems 0.0.7 and 3.6.26 that complete gentle algebras are g-tame. Thus the assertion follows from Proposition 3.6.6(2) since every complete special biserial algebras is a factor algebra of some complete gentle algebra by Proposition 3.6.9. □

3.6.5 Application to finite dimensional k-algebras

In this subsection, we introduce a class of special biserial algebras which is a common generalization of finite dimensional gentle algebras and Brauer graph algebras. It has the same geometric model of two-term silting theory as complete gentle algebras.

Let (S, M) be a marked surface and D a $\bullet$-dissection of (S, M). Remember that every $d \in D$ determines a special d-cycle for each endpoint v of $d^* \in D^*$ which is a puncture.

Definition 3.6.27. For a function $\mathfrak{m}\colon M_\circ \backslash \partial S \to \mathbb{Z}_{>0}$, we define a finite dimensional special biserial algebra $B(D) := A(D)/J$, where J is the closure of an ideal generated by

$$C_{u,d}^{\mathfrak{m}(u)} - C_{v,d}^{\mathfrak{m}(v)}$$

for all $d \in D$ and endpoints u, v of d^*. Here, $C_{v,d}^{\mathfrak{m}(v)}$ is an $\mathfrak{m}(v)$-th of $C_{v,d}$ if v is a puncture; otherwise, it is zero.

Example 3.6.28. Let D be a $\bullet$-dissection of a marked surface (S, M).

(a) If all $\bullet$-marked points lie on the boundary ∂S, then $A(D) = B(D)$ and this is precisely a finite dimensional gentle algebra.

(b) If all marked points are punctures (i.e., $\partial S = \emptyset$), then $B(D)$ is a Brauer graph algebra. In fact, the corresponding Brauer graph is given by the following data:

- The set of vertices corresponds bijectively with $M_\circ$;
- The set of edges corresponds bijectively with the dual dissection D^* of D;
- The cyclic ordering around each vertex is induced from the orientation of S;
- A multiplicity of a vertex v is $\mathfrak{m}(v)$

(see [Sch18] for the definition of Brauer graphs and Brauer graph algebras). Conversely, it is shown in [Lab13] that every Brauer graph algebra arises in this way. A Brauer graph with $\mathfrak{m}(v) = 1$ for all $v \in M_\circ$ is called a *ribbon graph*.

For these algebras, we have the following geometric model of two-term silting theory, which is compatible with that of complete gentle algebras. Remark that Adachi–Aihara–Chan [AAC18] studied two-term tilting theory over Brauer graph algebras by using the notion of signed walks on ribbon graphs, and our result can be understood as a generalization of theirs.

Proposition 3.6.29. *Let (S, M) be a marked surface and D a $\bullet$-dissection of (S, M). Let $B(D)$ be a special biserial algebra associated to D. Then there are bijections*

$$\{\textit{reduced } D\textit{-laminations}\} \to \text{2-presilt}\, B(D) \textit{ and } \{\textit{complete } D\textit{-laminations}\} \to \text{2-silt}\, B(D)$$

that preserve their g-vectors. In particular, we have $\mathcal{F}(B(D)) = \mathcal{F}(D)$.

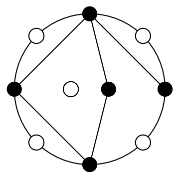
Proof. Let $A(D)$ be a complete gentle algebra associated to D. Let K be an ideal in $kQ(D)$ generated by all special cycles in $Q(D)$ and t a sum of all special cycles in $Q(D)$. We have maps

$$A(D) \to \frac{A(D)}{(t)} \to \frac{A(D)}{K} \cong \frac{B(D)}{K} \leftarrow \frac{B(D)}{(t)} \leftarrow B(D) \tag{3.6.1}$$

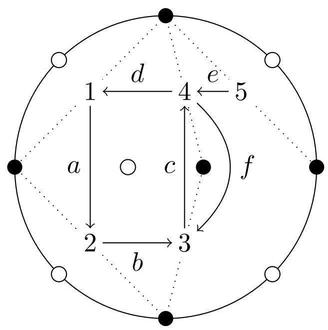
of algebras, where each map is a canonical surjection. It is easy to see that each factor algebra is given by an ideal satisfying the assumption of Proposition 3.6.7. In particular, we have a canonical bijection between 2-presilt $A(D)$ and 2-presilt $B(D)$ by Proposition 3.6.7. Finally, Theorem 3.6.26 yields the assertion. □

3.7. Examples for representation theory

(1) Let (S, M) be a disk with $|M| = 10$ such that one marked point in $M_\circ$ (resp., $M_\bullet$) is a puncture and the others lie on ∂S. For a $\bullet$-dissection of (S, M)

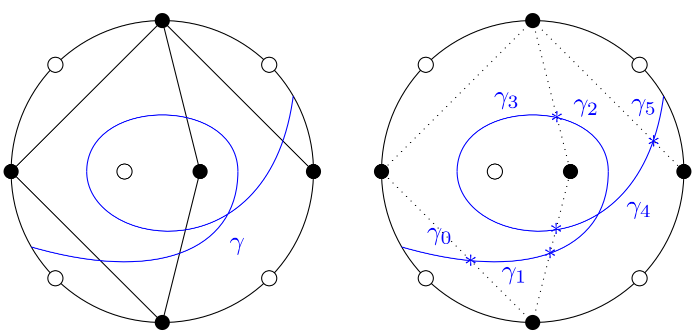
$D =$  ,

the quiver $Q(D)$ and the ideal $I(D)$ are given by

$Q(D) =$ 

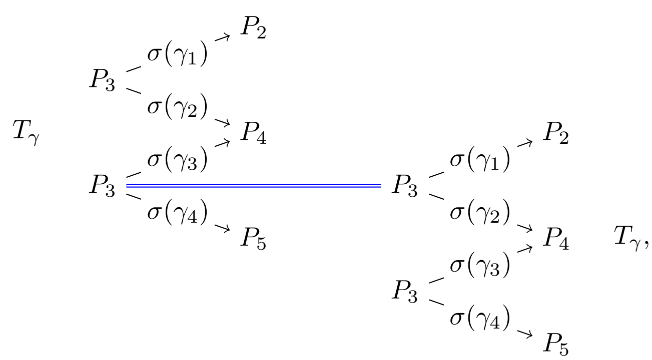
and $I(D) = \langle cf, fc, ed \rangle$.

We consider an NCG D-laminate γ, but not a D-laminate, that is decomposed into D-segments $\gamma_0, \ldots, \gamma_5$ as follows:



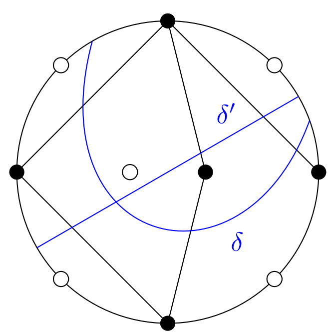
Then the corresponding two-term string complex T_γ is not presilting. In fact, there is a nonzero quasi-graph map representative in $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A(D))}(T_\gamma, T_\gamma[1])$

induced by the form



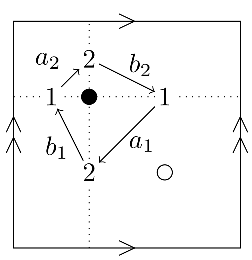
where $\sigma(\gamma_1)$ (resp., $\sigma(\gamma_2)$, $\sigma(\gamma_3)$, $\sigma(\gamma_4)$) is the short map in $\operatorname{proj} A(D)$ induced by a short path b (resp., f, dab, ef) in $Q(D)$. There is no short map p from P_4 to P_2 (resp., from P_5 to P_4) such that $\sigma(\gamma_1) = p\sigma(\gamma_3)$ (resp., $\sigma(\gamma_2) = p\sigma(\gamma_4)$). Therefore, T_γ is not presilting.

We consider two D-laminates δ and δ' as follows:



Then δ' is a positive position for δ, but δ is not a positive position for δ'. We observe whether $T_\delta \oplus T_{\delta'}$ is presilting. It is easy to see that T_δ and $T_{\delta'}$ are presilting respectively. In addition, we have $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(T_{\delta'}, T_\delta[1]) = 0$. However, there is a nonzero singleton single map from T_δ to $T_{\delta'}[1]$ induced by a short path b, as (d) in Definition 3.6.21. Thus $T_\delta \oplus T_{\delta'}$ is not presilting.

