Reconstruction methods for inverse scattering problems
(散乱逆問題における再構成手法について)

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Contents

1 Introduction

The inverse scattering problem is the problem to determine unknown scatterers by measuring scattered waves that is generated by sending incident waves far away from scatterers. It is of importance for many applications, for example medical imaging, nondestructive testing, remote exploration, and geophysical prospecting. Due to many applications, the inverse scattering problem has been studied in various ways. For further readings, we refer to the following books [11, 16, 18, 52, 78], which include the summary of classical and recent progress of the inverse scattering problem.

We begin with the mathematical formulation of the scattering problem. Let $k > 0$ be the wave number, and let $\theta \in \mathbb{S}^{d-1}$ be incident direction. We denote the incident field $u^{inc}(\cdot, \theta)$ with incident direction θ by the plane wave of the form

$$u^{inc}(x, \theta) := \mathrm{e}^{ikx\cdot\theta}, \; x \in \mathbb{R}^d. \tag{1.1}$$

Let $\Omega \subset \mathbb{R}^d$ (in particular we consider $d = 2, 3$) be a bounded open set with a smooth boundary $\partial\Omega$ such that the exterior $\mathbb{R}^d \setminus \overline{\Omega}$ is connected. In particular, we discuss the following two cases. The first case is that the scatterer Ω is a penetrable medium, and determine the total field $u = u^{sca} + u^{inc}$ such that

$$\Delta u + k^2(1 + q)u = 0 \text{ in } \mathbb{R}^2, \tag{1.2}$$

$$\lim_{r:=|x|\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^{sca}}{\partial r} - iku^{sca} \right) = 0, \tag{1.3}$$

where $q \in L^\infty(\mathbb{R}^d)$ has a compact support such that $\Omega = \mathrm{supp}\, q$. The *Sommerfeld radiation condition* (1.3) holds uniformly in all directions $\hat{x} := \frac{x}{|x|}$. The second case is that Ω is an impenetrable obstacle, and determine the total field $u = u^{sca} + u^{inc}$ such that

$$\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega}, \tag{1.4}$$

$$\mathcal{B}u = 0 \text{ on } \partial\Omega, \tag{1.5}$$

$$\lim_{r:=|x|\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^{sca}}{\partial r} - iku^{sca} \right) = 0, \tag{1.6}$$

where (1.5) means the boundary conditions, for example, the Dirichlet boundary condition $\mathcal{B}u = u$, the Neumann boundary condition $\mathcal{B}u = \frac{\partial u}{\partial \nu}$, and so on. In both problems (1.2)–(1.3) and (1.4)–(1.6), it is well known that there exists a unique solution u^{sca} and it has the following asymptotic behaviour (see e.g., [18]),

$$u^{sca}(x) = \frac{\mathrm{e}^{ikr}}{r^{\frac{d-1}{2}}} \Big\{ u^{\infty}(\hat{x}, \theta) + O\big(1/r\big) \Big\}, \ r \to \infty. \tag{1.7}$$

The function u^{∞} is called the *far field pattern* of the scattered field u^{sca}. For further details of the direct scattering problem, we refer to [18]. The inverse scattering problem we consider here is to extract information of the unknown scatterer Ω from the far field pattern u^{∞}.

The first question of inverse problems is uniqueness. It is well known that the far field pattern $u^{\infty}(\hat{x}, \theta)$ for all $\hat{x}, \theta \in \mathbb{S}^{d-1}$ and fixed $k > 0$ uniquely determines the unknown scatterer Ω (see e.g., [77, 81, 87]). However, the uniqueness when all directions $\hat{x}, \theta \in \mathbb{S}^{d-1}$ are not given, which is called as the partial data problem, is still open. For further readings of uniqueness, we refer to the following books [18, 45].

In Section 4 (original paper [25]), we discuss the direct and inverse scattering problem for the semilinear Schrödinger equation,

$$\Delta u + a(x, u) + k^2 u = 0 \text{ in } \mathbb{R}^d, \tag{1.8}$$

where $a : \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$ is a semilinear function under Assumption 4.1. This type of semilinear function $a(x, u)$ is the generalization of, in particular, the power type $q(x)u^m$ where $m \in \mathbb{N}$, and $q \in L^{\infty}(\mathbb{R}^d)$ with a compact support. The case of $m = 1$ corresponds to the linear Schrödinger equation (1.2). We prove the well-posedness of the direct scattering problem (1.8) by employing the Banach fixed point theorem, and prove the uniqueness of the inverse problem by some linearization technique. (For main results, we see Theorems 4.2 and 4.3.)

The second question is reconstruction, which is the problem to provide reconstruction algorithms for the unknown scatterer Ω from far field patterns. Existing methods for reconstruction can be roughly categorized into two group: the iterative optimization method (see e.g., [5, 18, 30, 42, 51]) and the sampling method (see e.g., [17, 33, 43, 44, 58, 85]). This thesis mainly deals with reconstruction and contributes to both groups.

In Sections 7 and 8 (original papers [27, 28]), we discuss reconstruction schemes for inverse medium scattering problem (1.2)–(1.3) based on the *Kalman filter* techniques, which is categorized into the iterative optimization method. The Kalman filter (see e.g., [50]) is the algorithm to estimate the unknown state in the dynamics system by employing the sequential measurements observed over time. It has many applications such as navigations and tracking objects, and for further readings, we refer to [31, 48, 50, 78]. By applying the Kalman filter to our inverse scattering problem, we provide algorithms to estimate the unknown scatterer Ω every time to observe the far field pattern $u^{\infty}(\cdot, \theta_n)$ with one incident direction $\theta_n \in \mathbb{S}^{d-1}$ without waiting all data $\{u^{\infty}(\cdot, \theta_n)\}_{n=1}^{N}$. (For main results, we see (7.44)–(7.46), (8.39)–(8.43), and (8.53)–(8.57), and see Theorems 7.4 and 8.4.)

In Section 2 (original paper [22]), we discuss the factorization method, which is categorized into the sampling method, for the case that the scatterer consists of two components ($\Omega = \Omega_1 \cup \Omega_2$) with different physical properties (for example, Ω_1 is an impenetrable obstacle with the Dirichlet boundary condition, and Ω_2 with the Neumann boundary condition). In order to apply the factorization method to such a complicated scatterer, a lot of a priori assumptions for the wave number $k > 0$ have been required (see e.g., [58, 59]). The contribution of Section 2 is to provide the reconstruction scheme of the factorization method without any a priori assumptions for the wave number $k > 0$, but instead, we have to know the topological properties of Ω (see Assumption 2.1 and Figure 1). (For main results, we see Theorems 2.2 and 2.4.)

In Section 3 (original paper [24]), we discuss the monotonicity method for the inverse crack scattering problem, which is the case when the scatterer Ω is a smooth arc, i.e $\Omega = \{\gamma(s) : s \in [-1, 1]\}$ where $\gamma : [-1, 1] \to \mathbb{R}^2$ is smooth. The monotonicity method is a similar method to the factorization method, and it has been originally introduced in Electrical impedance tomography ([40]). Recently, the monotonicity has been extended to inverse acoustic scattering problem in the case of the impenetrable obstacle and the penetrable medium ([2, 33]). However, it was not obvious to extend it to crack like not having volume. In Section 3, we extend the monotonicity method to the inverse crack scattering problem, and provide its reconstruction scheme. (For main results, we see Theorems 3.1 and 3.2.)

In Sections 5 and 6 (original papers [23, 26]), we discuss the direct and inverse scattering by a local perturbation in an infinite medium with periodicity in the upper half space $\mathbb{R} \times (0, \infty)$. In Section 5, we discuss the well-posedness of the following direct scattering problem.

$$\Delta u + k^2(1 + q)nu = 0 \text{ in } \mathbb{R} \times (0, \infty), \tag{1.9}$$

$$u = 0 \text{ on } \mathbb{R} \times \{0\}, \tag{1.10}$$

where $n \in L^{\infty}(\mathbb{R} \times (0, \infty))$ is real value, 2π-periodic with respect to x_1 (that is, $n(x_1 + 2\pi, x_2) = n(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2_+$), and equal to one for $x_2 > h$ where $h > 0$ is some positive number, and $q \in L^{\infty}(\mathbb{R} \times$

$(0, \infty))$ is real valued with the compact support in $\mathbb{R} \times (0, h)$ such that $\Omega = \text{supp}\, q$. The Sommerfeld radiation condition (1.3) can not be imposed in the case of the scattering in the half space $\mathbb{R} \times (0, \infty)$, so by imposing a suitable radiation condition recently introduced in [62], we showed the well-posedness of this perturbed scattering problem (1.9)–(1.10). Then, we become able to define the inverse problem of reconstruction of the support Ω of q from scattered fields. In Section 6, we discuss this inverse problem, and had two contributions. Firstly, we mention that there is a mistake in the factorization method of the earlier paper [72], which leads to the difficulty to apply the factorization method to our inverse problem. Secondly, we give the reconstruction scheme by employing the monotonicity method instead of the factorization method. (For main results, we see Theorems 5.2, 6.1, 6.2, and 6.11.)

Through our works, we conclude that the iterative optimization method and the sampling method complement each other. The iterative method does not need a lot of data, however it requires the initial guess which is the starting point of the optimization. It must be appropriately chosen by a priori knowledge of true Ω, otherwise, the iterative solution could not converge to true one. On the other hand, the sampling method does not require the initial guess, which is one of the advantages over the iterative method. However, the disadvantage is to need infinite data that can not be practically measured. In the future, sampling methods for finite measurements should be studied for more realistic problems, and it would be good to develop the combination of both methods. Although our papers mostly contribute to theoretical aspects of inverse acoustic scattering problems, Sections 3, 6, 7, and 8 present numerical experiments for reconstruction by using the Python programming language. (We see Figures 6, 7, 9, 11, 12, 17, and 18.)

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2 A modification of the factorization method for scatterers with different physical properties

2.1 Introduction

Sampling methods are proposed for reconstruction of shape and location in inverse acoustic scattering problems. In the last twenty years, sampling methods such as the Linear Sampling method of Colton and Kress [18], the Singular Sources Method of Potthast [85], the Factorization Method of Kirsch [53], have been introduced and intensively studied. As an advantage of these sampling methods, the numerical implementation are so simple and fast. However, as disadvantage of sampling methods except the Factorization Method, only sufficient conditions are given for the identification of unknown scatterers. To overcome this drawback, that is, to provide necessary and sufficient conditions, the Factorization Method was introduced and developed by a lot of researchers.

However, for rigorous justification of the original Factorization Method, we have to assume that the wave number of the incident wave is not an eigenvalue of the Laplacian on an obstacle with respect to the boundary condition of the scattering problem. Kirsch and Liu [63] eliminated this problem for the case of a single obstacle by assuming that a small ball is in the interior of the unknown obstacle. They modified the original far field operator by adding the far field operator corresponding to a small ball so that the Factorization Method can be applied to it. On the other hands, in the case of a scatterer consisting of two objects with different physical properties, this problem has been still open. For recent works discussing this case, we refer to [3, 7, 60, 65, 97].

In this section, we study the Factorization Method for a scatterer consisting of two objects with different physical properties. Especially, we consider the following two cases: One is the case when each object has the different boundary condition, and the other one is when different penetrability. For recent works discussing such a scatterer, we refer to [59, 64, 74]. We remark that these works have to assume that the wave number of the incident wave is not an eigenvalue of the Laplacian on impenetrable obstacles included in a scatterer. Our aim of this paper is to eliminate this restriction by developing the idea of [63].

We begin with the formulations of the scattering problems. Let $k > 0$ be the wave number and for $\theta \in \mathbb{S}^2$ be incident direction. Here, $\mathbb{S}^2 = \{x \in$

$\mathbb{R}^3 : |x| = 1\}$ denotes the unit spherer in $\mathbb{R}^3$. We set

$$u^i(x) := \mathrm{e}^{ik\theta\cdot x},\ x \in \mathbb{R}^3, \tag{2.1}$$

where i in the left hand side stands for *incident plane wave*. Let $\Omega \subset \mathbb{R}^3$ be a bounded open set and let its exterior $\mathbb{R}^3 \setminus \overline{\Omega}$ be connected. We assume that Ω consists of two bounded domains, i.e., $\Omega = \Omega_1 \cup \Omega_2$ such that $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$. We consider the following two cases.

The first case. Ω_1 **is an impenetrable obstacle with Dirichlet boundary condition, and** Ω_2 **with Neumann boundary condition.** Find $u^s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$ such that

$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, \tag{2.2}$$

$$u^s = -u^i \text{ on } \partial\Omega_1, \tag{2.3}$$

$$\frac{\partial u^s}{\partial \nu_{\Omega_2}} = -\frac{\partial u^i}{\partial \nu_{\Omega_2}} \text{ on } \partial\Omega_2, \tag{2.4}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{2.5}$$

where $r = |x|$, and (2.5) is the *Sommerfeld radiation condition*. Here, $H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega}) = \{u : \mathbb{R}^3 \setminus \overline{\Omega} \to \mathbb{C} : u\big|_B \in H^1(B) \text{ for all open balls } B\}$ denotes the local Sobolev space of one order. $\nu_{\Omega_2}(x)$ denotes the unit normal vector at $x \in \partial\Omega_2$. We refer to Theorem 7.15 in [76] for the well posedness of the problem (2.2)–(2.5), and refer to [59] and [74] for the factorization method in this case.

The second case. Ω_1 **is a penetrable medium modeled by a contrast function** $q \in L^\infty(\Omega_1)$ (that is, $\Omega_1 = \mathrm{supp} q$), **and** Ω_2 **is an impenetrable obstacle with Dirichlet boundary condition**. Find $u^s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega_2})$ such that

$$\Delta u^s + k^2(1+q)u^s = -k^2 q u^i \text{ in } \mathbb{R}^3 \setminus \overline{\Omega_2}, \tag{2.6}$$

$$u^s = -u^i \text{ on } \partial\Omega_2, \tag{2.7}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \tag{2.8}$$

Note that we extend q by zero outside Ω_1. The well posedness of the problem (2.6)–(2.8) and its factorization method was shown in [64].

In both cases, it is well known that the scattered wave u^s has the following asymptotic behavior:

$$u^s(x,\theta) = \frac{\mathrm{e}^{ik|x|}}{4\pi|x|}u^{\infty}(\hat{x},\theta) + O\left(\frac{1}{|x|^2}\right),\ |x| \to \infty,\quad \hat{x} := \frac{x}{|x|}. \tag{2.9}$$

The function u^{∞} is called the far field pattern of u^s. With the far field pattern u^{∞}, we define the far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$Fg(\hat{x}) := \int_{\mathbb{S}^2} u^{\infty}(\hat{x},\theta)g(\theta)ds(\theta),\ \hat{x} \in \mathbb{S}^2. \tag{2.10}$$

We write the far field operator of the problem (2.2)–(2.5) as $F = F^{Mix}_{\Omega_1,\Omega_2}$, and (2.6)–(2.8) as $F = F^{Mix}_{\Omega_1 q,\Omega_2}$, respectively. The inverse scattering problem we consider is to reconstruct Ω from the far field pattern $u^{\infty}(\hat{x},\theta)$ for all $\hat{x},\theta \in \mathbb{S}^2$. In other words, given the far field operator F, reconstruct Ω.

Our contribution in this section is, in both cases, to give the characterization of Ω_1 without a priori assumptions for the wave number $k > 0$. But we have to know the topological properties of Ω. More precisely, an *inner* domain B_1 of Ω_1 ([63]), and an *outer* domain B_2 of Ω_2 ([59]), have to be a priori known. Furthermore, we take an *additional* domain B_3 in the interior of B_2. By adding artificial far field operators corresponding to B_1, B_2, and B_3, we modify the original far field operator F.

In the first case, we give the following characterization:

Assumption 2.1. *Let bounded domain B_1 and B_2 be a prior known. Assume that $\overline{B_1} \subset \Omega_1$, $\overline{\Omega_2} \subset B_2$, $\overline{\Omega_1} \cap \overline{B_2} = \emptyset$.*

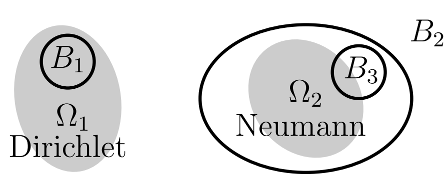


Figure 1: Assumption of Theorem 2.2

Theorem 2.2. *For $\hat{x} \in \mathbb{S}^2$, $z \in \mathbb{R}^3$, define*

$$\phi_z(\hat{x}) := \mathrm{e}^{-ikz\cdot\hat{x}}. \tag{2.11}$$

Let Assumption 2.1 *hold. Take a positive number $\lambda_0 > 0$, and a bounded domain B_3 with $\overline{B_3} \subset B_2$.* (*See* Figure 1.) *Then, for $z \in \mathbb{R}^3 \setminus \overline{B_2}$*

$$z \in \Omega_1 \Longleftrightarrow \sum_{n=1}^{\infty} \frac{\left|(\phi_z, \varphi_n)_{L^2(\mathbb{S}^2)}\right|^2}{\lambda_n} < \infty, \tag{2.12}$$

where (λ_n, φ_n) is a complete eigensystem of $F_{\#}$ given by

$$F_{\#} := \left|\mathrm{Re}F\right| + \left|\mathrm{Im}F\right|, \tag{2.13}$$

where $F := F^{Mix}_{\Omega_1,\Omega_2} + F^{Dir}_{B_2} + F^{Imp}_{B_1 \cup B_3, i\lambda_0}$. Here, $F^{Dir}_{B_2}$ and $F^{Imp}_{B_1 \cup B_3, i\lambda_0}$ are the far field operators for the pure Dirichlet boundary condition on B_2, and for the pure impedance boundary condition on $B_1 \cup B_3$ with an impedance function $i\lambda_0$, respectively.

Latter, we explain artificial far field operators $F^{Dir}_{B_2}$ and $F^{Imp}_{B_1 \cup B_3, i\lambda_0}$ in Section 2.2, and prove Theorem 2.2 in Section 2.3.

In the second case, we give the following characterization:

Assumption 2.3. *Let a bounded domain B_2 be a priori known. Assume the following assumptions:*

(i) *$q \in L^{\infty}(\Omega_1)$ with $\mathrm{Im}q \geq 0$ in Ω_1.*

(ii) *$|q|$ is locally bounded below in Ω_1, i.e., for every compact subset $M \subset \Omega_1$, there exists $c > 0$ (depend on M) such that $|q| \geq c$* in *M.*

(iii) *$\overline{\Omega_2} \subset B_2$, $\overline{\Omega_1} \cap \overline{B_2} = \emptyset$.*

(iv) *There exists $t \in (\pi/2, 3\pi/2)$ and $C > 0$ such that $\mathrm{Re}(\mathrm{e}^{-it}q) \geq C|q|$ a.e. in Ω_1.*

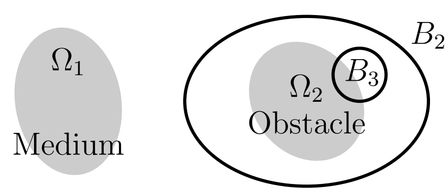


Figure 2: Assumption of Theorem 2.4

Theorem 2.4. *Let* Assumption 2.3 *hold. Take a positive number* $\lambda_0 > 0$, *and a bounded domain* B_3 *with* $\overline{B_3} \subset B_2$. (*See* Figure 2.) *Then, for* $z \in \mathbb{R}^3 \setminus \overline{B_2}$

$$z \in \Omega_1 \Longleftrightarrow \sum_{n=1}^{\infty} \frac{|(\phi_z, \varphi_n)_{L^2(\mathbb{S}^2)}|^2}{\lambda_n} < \infty, \tag{2.14}$$

where (λ_n, φ_n) *is a complete eigensystem of* $F_{\#}$ *given by*

$$F_{\#} := \left|\mathrm{Re}\left(\mathrm{e}^{-it} F\right)\right| + \left|\mathrm{Im} F\right|, \tag{2.15}$$

where $F := F^{Mix}_{\Omega_1 q, \Omega_2} + F^{Dir}_{B_2} + F^{Imp}_{B_3, i\lambda_0}$. *Here, the function* ϕ_z *is given by* (2.11).

We prove Theorem 2.4 in Section 2.4. We can also give the characterization by replacing (iv) in Assumption 2.3 with

(iv') *There exists* $t \in [0, \pi/2) \cup (3\pi/2, 2\pi]$ *and* $C > 0$ *such that* $\mathrm{Re}(\mathrm{e}^{-it} q) \geq C|q|$ *a.e. in* Ω_1.

For details, see Assumption 2.20 and Theorem 2.21.

Let us compare our works (Theorems 2.2 and 2.4) with previous works from the mathematical point of view of a priori assumptions. For Theorem 2.2 we refer to Theorem 2.5 of [74], and for Theorems 2.4 we refer to Theorem 3.9 (b) of [64]. These previous works also gave the characterization of Ω_1 by assuming the existence of outer domain B_2 of Ω_2 and that the wave number k^2 is not an eigenvalue on an obstacle, while, in our work we can choose arbitrary wave number $k > 0$ by introducing extra artificial domains such as B_1, B_2, and B_3, which are not so difficult topological assumptions.

This section is organized as follows. In Section 2.1, we recall a factorization of the far field operator and its properties. In Section 2.3 and Section 2.4, we prove Theorems 2.2 and 2.4, respectively.

2.2 A factorization for the far field operator

In Section 2.2, we briefly recall a factorization for the far field operators and its properties.

First, we consider a factorization of the far field operator for the pure boundary condition. Let B be a bounded open set and let $\mathbb{R}^3 \setminus \overline{B}$ be connected. Later, we will use the result of this section by regarding B as auxiliary domains, like B_1, B_2, and B_3 in Theorems 2.2 and 2.4. We define $G^{Dir}_B : H^{1/2}(\partial B) \to L^2(\mathbb{S}^2)$ by

$$G^{Dir}_B f := v^{\infty}, \tag{2.16}$$

where v^∞ is the far field pattern of a radiating solution v (that is, v satisfies the Sommerfeld radiation condition) such that

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B}, \tag{2.17}$$

$$v = f \text{ on } \partial B. \tag{2.18}$$

Let $\lambda_0 > 0$. We also define $G^{Imp}_{B,i\lambda_0} : H^{-1/2}(\partial B) \to L^2(\mathbb{S}^2)$ in the same way as G^{Dir}_B by replacing (2.18) with

$$\frac{\partial v}{\partial \nu_B} + i\lambda_0 v = f \text{ on } \partial B. \tag{2.19}$$

We define the boundary integral operators $S_B : H^{-1/2}(\partial B) \to H^{1/2}(\partial B)$ and $N_B : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ by

$$S_B\varphi(x) := \int_{\partial B} \varphi(y)\Phi(x,y)ds(y), \ x \in \partial B, \tag{2.20}$$

$$N_B\psi(x) := \frac{\partial}{\partial \nu_B(x)} \int_{\partial B} \psi(y)\frac{\partial \Phi(x,y)}{\partial \nu_B(y)}ds(y), \ x \in \partial B, \tag{2.21}$$

where $\Phi(x,y) := \dfrac{\mathrm{e}^{ik|x-y|}}{4\pi|x-y|}$. We also define $S_{B,i}$ and $N_{B,i}$ by the boundary integral operators (2.20) and (2.21), respectively, corresponding to the wave number $k = i$. It is well known that $S_{B,i}$ is self-adjoint and positive coercive, and $N_{B,i}$ is self-adjoint and negative coercive. For details of the boundary integral operators, we refer to [58] and [76].

The following properties of far field operators F^{Dir}_B and $F^{Imp}_{B,i\lambda_0}$ are given by previous works in [58] and [63]:

Lemma 2.5 (Lemma 1.14 in [58], Theorem 2.1 and Lemma 2.2 in [63]).

(a) *The far field operators F^{Dir}_B and $F^{Imp}_{B,i\lambda_0}$ have a factorization of the form*

$$F^{Dir}_B = -G^{Dir}_B S^*_B G^{Dir}_B{}^*, \qquad F^{Imp}_{B,i\lambda_0} = -G^{Imp}_{B,i\lambda_0} T^{Imp}_{B,i\lambda_0}{}^* G^{Imp}_{B,i\lambda_0}{}^*. \tag{2.22}$$

(b) *The operators $S_B : H^{-1/2}(\partial B) \to H^{1/2}(\partial B)$ and $T^{Imp}_{B,i\lambda_0} : H^{1/2}(\partial B) \to H^{-1/2}(\partial B)$ is of the form*

$$S_B = S_{B,i} + K, \qquad T^{Imp}_{B,i\lambda_0} = N_{B,i} + K', \tag{2.23}$$

where K and K' are some compact operators.

(c) $\mathrm{Im}\langle\varphi, S_B\varphi\rangle \leq 0$ *for all* $\varphi \in H^{-1/2}(\partial B)$*. Furthermore, if we assume that* k^2 *is not a Dirichlet eigenvalue of* $-\Delta$ *in* B*, then* $\mathrm{Im}\langle\varphi, S_B\varphi\rangle < 0$ *for all* $\varphi \in H^{-1/2}(\partial B)$ *with* $\varphi \neq 0$*.*

(d) $\mathrm{Im}\langle T^{Imp}_{B,i\lambda_0}\varphi, \varphi\rangle > 0$ *for all* $\varphi \in H^{1/2}(\partial B)$ *with* $\varphi \neq 0$*.*

Secondly, we consider the far field operator $F^{Mix}_{\Omega_1,\Omega_2}$ for the problem (2.2)–(2.5). Recall that $\Omega = \Omega_1 \cup \Omega_2$, and Ω_1 is an impenetrable obstacle with Dirichlet boundary condition, and Ω_2 with Neumann boundary condition. We define $G^{Mix}_{\Omega_1,\Omega_2} : H^{1/2}(\partial\Omega_1) \times H^{-1/2}(\partial\Omega_2) \to L^2(\mathbb{S}^2)$ by

$$G^{Mix}_{\Omega_1,\Omega_2}\begin{pmatrix} f \\ g \end{pmatrix} := v^\infty, \tag{2.24}$$

where v^∞ is the far field pattern of a radiating solution v such that

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, \tag{2.25}$$

$$v = f \text{ on } \partial\Omega_1, \qquad \frac{\partial v}{\partial \nu_{\Omega_2}} = g \text{ on } \partial\Omega_2. \tag{2.26}$$

The following properties of $F^{Mix}_{\Omega_1,\Omega_2}$ are given by previous works in [58]:

Lemma 2.6 (Theorem 3.4 in [58]). **(a)** *The far field operator* $F^{Mix}_{\Omega_1,\Omega_2}$ *has a factorization of the form*

$$F^{Mix}_{\Omega_1,\Omega_2} = -G^{Mix}_{\Omega_1,\Omega_2} T^{Mix}_{\Omega_1,\Omega_2}{}^{*} G^{Mix}_{\Omega_1,\Omega_2}{}^{*}. \tag{2.27}$$

(b) *The middle operator* $T^{Mix}_{\Omega_1,\Omega_2} : H^{-1/2}(\partial\Omega_1) \times H^{1/2}(\partial\Omega_2) \to H^{1/2}(\partial\Omega_1) \times H^{-1/2}(\partial\Omega_2)$ *is of the form*

$$T^{Mix}_{\Omega_1,\Omega_2} = \begin{pmatrix} S_{\Omega_1,i} & 0 \\ 0 & N_{\Omega_2,i} \end{pmatrix} + K, \tag{2.28}$$

where K *is some compact operator.*

(c) $\mathrm{Im}\langle T^{Mix}_{\Omega_1,\Omega_2}\varphi, \varphi\rangle \geq 0$ *for all* $\varphi \in H^{-1/2}(\partial\Omega_1) \times H^{1/2}(\partial\Omega_2)$*.*

Thirdly, we consider the far field operator $F^{Mix}_{\Omega_1 q,\Omega_2}$ for the problem (2.6)–(2.8). Here, Ω_1 is a penetrable medium modeled by a contrast function $q \in L^\infty(\Omega_1)$, and Ω_2 is an impenetrable obstacle with Dirichlet boundary condition. We define $G^{Mix}_{\Omega_1 q,\Omega_2} : L^2(\Omega_1) \times H^{1/2}(\partial\Omega_2) \to L^2(\mathbb{S}^2)$ by

$$G^{Mix}_{\Omega_1 q,\Omega_2}\begin{pmatrix} f \\ g \end{pmatrix} := v^\infty, \tag{2.29}$$

where v^∞ is the far field pattern of a radiating solution v such that

$$\Delta v + k^2(1+q)v = -k^2 \frac{q}{\sqrt{|q|}} f \text{ in } \mathbb{R}^3 \setminus \overline{\Omega_2}, \tag{2.30}$$

$$v = -g \text{ on } \partial\Omega_2. \tag{2.31}$$

The following properties of $F^{Mix}_{\Omega_1 q, \Omega_2}$ are given by previous works in [64]:

Lemma 2.7 (Theorem 3.2 and Theorem 3.3 in [64]). **(a)** *The far field operator $F^{Mix}_{\Omega_1 q, \Omega_2}$ has a factorization of the form*

$$F^{Mix}_{\Omega_1 q, \Omega_2} = G^{Mix}_{\Omega_1 q, \Omega_2} M^{Mix\,*}_{\Omega_1 q, \Omega_2} G^{Mix\,*}_{\Omega_1 q, \Omega_2}. \tag{2.32}$$

(b) *The middle operator $M^{Mix}_{\Omega_1 q, \Omega_2} : L^2(\Omega_1) \times H^{-1/2}(\partial\Omega_2) \to L^2(\Omega_1) \times H^{1/2}(\partial\Omega_2)$ is of the form*

$$M^{Mix}_{\Omega_1 q, \Omega_2} = \begin{pmatrix} \frac{|q|}{k^2 q} & 0 \\ 0 & -S_{\Omega_2, i} \end{pmatrix} + K, \tag{2.33}$$

where K is some compact operator.

(c) $\mathrm{Im}\langle \varphi, M^{Mix}_{\Omega_1 q, \Omega_2} \varphi \rangle \geq 0$ *for all* $\varphi \in L^2(\Omega_1) \times H^{-1/2}(\partial\Omega_2)$.

(d) *If* $M^{Mix}_{\Omega_1 q, \Omega_2} \varphi = 0$, $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in L^2(\Omega_1) \times H^{-1/2}(\partial\Omega_2)$, *then* $\varphi_1 = 0$.

Finally, we give the following functional analytic theorem behind the factorization method. The proof is completely analogous to previous works, e.g., Theorem 2.15 in [58].

Theorem 2.8. *Let $X \subset U \subset X^*$ be a Gelfand triple with a Hilbert space U and a reflexive Banach space X such that the imbedding is dense. Furthermore, let Y be a second Hilbert space and let $F : Y \to Y$, $G : X \to Y$, $T : X^* \to X$ be linear bounded operators such that*

$$F = GTG^*. \tag{2.34}$$

We make the following assumptions:

(1) *G is compact with dense range in Y.*

(2) *There exists $t \in [0, 2\pi]$ such that $\mathrm{Re}(\mathrm{e}^{it}T)$ has the form $\mathrm{Re}(\mathrm{e}^{it}T) = C + K$ with some compact operator K and some self-adjoint and positive coercive operator C, i.e., there exists $c > 0$ such that*

$$\langle \varphi, C\varphi \rangle \geq c \|\varphi\|^2 \text{ for all } \varphi \in X^*. \tag{2.35}$$

(3) $\mathrm{Im}\langle\varphi, T\varphi\rangle \geq 0$ *or* $\mathrm{Im}\langle\varphi, T\varphi\rangle \leq 0$ *for all* $\varphi \in X^*$.

Furthermore, we assume that one of the following assumptions:

(4) T *is injective.*

(5) $\mathrm{Im}\langle\varphi, T\varphi\rangle > 0$ *or* $\mathrm{Im}\langle\varphi, T\varphi\rangle < 0$ *for all* $\varphi \in \overline{\mathrm{Ran}(G^*)}$ *with* $\varphi \neq 0$.

Then, the operator $F_{\#} := \left|\mathrm{Re}(\mathrm{e}^{it}F)\right| + \left|\mathrm{Im}F\right|$ *is positive, and the ranges of* $G : X \to Y$ *and* $F_{\#}^{1/2} : Y \to Y$ *coincide with each other.*

Remark that, in this paper, the real part and the imaginary part of an operator A are self-adjoint operators given by

$$\mathrm{Re}(A) = \frac{A + A^*}{2} \quad \text{and} \quad \mathrm{Im}(A) = \frac{A - A^*}{2i}. \tag{2.36}$$

2.3 The first case

In section 2.3, we prove Theorem 2.2. Let Assumption 2.1 hold. We define $R_1 : H^{1/2}(\partial\Omega_1) \times H^{-1/2}(\partial\Omega_2) \to H^{1/2}(\partial\Omega_1) \times H^{1/2}(\partial B_2)$ by

$$R_1 \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} := \begin{pmatrix} f_1 \\ v_1\big|_{\partial B_2} \end{pmatrix}, \tag{2.37}$$

where v_1 is a radiating solution such that

$$\Delta v_1 + k^2 v_1 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, \tag{2.38}$$

$$v_1 = f_1 \text{ on } \partial\Omega_1, \qquad \frac{\partial v_1}{\partial \nu_{\Omega_2}} = g_1 \text{ on } \partial\Omega_2. \tag{2.39}$$

Then, from the definition of R_1, we obtain

$$G^{Mix}_{\Omega_1,\Omega_2} = G^{Dir}_{\Omega_1,B_2} R_1, \tag{2.40}$$

where $G^{Dir}_{\Omega_1,B_2} : H^{1/2}(\partial\Omega_1) \times H^{1/2}(\partial B_2) \to L^2(\mathbb{S}^2)$ is also defined for the pure Dirichlet boundary condition on Ω_1 and B_2 in the same way as $G^{Mix}_{\Omega_1,\Omega_2}$. (See (2.16).)

Next, we define $R_2 : H^{1/2}(\partial B_2) \to H^{1/2}(\partial\Omega_1) \times H^{1/2}(\partial B_2)$ by

$$R_2 f_2 := \begin{pmatrix} v_2\big|_{\partial\Omega_1} \\ f_2 \end{pmatrix}, \tag{2.41}$$

where v_2 is a radiating solution such that

$$\Delta v_2 + k^2 v_2 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B_2}, \tag{2.42}$$

$$v_2 = f_2 \text{ on } \partial B_2, \tag{2.43}$$

Then, from the definition of R_2, we obtain

$$G^{Dir}_{B_2} = G^{Dir}_{\Omega_1, B_2} R_2. \tag{2.44}$$

Here, take a positive number $\lambda_0 > 0$, and a bounded domain B_3 with $\overline{B_3} \subset B_2$. We define $R_3 : H^{-1/2}(\partial B_1 \cup \partial B_3) \to H^{1/2}(\partial \Omega_1) \times H^{1/2}(\partial B_2)$ by

$$R_3 f_3 := \begin{pmatrix} v_3\big|_{\partial \Omega_1} \\ v_3\big|_{\partial B_2} \end{pmatrix}, \tag{2.45}$$

where v_3 is a radiating solution such that

$$\Delta v_3 + k^2 v_3 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B_1 \cup B_3}, \tag{2.46}$$

$$\frac{\partial v_3}{\partial \nu_{B_1 \cup B_3}} + i\lambda_0 v_3 = f_3 \text{ on } \partial B_1 \cup \partial B_3. \tag{2.47}$$

Then, from the definition of R_3, we obtain

$$G^{Imp}_{B_1 \cup B_3, i\lambda_0} = G^{Dir}_{\Omega_1, B_2} R_3. \tag{2.48}$$

By (2.40), (2.44), (2.48), and the factorization of the far field operator in Section 2.2, we have

$$F^{Mix}_{\Omega_1, \Omega_2} + F^{Dir}_{B_2} + F^{Imp}_{B_1 \cup B_3, i\lambda_0} = G^{Dir}_{\Omega_1, B_2} T G^{Dir\ *}_{\Omega_1, B_2}, \tag{2.49}$$

where $T := \Big[-R_1 T^{Mix\ *}_{\Omega_1, \Omega_2} R_1^* - R_2 S^*_{B_2} R_2^* - R_3 T^{Imp\ *}_{B_1 \cup B_3, i\lambda_0} R_3^* \Big]$.

The following properties of $G^{Dir}_{\Omega_1, B_2}$ are given by the same argument in Theorem 1.12 and Lemma 1.13 in [58]:

Lemma 2.9. **(a)** *The operator $G^{Dir}_{\Omega_1, B_2} : H^{1/2}(\partial \Omega_1) \times H^{1/2}(\partial B_2) \to L^2(\mathbb{S}^2)$ is compact with dense range in $L^2(\mathbb{S}^2)$.*

(b) *For $z \in \mathbb{R}^3 \setminus \overline{B_2}$*

$$z \in \Omega_1 \Longleftrightarrow \phi_z \in \mathrm{Ran}(G^{Dir}_{\Omega_1, B_2}), \tag{2.50}$$

where the function ϕ_z is given by (2.11).

To prove Theorem 2.2, we apply Theorem 2.8 to this case. First of all, we show the following lemma:

Lemma 2.10. **(a)** $R_1 - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, *$R_2 - P_2$, R_3 are compact. Here, P_2 : $H^{1/2}(\partial B_2) \to H^{1/2}(\partial\Omega_1) \times H^{1/2}(\partial B_2)$ is defined by*

$$P_2 h := \begin{pmatrix} 0 \\ h \end{pmatrix}. \tag{2.51}$$

(b) *R_3^* is injective.*

Proof. (a) The mappings $R_1 - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : H^{1/2}(\partial\Omega_1) \times H^{-1/2}(\partial\Omega_2) \to H^1(\partial\Omega_1) \times H^1(\partial B_2)$, $R_2 - P_2 : H^{1/2}(\partial B_2) \to H^1(\partial\Omega_1) \times H^1(\partial B_2)$, and $R_3 : H^{-1/2}(\partial B_1 \cup \partial B_3) \to H^1(\partial\Omega_1) \times H^1(\partial B_2)$ are bounded since they are given by $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ v_1\big|_{\partial B_2} \end{pmatrix}$, $f_2 \mapsto \begin{pmatrix} v_2\big|_{\partial\Omega_1} \\ 0 \end{pmatrix}$, and $f_3 \mapsto \begin{pmatrix} v_3\big|_{\partial\Omega_1} \\ v_3\big|_{\partial B_2} \end{pmatrix}$, respectively. By Rellich theorem, they are compact.

(b) Let $\phi \in H^{-1/2}(\partial\Omega_1)$ and $\psi \in H^{-1/2}(\partial B_2)$. Assume that $R_3^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$. Using the same argument as done in Theorem 2.5 in [75], one knows the existence of a radiating solution w such that

$$\Delta w + k^2 w = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega_1 \cup B_2}, \tag{2.52}$$

$$\Delta w + k^2 w = 0 \text{ in } \Omega_1 \setminus \overline{B_1}, \text{ in } B_2 \setminus \overline{B_3}, \tag{2.53}$$

$$w_+ - w_- = 0, \qquad \frac{\partial w_+}{\partial \nu_{\Omega_1}} - \frac{\partial w_-}{\partial \nu_{\Omega_1}} = \overline{\phi} \text{ on } \partial\Omega_1, \tag{2.54}$$

$$w_+ - w_- = 0, \qquad \frac{\partial w_+}{\partial \nu_{B_2}} - \frac{\partial w_-}{\partial \nu_{B_2}} = \overline{\psi} \text{ on } \partial B_2, \tag{2.55}$$

$$\frac{\partial w}{\partial \nu_{B_1}} + i\lambda_0 w = 0 \text{ on } \partial B_1, \qquad \frac{\partial w}{\partial \nu_{B_3}} + i\lambda_0 w = 0 \text{ on } \partial B_3, \tag{2.56}$$

where the subscripts + and − denote the trace from the exterior and interior, respectively. (See Figure 3).