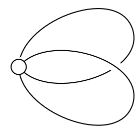
(2) We consider the marked surface (S, M) and the $\bullet$-dissection $D = \{1, 2\}$ in Section 3.2(4). Then the associated quiver $Q(D)$ and the ideal $I(D)$ are given by

$Q(D) =$  and $I(D) = \langle a_1b_1, b_1a_2, a_2b_2, b_2a_1 \rangle$.

Let $\mathfrak{m}(\circ) := 1$, then we have

$$J = \langle a_1b_1a_2b_2 - a_2b_2a_1b_1, b_1a_2b_2a_1 - b_2a_1b_1a_2 \rangle,$$

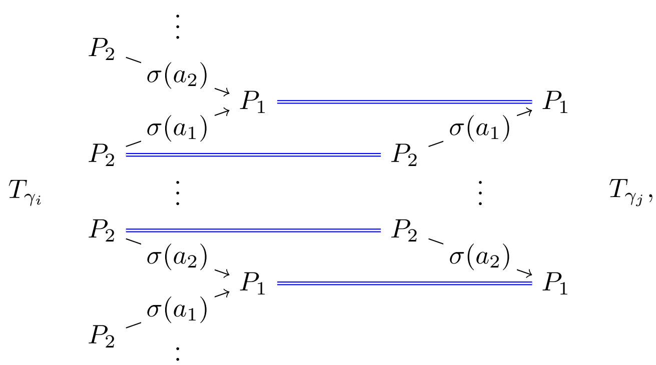
and the algebra $B(D)$ defined in Definition 3.6.27 is a Brauer graph algebra whose Brauer graph is



with multiplicity 1 on the vertex $\circ$. In Section 3.2(4), we gave the complete lists of D-laminates and complete D-laminations. For $i \in \mathbb{Z}_{>0}$ and the non-closed D-laminate γ_i, the corresponding two-term string complex T_{γ_i} is given by

$$\begin{array}{ccc} & & P_1 \\ & \nearrow^{\sigma(a_1)} & \\ P_2 & & \\ & \searrow_{\sigma(a_2)} & \\ & & P_1 \\ & \vdots & \\ P_2 & & \\ & \searrow_{\sigma(a_2)} & \\ & & P_1, \end{array}$$

where $\sigma(a_k)$ is a short map induced by a_k and P_1 only appears i times in degree 0 (resp., P_2 only appears $i-1$ times in degree -1). For $i, j \in \mathbb{Z}_{>0}$, it is easy to show that all nonzero maps between T_{γ_i} and $T_{\gamma_j}[1]$ are quasi-graph map representatives induced by the form



that is, $i > j + 1$. Therefore, $T_{\gamma_i} \oplus T_{\gamma_j}$ is two-term silting for $j = i, i \pm 1$; otherwise it's not.

Chapter 4

Brauer tree algebras have $\binom{2n}{n}$ two-term tilting complexes

In this chapter, we consider a marked surface (S, M) with $\partial S = \emptyset$, equivalently, all marked points are punctures. In this case, every dissection D of (S, M) can be regarded as a ribbon graph (Example 3.6.28). Moreover, the dual dissection D^* of D is precisely the dual graph each other. In addition, the cardinality $|D|$ of D is given by $|M| + 2g - 2$ by Remark 3.1.3, where g is the genus of S.

This chapter (Chapter 4) is based on the author's work [Aok]. The aim of this chapter is to prove the following results. Now, we say that a dissection of (S, M) is a *tree* if its underlying graph has no cycles.

Theorem 4.0.1. *Let D be a $\bullet$-dissection of (S, M) such that the dual dissection D^* is a tree with $n := |D| = |D^*|$. Then the number of complete D-lamination is $\binom{2n}{n}$. In particular, it is invariant on n.*

The Brauer graph algebra $B(D)$ associated with D is called *Brauer tree algebra* when D^* is a tree. The following result is a direct consequence of Theorem 4.0.1. Here, the latter equality of (4.0.1) was shown by [AMN20] recently in a completely different way.

Theorem 4.0.2. *Let D be a $\bullet$-dissection of (S, M) such that the dual dissection D^* is a tree with $n := |D| = |D^*|$. Let $A(D)$ be the complete gentle algebra associated with D and $B(D)$ the Brauer tree algebra associated with D. Then the following hold.*

$$\#\,2\text{-silt}\, A(D) = \#\,2\text{-tilt}\, B(D) = \binom{2n}{n}. \tag{4.0.1}$$

In particular, these numbers are invariant on n.

Proof. By Theorem 3.6.26 and Proposition 3.6.29,

$$\#\,2\text{-silt}\,A(D) = \#\,2\text{-tilt}\,B(D)$$

and it is equal to the number of complete D-laminations. By Theorem 4.0.1, this is equal to $\binom{2n}{n}$. $\square$

We prove Theorem 4.0.1 in Section 4.3.

4.1. Gluing laminations

Throughout this section, let (S, M) be a marked surface with $\partial S = \emptyset$. Remember that any dissection of (S, M) corresponds to a ribbon graph (Example 3.6.28).

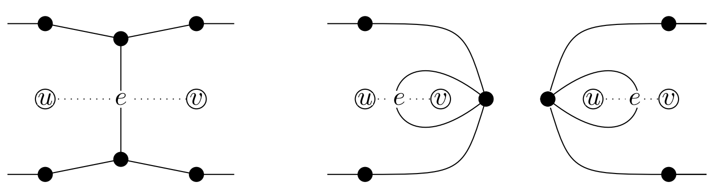
Let D be a $\bullet$-dissection of (S, M) and D^* the dual $\circ$-dissection of D. For $d \in D$, we write the corresponding arc by $e^* \in D^*$, vice versa. Assume that $n := |D| = |D^*| > 1$.

Definition 4.1.1. Let e^* be an arc of D^*.

(1) It is called *bridge* if $D^* \backslash \{e^*\}$ is a disjoint union of two connected components.

(2) It is called *external arc* if it has an endpoint $v \in M_\circ$ such that e^* is the unique arc incident to v. In this case, the polygon $\triangle_v$ is a monogon whose boundary segment is e.

Definition 4.1.2. Suppose that e^* is a bridge of D^*. Let D_1^*, D_2^* be subsets of D^* obtained from E, F by adding the arc e^* respectively, where E and F are connected components of $D^* \backslash \{e^*\}$. Namely, $D^* = D_1^* \cup D_2^*$ and $D_1^* \cap D_2^* = \{e^*\}$ hold. In this setting, we write $D^* = D_1^* \underset{e^*}{\times} D_2^*$ and call it a decomposition of D^* with respect to e^*.



(S, M, D) (S_1, M_1, D_1) (S_2, M_2, D_2)

Figure 4.1: Marked surfaces with dissections associated to a decomposition $D^* = D_1^* \underset{e^*}{\times} D_2^*$

Suppose that $D^* = D_1^* \underset{e^*}{\times} D_2^*$ is a decomposition of D^* with respect to a bridge $e^* \in D^*$. As D_1^* (resp., D_2^*) being a ribbon graph, it is regarded as a dissection of the marked surface (S_1, M_1) (resp., (S_2, M_2)), which is a subsurface of (S, M). In addition, let D_i be the dual dissection of D_i^*, then it is naturally regarded as a subset of D by the inclusion map $D_i^* \hookrightarrow D^*$ for $i \in \{1, 2\}$. See Figure 4.1. In this sense, it is easy to see that every D_i-lamination is a D-lamination for $i \in \{1, 2\}$. More generally, we have the following.

Proposition 4.1.3. *For $i \in \{1, 2\}$, a D_i-lamination is precisely a D-lamination which does not intersect with $\bullet$-arcs in $D \backslash D_i$.*

Proof. For $i \in \{1, 2\}$, it is easy to see that every D_i-lamination is a D-lamination as $D_i \subseteq D$. On the other hand, if γ is a D-laminate intersecting with no arcs in $D \backslash D_i$, then it has D-segments in the polygons of (S, M), which are naturally identified with D_i-segments in the polygons of (S_i, M_i). This means that γ provides a D_i-laminate. □

Since e^* is an external arc of D_1^* (resp., D_2^*) on the marked surface (S_1, M_1) (resp., (S_2, M_2)), we get the following. Now, we suppose that u, v are endpoints of e^* such that e^* is the unique arc of D_1^* (resp., D_2^*) incident to v (resp., u).

Lemma 4.1.4. *For $i \in \{1, 2\}$, if a D_i-laminate γ intersects with e, then it is non-closed and satisfies $|g_e(\gamma)| \leq 2$.*

Proof. Since e^* is an external arc of D_1^* (resp., D_2^*), it forms the monogon $\triangle_v$ (resp., $\triangle_u$) in (S_1, M_1) (resp., (S_2, M_2)). Then the assertion follows from the definition of D-laminates. □

In the following, we relate D-laminations and D_i-laminations. Firstly, we give a restriction of a D-lamination to a D_i-lamination.