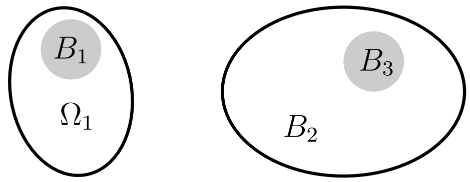


Figure 3: The inclusion relation of Ω_1, B_1, B_2, and B_3

By using the boundary conditions (2.47), (2.54), (2.55), (2.56), and Green's theorem, we have

$$
\begin{aligned}
0 &= \left\langle f_3, R_3^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} v_3\big|_{\partial\Omega_1} \\ v_3\big|_{\partial B_2} \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle \\
&= \int_{\partial\Omega_1} v_3\overline{\phi} ds + \int_{\partial B_2} v_3\overline{\psi} ds \\
&= \int_{\partial\Omega_1\cup\partial B_2} v_3\Big(\frac{\partial w_+}{\partial\nu} - \frac{\partial w_-}{\partial\nu}\Big) ds - \int_{\partial\Omega_1\cup\partial B_2} \frac{\partial v_3}{\partial\nu}(w_+ - w_-) ds \\
&= \int_{\partial\Omega_1\cup\partial B_2} \left[\frac{\partial v_3}{\partial\nu} w_- - v_3\frac{\partial w_-}{\partial\nu}\right] ds - \int_{\partial\Omega_1\cup\partial B_2} \left[\frac{\partial v_3}{\partial\nu} w_+ - v_3\frac{\partial w_+}{\partial\nu}\right] ds \\
&= \int_{\partial B_1} \left[\frac{\partial v_3}{\partial\nu_{B_1}} w - v_3\frac{\partial w}{\partial\nu_{B_1}}\right] ds + \int_{\partial B_3} \left[\frac{\partial v_3}{\partial\nu_{B_3}} w - v_3\frac{\partial w}{\partial\nu_{B_3}}\right] ds \\
&= \int_{\partial B_1\cup\partial B_3} f_3 w ds, \qquad (2.57)
\end{aligned}
$$

which proves that $w = 0$ in $\partial B_1 \cup \partial B_3$. Holmgren's uniqueness theorem implies that w vanishes in $\Omega_1 \setminus \overline{B_1}$ and $B_2 \setminus \overline{B_3}$. Equations (2.54) and (2.55) yield $w_+ = 0$ on $\partial\Omega_1 \cup \partial B_2$ which implies that w vanishes also outside of Ω_1 and B_2 by the uniqueness of the exterior Dirichlet problem. Therefore, equations (2.54) and (2.55) yield $\phi = 0$ and $\psi = 0$. □

By Lemma 2.10, the middle operator T of (2.49) has the following properties:

Lemma 2.11. (a) $\mathrm{Re}\big(\mathrm{e}^{i\pi}T\big)$ *has the form* $\mathrm{Re}\big(\mathrm{e}^{i\pi}T\big) = C + K$ *with some self-adjoint and positive coercive operator* C *and some compact operator* K.

(b) $\mathrm{Im}\langle\varphi, T\varphi\rangle < 0$ *for all* $\varphi \in H^{-1/2}(\partial\Omega_1) \times H^{-1/2}(\partial B_2)$ *with* $\varphi \neq 0$.

Proof. (a) By Lemma 2.5 (b), Lemma 2.6 (b), and Lemma 2.10 (a),

$$\begin{aligned}
\mathrm{Re}\big(\mathrm{e}^{i\pi}T\big) &= \mathrm{Re}\Big(R_1 T^{Mix}_{\Omega_1,\Omega_2}{}^{*} R_1^* + R_2 S^*_{B_2} R_2^* + R_3 T^{Imp}_{B_1\cup B_3, i\lambda_0}{}^{*} R_3^*\Big) \\
&= R_1 \begin{pmatrix} S_{\Omega_1,i} & 0 \\ 0 & N_{\Omega_2,i} \end{pmatrix} R_1^* + R_2 S_{B_2,i} R_2^* + K \\
&= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_{\Omega_1,i} & 0 \\ 0 & N_{\Omega_2,i} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + P_2 S_{B_2,i} P_2^* + K' \\
&= \begin{pmatrix} S_{\Omega_1,i} & 0 \\ 0 & S_{B_2,i} \end{pmatrix} + K', \qquad (2.58)
\end{aligned}$$

where K and K' are some compact operators. Since the boundary integral operators $S_{\Omega_1,i}$ and $S_{B_2,i}$ are self-adjoint and positive coercive, (a) holds.

(b) By Lemma 2.5 (c) (d), Lemma 2.6 (c), and Lemma 2.10 (b), especially, by the strictly positivity of the operator $\mathrm{Im}T^{Imp}_{B_1\cup B_3,i\lambda_0}$, and the injectivity of R_3^*, for all $\varphi \in H^{-1/2}(\partial\Omega_1)\times H^{-1/2}(\partial B_2)$ with $\varphi \neq 0$, we have

$$\begin{aligned}
\mathrm{Im}\langle\varphi, T\varphi\rangle &= -\mathrm{Im}\langle T^{Mix}_{\Omega_1,\Omega_2} R_1^*\varphi, R_1^*\varphi\rangle + \mathrm{Im}\langle R_2^*\varphi, S_{B_2} R_2^*\varphi\rangle \\
&\quad -\mathrm{Im}\langle T^{Imp}_{B_1\cup B_3,i\lambda_0} R_3^*\varphi, R_3^*\varphi\rangle < 0. \qquad (2.59)
\end{aligned}$$

□

Therefore, by Lemma 2.11, we can apply Theorem 2.8 to this case. From Lemma 2.9 (b), and applying Theorem 2.8, we obtain Theorem 1.2.

Remark 2.12. Unknown obstacle Ω_2 may consist of finitely many connected components whose closures are mutually disjoint. Furthermore, the boundary condition on Ω_2 can not be only Neumann but also Dirichlet, impedance, and not only impenetrable obstacles but also penetrable mediums, and their mixed situations by the same argument in Theorem 2.2. In all cases, we can choose arbitrary wave numbers $k > 0$.

Remark 2.13. If we assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in artificial domains B_1, B_2, then we do not need to take an additional domain B_3. In such a case, we only use $F^{Dir}_{B_1\cup B_2}$ as artificial far field operators since $F^{Dir}_{B_1\cup B_2}$ has a role to keep the strictly positivity of the imaginary part of the middle operator of F. (See Lemma 2.5 (c).) That is, we can give the following characterization by the same argument in Theorem 2.2:

Theorem 2.14. *In addition to* Assumption 2.1*, we assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B_1, B_2. Take a positive number $\lambda_0 > 0$.*

Then, for $z \in \mathbb{R}^3 \setminus \overline{B_2}$

$$z \in \Omega_1 \Longleftrightarrow \sum_{n=1}^{\infty} \frac{|(\phi_z, \varphi_n)_{L^2(\mathbb{S}^2)}|^2}{\lambda_n} < \infty, \tag{2.60}$$

where (λ_n, φ_n) is a complete eigensystem of $F_{\#}$ given by

$$F_{\#} := \left|\mathrm{Re}F\right| + \left|\mathrm{Im}F\right|, \tag{2.61}$$

where $F := F^{Mix}_{\Omega_1,\Omega_2} + F^{Dir}_{B_1 \cup B_2}$. Here, the function ϕ_z is given by (2.11).

Remark 2.15. We can also give the characterization of the Neumann part Ω_2 if we assume $\overline{\Omega_1} \subset B_1$, $\overline{B_2} \subset \Omega_2$, $\overline{B_1} \cap \overline{\Omega_2} = \emptyset$ by the same argument in Theorem 2.2 (See Figure 4).

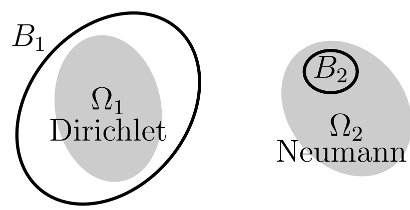


Figure 4: Assumption of Remark 2.15

2.4 The second case

In Section 2.4, we prove Theorem 2.4. Let Assumption 2.3 hold. We define $G^{Mix}_{\Omega_1 0, B_2} : L^2(\Omega_1) \times H^{1/2}(\partial B_2) \to L^2(\mathbb{S}^2)$ by

$$G^{Mix}_{\Omega_1 0, B_2} \begin{pmatrix} f \\ g \end{pmatrix} := v^{\infty}, \tag{2.62}$$

where v^{∞} is the far field pattern of a radiating solution v such that

$$\Delta v + k^2 v = -k^2 \frac{q}{\sqrt{|q|}} f \text{ in } \mathbb{R}^3 \setminus \overline{B_2}, \tag{2.63}$$

$$v = g \text{ on } \partial B_2. \tag{2.64}$$

Note that we extend q by zero outside Ω_1. Next, we define $R_1 : L^2(\Omega_1) \times H^{1/2}(\partial \Omega_2) \to L^2(\Omega_1) \times H^{1/2}(\partial B_2)$ by

$$R_1 \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} := \begin{pmatrix} f_1 + \sqrt{|q|} v_1 \\ v_1\big|_{\partial B_2} \end{pmatrix}, \tag{2.65}$$

where v_1 is a radiating solution such that

$$\Delta v_1 + k^2(1+q)v_1 = -k^2 \frac{q}{\sqrt{|q|}} f_1 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega_2}, \tag{2.66}$$

$$v_1 = -g_1 \text{ on } \partial\Omega_2. \tag{2.67}$$

Then, from the definition of R_1, we obtain

$$G^{Mix}_{\Omega_1 q, \Omega_2} = G^{Mix}_{\Omega_1 0, B_2} R_1. \tag{2.68}$$

We define $R_2 : H^{1/2}(\partial B_2) \to L^2(\Omega_1) \times H^{1/2}(\partial B_2)$ by

$$R_2 f_2 := \begin{pmatrix} 0 \\ f_2 \end{pmatrix}. \tag{2.69}$$

Then, from the definition of R_2, we obtain

$$G^{Dir}_{B_2} = G^{Mix}_{\Omega_1 0, B_2} R_2. \tag{2.70}$$

Here, take a positive number $\lambda_0 > 0$, and a bounded domain B_3 with $\overline{B_3} \subset B_2$. We define $R_3 : H^{-1/2}(\partial B_3) \to H^{1/2}(\partial B_2)$ by

$$R_3 f_3 := v_3\big|_{\partial B_2}, \tag{2.71}$$

where v_3 is a radiating solution such that

$$\Delta v_3 + k^2 v_3 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B_3}, \tag{2.72}$$

$$\frac{\partial v_3}{\partial \nu_{B_3}} + i\lambda_0 v_3 = f_3 \text{ on } \partial B_3. \tag{2.73}$$

Then, from the definition of R_3, and (2.70), we obtain

$$G^{Imp}_{B_3, i\lambda_0} = G^{Dir}_{B_2} R_3 = G^{Mix}_{\Omega_1 0, B_2} R_2 R_3. \tag{2.74}$$

By (2.68), (2.70), (2.74), and the factorization of the far field operator in Section 2.2, we have

$$F^{Mix}_{\Omega_1 q, \Omega_2} + F^{Dir}_{B_2} + F^{Imp}_{B_3, i\lambda_0} = G^{Mix}_{\Omega_1 0, B_2} M G^{Mix\ *}_{\Omega_1 0, B_2}, \tag{2.75}$$

where $M := \Big[R_1 M^{Mix\ *}_{\Omega_1 q, \Omega_2} R_1^* - R_2 S^*_{B_2} R_2^* - R_2 R_3 T^{Imp\ *}_{B_3, i\lambda_0} R_3^* R_2^* \Big]$.

The following properties are given by the same argument in Theorem 3.2 (c) in [64]:

Lemma 2.16. **(a)** *The operator* $G^{Mix}_{\Omega_1 0, B_2} : L^2(\Omega_1) \times H^{1/2}(\partial B_2) \to L^2(\mathbb{S}^2)$ *is compact with dense range in* $L^2(\mathbb{S}^2)$.

(b) *For* $z \in \mathbb{R}^3 \setminus \overline{B_2}$

$$z \in \Omega_1 \Longleftrightarrow \phi_z \in \mathrm{Ran}(G^{Mix}_{\Omega_1 0, B_2}), \tag{2.76}$$

where the function ϕ_z *is given by* (2.11).

To prove Theorem 2.4, we apply Theorem 2.8 to this case with $F = F^{Mix\ *}_{\Omega_1 q, \Omega_2} + F^{Dir\ *}_{B_2} + F^{Imp\ *}_{B_3, i\lambda_0}$. First, we show the following lemma:

Lemma 2.17. **(a)** $R_1 - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, R_3 *are compact.*

(b) R_1 *is injective.*

(c) R_3^* *is injective.*

Proof. (a) The mappings $R_1 - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : L^2(\Omega_1) \times H^{1/2}(\partial\Omega_2) \to H^1(\Omega_1) \times H^1(\partial B_2)$, and $R_3 : H^{-1/2}(\partial B_3) \to H^1(\partial B_2)$ are bounded since they are given by $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{|q|} v_1 \\ v_1\big|_{\partial B_2} \end{pmatrix}$, and $f_3 \mapsto v_3\big|_{\partial B_2}$, respectively. By Rellich theorem, they are compact.

(b) Assume that

$$R_1 \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = \begin{pmatrix} f_1 + \sqrt{|q|} v_1 \\ v_1\big|_{\partial B_2} \end{pmatrix} = 0. \tag{2.77}$$

Equation (2.66) yields that

$$\Delta v_1 + k^2 v_1 = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B_2}, \tag{2.78}$$

$$v_1 = 0 \text{ on } \partial B_2. \tag{2.79}$$

By the uniqueness of the exterior Dirichlet problem, v_1 vanishes outside of B_2. Therefore, $f_1 = 0$. Furthermore, the analyticity of v_1 yields that v_1 also vanishes in $B_2 \setminus \overline{\Omega_2}$, which implies that $g_1 = 0$.

(c) The injectivity of R_3^* follows from the same argument as done in the proof of Lemma 3.2 in [63]. $\square$

By Lemma 2.17, the middle operator M of (2.75) has the following properties:

Lemma 2.18. **(a)** $\mathrm{Re}\big(\mathrm{e}^{it}M^{*}\big)$ *has the form* $\mathrm{Re}\big(\mathrm{e}^{it}M^{*}\big)=C+K$ *with some self-adjoint and positive coercive operator* C*, and some compact operator* K*.*

(b) $\mathrm{Im}\langle\varphi, M^{*}\varphi\rangle \geq 0$ *for all* $\varphi \in L^{2}(\Omega_1)\times H^{-1/2}(\partial B_2)$.

(c) M^{*} *is injective.*

Proof. (a) By Lemma 2.5 (b), Lemma 2.7 (b), and Lemma 2.17 (a),

$$\begin{aligned}
\mathrm{Re}\big(\mathrm{e}^{it}M^{*}\big) &= \mathrm{Re}\Big(\mathrm{e}^{it}R_1 M^{Mix}_{\Omega_1 q,\Omega_2}R_1^{*} - \mathrm{e}^{it}R_2 S_{B_2}R_2^{*} - \mathrm{e}^{it}R_2R_3T^{Imp}_{B_3,i\lambda_0}R_3^{*}R_2^{*}\Big)\\
&= R_1\begin{pmatrix} \mathrm{Re}(\frac{\mathrm{e}^{it}|q|}{k^2 q}) & 0\\ 0 & -(\cos\,\mathrm{t})S_{\Omega_2,i}\end{pmatrix}R_1^{*} - R_2(\cos\,\mathrm{t})S_{B_2,i}R_2^{*}+K\\
&= \begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} \mathrm{Re}(\frac{\mathrm{e}^{it}|q|}{k^2 q}) & 0\\ 0 & -(\cos\,\mathrm{t})S_{\Omega_2,i}\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}\\
&\quad -R_2(\cos\,\mathrm{t})S_{B_2,i}R_2^{*}+K'\\
&= \begin{pmatrix} \mathrm{Re}(\frac{\mathrm{e}^{it}|q|}{k^2 q}) & 0\\ 0 & (-\cos\,\mathrm{t})S_{B_2,i}\end{pmatrix}+K', \qquad (2.80)
\end{aligned}$$

where K and K' are some compact operators. The first term of the right hand side in (2.80) is self-adjoint and positive coercive since $(-\cos\,\mathrm{t})>0$ when $t\in(\pi/2,3\pi/2)$, and Assumption 2.3 (iv) yields

$$\begin{aligned}
\Big\langle\varphi, \mathrm{Re}\big(\frac{\mathrm{e}^{it}|q|}{k^2 q}\big)\varphi\Big\rangle &= \int_{\Omega_1}|\varphi|^2\frac{\mathrm{Re}(\mathrm{e}^{-it}q)}{k^2|q|}dx\\
&\geq \int_{\Omega_1}|\varphi|^2\frac{C|q|}{k^2|q|}dx\\
&= \frac{C}{k^2}\left\|\varphi\right\|^2_{L^2(\Omega_1)}. \qquad (2.81)
\end{aligned}$$

(b) By Lemma 2.5 (c), Lemma 2.7 (c) (d), for all $\varphi\in L^2(\Omega_1)\times H^{-1/2}(\partial B_2)$

$$\begin{aligned}
\mathrm{Im}\langle\varphi, M^{*}\varphi\rangle &= \mathrm{Im}\langle R_1^{*}\varphi, M^{Mix}_{\Omega_1 q,\Omega_2}R_1^{*}\varphi\rangle - \mathrm{Im}\langle R_2^{*}\varphi, S_{B_2}R_2^{*}\varphi\rangle\\
&\quad +\mathrm{Im}\langle T^{Imp}_{B_3,i\lambda_0}R_3^{*}R_2^{*}\varphi, R_3^{*}R_2^{*}\varphi\rangle \geq 0. \qquad (2.82)
\end{aligned}$$

(c) Let $\phi \in L^2(\Omega_1)$ and $\psi \in H^{-1/2}(\partial B_2)$. Assume that $M^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$. Inequality (2.82) yields that

$$\mathrm{Im}\Big\langle T^{Imp}_{B_3,i\lambda_0} R_3^* R_2^* \begin{pmatrix} \phi \\ \psi \end{pmatrix}, R_3^* R_2^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} \Big\rangle = 0, \tag{2.83}$$

which implies that $R_3^* R_2^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$ from Lemma 2.5 (d). By Lemma 2.17 (c), and the definition of R_2, we have $\psi = 0$. Therefore,

$$M^* \begin{pmatrix} \phi \\ 0 \end{pmatrix} = R_1 M^{Mix}_{\Omega_1 q, \Omega_2} R_1^* \begin{pmatrix} \phi \\ 0 \end{pmatrix} = 0. \tag{2.84}$$

From Lemma 2.17 (b) and Lemma 2.7 (d), we obtain

$$R_1^* \begin{pmatrix} \phi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}. \tag{2.85}$$

Finally, we will show $\phi = 0$. Let $f_1 \in L^2(\Omega_1)$. Take radiating solutions v_1 and w such that

$$\Delta v_1 + k^2(1+q)v_1 = -k^2 \frac{q}{\sqrt{|q|}} f_1 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega_2}, \tag{2.86}$$

$$v_1 = 0 \text{ on } \partial\Omega_2, \tag{2.87}$$

$$\Delta w + k^2(1+q)w = \sqrt{|q|}\overline{\phi} \text{ in } \mathbb{R}^3 \setminus \overline{\Omega_2}, \tag{2.88}$$

$$w = 0 \text{ on } \partial\Omega_2. \tag{2.89}$$

By (2.85),

$$\begin{aligned} 0 &= \Big\langle \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, R_1^* \begin{pmatrix} \phi \\ 0 \end{pmatrix} \Big\rangle = \Big\langle \begin{pmatrix} f_1 + \sqrt{|q|} v_1 \\ v_1\big|_{\partial B_2} \end{pmatrix}, \begin{pmatrix} \phi \\ 0 \end{pmatrix} \Big\rangle \\ &= \int_{\Omega_1} f_1 \overline{\phi} dx + \int_{\Omega_1} v_1 \sqrt{|q|} \overline{\phi} dx. \end{aligned} \tag{2.90}$$

By (2.86) and (2.88),

$$\begin{aligned} \int_{\Omega_1} v_1 \sqrt{|q|}\overline{\phi} dx &= \int_{\Omega_1} v_1 \big(\Delta w + k^2(1+q)w\big) dx \\ &\quad - \int_{\Omega_1} \Big(\Delta v_1 + k^2(1+q)v_1 + k^2 \frac{q}{\sqrt{|q|}} f_1\Big) w dx \\ &= - \int_{\Omega_1} k^2 \frac{q}{\sqrt{|q|}} f_1 w dx \\ &\quad + \int_{\Omega_1} (\Delta w) v_1 - w(\Delta v_1) dx. \end{aligned} \tag{2.91}$$

By using Green's theorem, (2.87) and (2.89),

$$\begin{aligned}
\int_{\Omega_1} (\Delta w)v_1 - w(\Delta v_1)dx &= \int_{\mathbb{R}^3\setminus\overline{\Omega_2}} (\Delta w)v_1 - w(\Delta v_1)dx \\
&= -\int_{\partial\Omega_2} \left[\frac{\partial w}{\partial \nu_{\Omega_2}} v_1 - w \frac{\partial v}{\partial \nu_{\Omega_2}} \right] ds \\
&= 0. \qquad (2.92)
\end{aligned}$$

By (2.90)–(2.92),

$$\overline{\phi} = k^2 \frac{q}{\sqrt{|q|}} w \text{ in } \Omega_1. \qquad (2.93)$$

From (2.88), (2.89), and (2.93), we obtain

$$\Delta w + k^2 w = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega_2}, \qquad (2.94)$$

$$w = 0 \text{ on } \partial\Omega_2, \qquad (2.95)$$

which proves that w vanishes in $\mathbb{R}^3 \setminus \overline{\Omega_2}$ by the uniqueness of the exterior Dirichlet problem. Therefore, equation (2.93) yields that $\phi = 0$. $\square$

Therefore, by Lemma 2.18, we can apply Theorem 2.8 to this case with $F = F^{Mix\ *}_{\Omega_1 q, \Omega_2} + F^{Dir\ *}_{B_2} + F^{Imp\ *}_{B_3, i\lambda_0}$. From Lemma 2.16 (b), and applying Theorem 2.8, we obtain Theorem 2.4.

Remark 2.19. We can also consider various situations on Ω_2 like Remark 2.12, and replace the assumption of taking B_3 with that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in an artificial domain B_2 like Remark 2.15.

We can also give the characterization by replacing (iv) in Assumption 2.3 with

(iv') *There exists* $t \in [0, \pi/2) \cup (3\pi/2, 2\pi]$ *and* $C > 0$ *such that* $\mathrm{Re}(\mathrm{e}^{-it}q) \geq C|q|$ *a.e. in* Ω_1.

by the same argument in Theorem 2.4:

Assumption 2.20. *Let a bounded domain* B_2 *be a priori known. Assume the following assumptions:*

(i) $q \in L^\infty(\Omega_1)$ *with* $\mathrm{Im} q \geq 0$ *in* Ω_1.

(ii) $|q|$ *is locally bounded below in* Ω_1, *i.e., for every compact subset* $M \subset \Omega_1$, *there exists* $c > 0$ *(depend on* M*) such that* $|q| \geq c$ in M.

(iii) $\overline{\Omega_2} \subset B_2$, $\overline{\Omega_1} \cap \overline{B_2} = \emptyset$.

(iv') *There exists* $t \in [0, \pi/2) \cup (3\pi/2, 2\pi]$ *and* $C > 0$ *such that* $\mathrm{Re}(\mathrm{e}^{-it} q) \geq C|q|$ *a.e. in* Ω_1.

Theorem 2.21. *Let* Assumption 2.20 *hold. Take a positive number* $\lambda_0 > 0$. *Then, for* $z \in \mathbb{R}^3 \setminus \overline{B_2}$

$$z \in \Omega_1 \Longleftrightarrow \sum_{n=1}^{\infty} \frac{\left|(\phi_z, \varphi_n)_{L^2(\mathbb{S}^2)}\right|^2}{\lambda_n} < \infty, \tag{2.96}$$

where (λ_n, φ_n) *is a complete eigensystem of* $F_{\#}$ *given by*

$$F_{\#} := \left|\mathrm{Re}\left(\mathrm{e}^{-it} F\right)\right| + \left|\mathrm{Im} F\right|, \tag{2.97}$$

where $F := F^{Mix}_{\Omega_1 q, \Omega_2} + F^{Imp}_{B_2, i\lambda_0}$. *Here, the function* ϕ_z *is given by* (2.11).

3 The monotonicity method for the inverse crack scattering problem

3.1 Introduction

Let $\Gamma \subset \mathbb{R}^2$ be a smooth non-intersecting open arc, and we assume that Γ can be extended to an arbitrary smooth, simply connected, closed curve $\partial\Omega$ enclosing a bounded domain Ω in $\mathbb{R}^2$. Let $k > 0$ be the wave number, and let $\theta \in \mathbb{S}^1$ be incident direction, where $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ denotes the unit sphere in $\mathbb{R}^2$. We consider the following direct scattering problem: For $\theta \in \mathbb{S}^1$ determine u^s such that

$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^2 \setminus \Gamma, \tag{3.1}$$

$$u^s = -\mathrm{e}^{ik\theta\cdot x} \text{ on } \Gamma \tag{3.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{3.3}$$

where $r = |x|$, and (3.3) is the *Sommerfeld radiation condition.* Precisely, this problem is understood in the variational form, that is, determine $u^s \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{\Gamma})$ satisfying $u^s\big|_{\Gamma} = -\mathrm{e}^{ik\theta\cdot x}$, the Sommerfeld radiation condition (1.3), and

$$\int_{\mathbb{R}^2\setminus\Gamma} \left[\nabla u^s \cdot \nabla\overline{\varphi} - k^2 u^s \overline{\varphi}\right] dx = 0, \tag{3.4}$$

for all $\varphi \in H^1(\mathbb{R}^2 \setminus \overline{\Gamma})$, $\varphi\big|_{\Gamma} = 0$, with compact support. Here, $H^1_{loc}(\mathbb{R}^2 \setminus \overline{\Gamma}) = \{u : \mathbb{R}^2 \setminus \overline{\Gamma} \to \mathbb{C} : u\big|_{B\setminus\overline{\Gamma}} \in H^1(B\setminus\overline{\Gamma}) \text{ for all open balls } B \text{ including } \Gamma\}$ denotes the local Sobolev space of one order.

It is well known that there exists a unique solution u^s and it has the following asymptotic behaviour (see, e.g. [18]):

$$u^s(x) = \frac{\mathrm{e}^{ikr}}{\sqrt{r}}\Big\{u^\infty(\hat{x}, \theta) + O\big(1/r\big)\Big\},\ r \to \infty, \quad \hat{x} := \frac{x}{|x|}. \tag{3.5}$$

The function u^∞ is called the *far field pattern* of u^s. With the far field pattern u^∞, we define the *far field operator* $F : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ by

$$Fg(\hat{x}) := \int_{\mathbb{S}^1} u^\infty(\hat{x}, \theta) g(\theta) ds(\theta),\ \hat{x} \in \mathbb{S}^1. \tag{3.6}$$

The inverse scattering problem we consider is to reconstruct the unknown arc Γ from the far field pattern $u^\infty(\hat{x}, \theta)$ for all $\hat{x} \in \mathbb{S}^1$, all $\hat{x} \in \mathbb{S}^1$ with one $k > 0$. In other words, given the far field operator F, reconstruct Γ.

In order to solve such an inverse problem, we use the idea of the monotonicity method. The feature of this method is to understand the inclusion relation of an unknown object and artificial one by comparing the data operator with some operator corresponding to an artificial object. For electrical impedance tomography (EIT) we refer to [40], for the inverse boundary value problem for the Helmholtz equation we refer to [37, 38, 39], and for the inverse medium scattering problem we refer to [33, 69].

Our aim in this section is to provide the following two theorems.

Theorem 3.1. *Let $\sigma \subset \mathbb{R}^2$ be a smooth non-intersecting open arc. Then,*

$$\sigma \subset \Gamma \quad \Longleftrightarrow \quad H_\sigma^* H_\sigma \leq_{\mathrm{fin}} -\mathrm{Re}F, \tag{3.7}$$

where the Herglotz operator $H_\sigma : L^2(\mathbb{S}^1) \to L^2(\sigma)$ is given by

$$H_\sigma g(x) := \int_{\mathbb{S}^1} \mathrm{e}^{ik\theta\cdot x} g(\theta) ds(\theta),\ x \in \sigma, \tag{3.8}$$

and the inequality on the right-hand side in (3.7) denotes that $-\mathrm{Re}F - H_\sigma^* H_\sigma$ has only finitely many negative eigenvalues, and the real part of an operator A is self-adjoint operators given by $\mathrm{Re}(A) := \frac{1}{2}(A + A^*)$.

Theorem 3.2. *Let $B \subset \mathbb{R}^2$ be a bounded open set. Then,*

$$\Gamma \subset B \quad \Longleftrightarrow \quad -\mathrm{Re}F \leq_{\mathrm{fin}} \tilde{H}_{\partial B}^* \tilde{H}_{\partial B}, \tag{3.9}$$

where $\tilde{H}_{\partial B} : L^2(\mathbb{S}^1) \to H^{1/2}(\partial B)$ is given by

$$\tilde{H}_{\partial B} g(x) := \int_{\mathbb{S}^1} \mathrm{e}^{ik\theta\cdot x} g(\theta) ds(\theta),\ x \in \partial B. \tag{3.10}$$

Theorem 3.1 determines whether an artificial open arc σ is contained in Γ or not. While, Theorem 3.2 determines an artificial domain B contains Γ. In two theorems we can understand Γ from the inside and outside.

This section is organized as follows. In Section 3.2, we give a rigorous definition of the above inequality. Furthermore, we recall the properties of the far field operator and technical lemmas which are useful to prove main results. In Sections 3.3 and 3.4, we prove Theorems 3.1 and 3.2 respectively. In Section 3.5, we give numerical examples based on Theorem 3.1.

3.2 Preliminary

First, we give a rigorous definition of the inequality in Theorems 3.1 and 3.2.

Definition 3.3. Let $A, B : X \to X$ be self-adjoint compact linear operators on a Hilbert space X. We write

$$A \leq_{\text{fin}} B, \tag{3.11}$$

if $B - A$ has only finitely many negative eigenvalues.

The following lemma was shown in Corollary 3.3 of [38].

Lemma 3.4. *Let $A, B : X \to X$ be self-adjoint compact linear operators on a Hilbert space X with an inner product $\langle \cdot, \cdot \rangle$. Then, the following statements are equivalent:*

(a) $A \leq_{\text{fin}} B$

(b) *There exists a finite dimensional subspace V in X such that*

$$\langle (B - A)v, v \rangle \geq 0, \tag{3.12}$$

for all $v \in V^{\perp}$.

Secondly, we define several operators in order to mention properties of the far field operator F. The data-to-pattern operator $G : H^{1/2}(\Gamma) \to L^2(\mathbb{S}^1)$ is defined by

$$Gf := v^{\infty}, \tag{3.13}$$

where v^{∞} is the far field pattern of a radiating solution v (that is, v satisfies the Sommerfeld radiation condition) such that

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^2 \setminus \Gamma, \tag{3.14}$$

$$v = f \text{ on } \Gamma. \tag{3.15}$$

The following lemma was given by the same argument in Lemma 1.13 of [58].

Lemma 3.5. *The data-to-pattern operator G is compact and injective.*

We define the single layer boundary operator $S : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ by

$$S\varphi(x) := \int_\Gamma \varphi(y)\Phi(x,y)ds(y), \ x \in \Gamma, \tag{3.16}$$

where $\Phi(x,y)$ denotes the fundamental solution to Helmholtz equation in $\mathbb{R}^2$, i.e.,

$$\Phi(x,y) := \frac{i}{4}H_0^{(1)}(k|x-y|), \ x \neq y. \tag{3.17}$$

Here, we denote by

$$H^{1/2}(\Gamma) := \{u\big|_\Gamma : u \in H^{1/2}(\partial\Omega)\}, \tag{3.18}$$

$$\tilde{H}^{1/2}(\Gamma) := \{u \in H^{1/2}(\partial\Omega) : supp(u) \subset \overline{\Gamma}\}, \tag{3.19}$$

and $H^{-1/2}(\Gamma)$ and $\tilde{H}^{-1/2}(\Gamma)$ the dual spaces of $\tilde{H}^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ respectively. Then, we have the following inclusion relation:

$$\tilde{H}^{1/2}(\Gamma) \subset H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma). \tag{3.20}$$

For these details, we refer to [76]. The following two Lemmas was shown in Section 3 of [66].

Lemma 3.6. **(a)** *S is an isomorphism from $\tilde{H}^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$.*

(b) *Let S_i be the boundary integral operator* (3.16) *corresponding to the wave number $k = i$. The operator S_i is self-adjoint and coercive, i.e, there exists $c_0 > 0$ such that*

$$\langle \varphi, S_i\varphi \rangle \geq c_0 \|\varphi\|^2_{\tilde{H}^{-1/2}(\Gamma)} \ \ for\ all\ \varphi \in \tilde{H}^{-1/2}(\Gamma), \tag{3.21}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $\langle \tilde{H}^{-1/2}(\Gamma), H^{1/2}(\Gamma)\rangle$.

(c) *$S - S_i$ is compact.*

(d) *There exists a self-adjoint and positive square root $S_i^{1/2} : L^2(\Gamma) \to L^2(\Gamma)$ of S_i which can be extended such that $S_i^{1/2} : \tilde{H}^{-1/2}(\Gamma) \to L^2(\Gamma)$ is an isomorphism and $S_i^{1/2\,*} S_i^{1/2} = S_i$.*

Lemma 3.7. *The far field operator F has the following factorization:*

$$F = -GS^*G^*. \tag{3.22}$$

where $G^* : L^2(\mathbb{S}^1) \to \tilde{H}^{-1/2}(\Gamma)$ and $S^* : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ are the adjoints of G and S, respectively.

Thirdly, we recall the following technical lemmas which will be useful to prove Theorems 3.1 and 3.2. We refer to Lemma 4.6 and 4.7 in [38].

Lemma 3.8. *Let X, Y, and Z be Hilbert spaces, and let $A : X \to Y$ and $B : X \to Z$ be bounded linear operators. Then,*

$$\exists C > 0 : \ \|Ax\|^2 \leq C \|Bx\|^2 \ \textit{for all } x \in X \quad \Longleftrightarrow \quad \mathrm{Ran}(A^*) \subseteq \mathrm{Ran}(B^*). \tag{3.23}$$

Lemma 3.9. *Let X, Y, $V \subset Z$ be subspaces of a vector space Z. If*

$$X \cap Y = \{0\}, \quad \textit{and} \quad X \subseteq Y + V, \tag{3.24}$$

then $\dim(X) \leq \dim(V)$.

3.3 Proof of Theorem 3.1

In Section 3.3, we will show Theorem 3.1. Let $\sigma \subset \Gamma$. We denote by $R : L^2(\Gamma) \to L^2(\sigma)$ the restriction operator, $J : H^{1/2}(\Gamma) \to L^2(\Gamma)$ the compact embedding, and $H : L^2(\mathbb{S}^1) \to L^2(\Gamma)$, $\hat{H} : L^2(\mathbb{S}^1) \to H^{1/2}(\Gamma)$ the Herglotz operators, respectively. Since $e^{-ik\hat{x}\cdot y}$ is a far field pattern of $\Phi(x, y)$, we have by definitions of G and S

$$GS\varphi(\hat{x}) = \int_{\Gamma} e^{-ik\hat{x}\cdot y}\varphi(y)ds(y). \tag{3.25}$$

The right-hand side is identical with $\hat{H}^*\varphi(\hat{x})$ (see the proof of Lemma 3.4 in [66]). Then, we have $\hat{H}^* = GS$. By this equality we have

$$\begin{aligned} H_\sigma &= RH \\ &= RJ\hat{H} \\ &= RJS^*G^*. \end{aligned} \tag{3.26}$$

Using (3.25) and Lemmas 3.6 and 3.7, $-\mathrm{Re}F - H_\sigma^* H_\sigma$ has the following factorization:

$$\begin{aligned} -\mathrm{Re}F - H_\sigma^* H_\sigma &= G\big[\mathrm{Re}S - SJ^*R^*RJS^*\big]G^* \\ &= G\big[S_i + \mathrm{Re}(S - S_i) - SJ^*R^*RJS^*\big]G^* \\ &= \big[GW^*\big]W^{*\ -1}\big[S_i + \mathrm{Re}(S - S_i) - SJ^*R^*RJS^*\big]W^{-1}\big[GW^*\big]^* \\ &= \big[GW^*\big]\big[I_{L^2(\Gamma)} + K\big]\big[GW^*\big]^*, \end{aligned} \tag{3.27}$$

where $W := S_i^{1/2} : \tilde{H}^{-1/2}(\Gamma) \to L^2(\Gamma)$ is an extension of the square root of $S_i^{1/2}$, $K := W^{*\ -1}\big[\mathrm{Re}(S - S_i) - SJ^*R^*RJS^*\big]W^{-1} : L^2(\Gamma) \to L^2(\Gamma)$ is self-adjoint compact, and $I_{L^2(\Gamma)}$ is the identity operator on $L^2(\Gamma)$. Let V be the sum of eigenspaces of K associated to eigenvalues less than $-1/2$. Then, V is a finite dimensional and

$$\langle (I_{L^2(\Gamma)} + K\big)v, v\rangle \geq 0, \tag{3.28}$$

for all $v \in V^{\perp}$. Since for $g \in L^2(\mathbb{S}^1)$

$$\big[GW^*\big]^* g \in V^{\perp} \quad \Longleftrightarrow \quad g \in [(GW^*)V\big]^{\perp}, \tag{3.29}$$

and $\dim[(GW^*)V\big] \leq \dim(V) < \infty$, we have by (3.27) and Lemma 3.4 that $H_\sigma^* H_\sigma \leq_{\mathrm{fin}} -\mathrm{Re}F$.

Let now $\sigma \not\subset \Gamma$ and assume on the contrary $H_\sigma^* H_\sigma \leq_{\mathrm{fin}} -\mathrm{Re}F$, that is, by Lemma 3.4 there exists a finite dimensional subspace V in $L^2(\mathbb{S}^1)$ such that

$$\langle (-\mathrm{Re}F - H_\sigma^* H_\sigma)v, v\rangle \geq 0, \tag{3.30}$$

for all $v \in V^{\perp}$. Since $\sigma \not\subset \Gamma$, we can take a small open arc $\sigma_0 \subset \sigma$ such that $\sigma_0 \cap \Gamma = \emptyset$, which implies that for all $v \in V^{\perp}$

$$\begin{aligned} \|H_{\sigma_0} v\|^2_{L^2(\sigma_0)} &\leq \|H_\sigma v\|^2_{L^2(\sigma)} \\ &\leq \langle (-\mathrm{Re}F)v, v\rangle_{L^2(\mathbb{S}^1)} \\ &= \langle (\mathrm{Re}S^*)G^* v, G^* v\rangle \\ &\leq \|\mathrm{Re}S^*\| \, \|G^* v\|^2 . \end{aligned} \tag{3.31}$$

Before showing a contradiction with (3.31), we will show the following lemma.

Lemma 3.10. **(a)** $\dim(\mathrm{Ran}(H_{\sigma_0}^*)) = \infty$

(b) $\mathrm{Ran}(G) \cap \mathrm{Ran}(H_{\sigma_0}^*) = \{0\}$.

Proof of Lemma 3.10. **(a)** By the same argument in (3.25) we have

$$H_{\sigma_0} = J_{\sigma_0} \hat{H}_{\sigma_0} = J_{\sigma_0} S_{\sigma_0}^* G_{\sigma_0}^*, \tag{3.32}$$

where $G_{\sigma_0} : H^{1/2}(\sigma_0) \to L^2(\mathbb{S}^1)$, $S_{\sigma_0} : \tilde{H}^{-1/2}(\sigma_0) \to H^{1/2}(\sigma_0)$, and $J_{\sigma_0} : H^{1/2}(\sigma_0) \to L^2(\sigma_0)$ are the data-to-pattern operator, the single layer boundary operator, and the compact embedding, respectively, corresponding to σ_0. Since $H_{\sigma_0}^* = G_{\sigma_0} S_{\sigma_0} J_{\sigma_0}^*$, $\mathrm{Ran}(J_{\sigma_0}^*)$ is dense, and $G_{\sigma_0} S_{\sigma_0}$ is injective, we have $\dim(\mathrm{Ran}(H_{\sigma_0}^*)) = \dim(\mathrm{Ran}(J_{\sigma_0}^*)) = \infty$.