Definition-Proposition 4.1.5. Let $\mathcal{X}$ be a D-lamination. If $g_e(\mathcal{X}) = 0$, then let $\mathcal{X}|_{D_1}$ be a subset of $\mathcal{X}$ consisting of all D_1-laminates in $\mathcal{X}$. Assume that $g_e(\mathcal{X}) > 0$. Let $\mathcal{X}|_{D_1}$ be a collection of $\circ$-laminates constructed from $\mathcal{X}$ in the following way: For each intersection point p of $\gamma \in \mathcal{X}$ and e, we replace a D-segment of γ in the polygon $\triangle_v$ starting at p with a D-segment in $\triangle_v$ starting at p and ending at v as a spiral in the counterclockwise direction. After that, we remove all D-segments of $\mathcal{X}$ which are not D_1-segments. Similarly, we define $\mathcal{X}|_{D_2}$ of $\mathcal{X}$.

In the above, for $i \in \{1, 2\}$, a collection $\mathcal{X}|_{D_i}$ is a D_i-lamination satisfying $g_d(\mathcal{X}|_{D_i}) = g_d(\mathcal{X})$ for all $d \in D_i$.

Proof. First, if $g(\mathcal{X}) = 0$, then it is clear. Second, we assume $g_e(\mathcal{X}) > 0$. Let $\mathcal{X}'$ be a collection of D-segments obtained from $\mathcal{X}$ by removing all D_2-segments. By Lemma 3.6.19, $\mathcal{X}'$ gives rise to a unique D-lamination $\mathcal{X}''$ satisfying $g_d(\mathcal{X}'') = g_d(\mathcal{X})$ for all $d \in D_1$ and 0 otherwise, which coincides with $\mathcal{X}|_{D_1}$ by our construction. By Proposition 4.1.3, it is the desired D_1-lamination. Similarly, we get the assertion for $\mathcal{X}|_{D_2}$. □

Secondly, we construct a D-lamination by gluing a certain pair of a D_1-lamination and a D_2-lamination.

Definition-Proposition 4.1.6. *Let $\mathcal{X}$ be a (not necessarily reduced) D_1-lamination and $\mathcal{Y}$ a (not necessarily reduced) D_2-lamination such that $m := g_e(\mathcal{X}) = g_e(\mathcal{Y})$. Assume that m is positive. Let $p_1, \ldots, p_m$ (resp., $q_1, \ldots, q_m$) be intersection points of $\mathcal{X}$ and e (resp., $\mathcal{Y}$ and e). Reordering the numbering if necessary, we may assume that $p_1, \ldots, p_m$ (resp., $q_1, \ldots, q_m$) lie on the side e of the polygon $\triangle_v$ in the counterclockwise direction. Let $\mathcal{X}'$ (resp., $\mathcal{Y}'$) be a collection of curves obtained from curves of $\mathcal{X}$ (resp., $\mathcal{Y}$) by removing all D_1-segments in the polygon $\triangle_v$ (reps., D_2-segments in $\triangle_u$). Then $\mathcal{X} \underset{e}{\times} \mathcal{Y}$ is defined to be a collection of $\circ$-laminates of (S, M) obtained by gluing curves of $\mathcal{X}'$ and $\mathcal{Y}'$ at their endpoints p_i and q_i on e for each $i \in \{1, \ldots, m\}$. Then it is a D-lamination satisfying*

$$(\mathcal{X} \underset{e}{\times} \mathcal{Y})|_{D_1} = \mathcal{X}, \quad (\mathcal{X} \underset{e}{\times} \mathcal{Y})|_{D_2} = \mathcal{Y}$$

and

$$g_d(\mathcal{X} \underset{e}{\times} \mathcal{Y}) = \begin{cases} g_d(\mathcal{X}) & \text{if } d \in D_1, \\ g_d(\mathcal{Y}) & \text{if } d \in D \backslash D_1. \end{cases}$$

Proof. From our construction, $\mathcal{X} \underset{e}{\times} \mathcal{Y}$ consists of pairwise compatible D-laminates. Moreover, the set of D-segments of $\mathcal{X} \underset{e}{\times} \mathcal{Y}$ is a union of D_1-segments of $\mathcal{X}'$ and D_2-segments of $\mathcal{Y}'$. This means that it satisfies the desired properties. □

Example 4.1.7. Let γ be a D_1-laminate and δ a D_2-laminate such that $g_e(\gamma) = g_e(\delta) = 1$. Then we get a non-closed D-laminate $\gamma \underset{e}{\times} \delta$ by Proposition 4.1.6.

Finally, we would like to construct pairwise distinct and pairwise compatible D-laminates as many as possible by gluing a given pair of reduced laminations. This is achieved by using a combinatorial object called lattice paths.

Definition 4.1.8. Let s, t be a positive integers. For two elements (j_1, k_1) and (j_2, k_2) in $\{1, \ldots, s\} \times \{1, \ldots, t\}$, we say that they are *compatible* if one of (i) $j_1 \geq j_2$ and $k_1 \geq k_2$ or (ii) $j_1 \leq j_2$ and $k_2 \leq k_1$ holds. A *lattice sub-path* of $\{1, \ldots, s\} \times \{1, \ldots, t\}$ is a set of pairwise compatible elements of $\{1, \ldots, s\} \times \{1, \ldots, t\}$. A *lattice path* is a lattice sub-path which is maximal as a set. We denote by $\mathrm{P}(s, t)$ the set of all lattice paths of $\{1, \ldots, s\} \times \{1, \ldots, t\}$. For a given lattice sub-path of $\{1, \ldots, s\} \times \{1, \ldots, t\}$, let $P(j, -)$ (resp, $P(-, k)$) be the number of elements of P of the form (j, b) (resp., (a, k)) for $b \in \{1, \ldots, t\}$ (resp., $a \in \{1, \ldots, s\}$).

It is well-known that every lattice path P of $\{1, \ldots, s\} \times \{1, \ldots, t\}$ consists of precisely $s + t - 1$ elements. In particular, we have $\sum_{j=1}^{s} P(j, -) = \sum_{k=1}^{t} P(-, k) = s + t - 1$. Remember that P must contain $(1, 1)$ and (s, t) from its maximality. On the other hand, the cardinality of $\mathrm{P}(s, t)$ is $\binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}$.

Definition-Proposition 4.1.9. *Let $\mathcal{X}$ (resp., $\mathcal{Y}$) be a reduced D_1-lamination (resp., reduced D_2-lamination) consisting of non-closed laminates intersecting with e at most once positively. Let $s := g_e(\mathcal{X})$ and $t := g_e(\mathcal{Y})$. Assume that s, t are positive. Let $\gamma_1, \dots, \gamma_s$ (resp., $\delta_1, \dots, \delta_t$) be the set of arcs of $\mathcal{X}$ (resp., $\mathcal{Y}$) intersecting with e, and let p_j (resp., q_k) be a positive intersection point of γ_j and e (resp., δ_k and e) for $j \in \{1, \dots, s\}$ (resp., $k \in \{1, \dots, t\}$). Reordering the numbering if necessary, we assume that $p_1, \dots, p_s$ (resp., $q_1, \dots, q_t$) lie on the side e of the polygon $\triangle_v$ in the counterclockwise direction. For each lattice sub-path P of $\{1, \dots, s\} \times \{1, \dots, t\}$, define*

$$\mathcal{X} \underset{e_P}{\times} \mathcal{Y} := \{\gamma_j \underset{e}{\times} \delta_k \mid (j,k) \in P\} \cup \mathcal{X} \backslash \{\gamma_j\}_{j=1}^{s} \cup \mathcal{Y} \backslash \{\delta_k\}_{k=1}^{t}.$$

Then it is a reduced D-lamination satisfying

$$(\mathcal{X} \underset{e_P}{\times} \mathcal{Y})|_{D_1} = \mathcal{X}, \quad (\mathcal{X} \underset{e_P}{\times} \mathcal{Y})|_{D_2} = \mathcal{Y} \tag{4.1.1}$$

and

$$g_d(\mathcal{X} \underset{e_P}{\times} \mathcal{Y}) = \begin{cases} g_d(\mathcal{X}) + \displaystyle\sum_{j=1}^{s} g_d(\gamma_j)(P(j,-) - 1) & \text{if } d \in D_1, \\ g_d(\mathcal{Y}) + \displaystyle\sum_{k=1}^{t} g_d(\delta_k)(P(-,k) - 1) & \text{if } d \in D \backslash D_1. \end{cases} \tag{4.1.2}$$

Moreover, it is complete if all the following conditions are satisfied:

- *$\mathcal{X}$ is a complete D_1-lamination;*
- *$\mathcal{Y}$ is a complete D_2-lamination;*
- *P is a lattice path.*

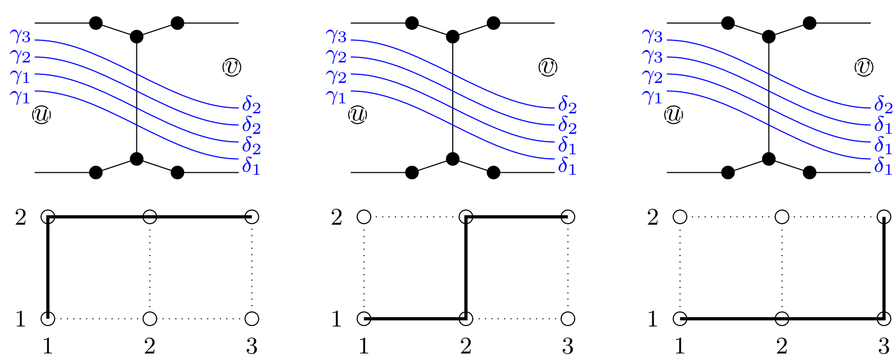


Figure 4.2: Gluing laminations via lattice paths of $\{1, 2, 3\} \times \{1, 2\}$

Proof. Consider collections

$$\begin{aligned}\mathcal{X}^P &:= \{\gamma_1^1,\dots,\gamma_1^{P(1,-)},\dots,\gamma_s^1,\dots,\gamma_s^{P(s,-)}\}\cup\mathcal{X}\backslash\{\gamma_1,\dots,\gamma_s\},\\ \mathcal{Y}^P &:= \{\delta_1^1,\dots,\delta_1^{P(-,1)},\dots,\delta_t^1,\dots,\delta_t^{P(-,t)}\}\cup\mathcal{Y}\backslash\{\delta_1,\dots,\delta_t\},\end{aligned}$$

where $\gamma_j^1,\dots,\gamma_j^{P(j,-)}$ (resp., $\delta_k^1,\dots,\delta_k^{P(-,k)}$) are copies of γ_j for $j\in\{1,\dots,s\}$ (resp., δ_k for $k\in\{1,\dots,t\}$). Since they are D_1-laminations and D_2-laminations respectively satisfying $g_e(\mathcal{X}^P)=g_e(\mathcal{Y}^P)=|P|$, we get a D-lamination $\mathcal{X}^P\underset{e}{\times}\mathcal{Y}^P$ as in Definition 4.1.6. By the ordering of intersection points on the side e, the non-closed laminate $\gamma_j\underset{e}{\times}\delta_k$ lies in $\mathcal{X}^P\underset{e}{\times}\mathcal{Y}^P$ if and only if $(j,k)\in P$. That is, $\mathcal{X}\underset{e_P}{\times}\mathcal{Y}=\mathcal{X}^P\underset{e}{\times}\mathcal{Y}^P$. By our construction, it is a reduced D-lamination whose g-vector is given by (4.1.2). In addition, if $\mathcal{X}$ is a complete D_1-lamination, $\mathcal{Y}$ is a complete D_2-lamination, and P is a lattice path, then $\mathcal{X}\underset{e_P}{\times}\mathcal{Y}$ is complete since

$$|\mathcal{X}\underset{e_P}{\times}\mathcal{Y}|=(|\mathcal{X}|-s)+(|\mathcal{Y}|-t)+(s+t-1)=|D_1|+|D_2|-1=|D|. \qquad \square$$

4.2. Laminations for trees

In this section, we mainly study a $\bullet$-dissection D of (S,M) with $\partial S=\emptyset$ such that its dual dissection D^* is a tree. In this case, S must be a sphere with $|M_\bullet|=1$ and $|M_\circ|=|D|+1=|D^*|+1$.

For a $\bullet$-dissection D of (S,M), we denote by $\mathcal{L}(D)$ the set of complete D-laminations. For a $\bullet$-arc $d\in D$ and an integer $j\in\mathbb{Z}$, let

$$\mathcal{L}(D)_d^j := \{\text{complete } D\text{-laminations } \mathcal{X} \text{ such that } g_d(\mathcal{X})=j\}.$$

Notice that $\mathcal{L}(D)_d^0=\emptyset$ since the set $\{g(\gamma)\mid\gamma\in\mathcal{X}\}$ for a complete D-lamination $\mathcal{X}$ forms a basis of $\mathbb{Z}^{|D|}$ by Proposition 3.1.8. In particular, the set $\mathcal{L}(D)_d^j$ is a disjoint union of $\mathcal{L}(D)_d^{>0}$ and $\mathcal{L}(D)_d^{<0}$, where

$$\mathcal{L}(D)_d^{>j}:=\bigcup_{k>j}\mathcal{L}(D)_d^k \quad\text{and}\quad \mathcal{L}(D)_d^{<j}:=\bigcup_{k<j}\mathcal{L}(D)_d^k.$$

4.2.1 Reversing the orientation of marked surfaces

We consider a marked surface (S',M') obtained from (S,M) by reversing the orientation of the surface S. For a given $\bullet$-dissection D of (S,M), we denote by D' the $\bullet$-dissection of (S',M') corresponding to D.

By symmetry, we get the following.

Proposition 4.2.1. *We have a canonical bijection between $\mathcal{L}(D)$ and $\mathcal{L}(D')$. Under this map, if $\mathcal{X}'$ is a D'-lamination corresponding to a D-lamination $\mathcal{X}$, then we have $g(\mathcal{X}')=-g(\mathcal{X})$. In particular, it gives bijections*

$$\mathcal{L}(D)_d^{>0}\to\mathcal{L}(D')_d^{<0} \quad\textit{and}\quad \mathcal{L}(D)_d^{<0}\to\mathcal{L}(D)_d^{>0}$$

for any $d \in D$.

Proof. By symmetry, all D-laminates (resp., D-laminations and complete D-laminations) are naturally identified with D'-laminations (resp., D'-laminations and complete D'-laminations). However, a role of positive intersection points and negative intersection points is swapped, that is, if p is a positive (resp., negative) intersection point of a D-laminate γ and $d \in D$, then p is negative (resp., positive) in the other side. □

4.2.2 Basic properties of laminates for trees

Let D be a $\bullet$-dissection of (S, M) and D^* its dual dissection. Assume that D^* is a tree with $n := |D| = |D^*|$.

Proposition 4.2.2. *(1) There is a two-to-one correspondence*

$$\{D\text{-laminates}\} \to \{\{u, v\} \mid u, v \in M_\circ, u \neq v\}$$

mapping a D-laminate γ to its ends. In particular, the number of D-laminates is $\binom{n+1}{2}$.

(2) If γ and δ are two D-laminates having common ends, then we have $g(\gamma) = -g(\delta)$.

(3) Every D-laminate γ is non-closed and intersects with d at most once for each $d \in D$, that is, $|g_d(\gamma)| \leq 1$.

Proof. Let u, v be distinct marked points in $M_\circ$. Since D^* is a tree, there exists a unique walk on D^* starting at u and ending at v. More precisely, we have a sequence $d_1^*, \ldots, d_m^*$ of pairwise distinct arcs in D^* for which there is a sequence $v_1, \ldots, v_{k+1}$ of punctures in $M_\circ$ such that v_i, v_{i+1} are endpoints of d_i^* for $i \in \{1, \ldots, k\}$. Every consecutive pair d_i^*, d_{i+1}^* provides D-segments $\gamma_i^\pm$ such that v_i is left to γ_i^+ and is right to γ_i^-. By Lemma 3.6.19, a collection $\gamma_1^+, \gamma_2^-, \ldots, \gamma_m^{(-1)^{m-1}}$ (resp., $\gamma_1^-, \gamma_2^+, \ldots, \gamma_m^{(-1)^m}$) gives rise to a D-laminate γ (resp., δ) satisfying $g_{d_i}(\gamma) = -g_{d_i}(\delta) = (-1)^{i-1}$ for $i \in \{1, \ldots, m\}$ and 0 otherwise. On the other hand, it is easy to check that all D-laminates can be obtained in this way. This implies (1) and (2). The assertion (3) is clear because D^* has no cycles. □

Fix an arc $e \in D$. Let $e^* \in D^*$ be the corresponding element of e. We consider a decomposition $D^* = D_1^* \underset{e^*}{\times} D_2^*$ with respect to e^*. Here, we regard D_1^*, D_2^* as $\circ$-dissections of (S_1, M_1) and (S_2, M_2) with dual dissections D_1, D_2 respectively. Clearly, D_1^* and D_2^* are trees and $n = |D_1| + |D_2| - 1$.