(b) By (3.7), we have $\mathrm{Ran}(H_{\sigma_0}^*) \subset \mathrm{Ran}(G_{\sigma_0})$. Let $h \in \mathrm{Ran}(G) \cap \mathrm{Ran}(G_{\sigma_0})$, i.e. $h = v_\Gamma^\infty = v_{\sigma_0}^\infty$ where v_Γ^∞ and $v_{\sigma_0}^\infty$ are far field patterns of the scattered field v_Γ and v_{σ_0} associated to scatterers Γ and σ_0, respectively. Then by Rellich lemma and unique continuation we have $v_\Gamma = v_{\sigma_0}$ in $\mathbb{R}^2 \setminus (\Gamma \cup \sigma_0)$. Hence, we can define $v \in H^1_{loc}(\mathbb{R}^2)$ by

$$v := \begin{cases} v_\Gamma = v_{\sigma_0} & \text{in } \mathbb{R}^2 \setminus (\Gamma \cup \sigma_0) \\ v_\Gamma & \text{on } \sigma_0 \\ v_{\sigma_0} & \text{on } \Gamma \end{cases} \tag{3.33}$$

and v is a radiating solution to

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^2. \tag{3.34}$$

Thus $v = 0$ in $\mathbb{R}^2$, which implies that $h = 0$. $\square$

By the above lemma we have $\infty = \dim(\mathrm{Ran}(H^*_{\sigma_0})) \not\leq \dim V < \infty$ and $\mathrm{Ran}(H^*_{\sigma_0}) \cap \mathrm{Ran}(G) = \{0\}$. By a contraposition of Lemma 3.9, we have

$$\mathrm{Ran}(H^*_{\sigma_0}) \not\subseteq \mathrm{Ran}(G) + V = \mathrm{Ran}(G,\ P_V), \tag{3.35}$$

where $P_V : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is the orthognal projection on V. Lemma 3.8 implies that for any $C > 0$ there exists a v_c such that

$$\|H_{\sigma_0} v_c\|^2 > C^2 \left\| \begin{pmatrix} G^* \\ P_V \end{pmatrix} v_c \right\|^2 = C^2(\|G^* v_c\|^2 + \|P_V v_c\|^2). \tag{3.36}$$

Hence, there exists a sequence $(v_m)_{m\in\mathbb{N}} \subset L^2(\mathbb{S}^1)$ such that $\|H_{\sigma_0} v_m\| \to \infty$ and $\|G^* v_m\|^2 + \|P_V v_m\| \to 0$ as $m \to \infty$. Setting $\tilde{v}_m := v_m - P_V v_m \in V^\perp$ we have as $m \to \infty$,

$$\|H_{\sigma_0} \tilde{v}_m\| \geq \|H_{\sigma_0} v_m\| - \|H_{\sigma_0}\| \, \|P_V v_m\| \to \infty, \tag{3.37}$$

$$\|G^* \tilde{v}_m\| \leq \|G^* v_m\| + \|G^*\| \, \|P_V v_m\| \to 0. \tag{3.38}$$

This contradicts (3.31). Therefore, we have $H^*_\sigma H_\sigma \not\leq_{\mathrm{fin}} -\mathrm{Re}F$. Theorem 3.1 has been shown. $\square$

3.4 Proof of Theorem 3.2

In Section 3.4, we will show Theorem 3.2. Let $\Gamma \subset B$. We denote by $G_{\partial B} : H^{1/2}(\partial B) \to L^2(\mathbb{S}^1)$ and $S_{\partial B} : H^{-1/2}(\partial B) \to H^{1/2}(\partial B)$ are the data-to-pattern operator and the single layer boundary operator, respectively corresponding to closed curve ∂B. They have the same properties like Lemmas 3.5 and 3.6 and we have $\tilde{H}^*_{\partial B} = G_{\partial B} S_{\partial B}$. (See, e.g., Lemma 1.14, Theorem 1.15 in [58].) We define $T : H^{1/2}(\Gamma) \to H^{1/2}(\partial B)$ by

$$Tf := v\big|_{\partial B}, \tag{3.39}$$

where v is a radiating solution such that

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^2 \setminus \Gamma, \tag{3.40}$$

$$v = f \text{ on } \Gamma. \tag{3.41}$$

T is compact since its mapping is from $H^{1/2}(\Gamma)$ to $C^\infty(\partial B)$. Furthermore, by the definition of T we have that $G = G_{\partial B} T$. Thus, we have

$$\begin{aligned} \tilde{H}^*_{\partial B}\tilde{H}_{\partial B} + \mathrm{Re}F &= G_{\partial B} S_{\partial B} S^*_{\partial B} G^*_{\partial B} + G_{\partial B}\big[-T\mathrm{Re}(S)T^*\big]G^*_{\partial B} \\ &= G_{\partial B}\big[S_{\partial B,i}S^*_{\partial B,i} + K\big]G^*_{\partial B} \\ &= \big[G_{\partial B}W^*\big]\big[W^{*\ -1}S_{\partial B,i}S^*_{\partial B,i}W^{-1} + K'\big]\big[G_{\partial B}W^*\big]^*, \end{aligned} \tag{3.42}$$

where K and K' are some self-adjoint compact operators, and $W := S^{1/2}_{\partial B,i} : H^{-1/2}(\partial B) \to L^2(\partial B)$ is an extension of the square root of $S_{\partial B,i}$ where $S_{\partial B,i} : \tilde{H}^{-1/2}(\partial B) \to H^{1/2}(\partial B)$ is the single layer boundary operator corresponding to ∂B and the wave number $k = i$. Let V be the sum of eigenspaces of K' associated to eigenvalues less than $-\frac{1}{2}\left\|\left(S^*_{\partial B,i}W^{-1}\right)^{-1}\right\|^{-2}$. Then V is a finite dimensional, and for all $g \in \big[(G_{\partial B}W^*)V\big]^\perp$ we have

$$\begin{aligned} &\langle(\tilde{H}^*_{\partial B}\tilde{H}_{\partial B} + \mathrm{Re}F)g, g\rangle \\ = \ & \big\|(S^*_{\partial B,i}W^{-1})\big[G_{\partial B}W^*\big]^* g\big\|^2_{H^{1/2}(\partial B)} + \langle K'\big[G_{\partial B}W^*\big]^* g, \big[G_{\partial B}W^*\big]^* g\rangle_{L^2(\partial B)} \\ \geq \ & \big\|(S^*_{\partial B,i}W^{-1})^{-1}\big\|^{-2}\,\big\|\big[G_{\partial B}W^*\big]^* g\big\|^2 - \frac{1}{2}\,\big\|(S^*_{\partial B,i}W^{-1})^{-1}\big\|^{-2}\,\big\|\big[G_{\partial B}W^*\big]^* g\big\|^2 \\ \geq \ & 0. \end{aligned} \tag{3.43}$$

Therefore, $-\mathrm{Re}F \leq_{\mathrm{fin}} \tilde{H}^*_{\partial B}\tilde{H}_{\partial B}$.

Let now $\Gamma \not\subset B$ and assume on the contrary $-\mathrm{Re}F \leq_{\mathrm{fin}} \tilde{H}^*_{\partial B}\tilde{H}_{\partial B}$, i.e., by Lemma 3.4 there exists a finite dimensional subspace V in $L^2(\mathbb{S}^1)$ such that

$$\langle(\tilde{H}^*_{\partial B}\tilde{H}_{\partial B} + \mathrm{Re}F)v, v\rangle \geq 0, \tag{3.44}$$

for all $v \in V^\perp$. Since $\Gamma \not\subset B$, we can take a small open arc $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \cap B = \emptyset$. We define $L : H^{1/2}(\Gamma_0) \to H^{1/2}(\Gamma)$ by

$$Lf := v\big|_{\Gamma}, \tag{3.45}$$

where v is a radiating solution such that

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^2 \setminus \Gamma_0, \tag{3.46}$$

$$v = f \text{ on } \Gamma_0. \tag{3.47}$$

By the definition of L, we have $G_{\Gamma_0} = GL$ where $G_{\Gamma_0} : H^{1/2}(\Gamma_0) \to L^2(\mathbb{S}^1)$ is the data-to-pattern operator corresponding to Γ_0. We denote by $S_{\Gamma_0} : \tilde{H}^{-1/2}(\Gamma_0) \to H^{1/2}(\Gamma_0)$ the single layer boundary operator corresponding to Γ_0, and $H_{\Gamma_0} : L^2(\mathbb{S}^1) \to L^2(\Gamma_0)$, $\hat{H}_{\Gamma_0} : L^2(\mathbb{S}^1) \to H^{1/2}(\Gamma_0)$ the Herglotz operators corresponding to Γ_0, respectively. By the same argument in (3.25) we have $\hat{H}_{\Gamma_0} = S^*_{\Gamma_0} G^*_{\Gamma_0}$. Then, we have

$$\begin{aligned} \|H_{\Gamma_0} x\|^2_{L^2(\Gamma_0)} &\leq \left\|\hat{H}_{\Gamma_0} x\right\|^2_{H^{1/2}(\Gamma_0)} \\ &\leq \left\|S^*_{\Gamma_0}\right\|^2 \left\|G^*_{\Gamma_0} x\right\|^2 \\ &\leq \left\|S^*_{\Gamma_0}\right\|^2 \|L^*\|^2 \|G^* x\|^2 , \end{aligned} \tag{3.48}$$

for $x \in L^2(\mathbb{S}^1)$. Since $\mathrm{Re}S$ is of the form $\mathrm{Re}S = S_i + \mathrm{Re}(S - S_i)$, by the similar argument in (3.26)–(3.27) and (3.42)–(3.43), there exists a finite dimensional subspace W in $L^2(\mathbb{S}^1)$ such that for $x \in W^\perp$

$$\|G^* x\|^2 \leq C\langle (\mathrm{Re}S) G^* x, G^* x\rangle = C\langle (-\mathrm{Re}F)x, x\rangle. \tag{3.49}$$

Collecting (3.48), (3.49), and $\tilde{H}^*_{\partial B} = G_{\partial B} S_{\partial B}$, we have

$$\begin{aligned} \|H_{\Gamma_0} x\|^2 &\leq C\langle (-\mathrm{Re}F)x, x\rangle \leq C \left\|\tilde{H}_{\partial B} x\right\|^2 \\ &\leq C \|S^*_{\partial B}\|^2 \|G^*_{\partial B} x\|^2_{H^{-1/2}(\partial B)} . \end{aligned} \tag{3.50}$$

for $x \in (V \cup W)^\perp$.

Lemma 3.11. **(a)** $\dim(\mathrm{Ran}(H^*_{\Gamma_0})) = \infty$

(b) $\mathrm{Ran}(G_{\partial B}) \cap \mathrm{Ran}(H^*_{\Gamma_0}) = \{0\}$.

Proof of Lemma 3.11. (a) is given by the same argument in Lemma 3.10.

(b) Since (3.31) replacing σ_0 by Γ_0 holds, by taking a conjugation in (3.31) we have $\mathrm{Ran}(H^*_{\Gamma_0}) \subset \mathrm{Ran}(G_{\Gamma_0})$. Let $h \in \mathrm{Ran}(G_{\partial B}) \cap \mathrm{Ran}(G_{\Gamma_0})$, i.e., $h = v^\infty_B = v^\infty_{\Gamma_0}$ where v^∞_B and $v^\infty_{\Gamma_0}$ are far field patterns of the scattered field v_B and v_{Γ_0} associated to scatterers B and Γ_0, respectively. Then by Rellich lemma and unique continuation we have $v_B = v_{\Gamma_0}$ in $\mathbb{R}^2 \setminus (B \cup \Gamma_0)$. Hence, we can define $v \in H^1_{loc}(\mathbb{R}^2)$ by

$$v := \begin{cases} v_B = v_{\Gamma_0} & \text{in } \mathbb{R}^2 \setminus (B \cup \Gamma_0) \\ v_{\Gamma_0} & \text{on } B \\ v_B & \text{on } \Gamma_0 \end{cases} \tag{3.51}$$

and v is a radiating solution to

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^2. \tag{3.52}$$

Thus $v = 0$ in $\mathbb{R}^2$, which implies that $h = 0$. $\square$

By the above lemma we have $\infty = \dim(\mathrm{Ran}(H^*_{\Gamma_0})) \nleq \dim(V \cup W) < \infty$ and $\mathrm{Ran}(H^*_{\Gamma_0}) \cap \mathrm{Ran}(G_{\partial B}) = \{0\}$. By a contraposition of Lemma 3.9, we have

$$\mathrm{Ran}(H^*_{\Gamma_0}) \nsubseteq \mathrm{Ran}(G_{\partial B}) + (V \cup W) = \mathrm{Ran}(G_{\partial B},\ P_{V\cup W}), \tag{3.53}$$

where $P_{V \cup W} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is the orthognal projection on $V \cup W$. Lemma 3.8 implies that for any $C > 0$ there exists a x_c such that

$$\|H_{\Gamma_0} x_c\|^2 > C^2 \left\| \begin{pmatrix} G^*_{\partial B} \\ P_{V\cup W} \end{pmatrix} x_c \right\|^2 = C^2(\|G^*_{\partial B} x_c\|^2 + \|P_{V\cup W} x_c\|^2). \tag{3.54}$$

Hence, there exists a sequence $(x_m)_{m\in\mathbb{N}} \subset L^2(\mathbb{S}^1)$ such that $\|H_{\Gamma_0} x_m\| \to \infty$ and $\|G^*_{\partial B} x_m\|^2 + \|P_{V\cup W} x_m\| \to 0$ as $m \to \infty$. Setting $\tilde{x}_m := x_m - P_{V\cup W} x_m \in (V \cup W)^\perp$ we have as $m \to \infty$,

$$\|H_{\Gamma_0} \tilde{x}_m\| \geq \|H_{\Gamma_0} x_m\| - \|H_{\Gamma_0}\| \, \|P_{V\cup W} x_m\| \to \infty, \tag{3.55}$$

$$\|G^*_{\partial B} \tilde{x}_m\| \leq \|G^*_{\partial B} x_m\| + \|G^*_{\partial B}\| \, \|P_{V\cup W} x_m\| \to 0. \tag{3.56}$$

This contradicts (3.50). Therefore, we have $-\mathrm{Re}F \not\leq_{\mathrm{fin}} \tilde{H}^*_{\partial B} \tilde{H}_{\partial B}$. Theorem 3.2 has been shown. $\square$

3.5 Numerical examples

In Section 3.5, we discuss the numerical examples based on Theorem 3.1. The following three open arcs Γ_j $(j = 1, 2, 3)$ are considered. (see Figure 5)

(a) $\Gamma_1 = \{(s, s) | -1 \leq s \leq 1\}$

(b) $\Gamma_2 = \left\{ \left(2\sin\left(\frac{\pi}{8} + (1+s)\frac{3\pi}{8}\right) - \frac{2}{3},\ \sin\left(\frac{\pi}{4} + (1+s)\frac{3\pi}{4}\right) \right) \middle| -1 \leq s \leq 1 \right\}$

(c) $\Gamma_3 = \left\{ \left(s,\ \sin\!\left(\frac{\pi}{4} + (1+s)\frac{3\pi}{4} \right) \right) \middle| -1 \leq s \leq 1 \right\}$

Based on Theorem 3.1, the indicator function in our examples is given by

$$I(\sigma) := \#\left\{\text{negative eigenvalues of } -\mathrm{Re}F - H_\sigma^* H_\sigma\right\}. \tag{3.57}$$

The idea to reconstruct Γ_j is to plot the value of $I(\sigma)$ for many of small σ in the sampling region. Then, we expect from Theorem 3.1 that the value of the function $I(\sigma)$ is low if σ is close to Γ_j.

Here, σ is chosen in two ways; One is the vertical line segment $\sigma_{i,j}^{ver} := z_{i,j} + \{0\} \times [-\frac{R}{2M}, \frac{R}{2M}]$ where $z_{i,j} := (\frac{Ri}{M}, \frac{Rj}{M})$ $(i, j = -M, -M+1, ..., M)$ denote the center of $\sigma_{i,j}^{ver}$, and $\frac{R}{M}$ is the length of $\sigma_{i,j}^{ver}$, and $R > 0$ is length of sampling square region $[-R, R]^2$, and $M \in \mathbb{N}$ is large to take a small segment. The other is horizontal one $\sigma_{i,j}^{hor} := z_{i,j} + [-\frac{R}{2M}, \frac{R}{2M}] \times \{0\}$.

The far field operator F is approximated by the matrix

$$F \approx \frac{2\pi}{N} \big(u^\infty(\hat{x}_l, \theta_m) \big)_{1 \leq l,m \leq N} \in \mathbb{C}^{N \times N}, \tag{3.58}$$

where $\hat{x}_l = \big(\cos(\frac{2\pi l}{N}), \sin(\frac{2\pi l}{N}) \big)$ and $\theta_m = \big(\cos(\frac{2\pi m}{N}), \sin(\frac{2\pi m}{N}) \big)$. The far field pattern u^∞ of the problem (3.1)–(3.3) is computed by the Nyström method in [67]. The operator $H_\sigma^* H_\sigma$ is approximated by

$$H_\sigma^* H_\sigma \approx \frac{2\pi}{N} \left(\int_\sigma e^{iky\cdot(\theta_m - \hat{x}_l)} dy \right)_{1 \leq l,m \leq N} \in \mathbb{C}^{N \times N}. \tag{3.59}$$

When σ is given by the vertical and horizontal line segment, we can compute the integrals

$$\int_{\sigma_{i,j}^{ver}} e^{iky\cdot(\theta_m - \hat{x}_l)} dy = \frac{R}{M} e^{ik(\theta_m - \hat{x}_l)\cdot z_{i,j}} \mathrm{sinc}\left(\frac{kR}{2M\pi} \Big(\sin(\frac{2\pi m}{N}) - \sin(\frac{2\pi l}{N}) \Big) \right), \tag{3.60}$$

$$\int_{\sigma_{i,j}^{hor}} e^{iky\cdot(\theta_m - \hat{x}_l)} dy = \frac{R}{M} e^{ik(\theta_m - \hat{x}_l)\cdot z_{i,j}} \mathrm{sinc}\left(\frac{kR}{2M\pi} \Big(\cos(\frac{2\pi m}{N}) - \cos(\frac{2\pi l}{N}) \Big) \right). \tag{3.61}$$

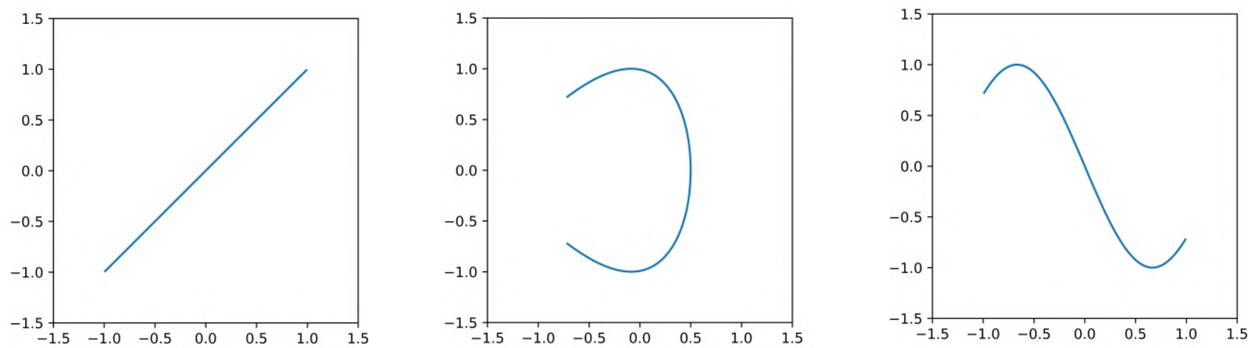
In our examples we fix $R = 1.5$, $M = 100$, $N = 60$, and wavenumber $k = 1$. Figure 6 is given by plotting the values of the vertical indicator function

$$I_{ver}(z_{i,j}) := I(\sigma_{i,j}^{ver}), \tag{3.62}$$

for each $i, j = -100, -99, ..., 100$. Figure 7 is given by plotting the values of the horizontal indicator function

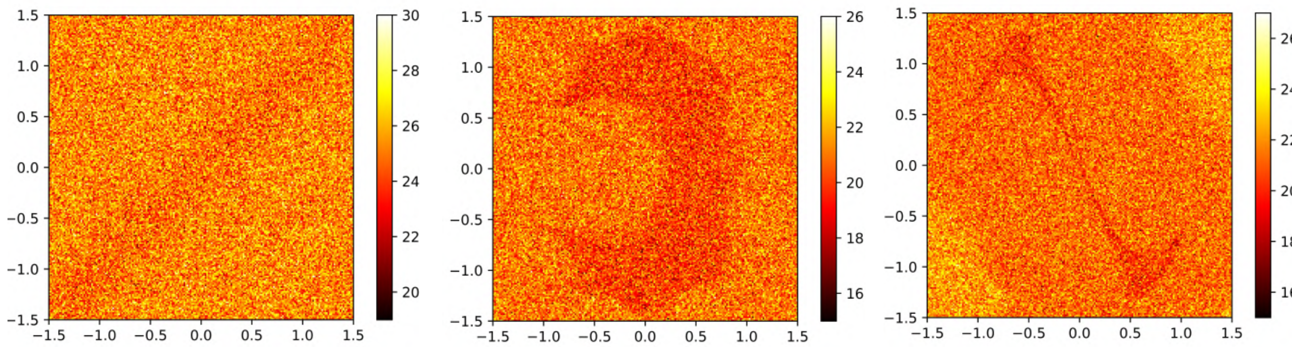
$$I_{hor}(z_{i,j}) := I(\sigma_{i,j}^{hor}), \tag{3.63}$$

for each $i, j = -100, -99, ..., 100$. We obverse that Γ_j seems to be reconstructed independently of the direction of linear segment.



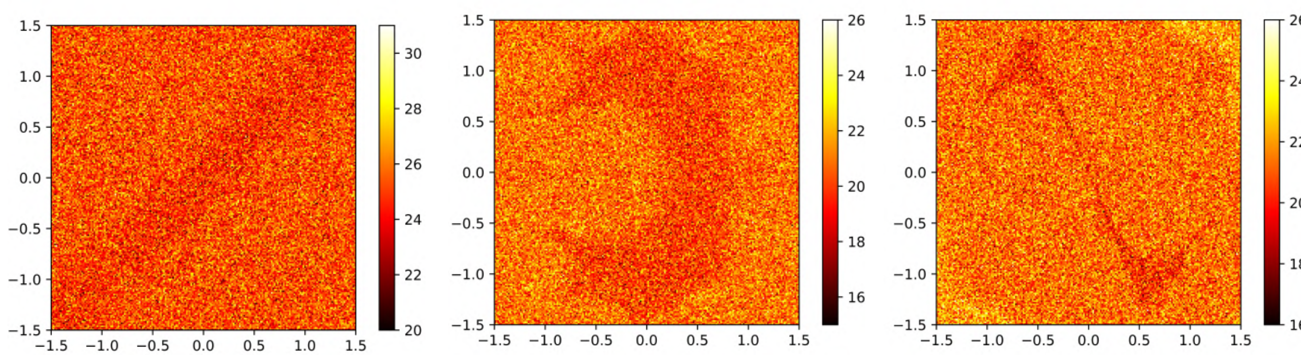
Γ_1 Γ_2 Γ_3

Figure 5: The original open arc



Γ_1 Γ_2 Γ_3

Figure 6: Reconstruction by the vertical indicator function I_{ver}



Γ_1 Γ_2 Γ_3

Figure 7: Reconstruction by the horizontal indicator function I_{hor}

4 The direct and inverse scattering problem for the semilinear Schrödinger equation

4.1 Introduction

In this section, we study the direct and inverse scattering problem for the semilinear Schrödinger equation

$$\Delta u + a(x,u) + k^2 u = 0 \text{ in } \mathbb{R}^d, \tag{4.1}$$

where $d \geq 2$, and $k > 0$. Throughout this section, we make the following assumptions for the semilinear function $a : \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$.

Assumption 4.1. *We assume that*

(i) $a(x,0) = 0$ *for all* $x \in \mathbb{R}^d$.

(ii) $a(x,z)$ *is holomorphic at* $z = 0$ *for each* $x \in \mathbb{R}^d$*, that is, there exists* $\eta > 0$ *such that* $a(x,z) = \sum_{l=1}^{\infty} \frac{\partial_z^l a(x,0)}{l!} z^l$ *for* $|z| < \eta$.

(iii) $\partial_z^l a(\cdot,0) \in L^\infty(\mathbb{R}^d)$ *for all* $l \geq 1$*. Furthermore, there exists* $c_0 > 0$ *such that* $\left\|\partial_z^l a(\cdot,0)\right\|_{L^\infty(\mathbb{R}^d)} \leq c_0^l$ *for all* $l \geq 1$.

(iv) *There exists* $R > 0$ *such that* $\mathrm{supp}\partial_z^l a(\cdot,0) \subset B_R$ *for all* $l \geq 1$ *where* $B_R \subset \mathbb{R}^d$ *is a open ball with center 0 and radius* $R > 0$.

The inverse scattering problems for non-linear Schrödinger equations have been studied in various ways. For the time dependent case, we refer to [93, 94, 95], and for the stationary case, we refer to [1, 36, 47, 88, 89, 90]. In stationary case, [36, 47, 89] have studied the general non-linear function of the form $a(x,|u|)u$, which does not include our no-linear function $a(x,u)$. The function $a(x,u)$ which satisfies Assumption 4.1 is the generalization of, in particular, the power type $q(x)u^m$ where $m \in \mathbb{N}$ where $q \in L^\infty(\mathbb{R}^d)$ with compact support. If $m = 1$, the problem is for linear Schrödinger equations, which has been well understood so far by many authors. (see e.g., [29, 77, 81, 87])

Recently in [20, 70, 71], the generalization of a power type has been studied in inverse boundary value problems via using the Dirichlet-to-Neumann map. [46] also has studied the similar type of this nonlinearity. However in inverse scattering problems, only [1] has studied it in one dimension, which the non-linear function is of the form $a(x,u) = \sum_{n=1}^{\infty} q_n(x)u^n$. Motivated by these previous studies, our aim in this section is to study the type of this nonlinearity in the case of higher dimensions $d \geq 2$, and a more general form $a(x,u)$ than [1].

We consider the incident field u_g^{in} as the *Herglotz wave function*

$$u_g^{in}(x) := \int_{\mathbb{S}^{d-1}} e^{ikx\cdot\theta} g(\theta) ds(\theta),\ x \in \mathbb{R}^d,\ g \in L^2(\mathbb{S}^{d-1}), \tag{4.2}$$

which solves the free Schrödinger equation $\Delta u_g^{in} + k^2 u_g^{in} = 0$ in $\mathbb{R}^d$. The scattered field u_g^{sc} corresponding to the incident field u_g^{in} is a solution of the following Schrödinger equation perturbed by the semilinear function $a(x,z)$

$$\Delta u_g + a(x, u_g) + k^2 u_g = 0 \text{ in } \mathbb{R}^d, \tag{4.3}$$

where u_g is total field that is of the form $u_g = u_g^{sc} + u_g^{in}$, and the scattered field u^{sc} satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^{sc}}{\partial r} - iku^{sc} \right) = 0, \tag{4.4}$$

where $r = |x|$.

Since support of the function $a(x,z)$ is compact, the direct scattering problem (4.3)–(4.4) is equivalent to the following integral equation. (see e.g., the argument of Theorem 8.3 in [18].)

$$u_g(x) = u_g^{in} + \int_{\mathbb{R}^d} \Phi(x,y) a(y, u_g(y)) dy, \quad x \in\ \mathbb{R}^d, \tag{4.5}$$

where $\Phi(x,y)$ is the fundamental solution for $-\Delta - k^2$ in $\mathbb{R}^d$. In the following theorem, we find a small solution u_g^{sc} of (4.5) for small $g \in L^\infty(\mathbb{S}^{d-1})$.

Theorem 4.2. *We assume that $a(x,z)$ satisfies Assumption 4.1. Then, there exists $\delta_0 \in (0,1)$ such that for all $\delta \in (0,\delta_0)$ and $g \in L^\infty(\mathbb{S}^{d-1})$ with $\|g\|_{L^\infty(\mathbb{S}^{d-1})} < \delta^2$, there exists a unique solution $u_g^{sc} \in L^\infty(\mathbb{R}^d)$ with $\|u_g^{sc}\|_{L^\infty(\mathbb{R}^d)} \leq \delta$ such that*

$$u_g^{sc}(x) = \int_{\mathbb{R}^d} \Phi(x,y) a(y, u_g^{sc}(y) + u_g^{in}(y)) dy,\ x \in \mathbb{R}^d. \tag{4.6}$$

Theorem 4.2 is proved by the Banach fixed point theorem. By the same argument in Section 19 of [19], the solution u_g^{sc} of (4.6) has the following asymptotic behavior

$$u_g^{sc}(x) = C_d \frac{\mathrm{e}^{ikr}}{r^{\frac{d-1}{2}}} u_g^\infty(\hat{x}) + O\left(\frac{1}{r^{\frac{d+1}{2}}} \right),\ r := |x| \to \infty,\ \ \hat{x} := \frac{x}{|x|}. \tag{4.7}$$

where $C_d := k^{\frac{d-3}{2}} e^{-i\frac{\pi}{4}(d-3)}/2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}$. The function u_g^∞ is called the *scattering amplitude*, which is of the form

$$u_g^\infty(\hat{x}) = \int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} a(y, u_g(y))dy, \ \hat{x} \in \mathbb{S}^{d-1}. \tag{4.8}$$

We remark that in the standard linear case, that is, $a(x, u) = q(x)u$, the scattering amplitude corresponding to the Herglotz wave function (4.8) can be of the form

$$u_g^\infty(\hat{x}) = \int_{\mathbb{S}^{d-1}} \tilde{u}^\infty(\hat{x}, \theta)g(\theta)ds(\theta), \ \hat{x} \in \mathbb{S}^{d-1}. \tag{4.9}$$

where $\tilde{u}^\infty(\hat{x}, \theta)$ is the scattering amplitude corresponding to plane waves $e^{ikx\cdot\theta}$. This tells us that in standard linear case, the scattering amplitude of the Herglotz wave function is equivalent to that of the plane wave.

Now, we are ready to consider the inverse problem to determine the semilinear function $a(x, z)$ from scattering amplitudes $u_g^\infty(\hat{x})$ for all $g \in L^2(\mathbb{S}^{d-1})$ with $\|g\|_{L^2(\mathbb{S}^{d-1})} < \delta$ where $\delta > 0$ is a sufficiently small. We will show the following theorem.

Theorem 4.3. *We assume that $a_j(x, z)$ satisfies Assumption 4.1. ($j = 1, 2$.) Let $u_{g,j}^\infty$ be the scattering amplitude for the following problem*

$$\Delta u_{j,g} + a_j(x, u_{j,g}) + k^2 u_{j,g} = 0 \text{ in } \mathbb{R}^d, \tag{4.10}$$

$$u_{j,g} = u_{j,g}^{sc} + u_g^{in}, \tag{4.11}$$

where $u_{j,g}^{sc}$ satisfies the Sommerfeld radiation (4.4), and u_g^{in} is given by (4.2), and we assume that

$$u_{1,g}^\infty = u_{2,g}^\infty, \tag{4.12}$$

for any $g \in L^2(\mathbb{S}^{d-1})$ with $\|g\|_{L^2(\mathbb{S}^{d-1})} < \delta$ where $\delta > 0$ is sufficiently small. Then, we have

$$a_1(x, z) = a_2(x, z), \ x \in \mathbb{R}^d, \ |z| < \eta \tag{4.13}$$

The idea of the proof is the linearization, which by using sources with several parameters we differentiate the nonlinear equation with respect to these parameter in order to get the linear equation. (For such ideas, we refer to [20, 70, 71].)

There are few previous studies that the general nonlinear function is uniquely determined from the scattering amplitude with fixed $k > 0$. [47] has shown it from behaviour of scattering amplitude corresponding to plane waves $\lambda e^{ikx\dot{\theta}}$ as $\lambda \to 0$. [88] has done from the scattering amplitude with fixed $\lambda = 1$, but the additional assumptions are needed. Our work shows it from the scattering amplitude corresponding to Herglotz wave functions u_g^{in} for all small g instead of using plane waves.

This section is organized as follows. In Section 4.2, we recall the Green function for the Helmholtz equation and its properties. We also prepare the several lemmas required in the forthcoming argument. In Section 4.3, we prove Theorem 4.2 based on the Banach fixed point theorem. In Section 4.4, we consider the special solution of (4.3)–(4.4) corresponding to the incident field with several parameters in order to linearize problems. Finally in Section 4.5, we prove Theorem 4.3.

4.2 Preliminary

First, we recall the Green functions for the Helmholtz equation and its properties. We denote the Green function for $-\Delta - k^2$ in $\mathbb{R}^d$ by $\Phi(x,y)$, that is, $\Phi(x,y)$ satisfies

$$(-\Delta - k^2)\Phi(x,y) = \delta(x-y), \tag{4.14}$$

for $x, y \in \mathbb{R}^d$, $x \neq y$. In the case of $d = 2, 3$, $\Phi(x,y)$ is of the form

$$\Phi(x,y) = \begin{cases} \frac{i}{4}H_0^{(1)}(k|x-y|) & \text{for } x,y \in \mathbb{R}^2,\ x \neq y, \\ \dfrac{e^{ik|x-y|}}{4\pi|x-y|} & \text{for } x,y \in \mathbb{R}^3,\ x \neq y, \end{cases} \tag{4.15}$$

respectively. Let $q \in L^\infty(\mathbb{R}^d)$ with compact support. We denote the Green function for $-\Delta - k^2 - q$ in $\mathbb{R}^d$ by $\Phi_q(x,y)$, that is, $\Phi_q(x,y)$ satisfies

$$(-\Delta - k^2 - q)\Phi_q(x,y) = \delta(x-y). \tag{4.16}$$

for $x, y \in \mathbb{R}^d$, $x \neq y$. It is well known that for every fixed y, $\Phi(x,y)$ and $\Phi_q(x,y)$ satisfy the Sommerfeld radiation condition.

We also recall the asymptotics behavior of $\Phi(x,y)$ as $|x| \to \infty$. In Lemma 19.3 of [19], $\Phi(x,y)$ has the following asymptotics behavior for every fixed y,

$$\Phi(x,y) = C_d \frac{\mathrm{e}^{ik|x-y|}}{|x-y|^{\frac{d-1}{2}}} + O\left(\frac{1}{|x-y|^{\frac{d+1}{2}}}\right),\ |x| \to \infty, \tag{4.17}$$

and (see the proof of Theorem 19.5 in [19])

$$\Phi(x,y) = \begin{cases} O\left(\frac{1}{|x-y|^{d-2}}\right) & d \geq 3,\ x \neq y \\ O\left(\left|\ln|x-y|\right|\right) & d = 2,\ x \neq y \end{cases} \tag{4.18}$$

In Theorem 19.5 of [19], for every $f \in L^{\infty}(\mathbb{R}^d)$ with compact support, $u(x) = \int_{\mathbb{R}^d} \Phi(x, y) f(y) dy$ is a unique radiating solution (that is, u satisfies the Sommerfeld radiation condition (4.4)). Furthermore, u has the following asymptotic behavior

$$u(x) = C_d \frac{\mathrm{e}^{ikr}}{r^{\frac{d-1}{2}}} u^{\infty}(\hat{x}) + O\left(\frac{1}{r^{\frac{d+1}{2}}}\right), \ r = |x| \to \infty, \quad \hat{x} := \frac{x}{|x|}, \tag{4.19}$$

where the scattering amplitude u^{∞} is of the form

$$u^{\infty}(\hat{x}) = \int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} f(y) dy, \ \hat{x} \in \mathbb{S}^{d-1}. \tag{4.20}$$

The following lemma is given by the same argument as in Lemma 10.4 of [18] or Proposition 2.4 of [82].

Lemma 4.4. *Let $q \in L^{\infty}(\mathbb{R}^d)$ with compact support in $B_R \subset \mathbb{R}^d$ where some $R > 0$. We define the Helglotz operator $H : L^2(\mathbb{S}^{d-1}) \to L^2(B_R)$ by*

$$Hg(x) := \int_{\mathbb{S}^{d-1}} e^{ikx\cdot\theta} g(\theta) ds(\theta), \ x \in B_R, \tag{4.21}$$

and define the operator $T_q : L^2(B_R) \to L^2(B_R)$ by $T_q f := f + w\Big|_{B_R}$ where w is a radiating solution such that

$$\Delta w + k^2 w + qw = -qf \text{ in } \mathbb{R}^d. \tag{4.22}$$

We define the subspace V of $L^2(B_R)$ by

$$V := \overline{\left\{ v\Big|_{B_R}; v \in L^2(B_{R+1}), \ \Delta v + k^2 v + qv = 0 \text{ in } \mathrm{B}_{R+1} \right\}}^{\|\cdot\|_{L^2(B_R)}}. \tag{4.23}$$

Then, the range of the operator $T_q H$ is dense in V with respect to the norm $\|\cdot\|_{L^2(B_R)}$, that is,

$$\overline{T_q H\left(L^2(\mathbb{S}^{d-1})\right)}^{\|\cdot\|_{L^2(B_R)}} = V. \tag{4.24}$$

The following result is well known. For $d = 2$ we refer to [10], and for $d \geq 3$ we refer to [91], which corresponds to real functions. For complex functions, see Theorem 6.2 in [92].

Lemma 4.5. *Let $f, q_1, q_2 \in L^{\infty}(\mathbb{R}^d)$ with compact support in $B_R \subset \mathbb{R}^d$. We assume that*

$$\int_{B_R} f v_1 v_2 dx = 0, \tag{4.25}$$

for all $v_1, v_2 \in L^2(B_{R+1})$ with $\Delta v_j + k^2 v_j + q_j v_j = 0$ in B_{R+1}. ($j = 1, 2$.) Then, $f = 0$ in B_R.

4.3 Proof of Theorem 4.2

In Section 4.3, we will show Theorem 4.2 based on the Banach fixed point theorem. We denote the Herglotz wave function by

$$v_g(x) := \int_{\mathbb{S}^{d-1}} e^{ikx\cdot\theta} g(\theta) ds(\theta),\ x \in \mathbb{R}^d,\ g \in L^2(\mathbb{S}^{d-1}). \tag{4.26}$$

Let $q := \partial_z a(\cdot, 0)$. We define the operator $T : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ by

$$\begin{aligned} Tw(x) &:= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[a\big(y, w(y) + v_g(y)\big) - q(y)w(y)\right] dy \\ &= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[\sum_{l\geq 2} \frac{\partial_z^l a(y,0)}{l!} \big(w(y) + v_g(y)\big)^l + q(y)v_g(y)\right] dy,\ x \in \mathbb{R}^d. \end{aligned} \tag{4.27}$$

Let $X_\delta := \left\{u \in L^\infty(\mathbb{R}^d) : \|u\|_{L^\infty(\mathbb{R}^d)} \leq \delta\right\}$. We remark that $L^\infty(\mathbb{R}^d)$ is a Banach space, and X_δ is closed subspace in $L^\infty(\mathbb{R}^d)$. To find an unique fixed point of T in X, we will show that $T : X_\delta \to X_\delta$ and T is a contraction. Let $w \in X_\delta$, and let $\delta \in (0, \delta_0)$, and let $\|g\|_{L^\infty(\mathbb{S}^{d-1})} < \delta^2$. Later, we will choose a appropriate $\delta_0 > 0$.

By $\|g\|_{L^\infty(\mathbb{S}^{d-1})} < \delta^2$, we have

$$\|v_g\|_{L^\infty(\mathbb{R}^d)} \leq C \|g\|_{L^\infty(\mathbb{S}^{d-1})} \leq C\delta^2, \tag{4.28}$$

where $C > 0$ is constant only depending on g. By (iii) and (iv) of Assumption 4.1, we have

$$\begin{aligned} |Tw(x)| &\leq \int_{B_R} |\Phi_q(x,y)| \left[\sum_{l\geq 2} \frac{c_0^l}{l!} \big(C_1\delta\big)^l + C_1\delta^2\right] dy \\ &\leq C_2\delta^2 \left(\sum_{l\geq 0} \big(C_1 c_0 \delta\big)^l\right) \int_{B_R} |\Phi_q(x,y)| dy, \end{aligned} \tag{4.29}$$

where $C_j > 0$ ($j = 1, 2$) is constant independent of u and δ, and so is $\left(\sum_{l\geq 0} \big(C_1 c_0 \delta\big)^l\right)$ when $\delta > 0$ is sufficiently small. Furthermore, by the continuity of difference $\Phi(x,y) - \Phi_q(x,y)$ in x and y (see the proof of Theorem

31.6 in [19]), and the estimation (4.18), we have for $x \in \mathbb{R}^d$

$$\begin{aligned}\int_{B_R} |\Phi_q(x,y)|dy &\leq \int_{B_R} \big(|\Phi(x,y)| + |\Phi_q(x,y) - \Phi(x,y)|\big)dy \\ &\leq \int_{B_R} \big(|\Phi(x,y)| + C_3\big)dy \leq C_4, \qquad (4.30)\end{aligned}$$

which implies that $|Tw(x)| \leq C\delta^2$ where $C, C_j > 0$ $(j = 3, 4)$ is constant independent of u and δ. By choosing $\delta_0 \in (0, 1/C)$, we conclude that $\|Tw\| \leq \delta$, which means $Tw \in X_\delta$.