The following is immediate from Proposition 4.1.9.

Proposition 4.2.3. *For any $s \in \{1, \ldots, |D_1|\}$ and $t \in \{1, \ldots, |D_2|\}$, we have injective maps*

$$\rho_e^{s,t} \quad : \quad \mathcal{L}(D_1)_e^s \times \mathcal{L}(D_2)_e^t \times \mathrm{P}(s, t) \to \mathcal{L}(D)_e^{s+t-1} \tag{4.2.1}$$
$$\rho_e^{-s,-t} \quad : \quad \mathcal{L}(D_1)_e^{-s} \times \mathcal{L}(D_2)_e^{-t} \times \mathrm{P}(s, t) \to \mathcal{L}(D)_e^{-s-t+1} \tag{4.2.2}$$

mapping $(\mathcal{X}, \mathcal{Y}, P) \mapsto \mathcal{X} \underset{e_P}{\times} \mathcal{Y}$. *Moreover, they provide decompositions*

$$\mathcal{L}(D)_e^{>0} = \bigsqcup_{\substack{1 \leq s \leq |D_1| \\ 1 \leq t \leq |D_2|}} \operatorname{Im} \rho_e^{s,t} \quad \textit{and} \quad \mathcal{L}(D)_e^{<0} = \bigsqcup_{\substack{1 \leq s \leq |D_1| \\ 1 \leq t \leq |D_2|}} \operatorname{Im} \rho_e^{-s,-t}. \tag{4.2.3}$$

Proof. By Proposition 4.2.2(3), every D-laminate γ is non-closed and intersecting with e at most once. Thus, we have the map (4.2.1), which is injective by (4.1.1). We also get the injective map (4.2.2) by Proposition 4.2.1. The equalities (4.2.3) follow from the fact that every $\mathcal{X} \in \mathcal{L}(D)$ can be written as $(\mathcal{X}|_{D_1}) \underset{e_P}{\times} (\mathcal{X}|_{D_2})$ for some lattice path P. □

4.2.3 Flipping a tree

We study a combinatorial operation called flip on dissections of a given marked surface (S, M). A flip of ribbon graphs is introduced by Kauer [Kau98] and Aihara [Aih14, Aih15]. We slightly modify their definitions.

Definition 4.2.4. Let D be a $\bullet$-dissection of (S, M) and D^* the dual dissection of D. Assume that D^* is a tree. For $d^* \in D$, let $u, v \in M_\circ$ be endpoints of d^*. If d^* is an external arc and it is the unique arc incident to u, then we define $\vec{d_u^*} := \emptyset$. If d^* is followed by e^* around u in the counterclockwise direction, then we define $\vec{d_u^*} := e^*$.

A *flip* $\mu_{d^*}(D^*)$ of D^* with respect to d^* is a $\circ$-dissection obtained from $D^* \backslash \{d^*\}$ by adding the $\circ$-arc $\mu(d^*)$ constructed in the following way (see Figure 4.3):

- (i) If d^* is an external arc and $\vec{d_u^*} = \emptyset$, then $\mu(d^*)$ is a $\circ$-arc obtained by gluing two curves d^* and $\vec{d_v^*}$ at v so that $\mu(d^*)$ starts at u, and v is to its left.
- (ii) If d^* is not external, then let p be a middle point of d^*. The arc $\mu(d^*)$ is constructed by gluing the following two curves at a common starting point p:
  - a curve obtained by gluing a line segment between p, v and the arc $\vec{d_v^*}$ so that v is to its left,
  - a curve obtained by gluing a line segment between p, u and the arc $\vec{d_u^*}$ so that u is to its left.

In the above, let E be the dual $\bullet$-dissection of a flip $\mu_{d^*}(D^*)$ of D^* with respect to d^*. Fix the bijection from D to E given by a composition of natural bijections

$$D \to D^* \to \mu_{e^*}(D^*) \to E.$$

Under this map, the arc $d \in D$ clearly corresponds to $d \in E$. Let u, v be endpoints of d^*. If $\vec{d_u^*} = \emptyset$, then let $f_u = \emptyset$ and $f'_u = \emptyset$. If $\vec{d_u^*} = f^*$ for $f^* \in D^*$, then let $f_u \in D$ (resp., $f'_u \in E$) be its corresponding element in D^* (resp., in $\mu_{e^*}(D^*)$). Namely, $f_u \mapsto f'_u$ under the above bijection $D \to E$.

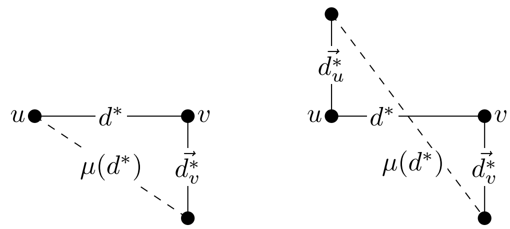


Figure 4.3: A local figure a flip with respect to d^*

Proposition 4.2.5. *In the above, there is a bijection between*

(1) the set of D-laminates intersecting with e non-negatively, and

(2) the set of E-laminates intersecting with e non-positively.

Moreover, it induces a bijection between $\mathcal{L}(D)_e^{\geq 0}$ and $\mathcal{L}(E)_e^{\leq 0}$. Under this map, if $\mathcal{X}'$ is a complete E-lamination corresponding to a complete D-lamination $\mathcal{X}$, then we have

$$g_d(\mathcal{X}') = \begin{cases} -g_e(\mathcal{X}) & \text{if } d = e, \\ g_{f_w}(\mathcal{X}) + g_e(\mathcal{X}) & \text{if } d = f'_w \text{ for } w \in \{u, v\}, \\ g_d(\mathcal{X}) & \text{otherwise.} \end{cases} \tag{4.2.4}$$

Proof. We only show the case when e^* is not an external arc of D^* because another case can be proved similarly. We show that any D-laminate γ intersecting with e non-negatively provides an E-laminate. In fact, if γ and e have no intersection points (i.e., $g_e(\gamma) = 0$), then it is clear by definition (Definition 3.1.6). Otherwise, we find that whenever γ intersects with $e \in D$ positively, it intersects with both $f'_u, f'_v \in E$ positively and with e negatively on E, see Figure 4.4. Moreover, other intersection points are not changed. By the ordering of arcs of E, this means that γ is an E-laminate whose g-vector on E is given by the formula (4.2.4). By symmetry (Proposition 4.2.1), the correspondence $\gamma \mapsto \gamma$ is a bijection between the sets (1) and (2). In addition, it preserves the property of being compatible.

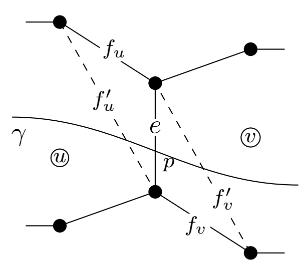


Figure 4.4: Intersection points of γ and $\bullet$-arcs by a flip

□

4.3. Proof of main theorems

Theorem 4.0.1 is deduced from the following claim.

Theorem 4.3.1. *Let (S, M) be a marked surface with $\partial S = \emptyset$. Let D be a $\bullet$-dissection of (S, M) such that its dual dissection D^* is a tree and $n := |D| = |D^*|$. For any external arc $d^* \in D^*$ and any integer $j \in \mathbb{Z}$, we have*

$$\#\mathcal{L}(D)_d^j = \begin{cases} \binom{2n-|j|-1}{n-1} & \text{if } j \in \{\pm 1, \dots \pm n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\#\mathcal{L}(D) = 2\sum_{j=1}^{n} \binom{2n-j-1}{n-1} = \binom{2n}{n}.$$

4.3.1 Proof of Theorem 4.3.1

Let (S, M) be a marked surface with $\partial S = \emptyset$. Let D be a $\bullet$-dissection of (S, M) and D^* its dual dissection. Assume that D^* is a tree with $n := |D| = |D^*|$. In the following, we show Theorem 4.3.1 by induction on n. Namely, we show

$$\#\mathcal{L}(D)_d^j = \#\mathcal{L}(D)_d^{-j} = \binom{2n-j-1}{n-1}$$

for any arc $d \in D$ such that $d^* \in D^*$ is external and any positive integer j.