Let $w_1, w_2 \in X_\delta$. Since we have

$$\begin{aligned}&\big(w_1(y) + v_g(y)\big)^l - \big(w_2(y) + v_g(y)\big)^l \\ =\ & \sum_{m=1}^{l} \frac{l!}{(l-m)!m!}\big(w_1^m(y) - w_2^m(y)\big)v_g^{l-m}(y) \\ \leq\ & \sum_{m=1}^{l} \frac{l!}{(l-m)!m!} \left(\sum_{h=0}^{m-1} w_1^{m-1-h}(y) w_2^h(y)\right) \big(w_1(y) - w_2(y)\big)v_g^{l-m}(y), \qquad (4.31)\end{aligned}$$

and $|w_j(x)| \leq \delta$, then

$$\begin{aligned}&|Tw_1(x) - Tw_2(x)| \\ =\ & \left|\int_{B_R} \Phi_q(x,y) \sum_{l\geq 2} \frac{\partial_z^l a(y,0)}{l!} \left[\big(w_1(y) + v_g(y)\big)^l - \big(w_2(y) + v_g(y)\big)^l\right] dy\right| \\ \leq\ & \left(\int_{B_R} |\Phi_q(x,y)|\, dy\right) \sum_{l\geq 2} \frac{c_0^l}{l!} \sum_{m=1}^{l} \frac{l!}{(l-m)!m!} \left(\sum_{h=0}^{m-1} \delta^{m-1}\right) \big(C_1'\delta\big)^{l-m} \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\ \leq\ & C_2' \sum_{l\geq 2} \sum_{m=1}^{l} \frac{m}{(l-m)!m!} \big(c_0 C_1'\delta\big)^{l-1} \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\ \leq\ & C_2' \sum_{l\geq 2} \left(\sum_{m=1}^{\infty} \frac{1}{(m-1)!}\right) \big(c_0 C_1'\delta\big)^{l-1} \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\ \leq\ & C_3' \sum_{l\geq 2} \big(c_0 C_1'\delta\big)^{l-1} \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\ \leq\ & C_3' \left(\sum_{l\geq 0} \big(c_0 C_1'\delta\big)^{l}\right) \delta \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\ \leq\ & C'\delta \|u_1 - u_2\|_{L^\infty(\mathbb{R}^d)}, \ x \in \mathbb{R}^d. \qquad (4.32)\end{aligned}$$

where $C', C'_j > 0$ $(j = 1, 2, 3)$ is constant independent of w_1, w_2 and δ. (We remark that $\left(\sum_{l\geq 0}(c_0 C'_1\delta)^l\right)$ is also constant when $\delta > 0$ is sufficiently small.) By choosing $\delta_0 \in (0, 1/C')$, we have $\|Tw_1 - Tw_2\|_{L^\infty(\mathbb{R}^d)} < \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)}$. Choosing sufficiently small $\delta_0 \in (0, \min(1/C, 1/C'))$ we conclude that T has a unique fixed point in X_δ.

Let $w \in X_\delta$ be a unique fixed point, that is, w satisfies

$$w(x) = \int_{\mathbb{R}^d} \Phi_q(x,y)\left[a\big(y, w(y) + v_g(y)\big) - q(y)w(y)\right] dy, \ x \in \mathbb{R}^d. \quad (4.33)$$

Since $\Phi_q(x,y)$ satisfy the Sommerfeld radiation condition (e.g., see Theorem 31.6 in [19]), w is a radiating solution of $\Delta w + a(x, w + v_g) + k^2 w = 0$ in $\mathbb{R}^d$. By the same argument as in Theorem 8.3 of [18], this is equivalent to the integral equation

$$w(x) = \int_{\mathbb{R}^d} \Phi(x,y) a\big(y, w(y) + v_g(y)\big) dy, \ x \in \mathbb{R}^d, \quad (4.34)$$

which means (4.6). Therefore, Theorem 4.2 has been shown.

4.4 The special solution

In Section 4.4, we consider the special solution of (4.3)–(4.4) corresponding to the incident field with several parameters in order to linearize problems. Let $N \in \mathbb{N}$ be fixed and let $g_j \in L^2(\mathbb{S}^{d-1})$ be fixed $(j = 1, 2, ..., N+1)$. We set

$$v_\epsilon := \sum_{j=1}^{N+1} \epsilon_j \delta^2 v_{g_j} = v_{\left(\delta^2 \sum_{j=1}^{N+1} \epsilon_j g_j\right)}, \quad (4.35)$$

where v_{g_j} is the Herglotz wave function defined by (4.2), and $\epsilon_j \in (0, \delta)$. Later, we will choose a appropriate $\delta = \delta_{g_j, N} > 0$. We remark that we can estimate that

$$\|v_\epsilon\|_{L^\infty(\mathbb{R}^d)} \leq C\delta^2 \sum_{j=1}^{N+1} \epsilon_j, \quad (4.36)$$

where $C > 0$ is constant only depending on g_j. We denote by $\epsilon = (\epsilon_1, ..., \epsilon_{N+1}) \in \mathbb{R}^{N+1}$. We will find a small solution u_ϵ of (4.6) that is of the form

$$u_\epsilon = r_\epsilon + v_\epsilon. \quad (4.37)$$

This problem is equivalent to

$$r_\epsilon(x) = \int_{\mathbb{R}^d} \Phi_q(x,y)\left[a\big(y, r_\epsilon(y) + v_\epsilon(y)\big) - q(y) r_\epsilon(y)\right] dy, \ x \in \mathbb{R}^d, \quad (4.38)$$

where $q := \partial_z a(\cdot, 0)$.

We define the space for $\delta > 0$

$$\tilde{X}_\delta := \left\{ r \in L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1}); \; \begin{array}{c} \text{ess.sup}_{x\in\mathbb{R}^d} |r(x,\epsilon)| \le \sum_{j=1}^{N+1} \epsilon_j, \\ \|r\|_{L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})} \le \delta, \end{array} \right\}, \tag{4.39}$$

where the norm $\|\cdot\|_{L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})}$ is defined by

$$\|r\|_{L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})} := \sum_{|\alpha| \le N+1} \sup_{\epsilon \in (0,\delta)^{N+1}} \text{ess.sup}_{x\in\mathbb{R}^d} \left| \partial_\epsilon^\alpha r(x,\epsilon) \right|. \tag{4.40}$$

We remark that $L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})$ is a Banach space, and $\tilde{X}_\delta$ is closed subspace in $L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})$. We will show that following lemma in the same way of Theorem 4.2.

Lemma 4.6. *We assume that $a(x,z)$ satisfies Assumption 4.1. Then, there exists $\tilde{\delta}_0 = \tilde{\delta}_{0,g_j,N} \in (0,1)$ such that for all $\delta \in (0, \tilde{\delta}_0)$ there exists an unique solution $r \in \tilde{X}_\delta$ such that*

$$r(x,\epsilon) = \int_{\mathbb{R}^d} \Phi_q(x,y) \left[a\big(y, r(y,\epsilon) + v_\epsilon(y)\big) - q(y) r(y,\epsilon) \right] dy, \; x \in \mathbb{R}^d, \; \epsilon \in (0,\delta)^{N+1}. \tag{4.41}$$

Proof. We define the operator $\tilde{T}$ from $L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})$ into itself by

$$\begin{aligned} \tilde{T} r(x,\epsilon) &:= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[a\big(y, r(y,\epsilon) + v_\epsilon(y)\big) - q(y) r(y,\epsilon) \right] dy \\ &= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[\sum_{l \ge 2} \frac{\partial_z^l a(y,0)}{l!} \big(r(y,\epsilon) + v_\epsilon(y) \big)^l + q(y) v_\epsilon(y) \right] dy \\ &= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[\sum_{l \ge 2} \frac{\partial_z^l a(y,0)}{l!} \sum_{m=0}^{l} \frac{l!}{(l-m)! m!} r^{l-m}(y,\epsilon) v_\epsilon^m(y) + q(y) v_\epsilon(y) \right] dy. \end{aligned} \tag{4.42}$$

Let $r \in \tilde{X}_\delta$. With (4.36) we have

$$\begin{aligned}
&\left|\tilde{T}r(x,\epsilon)\right| \\
\leq\ & \left(\int_{B_R} |\Phi_q(x,y)|\, dy\right) \left[\sum_{l\geq 2} c_0^l \sum_{m=0}^{l} \frac{1}{m!} \left(\sum_{j=1}^{N+1} \epsilon_j\right)^{l-m} \left(\tilde{C}_1\delta^2 \sum_{j=1}^{N+1} \epsilon_j\right)^m + \tilde{C}_1\delta^2 \sum_{j=1}^{N+1} \epsilon_j\right] \\
\leq\ & \tilde{C}_2 \left[\sum_{l\geq 2} c_0^l \left(\sum_{m=0}^{\infty} \frac{\tilde{C}_1^m}{m!}\right) \left(\sum_{j=1}^{N+1} \epsilon_j\right)^l + \tilde{C}_1\delta^2 \sum_{j=1}^{N+1} \epsilon_j\right] \\
\leq\ & \tilde{C}_3 \left(\sum_{j=1}^{N+1} \epsilon_j\right)^2 \sum_{l\geq 2} c_0^l \left(\sum_{j=1}^{N+1} \epsilon_j\right)^{l-2} + \tilde{C}_3\delta \left(\sum_{j=1}^{N+1} \epsilon_j\right) \\
\leq\ & \tilde{C}\delta \left(\sum_{j=1}^{N+1} \epsilon_j\right), \qquad (4.43)
\end{aligned}$$

where $\tilde{C}, \tilde{C}_j > 0$ $(j = 1, 2)$ is constant independent of r, δ, ϵ (but, depending on g_j and N). Furthermore, we consider for $\alpha \in \mathbb{N}^{N+1}$ with $|\alpha| \leq N+1$

$$\begin{aligned}
&\partial_\epsilon^\alpha \tilde{T}r(x,\epsilon) \\
=\ & \int_{\mathbb{R}^d} \Phi_q(x,y) \partial_\epsilon^\alpha \left[\sum_{l\geq 2} \frac{\partial_z^l a(y,0)}{l!} \sum_{m=0}^{l} \frac{l!}{(l-m)!m!} r^{l-m}(y,\epsilon) v_\epsilon^m(y) + q(y)v_\epsilon(y)\right] dy. \qquad (4.44)
\end{aligned}$$

Since $|\partial_{\epsilon_j} v_\epsilon(x)| \leq \tilde{C}_1'\delta^2$ and $|\partial_\epsilon^\alpha r^{l-m}(x,\epsilon) v_\epsilon^m(x)| \leq \tilde{C}_2'(l-m)!m!\delta^{l-m}(\tilde{C}_2'\delta^2)^m$, we have

$$\begin{aligned}
\left|\partial_\epsilon^\alpha \tilde{T}r(x,\epsilon)\right| \ \leq\ & \left(\int_{B_R} |\Phi_q(x,y)|\, dy\right) \left[\sum_{l\geq 2} \frac{c_0^l}{l!} \sum_{m=0}^{l} \frac{l!m!(l-m)!}{(l-m)!m!} \delta^{l+m} (\tilde{C}_2')^m + \tilde{C}_3'\delta^2\right] \\
\leq\ & \tilde{C}_4'\delta^2 \left(\sum_{l\geq 2} (c_0\delta)^{(l-2)} \sum_{m=0}^{\infty} (\tilde{C}_2'\delta)^m\right) + \tilde{C}_4'\delta^2 \leq \tilde{C}_5'\delta^2, \qquad (4.45)
\end{aligned}$$

where $\tilde{C}_j' > 0$ $(j = 3, 4, 5)$ is also constant independent of r, δ, ϵ (but depending on α). Then, we have

$$\sum_{|\alpha|\leq N+1} \sup_{\epsilon\in(0,\delta)^{N+1}} \operatorname{ess.sup}_{x\in\mathbb{R}^d} \left|\partial_\epsilon^\alpha \tilde{T}r(x,\epsilon)\right| \leq \tilde{C}'\delta^2, \qquad (4.46)$$

where $\tilde{C}'$ is constant independent of r, δ, ϵ. (depending on g_j and N.) By choosing $\tilde{\delta}_0 \in \left(0, \min(1/\tilde{C}, 1/\tilde{C}')\right)$, we conclude that $\tilde{T}r \in \tilde{X}_\delta$.

Let $r_1, r_2 \in \tilde{X}_\delta$. By similar argument in (4.29) we have

$$\begin{aligned}
& \tilde{T}r_1(x,\epsilon) - \tilde{T}r_2(x,\epsilon) \\
= & \int_{B_R} \Phi_q(x,y) \sum_{l\geq 2} \frac{\partial_z^l a(y,0)}{l!} \left[\left(r_1(y,\epsilon) + v_\epsilon(y)\right)^l - \left(r_2(y,\epsilon) + v_\epsilon(y)\right)^l \right] dy \\
= & \int_{B_R} \Phi_q(x,y) \sum_{l\geq 2} \frac{\partial_z^l a(y,0)}{l!} \sum_{m=1}^{l} \frac{l!}{(l-m)!m!} v_\epsilon^{l-m}(y) \\
& \qquad \times \sum_{h=0}^{m-1} r_1^{m-1-h}(y,\epsilon) r_2^h(y,\epsilon) \left(r_1(y,\epsilon) - r_2(y,\epsilon)\right) dy.
\end{aligned} \tag{4.47}$$

Then, we have for $\alpha \in \mathbb{N}^{N+1}$ with $|\alpha| \leq N+1$

$$\begin{aligned}
& \left| \partial_\epsilon^\alpha \left(\tilde{T}r_1(x) - \tilde{T}r_2(x) \right) \right| \\
\leq & \int_{B_R} |\Phi_q(x,y)| \sum_{\beta\leq\alpha} \frac{\alpha!}{(\alpha-\beta)!\beta!} \sum_{l\geq 2} \frac{\left|\partial_z^l a(y,0)\right|}{l!} \sum_{m=1}^{l} \frac{l!}{(l-m)!m!} \\
\times & \sum_{h=0}^{m-1} \left| \partial_\epsilon^\beta \left(v_\epsilon^{l-m}(y) r_1^{m-1-h}(y,\epsilon) r_2^h(y,\epsilon) \right) \right| \left| \partial_\epsilon^{\alpha-\beta} \left(r_1(y,\epsilon) - r_2(y,\epsilon)\right) \right| dy.
\end{aligned} \tag{4.48}$$

Since

$$\left| \partial_\epsilon^\beta \left(v_\epsilon^{l-m}(y) r_1^{m-1-h}(y,\epsilon) r_2^h(y,\epsilon) \right) \right| \leq \tilde{C}_1''(l-m)!(m-1-h)!h!(\tilde{C}_1''\delta^2)^{l-m}\delta^{m-1-h}\delta^h, \tag{4.49}$$

where $\tilde{C}_1''$ is constant independent of r_1, r_2 and δ (depending on β), we have that

$$
\begin{aligned}
&\left|\partial_\epsilon^\alpha\left(\tilde{T}r_1(x)-\tilde{T}r_2(x)\right)\right| \\
\leq\;& \tilde{C}_2''\left(\sum_{\beta\leq\alpha}\frac{\alpha!}{(\alpha-\beta)!\beta!}\sum_{l\geq 2}\frac{c_0^l}{l!}\sum_{m=1}^{l}\sum_{h=0}^{m-1}\frac{l!(l-m)!(m-1-h)!h!}{(l-m)!m!}\delta^{2l-m-1}(\tilde{C}_1'')^{l-m}\right)\|r_1-r_2\| \\
\leq\;& \tilde{C}_3''\delta\left(\sum_{l\geq 2}(c_0\delta)^{l-2}\sum_{m=1}^{l}(\tilde{C}_1''\delta)^{l-m}\sum_{h=0}^{m-1}\frac{(m-1-h)!h!}{m!}\right)\|r_1-r_2\| \\
\leq\;& \tilde{C}_4''\delta\left(\sum_{l\geq 2}(c_0\delta)^{l-2}\sum_{p=0}^{\infty}(\tilde{C}_1''\delta)^{p}\right)\|r_1-r_2\| \leq \tilde{C}_5''\delta\,\|r_1-r_2\|_{L^\infty(\mathbb{R}^d;C^{N+1}(0,\delta)^{N+1})}\,, \qquad (4.50)
\end{aligned}
$$

which implies that

$$
\sum_{|\alpha|\leq N+1}\sup_{\epsilon\in(0,\delta)^{N+1}}\mathrm{ess.sup}_{x\in\mathbb{R}^d}\left|\partial_\epsilon^\alpha\left(\tilde{T}r_1(x,\epsilon)-\tilde{T}r_2(x,\epsilon)\right)\right| \leq \tilde{C}''\delta\,\|r_1-r_2\|\,, \tag{4.51}
$$

where $\tilde{C}_j'', \tilde{C}'' > 0$ $(j = 2,3,4)$ is constant independent of r_1, r_2 and δ. By choosing $\tilde{\delta}_0 \in \left(0, \min(1/\tilde{C}, 1/\tilde{C}', 1/\tilde{C}'')\right)$, we have $\|Tr_1 - Tr_2\| < \|r_1 - r_2\|$, which implies that $\tilde{T}$ has a unique fixed point in $\tilde{X}_\delta$. Lemma 4.6 has been shown. □

4.5 Proof of Theorem 4.3

In Section 4.5, we will show Theorem 4.3. Since $a(x,z)$ is holomorphic at $z = 0$ by (ii) of Assumption 4.1, it is sufficient to show that

$$
\partial_z^l a_1(x,0) = \partial_z^l a_2(x,0),\ x\in\mathbb{R}^d, \tag{4.52}
$$

for all $l \in \mathbb{N}$. Let $N \in \mathbb{N}$ and let $g_j \in L^2(\mathbb{S}^{d-1})$ $(j = 1,2,...,N+1)$. Let $\delta \in \left(0, \min(\delta_0, \tilde{\delta}_0)\right)$ be chosen as sufficiently small and depending on N and g_j. ($\delta_0, \tilde{\delta}_0$ are corresponding to Theorem 4.2 and Lemma 4.6, respectively.) From Section 4.4, we obtain the unique solution $r_{\epsilon,j} \in \tilde{X}_\delta$ $(j = 1,2)$ such that

$$
\Delta r_{\epsilon,j} + a_j(x, r_{\epsilon,j} + v_\epsilon) + k^2 r_{\epsilon,j} = 0 \text{ in } \mathbb{R}^d, \tag{4.53}
$$

where $r_{\epsilon,j}$ satisfies the Sommerfeld radiation, and v_ϵ is given by (4.35). The solution $r_{\epsilon,j}$ has the form

$$
r_{\epsilon,j}(x) = \int_{\mathbb{R}^d}\Phi(x,y)a_j(y, r_{\epsilon,j}(y) + v_\epsilon(y))dy,\ x\in\mathbb{R}^d,\ \epsilon\in(0,\delta)^{N+1}. \tag{4.54}
$$

By the assumption of Theorem 4.3 we have

$$r_{\epsilon,1}^{\infty}(\hat{x}) = r_{\epsilon,2}^{\infty}(\hat{x}),\ \hat{x} \in \mathbb{S}^{d-1},\ \epsilon \in (0,\delta)^{N+1}, \tag{4.55}$$

where $r_{\epsilon,j}^{\infty}$ is a scattering amplitude for $r_{\epsilon,j}$, and it has the form

$$r_{\epsilon,j}^{\infty}(\hat{x}) = \int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} a_j(y, r_{\epsilon,j}(y) + v_{\epsilon}(y))dy,\ \hat{x} \in \mathbb{S}^{d-1},\ \epsilon \in (0,\delta)^{N+1}. \tag{4.56}$$

In order to linearize (4.54), we will differentiate it with respect to ϵ_l ($l = 1, ..., N+1$), which is possible because $r_{\epsilon,j} \in \tilde{X}_{\delta}$. Then, we have

$$\partial_{\epsilon_l} r_{\epsilon,j}(x) = \int_{\mathbb{R}^d} \Phi(x,y)\partial_z a_j(y, r_{\epsilon,j}(y) + v_{\epsilon}(y))(\partial_{\epsilon_l} r_{\epsilon,j}(y) + \delta^2 v_{g_l}(y))dy. \tag{4.57}$$

As $\epsilon \to +0$ we have by setting $q_j := \partial_z a_j(y,0)$

$$w_{l,j}(x) := \partial_{\epsilon_l} r_{\epsilon,j}\Big|_{\epsilon=0}(x) = \int_{\mathbb{R}^d} \Phi(x,y) q_j(y)(w_{l,j}(y) + \delta^2 v_{g_l}(y))dy, \tag{4.58}$$

which implies that

$$\Delta w_{l,j} + k^2 w_{l,j} = -q_j(w_{l,j} + \delta^2 v_{g_l}) \text{ in } \mathbb{R}^d. \tag{4.59}$$

By setting $u_{l,j} := w_{l,j} + \delta^2 v_{g_l}$ we have

$$\Delta u_{l,j} + k^2 u_{l,j} + q_j u_{l,j} = 0 \text{ in } \mathbb{R}^d. \tag{4.60}$$

By setting $u_l := u_{l,1} - u_{l,2} (= w_{l,1} - w_{l,2})$ we have

$$\Delta u_l + k^2 u_l + q_1 u_l = (q_2 - q_1) u_{l,2} \text{ in } \mathbb{R}^d, \tag{4.61}$$

and we also have

$$(q_2 - q_1) u_{h,1} u_{l,2} = u_{h,1} \Delta u_l - u_l \Delta u_{h,1} \text{ in } \mathbb{R}^d. \tag{4.62}$$

Differentiating (4.54) with respect to ϵ_l and as $\epsilon \to 0$ we have

$$\int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} q_1(y)(w_{l,1}(y) + \delta^2 v_{g_l}(y))dy = \int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} q_2(y)(w_{l,2}(y) + \delta^2 v_{g_l}(y))dy, \tag{4.63}$$

which means that $w_{l,1}^{\infty} = w_{l,2}^{\infty}$, where $w_{l,j}^{\infty}$ is a scattering amplitude of $w_{l,j}$. By setting $\hat{w}_l := w_{l,1} - w_{l,2}$ we have

$$\Delta \hat{w}_l + k^2 \hat{w}_l = 0 \text{ in } \mathbb{R} \setminus \overline{B_R}, \tag{4.64}$$

where $\hat{w}_l$ satisfies the Sommerfeld radiation condition, and the scattering amplitude $\hat{w}_l^{\infty}$ of $\hat{w}_l$ vanishes. Then, we have $\hat{w}_l = 0$ (that is, $u_l = 0$) in $\mathbb{R} \setminus \overline{B_R}$, which implies that by the Green's second theorem we have ($l, h = 1, ..., N+1$)

$$\begin{aligned} 0 &= \int_{\partial B_{R+1}} u_{h,1}\partial_\nu u_l - u_l \partial_\nu u_{h,1} ds \\ &= \int_{B_{R+1}} u_{h,1}\Delta u_l - u_l \Delta u_{h,1} dx \\ &= \int_{B_R} (q_2 - q_1) u_{h,1} u_{l,2} dx. \end{aligned} \tag{4.65}$$

By (4.59), and definition of H and T_{q_j} in Section 4.2, $u_{l,j}$ can be of the form

$$u_{l,j} = \delta^2 T_{q_j} H g_l, \tag{4.66}$$

and dividing by $\delta^4 > 0$,

$$0 = \int_{B_R} (q_2 - q_1) T_{q_1} H g_h T_{q_2} H g_l dx. \tag{4.67}$$

Combining Lemma 4.4 with Lemma 4.5, we conclude that $q_1 = q_2$.

By induction, we will show (4.53). In the first part of this section, the case of $l = 1$ has been shown. We assume that

$$\partial_z^l a_1(x, 0) = \partial_z^l a_2(x, 0), \tag{4.68}$$

for all $l = 1, 2, ..., N$. We will show the case of $l = N+1$. We alredy have shown that $q_1 = q_2$ and $w_{l,1}^{\infty} = w_{l,2}^{\infty}$, which implies that by the uniqueness of the linear Schrödinger equation (4.53) we have

$$w_{l,1} = w_{l,2} \text{ in } \mathbb{R}^d, \tag{4.69}$$

for all $l = 1, ..., N+1$.

We set $q := q_1 = q_2$ and $w_l := w_{l,1} = w_{l,2}$. By subinduction we will show that for all $h \in \mathbb{N}$ with $1 \le h \le N$

$$\partial^h_{\epsilon_{l_1} ... \epsilon_{l_h}} r_{\epsilon,1}\Big|_{\epsilon=0} = \partial^h_{\epsilon_{l_1} ... \epsilon_{l_h}} r_{\epsilon,2}\Big|_{\epsilon=0}, \tag{4.70}$$

where $l_1, ... l_h \in \{1, ..., N+1\}$. We already have shown that (4.70) holds for $h = 1$. We assume that (4.70) holds for all $h \le K \le N-1$. (If $N = 1$, this subinduction is skipped.) By differentiating (4.54) with respect to $\partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}}$ we have

$$\partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}(x) = \int_{\mathbb{R}^d} \Phi(x,y) \Bigg\{ \partial_z^{K+1} a_j(y, r_{\epsilon,j}(y) + v_\epsilon(y)) \prod_{h=1}^{K+1} (\partial_{\epsilon_{l_h}} r_{\epsilon,j}(y) + \delta^2 v_{g_{l_h}}(y))$$
$$+\partial_z a_j(y, r_{\epsilon,j}(y) + v_\epsilon(y)) \partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}(y) + R_{K,j}(y,\epsilon) \Bigg\} dy, \quad (4.71)$$

where $R_{K,j}(y,\epsilon)$ is a polynomial of $\partial_z^h a_j(y, r_{\epsilon,j}(y)+v_\epsilon(y))$ and $\partial^h_{\epsilon_{l_1}\cdots\epsilon_{l_h}} (r_{\epsilon,j}(y) + v_\epsilon(y))$ for $1 \leq h \leq K$. As $\epsilon \to 0$ we have

$$\partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}\Big|_{\epsilon=0}(x) = \int_{\mathbb{R}^d} \Phi(x,y) \Bigg\{ \partial_z^{K+1} a_j(y,0) \prod_{h=1}^{K+1} (w_{l_h}(y) + \delta^2 v_{g_{l_h}}(y))$$
$$+q(y) \partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}\Big|_{\epsilon=0}(y) + R_{K,j}(y,0) \Bigg\} dy. \quad (4.72)$$

We set $\tilde{w}_{K+1,j} := \partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}\Big|_{\epsilon=0}$ and set $\tilde{w}_{K+1} := \tilde{w}_{K+1,1} - \tilde{w}_{K+1,2}$. By assumptions of induction and subinduction we have $R_{K,1}(y,0) = R_{K,2}(y,0)$ and $\partial_z^{K+1} a_1(\cdot,0) = \partial_z^{K+1} a_2(\cdot,0)$, which implies that

$$\tilde{w}_{K+1}(x) = \int_{\mathbb{R}^d} \Phi(x,y) q(y) \tilde{w}_{K+1}(y) dy, \quad (4.73)$$

which is equivalent to

$$\Delta \tilde{w}_{K+1} + k^2 \tilde{w}_{K+1} + q \tilde{w}_{K+1} = 0 \text{ in } \mathbb{R}^d, \quad (4.74)$$

where $\tilde{w}_{K+1}$ satisfies Sommerfeld radiation condition. By differentiating (4.55) with respect to $\partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}}$ and as $\epsilon \to 0$ we have

$$\tilde{w}^\infty_{K+1,1} = \tilde{w}^\infty_{K+1,2}, \quad (4.75)$$

where $\tilde{w}^\infty_{K+1,j}$ is a scattering amplitude of $\tilde{w}_{K+1,j}$. (4.75) means that $\tilde{w}^\infty_{K+1} = 0$, which implies that by Rellich theorem, we conclude that $\tilde{w}_{K+1} = 0$ in $\mathbb{R}^d$. (4.70) for the case of $K+1$ has been shown, and the claim (4.70) holds for all $h = 1, ..., N$ by subinduction.

By differentiating (4.54) with respect to $\partial^{N+1}_{\epsilon_1\ldots\epsilon_{K+1}}$, and as $\epsilon \to 0$ (the same argument in (4.71)–(4.73)) we have

$$\tilde{w}_{N+1}(x) = \int_{\mathbb{R}^d} \Phi(x,y) \Bigg\{ \left(\partial_z^{N+1} a_1(x,0) - \partial_z^{N+1} a_2(x,0)\right) \prod_{h=1}^{N+1} (w_h(y) + \delta^2 v_{g_h}(y))$$

$$+q(y)\tilde{w}_{N+1}(y)\Bigg\}dy. \tag{4.76}$$

where $\tilde{w}_{N+1,j} := \partial^{N+1}_{\epsilon_1 \dots \epsilon_{l_{N+1}}} r_{\epsilon,j}\Big|_{\epsilon=0}$ and set $\tilde{w}_{N+1} := \tilde{w}_{N+1,1} - \tilde{w}_{N+1,2}$. This is equivalent to

$$\Delta\tilde{w}_{N+1} + k^2\tilde{w}_{N+1} + q\tilde{w}_{N+1} = -f\prod_{h=1}^{N+1}\delta^2 T_q H g_h \text{ in } \mathbb{R}^d, \tag{4.77}$$

where $f(x) := \partial_z^{N+1}a_1(x,0) - \partial_z^{N+1}a_2(x,0)$. By differentiating (4.55) with respect to $\partial^{N+1}_{\epsilon_1 \dots \epsilon_{K+1}}$ and as $\epsilon \to 0$ (the same argument in (4.75)) we have

$$\tilde{w}^{\infty}_{N+1} = 0, \tag{4.78}$$

where $\tilde{w}^{\infty}_{N+1}$ is a scattering amplitude of $\tilde{w}_{N+1}$. Then, we have $\tilde{w}_{N+1} = 0$ in $\mathbb{R} \setminus \overline{B_R}$.

Let $\tilde{v} \in L^2(B_{R+1})$ be a solution of $\Delta\tilde{v} + k^2\tilde{v} + q\tilde{v} = 0$ in B_{R+1}. By the Green's second theorem and (4.77) we have

$$\begin{aligned} 0 &= \int_{\partial B_{R+1}} \tilde{v}\partial_\nu \tilde{w}_{N+1} - \tilde{v}\partial_\nu \tilde{w}_{N+1} ds \\ &= \int_{B_{R+1}} \tilde{v}\Delta\tilde{w}_{N+1} - \tilde{w}_{N+1}\Delta\tilde{v} dx \\ &= \int_{B_{R+1}} -f\prod_{h=1}^{N+1}\delta^2 T_q H g_h \tilde{v} dx, \end{aligned} \tag{4.79}$$

which implies that dividing by $\delta^2 > 0$

$$\int_{B_{R+1}} f\prod_{h=1}^{N+1} T_q H g_h \tilde{v} dx = 0. \tag{4.80}$$

Let $v \in L^2(B_{R+1})$ be a solution of $\Delta v + k^2 v + qv = 0$ in B_{R+1}. By Lemma 4.4 we can choose g_{N+1} as $g_{N+1,j} \in L^2(B_{R+1})$ such that $T_q H g_{N+1,j} \to v$ in $L^2(B_R)$ as $j \to \infty$. Then, we have that

$$\int_{B_{R+1}} f\prod_{h=1}^{N} T_q H g_h v\tilde{v} dx = 0. \tag{4.81}$$

which implies that by Lemma 4.5

$$f\prod_{h=1}^{N} T_q H g_h = 0. \tag{4.82}$$

By Theorem 5.1 of [92], we can choose a solution $u_h \in L^2(B_{R+1})$ ($h = 1, ..., N$) of $\Delta u_h + k^2 u_h + q u_h = 0$ in B_{R+1}, which is of the form

$$u_h(x) = e^{x \cdot p_h}(1 + \psi_h(x, p_h)), \tag{4.83}$$

with $\|\psi_h(\cdot, p_h)\|_{L^2(B_{R+1})} \leq \frac{C}{|p_h|}$ where $C > 0$ is a constant, and $p_h = a_h + i b_h$, $a_h, b_h \in \mathbb{R}^d$ such that $|a_h| = |b_h|$ and $a_h \cdot b_h = 0$ (which implies that $p_h \cdot p_h = 0$), and $a_h \neq a_{h'}$, $b_h \neq b_{h'}$.

Multiplying (4.82) by $\overline{f} \prod_{h=1}^{N+1} e^{-x \cdot p_h}$ we have

$$|f|^2 \prod_{h=1}^{N} e^{-x \cdot p_h} T_q H g_h = 0, \tag{4.84}$$

which implies that

$$\int_{B_R} |f|^2 \left(\prod_{h=1}^{N-1} e^{-x \cdot p_h} T_q H g_h \right) e^{-x \cdot p_N} T_q H g_N dx = 0. \tag{4.85}$$

By Lemma 4.4, there exists a sequence $\{g_{N,j}\}_{j \in \mathbb{N}} \subset L^2(\mathbb{S}^{d-1})$ such that $T_q H g_{N,j} \to u_N = e^{x \cdot p_N}(1 + \psi_N(x, p_N))$ in $L^2(B_R)$, which implies that

$$\int_{B_R} |f|^2 \left(\prod_{h=1}^{N-1} e^{-x \cdot p_h} T_q H g_h \right) (1 + \psi(x, p_N)) dx = 0. \tag{4.86}$$

As $|a_N| = |b_N| \to \infty$ in (4.86) we have

$$\int_{B_R} |f|^2 \prod_{h=1}^{N-1} e^{-x \cdot p_h} T_q H g_h dx = 0. \tag{4.87}$$

Repeating the operation (4.85)–(4.87) $(N-1)$ times, we have that

$$\int_{B_R} |f|^2 dx = 0, \tag{4.88}$$

which conclude that $f = 0$. By induction, we conclude that (4.52) for all $l \in \mathbb{N}$. Therefore, Theorem 4.3 has been shown.

5 Scattering by the local perturbation of an open periodic waveguide in the half plane

5.1 Introduction

Let $k > 0$ be the wave number, and let $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$ be the upper half plane, and let $W := \mathbb{R} \times (0, h)$ be the waveguide in $\mathbb{R}^2_+$. We denote by $\Gamma_a := \mathbb{R} \times \{a\}$ for $a > 0$. Let $n \in L^\infty(\mathbb{R}^2_+)$ be real value, 2π-periodic with respect to x_1 (that is, $n(x_1 + 2\pi, x_2) = n(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2_+$), and equal to one for $x_2 > h$. We assume that there exists a constant $n_0 > 0$ such that $n \geq n_0$ in $\mathbb{R}^2_+$. Let $q \in L^\infty(\mathbb{R}^2_+)$ be real valued with the compact support supp q in W. We denote by $Q :=$ supp q. In this paper, we consider the following scattering problem: For fixed $y \in \mathbb{R}^2_+ \setminus \overline{W}$, determine the scattered field $u^s \in H^1_{loc}(\mathbb{R}^2_+)$ such that

$$\Delta u^s + k^2(1 + q)nu^s = -k^2 qnu^i(\cdot, y) \text{ in } \mathbb{R}^2_+, \tag{5.1}$$

$$u^s = 0 \text{ on } \Gamma_0, \tag{5.2}$$

Here, the incident field u^i is given by $u^i(x, y) = G_n(x, y)$, where G_n is the Dirichlet Green's function in the upper half plane $\mathbb{R}^2_+$ for $\Delta + k^2 n$, that is,

$$G_n(x, y) := G(x, y) + \tilde{u}^s(x, y), \tag{5.3}$$

where $G(x, y) := \Phi_k(x, y) - \Phi_k(x, y^*)$ is the Dirichlet Green's function in $\mathbb{R}^2_+$ for $\Delta + k^2$, and $y^* = (y_1, -y_2)$ is the reflected point of y at $\mathbb{R} \times \{0\}$. Here, $\Phi_k(x, y)$ is the fundamental solution to Helmholtz equation in $\mathbb{R}^2$, that is,

$$\Phi_k(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|), \ x \neq y, \tag{5.4}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order one. $\tilde{u}^s$ is the scattered field of the unperturbed problem by the incident field $G(x, y)$, that is, $\tilde{u}^s$ vanishes for $x_2 = 0$ and solves

$$\Delta \tilde{u}^s + k^2 n \tilde{u}^s = k^2(1 - n)G(\cdot, y) \text{ in } \mathbb{R}^2_+. \tag{5.5}$$

If we impose a suitable radiation condition introduced in [62], the unperturbed solution $\tilde{u}^s$ is uniquely determined. Later, we will explain the exact definition of this radiation condition (see Definition 5.6).

In order to show the well-posedness of the perturbed scattering problem (5.1)–(5.2), we make the following assumption.

Assumption 5.1. *We assume that k^2 is not the point spectrum of $\frac{1}{(1+q)n}\Delta$ in $H^1_0(\mathbb{R}^2_+)$, that is, every $v \in H^1(\mathbb{R}^2_+)$ which satisfies*

$$\Delta v + k^2(1+q)nv = 0 \text{ in } \mathbb{R}^2_+, \tag{5.6}$$

$$v = 0 \text{ on } \Gamma_0, \tag{5.7}$$

has to vanish for $x_2 > 0$.

If we assume that q and n satisfy in addition that $\partial_2\big((1+q)n\big) \geq 0$ in W, then v which satisfies (5.6)–(5.7) vanishes, that is, under this assumption all of k^2 is not the point spectrum of $\frac{1}{(1+q)n}\Delta$ (see Section 5.6). Our aim in this section is to show the following theorem.

Theorem 5.2. *Let Assumptions 5.1 and 5.3 hold and let $k > 0$ be regular in the sense of Definition 5.5 and let $f \in L^2(\mathbb{R}^2_+)$ such that* $\mathrm{supp} f = Q$. *Then, there exists a unique solution $u \in H^1_{loc}(\mathbb{R}^2_+)$ such that*

$$\Delta u + k^2(1+q)nu = f \text{ in } \mathbb{R}^2_+, \tag{5.8}$$

$$u = 0 \text{ on } \Gamma_0, \tag{5.9}$$

and u satisfies the radiation condition in the sense of Definition 5.6.

Roughly speaking, the radiation condition of Definition 5.6 requires that we have a decomposition of the solution u into $u^{(1)}$ which decays in the direction of x_1, and a finite combination $u^{(2)}$ of *propagative modes* which does not decay, but it exponentially decays in the direction of x_2.

This section is organized as follows. In Section 5.2, we briefly recall a radiation condition introduced in [62]. Under the radiation condition in the sense of Definition 5.6, we show the uniqueness of $u^{(2)}$ and $u^{(1)}$ in Section 5.3 and 5.4, respectively. In Section 5.5, we show the existence of u. In Section 5.6, we give an example of n and q with respect to Assumption 5.1.

5.2 A radiation condition

In Section 5.2, we briefly recall a radiation condition introduced in [62]. Let $f \in L^2(\mathbb{R}^2_+)$ have the compact support in W. First, we consider the following problem: Find $u \in H^1_{loc}(\mathbb{R}^2_+)$ such that

$$\Delta u + k^2 nu = f \text{ in } \mathbb{R}^2_+, \tag{5.10}$$

$$u = 0 \text{ on } \Gamma_0. \tag{5.11}$$

(5.10) is understood in the variational sense, that is,

$$\int_{\mathbb{R}^2_+} \left[\nabla u \cdot \nabla \overline{\varphi} - k^2 n u \overline{\varphi}\right] dx = -\int_W f \overline{\varphi} dx, \tag{5.12}$$

for all $\varphi \in H^1(\mathbb{R}^2_+)$, with compact support. In such a problem, it is natural to impose the *upward propagating radiation condition*, that is, $u(\cdot, h) \in L^\infty(\mathbb{R})$ and

$$u(x) = 2\int_{\Gamma_h} u(y) \frac{\partial \Phi_k(x,y)}{\partial y_2} ds(y) = 0, \ x_2 > h. \tag{5.13}$$

However, even with this condition we can not expect the uniqueness of this problem. (see Example 2.3 of [62].) In order to introduce a *suitable radiation condition*, [62] discussed limiting absorption solution of this problem, that is, the limit of the solution u_ϵ of $\Delta u_\epsilon + (k+i\epsilon)^2 n u_\epsilon = f$ as $\epsilon \to 0$. For the details of an introduction of this radiation condition, we refer to [55, 56, 61, 62].

Let us prepare for the exact definition of the radiation condition. First we recall that the *Floquet Bloch transform* $T_{per} : L^2(\mathbb{R}) \to L^2\big((0, 2\pi) \times (-1/2, 1/2)\big)$ is defined by

$$T_{per} f(t, \alpha) = \tilde{f}_\alpha(t) := \sum_{m \in \mathbb{Z}} f(t + 2\pi m) e^{-i\alpha(t+2\pi m)}, \tag{5.14}$$

for $(t, \alpha) \in (0, 2\pi) \times (-1/2, 1/2)$. The inverse transform is given by

$$T^{-1}_{per} g(t) = \int_{-1/2}^{1/2} g(t, \alpha) e^{i\alpha t} d\alpha, \ t \in \mathbb{R}. \tag{5.15}$$

By taking the Floquet Bloch transform with respect to x_1 in (5.10)–(5.11), we have for $\alpha \in (-1/2, 1/2]$

$$\Delta \tilde{u}_\alpha + 2i\alpha \frac{\partial \tilde{u}_\alpha}{\partial x_1} + (k^2 n - \alpha^2)\tilde{u}_\alpha = \tilde{f}_\alpha \text{ in } (0, 2\pi) \times (0, \infty). \tag{5.16}$$

$$\tilde{u}_\alpha = 0 \text{ on } (0, 2\pi) \times \{0\}. \tag{5.17}$$

By taking the Floquet Bloch transform with respect to x_1 in (5.13), $\tilde{u}_\alpha$ satisfies the *Rayleigh expansion* of the form

$$\tilde{u}_\alpha(x) = \sum_{n \in \mathbb{Z}} u_n(\alpha) e^{inx_1 + i\sqrt{k^2 - (n+\alpha)^2}(x_2 - h)}, \ x_2 > h, \tag{5.18}$$

where $u_n(\alpha) := (2\pi)^{-1} \int_0^{2\pi} u_\alpha(x_1, h) e^{-inx_1} dx_1$ are the Fourier coefficients of $u_\alpha(\cdot, h)$, and $\sqrt{k^2 - (n+\alpha)^2} = i\sqrt{(n+\alpha)^2 - k^2}$ if $n + \alpha > k$.

We denote by $C_R := (0, 2\pi) \times (0, R)$ for $R \in (0, \infty]$, and $H^1_{per}(C_R)$ the subspace of the 2π-periodic function in $H^1(C_R)$. We also denote by $H^1_{0,per}(C_R) := \{u \in H^1_{per}(C_R) : u = 0 \text{ on } (0, 2\pi) \times \{0\}\}$ that is equipped with $H^1(C_R)$ norm. The space $H^1_{0,per}(C_R)$ has the inner product of the form

$$\langle u, v \rangle_* = \int_{C_h} \nabla u \cdot \nabla \overline{v} dx + 2\pi \sum_{n \in \mathbb{Z}} \sqrt{n^2 + 1} u_n \overline{v_n}, \tag{5.19}$$

where $u_n = (2\pi)^{-1} \int_0^{2\pi} u(x_1, R) e^{-inx_1} dx_1$. The problem (5.16)–(5.18) is equivalent to the following operator equation (see section 3 in [62]),

$$\tilde{u}_\alpha - K_\alpha \tilde{u}_\alpha = \tilde{f}_\alpha \text{ in } H^1_{0,per}(C_h), \tag{5.20}$$

where the operator $K_\alpha : H^1_{0,per}(C_h) \to H^1_{0,per}(C_h)$ is defined by

$$\begin{aligned} \langle K_\alpha u, v \rangle_* &= -\int_{C_h} \left[i\alpha \left(u \frac{\partial \overline{v}}{\partial x_1} - \overline{v} \frac{\partial \overline{u}}{\partial x_1} \right) + (\alpha^2 - k^2 n) u \overline{v} \right] dx \\ &+ 2\pi i \sum_{|n+\alpha| \leq k} u_n \overline{v_n} \big(\sqrt{k^2 - (n+\alpha)^2} - i\sqrt{n^2+1} \big) \\ &+ 2\pi \sum_{|n+\alpha| > k} u_n \overline{v_n} \big(\sqrt{n^2+1} - \sqrt{(n+\alpha)^2 - k^2} \big). \end{aligned} \tag{5.21}$$

For several $\alpha \in (-1/2, 1/2]$, the uniqueness of this problem fails. We call these α *exceptional values* if the operator $I - K_\alpha$ fails to be injective. For the difficulty of treatment of α such that $|\alpha + l| = k$ for some $l \in \mathbb{Z}$ in an periodic scattering problem, we set $A_k := \{\alpha \in (-1/2, 1/2] : \exists l \in \mathbb{Z} \ s.t. \ |\alpha + l| = k\}$, and make the following assumption:

Assumption 5.3. *For every $\alpha \in A_k$, $I - K_\alpha$ has to be injective.*

The following properties of exceptional values was shown in Lemmas 4.2 and 5.6 of [62].

Lemma 5.4. *Let Assumption 5.3 hold. Then, there exists only finitely many exceptional values $\alpha \in (-1/2, 1/2]$. Furthermore, if α is an exceptional value, then so is $-\alpha$. Therefore, the set of exceptional values can be described by $\{\alpha_j : j \in J\}$ where some $J \subset \mathbb{Z}$ is finite and $\alpha_{-j} = -\alpha_j$ for $j \in J$. For each exceptional value α_j we define*

$$X_j := \left\{ \phi \in H^1_{loc}(\mathbb{R}^2_+) : \begin{array}{c} \Delta\phi + 2i\alpha_j \frac{\partial \phi}{\partial x_1} + (k^2 n - \alpha^2)\phi = 0 \text{ in } \mathbb{R}^2_+, \\ \phi = 0 \text{ for } x_2 = 0, \quad \phi \text{ is } 2\pi\text{−periodic for } x_1, \\ \phi \text{ satisfies the Rayleigh expansion (5.18)} \end{array} \right\}$$

Then, X_j are finite dimensional. We set $m_j = \dim X_j$. Furthermore, $\phi \in X_j$ is evanescent, that is, there exists $c > 0$ and $\delta > 0$ such that $|\phi(x)|,\ |\nabla\phi(x)| \leq ce^{-\delta|x_2|}$ for all $x \in \mathbb{R}^2_+$.

Next, we consider the following eigenvalue problem in X_j: Determine $d \in \mathbb{R}$ and $\phi \in X_j$ such that

$$\int_{C_\infty} \left[-i\frac{\partial\phi}{\partial x_1} + \alpha_j\phi \right] \overline{\psi} dx = dk \int_{C_\infty} n\phi\overline{\psi} dx, \tag{5.22}$$

for all $\psi \in X_j$. We denote by the eigenvalues $d_{l,j}$ and the eigenfunction $\phi_{l,j}$ of this problem, that is,

$$\int_{C_\infty} \left[-i\frac{\partial\phi_{l,j}}{\partial x_1} + \alpha_j\phi_{l,j} \right] \overline{\psi} dx = d_{l,j}k \int_{C_\infty} n\phi_{l,j}\overline{\psi} dx, \tag{5.23}$$

for every $l = 1, ..., m_j$ and $j \in J$. We normalize the eigenfunction $\{\phi_{l,j} : l = 1, ..., m_j\}$ such that

$$k \int_{C_\infty} n\phi_{l,j}\overline{\phi_{l',j}} dx = \delta_{l,l'}, \tag{5.24}$$

for all l, l'. We will assume that the wave number $k > 0$ is *regular* in the following sense.

Definition 5.5. $k > 0$ is *regular* if $d_{l,j} \neq 0$ for all $l = 1, ...m_j$ and $j \in J$.

Now we are ready to define the radiation condition.

Definition 5.6. Let Assumptions 5.3 hold, and let $k > 0$ be regular in the sense of Definition 5.5. We set

$$\psi^{\pm}(x_1) := \frac{1}{2}\left[1 \pm \frac{2}{\pi}\int_0^{x_1/2} \frac{sint}{t} dt \right],\ x_1 \in \mathbb{R}. \tag{5.25}$$

Then, $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfies the *radiation condition* if u satisfies the upward propagating radiation condition (5.13), and has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)}\big|_{\mathbb{R}\times(0,R)} \in H^1(\mathbb{R} \times (0, R))$ for all $R > 0$, and $u^{(2)} \in L^\infty(\mathbb{R}^2_+)$ has the following form

$$u^{(2)}(x) = \psi^+(x_1) \sum_{j\in J} \sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \psi^-(x_1) \sum_{j\in J} \sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x), \tag{5.26}$$

where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, ..., m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (5.23).

Remark 5.7. We can replace $\psi^{\pm}$ by any smooth functions $\tilde{\psi}^{\pm}$ such that $\left|\psi^{\pm}(x_1) - \tilde{\psi}^{\pm}(x_1)\right| \to 0$, and $\left|\frac{d}{dx_1}\psi^{\pm}(x_1) - \frac{d}{dx_1}\tilde{\psi}^{\pm}(x_1)\right| \to 0$ as $|x_1| \to \infty$ because (5.26) is of the form

$$u^{(2)}(x) = \tilde{\psi}^{+}(x_1)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \tilde{\psi}^{-}(x_1)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x)$$
$$+\Big(\psi^{+}(x_1) - \tilde{\psi}^{+}(x_1)\Big)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \Big(\psi^{-}(x_1) - \tilde{\psi}^{-}(x_1)\Big)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x), \tag{5.27}$$

where the second term in the right-hand side of (5.27) is a H^1-function, which is the role of $u^{(1)}$.

The following was shown in Theorems 2.2, 6.6, and 6.8 of [62].

Theorem 5.8. *Let Assumptions 5.3 hold and let $k > 0$ be regular in the sense of Definition 5.5. For every $f \in L^2(\mathbb{R}^2_+)$ with the compact support in W, there exists a unique solution $u_{k+i\epsilon} \in H^1(\mathbb{R}^2_+)$ of the problem (5.10)–(5.11) replacing k by $k + i\epsilon$. Furthermore, $u_{k+i\epsilon}$ converge as $\epsilon \to +0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to some $u \in H^1_{loc}(\mathbb{R}^2_+)$ which satisfy (5.10)–(5.11) and the radiation condition in the sense of Definition 5.6. Furthermore, the solution u of this problem is uniquely determined.*

Finally in this section, we will show the following integral representation.

Lemma 5.9. *Let $f \in L^2(\mathbb{R}^2_+)$ have a compact support in W, and let u be a solution of (5.10)–(5.11) which satisfying the radiation condition in the sense of Definition 5.6. Then, u has an integral representation of the form*

$$u(x) = k^2\int_W (n(y)-1)u(y)G(x,y)dy - \int_W f(y)G(x,y)dy, \quad x \in \mathbb{R}^2_+ \tag{5.28}$$

Proof of Lemma 5.9. Let $\epsilon > 0$ be small enough and let $u_\epsilon \in H^1(\mathbb{R}^2_+)$ be a solution of the problem (5.10)–(5.11) replacing k by $k + i\epsilon$, that is, u_ϵ satisfies

$$\Delta u_\epsilon + (k+i\epsilon)^2 n u_\epsilon = f \text{ in } \mathbb{R}^2_+, \tag{5.29}$$

$$u_\epsilon = 0 \text{ on } \Gamma_0. \tag{5.30}$$

Let $G_\epsilon(x,y)$ be the Dirichlet Green's function in the upper half plane $\mathbb{R}^2_+$ for $\Delta + (k+i\epsilon)^2$. Let $x \in \mathbb{R}^2_+$ be always fixed such that $x_2 > R$. Let $r > 0$ be large enough such that $x \in B_r(0)$ where $B_r(0) \subset \mathbb{R}^2$ be a open ball with center 0 and radius $r > 0$. By Green's representation theorem in $B_r(0) \cap \mathbb{R}^2_+$ we have

$$\begin{aligned}
u_\epsilon(x) &= \int_{\partial B_r(0)\cap\mathbb{R}^2_+} \big[\frac{\partial u_\epsilon}{\partial \nu}(y) G_\epsilon(x,y) - u_\epsilon(y)\frac{\partial G_\epsilon}{\partial \nu}(x,y)\big] ds(y) \\
&- \int_{B_r(0)\cap\mathbb{R}^2_+} \big[\Delta u_\epsilon(y) + (k+i\epsilon)^2 u_\epsilon(y)\big] G_\epsilon(x,y) dy \\
&= \int_{\partial B_r(0)\cap\mathbb{R}^2_+} \big[\frac{\partial u_\epsilon}{\partial \nu}(y) G_\epsilon(x,y) - u_\epsilon(y)\frac{\partial G_\epsilon}{\partial \nu}(x,y)\big] ds(y) \\
&+ (k+i\epsilon)^2 \int_{B_r(0)\cap\mathbb{R}^2_+} (n(y)-1) u_\epsilon(y) G_\epsilon(x,y) dy \\
&- \int_{B_r(0)\cap\mathbb{R}^2_+} f(y) G_\epsilon(x,y) dy. \qquad (5.31)
\end{aligned}$$

Since $u_\epsilon \in H^1(\mathbb{R}^2_+)$, the first term of the right hand side converges to zero as $r \to \infty$. Therefore, as $r \to \infty$ we have for $x \in \mathbb{R}^2_+$

$$u_\epsilon(x) = (k+i\epsilon)^2 \int_W (n(y)-1) u_\epsilon(y) G_\epsilon(x,y) dy - \int_W f(y) G_\epsilon(x,y) dy. \quad (5.32)$$

We will show that (5.32) converges as $\epsilon \to 0$ to

$$u(x) = k^2 \int_W (n(y)-1) u(y) G(x,y) dy - \int_W f(y) G(x,y) dy. \qquad (5.33)$$

Indeed, by the argument in (3.8) and (3.9) of [13], $G_\epsilon(x,y)$ is of the estimation

$$|G_\epsilon(x,y)| \leq C \frac{x_2 y_2}{1+|x-y|^{3/2}}, \ |x-y| > 1, \qquad (5.34)$$

where above C is independent of $\epsilon > 0$. Then, by Lebesgue dominated convergence theorem we have the second integral in (5.32) converges as $\epsilon \to 0$ to one in (5.33). So, we will consider the convergence of the first integral in (5.32).

By the beginning of the proof of Theorem 6.6 in [62], u_ϵ can be of the form $u_\epsilon = u^{(1)}_\epsilon + u^{(2)}_\epsilon$ where $u^{(1)}_\epsilon$ converges to $u^{(1)}$ in $H^1(W)$, and $u^{(2)}_\epsilon$ is of the form for $x \in W$

$$u^{(2)}_\epsilon(x) = \sum_{j \in J} \sum_{l=1}^{m_j} y_{l,j} \int_{-1/2}^{1/2} \frac{e^{i\alpha x_1}}{i\epsilon - d_{l,j}\alpha} d\alpha \ \phi_{l,j}(x), \qquad (5.35)$$

which converges pointwise to $u^{(2)}(x)$. Here, $y_{l,j} \in \mathbb{C}$ is some constant. From the convergence of $u_{\epsilon}^{(1)}$ in $H^1(W)$ we obtain that $\int_W (n(y)-1)u_{\epsilon}^{(1)}(y)G_{\epsilon}(x,y)dy$ converges $\int_W (n(y)-1)u^{(1)}(y)G(x,y)dy$ as $\epsilon \to 0$.

By the argument of (b) in Lemma 6.1 of [62] we have

$$\begin{aligned} \psi_{l,j,\epsilon}(x_1) &:= \int_{-1/2}^{1/2} \frac{e^{i\alpha x_1}}{i\epsilon - d_{l,j}\alpha} d\alpha \\ &= -\frac{i}{|d_{l,j}|}\int_{-|d_{l,j}|/(2\epsilon)}^{|d_{l,j}|/(2\epsilon)} \frac{\cos(t\epsilon x_1/|d_{l,j}|)}{1+t^2}dt - 2id_{l,j}\int_0^{x_1/2}\frac{t\mathrm{sin}t}{x_1^2\epsilon^2 + d_{l,j}^2 t^2}dt, \end{aligned} \tag{5.36}$$

which implies that for all $x_1 \in \mathbb{R}$

$$\begin{aligned} \big|\psi_{l,j,\epsilon}(x_1)\big| &\leq C\bigg(\int_{-\infty}^{\infty}\frac{dt}{1+t^2} + \int_0^{|x_1|/2}\bigg|\frac{\mathrm{sin}t}{t}\bigg|dt\bigg) \\ &\leq C\bigg(\int_{-\infty}^{\infty}\frac{dt}{1+t^2}dt + \int_0^1\bigg|\frac{\mathrm{sin}t}{t}\bigg|dt + \int_1^{|x_1|+1}\frac{1}{t}dt\bigg) \\ &\leq C\big(1+\log(|x_1|+1)\big), \end{aligned} \tag{5.37}$$

where above C is independent of $\epsilon > 0$. Then, we have that for $y \in W$

$$\big|(n(y)-1)u_{\epsilon}^{(2)}(y)G_{\epsilon}(x,y)\big| \leq \frac{C\big(1+\log(|y_1|+1)\big)}{1+|x-y|^{3/2}}, \tag{5.38}$$

where above C is independent of y and ϵ. Then, right hand side of (5.38) is an integrable function in W with respect to y. Then, by Lebesgue dominated convergence theorem $\int_W (n(y)-1)u_{\epsilon}^{(2)}(y)G_{\epsilon}(x,y)dy$ converges to $\int_W (n(y)-1)u^{(2)}(y)G(x,y)dy$ as $\epsilon \to 0$. Therefore, (5.33) has been shown. □

5.3 Uniqueness of $u^{(2)}$

In Section 5.3, we will show the uniqueness of $u^{(2)}$ in Theorem 5.2.

Lemma 5.10. *Let Assumptions 5.3 hold and let $k > 0$ be regular in the sense of Definition 5.5. If $u \in H^1_{loc}(\mathbb{R}^2_+)$ such that*

$$\Delta u + k^2(1+q)nu = 0, \text{ in } \mathbb{R}^2_+, \tag{5.39}$$

$$u = 0 \text{ on } \Gamma_0, \tag{5.40}$$

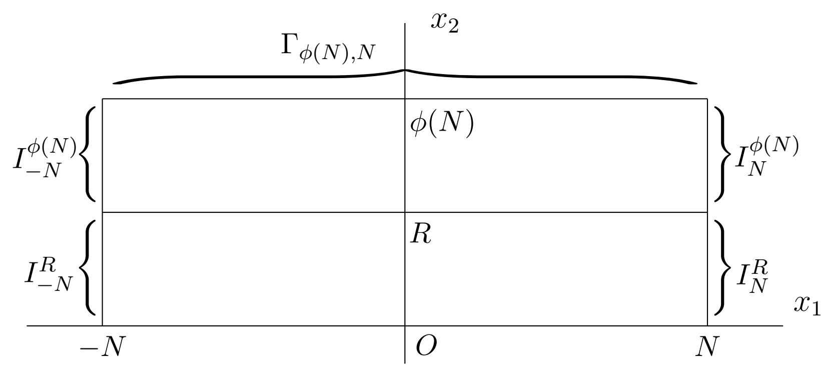
and u satisfies the radiation condition in the sense of Definition 5.6, then $u^{(2)} = 0$ in $\mathbb{R}^2_+$.

Proof of Lemma 5.10. By the definition of the radiation condition, u is of the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)}\big|_{\mathbb{R}\times(0,R)} \in H^1(\mathbb{R}\times(0,R))$ for all $R > 0$, and $u^{(2)} \in L^\infty(\mathbb{R}^2_+)$ has the form

$$u^{(2)}(x) = \psi^+(x_1)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \psi^-(x_1)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x), \quad (5.41)$$

where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, ..., m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (5.23). Here, by Remark 5.7 the function ψ^+ is chosen as a smooth function such that $\psi^+(x_1) = 1$ for $x_1 \geq \eta$ and $\psi^+(x_1) = 0$ for $x_1 \leq -\eta$, and $\psi^- := 1 - \psi^+$ where $\eta > 0$ is some positive number.

Step1 (Green's theorem in Ω_N): We set $\Omega_N := (-N, N) \times (0, \phi(N))$ where $\psi(N) := N^s$. Later we will choose a appropriate $s \in (0, 1)$. Let $R > h$ be large and always fixed, and let N be large enough such that $\phi(N) > R$. We denote by $I^R_{\pm N} := \{\pm N\} \times (0, R)$, $I^{\phi(N)}_{\pm N} := \{\pm N\} \times (R, \phi(N))$, and $\Gamma_{\phi(N),N} := (-N, N) \times \{\phi(N)\}$. (see the figure below.) We set $I_{\pm N} := I^R_{\pm N} \cup I^{\phi(N)}_{\pm N}$.



By Green's first theorem in Ω_N and $u = 0$ on $(-N, N) \times \{0\}$, we have

$$\begin{aligned}
\int_{\Omega_N} \{-k^2(1+q)n|u|^2 + |\nabla u|^2\}dx &= \int_{\Omega_N} \{\overline{u}\Delta u + |\nabla u|^2\}dx \\
&= \int_{I_N} \overline{u}\frac{\partial u}{\partial x_1}ds - \int_{I_{-N}} \overline{u}\frac{\partial u}{\partial x_1}ds + \int_{\Gamma_{\phi(N),N}} \overline{u}\frac{\partial u}{\partial x_2}ds \\
&= \int_{I_N} \overline{u^{(2)}}\frac{\partial u^{(2)}}{\partial x_1}ds - \int_{I_{-N}} \overline{u^{(2)}}\frac{\partial u^{(2)}}{\partial x_1}ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{I_N} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds + \int_{I_N} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds + \int_{I_N} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
&- \int_{I_{-N}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds - \int_{I_{-N}} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds - \int_{I_{-N}} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
&+ \int_{\Gamma_{\phi(N),N}} \overline{u} \frac{\partial u}{\partial x_2} ds.
\end{aligned} \tag{5.42}
$$

By the same argument in Theorem 4.6 of [61] and Lemma 6.3 of [62], we can show that

$$
\begin{aligned}
&\int_{I_N} \overline{u^{(2)}} \frac{\partial u^{(2)}}{\partial x_1} ds - \int_{I_{-N}} \overline{u^{(2)}} \frac{\partial u^{(2)}}{\partial x_1} ds \\
&+ \int_{I_N^R} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds + \int_{I_N^R} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds + \int_{I_N^R} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
&- \int_{I_{-N}^R} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds - \int_{I_{-N}^R} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds - \int_{I_{-N}^R} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
&= \frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} > 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \\
&- \frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} < 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx + o(1),
\end{aligned} \tag{5.43}
$$

and the first and second term in the right hand side converge as $N \to \infty$ to $\frac{ik}{2\pi} \sum_{j \in J} \sum_{d_{l,j} > 0} |a_{l,j}|^2 d_{l,j}$ and $-\frac{ik}{2\pi} \sum_{j \in J} \sum_{d_{l,j} < 0} |a_{l,j}|^2 d_{l,j}$ respectively. Therefore, taking an imaginary part in (5.42) yields that

$$
\begin{aligned}
0 = &\,\mathrm{Im}\left[\frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} > 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \right] \\
&- \mathrm{Im}\left[\frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} < 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \right] \\
&+ \mathrm{Im} \int_{I_N^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds + \mathrm{Im} \int_{I_N^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds + \mathrm{Im} \int_{I_N^{\phi(N)}} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
&- \mathrm{Im} \int_{I_{-N}^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds - \mathrm{Im} \int_{I_{-N}^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds - \mathrm{Im} \int_{I_{-N}^{\phi(N)}} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
&+ \mathrm{Im} \int_{\Gamma_{\phi(N),N}} \overline{u} \frac{\partial u}{\partial x_2} ds + o(1).
\end{aligned} \tag{5.44}
$$

We set

$$J_{\pm}(N) := \pm \mathrm{Im} \int_{I_{\pm N}^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds \pm \mathrm{Im} \int_{I_{\pm N}^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds \pm \mathrm{Im} \int_{I_{\pm N}^{\phi(N)}} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds, \tag{5.45}$$

and we will show that $\mathrm{limsup}_{\mathrm{N}\to\infty} J_{\pm}(N) \geq 0$.

Step2 ($\mathrm{limsup}_{\mathrm{N}\to\infty} J_{\pm}(N) \geq 0$**):** By the Cauchy Schwarz inequality we have

$$\begin{aligned}
|J_{+}(N)| \leq & \Big(\int_R^{\phi(N)} |u^{(1)}(N,x_2)|^2 dx_2 \Big)^{1/2} \Big(\int_R^{\phi(N)} \Big| \frac{\partial u^{(1)}}{\partial x_1}(N,x_2) \Big|^2 dx_2 \Big)^{1/2} \\
+ & \Big(\int_R^{\phi(N)} |u^{(1)}(N,x_2)|^2 dx_2 \Big)^{1/2} \Big(\int_R^{\phi(N)} \Big| \frac{\partial u^{(2)}}{\partial x_1}(N,x_2) \Big|^2 dx_2 \Big)^{1/2} \\
+ & \Big(\int_R^{\phi(N)} |u^{(2)}(N,x_2)|^2 dx_2 \Big)^{1/2} \Big(\int_R^{\phi(N)} \Big| \frac{\partial u^{(1)}}{\partial x_1}(N,x_2) \Big|^2 dx_2 \Big)^{1/2} \\
\leq & \Big(\int_R^{\phi(N)} |u^{(1)}(N,x_2)|^2 dx_2 \Big)^{1/2} \Big(\int_R^{\phi(N)} \Big| \frac{\partial u^{(1)}}{\partial x_1}(N,x_2) \Big|^2 dx_2 \Big)^{1/2} \\
+ & C(\phi(N)-R)^{1/2} \Big(\int_R^{\phi(N)} |u^{(1)}(N,x_2)|^2 dx_2 \Big)^{1/2} \\
+ & C(\phi(N)-R)^{1/2} \Big(\int_R^{\phi(N)} \Big| \frac{\partial u^{(1)}}{\partial x_1}(N,x_2) \Big|^2 dx_2 \Big)^{1/2}.
\end{aligned} \tag{5.46}$$

In order to estimate $u^{(1)}$, we will show the following lemma.

Lemma 5.11. $u^{(1)}$ *has an integral representation of the form*

$$\begin{aligned}
u^{(1)}(x) = & \int_{y_2>0} \sigma(y) G(x,y) dy + k^2 \int_W \big(n(y)(1+q(y)) - 1 \big) u^{(1)}(y) G(x,y) dy \\
+ & k^2 \int_Q n(y) q(y) u^{(2)}(y) G(x,y) dy, \quad x_2 > 0,
\end{aligned} \tag{5.47}$$

where $\sigma := \Delta u^{(2)} + k^2 n u^{(2)}$.

Proof of Lemma 5.11. First, we will consider an integral representation of $u^{(2)}$. Let $N > 0$ be large enough. By Green's representation theorem in $(-N,N) \times (0, N^{1/4})$, we have

$$u^{(2)}(x) = \int_{(-N,N)\times\{N^{1/4}\}} \big[u^{(2)}(y) \frac{\partial G}{\partial y_2}(x,y) - G(x,y) \frac{\partial u^{(2)}}{\partial y_2}(y) \big] ds(y)$$

$$
\begin{aligned}
&+ \quad \Big(\int_{\{N\}\times(0,N^{1/4})} - \int_{\{-N\}\times(0,N^{1/4})}\Big)\big[u^{(2)}(y)\frac{\partial G}{\partial y_1}(x,y) - G(x,y)\frac{\partial u^{(2)}}{\partial y_1}(y)\big]ds(y) \\
&- \quad \int_{(-N,N)\times(0,N^{1/4})}\big[\sigma(y) + k^2(1-n(y))u^{(2)}(y)\big]G(x,y)dy. \qquad (5.48)
\end{aligned}
$$

By Lemma 3.1 of [13], the Dirichlet Green's function $G(x,y)$ is of the estimation

$$
|G(x,y)|,\ |\nabla_y G(x,y)| \le C\frac{x_2y_2}{1+|x-y|^{3/2}},\ |x-y|>1. \qquad (5.49)
$$

By Lemma 5.4 we have that $|u^{(2)}(x)|$, $\big|\frac{\partial u^{(2)}(x)}{\partial x_2}\big| \le ce^{-\delta|x_2|}$ for all $x \in \mathbb{R}^2_+$, and some $c, \delta > 0$. Then, we obtain

$$
\begin{aligned}
&\left|\int_{(-N,N)\times\{N^{1/4}\}}\big[u^{(2)}(y)\frac{\partial G}{\partial y_2}(x,y) - G(x,y)\frac{\partial u^{(2)}}{\partial y_2}(y)\big]ds(y)\right| \\
&\le \quad C\int_{-N}^{N}\frac{x_2e^{-\delta N^{1/4}}}{|N^{1/4}-x_2|^{3/2}}dy_2 \le C\frac{x_2Ne^{-\delta N^{1/4}}}{|N^{1/4}-x_2|^{3/2}}. \qquad (5.50)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\left|\int_{\{\pm N\}\times(0,N^{1/4})}\big[u^{(2)}(y)\frac{\partial G}{\partial y_1}(x,y) - G(x,y)\frac{\partial u^{(2)}}{\partial y_1}(y)\big]ds(y)\right| \\
&\le \quad C\int_{0}^{N^{1/4}}\frac{x_2y_2}{|\pm N-x_1|^{3/2}}dy_2 \le C\frac{x_2N^{1/2}}{|\pm N-x_1|^{3/2}}. \qquad (5.51)
\end{aligned}
$$

Therefore, as $N \to \infty$ in (5.48) we get

$$
u^{(2)}(x) = -\int_{y_2>0}\sigma(y)G(x,y)dy + k^2\int_W(n(y)-1)u^{(2)}(y)G(x,y)dy. \quad (5.52)
$$

By Lemma 5.9, we have (substitute $-k^2qnu$ for f in (5.28))

$$
u(x) = k^2\int_W\big(n(y)-1\big)u(y)G(x,y)dy + k^2\int_Q q(y)n(y)u(y)G(x,y)dy. \quad (5.53)
$$

Combining (5.52) with (5.53) we have

$$
u^{(1)}(x) \quad = \quad -u^{(2)}(x) + k^2\int_W\big(n(y)-1\big)u(y)G(x,y)dy + k^2\int_Q q(y)n(y)u(y)G(x,y)dy
$$

$$
\begin{aligned}
&= \int_{y_2>0} \sigma(y)G(x,y)dy - k^2 \int_W (n(y)-1)u^{(2)}(y)G(x,y)dy \\
&+ k^2 \int_W \big(n(y)-1\big)u(y)G(x,y)dy + k^2 \int_Q q(y)n(y)u(y)G(x,y)dy \\
&= \int_{\mathbb{R}^2_+} \sigma(y)G(x,y)dy + k^2 \int_W \big(n(y)(1+q(y))-1\big)u^{(1)}(y)G(x,y)dy \\
&+ k^2 \int_Q n(y)q(y)u^{(2)}(y)G(x,y)dy. \qquad (5.54)
\end{aligned}
$$

Therefore, Lemma 5.11 has been shown. □

We set $u^{\pm}(x) := \sum_{j\in J}\sum_{d_{l,j}\lessgtr 0} a_{l,j}\phi_{l,j}(x)$. Then, by a simple calculation we can show

$$
\sigma(y) = \frac{d^2\psi^+(y_1)}{dy_1^2}u^+(y)+2\frac{d\psi^+(y_1)}{dy_1}\frac{\partial u^+(y)}{\partial y_1}+\frac{d^2\psi^-(y_1)}{dy_1^2}u^-(y)+2\frac{d\psi^-(y_1)}{dy_1}\frac{\partial u^-(y)}{\partial y_1}, \qquad (5.55)
$$

which implies that supp $\sigma \subset (-\eta,\eta)\times(0,\infty)$. By Lemma 5.11 we have for $R < x_2 < \phi(N)$

$$
\begin{aligned}
&|u^{(1)}(N,x_2)|,\ \left|\frac{\partial u^{(1)}}{\partial x_1}(N,x_2)\right| \leq C\int_{(-\eta,\eta)\times(0,\infty)} |\sigma(y)|\frac{\phi(N)y_2}{|N-\eta|^{3/2}}dy \\
&+ \ C\int_W |u^{(1)}(y)|\frac{\phi(N)h}{(1+|N-y_1|)^{3/2}}dy + C\int_Q \frac{\phi(N)|u^{(2)}(y)|}{|N-y_1|^{3/2}}dy \\
&\leq \ C\frac{\phi(N)}{N^{3/2}} + C\phi(N)\int_W \frac{|u^{(1)}(y)|}{(1+|N-y_1|)^{3/2}}dy. \qquad (5.56)
\end{aligned}
$$

We have to estimate the second term in right hand side. The following lemma was shown in Lemma 4.12 of [12].

Lemma 5.12. *Assume that $\varphi \in L^2_{loc}(\mathbb{R})$ such that*

$$
\sup_{A>0}\Big\{(1+A^2)^{-\epsilon}\int_{-A}^{A}|\varphi(t)|^2dt\Big\} < \infty, \qquad (5.57)
$$

for some $\epsilon > 0$. Then, for every $\alpha \in [0, \frac{1}{2}-\epsilon)$ there exists a constant $C > 0$ and a sequence $\{A_m\}_{m\in\mathbb{N}}$ such that $A_m \to \infty$ as $m \to \infty$ and

$$
\int_{K_{A_m}} |\varphi(t)|^2dt \leq CA_m^{-\alpha},\ m \in \mathbb{N}, \qquad (5.58)
$$

where $K_A := K_A^+ \cup K_A^-$*,* $K_A^+ := (-A^+, A^+) \setminus (-A, A)$*,* $K_A^- := (-A, A) \setminus (-A^-, A^-)$*, and* $A^\pm := A \pm A^{1/2}$ *for* $A \in [1, \infty)$*.*

Applying Lemma 5.12 to $\varphi = \left(\int_0^h \left|u^{(1)}(\cdot, y_2)\right|^2 dy_2\right)^{1/2} \in L^2(\mathbb{R})$, there exists a sequence $\{N_m\}_{m\in\mathbb{N}}$ such that $N_m \to \infty$ as $m \to \infty$ and

$$\int_{K_{N_m}} \int_0^h |u^{(1)}(y_1, y_2)|^2 dy_1 dy_2 \le C N_m^{-1/4},\ m \in \mathbb{N}. \tag{5.59}$$

Then, by the Cauchy Schwarz inequality we have

$$\begin{aligned}
\int_W \frac{|u^{(1)}(y)|}{(1+|N-y_1|)^{3/2}} dy &= \left(\int_{-N_m^-}^{N_m^-} + \int_{K_{N_m}} + \int_{\mathbb{R}\setminus[-N_m^+, N_m^+]}\right) \int_0^h \frac{|u^{(1)}(y)|}{(1+|N_m-y_1|)^{3/2}} dy \\
&\le C\left(\int_{-N_m^-}^{N_m^-} \frac{dy_1}{(1+N_m-|y_1|)^3}\right)^{1/2} + C\left(\int_{K_{N_m}} \int_0^h |u^{(1)}(y_1, y_2)|^2 dy_1 dy_2\right)^{1/2} \\
&+ C\left(\int_{\mathbb{R}\setminus[-N_m^+, N_m^+]} \frac{dy_1}{(1+|y_1|-N_m)^3}\right)^{1/2} \\
&\le C\left(\int_0^{N_m^-} \frac{dy_1}{(1+N_m-y_1)^3}\right)^{1/2} + C N_m^{-1/8} + C\left(\int_{N_m^+}^\infty \frac{dy_1}{(1+y_1-N_m)^3}\right)^{1/2} \\
&\le C N_m^{-1/8}.
\end{aligned} \tag{5.60}$$

With (5.56) we have for $m \in \mathbb{N}$,

$$|u^{(1)}(N_m, x_2)|,\ \left|\frac{\partial u^{(1)}}{\partial x_1}(N_m, x_2)\right| \le C \frac{\phi(N_m)}{N_m^{1/8}}. \tag{5.61}$$

Therefore, by (5.46) we have

$$\begin{aligned}
|J_+(N_m)| &\le C(\phi(N_m) - R)\frac{\phi(N_m)^2}{N_m^{1/4}} + C(\phi(N_m) - R)\frac{\phi(N_m)}{N_m^{1/8}} \\
&\le C(\phi(N_m) - R)\frac{\phi(N_m)^2}{N_m^{1/8}} \le C\frac{\phi(N_m)^3}{N_m^{1/8}}.
\end{aligned} \tag{5.62}$$

Since $\phi(N) = N^s$, if we choose $s \in (0, 1)$ such that $3s < \frac{1}{8}$, that is, $0 < s < \frac{1}{24}$ the right hand side in (5.62) converges to zero as $m \to \infty$. Therefore, $\mathrm{limsup}_{N\to\infty} J_+(N) \ge 0$. By the same argument of J_+, we can show that $\mathrm{limsup}_{N\to\infty} J_-(N) \ge 0$, which yields Step 2.

Next, we discuss the last term in (5.44). By the same argument in Lemma

5.11 that we apply Green's representation theorem in $x_2 > h$ and use the Dirichlet Green's function G_h of $\mathbb{R}^2_{x_2>h}(:= \mathbb{R}\times(h,\infty))$ instead of G, $u^{(1)}$ can also be of another integral representation for $x_2 > h$

$$\begin{aligned} u^{(1)}(x) &= \int_{y_2>h}\sigma(y)G_h(x,y)dy + 2\int_{\Gamma_h} u^{(1)}(y)\frac{\partial\Phi_k(x,y)}{\partial y_2}ds(y) \\ &=: v^1(x) + v^2(x), \end{aligned} \tag{5.63}$$

where G_h is defined by $G_h(x,y) := \Phi_k(x,y) - \Phi_k(x,y_h^*)$ where $y_h^* = (y_1, 2h - y_2)$. We define approximation $u_N^{(1)}$ of $u^{(1)}$ by

$$\begin{aligned} u_N^{(1)}(x) &:= \int_{y_2>0}\chi_{\phi(N)-1}(y_2)\sigma(y)G(x,y)dy + 2\int_{\Gamma_h}\chi_N(y_1)u^{(1)}(y)\frac{\partial\Phi_k(x,y)}{\partial y_2}ds(y) \\ &=: v_N^1(x) + v_N^2(x), \quad x_2 > h, \end{aligned} \tag{5.64}$$

where χ_a is defined by for $a > 0$,

$$\chi_a(t) := \begin{cases} 1 & \text{for } |t| \le a \\ 0 & \text{for } |t| > a. \end{cases} \tag{5.65}$$

By Lemma 3.4 of [15] and Lemma 2.1 of [14] we can show that v_N^1 and v_N^2 satisfy the upward propagating radiation condition, which implies that so does $u_N^{(1)}$. Furthermore, by the definition of $u_N^{(1)}$ we can show that $u_N^{(1)}(\cdot,\phi(N)-1) \in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. Then, by Lemma 6.1 of [15] we have that

$$\mathrm{Im}\int_{\Gamma_{\phi(N)}}\overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds \ge 0. \tag{5.66}$$

Combining (5.44) with (5.66) we have

$$\begin{aligned} 0 &\ge -\mathrm{Im}\int_{\Gamma_{\phi(N)}}\overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds \\ &= \mathrm{Im}\left[\frac{1}{2\pi}\sum_{j\in J}\sum_{d_{l,j},d_{l',j}>0}\overline{a_{l,j}}a_{l',j}\int_{C_{\phi(N)}}\overline{\phi_{l,j}}\frac{\partial\phi_{l,j}}{\partial x_1}dx\right] \\ &- \mathrm{Im}\left[\frac{1}{2\pi}\sum_{j\in J}\sum_{d_{l,j},d_{l',j}<0}\overline{a_{l,j}}a_{l',j}\int_{C_{\phi(N)}}\overline{\phi_{l,j}}\frac{\partial\phi_{l,j}}{\partial x_1}dx\right] + J_+(N) + J_-(N) \\ &+ \mathrm{Im}\int_{\Gamma_{\phi(N),N}}\overline{u}\frac{\partial u}{\partial x_2} - \mathrm{Im}\int_{\Gamma_{\phi(N)}}\overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds + o(1). \end{aligned} \tag{5.67}$$

We observe the last term

$$\mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u}\frac{\partial u}{\partial x_2} - \mathrm{Im}\int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds =: L(N) + M(N), \tag{5.68}$$

where

$$L(N) := \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_2}ds - \mathrm{Im}\int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds, \tag{5.69}$$

$$M(N) := \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u^{(2)}}{\partial x_2}ds + \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(2)}}\frac{\partial u^{(1)}}{\partial x_2}ds + \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(2)}}\frac{\partial u^{(2)}}{\partial x_2}ds. \tag{5.70}$$

By Lemma 5.11 we can show $|u^{(1)}(x_1,\phi(N))|$, $|\frac{\partial u^{(1)}}{\partial x_2}(x_1,\phi(N))| \leq C\phi(N)$ for $x_1 \in \mathbb{R}$, and by Lemma 5.4 we have $|u^{(2)}(x_1,\phi(N))|$, $|\frac{\partial u^{(2)}}{\partial x_2}(x_1,\phi(N))| \leq Ce^{-\delta\phi(N)}$ for $x_1 \in \mathbb{R}$. Then, we have

$$\begin{aligned}
|M(N)| &\leq \int_{-N}^{N} |u^{(1)}(x_1,\phi(N))|\Big|\frac{\partial u^{(2)}}{\partial x_2}(x_1,\phi(N))\Big|dx_1 \\
&+ \int_{-N}^{N} |u^{(2)}(x_1,\phi(N))|\Big|\frac{\partial u^{(1)}}{\partial x_2}(x_1,\phi(N))\Big|dx_1 \\
&+ \int_{-N}^{N} |u^{(2)}(x_1,\phi(N))|\Big|\frac{\partial u^{(2)}}{\partial x_2}(x_1,\phi(N))\Big|dx_1 \\
&\leq C(N\phi(N)e^{-\delta\phi(N)} + Ne^{-2\delta\phi(N)}) \\
&\leq CN\phi(N)e^{-\delta\phi(N)},
\end{aligned} \tag{5.71}$$

which implies that $M(N) = o(1)$ as $N \to \infty$. Hence, we will show that $\mathrm{limsup}_{N\to\infty} L(N) \geq 0$.