If $n = 1$, then the assertion is clear. So, we assume that $n > 1$. We prepare the following induction hypothesis:

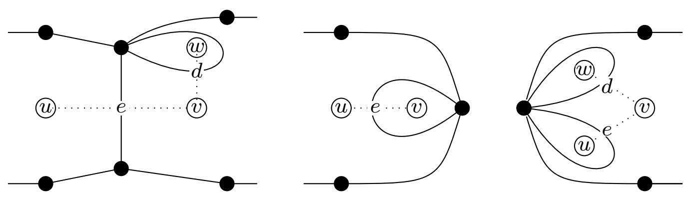
($*$) Theorem 4.3.1 holds for any $\bullet$-dissection whose dual dissection is a tree with less than n arcs.

Fix a $\bullet$-arc $d \in D$ such that d^* is an external arc of D^*. Suppose that v, w are endpoints of d^* such that d^* is the unique $\circ$-arc of D^* incident to w. Let $e^* := \vec{d_v^*}$ be the $\circ$-arc of D^* following d^* around v in the counterclockwise direction (see Definition 4.2.4). We have a decomposition $D^* = D_1^* \underset{e^*}{\times} D_2^*$ with respect to e^*. Here, we regard D_1^* (resp., D_2^*) as a $\circ$-dissection of (S_1, M_1) (resp., (S_2, M_2)) with the dual dissection D_1 (resp., D_2) respectively. See Figure 4.5. Without loss of generality, we may assume that $d^* \in D_2^*$, equivalently, $d \in D_2$. Let $n_1 := |D_1|$ and $n_2 := |D_2|$.

We observe the set $\mathcal{L}(D_2)$. Now, for any integers $s, t \in \mathbb{Z}$, let $\mathcal{L}(D_2)_{d,e}^{s,t} := \mathcal{L}(D_2)_d^s \cap \mathcal{L}(D_2)_e^t$. In addition, let $\mathcal{L}(D_2)_{d,e}^{>0,>0} := \mathcal{L}(D_2)_d^{>0} \cap \mathcal{L}(D_2)_e^{>0}$, and so on.

Lemma 4.3.2. *In the above, the set $\mathcal{L}(D_2)_d^{>0}$ is a disjoint union of the following subsets:*

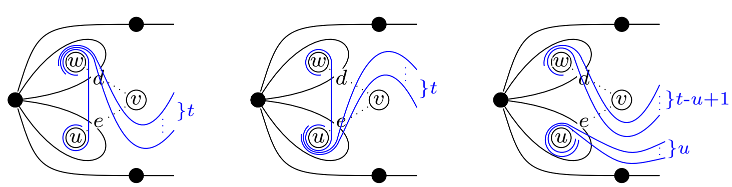
$$\bigsqcup_{1 \le t \le n_2-1} \mathcal{L}(D_2)_{d,e}^{t+1,-1}, \quad \bigsqcup_{1 \le t \le n_2-1} \mathcal{L}(D_2)_{d,e}^{1,-t-1}, \quad \bigsqcup_{1 \le u \le t \le n_2-1} \mathcal{L}(D_2)_{d,e}^{t-u+1,u}. \tag{4.3.1}$$



(S, M, D) (S_1, M_1, D_1) (S_2, M_2, D_2)

Figure 4.5: A local figure for $D^* = D_1^* \underset{e^*}{\times} D_2^*$ with respect to $e^* := \vec{d}_v^*$

Proof. By definition, the set $\mathcal{L}(D_2)_d^{>0}$ is a disjoint union of $\mathcal{L}(D_2)_{d,e}^{>0,<0}$ and $\mathcal{L}(D_2)_{d,e}^{>0,>0}$. First, we observe the former subset. There is the unique D_2-laminate which intersects with d positively and e negatively. We denote this laminate by γ_0, namely, $g_d(\gamma_0) = -g_e(\gamma_0) = 1$ and 0 otherwise. Since d^* is followed by e^* in counterclockwise direction, we find that γ_0 is compatible with any D_2-laminate γ intersecting with d non-negatively and e non-positively. From the maximality, this means that any complete D_2-lamination $\mathcal{B} \in \mathcal{L}(D_2)_{d,e}^{>0,<0}$ contains γ_0. Furthermore, $\mathcal{B}$ never contain D_2-laminates $\gamma \neq \gamma_0$ intersecting with d positively and $\delta \neq \gamma_0$ intersecting with e negatively simultaneously. In fact, such laminates γ, δ must intersect each other since d^* is followed by e^*. Thus, $\mathcal{B}$ is contained in subsets of the form $\mathcal{L}(D_2)_{d,e}^{t+1,-1}$ or $\mathcal{L}(D_2)_{d,e}^{1,-t-1}$ for some $t \in \{1, \ldots, n_2 - 1\}$. See the left and center figures in Figure 4.6 respectively. Second, it is clear that any $\mathcal{B} \in \mathcal{L}(D_2)_{d,e}^{>0,<0}$ is contained in the subset of the form $\mathcal{L}(D_2)_{d,e}^{t-u+1,u}$ for some $1 \leq u \leq t \leq n_2 - 1$. See the right figure in Figure 4.6. We finish a proof. $\square$



$\mathcal{L}(D_2)_{d,e}^{t+1,-1}$ $\mathcal{L}(D_2)_{d,e}^{1,-t-1}$ $\mathcal{L}(D_2)_{d,e}^{t-u+1,u}$

Figure 4.6: Three types of laminations in $\mathcal{L}(D_2)_d^{>0}$

A proof of Theorem 4.3.1 is achieved by the following lemmas. The next result follows from Lemma 4.3.2.

Lemma 4.3.3. *(1) For any $j \in \{1, \ldots, n\}$, the set $\mathcal{L}(D)_d^j$ is a disjoint union of the following subsets:*

$$\bigsqcup_{\substack{1 \le s \le n_1 \\ 1 \le t \le n_2 - 1 \\ s+t=j}} \rho_e^{-s,-1}(\mathcal{L}(D_1)_e^{-s}, \mathcal{L}(D_2)_{d,e}^{t+1,-1}, \mathrm{P}(s,1)), \tag{4.3.2}$$

$$\bigsqcup_{\substack{1 \le s \le n_1 \\ 1 \le t \le n_2 - 1}} \rho_e^{-s,-t-1}(\mathcal{L}(D_1)_e^{-s}, \mathcal{L}(D_2)_{d,e}^{1,-t-1}, \mathrm{P}^j(s,t+1)), \tag{4.3.3}$$

$$\bigsqcup_{\substack{1 \le s \le n_1 \\ 1 \le u \le t \le n_2 - 1 \\ t-u+1=j}} \rho_e^{s,u}(\mathcal{L}(D_1)_e^{s}, \mathcal{L}(D_2)_{d,e}^{t-u+1,u}, \mathrm{P}(s,u)), \tag{4.3.4}$$

where $\mathrm{P}^j(s,t+1)$ is a subset of $P(s,t+1)$ consisting of all lattice path P satisfying $P(1,-) = j$.

(2) Under the induction hypothesis $(*)$, the cardinality of (4.3.2), (4.3.3) and (4.3.4) are

$$A_{n_1,n_2-1}(j) := \sum_{\substack{1 \le s \le n_1 \\ 1 \le t \le n_2 - 1 \\ s+t=j}} \binom{2n_1 - s - 1}{n_1 - 1}\binom{2n_2 - t - 3}{n_2 - 2}, \tag{4.3.5}$$

$$B_{n_1,n_2-1}(j) := \sum_{\substack{1 \le s \le n_1 \\ 1 \le t \le n_2 - 1}} \binom{2n_1 - s - 1}{n_1 - 1}\binom{2n_2 - t - 3}{n_2 - 2}\binom{s + t - j - 1}{s - j}, \tag{4.3.6}$$

$$C_{n_1,n_2-1}(j) := \sum_{\substack{1 \le s \le n_1 \\ 1 \le u \le t \le n_2 - 1 \\ t-u+1=j}} \binom{2n_1 - s - 1}{n_1 - 1}\binom{2n_2 - t - 3}{n_2 - 2}\binom{s + t - j - 1}{t - j} \tag{4.3.7}$$

respectively.

Thus, it remains to show the next claim.