Step3 ($\mathrm{limsup}_{N\to\infty} L(N) \geq 0$): First, we observe that

$$\begin{aligned}
|L(N)| &\leq \Big|\mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_2}ds - \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds\Big| \\
&+ \Big|\mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds - \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds\Big| \\
&+ \Big|\mathrm{Im}\int_{\Gamma_{\phi(N)}\setminus\Gamma_{\phi(N),N}} \overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds\Big|
\end{aligned}$$

$$\leq \int_{-N}^{N} |u^{(1)}(x_1,\phi(N))| \Big| \frac{\partial u^{(1)}}{\partial x_2}(x_1,\phi(N)) - \frac{\partial u_N^{(1)}}{\partial x_2}(x_1,\phi(N)) \Big| ds$$
$$+ \int_{-N}^{N} |u^{(1)}(x_1,\phi(N)) - u_N^{(1)}(x_1,\phi(N))| \Big| \frac{\partial u_N^{(1)}}{\partial x_2}(x_1,\phi(N)) \Big| ds$$
$$+ \int_{\mathbb{R}\setminus(-N,N)} |u_N^{(1)}(x_1,\phi(N))| \Big| \frac{\partial u_N^{(1)}}{\partial x_2}(x_1,\phi(N)) \Big| ds. \tag{5.72}$$

By Lemma 5.4, σ has a exponential decay in y_2. Then, we have for $x_1 \in \mathbb{R}$,

$$|v^1(x_1,\phi(N))|, \ \Big| \frac{\partial v^1}{\partial x_2}(x_1,\phi(N)) \Big|, \ |v_N^1(x_1,\phi(N))|, \ \Big| \frac{\partial v_N^1}{\partial x_2}(x_1,\phi(N)) \Big|$$
$$\leq C \int_{(-\eta,\eta)\times(0,\infty)} \frac{e^{-\delta y_2}\phi(N) y_2}{(1+|x_1-y_1|)^{3/2}} dy \leq C \frac{\phi(N)}{(1+|x_1|)^{3/2}}, \tag{5.73}$$

and

$$|v^1(x_1,\phi(N)) - v_N^1(x_1,\phi(N))|, \ \Big| \frac{\partial v^1}{\partial x_2}(x_1,\phi(N)) - \frac{\partial v_N^1}{\partial x_2}(x_1,\phi(N)) \Big|$$
$$\leq C \int_{(-\eta,\eta)\times(\phi(N)-1,\infty)} \frac{e^{-\delta y_2}\phi(N) y_2}{(1+|x_1-y_1|)^{3/2}} dy$$
$$\leq C \Big(\int_{\phi(N)}^{\infty} e^{-\delta y_2} y_2 dy_2 \Big) \frac{\phi(N)}{(1+|x_1|)^{3/2}} dy \leq \frac{e^{-\delta \phi(N)}\phi(N)}{(1+|x_1|)^{3/2}}. \tag{5.74}$$

Since the fundamental solution to Helmholtz equation $\Phi(x,y)$ is of the following estimation (see e.g., [13]) for $|x-y| \geq 1$

$$\Big| \frac{\partial \Phi}{\partial y_2}(x,y) \Big| \leq C \frac{|x_2-y_2|}{1+|x-y|^{3/2}}, \quad \Big| \frac{\partial^2 \Phi}{\partial x_2 \partial y_2}(x,y) \Big| \leq C \frac{|x_2-y_2|^2}{1+|x-y|^{3/2}}, \tag{5.75}$$

we can show that for $x_1 \in \mathbb{R}$

$$|v^2(x_1,\phi(N))| \leq C\phi(N) W_\infty(x_1), \quad |v_N^2(x_1,\phi(N))| \leq C\phi(N) W_N(x_1), \tag{5.76}$$

and

$$\Big| \frac{\partial v^2}{\partial x_2}(x_1,\phi(N)) \Big| \leq C\phi(N)^2 W_\infty(x_1), \quad \Big| \frac{\partial v_N^2}{\partial x_2}(x_1,\phi(N)) \Big| \leq C\phi(N)^2 W_N(x_1), \tag{5.77}$$

and

$$|v^2(x_1,\phi(N)) - v_N^2(x_1,\phi(N))| \leq C\phi(N) \big(W_\infty(x_1) - W_N(x_1) \big), \tag{5.78}$$

and

$$\left|\frac{\partial v^2}{\partial x_2}(x_1, \phi(N)) - \frac{\partial v_N^2}{\partial x_2}(x_1, \phi(N))\right| \le C\phi(N)^2\big(W_\infty(x_1) - W_N(x_1)\big), \quad (5.79)$$

where W_N is defined by for $N \in (0, \infty]$

$$W_N(x_1) := \int_{-N}^{N} \frac{|u^{(1)}(y_1, h)|}{(1+|x_1-y_1|)^{3/2}} dy_1, \quad x_1 \in \mathbb{R}. \quad (5.80)$$

Using (5.73)–(5.79), we continue to estimate (5.72). By the Cauchy Schwarz inequality we have

$$\begin{aligned}
|L(N)| \le C \int_{-N}^{N} &\Big\{ \frac{\phi(N)}{(1+|x_1|)^{3/2}} + \phi(N) W_\infty(x_1) \Big\} \\
&\times \Big\{ \frac{\phi(N) e^{-\sigma\phi(N)}}{(1+|x_1|)^{3/2}} + \phi(N)^2 \big(W_\infty(x_1) - W_N(x_1)\big) \Big\} dx_1 \\
+ \int_{-N}^{N} &\Big\{ \frac{\phi(N) e^{-\sigma\phi(N)}}{(1+|x_1|)^{3/2}} + \phi(N) \big(W_\infty(x_1) - W_N(x_1)\big) \Big\} \\
&\times \Big\{ \frac{\phi(N)}{(1+|x_1|)^{3/2}} + \phi(N)^2 W_N(x_1) \Big\} dx_1 \\
+ \int_{\mathbb{R}\setminus(-N,N)} &\Big\{ \frac{\phi(N)}{(1+|x_1|)^{3/2}} + \phi(N) W_N(x_1) \Big\} \Big\{ \frac{\phi(N)}{(1+|x_1|)^{3/2}} + \phi(N)^2 W_N(x_1) \Big\} dx_1 \\
\le\ & C\phi(N)^3 \int_{-N}^{N} W_\infty(x_1)\big(W_\infty(x_1) - W_N(x_1)\big) dx_1 \\
+\ & C\phi(N)^3 \int_{-N}^{N} \frac{1}{(1+|x_1|)^{3/2}} \big(W_\infty(x_1) - W_N(x_1)\big) dx_1 \\
+\ & C\phi(N)^2 \int_{\mathbb{R}\setminus(-N,N)} \frac{1}{(1+|x_1|)^{3}} dx_1 + C\phi(N)^2 \int_{\mathbb{R}\setminus(-N,N)} \frac{1}{(1+|x_1|)^{3/2}} W_N(x_1) dx_1 \\
+\ & C\phi(N)^3 \int_{\mathbb{R}\setminus(-N,N)} |W_N(x_1)|^2 dx_1 + o(1) \\
\le\ & C\phi(N)^3 \Big\{ \Big(\int_{-N}^{N} \big(W_\infty(x_1) - W_N(x_1)\big)^2 dx_1 \Big)^{1/2} + \Big(\int_{\mathbb{R}\setminus(-N,N)} W_N(x_1)^2 dx_1 \Big)^{1/2} \Big\} \\
& + o(1). \quad (5.81)
\end{aligned}$$

Finally, we will estimate $\big(W_\infty(x_1) - W_N(x_1)\big)$ and $W_N(x_1)$. Since $u^{(1)}(\cdot, h) \in L^2(\mathbb{R})$, by Lemma 5.12 there exists a sequence $\{N_m\}_{m\in\mathbb{N}}$ such that $N_m \to \infty$

as $m \to \infty$ and

$$\int_{K_{N_m}} |u^{(1)}(y_1, h)|^2 dy_1 \leq C N_m^{-\frac{1}{4}}, \ m \in \mathbb{N}, \tag{5.82}$$

where $K_A := K_A^+ \cup K_A^-$, $K_A^+ := (-A^+, A^+) \setminus (-A, A)$, $K_A^- := (-A, A) \setminus (-A^-, A^-)$, and $A^\pm := A \pm A^{1/2}$ for $A \in [1, \infty)$.

By the Cauchy Schwarz inequality we have for $|x_1| > N_m$,

$$\begin{aligned} \int_{-N_m^-}^{N_m^-} \frac{|u^{(1)}(y_1, h)|}{(1 + |x_1 - y_1|)^{3/2}} dy_1 &\leq \left(\int_{-N_m^-}^{N_m^-} |u^{(1)}(y_1, h)|^2 dy_1 \right)^{1/2} \left(\int_{-N_m^-}^{N_m^-} \frac{dy_1}{(1 + |x_1| - y_1)^3} \right)^{1/2} \\ &\leq \frac{C}{1 - |x_1| - N_m^-}, \end{aligned} \tag{5.83}$$

and

$$\begin{aligned} \int_{K_{N_m^-}} \frac{|u^{(1)}(y_1, h)|}{(1 + |x_1 - y_1|)^{3/2}} dy_1 &\leq \left(\int_{K_{N_m}} |u^{(1)}(y_1, h)|^2 dy_1 \right)^{1/2} \left(\int_{K_{N_m}^-} \frac{dy_1}{(1 + |x_1| - y_1)^3} \right)^{1/2} \\ &\leq \frac{C}{N_m^{1/8}(1 + |x_1| - N_m)}. \end{aligned} \tag{5.84}$$

Therefore, we obtain

$$\begin{aligned} &\int_{\mathbb{R} \setminus (-N_m, N_m)} W_N(x_1)^2 dx_1 \\ &\leq C \int_{N_m}^{\infty} \frac{dx_1}{(1 - |x_1| - N_m^-)^2} + \frac{C}{N_m^{1/4}} \int_{N_m}^{\infty} \frac{dx_1}{(1 - |x_1| - N_m)^2} \\ &\leq \frac{C}{1 + N_m^{1/2}} + \frac{C}{N_m^{1/4}} \leq \frac{C}{N_m^{1/4}}. \end{aligned} \tag{5.85}$$

By the Cauchy Schwarz inequality we have for $|x_1| < N_m$,

$$\begin{aligned} &\int_{\mathbb{R} \setminus (-N_m^+, N_m^+)} \frac{|u^{(1)}(y_1, h)|}{(1 + |x_1 - y_1|)^{3/2}} dy_1 \\ &\leq \left(\int_{\mathbb{R} \setminus (-N_m^+, N_m^+)} |u^{(1)}(y_1, h)|^2 dy_1 \right)^{1/2} \left(\int_{\mathbb{R} \setminus (-N_m^+, N_m^+)} \frac{dy_1}{(1 + y_1 - |x_1|)^3} \right)^{1/2} \\ &\leq \frac{C}{1 + N_m^+ - |x_1|}, \end{aligned} \tag{5.86}$$

and

$$\int_{K_{N_m^+}} \frac{|u^{(1)}(y_1,h)|}{(1+|x_1-y_1|)^{3/2}}dy_1 \;\leq\; \Big(\int_{K_{N_m}} |u^{(1)}(y_1,h)|^2 dy_1\Big)^{1/2}\Big(\int_{K^+_{N_m}} \frac{dy_1}{(1+y_1-|x_1|)^3}\Big)^{1/2}$$
$$\leq \frac{C}{N_m^{1/8}(1+N_m-|x_1|)}. \tag{5.87}$$

Therefore, we obtain

$$\begin{aligned}
&\int_{-N_m}^{N_m} \big(W_\infty(x_1)-W_N(x_1)\big)^2 dx_1\\
&\leq\; C\int_{-N_m}^{N_m}\frac{dx_1}{(1+N_m^+-|x_1|)^2}+\frac{C}{N_m^{1/4}}\int_{-N_m}^{N_m}\frac{dx_1}{(1+N_m-|x_1|)^2}\\
&\leq\; \frac{C}{1+N_m^{1/2}}+\frac{C}{N_m^{1/4}} \;\leq\; \frac{C}{N_m^{1/4}}.
\end{aligned} \tag{5.88}$$

Therefore, Collecting (5.81), (5.85), and (5.88) we conclude that $|L(N_m)| \leq C\frac{\phi(N_m)^3}{N_m^{1/8}}$. Since $\phi(N) = N^s$, if we choose $s \in (0,1)$ such that $3s < \frac{1}{8}$, that is, $0 < s < \frac{1}{24}$, the term $\frac{\phi(N_m)^3}{N_m^{1/8}}$ converges to zero as $m \to \infty$. Therefore, $\mathrm{limsup}_{\mathrm{N}\to\infty} L(N) \geq 0$, which yields Step 3.

By taking $\mathrm{limsup}_{\mathrm{N}\to\infty}$ in (5.67) we have that

$$\begin{aligned}
0 \;\geq\;& \frac{k}{2\pi}\sum_{j\in J}\Bigg[\sum_{d_{l,j}>0}|a_{l,j}|^2 d_{l,j}-\sum_{d_{l,j}<0}|a_{l,j}|^2 d_{l,j}\Bigg]\\
+\;& \mathrm{limsup}_{N\to\infty}\Big(J_+(N)+J_-(N)+L(N)\Big).
\end{aligned} \tag{5.89}$$

By Steps 2 and 3 and choosing $0 < s < \frac{1}{24}$ the right hand side is non-negative. Therefore, $a_{l,j} = 0$ for all l, j, which yields $u^{(2)} = 0$. Lemma 5.10 has been shown, and in next section we will show the uniqueness of $u^{(1)}$. □

5.4 Uniqueness of $u^{(1)}$

In Section 5.4, we will show the following lemma.

Lemma 5.13. *If $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfies*

(i) *$u \in H^1(\mathbb{R}\times(0,R))$ for all $R > 0$,*

(ii) $\Delta u + k^2(1+q)nu = 0$ in $\mathbb{R}^2_+$,

(iii) *u vanishes for $x_2 = 0$,*

(iv) *There exists $\phi \in L^\infty(\Gamma_h) \cap H^{1/2}(\Gamma_h)$ with $u(x) = 2\int_{\Gamma_h} \phi(y) \frac{\partial \Phi_k(x,y)}{\partial y_2} ds(y)$ for $x_2 > h$,*

then, $u \in H^1_0(\mathbb{R}^2_+)$.

If we can use Lemma 5.13, we have the uniqueness of the solution in Theorem 5.2.

Theorem 5.14. *Let Assumptions 5.1 and 5.3 hold and let $k > 0$ be regular in the sense of Definition 5.5. If $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfies (5.39), (5.40), and the radiation condition in the sense of Definition 5.6, then u vanishes for $x_2 > 0$.*

Proof of Theorem 5.14. Let $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfy (5.39), (5.40), and the radiation condition in the sense of Definition 5.6. By Lemma 5.9, $u^{(2)} = 0$ for $x_2 > 0$. Then, $u^{(1)}$ satisfies the assumptions (i)–(iv) of Lemma 5.13, which implies that $u^{(1)} \in H^1_0(\mathbb{R}^2_+)$. By Assumption 5.1, $u^{(1)}$ vanishes for $x_2 > 0$, which yields the uniqueness. □

Finally in this section we will show Lemma 5.13.

Proof of Lemma 5.13. Let $R > h$ be fixed. We set $\Omega_{N,R} := (-N, N) \times (0, R)$ where $N > 0$ is large enough. We denote by $I^R_{\pm N} := \{\pm N\} \times (0, R)$, $\Gamma_{R,N} := (-N, N) \times \{R\}$, and $\Gamma_R := (-\infty, \infty) \times \{R\}$. By Green's first theorem in $\Omega_{N,R}$ and assumptions **(ii)**, **(iii)** we have

$$\begin{aligned}\int_{\Omega_{N,R}} \{-k^2(1+q)n|u|^2 + |\nabla u|^2\} dx &= \int_{\Omega_{N,R}} \{\overline{u}\Delta u + |\nabla u|^2\} dx \\ = \int_{I^R_N} \overline{u} \frac{\partial u}{\partial x_1} ds &- \int_{I^R_{-N}} \overline{u} \frac{\partial u}{\partial x_1} ds + \int_{\Gamma_{R,N}} \overline{u} \frac{\partial u}{\partial x_2} ds. \end{aligned} \tag{5.90}$$

By the assumption **(i)**, the first and second term in the right hands side of (5.90) go to zero as $N \to \infty$. Then, by taking an imaginary part and as $N \to \infty$ in (5.90) we have

$$\mathrm{Im} \int_{\Gamma_R} \overline{u} \frac{\partial u}{\partial x_2} ds = 0. \tag{5.91}$$

By considering the Floquet Bloch transform with respect to x_1, we can show that

$$\int_{\Gamma_R} \overline{u}\frac{\partial u}{\partial x_2} ds = \int_{-1/2}^{1/2}\int_0^{2\pi} \overline{\tilde{u}_\alpha}(x_1, R)\frac{\partial \tilde{u}_\alpha(x_1, R)}{\partial x_2} dx_1 d\alpha. \tag{5.92}$$

Since the upward propagating radiation condition is equivalent to the Rayleigh expansion by the Floquet Bloch transform (see the proof of Theorem 6.8 in [62]), we can show that

$$\tilde{u}_\alpha(x) = \sum_{n\in\mathbb{Z}} u_n(\alpha) e^{inx_1 + i\sqrt{k^2-(n+\alpha)^2}(x_2-h)}, \ x_2 > h, \tag{5.93}$$

where $u_n(\alpha) := (2\pi)^{-1}\int_0^{2\pi} u_\alpha(x_1, h)e^{-inx_1}dx_1$. From (5.91)–(5.93) we obtain that

$$\begin{aligned} 0 &= \mathrm{Im}\int_{-1/2}^{1/2}\int_0^{2\pi} \overline{\tilde{u}_\alpha}(x_1, R)\frac{\partial \tilde{u}_\alpha(x_1, R)}{\partial x_2} dx_1 d\alpha \\ &= \mathrm{Im}\sum_{n\in\mathbb{Z}}\int_{-1/2}^{1/2} 2\pi|u_n(\alpha)|^2 i\sqrt{k^2-(n+\alpha)^2}, \end{aligned} \tag{5.94}$$

Here, we denote by $k = n_0 + r$ where $n_0 \in \mathbb{N}_0$ and $r \in [-1/2, 1/2)$. Then by (5.94) we have

$$\begin{gathered} u_n(\alpha) = 0 \text{ for } |n| < n_0, \text{ a.e. } \alpha \in (-1/2, 1/2), \\ u_{n_0}(\alpha) = 0 \text{ for } \alpha \in (-1/2, r), \\ u_{-n_0}(\alpha) = 0 \text{ for } \alpha \in (-r, 1/2). \end{gathered} \tag{5.95}$$

By (5.95) we have

$$\begin{aligned} &\int_{-1/2}^{1/2}\int_0^{2\pi}\int_R^{\infty} |\tilde{u}_\alpha(x)|^2 dx_2 dx_1 d\alpha \\ = \ & 2\pi\int_{-1/2}^{1/2}\sum_{|n|>n_0}|u_n(\alpha)|^2\int_R^{\infty} e^{-\sqrt{(n+\alpha)^2-k^2}(x_2-h)}dx_2 d\alpha \\ + \ & 2\pi\int_r^{1/2}|u_{n_0}(\alpha)|^2\int_R^{\infty} e^{-\sqrt{(n_0+\alpha)^2-k^2}(x_2-h)}dx_2 d\alpha \\ + \ & 2\pi\int_{-1/2}^{-r}|u_{-n_0}(\alpha)|^2\int_R^{\infty} e^{-\sqrt{(-n_0+\alpha)^2-k^2}(x_2-h)}dx_2 d\alpha \end{aligned}$$

$$
\begin{aligned}
&\leq \quad 2\pi \sum_{|n|>n_0} \int_{-1/2}^{1/2} \frac{|u_n(\alpha)|^2 e^{-\sqrt{(n+\alpha)^2-k^2}(R-h)}}{\sqrt{(n+\alpha)^2-k^2}} d\alpha \\
&+ \quad 2\pi \int_{r}^{1/2} \frac{|u_{n_0}(\alpha)|^2 e^{-\sqrt{(n_0+\alpha)^2-k^2}(R-h)}}{\sqrt{(n_0+\alpha)^2-k^2}} d\alpha \\
&+ \quad 2\pi \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2 e^{-\sqrt{(-n_0+\alpha)^2-k^2}(R-h)}}{\sqrt{(-n_0+\alpha)^2-k^2}} d\alpha \\
&\leq \quad C \sum_{|n|>n_0} \int_{-1/2}^{1/2} |u_n(\alpha)|^2 d\alpha \\
&+ \quad C \int_{r}^{1/2} \frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha - r}} d\alpha + C \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha - r}} d\alpha,
\end{aligned} \tag{5.96}
$$

and

$$
\begin{aligned}
&\int_{-1/2}^{1/2} \int_{0}^{2\pi} \int_{R}^{\infty} |\partial_{x_1} \tilde{u}_\alpha(x)|^2 dx_2 dx_1 d\alpha \\
= \quad & 2\pi \sum_{|n|>n_0} \int_{-1/2}^{1/2} \frac{|u_n(\alpha)|^2 n^2 e^{-\sqrt{(n+\alpha)^2-k^2}(R-h)}}{\sqrt{(n+\alpha)^2-k^2}} d\alpha \\
+ \quad & 2\pi \int_{r}^{1/2} \frac{|u_{n_0}(\alpha)|^2 n_0^2 e^{-\sqrt{(n_0+\alpha)^2-k^2}(R-h)}}{\sqrt{(n_0+\alpha)^2-k^2}} d\alpha \\
+ \quad & 2\pi \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2 n_0^2 e^{-\sqrt{(-n_0+\alpha)^2-k^2}(R-h)}}{\sqrt{(-n_0+\alpha)^2-k^2}} d\alpha \\
\leq \quad & C \sum_{|n|>n_0} \int_{-1/2}^{1/2} |u_n(\alpha)|^2 d\alpha \\
+ \quad & C \int_{r}^{1/2} \frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha - r}} d\alpha + C \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha - r}} d\alpha.
\end{aligned} \tag{5.97}
$$

By the same argument in (5.97) we have

$$
\begin{aligned}
\int_{-1/2}^{1/2} \int_{0}^{2\pi} \int_{R}^{\infty} |\partial_{x_2} \tilde{u}_\alpha(x)|^2 dx_2 dx_1 d\alpha \leq C \sum_{|n|>n_0} \int_{-1/2}^{1/2} |u_n(\alpha)|^2 d\alpha \\
+C \int_{r}^{1/2} \frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha - r}} d\alpha + C \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha - r}} d\alpha.
\end{aligned} \tag{5.98}
$$

It is well known that the Floquet Bloch Transform is an isomorphism between $H^1(\mathbb{R}^2_+)$ and $L^2\big((-1/2,1/2)_\alpha; H^1((0,2\pi)\times\mathbb{R})_x\big)$ (e.g., see Theorem 4 in [73]). Therefore, we obtain from (5.96)–(5.98)

$$\begin{aligned}
\|u\|^2_{H^1(\mathbb{R}\times(R,\infty))} &\leq C\int_{-1/2}^{1/2}\int_0^{2\pi}\int_R^\infty |\tilde{u}_\alpha(x)|^2+|\partial_{x_1}\tilde{u}_\alpha(x)|^2+|\partial_{x_2}\tilde{u}_\alpha(x)|^2 dx_2dx_1d\alpha \\
&\leq C\sum_{|n|>n_0}\int_{-1/2}^{1/2}|u_n(\alpha)|^2 d\alpha \\
&+ C\int_r^{1/2}\frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}}d\alpha + C\int_{-1/2}^{-r}\frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}}d\alpha. \\
&\leq C\int_{-1/2}^{1/2}\int_0^{2\pi}|\tilde{u}_\alpha(x_1,h)|^2 dx_1 d\alpha \\
&+ C\int_r^{1/2}\frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}}d\alpha + C\int_{-1/2}^{-r}\frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}}d\alpha.
\end{aligned} \tag{5.99}$$

If we can show that

$$\exists\delta>0 \text{ and } \exists C>0 \text{ s.t. } |u_{\pm n_0}(\alpha)|\leq C \text{ for all } \alpha\in(-\delta\pm r,\delta\pm r), \tag{5.100}$$

then the right hands side of (5.99) is finite, which yields Lemma 5.13.

Finally, we will show (5.100). By the same argument in section 3 of [62] we have

$$(I-K_\alpha)\tilde{u}_\alpha = f_\alpha \text{ in } H^1_{0,per}(C_h), \tag{5.101}$$

where the operator K_α is defined by (5.21) and $f_\alpha := -(T_{per}k^2nqu)(\cdot,\alpha)$. Since the function k^2nqu has a compact support, $\|f_\alpha\|^2_{H^1(C_h)}$ is bounded with respect to α. By Assumption 5.3 and the operator K_α is compact, $(I-K_\alpha)$ is invertible if $\alpha\in A_k$. Since $\pm r\in A_k$, $(I-K_\pm)$ is invertible. Since the exceptional values are finitely many (see Lemma 5.4), $(I-K_\alpha)$ is also invertible if α is close to $\pm r$. Therefore, there exists $\delta>0$ such that $(I-K_\alpha)$ is invertible for all $\alpha\in(-\delta+r,\delta+r)\cup(-\delta-r,\delta-r)$.

The operator $(I-K_\alpha)$ is of the form

$$(I-K_\alpha) = (I-K_{\pm r})\Big(I-(I-K_{\pm r})^{-1}[I-K_{\pm r}-(I-K_\alpha)]\Big) = (I-K_{\pm r})(I-M_\alpha), \tag{5.102}$$

where $M_\alpha := (I-K_{\pm r})^{-1}(K_\alpha-K_{\pm r})$. Next, we will estimate $(K_\alpha-K_{\pm r})$.

By the definition of K_α we have for all $v, w \in H^1_{0,per}(C_h)$,

$$\begin{aligned} \langle (K_\alpha - K_{\pm r})v, w\rangle_* &= -\int_{C_h} \left[i(\alpha \mp r)\left(v\frac{\partial \overline{w}}{\partial x_1} - \overline{v}\frac{\partial \overline{w}}{\partial x_1}\right) + (\alpha^2 - r^2)v\overline{w}\right] dx \\ &+ 2\pi i \sum_{|n|\neq n_0} v_n \overline{w_n}\big(\sqrt{k^2-(n+\alpha)^2} - \sqrt{k^2-(n\pm r)^2}\big) \\ &+ 2\pi i \sum_{|n|= n_0} v_n \overline{w_n}\big(\sqrt{k^2-(n+\alpha)^2} - \sqrt{k^2-(n\pm r)^2}\big). \end{aligned} \tag{5.103}$$

Since

$$|\sqrt{k^2-(n+\alpha)^2} - \sqrt{k^2-(n\pm r)^2}| = \left| \frac{\pm 2nr + r^2 - 2n\alpha - \alpha^2}{\sqrt{k^2-(n+\alpha)^2} + \sqrt{k^2-(n\pm r)^2}} \right|$$

$$\leq \begin{cases} \frac{|n||\alpha \pm r| + |r^2-\alpha^2|}{\sqrt{|k^2-(n\pm r)^2|}} & \text{for } |n| \neq n_0 \\ \frac{|n||\alpha \pm r| + |r^2-\alpha^2|}{\sqrt{|r+\alpha||r-\alpha|}} & \text{for } |n| = n_0, \end{cases} \tag{5.104}$$

we have for all $\alpha \in (-\delta + r, \delta + r) \cup (-\delta - r, \delta - r)$

$$\begin{aligned} |\langle (K_\alpha - K_{\pm r})v, w\rangle_*| &\leq C|\alpha \mp r| \, \|v\|_{H^1(C_h)} \, \|w\|_{H^1(C_h)} \\ &+ C\sum_{|n|\neq n_0} |v_n||w_n| \frac{|n||\alpha \mp r|}{\sqrt{|k^2-(n\pm r)^2|}} \\ &+ C\sum_{|n|= n_0} |v_n||w_n| n_0 \sqrt{|\alpha \mp r|} \\ &\leq C\sqrt{|\alpha \mp r|} \, \|v\|_{H^1(C_h)} \, \|w\|_{H^1(C_h)}. \end{aligned} \tag{5.105}$$

(we retake very small $\delta > 0$ if needed.) This implies that there is a constant number $C > 0$ which is independent of α such that $\|K_\alpha - K_{\pm r}\| \leq C\sqrt{|\alpha \mp r|}$. Therefore, by the property of Neumann series, there is a small $\delta > 0$ such that for all $\alpha \in (-\delta + r, \delta + r) \cup (-\delta - r, \delta - r)$

$$(I - M_\alpha)^{-1} = \sum_{n=0}^{\infty} M_\alpha^n \quad \text{and} \quad \|M_\alpha\| \leq 1/2. \tag{5.106}$$

By the Cauchy Schwarz inequality, the boundedness of trace operator, and

(5.106) we have

$$
\begin{aligned}
|u_{\pm n_0}(\alpha)| &\leq \int_0^{2\pi} |\tilde{u}_\alpha(x_1, h)| dx_1 \leq C \left\| \tilde{u}_\alpha \right\|_{H^1(C_h)} \\
&= C \left\| (I - M_\alpha)^{-1} (I - K_{\pm r})^{-1} f_\alpha \right\|_{H^1(C_h)} \\
&\leq C \left\| (I - M_\alpha)^{-1} \right\| \left\| (I - K_{\pm r})^{-1} f_\alpha \right\| \\
&\leq C \sum_{n=0}^{\infty} \| M_\alpha \|^n < C \sum_{n=0}^{\infty} (1/2)^j < \infty,
\end{aligned}
\tag{5.107}
$$

where constant number $C > 0$ is independent of α. Therefore, we have shown (5.100). □

5.5 Existence

In previous sections we discussed the uniqueness of Theorem 5.2. In Section 5.5, we will show the existence. Let Assumptions 5.1 and 5.3 hold and let $k > 0$ be regular in the sense of Definition 5.5. Let $f \in L^2(\mathbb{R}^2_+)$ such that $\mathrm{supp} f = Q$. We define the solution operator $S : L^2(Q) \to L^2(Q)$ by $Sg := v\big|_Q$ where v satisfies the radiation condition and

$$\Delta v + k^2 n v = g, \text{ in } \mathbb{R}^2_+, \tag{5.108}$$

$$v = 0 \text{ on } \Gamma_0. \tag{5.109}$$

Remark that by Theorem 5.8 we can define such a operator S, and S is a compact operator since the restriction to Q of the solution v is in $H^1(Q)$. We define the multiplication operator $M : L^2(Q) \to L^2(Q)$ by $Mh := k^2 nqh$. We will show the following lemma.

Lemma 5.15. *$I_{L^2(Q)} + SM$ is invertible.*

Proof of Lemma 5.15. By the definition of operators S and M we have $SMg = v\big|_Q$ where v is a radiating solution of (5.108)–(5.109) replacing g by $k^2 nqg$. If we assume that $(I_{L^2(Q)} + SM)g = 0$, then $g = -v\big|_Q$, which implies that v satisfies $\Delta v + k^2 n(1 + q)v = 0$ in $\mathbb{R}^2_+$. By the uniqueness we have $v = 0$ in $\mathbb{R}^2_+$, which implies that $I_{L^2(Q)} + SM$ is injective. Since the operator SM is compact, by Fredholm theory we conclude that $I_{L^2(Q)} + SM$ is invertible. □

We define u as the solution of

$$\Delta u + k^2 nu = f - M(I_{L^2(Q)} + SM)^{-1} Sf, \text{ in } \mathbb{R}^2_+. \tag{5.110}$$

satisfying the radiation condition and $u = 0$ on Γ_0. Since

$$\begin{aligned} u\big|_Q &= S(f - M(I_{L^2(Q)} + SM)^{-1}Sf) \\ &= (I_{L^2(Q)} + SM)(I_{L^2(Q)} + SM)^{-1}Sf - SM(I_{L^2(Q)} + SM)^{-1}Sf \\ &= (I_{L^2(Q)} + SM)^{-1}Sf, \end{aligned} \tag{5.111}$$

we have that

$$\Delta u + k^2 nu = f - k^2 nqu, \text{ in } \mathbb{R}^2_+, \tag{5.112}$$

and u is a radiating solution of (5.8)–(5.9). Therefore, Theorem 5.2 has been shown.

5.6 Example of Assumption 5.1

In Section 5.6, we will show the following lemma in order to give one of the example of Assumption 5.1.

Lemma 5.16. *Let q and n satisfy that $\partial_2\big((1+q)n\big) \geq 0$ in W, and let $v \in H^1(\mathbb{R}^2_+)$ satisfy (5.6)–(5.7). Then, v vanishes for $x_2 > 0$.*

Proof of Lemma 5.16. Let $R > h$ be fixed. For $N > 0$ we set $\Omega_{N,R} := (-N, N) \times (0, R)$ and $I^R_{\pm N} := \{\pm N\} \times (0, R)$ and $\Gamma_{R,N} := (-N, N) \times \{R\}$. By Green's first theorem in $\Omega_{N,R}$ we have

$$\begin{aligned} \int_{\Omega_{N,R}} \{-k^2(1+q)n|v|^2 + |\nabla v|^2\} dx &= \int_{\Omega_{N,R}} \{\overline{v}\Delta v + |\nabla u|^2\} dx \\ = \int_{I^R_N} \overline{v}\partial_1 v ds &- \int_{I^R_{-N}} \overline{v}\partial_1 v ds + \int_{\Gamma_{R,N}} \overline{v}\partial_2 v ds. \end{aligned} \tag{5.113}$$

Since $v \in H^1(\mathbb{R}^2_+)$ the first and second term in the right hand side of (5.6) go to zero as $N \to \infty$. Then, by taking an imaginary part in (5.113) and as $N \to \infty$ we have

$$\mathrm{Im} \int_{\Gamma_R} \overline{v}\partial_2 v ds = 0. \tag{5.114}$$

By the simple calculation, we have

$$\begin{aligned} &2\mathrm{Re}\big(\partial_2\overline{v}(\Delta v + k^2(1+q)nv)\big) \\ &\quad = 2\mathrm{Re}\big(\nabla \cdot (\partial_2\overline{v}\nabla v)\big) - \partial_2(|\nabla v|^2) + k^2(1+q)n\partial_2(|v|^2), \end{aligned} \tag{5.115}$$

which implies that

$$
\begin{aligned}
0 &= 2\mathrm{Re}\int_{\Omega_{N,R}} \partial_2\overline{v}\big(\Delta v + k^2(1+q)nv)\big)dx = 2\mathrm{Re}\int_{\Omega_{N,R}} \nabla\cdot(\partial_2\overline{v}\nabla v)dx \\
&- \int_{\Omega_{N,R}} \partial_2(|\nabla v|^2)dx + \int_{\Omega_{N,R}} k^2(1+q)n\partial_2(|v|^2)dx \\
&= 2\mathrm{Re}\Big(-\int_{\Gamma_{0,N}} \partial_2\overline{v}\partial_2 v ds + \int_{I_N^R} \partial_2\overline{v}\partial_1 v ds - \int_{I_{-N}^R} \partial_2\overline{v}\partial_1 v ds + \int_{\Gamma_{R,N}} \partial_2\overline{v}\partial_2 v ds\Big) \\
&- \Big(-\int_{\Gamma_{0,N}} |\nabla v|^2 ds + \int_{\Gamma_{R,N}} |\nabla v|^2 ds\Big) \\
&- \int_{\Gamma_{0,N}} k^2(1+q)n|v|^2 ds + \int_{\Gamma_{R,N}} k^2(1+q)n|v|^2 ds - \int_{\Omega_{N,R}} k^2\partial_2\big((1+q)n\big)|v|^2 dx \\
&= -\int_{\Gamma_{0,N}} |\partial_2 v|^2 ds + \int_{\Gamma_{R,N}} \big(|\partial_2 v|^2 - |\partial_1 v|^2 + k^2|v|^2\big)ds \\
&- \int_{\Omega_{N,R}\cap W} k^2\partial_2\big((1+q)n\big)|v|^2 dx + o(1).
\end{aligned}
\tag{5.116}
$$

Since $\partial_2\big((1+q)n\big) \geq 0$ in W, we have

$$
\int_{\Gamma_{0,N}} |\partial_2 v|^2 ds \leq \int_{\Gamma_{R,N}} \big(|\partial_2 v|^2 - |\partial_1 v|^2 + k^2|v|^2\big)ds + o(1). \tag{5.117}
$$

By taking limit as $N \to \infty$ we have

$$
\int_{\Gamma_R} |\partial_2 v|^2 ds \leq \int_{\Gamma_R} \big(|\partial_2 v|^2 - |\partial_1 v|^2 + k^2|v|^2\big)ds. \tag{5.118}
$$

By Lemma 6.1 of [15] we have

$$
\int_{\Gamma_R} \big(|\partial_2 v|^2 - |\partial_1 v|^2 + k^2|v|^2\big)ds \leq 2\mathrm{Im}\int_{\Gamma_R} \overline{v}\partial_2 v ds. \tag{5.119}
$$

From (5.114) and (5.118) we obtain that $\partial_2 v = 0$ on Γ_0. We also have $v = 0$ on Γ_0, which implies that by the Holmgren's theorem and the unique continuation principle we conclude that $v = 0$ in $\mathbb{R}^2_+$. □

6 The factorization and monotonicity method for the defect in an open periodic waveguide

6.1 Introduction

In this section, we consider the inverse scattering problem to reconstruct the defect in an infinite medium with periodicity in the upper half space from near field data. This scattering problem is motivated by applications of open waveguides, e.g., optical fibers, planar waveguides, and so far, it has been often studied from a mathematical perspective. (see e.g., [8, 14, 41, 55, 56, 62, 80]).

The contributions of this paper are followings.

- We mention that there is a mistake in factorization method of the earlier paper [72], and give the correct one (Theorem 6.11).
- We give two reconstruction algorithms (Theorems 6.1 and 6.2) for the unknown defect by a combination of the factorization and the monotonicity method.

[72] has provided the general functional analysis theorem for the factorization method (Theorem 2.1 of [72]) under weaker assumptions than previous ones (Theorem 2.15 of [58]), and mentioned that by this relaxation one can avoid the assumption corresponding to *transmission eigenvalue* in the case of inverse medium scattering problems. This general theorem of [72] has been mainly used when the factorization method has been discussed. (see e.g., [3, 9, 7, 22, 64]) However, there is a mistake in this theorem, which leads to the difficulty to apply the factorization method to inverse medium scattering problems without the assumption of transmission eigenvalue. (Remark 6.12). Firstly in this paper, we give the correct functional analysis theorem with its proof.

A new functional analysis theorem (Theorem 6.11) needs the assumption that an imaginary part of the middle operator T of the data operator F is strictly positive. However, the middle operator T corresponding to our case does not have such a property. (see Lemma 6.15 and Remark 6.16). Due to this failure of factorization method, we give alternative reconstruction algorithms (Theorems 6.1 and 6.2) by employing the idea of monotonicity method. Recently in [23, 33, 37, 38, 39, 40, 69], the monotonicity method has been studied by many authrs, and it has advantage over the factorization method that reconstruction algorithms are given under weaker assumptions. For example, [33] has studied the inverse medium scattering problems without the assumption of transmission eigenvalues. Theorems 6.1 and 6.2 are proved by a combination of techniques of the factorization and monotonicity method. Very recently, [23] has employed such an idea in the study of inverse crack scattering problem.

We begin with formulation of our scattering problem. Let $k > 0$ be the wave number, and let $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$ be the upper half plane, and let $W := \mathbb{R} \times (0, h)$ be the waveguide in $\mathbb{R}^2_+$. We denote by $\Gamma_a := \mathbb{R} \times \{a\}$ for $a \geq 0$. Let $n \in L^\infty(\mathbb{R}^2_+)$ be real valued, 2π-periodic with respect to x_1 (that is, $n(x_1 + 2\pi, x_2) = n(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2_+$), and equal to one for $x_2 > h$. We assume that there exists a constant $n_{max} > 0$ and $n_{min} > 0$ such that $n_{min} \leq n \leq n_{max}$ in $\mathbb{R}^2_+$. Let $q \in L^\infty(\mathbb{R}^2_+)$ be real valued with the compact support supp q in W. We denote by $Q :=$ supp q, and assume that $\mathbb{R}^2_+ \setminus \overline{Q}$ is connected. First of all, we consider the following direct scattering problem: For fixed $y \in \mathbb{R}^2_+ \setminus \overline{W}$, determine the scattered field $u^s \in H^1_{loc}(\mathbb{R}^2_+)$ such that

$$\Delta u^s + k^2(1+q)nu^s = -k^2 qnu^i(\cdot, y) \text{ in } \mathbb{R}^2_+, \tag{6.1}$$

$$u^s = 0 \text{ on } \Gamma_0, \tag{6.2}$$

where the incident field u^i is given by $u^i(x, y) = \overline{G_n(x, y)}$, where G_n is the Dirichlet Green's function in the upper half plane $\mathbb{R}^2_+$ for $\Delta + k^2 n$, that is,

$$G_n(x, y) := G(x, y) + \tilde{u}^s(x, y), \tag{6.3}$$

where $G(x, y) := \Phi_k(x, y) - \Phi_k(x, y^*)$ is the Dirichlet Green's function for $\Delta + k^2$, and $y^* = (y_1, -y_2)$ is the reflected point of y at $\mathbb{R} \times \{0\}$. Here, $\Phi_k(x, y)$ is the fundamental solution to Helmholtz equation in $\mathbb{R}^2$, that is,

$$\Phi_k(x, y) := \frac{i}{4} H^{(1)}_0(k|x - y|), \ x \neq y, \tag{6.4}$$

where $H^{(1)}_0$ is the Hankel function of the first kind of order one. $\tilde{u}^s$ is the scattered field of the unperturbed problem by the incident field $G(x, y)$, that is, $\tilde{u}^s$ vanishes for $x_2 = 0$ and solves

$$\Delta \tilde{u}^s + k^2 n \tilde{u}^s = k^2(1 - n)G(\cdot, y) \text{ in } \mathbb{R}^2_+. \tag{6.5}$$

If we impose a suitable radiation condition introduced in [62], the unperturbed solution $\tilde{u}^s$ is uniquely determined. Later, we will explain the exact definition of this radiation condition (Definition 6.6). Furthermore, with this radiation condition and an additional assumption (Assumption 6.9) the well-posedness of the problem (6.1)–(6.2) was show in [23].

By the well-posedness of this perturbed scattering problem, we are able to consider the inverse problem of determining the support of q from the measured scattered field u^s by the incident field u^i. Let $M := \{(x_1, m) : a < x_1 < b\}$ for $a < b$ and $m > h$. With the scattered field u^s, we define the *near field operator* $N : L^2(M) \to L^2(M)$ by

$$Ng(x) := \int_M u^s(x,y)g(y)ds(y), \ x \in M. \tag{6.6}$$

The inverse problem we consider in this section is to determine the support Q of q from the scattered field $u^s(x, y)$ for all x and y in M with one $k > 0$. In other words, given the near field operator N, determine Q. Accordingly, we will prove the following two theorems.

Theorem 6.1. *Let $B \subset \mathbb{R}^2$ be a bounded open set. We assume that there exists $q_{min} > 0$ such that $q \geq q_{min}$ a.e. in Q. Then, for $0 < \alpha < k^2 n_{min} q_{min}$,*

$$B \subset Q \quad \Longleftrightarrow \quad \alpha H_B^* H_B \leq_{\text{fin}} \text{Re} N, \tag{6.7}$$

where the operator $H_B : L^2(M) \to L^2(B)$ is given by

$$H_B g(x) := \int_M \overline{G_n(x,y)} g(y)ds(y), \ x \in B, \tag{6.8}$$

and the inequality on the right hand side in (6.7) denotes that $\text{Re}N - \alpha H_B^* H_B$ has only finitely many negative eigenvalues, and the real part of an operator A is self-adjoint operators given by $\text{Re}A := \frac{1}{2}(A + A^*)$.

Theorem 6.2. *Let $B \subset \mathbb{R}^2$ be a bounded open set. We assume that there exists $q_{min} > 0$ and $q_{max} > 0$ such that $q_{min} \leq q \leq q_{max}$ a.e. in Q. Then, for $\alpha > k^2 n_{max} q_{max}$,*

$$Q \subset B \quad \Longleftrightarrow \quad \text{Re} N \leq_{\text{fin}} \alpha H_B^* H_B, \tag{6.9}$$

This section is organized as follows. In Section 6.2, we recall a radiation condition introduced in [62], and the well-posedness of the problem (6.1)–(6.2). In Section 6.3, we give the correct functional analysis theorem for the factorization method. In Section 6.4, we study a factorization of the near field operator N and its properties. In Sections 6.5 and 6.6, we prove Theorems 6.1 and 6.2, respectively. Finally in Section 6.7, we give numerical examples based on Theorem 6.1.

6.2 A radiation condition

In Section 6.2, we recall a radiation condition introduced in [62]. Let $f \in L^2(\mathbb{R}^2_+)$ have the compact support supp$f$ in W. First, we consider the following direct scattering problem: Determine the scattered field $u \in H^1_{loc}(\mathbb{R}^2_+)$ such that

$$\Delta u + k^2 n u = f \text{ in } \mathbb{R}^2_+, \tag{6.10}$$

$$u = 0 \text{ on } \Gamma_0. \tag{6.11}$$

(6.10) is understood in the variational sense, that is,

$$\int_{\mathbb{R}^2_+} \left[\nabla u \cdot \nabla \overline{\varphi} - k^2 n u \overline{\varphi} \right] dx = - \int_W f \overline{\varphi} dx, \tag{6.12}$$

for all $\varphi \in H^1(\mathbb{R}^2_+)$, with compact support. In such a problem, it is natural to impose the *upward propagating radiation condition*, that is, $u(\cdot, h) \in L^\infty(\mathbb{R})$ and

$$u(x) = 2 \int_{\Gamma_h} u(y) \frac{\partial \Phi_k(x,y)}{\partial y_2} ds(y), \;\; x_2 > h. \tag{6.13}$$

However, even with this condition we can not expect the uniqueness of this problem. (see Example 2.3 of [62].) In order to introduce a *suitable radiation condition*, [62] discussed limiting absorption solution of this problem, that is, the limit of the solution u_ϵ of $\Delta u_\epsilon + (k+i\epsilon)^2 n u_\epsilon = f$ as $\epsilon \to 0$. For the details of an introduction of this radiation condition, we refer to [55, 56, 61, 62].

Let us prepare for the exact definition of the radiation condition in this problem. We denote by $C_R := (0, 2\pi) \times (0, R)$ for $R \in (0, \infty]$. The function $u \in H^1(C_R)$ is called α-quasi periodic if $u(2\pi, x_2) = e^{2\pi i \alpha} u(0, x_2)$. We denote by $H^1_\alpha(C_R)$ the subspace of the α-quasi periodic function in $H^1(C_R)$, and denote by $H^1_{\alpha,loc}(C_\infty) := \{ u \in H^1_{loc}(C_\infty) : u\big|_{C_R} \in H^1_\alpha(C_R) \text{ for all R} > 0 \}$. Then, we consider the following problem, which arises from taking the quasi-periodic Floquet Bloch transform (see, e.g., [73].) in (6.10)–(6.13): For $\alpha \in (-1/2, 1/2]$, determine $u_\alpha \in H^1_{\alpha,loc}(C_\infty)$ such that

$$\Delta u_\alpha + k^2 n u_\alpha = f_\alpha \text{ in } C_\infty, \tag{6.14}$$

$$u_\alpha = 0 \text{ on } (0, 2\pi) \times \{0\}, \tag{6.15}$$

$$u_\alpha(x) = \sum_{n \in \mathbb{Z}} u_n(\alpha) e^{inx_1 + i\sqrt{k^2 - (n+\alpha)^2}(x_2 - h)}, \;\; x_2 > h, \tag{6.16}$$

where $u_n(\alpha) := (2\pi)^{-1} \int_0^{2\pi} u_\alpha(x_1, h) e^{-inx_1} dx_1$ are the Fourier coefficients of $u_\alpha(\cdot, h)$, and $\sqrt{k^2 - (n+\alpha)^2} = i\sqrt{(n+\alpha)^2 - k^2}$ if $n+\alpha > k$. (6.16) is called the *Rayleigh expansion*. But even with this condition the uniqueness of this problem fails for some $\alpha \in (-1/2, 1/2]$. Then, we call these α *exceptional values* if there exists non-trivial solutions $u_\alpha \in H^1_{\alpha,loc}(C_\infty)$ of (6.14)–(6.16) with $f_\alpha = 0$. We set $A_k := \{\alpha \in (-1/2, 1/2] : \exists l \in \mathbb{Z} \ s.t.\ |\alpha + l| = k\}$, and make the following assumption:

Assumption 6.3. *For every $\alpha \in A_k$ the solution of $u_\alpha \in H^1_{\alpha,loc}(C_\infty)$ of (6.14)–(6.16) with $f_\alpha = 0$ has to be zero.*

The following properties of exceptional values was shown in Lemmas 4.2 and 5.6 of [62].

Lemma 6.4. *Let Assumption 6.3 hold. Then, there exists only finitely many exceptional values $\alpha \in (-1/2, 1/2]$. Furthermore, if α is an exceptional value, then so is $-\alpha$. Therefore, the set of exceptional values can be described by $\{\alpha_j : j \in J\}$ where some $J \subset \mathbb{Z}$ is finite and $\alpha_{-j} = -\alpha_j$ for $j \in J$. For each exceptional value α_j we define*

$$X_j := \left\{ \phi \in H^1_{\alpha_j,loc}(C_\infty) : \begin{array}{c} \Delta\phi + k^2 n\phi = 0 \text{ in } C_\infty, \quad \phi = 0 \text{ for } x_2 = 0, \\ \phi \text{ satisfies the Rayleigh expansion (6.16)} \end{array} \right\}$$

Then, X_j are finite dimensional. We set $m_j = \dim X_j$. Furthermore, $\phi \in X_j$ is evanescent, that is, there exists $c > 0$ and $\delta > 0$ such that $|\phi(x)|,\ |\nabla\phi(x)| \leq ce^{-\delta|x_2|}$ for all $x \in C_\infty$.

Next, we consider the following eigenvalue problem in X_j: Determine $d \in \mathbb{R}$ and $\phi \in X_j$ such that

$$-i\int_{C_\infty} \frac{\partial\phi}{\partial x_1}\overline{\psi}dx = dk\int_{C_\infty} n\phi\overline{\psi}dx, \tag{6.17}$$

for all $\psi \in X_j$. We denote by the eigenvalues $d_{l,j}$ and eigenfunction $\phi_{l,j}$ of this problem, that is,

$$-i\int_{C_\infty} \frac{\partial\phi_{l,j}}{\partial x_1}\overline{\psi}dx = d_{l,j}k\int_{C_\infty} n\phi_{l,j}\overline{\psi}dx, \tag{6.18}$$

for every $l = 1, ..., m_j$ and $j \in J$. We normalize the eigenfunction $\{\phi_{l,j} : l = 1, ..., m_j\}$ such that

$$k\int_{C_\infty} n\phi_{l,j}\overline{\phi_{l',j}}dx = \delta_{l,l'}, \tag{6.19}$$

for all l, l'. We assume that the wave number $k > 0$ is *regular* in the following sense.

Definition 6.5. $k > 0$ is *regular* if $d_{l,j} \neq 0$ for all $l = 1, ...m_j$ and $j \in J$.

Now we are ready to define the radiation condition.

Definition 6.6. Let Assumption 6.3 hold, and let $k > 0$ be regular in the sense of Definition 6.5. We set

$$\psi^{\pm}(x_1) := \frac{1}{2}\left[1 \pm \frac{2}{\pi}\int_0^{x_1/2} \frac{sint}{t}dt\right],\ x_1 \in \mathbb{R}. \tag{6.20}$$

Then, $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfies the *radiation condition* if u satisfies the upward propagating radiation condition (6.13), and has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)}\big|_{\mathbb{R}\times(0,R)} \in H^1(\mathbb{R} \times (0, R))$ for all $R > 0$, and $u^{(2)} \in L^{\infty}(\mathbb{R}^2_+)$ has the following form

$$u^{(2)}(x) = \psi^{+}(x_1)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \psi^{-}(x_1)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x), \tag{6.21}$$

where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, ..., m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (6.18).

Remark 6.7. We can replace $\psi^{\pm}$ by any smooth functions $\tilde{\psi}^{\pm}$ such that $\left|\psi^{\pm}(x_1) - \tilde{\psi}^{\pm}(x_1)\right| \to 0$, and $\left|\frac{d}{dx_1}\psi^{\pm}(x_1) - \frac{d}{dx_1}\tilde{\psi}^{\pm}(x_1)\right| \to 0$ as $|x_1| \to \infty$ because (6.21) is of the form

$$\begin{aligned} u^{(2)}(x) &= \tilde{\psi}^{+}(x_1)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \tilde{\psi}^{-}(x_1)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x) \\ &+ \Big(\psi^{+}(x_1) - \tilde{\psi}^{+}(x_1)\Big)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \Big(\psi^{-}(x_1) - \tilde{\psi}^{-}(x_1)\Big)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x), \end{aligned} \tag{6.22}$$

where the second term in the right-hand side of (6.22) is a H^1-function, which is the role of $u^{(1)}$.

The following was shown in Theorems 2.2, 6.6, and 6.8 of [62].

Theorem 6.8. *For every $f \in L^2(\mathbb{R}^2_+)$ with the compact support* supp$f$ *in W, there exists a unique solution $u_{k+i\epsilon} \in H^1(\mathbb{R}^2_+)$ of the problem (6.10)–(6.11) replacing k by $k+i\epsilon$. Furthermore, $u_{k+i\epsilon}$ converge as $\epsilon \to +0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to some $u \in H^1_{loc}(\mathbb{R}^2_+)$ which satisfy (6.10)–(6.11) and the radiation condition in the sense of Definition 6.6. Furthermore, the solution u of this problem is uniquely determined.*

Furthermore, with the same radiation condition and the following additional assumption, the well-posedness of the perturbed scattering problem of (6.10)–(6.11) was show in [23].

Assumption 6.9. *We assume that k^2 is not the point spectrum of $\frac{1}{(1+q)n}\Delta$ in $H^1_0(\mathbb{R}^2_+)$, that is, every $v \in H^1(\mathbb{R}^2_+)$ which satisfies*

$$\Delta v + k^2(1+q)nv = 0 \text{ in } \mathbb{R}^2_+, \tag{6.23}$$

$$v = 0 \text{ on } \Gamma_0, \tag{6.24}$$

has to vanishes for $x_2 > 0$.

Theorem 6.10. *Let Assumption 6.9 hold and let $f \in L^2(\mathbb{R}^2_+)$ such that* $\mathrm{supp} f = Q$*. Then, there exists a unique solution $u \in H^1_{loc}(\mathbb{R}^2_+)$ such that*

$$\Delta u + k^2(1+q)nu = f \text{ in } \mathbb{R}^2_+, \tag{6.25}$$

$$u = 0 \text{ on } \Gamma_0, \tag{6.26}$$

and u satisfies the radiation condition in the sense of Definition 6.6.

By Theorem 6.10, the well-posedness of the perturbed scattering problem (6.1)–(6.2) with the radiation condition follows. Then, we are able to consider the inverse problem of determining the support of q from the measured scattered field u^s by the incident field $u^i(x,y) = \overline{G_n(x,y)}$. In the following sections, we will discuss the inverse problem.

6.3 The factorization method

In Section 6.3, we mention the correct functional analysis theorem for the factorization method. The following functional analytic theorem is given by the almost same argument in Theorem 2.15 of [58].

Theorem 6.11. *Let $X \subset U \subset X^*$ be a Gelfand triple with a Hilbert space U and a reflexive Banach space X such that the imbedding is dense. Furthermore, let Y be a second Hilbert space and let $F : Y \to Y$, $G : X \to Y$, $T : X^* \to X$ be linear bounded operators such that*

$$F = GTG^*. \tag{6.27}$$

We make the following assumptions:

(1) *G is compact with dense range in Y.*

(2) *There exists* $t \in [0, 2\pi]$ *such that* $\mathrm{Re}(\mathrm{e}^{it}T)$ *has the form* $\mathrm{Re}(\mathrm{e}^{it}T) = C + K$ *with some compact operator* K *and some self-adjoint and positive coercive operator* C*, i.e., there exists* $c > 0$ *such that*

$$\langle \varphi, C\varphi \rangle \geq c \|\varphi\|^2 \quad for\ all\ \varphi \in X^*. \tag{6.28}$$

(3) $\mathrm{Im}\langle \varphi, T\varphi \rangle > 0$ *for all* $\varphi \in \overline{\mathrm{Ran}(G^*)}$ *with* $\varphi \neq 0$.

Then, the operator $F_{\#} := \left|\mathrm{Re}(\mathrm{e}^{it}F)\right| + \mathrm{Im}F$ *is non-negative, and the ranges of* $G : X \to Y$ *and* $F_{\#}^{1/2} : Y \to Y$ *coincide with each other, that is, we have the following range identity;*

$$\mathrm{Ran}(G) = \mathrm{Ran}(F_{\#}^{1/2}). \tag{6.29}$$

Here, the real part and the imaginary part of an operator A are self-adjoint operators given by

$$\mathrm{Re}A = \frac{A + A^*}{2} \quad \text{and} \quad \mathrm{Im}A = \frac{A - A^*}{2i}. \tag{6.30}$$

Remark 6.12. Here, we will mention a mistake in Theorem 2.1 of [72]. It was claimed that one can replace the assumption for strongly positivity of $\mathrm{Im}T$ by that for the injectivity of T, which is related to the independence of transmission eigenvalues in inverse medium scattering problem. However, this relacement of assumptions is not correct.

Here, we observe the following counterexample for Theorem 2.1 of [72]. Let $G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ be a 2×4 matrix, and let $T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ be a 4×4 matrix. Then, T is injective, but $\mathrm{Im}T = 0$. We calculate

$$F = GTG^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.31}$$

which leads to

$$\mathrm{Ran}(G) = \mathrm{Ran}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \neq \mathrm{Ran}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathrm{Ran}(F_{\#}^{1/2}). \tag{6.32}$$

In this section, we will prove Theorem 6.11 based on Theorem 2.15 of [58]. We also remark that Theorem 2.15 of [58] assumes the compactness of $\mathrm{Im}T$, while Theorem 6.11 of this section do not assume its compactness. The independence of its compactness is important because the operator ImT is not always compact in the case of inverse medium scattering problem with complex valued contrast function. (See Theorem 4.5 of [58])

Before the proof of Theorem 6.11, we show the following lemma.

Lemma 6.13. *Let X be a Hilbert space, and let $T : X \to X$ be linear bounded, and let $K : X \to X$ be linear bounded injective. We assume that*

$$\mathrm{Ran}(T) \text{ is closed subspace in } X, \quad \text{and} \quad \dim\mathrm{Ker}(T) < \infty. \tag{6.33}$$

Then, there is a constant $C > 0$ such that

$$\|u\|_X^2 \leq C(\|Tu\|_X^2 + \|Ku\|_X^2) \quad \textit{for all } u \in X. \tag{6.34}$$

Proof of Lemma 6.13. Assume that on contrary for any $C > 0$, there exists a $u_c \in X$ such that

$$\|u_c\|_X^2 > C(\|Tu_c\|_X^2 + \|Ku_c\|_X^2). \tag{6.35}$$

Then, we can choose a sequence $(u_m)_{m\in\mathbb{N}}$ in X such that $\|u_m\|^2 = 1$ and $\|Tu_m\|^2 + \|Ku_m\|^2$ converges to zero as $m \to \infty$. Since $\mathrm{Ker}(T)$ is a finite dimensional subspace in X, there exists an orthogonal complement $\mathrm{Ker}(T)^\perp$ of $\mathrm{Ker}(T)$ such that $X = \mathrm{Ker}(T) \oplus \mathrm{Ker}(T)^\perp$. We denote P by an orthogonal projection onto $\mathrm{Ker}(T)^\perp$. Since $\mathrm{Ker}(T)^\perp$ and $\mathrm{Ran}(T)$ are closed subspaces in X, the restrict operator $T\big|_{\mathrm{Ker}(T)^\perp}$ is injective and surjective from the Banach space $\mathrm{Ker}(T)^\perp$ to the Banach space $\mathrm{Ran}(T)$. Then by the closed graph theorem, $T\big|_{\mathrm{Ker}(T)^\perp}$ is invertible bounded, which implies that there is a constant $C > 0$ such that

$$\|Pu_m\|^2 \leq C \left\| T\big|_{\mathrm{Ker}(T)^\perp} Pu_m \right\|^2 = C \|Tu_m\|^2 . \tag{6.36}$$

Since K is injective and $\mathrm{Ker}(T)$ is a finite dimensional subspace in X, there is a constant $C > 0$ such that

$$\|v\| \leq C \|Kv\| \quad \textit{for all } v \in \mathrm{Ker}(T). \tag{6.37}$$

Then, there is a constant $C > 0$ such that

$$\begin{aligned} \|(I-P)u_m\|^2 &\leq C \|K(I-P)u_m\|^2 \leq 2(C \|Ku_m\|^2 + \|KPu_m\|^2) \\ &\leq 2C(\|Ku_m\|^2 + \|K\|^2 \|Pu_m\|^2). \end{aligned} \tag{6.38}$$

Therefore, by (6.36) and (6.38) there exists a constant $C' > 0$ such that

$$1 = \|Pu_m\|^2 + \|(I-P)u_m\|^2 \leq C'(\|Tu_m\|^2 + \|Ku_m\|^2). \tag{6.39}$$

As $m \to \infty$, the right-hand side of above inequality converges to zero, which is a contradiction. $\square$

We will show Theorem 6.11.

Proof of Theorem 6.11. By the same argument of Part A (Reduction) in the proof of Theorem 2.15 of [58], we can restrict ourselves to the case $X = U$ and $C = I$. Furthermore, we can also restrict ourselves to the case G is injective. Indeed, let $P : U \to U$ be the orthogonal projection onto $\hat{U} := \overline{\mathrm{Ran}(G^*)}$. Then, $PG^* = G^*$ and $G = GP$. By this, we can have the factorization of the form

$$F = GPTPG^* = \hat{G}\hat{T}\hat{G}^*, \tag{6.40}$$

where $\hat{G}\big|_{\hat{U}} \colon \hat{U} \to Y$ and $\hat{T} = PT\big|_{\hat{U}} \colon \hat{U} \to \hat{U}$. Therefore, all of assumptions (1)–(3) are satisfied. We remark that $\hat{T}$ is not injective even if T is injective, which leads to error in Theorem 2.1 of [72].

By the same argument in Part B, C, and D in the proof of Theorem 2.15 of [58], we can show that

$$F_{\#} = GT_{\#}G^*, \tag{6.41}$$

where $T_{\#} = \mathrm{Re}(\mathrm{e}^{it}T)D + \mathrm{Im}T$ and D is some isomorphism from U onto itself. It was also shown that the operator $T_{\#}$ is non-negative on U in its proof. By applying the inequality (4.5) of [57] to the non-negative operators $\mathrm{Re}(\mathrm{e}^{it}T)D$ and $\mathrm{Im}T$, there is a constant $C > 0$ such that

$$\begin{aligned} \langle \varphi, T_{\#}\varphi \rangle &= \langle \varphi, \mathrm{Re}(\mathrm{e}^{it}T)D\varphi \rangle + \langle \varphi, \mathrm{Im}T\varphi \rangle \\ &\geq C(\left\|\mathrm{Re}(\mathrm{e}^{it}T)D\varphi\right\|^2 + \|\mathrm{Im}T\varphi\|^2) \quad \textit{for all } \varphi \in U. \end{aligned} \tag{6.42}$$

By assumption (2), $\mathrm{Re}(\mathrm{e}^{it}T)D$ is a Fredholm operator, and by assumption (3), $\mathrm{Im}T$ is injective. Therefore by applying Lemma 6.13 to our operators, there is a constant $C' > 0$ such that

$$C'(\left\|\mathrm{Re}(\mathrm{e}^{it}T)D\varphi\right\|^2 + \|\mathrm{Im}T\varphi\|^2) \geq \|\varphi\|^2 \quad \textit{for all } \varphi \in U, \tag{6.43}$$

which implies that the operator $T_{\#} : U \to U$ is positive coercive. Since we can write

$$F_{\#} = F_{\#}^{1/2}(F_{\#}^{1/2})^* = GT_{\#}G^*, \tag{6.44}$$

then by applying Theorem 1.21 of [58], we conclude (6.29). We have shown Theorem 6.11. $\square$

6.4 A factorization of the near field operator

In Section 6.4, we discuss a factorization of the near field operator N. We define the operator $L : L^2(Q) \to L^2(M)$ by $Lf := v\big|_M$ where v is a radiating solution (that is, v satisfies the radiation condition in the sense of Definition 6.6) such that

$$\Delta v + k^2(1+q)nv = -k^2 qnf, \text{ in } \mathbb{R}^2_+, \tag{6.45}$$

$$v = 0 \text{ on } \mathbb{R} \times \{0\}. \tag{6.46}$$

We define $H_Q : L^2(M) \to L^2(Q)$ by

$$H_Q g(x) := \int_M \overline{G_n(x,y)} g(y) ds(y), \ x \in Q. \tag{6.47}$$

Then, by these we have $N = LH_Q$. In order to obtain a symmetric factorization of the near field operator N, we show the following symmetry of the Green's function G_n.

Lemma 6.14.

$$G_n(x,y) = G_n(y,x), \ x \neq y. \tag{6.48}$$

Proof of Lemma 6.14. We take a small $\eta > 0$ such that $B_{2\eta}(x) \cap B_{2\eta}(y) = \emptyset$ where $B_\epsilon(z) \subset \mathbb{R}^2$ is some open ball with center z and radius $\epsilon > 0$. We recall that $G_n(z,y) = G(z,y) + \tilde{u}^s(z,y)$ where $G(z,y) = \Phi_k(z,y) - \Phi_k(z,y^*)$ and $\tilde{u}^s(z,y)$ is a radiating solution of the problem (6.5) such that $\tilde{u}^s(z,y) = 0$ for $z_2 = 0$. In the introduction of [62] $\tilde{u}^s$ is given by $\tilde{u}^s(z,y) = u(z,y) - \chi(|z-y|)G(z,y)$ where $\chi \in C^\infty(\mathbb{R}_+)$ satisfying $\chi(t) = 0$ for $0 \leq t \leq \eta/2$ and $\chi(t) = 1$ for $t \geq \eta$, and u is a radiating solution such that $u = 0$ on $\mathbb{R} \times \{0\}$ and

$$\Delta u + k^2 nu = f(\cdot, y) \text{ in } \mathbb{R}^2_+, \tag{6.49}$$

$$u = 0 \text{ on } \mathbb{R} \times \{0\}, \tag{6.50}$$

where

$$f(\cdot,y) := \Big[k^2\big(1-n\big)\big(1-\chi(|\cdot-y|)\big) + \Delta\chi(|\cdot-y|)\Big] G(\cdot,y) + 2\nabla\chi(|\cdot-y|)\cdot\nabla G(\cdot,y). \tag{6.51}$$

Then, we have $G_n(z,y) = u(z,y) + (1-\chi(|z-y|))G(z,y)$. By Theorem 6.8 we can take an solution $u_\epsilon \in H^1(\mathbb{R}^2_+)$ of the problem (6.49)–(6.50) replacing k by $(k+i\epsilon)$ satisfying u_ϵ converges as $\epsilon \to +0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to u. We set $G_{n,\epsilon}(z,y) := u_\epsilon(z,y) + (1-\chi(|z-y|))G(z,y)$, and $G_{n,\epsilon}(\cdot,y)$ converges as $\epsilon \to +0$ to $G_n(\cdot,y)$ in $H^1_{loc}(\mathbb{R}^2_+)$. By a simple calculation, we have

$$\big[\Delta_z + (k+i\epsilon)^2 n(z)\big] G_{n,\epsilon}(z,y) = -\delta(z,y) + (2k\epsilon i - \epsilon^2)n(z)\big(1-\chi(|z-y|)\big)G(z,y). \tag{6.52}$$

Let $r > 0$ be large enough such that $x, y \in B_r(0)$. By Green's second theorem with $G_{n,\epsilon}(\cdot, x)$ and $G_{n,\epsilon}(\cdot, y)$ in $B_r(0) \cap \mathbb{R}^2_+$ we have

$$\begin{aligned}
&-G_{n,\epsilon}(y,x) + (2k\epsilon i - \epsilon^2)\int_{B_{2\eta}(y)} u_\epsilon(z,x)n(z)(1-\chi(|z-y|))G(z,y)dz \\
&+ \quad G_{n,\epsilon}(x,y) - (2k\epsilon i - \epsilon^2)\int_{B_{2\eta}(x)} u_\epsilon(z,y)n(z)(1-\chi(|z-x|))G(z,x)dz \\
&= \quad \int_{B_r(0)\cap\mathbb{R}^2_+} G_{n,\epsilon}(z,x)\big[\Delta_z + (k+i\epsilon)^2 n(z)\big]G_{n,\epsilon}(z,y)dz \\
&- \quad \int_{B_r(0)\cap\mathbb{R}^2_+} G_{n,\epsilon}(z,y)\big[\Delta_z + (k+i\epsilon)^2 n(z)\big]G_{n,\epsilon}(z,x)dz \\
&= \quad \int_{\partial B_r(0)\cap\mathbb{R}^2_+} u_\epsilon(z,x)\frac{\partial u_\epsilon(z,y)}{\partial \nu_z} - u_\epsilon(z,y)\frac{\partial u_\epsilon(z,x)}{\partial \nu_z} ds(z). \qquad (6.53)
\end{aligned}$$

Since $u_\epsilon \in H^1(\mathbb{R}^2_+)$, the right hand side of (6.53) converges as $r \to \infty$ to zero. Then, as $r \to \infty$ in (6.53) we have

$$\begin{aligned}
&G_{n,\epsilon}(x,y) - G_{n,\epsilon}(y,x) \\
&\quad = \quad (2k\epsilon i - \epsilon^2)\int_{B_{2\eta}(x)} u_\epsilon(z,y)n(z)(1-\chi(|z-x|))G(z,x)dz \\
&\quad - \quad (2k\epsilon i - \epsilon^2)\int_{B_{2\eta}(y)} u_\epsilon(z,x)n(z)(1-\chi(|z-y|))G(z,y)dz \qquad (6.54)
\end{aligned}$$

Since u_ϵ converges as $\epsilon \to +0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to u, the right hand side of (6.54) converges to zero as $\epsilon \to +0$. Therefore, we conclude that $G_n(x,y) = G_n(y,x)$ for $x \neq y$. □

By the symmetricity of G_n,

$$\begin{aligned}
\langle H_Q g, f\rangle &= \int_Q \left\{\int_M \overline{G_n(x,y)}g(y)ds(y)\right\}\overline{f(x)}dx \\
&= \int_M g(y)\overline{\left\{\int_Q G_n(x,y)f(x)dx\right\}}ds(y) \\
&= \int_M g(y)\overline{\left\{\int_Q G_n(y,x)f(x)dx\right\}}ds(y), \qquad (6.55)
\end{aligned}$$

which implies that

$$H_Q^* f(x) = \int_Q G_n(x,y)f(y)dy, \ x \in M. \qquad (6.56)$$

We define $T : L^2(Q) \to L^2(Q)$ by

$$Tf := k^2 qnf + k^2 qnv\big|_Q, \tag{6.57}$$

where v is a radiating solution of (6.45)–(6.46). Since $Lf(x)$ is of the form

$$Lf(x) = v(x) = \int_Q G_n(x,y) k^2 q(y) n(y) \left(f(y) + v(y)\right) dy, \tag{6.58}$$

we have

$$L = H_Q^* T. \tag{6.59}$$

Therefore, we have the following symmetric factorization of N:

$$N = H_Q^* T H_Q. \tag{6.60}$$

We show the following lemma corresponding to assumptions of Theorem 6.11.

Lemma 6.15. **(a)** *H_Q is compact and injective.*

(b) *If there exists a constant $q_{min} > 0$ such that $q_{min} \leq q$ a.e. in Q, then T has the form $T = C + K$ where C is a self-adjoint and positive coercive operator of the form $Cf := k^2 qnf$, and K is a compact operator of the form $Kf := k^2 qnv\big|_Q$.*

(c) $\mathrm{Im}\langle f, Tf\rangle \geq 0$ *for all $f \in L^2(Q)$.*

(d) *T is injective.*

Proof of Lemma 6.15. **(d)** Let $f \in L^2(Q)$ and assume that $Tf = 0$, i.e., $k^2 qnf + k^2 qnv = 0$ in Q. By this and (6.45), $\Delta v + k^2 nv = 0$. By the uniqueness (Theorem 6.8), $v = 0$ in $\mathbb{R}^2_+$ which implies that $f = 0$. Therefore, T is injective.

(b) By the definition of T, it is obvious that T has such a form. Since n and q are bounded below (that is, $n \geq n_{min} > 0$ and $q \geq q_{min} > 0$), C is a self-adjoint and positive coercive operator. The compactness of the operator $K : L^2(Q) \to L^2(Q)$ arises from $v\big|_Q \in H^1(Q)$.

(a) From (d), (b), and the Fredholm theorem, we obtain that T is bounded invertible. By this, it is sufficient to show that the operator L is compact. By the trace theorem and $v \in H^1_{loc}(\mathbb{R}^2_+)$, $Lf = v\big|_M \in H^{1/2}(M)$, which implies that the operator $L : L^2(Q) \to L^2(M)$ is compact.

To show the injectivity of H_Q, let $g \in L^2(M)$ and assume that $H_Q g(x) = \int_M \overline{G_n(x,y)} g(y) ds(y) = 0$ for $x \in Q$. We set $w(x) := \int_M \overline{G_n(x,y)} g(y) ds(y)$. By the definition of w we have

$$\Delta w + k^2 n w = 0, \text{ in } \mathbb{R}^2_+ \setminus M, \tag{6.61}$$

By unique continuation principle we have $w = 0$ in $\mathbb{R}^2_+ \setminus M$. By the jump relation (see e.g., Theorem 6.11 of [76]) we have $0 = \frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} = g$, which conclude that the operator H_Q is injective.

(c) For the proof of (c) we refer to Theorem 3.1 of [23]. By the definition of T we have

$$\mathrm{Im}\langle f, Tf \rangle = \mathrm{Im} \int_Q f k^2 q n \overline{v} dx = \mathrm{Im} \int_Q \overline{v} [\Delta + k^2 n] v dx, \tag{6.62}$$

where v is a radiating solution of the problem (6.45)–(6.46). We set $\Omega_N := (-N, N) \times (0, N^s)$ where $s > 0$ is small enough and $N > 0$ is large enough. By the same argument in Theorem 3.1 of [23] we have

$$\begin{aligned} \mathrm{Im}\langle f, Tf \rangle &= \mathrm{Im} \int_{\Omega_N} \overline{v} [\Delta + k^2 n] v dx = \mathrm{Im} \int_{\Omega_N} \overline{v} \Delta v dx \\ &\geq \left[\frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} > 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \right] \\ &- \mathrm{Im} \left[\frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} < 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \right] + o(1), \end{aligned} \tag{6.63}$$

where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, ..., m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (6.18). By Lemmas 6.3 and 6.4 of [62], as $N \to \infty$ in (6.63) we have

$$\mathrm{Im}\langle f, Tf \rangle \geq \frac{k}{2\pi} \sum_{j \in J} \left[\sum_{d_{l,j} > 0} |a_{l,j}|^2 d_{l,j} - \sum_{d_{l,j} < 0} |a_{l,j}|^2 d_{l,j} \right] \geq 0, \tag{6.64}$$

which concludes (c). $\square$

Remark 6.16. The strictly positivity of ImT is missing in Lemma 6.15 although we have the injectivity of T. From the viewpoint of Section 6.3, we have to show the strictly positivity of ImT if we use the factorization method, and we would expect the assumption of transmission eigenvalue for Q in this case. However, even with its assumption the author of this paper do not understand how to prove Im$T > 0$.

We will show the following lemma.

Lemma 6.17. *Let B and Q be bounded open sets in $\mathbb{R}^2_+$ such that $\mathbb{R}^2_+ \setminus \overline{B}$ and $\mathbb{R}^2_+ \setminus \overline{Q}$ is connected. Then,*

(a) $\dim(\mathrm{Ran}(H_B^*)) = \infty$.

(b) *If $B \cap Q = \emptyset$, then* $\mathrm{Ran}(H_B^*) \cap \mathrm{Ran}(H_Q^*) = \{0\}$.

Proof of Lemma 6.17. **(a)** By the same argument of the injectivity of H_Q in (a) of Lemma 6.15, one can show that H_B^* is injective for general B. Therefore, H_B^* has dense range in $L^2(M)$.

(b) Let $h \in \mathrm{Ran}(H_B^*) \cap \mathrm{Ran}(H_Q^*)$. Then, there exists f_B, f_Q such that $h = H_B^* f_B = H_Q^* f_Q$. We set

$$v_B(x) := \int_B G_n(x,y) f_B(y) dy, \ x \in \mathbb{R}^2_+ \tag{6.65}$$

$$v_Q(x) := \int_Q G_n(x,y) f_Q(y) dy, \ x \in \mathbb{R}^2_+ \tag{6.66}$$

then, v_B and v_Q satisfies $\Delta v_B + k^2 n v_B = -f_B$, and $\Delta v_Q + k^2 n v_Q = -f_Q$, respectively, and $v_B = v_Q$ on M. By the unique continuation we have $v_B = v_Q$ in $\mathbb{R}^2_+ \setminus (\overline{B \cap Q})$. Hence, we can define $v \in H^1_{loc}(\mathbb{R}^2_+)$ by

$$v := \begin{cases} v_B = v_Q & \text{in } \mathbb{R}^2_+ \setminus (\overline{B \cap Q}) \\ v_B & \text{in } Q \\ v_Q & \text{in } B \end{cases} \tag{6.67}$$

and v is a radiating solution such that $v = 0$ for $x_2 = 0$ and

$$\Delta v + k^2 n v = 0 \text{ in } \mathbb{R}^2_+. \tag{6.68}$$

By the uniqueness (Theorem 6.8), we have $v = 0$ in $\mathbb{R}^2_+$, which implies that $h = 0$. $\square$

In the following sections, we will show Theorems 6.1 and 6.2 by using these properties of the factorization of the near field operator N.

6.5 Proof of Theorem 6.1

In Section 6.5, we will show Theorem 6.1. Let $B \subset Q$, and let $K : L^2(Q) \to L^2(Q)$ be a compact operator defined in (b) of Lemma 6.15. Let V be the sum of eigenspaces of $\mathrm{Re}K$ associated to eigenvalues less than

$\alpha - k^2 n_{min} q_{min}$. Since $\alpha - k^2 n_{min} q_{min} < 0$, V is a finite dimensional subspace, and for $H_Q g \in V^\perp$

$$\begin{aligned}\langle \mathrm{Re} N g, g \rangle &= \int_Q k^2 nq |H_Q g|^2 dx + \langle (\mathrm{Re} K) H_Q g, H_Q g \rangle \\ &\geq k^2 n_{min} q_{min} \|H_Q g\|^2 + (\alpha - k^2 n_{min} q_{min}) \|H_Q g\|^2 \\ &\geq \alpha \|H_Q g\|^2 \geq \alpha \|H_B g\|^2 . \end{aligned} \tag{6.69}$$

Since for $g \in L^2(M)$

$$H_Q g \in V^\perp \quad \Longleftrightarrow \quad g \in (H_Q^* V)^\perp, \tag{6.70}$$

and $\dim(H_Q^* V) \leq \dim(V) < \infty$, we have by Corollary 3.3 of [38] that $\alpha H_B^* H_B \leq_{\text{fin}} \mathrm{Re} N$.

Let now $B \not\subset Q$ and assume on the contrary $\alpha H_B^* H_B \leq_{\text{fin}} \mathrm{Re} N$, that is, by Corollary 3.3 of [38] there exists a finite dimensional subspace W in $L^2(M)$ such that

$$\langle (\mathrm{Re} N - \alpha H_B^* H_B) w, w \rangle \geq 0, \tag{6.71}$$

for all $w \in W^\perp$. Since $B \not\subset Q$, we can take a small open domain $B_0 \subset B$ such that $B_0 \cap Q = \emptyset$, which implies that for all $w \in W^\perp$

$$\begin{aligned}\alpha \|H_{B_0} w\|^2 &\leq \alpha \|H_B w\|^2 \\ &\leq \langle (\mathrm{Re} N) w, w \rangle \\ &= \langle (\mathrm{Re} T) H_Q w, H_Q w \rangle \\ &\leq \|\mathrm{Re} T\| \, \|H_Q w\|^2 . \end{aligned} \tag{6.72}$$

By (a) of Lemma 4.7 in [38], we have

$$\mathrm{Ran}(H_{B_0}^*) \not\subseteq \mathrm{Ran}(H_Q^*) + W = \mathrm{Ran}(H_Q^*,\ P_W), \tag{6.73}$$

where the operator $(H_Q^*,\ P_W) : L^2(Q) \times L^2(M) \to L^2(M)$ is defined by $(H_Q^*,\ P_W) \begin{pmatrix} f \\ g \end{pmatrix} := H_Q^* f + P_W g$, and $P_W : L^2(M) \to L^2(M)$ is the orthogonal projection onto W. Lemma 4.6 of [38] implies that for any $C > 0$ there exists a w_c such that

$$\|H_{B_0} w_c\|^2 > C^2 \left\| \begin{pmatrix} H_Q \\ P_V \end{pmatrix} w_c \right\|^2 = C^2 (\|H_Q w_c\|^2 + \|P_W w_c\|^2). \tag{6.74}$$

Hence, there exists a sequence $(w_m)_{m \in \mathbb{N}}$ in $L^2(M)$ such that $\|H_{B_0} w_m\| \to \infty$ and $\|H_Q w_m\| + \|P_V w_m\| \to 0$ as $m \to \infty$. Setting $\tilde{w}_m := w_m - P_W w_m \in W^\perp$ we have as $m \to \infty$,

$$\|H_{B_0} \tilde{w}_m\| \geq \|H_{B_0} w_m\| - \|H_{B_0}\| \, \|P_W w_m\| \to \infty, \tag{6.75}$$

$$\|H_Q\tilde{w}_m\| \leq \|H_Q w_m\| + \|H_Q\| \, \|P_W w_m\| \to 0. \tag{6.76}$$

This contradicts (6.72). Therefore, we have $\alpha H_B^* H_B \not\leq_{\mathrm{fin}} \mathrm{Re}N$. Theorem 6.1 has been shown. □

By the same argument in Theorem 6.1 we can show the following.