Lemma 4.3.4. *Let $m = p + q - 1$, where p, q are integers with $p > 0$ and $q > 1$. For any $j \in \{1, \ldots, m\}$, we have*

$$\binom{2m - j - 1}{m - 1} = A_{p,q-1}(j) + B_{p,q-1}(j) + C_{p,q-1}(j). \tag{4.3.8}$$

Before proving these lemmas, we prove Theorem 4.3.1. Proofs of Lemma 4.3.3 and Lemma 4.3.4 will be appeared in Section 4.3.2 and 4.3.3 respectively.

Proof of Theorem 4.3.1. Let (S', M') be a marked surface obtained from (S, M) by reversing the orientation and D' the •-dissection of (S', M') corresponding to D. By Proposition 4.2.1, we have

$$\#\mathcal{L}(D)_d^{-j} = \#\mathcal{L}(D')_d^j.$$

for $j \in \{1, \ldots, n\}$. Applying Lemma 4.3.3 and 4.3.4 to D and to D', we have

$$\#\mathcal{L}(D)_d^j = \binom{2n-j-1}{n-1} \quad \text{and} \quad \#\mathcal{L}(D)_d^{-j} = \#\mathcal{L}(D')_d^j = \binom{2n-j-1}{n-1} \tag{4.3.9}$$

as desired. It finishes a proof. □

4.3.2 Proof of Lemma 4.3.3

In this subsection, we show Lemma 4.3.3.

Proof of Lemma 4.3.3. (1) Remember that D_1^* and D_2^* are trees having e^* as their external arc on the respective marked surface. In addition, they satisfy $D^* = D_1^* \cup D_2^*$ and $D_1^* \cap D_2^* = \{e^*\}$. By Proposition 4.2.3 and 4.3.2, the set $\mathcal{L}(D)_d^{>0}$ can be written as a disjoint union of the following subsets:

$$\bigsqcup_{\substack{1 \le s \le n_1 \\ 1 \le t \le n_2 - 1}} \rho_e^{-s,-1}(\mathcal{L}(D_1)_e^{-s}, \mathcal{L}(D_2)_{d,e}^{t+1,-1}, \mathrm{P}(s,1)), \tag{4.3.10}$$

$$\bigsqcup_{\substack{1 \le s \le n_1 \\ 1 \le t \le n_2 - 1}} \rho_e^{-s,-t-1}(\mathcal{L}(D_1)_e^{-s}, \mathcal{L}(D_2)_{d,e}^{1,-t-1}, \mathrm{P}(s,t+1)), \tag{4.3.11}$$

$$\bigsqcup_{\substack{1 \le s \le n_1 \\ 1 \le u \le t \le n_2 - 1}} \rho_e^{s,u}(\mathcal{L}(D_1)_e^{s}, \mathcal{L}(D_2)_{d,e}^{t-u+1,u}, \mathrm{P}(s,u)). \tag{4.3.12}$$

In the following, we determine the intersections of $\mathcal{L}(D)_d^j$ and (4.3.10)-(4.3.12) respectively.

(i) Suppose that $\mathcal{X}$ is an element of (4.3.10). Namely, $\mathcal{X} = \mathcal{A} \underset{e_{P_0}}{\times} \mathcal{B}$, where $\mathcal{A} \in \mathcal{L}(D_1)_e^{-s}$, $\mathcal{B} \in \mathcal{L}(D_2)_{d,e}^{t+1,-1}$ and $P_0 := \{(1,1), \ldots, (s,1)\}$ is a unique lattice path in $\{1, \ldots, s\} \times \{1\}$. We denote by γ_0 the D_2-laminate that intersects with d positively and with e negatively, in other words, $g_d(\gamma_0) = -g_e(\gamma_0) = 1$ and 0 otherwise. By the maximality of $\mathcal{B}$, γ_0 is contained in $\mathcal{B}$. Moreover, this is the unique D_2-laminate of $\mathcal{B}$ intersecting with both d and e. By Proposition 4.1.9, we have

$$g_d(\mathcal{X}) = g_d(\gamma_0) \cdot s + (g_d(\mathcal{B}) - g_d(\gamma_0)) = s + t.$$

Therefore, $\mathcal{X}$ lies in $\mathcal{L}(D)_d^j$ if and only if $s + t = j$.

(ii) Suppose that $\mathcal{X} := \mathcal{A} \underset{e_P}{\times} \mathcal{B}$, where $\mathcal{A} \in \mathcal{L}(D_1)_e^{-s}$, $\mathcal{B} \in \mathcal{L}(D_2)_{d,e}^{1,-t-1}$ and $P \in \mathrm{P}(s, t+1)$. From the same reason as (i), γ_0 is contained in $\mathcal{B}$, and hence

$$g_d(\mathcal{X}) = g_d(\gamma_0) \cdot P(1, -) = P(1, -).$$

Thus, $\mathcal{X}$ lies in $\mathcal{L}(D)_d^j$ if and only if $P \in P^j(s, t+1)$.

(iii) Suppose that $\mathcal{X} := \mathcal{A} \underset{e_P}{\times} \mathcal{B}$, where $\mathcal{A} \in \mathcal{L}(D_1)_e^{-s}$, $\mathcal{B} \in \mathcal{L}(D_2)_{d,e}^{t-u+1,u}$ and $P \in \mathrm{P}(s, u)$. Since $\mathcal{B}$ has no D_2-laminates intersecting with both d and e, we have

$$g_d(\mathcal{X}) = g_d(\mathcal{B}) = t - u + 1.$$

Thus, $\mathcal{X}$ lies in $\mathcal{L}(D)_d^j$ if and only if $t - u + 1 = j$.

From (i)-(iii), we get the assertion (1).

(2) We determine the cardinality of the sets (4.3.2)-(4.3.4) under the induction hypothesis $(*)$. Since e^* is an external arc of D_1^*, we have

$$\#\mathcal{L}(D_1)_e^s = \#\mathcal{L}(D_1)_e^{-s} = \binom{2n_1 - s - 1}{n_1 - 1} \tag{4.3.13}$$

for any $s \in \{1, \dots, n_1\}$.

By (4.3.1), we are enough to show that

$$\#\mathcal{L}(D_2)_{d,e}^{t+1,-1} = \#\mathcal{L}(D_2)_{d,e}^{1,-t-1} = \#\mathcal{L}(D_2)_{d,e}^{t-u+1,u} = \binom{2n_2 - t - 3}{n_2 - 2} \tag{4.3.14}$$

for any $t \in \{1 \dots, n_2 - 1\}$ and $u \in \{1 \dots, t\}$. To prove this, we first consider a right flip $E^* := \mu_{d^*}(D^*)$ of D^* with respect to d^*. After that, we take a decomposition $E^* = E_1^* \underset{e^*}{\times} E_2^*$ of E^* with respect to e^*. Here, we regard E^* (resp., E_1^* and E_2^*) as a $\circ$-dissections of a marked surface (S', M') (resp., (S_1', M_1') and (S_2', M_2')) with the dual $\bullet$-dissection E (resp., E_1 and E_2). See Figure 4.7 for its local figure. In this situation, we have $E_1^* = \{\mu(d^*), e^*\}$ and $E_2^* = E^* \backslash \{\mu(d^*)\}$. As $E_1, E_2 \subset E$, we have $E_1 = \{d = \mu(d^*)^*, e\}$ and $E_2 = E \backslash \{d\}$.

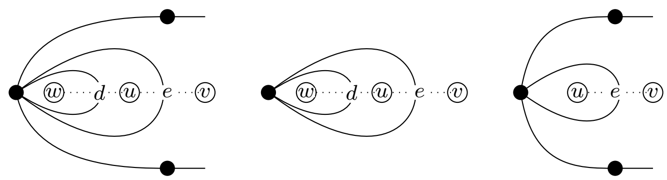
By flipping a dissection (Proposition 4.2.5), we have bijections

$$\mathcal{L}(D_2)_{d,e}^{t+1,-1} \overset{\sim}{\to} \mathcal{L}(E)_{d,e}^{-t-1,t}, \tag{4.3.15}$$

$$\mathcal{L}(D_2)_{d,e}^{1,-t-1} \overset{\sim}{\to} \mathcal{L}(E)_{d,e}^{-1,-t}, \tag{4.3.16}$$

$$\mathcal{L}(D_2)_{d,e}^{t-u+1,u} \overset{\sim}{\to} \mathcal{L}(E)_{d,e}^{-t-1,u} \tag{4.3.17}$$

We calculate the cardinality of the sets in the right-hand side from that of E_1^* and of E_2^*. We observe that E_1^* is a tree with 2 arcs, so there are 6 complete E_1-laminations $\mathcal{A}_{\pm 1}$, $\mathcal{A}_{\pm 2}, \mathcal{A}_{\pm 3}$ characterized by $(g_d(\mathcal{A}_1), g_e(\mathcal{A}_1)) = (-2, 1)$, $(g_d(\mathcal{A}_2), g_e(\mathcal{A}_2)) = (-1, -1)$, $(g_d(\mathcal{A}_3), g_e(\mathcal{A}_3)) = (-1, 2)$ and $g(\mathcal{A}_{-i}) = -g(\mathcal{A}_i)$



(S', M', E) (S_1', M_1', E_1) (S_2', M_2', E_2)

Figure 4.7: A local figure for $E^* = E_1^* \underset{e^*}{\times} E_2^*$ with respect to e^*

for $i \in \{1, 2, 3\}$, see Section 3.2(1). From this fact, it is not difficult to see that the bijections in Proposition 4.2.3 restricts to bijections

$$\rho_e^{-t-1,t} \quad : \{\mathcal{A}_1\} \times \mathcal{L}(E_2)_e^t \times \{P_0'\} \overset{\sim}{\to} \mathcal{L}(E)_{d,e}^{-t-1,t} \tag{4.3.18}$$

$$\rho_e^{-1,-t} \quad : \{\mathcal{A}_2\} \times \mathcal{L}(E_2)_e^{-t} \times \{P_0'\} \overset{\sim}{\to} \mathcal{L}(E_2)_e^{-t} \tag{4.3.19}$$

$$\rho_e^{-t-1,u} \quad : \{\mathcal{A}_3\} \times \mathcal{L}(E_2)_e^t \times \{P_u'\} \overset{\sim}{\to} \mathcal{L}(E_2)_e^t \tag{4.3.20}$$

for any $t \in \{1, \ldots, n_2 - 1\}$ and any $u \in \{1, \ldots, t\}$. Here, P_0' is a unique lattice path $\{(1, 1), \ldots, (1, t)\}$ of $\{1\} \times \{1, \ldots, t\}$, and P_u' is a lattice path $\{(1, 1), \ldots, (1, u), (2, u), \ldots, (2, t)\}$ of $\{1, 2\} \times \{1, \ldots, t\}$. We know that E_2^* is a tree having $n_2 - 1$ arcs and has e^* as its external arc, so

$$\#\mathcal{L}(E_2)_e^t = \#\mathcal{L}(E_2)_e^{-t} = \binom{2n_2 - t - 3}{n_2 - 2} \tag{4.3.21}$$

holds by induction hypothesis $(*)$. Combining all the above argument, we get the desired equation (4.3.14). It finishes a proof of Lemma 4.3.3. □

4.3.3 Proof of Lemma 4.3.4

We give a proof of Lemma 4.3.4.

Proof of Lemma 4.3.4. Fix $m = p + q - 1$, where p, q are integers $p > 0$ and $q > 1$. Let $A_{q-1} := A_{p,q-1}$, $B_{q-1} := B_{p,q-1}$ and $C_{q-1} := C_{p,q-1}$. In addition, let $F_n(s) := \binom{2n-s-1}{n-1}$ and $P(s,t) := \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}$ for positive integers s, t and n.

We show the claim by induction on q. First, we assume that $q = 2$. By definition, we have

$$A_1(j) = \begin{cases} 0 & \text{if } j = 1 \\ F_p(j-1) & \text{else,} \end{cases} \qquad B_1(j) = \sum_{s=j}^{p} F_p(s), \quad C_1(j) = \begin{cases} F_{p+1}(2) & \text{if } j = 1 \\ 0 & \text{else.} \end{cases}$$

So, we have

$$A_1(1) + B_1(1) + C_1(1) = 2F_{p+1}(2) = F_{p+1}(1)$$

for $j = 1$, and

$$A_1(j) + B_1(j) + C_1(j) = \sum_{s=j-1}^{p} F_p(s) = F_{p+1}(j)$$

for $j \in \{2, \ldots, p+1\}$. Therefore, the desired equation holds for $q = 2$.

Second, we assume that $q > 2$ and the assertion holds for $q - 1$. Remember that the desired equation is

$$F_{p+q}(j) = A_q(j) + B_q(j) + C_q(j) \tag{4.3.22}$$

for all $j \in \{1, \ldots, p+q\}$. Firstly, we consider $j > 1$. By induction hypothesis, we have

$$F_{p+q}(j) = \sum_{i=j-1}^{p+q-1} F_{p+q-1}(i) = \sum_{i=j-1}^{p+q-1} A_{q-1}(i) + \sum_{i=j-1}^{p+q-1} B_{q-1}(i) + \sum_{i=j-1}^{p+q-1} C_{q-1}(i).$$

We calculate each summand in the right-hand side.

(a) By definition, we have

$$\begin{aligned}
\sum_{i=j-1}^{p+q-1} A_{q-1}(i) &= \sum_{\substack{1\le s\le p\\ 1\le t\le q-1\\ p+q-1=s+t}} F_p(s)F_{q-1}(t) + \cdots + \sum_{\substack{1\le s\le p\\ 1\le t\le q-1\\ j-1=s+t}} F_p(s)F_{q-1}(t)\\
&= \sum_{s=j-1}^{p} F_p(s)\sum_{t=1}^{p-1} F_{q-1}(t) + \sum_{s=1}^{j-2} F_p(s) \sum_{t=j-s-1}^{q-1} F_{q-1}(t).\\
&= F_{p+1}(j)F_q(2) + \sum_{s=1}^{j-2} F_p(s)F_q(j-s)\\
&= A_q(j) - F_p(j-1)F_q(1) + F_{p+1}(j)F_q(2). \qquad (4.3.23)
\end{aligned}$$

(b) We continue our calculation.

$$\begin{aligned}
\sum_{i=j-1}^{p+q-1} B_{q-1}(i) &= \sum_{t=1}^{q-1} F_{q-1}(t)\Big\{\sum_{i=j-1}^{p+q-1}\sum_{s=i}^{p} F_p(s)P(s-i,t-1)\Big\} \\
&= \sum_{t=1}^{q-1} F_{q-1}(t)\Big\{P(0,t-1)\sum_{s=j-1}^{p} F_p(s) + \sum_{s=l}^{p} F_p(s)\sum_{\lambda=1}^{s-l+1} P(\lambda,t-1)\Big\}. \\
&= F_{p+1}(j)F_q(2) + \sum_{t=1}^{q-1} F_{q-1}(t)\Big\{\sum_{s=j}^{p} F_p(s)\sum_{\lambda=1}^{s-j+1} P(\lambda,t-1)\Big\} \\
&= F_{p+1}(j)F_q(2) + \sum_{t=1}^{q-1} F_{q-1}(t)\Big\{\sum_{s=j}^{p} F_p(s)\sum_{\mu=1}^{t} P(s-j,\mu)\Big\} \\
&= F_{p+1}(j)F_q(2) + \sum_{t=1}^{q-1}\sum_{u=t}^{q-1} F_{q-1}(u)\sum_{s=j}^{p} F_p(s)P(s-j,t) \\
&= F_{p+1}(j)F_q(2) + \sum_{t=1}^{q-1}\sum_{s=j}^{p} F_q(t+1)F_p(s)P(s-j,t) \\
&= B_q(i) - F_{p+1}(i+1)F_q(1) + F_{p+1}(i)F_q(2), \qquad (4.3.24)
\end{aligned}$$

where the last two equalities are obtained by replacing $t \to t+1$.

(c) Finally, we get

$$\begin{aligned}
\sum_{i=j-1}^{p+q-1} C_{q-1}(i) &= \sum_{s=1}^{p} F_p(s)\Big\{\sum_{i=j-1}^{p+q-1}\sum_{t=1}^{q-i} F_{q-1}(t+i-1)P(s-1,t-1)\Big\} \\
&= \sum_{s=1}^{p} F_p(s)\sum_{t=1}^{q-j+1} F_q(t+j-1)P(s-1,t-1) \\
&= C_q(j). \qquad (4.3.25)
\end{aligned}$$

Adding (4.3.23)–(4.3.25), we get the desired equation (4.3.22) for $j \in \{2,\ldots,p+q\}$ by

$$F_{p+1}(j)F_q(2) - F_p(j-1)F_q(1) - F_{p+1}(j+1)F_q(1) + F_{p+1}(j)F_q(2) = 0.$$

On the other hand, we get the case $j = 1$ from the previous result:

$$F_{p+q}(1) = 2F_{p+q}(2) = 2A_q(2) + 2B_q(2) + 2C_q(2) = A_q(1) + B_q(1) + C_q(1).$$

We finish the proof of Lemma 4.3.4. □

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