Corollary 6.18. *Let $B \subset \mathbb{R}^2$ be a bounded open set. We assume that there exists $q_{max} < 0$ such that $q \leq q_{max}$ a.e. in Q. Then for $0 < \alpha < k^2 n_{min}|q_{max}|$,*

$$B \subset Q \quad \Longleftrightarrow \quad \alpha H_B^* H_B \leq_{\mathrm{fin}} -\mathrm{Re}N, \tag{6.77}$$

6.6 Proof of Theorem 6.2

In Section 6.6, we will show Theorem 6.2. Let $Q \subset B$. Let V be the sum of eigenspaces of $\mathrm{Re}K$ associated to eigenvalues larger than $\alpha - k^2 n_{max} q_{max}$. Since $\alpha - k^2 n_{max} q_{max} > 0$, V is a finite dimensional subspace and for $H_Q g \in V^{\perp}$

$$\begin{aligned} \langle \mathrm{Re}Ng, g \rangle &= \int_Q k^2 nq|H_Q g|^2 dx + \langle (\mathrm{Re}K) H_Q g, H_Q g \rangle \\ &\leq k^2 n_{max} q_{max} \|H_Q g\|^2 + (\alpha - k^2 n_{max} q_{max}) \|H_Q g\|^2 \\ &\leq \alpha \|H_Q g\|^2 \leq \alpha \|H_B g\|^2 . \end{aligned} \tag{6.78}$$

Since for $g \in L^2(M)$

$$H_Q g \in V^{\perp} \quad \Longleftrightarrow \quad g \in (H_Q^* V)^{\perp}, \tag{6.79}$$

and $\dim(H_Q^* V) \leq \dim(V) < \infty$, we have by Corollary 3.3 of [38] that $\mathrm{Re}N \leq_{\mathrm{fin}} \alpha H_B^* H_B$.

Let now $Q \not\subset B$ and assume on the contrary $\mathrm{Re}N \leq_{\mathrm{fin}} \alpha H_B^* H_B$, that is, by Corollary 3.3 of [38] there exists a finite dimensional subspace W in $L^2(M)$ such that

$$\langle (\alpha H_B^* H_B - \mathrm{Re}N) w, w \rangle \geq 0, \tag{6.80}$$

for all $w \in W^{\perp}$. Since $Q \not\subset B$, we can take a small open domain $Q_0 \subset Q$ such that $Q_0 \cap B = \emptyset$. Let V be the sum of eigenspaces of $\mathrm{Re}K$ associated to eigenvalues less than $-(k^2 n_{min} q_{min}/2)$. Then, V is a finite dimensional subspace and for $w \in (H_Q^* V)^\perp \cap W^\perp = (H_Q^* V \cup W)^\perp$ we have

$$\begin{aligned}
& \alpha \left\| H_B w \right\|^2 \\
\geq\ & \langle (\mathrm{Re} N) w, w \rangle \\
=\ & \int_Q k^2 n q |H_Q w|^2 dx + \langle (\mathrm{Re} K) H_Q w, H_Q w \rangle \\
\geq\ & k^2 n_{min} q_{min} \left\| H_Q w \right\|^2 - (k^2 n_{min} q_{min}/2) \left\| H_Q w \right\|^2 \\
=\ & (k^2 n_{min} q_{min}/2) \left\| H_Q w \right\|^2 \\
\geq\ & (k^2 n_{min} q_{min}/2) \left\| H_{Q_0} w \right\|^2, \qquad (6.81)
\end{aligned}$$

and $\dim(H_Q^* V \cup W) < \infty$. By the same argument replacing Q, B_0, and W in the proof of Theorem 6.1 by B, Q_0, and $(H_Q^* V \cup W)$, respectively, there exists a sequence $(\tilde{w}_m)_{m \in \mathbb{N}}$ in $(H_Q^* V \cup W)^\perp$ such that $\|H_{Q_0} \tilde{w}_m\| \to \infty$ and $\|H_B \tilde{w}_m\| \to 0$ as $m \to \infty$, which contradicts (6.81). Therefore, we have $\mathrm{Re} N \not\leq_{\mathrm{fin}} \alpha H_B^* H_B$. Theorem 6.2 has been shown. $\square$

By the same argument in Theorem 6.2 we can show the following.

Corollary 6.19. *Let $B \subset \mathbb{R}^2$ be a bounded open set. We assume that there exists $q_{min} < 0$ and $q_{max} < 0$ such that $q_{min} \leq q \leq q_{max}$ a.e. in Q. Then for $\alpha > k^2 n_{max} |q_{min}|$,*

$$Q \subset B \quad \Longleftrightarrow \quad -\mathrm{Re} N \leq_{\mathrm{fin}} \alpha H_B^* H_B, \qquad (6.82)$$

6.7 Numerical examples

In Section 6.7, we give the numerical examples based on Theorem 6.1. We consider the following two supports Q_1 and Q_2 of functions q_1, q_2 (see Figure 8):

(1) $Q_1 = \left\{ (x_1, x_2) | (x_1 - 0.5)^2 + (x_2 - 0.5)^2 < (0.2)^2 \right\}$

(2) $Q_2 = \left\{ (x_1, x_2) | \left((x_1 - 0.5)/0.15 \right)^2 + \left((x_2 - 0.6)/0.3 \right)^2 < 1 \right\}$

where q_1 and q_2 are defined by

$$q_j(x) := \begin{cases} 1 & \text{for } x \in Q_j \\ 0 & \text{for } x \notin Q_j \end{cases} \qquad (6.83)$$

Based on Theorem 6.1, the indicator function in our examples is given by

$$I(B) := \# \left\{ \text{negative eigenvalues of } \mathrm{Re} N - \alpha H_B^* H_B \right\} \qquad (6.84)$$

We consider the sampling region by $[0, R] \times [0, R]$ with some $R > 0$. The test domain B is given by the small square $B_{i,j} := z_{i,j} + [-R/2M, R/2M]^2$ where the location $z_{i,j} = (Ri/M, Rj/M)$ $(i, j = 1, ..., M)$ and M is some large number.

The near field operator N is discretized by the matrix

$$N \approx \frac{b-a}{d} \big(u^s(x_l, x_p)\big)_{1 \le l,p \le d} \in \mathbb{C}^{d \times d} \tag{6.85}$$

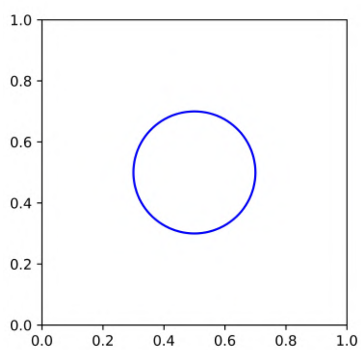
where $x_l = \big(a + \frac{(b-a)l}{d}, m\big)$, and $x_p = \big(a + \frac{(b-a)p}{d}, m\big)$, and the scattered field u^s is given by solving the following integral equation

$$u^s(x, z) = k^2 \int_Q q(y)n(y)u^s(y, z)G_n(x, y)dy + k^2 \int_Q q(y)n(y)\overline{G_n(y, z)}G_n(x, y)dy. \tag{6.86}$$

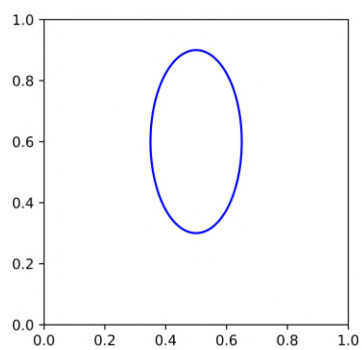
In our examples we fix $R = 1$, $M = 100$, $d = 30$, $a = -25$, $b = 25$, $m = 20$, and $n \equiv 1$. Figure 9 is given by plotting the values of the indicator function

$$I_{square}(z_{i,j}) := I(B_{i,j}),\ \ i, j = 1, ..., 100, \tag{6.87}$$

for two different supports Q_1 and Q_2 of true functions q_1 and q_2, and for two different parameters $\alpha = 10, 20$ in the case of wavenumber $k = 5$.

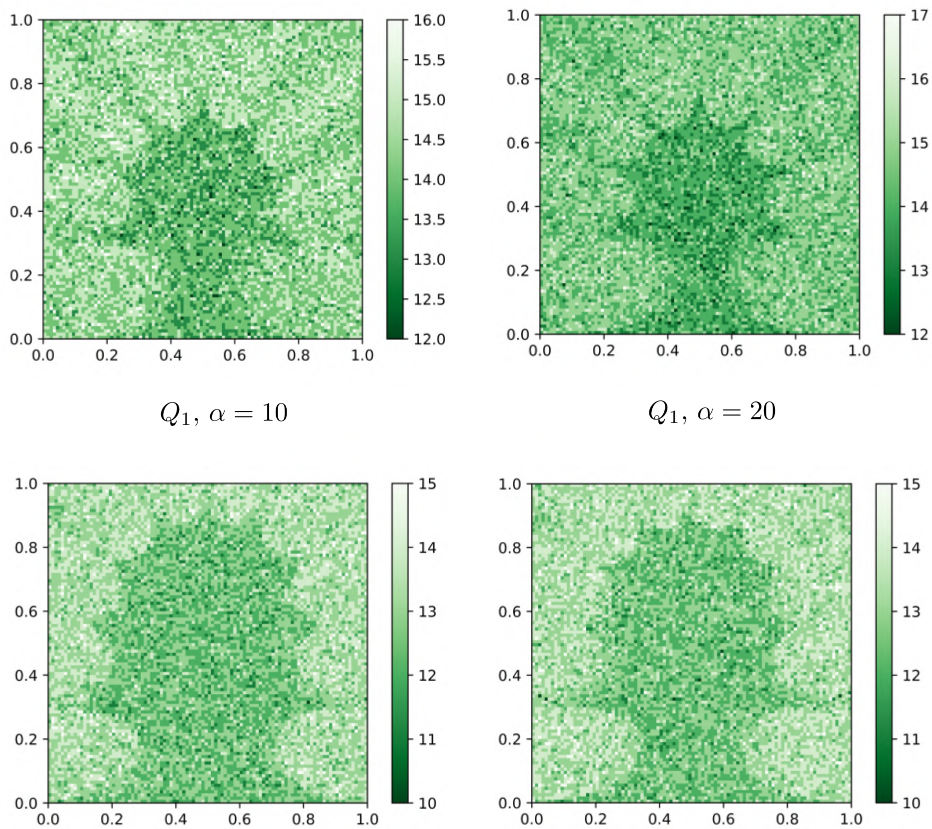


Q_1



Q_2

Figure 8: The original shape



Q_1, $\alpha = 10$

Q_1, $\alpha = 20$

Q_2, $\alpha = 10$

Q_2, $\alpha = 20$

Figure 9: Reconstruction by the indicator function I_{square} in the case of wavenumber $k = 5$

7 Inverse medium scattering problems with Kalman filter techniques I. Linear case

7.1 Introduction

The inverse scattering problem is the problem to determine unknown scatterers by measuring scattered waves that is generated by sending incident waves far away from scatterers. It is of importance for many applications, for example medical imaging, nondestructive testing, remote exploration, and geophysical prospecting. Due to many applications, the inverse scattering problem has been studied in various ways. For further readings, we refer to the following books [11, 16, 18, 52, 78], which include the summary of classical and recent progress of the inverse scattering problem.

We begin with the mathematical formulation of the scattering problem. Let $k > 0$ be the wave number, and let $\theta \in \mathbb{S}^1$ be incident direction. We denote the incident field $u^{inc}(\cdot, \theta)$ with the direction θ by the plane wave of the form

$$u^{inc}(x, \theta) := \mathrm{e}^{ikx\cdot\theta}, \ x \in \mathbb{R}^2. \tag{7.1}$$

Let Q be a bounded open set and let its exterior $\mathbb{R}^2 \setminus \overline{Q}$ be connected. Let $q \in L^\infty(\mathbb{R}^d)$ be real valued with a compact support such that $Q = \operatorname{supp} q$. Then, the direct scattering problem is to determine the total field $u = u^{sca} + u^{inc}$ such that

$$\Delta u + k^2(1 + q)u = 0 \text{ in } \mathbb{R}^2, \tag{7.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^{sca}}{\partial r} - iku^{sca}\right) = 0, \tag{7.3}$$

where $r = |x|$. The *Sommerfeld radiation condition* (7.3) holds uniformly in all directions $\hat{x} := \frac{x}{|x|}$. Furthermore, the problem (7.2)–(7.3) is equivalent to the *Lippmann-Schwinger integral equation*

$$u(x, \theta) = u^{inc}(x) + k^2 \int_Q q(y)u(y)\Phi(x, y)dy, \tag{7.4}$$

where $\Phi(x, y)$ denotes the fundamental solution to Helmholtz equation in $\mathbb{R}^2$, that is,

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|), \ x \neq y, \tag{7.5}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order one. It is well known that there exists a unique solution u^{sca} of the problem (7.2)–(7.3), and it has the following asymptotic behaviour (see e.g., Chapter 8 of [18]),

$$u^{sca}(x) = \frac{\mathrm{e}^{ikr}}{\sqrt{r}}\Big\{u^{\infty}(\hat{x},\theta) + O\big(1/r\big)\Big\},\ r \to \infty, \quad \hat{x} := \frac{x}{|x|}. \tag{7.6}$$

The function u^{∞} is called the *far field pattern* of u^{sca}, and it has the form

$$u^{\infty}(\hat{x},\theta) = \frac{k^2}{4\pi}\int_Q \mathrm{e}^{-ik\hat{x}\cdot y}u(y,\theta)q(y)dy =: \mathcal{F}q(\hat{x},\theta), \tag{7.7}$$

where the far field mapping $\mathcal{F} : L^2(Q) \to L^2(\mathbb{S}^1 \times \mathbb{S}^1)$ is defined in the second equality. For further details of these direct scattering problems, we refer to Chapter 8 of [18]. The inverse scattering problem we consider here is to reconstruct the function q from the far field pattern $u^{\infty}(\hat{x},\theta_n)$ for all $\hat{x} \in \mathbb{S}^1$, several incident directions $\{\theta_n\}_{n=1}^{N} \subset \mathbb{S}^1$ with some $N \in \mathbb{N}$, and one fixed wave number $k > 0$.

The equation (7.7) is nonlinear, that is, the far field mappings $\mathcal{F}$ is nonlinear because the function $u(y,\theta)$ depends on q. Existing methods for solving nonlinear inverse problem can be roughly categorized into two groups: iterative optimization methods and qualitative methods. The iterative optimization method (see e.g., [5, 18, 30, 42, 51]) does not require a lot of data, however it require the initial guess which is the starting point of optimization. It must be appropriately chosen by a priori knowledge of the unknown function q, otherwise, the iterative solution could not converge to the true function. On the other hand, the qualitative method (see e.g., [17, 33, 43, 44, 58, 78, 85]) such as the linear sampling method, the monotonicity method, the no-response test, the probe method, the factorization method, and the singular sources method, does not require the initial guess and it is computationally faster than the iterative method. However, the disadvantage of the qualitative method is to require a lot of data and to have difficulty in the case of the scatterer consisting of several components with different physical properties (see e.g., [22, 64]).

If the total field u in (7.7) is replaced by the incident field u^{inc}, the nonlinear equation (7.7) is transformed into the linear equation

$$u^{\infty}_{B}(\hat{x},\theta) = \frac{k^2}{4\pi}\int_Q \mathrm{e}^{-ik\hat{x}\cdot y}u^{inc}(y,\theta)q(y)dy =: \mathcal{F}_B q(\hat{x},\theta), \tag{7.8}$$

which is known as the *Born approximation*. The function u^{∞}_{B} is a good approximation of the far field pattern u^{∞} when $k > 0$ and the value of q are very small. Another interpretation is that the Born approximation is the

Fréchet derivative of the far field mapping $\mathcal{F}$ at $q = 0$. For further readings of the inverse scattering problem with the Born approximation, we refer to [5, 6, 18, 54, 83]. In this section, we study the linear integral equation (7.8) instead of the nonlinear one (7.7).

Although the inverse scattering problem became linear by the Born approximation, the linear equation (7.8) is ill-posed, which means there does not exist the inverse $\mathcal{F}_B^{-1}$ of the operator $\mathcal{F}_B$. A common technique to solve linear and ill-posed inverse problems is the *Tikhonov regularization method* (see e.g., [11, 35, 67, 78]). A natural approach applying regularization method to our situation is to put all available measurements $\{u_{B,n}^{\infty}\}_{n=1}^{N}$ and all far field mappings $\{\mathcal{F}_{B,n}\}_{n=1}^{N}$, where the index n corresponds to some incident direction θ_n, into one long vector $\vec{u}^{\infty}$ and $\vec{\mathcal{F}}_B$, respectively, and to apply the Tikhonov regularization method to the big system $\vec{u}^{\infty} = \vec{\mathcal{F}}_B q$. We shall call this way the *Full data Tikhonov*.

In this section, we propose the reconstruction scheme based on *Kalman filter* techniques. The Kalman filter (see the original paper [50]) is the algorithm to estimate the unknown state in the dynamics system by using the sequential measurements observed over time. It has many applications such as navigations and tracking objects, and for further readings, we refer to [31, 48, 50, 78].

The contributions of this section are followings.

(A) We propose the reconstruction algorithm for solving the linear inverse scattering problem (7.8) based on the Kalman Filter (see (7.44)–(7.46)).

(B) We show that in the linear problem, the Full data Tikhonov is equivalent to the Kalman Filter (see Theorem 7.4).

(A) means that we can estimate the unknown function q every time to observe the far field pattern $u_{B,n}^{\infty}$ with one incident direction θ_n without waiting for all data $\{u_{B,n}^{\infty}\}_{n=1}^{N}$. Furthermore, (B) means that the solution q_N^{KF} of the Kalman filter after giving all data coincides with the solution q_N^{FT} of the Full data Tikhonov with the same initial guess. The advantage of the Kalman Filter over the Full data Tikhonov is that we do not require to construct the big system $\vec{u}^{\infty} = \vec{\mathcal{F}}_B q$, which reduces computational costs. Instead of the big system, we update not only state, but also the norm of the state space, which is associated with the update of the covariance matrices of the state in the statistical viewpoint (see e.g., Chapter 5 of [21]).

This section is organized as follows. In Section 7.2, we briefly recall the Tikhonov regularization theory. In Sections 7.3, we give the algorithm of the Full data Tikhonov. In Section 7.4, we give the algorithm of the Kalman filter, and show that it is equivalent to the Full data Tikhonov discussed in the previous section. Finally in Section 7.5, we give numerical examples to demonstrate our theoretical results.

7.2 Tikhonov regularization method

Tikhonov regularization is the method to provide the stable approximate solution for linear and ill-posed inverse problem. In this section, we briefly recall the regularized approach. For further readings, we refer to [11, 35, 67, 78]. In Sections 7.2–7.5, we consider the general functional analytic situation of our inverse scattering problem.

Let X and Y be Hilbert spaces over complex variables $\mathbb{C}$, which are associated with the state space $L^2(Q)$ of the inhomogeneous medium function q, and the observation space $L^2(\mathbb{S}^1)$ of the far field pattern u^∞, respectively, and let $A : X \to Y$ be a compact linear operator from X to Y, which is the observation operator $\mathcal{F}_B : L^2(Q) \to L^2(\mathbb{S}^1)$ defined in (7.8) as the far field mapping. We consider the following problem to determine $\varphi \in X$ given $f \in Y$.

$$A\varphi = f. \tag{7.9}$$

Since the observation operator A is not generally invertible, the equation (7.9) is replaced by

$$\alpha\varphi + A^*A\varphi = A^*f, \tag{7.10}$$

which was derived from the multiplication with the adjoint A^* of the operator A and the addition of $\alpha\varphi$ where the regularization parameter $\alpha > 0$ in (7.9). We call the solution φ_α of the equation (7.10) the regularized solution of (7.9). The following lemma is well known as the properties of the regularized solution φ_α (see e.g., Chapter 4 of [18] and Chapter 3 of [78]).

Lemma 7.1. *Let X and Y be Hilbert spaces and let $A : X \to Y$ be a compact linear operator from X to Y. Then, followings hold.*

(i) *The operator $(\alpha I + A^*A)$ is bounded invertible, and*

$$\varphi_\alpha := (\alpha I + A^*A)^{-1}A^*f, \tag{7.11}$$

is the unique solution of (7.10).

(ii) *The solution φ_α defined in (7.11) is the unique solution of the following minimization problem.*

$$\min_{\varphi\in X} \left\{ \alpha \|\varphi\|_X^2 + \|f - A\varphi\|_Y^2 \right\}. \tag{7.12}$$

(iii) *If $f \in \mathrm{R}(A)$, then there exists $C = C_f$ such that*

$$\|\varphi_\alpha\| \leq C, \ \alpha > 0, \tag{7.13}$$

and if $f \notin \mathrm{R}(A)$, then $\|\varphi_\alpha\|_X \to \infty$ as $\alpha \to 0$.

We observe the above Lemma in the case when X is finite-dimensional and f is of the form $f = A\varphi^{true}$ where φ^{true} is the true solution of the problem (7.9). In this case, the regularized solution φ_α of (7.9) converges as $\alpha \to 0$ to φ^{least} defined by

$$\varphi^{least} := A^\dagger A\varphi^{true}, \tag{7.14}$$

where the operator $A^\dagger$ is the *pseudo inverse* of the operator A defined by $A^\dagger := (A^*A)^{-1}A^*$. In finite-dimensional case of the space X, the operator $(A^*A)^{-1}A^* : Y \to R(A^*)$ is well defined since $A^*A : R(A^*) \to R(A^*)$ is bijective. φ^{least} is known as the *least squares solution*, that is, it satisfies

$$\left\|A\varphi^{least} - f\right\| = \min_{\varphi \in X} \left\{\|A\varphi - f\|_Y\right\}, \tag{7.15}$$

which means that in the ill-posed problem of (7.9), φ^{least} is the best possible solution in the sense of a taking a smallest norm of $\|A\varphi - f\|_Y$. Furthermore, if the operator A is injective, then the least squares solution φ^{least} coincides with the true solution φ^{true} because $A^\dagger A$ is an identity operator. For details of the least squares solution, we refer to Section 4.3 of [34] and Section 3.2 of [78].

7.3 Full data Tikhonov

The natural approach for solving the equation (7.8) is to put all available measurements $\{u^\infty_{B,n}\}_{n=1}^N$ and all far field mappings $\{\mathcal{F}_{B,n}\}_{n=1}^N$, where the index n is associated with some incident angle $\theta_n \in \mathbb{S}^1$, into one long vector $\vec{u}^\infty_B$ and $\vec{\mathcal{F}}_B$, respectively, and to employ the regularized approach. In order to study the above general situation, let $f_1, ..., f_N \in Y$ be measurements, let $A_1, ..., A_N$ be observation operators, and let us consider the problem to determine $\varphi \in X$ such that

$$A_n\varphi = f_n, \tag{7.16}$$

for all $n = 1, ..., N$. Now, we assume that we have the initial guess $\varphi_0 \in X$, which is the starting point of the algorithm, and is usually determined by a priori information of the true solution φ^{true}. Then, we consider the minimization problem of the following functional.

$$
\begin{aligned}
J_{Full,N}(\varphi) &:= \alpha \left\|\varphi - \varphi_0\right\|_X^2 + \left\|\vec{f} - \vec{A}\varphi\right\|_{Y^N,R^{-1}}^2 \\
&= \alpha \left\|\varphi - \varphi_0\right\|_X^2 + \sum_{n=1}^{N} \left\|f_n - A_n\varphi\right\|_{Y,R^{-1}}^2 , \qquad (7.17)
\end{aligned}
$$

where $\vec{f} := \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$, and $\vec{A} := \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix}$. The norm $\|\cdot\|_{Y,R^{-1}}^2 := \langle \cdot, R^{-1} \cdot \rangle_Y$ is a weighted norm with a positive definite symmetric invertible operator $R : Y \to Y$, which is interpreted as the error covariance matrices of the observation distribution from a statistical viewpoint in the case when Y is the Euclidean space (see e.g., Chapter 5 of [21]). With $\tilde{\varphi} = \varphi - \varphi_0$, the problem (7.17) is transformed into

$$
\tilde{J}_{Full,N}(\tilde{\varphi}) := \alpha \left\|\tilde{\varphi}\right\|_X^2 + \left\|(\vec{f} - \vec{A}\varphi_0) - \vec{A}\tilde{\varphi}\right\|_{Y^N}^2 . \qquad (7.18)
$$

By Lemma 7.1, the minimizer $\tilde{\varphi}_\alpha$ of (7.18) is given by

$$
\tilde{\varphi}_\alpha = (\alpha I + \vec{A}^*\vec{A})^{-1}\vec{A}^*\left(\vec{f} - \vec{A}\varphi_0\right), \qquad (7.19)
$$

which implies that

$$
\varphi_N^{FT} := \varphi_0 + (\alpha I + \vec{A}^*\vec{A})^{-1}\vec{A}^*\left(\vec{f} - \vec{A}\varphi_0\right), \qquad (7.20)
$$

is the minimizer of (7.17). We call this the *Full data Tikhonov*. Here, $\vec{A}^*$ is the adjoint operator with respect to $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_{Y^N,R^{-1}}$. We calculate

$$
\begin{aligned}
\langle \vec{f}, \vec{A}\varphi \rangle_{Y^N,R^{-1}} &= \sum_{n=1}^{N} \langle f_n, R^{-1}A_n\varphi \rangle_Y \\
&= \sum_{n=1}^{N} \langle A_n^H R^{-1} f_n, \varphi \rangle_X, \qquad (7.21)
\end{aligned}
$$

which implies that

$$
\vec{A}^* = \left(A_1^H R^{-1}, ..., A_N^H R^{-1}\right). \qquad (7.22)
$$

where A_n^H is the adjoint operator with respect to usual scalar products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$. Then, the Full data Tikhonov solution in (7.20) is of the form

$$
\varphi_N^{FT} = \varphi_0 + \left(\alpha I + \sum_{n=1}^{N} A_n^H R^{-1} A_n\right)^{-1} \left(\sum_{n=1}^{N} A_n^H R^{-1}\left(f_n - A_n\varphi_0\right)\right). \qquad (7.23)
$$

7.4 Kalman filter

The Kalman filter is the algorithm to estimate the unknown state in the dynamics system by using the sequential measurements observed over time. In the usual Kalman filter, the model operator to describe the process of the state in the dynamics system is defined (see e.g., Chapter 5 of [78]). In our problem, it corresponds to the identity mapping because unknown function q does not develop over time.

Let us formulate the Kalman filter algorithm based on the functional analytic situation using the same notation described in Sections 7.2 and 7.3. In [21], the similar argument of followings was discussed in the special case when X and Y are the Euclidean spaces. In this section, we discuss more general situation, that is, the Hilbert space over complex variables $\mathbb{C}$, which is applicable to our inverse scattering problem. First, we consider the following minimization problem when one measurement $f_1 \in Y$, observation operator A_1, and the initial guess $\varphi_0 \in X$ are given.

$$J_1(\varphi) := \alpha \left\|\varphi - \varphi_0\right\|_X^2 + \left\|f_1 - A_1\varphi\right\|_{Y,R^{-1}}^2. \tag{7.24}$$

By using a weighted norm $\left\|\cdot\right\|_{X,B_0^{-1}}^2 := \langle \cdot, B_0^{-1}\cdot\rangle_X$ where $B_0 := \frac{1}{\alpha}I$, the functional J_1 can be of the form

$$J_1(\varphi) = \left\|\varphi - \varphi_0\right\|_{X,B_0^{-1}}^2 + \left\|f_1 - A_1\varphi\right\|_{Y,R^{-1}}^2, \tag{7.25}$$

and its unique minimizer φ_1 is given by

$$\varphi_1 := \varphi_0 + (I + A_1^* A_1)^{-1} A_1^* \left(f - A_1\varphi_0\right), \tag{7.26}$$

where A_1^* is the adjoint operator with respect to weighted scalar products $\langle \cdot, \cdot \rangle_{X,B_0^{-1}}$ and $\langle \cdot, \cdot \rangle_{Y,R^{-1}}$. We calculate

$$\begin{aligned} \langle f, A_1\varphi\rangle_{Y,R^{-1}} &= \langle f, R^{-1}A_1\varphi\rangle_Y \\ &= \langle A_1^H R^{-1} f, \varphi\rangle_X \\ &= \langle B_0 A_1^H R^{-1} f, \varphi\rangle_{X,B_0^{-1}}, \end{aligned} \tag{7.27}$$

which implies that

$$A_1^* = B_0 A_1^H R^{-1}, \tag{7.28}$$

where A_1^H is the adjoint operator with respect to usual scalar products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$. Then, we have

$$\begin{aligned} \varphi_1 &= \varphi_0 + (I + B_0 A_1^H R^{-1} A_1)^{-1} B_0 A_1^H R^{-1} \left(f - A_1\varphi_0\right) \\ &= \varphi_0 + (B_0^{-1} + A_1^H R^{-1} A_1)^{-1} A_1^H R^{-1} \left(f_1 - A_1\varphi_0\right). \end{aligned} \tag{7.29}$$

Next, we assume that one more measurement $f_2 \in Y$ and observation operator H_2 are given. The functional for two measurements is given by

$$
\begin{aligned}
J_{Full,2}(\varphi) &:= \|\varphi - \varphi_0\|^2_{X,B_0^{-1}} + \|f_1 - A_1\varphi\|^2_{Y,R^{-1}} + \|f_2 - A_2\varphi\|^2_{Y,R^{-1}} . \\
&= J_1(\varphi) + \|f_2 - A_2\varphi\|^2_{Y,R^{-1}} . \qquad (7.30)
\end{aligned}
$$

The question is whether we can find B_1 such that $J_{Full,2}(\varphi) = J_2(\varphi) + c$ where c is a constant number independently of φ, and the functional $J_2(\varphi)$ is defined by

$$
J_2(\varphi) = \|\varphi - \varphi_1\|^2_{X,B_1} + \|f_2 - A_2\varphi\|^2_{Y,R^{-1}} , \qquad (7.31)
$$

where φ_1 is defined by (7.29). To answer this question, we show the following lemma.

Lemma 7.2. *Set* $B_1 := \left(B_0^{-1} + A_1^H R^{-1} A_1\right)^{-1}$. *Then,*

$$
J_1(\varphi) = \|\varphi - \varphi_1\|^2_{X,B_1^{-1}} + c, \qquad (7.32)
$$

where c is some constant independently of φ.

Proof. We calculate

$$
\begin{aligned}
J_1(\varphi) &= \left\langle \varphi - \varphi_0, B_0^{-1}(\varphi - \varphi_0)\right\rangle_X + \left\langle f_1 - A_1\varphi, R^{-1}(f_1 - A_1\varphi)\right\rangle_Y \\
&= \left\langle \varphi, B_0^{-1}\varphi\right\rangle_X - 2\mathrm{Re}\left\langle \varphi, B_0^{-1}\varphi_0\right\rangle_X + \left\langle \varphi_0, B_0^{-1}\varphi_0\right\rangle_X \\
&\quad + \left\langle f_1, R^{-1}f_1\right\rangle_Y - 2\mathrm{Re}\left\langle \varphi, A_1^H R^{-1} f_1\right\rangle_X + \left\langle \varphi, A_1^H R^{-1} A_1\varphi\right\rangle_X . \\
&= \left\langle \varphi, B_0^{-1}\varphi\right\rangle_X - 2\mathrm{Re}\left\langle \varphi, B_0^{-1}\varphi_0\right\rangle_X - 2\mathrm{Re}\left\langle \varphi, A_1^H R^{-1} f_1\right\rangle_X \\
&\quad + \left\langle \varphi, A_1^H R^{-1} A_1\varphi\right\rangle_X + c_0 \\
&= \left\langle \varphi, B_1^{-1}\varphi\right\rangle_X - 2\mathrm{Re}\left\langle \varphi, B_0^{-1}\varphi_0\right\rangle_X - 2\mathrm{Re}\left\langle \varphi, A_1^H R^{-1} f_1\right\rangle_X + c_0, \\
&\qquad (7.33)
\end{aligned}
$$

where we used $B_1^{-1} = \left(B_0^{-1} + A_1^H R^{-1} A_1\right)$. By (7.29), we have

$$
\begin{aligned}
B_1^{-1}(\varphi - \varphi_1) &= B_1^{-1}\varphi - B_1^{-1}\varphi_1 \\
&= B_1^{-1}\varphi - \left(B_0^{-1} + A_1^H R^{-1} A_1\right)\varphi_0 - A_1^H R^{-1}(f - A_1\varphi_0) \\
&= B_1^{-1}\varphi - B_0^{-1}\varphi_0 - A_1^H R^{-1} f_1. \qquad (7.34)
\end{aligned}
$$

By using (7.34) and the self-adjointness of B_1^{-1}, we have

$$\begin{aligned}
& \left\langle \varphi - \varphi_1, B_1^{-1}(\varphi - \varphi_1)\right\rangle_X \\
= & \left\langle \varphi - \varphi_1, B_1^{-1}\varphi - B_0^{-1}\varphi_0 - A_1^H R^{-1} f_1\right\rangle_X \\
= & \left\langle B_1^{-1}(\varphi - \varphi_1), \varphi\right\rangle_X - \left\langle \varphi, B_0^{-1}\varphi_0\right\rangle_X - \left\langle \varphi, A_1^H R^{-1} f\right\rangle_X + c_1 \\
= & \left\langle B_1^{-1}\varphi - B_0^{-1}\varphi_0 - A_1^H R^{-1} f, \varphi\right\rangle_X \\
& \qquad - \left\langle \varphi, B_0^{-1}\varphi_0\right\rangle_X - \left\langle \varphi, A_1^H R^{-1} f\right\rangle_X + c_1 \\
= & \left\langle \varphi, B_1^{-1}\varphi\right\rangle_X - 2\mathrm{Re}\left\langle \varphi, B_0^{-1}\varphi_0\right\rangle_X - 2\mathrm{Re}\left\langle \varphi, A_1^H R^{-1} f_1\right\rangle_X + c_1.
\end{aligned} \tag{7.35}$$

With (7.33) and (7.35), $J_1(\varphi)$ is of the form

$$J_1(\varphi) = \left\langle \varphi - \varphi_1, B_1^{-1}(\varphi - \varphi_1)\right\rangle_X + c_2. \tag{7.36}$$

where c_0, c_1, and c_2 are some constant numbers independently of φ. Lemma 7.2 has been shown. □

This lemma tells us that $J_{Full,2}(\varphi)$ is equivalent to $J_2(\varphi)$ in the sense of minimization with respect to φ. By the same argument in (7.25)–(7.29), its unique minimizer φ_2 is given by

$$\varphi_2 := \varphi_1 + (B_1^{-1} + A_2^H R^{-1} A_2)^{-1} A_2^H R^{-1}(f_2 - A_2\varphi_1). \tag{7.37}$$

We can repeat the above argument (7.24)–(7.37) until measurements $f_1, ..., f_n$ and observation operators $A_1, ..., A_n$ are given. Then, we have following algorithms

$$\varphi_n := \varphi_{n-1} + K_n(f_n - A_n\varphi_{n-1}), \tag{7.38}$$

where the operator

$$K_n := \left(B_{n-1}^{-1} + A_n^H R^{-1} A_n\right)^{-1} A_n^H R^{-1}, \tag{7.39}$$

is called the *Kalman gain matrix*, and B_n is defined by

$$B_n := \left(B_{n-1}^{-1} + A_n^H R^{-1} A_n\right)^{-1}. \tag{7.40}$$

Since we have

$$\begin{aligned}
\left(B_{n-1}^{-1} + A_n^H R^{-1} A_n\right) B_{n-1} A_n^H &= A_n^H + A_n^H R^{-1} A_n B_{n-1} A_n^H \\
&= A_n^H R^{-1}\left(R + A_n B_{n-1} A_n^H\right),
\end{aligned}$$

the Kalman gain matrix K_n can be of the form

$$K_n = B_{n-1} A_n^H \left(R + A_n B_{n-1} A_n^H\right)^{-1}.$$

Here, we show the following lemma that the operator B_n has another form.

Lemma 7.3. *Let K_n be the Kalman gain matrix defined in (7.39). Then, the operator B_n has the following form*

$$B_n = (I - K_n A_n) B_{n-1}. \tag{7.41}$$

Proof. By multiplying (7.39) by $\left(B_{n-1}^{-1} + A_n^H R^{-1} A_n\right)$ from the left hand side, and by A_n from right hand side, we have

$$\left(B_{n-1}^{-1} + A_n^H R^{-1} A_n\right) K_n A_n = A_n^H R^{-1} A_n, \tag{7.42}$$

which implies that by using (7.40)

$$\begin{aligned} B_n^{-1} (I - K_n A_n) &= \left(B_{n-1}^{-1} + A_n^H R^{-1} A_n\right) (I - K_n A_n) \\ &= \left(B_{n-1}^{-1} + A_n^H R^{-1} A_n\right) - A_n^H R^{-1} A_n \\ &= B_{n-1}^{-1}. \end{aligned} \tag{7.43}$$

Multiplying (7.43) by B_n from the left hand side, and by B_{n-1} from the right hand side, we finally get (7.41). □

We summarize the update formula in the following.

$$\varphi_n^{KF} := \varphi_{n-1}^{KF} + K_n \left(f_n - A_n \varphi_{n-1}^{KF}\right), \tag{7.44}$$

$$K_n := B_{n-1} A_n^H \left(R + A_n B_{n-1} A_n^H\right)^{-1}, \tag{7.45}$$

$$B_n := \left(I - K_n A_n^H\right) B_{n-1}, \tag{7.46}$$

for $n = 1, ..., N$, where $\varphi_0^{KF} := \varphi_0$ and $B_0 := \frac{1}{\alpha} I$. We call this the *Kalman filter*.

We observe the above algorithm. It means that we can estimate the state φ every time n to observe one measurement f_n without waiting all measurements $\{f_n\}_{n=1}^N$. It includes not only the update (7.44) of the state φ, but also the update (7.46) of the weight B of the norm, which plays the role of keeping the information of the previous state. The weight B is also interpreted as the error covariance matrices of the state distribution from statistical viewpoint (see e.g., Chapter 5 of [21]).

Finally in this section, we show the equivalence of Full data Tikhonov and Kalman filter when all observation operators A_n are linear.

Theorem 7.4. *For measurements $f_1, ..., f_N$, linear operators $A_1, ..., A_N$, and the initial guess $\varphi_0 \in X$, the final sate of the Kalman filter given by (7.44)–(7.46) is equivalent to the state of the Full data Tikhonov given by (7.23), that is*

$$\varphi_N^{KF} = \varphi_N^{FT}. \tag{7.47}$$

Proof. It is sufficient to show that

$$J_{Full,N}(\varphi) = \left\|\varphi - \varphi_N^{KF}\right\|^2_{X,B_N^{-1}} + c_N, \tag{7.48}$$

where c_N is some constant independently of φ. We will prove (7.48) by the induction. The case of $N = 1$ has already been shown in Lemma 7.2.

We assume that (7.48) in the case of $n \in \mathbb{N}$ with $1 \leq n \leq N-1$ holds, that is,

$$J_{Full,n}(\varphi) = \left\|\varphi - \varphi_n^{KF}\right\|^2_{X,B_n^{-1}} + c_n, \tag{7.49}$$

where c_{N-1} is some constant. Then, we have

$$\begin{aligned} J_{Full,n+1}(\varphi) &= J_{Full,n}(\varphi) + \|f_{n+1} - A_{n+1}\varphi\|^2_{Y,R^{-1}} \\ &= \left\|\varphi - \varphi_n^{KF}\right\|^2_{X,B_n^{-1}} + \|f_{n+1} - A_{n+1}\varphi\|^2_{Y,R^{-1}} + c_n \end{aligned} \tag{7.50}$$

By the same argument in Lemma 7.2 replacing B_0, φ_0, f_1, A_1 by B_n, φ_n, f_{n+1}, A_{n+1}, respectively, we have that $J_{Full,n+1}(\varphi) = \left\|\varphi - \varphi_{n+1}^{KF}\right\|^2_{X,B_{n+1}^{-1}} + c_{n+1}$. Theorem 7.4 has been shown. □

7.5 Numerical examples

In this section, we give numerical examples of the algorithm which have been discussed in above sections. We recall that our inverse scattering problem is to solve the linear integral equation (7.8) with respect to q when the measurements $u^\infty_{B,n} := u^\infty_B(\cdot, \theta_n)$ for $n = 1, ..., N$ are given.

$$\mathcal{F}_{B,n} q = u^\infty(\cdot, \theta_n), \tag{7.51}$$

where the operator $\mathcal{F}_{B,n} : L^2(Q) \to L^\infty(\mathbb{S}^1)$ is defined by

$$\mathcal{F}_{B,n} q(\hat{x}) = \mathcal{F}_B q(\hat{x}, \theta_n) := \frac{k^2}{4\pi} \int_Q \mathrm{e}^{-ik\hat{x}\cdot y} u^{inc}(y, \theta_n) q(y) dy. \tag{7.52}$$

Here, the incident direction is given by $\theta_n := (\cos(2\pi n/N), \sin(2\pi n/N))$ for each $n = 1, ..., N$. The following discretizations are employed.

$$u^\infty_B(\cdot, \theta) \approx \left(u^\infty_B(\hat{x}_j, \theta)\right)_{j=1,...,J} \in \mathbb{R}^J, \tag{7.53}$$

where $\hat{x}_j := (\cos(2\pi j/J), \sin(2\pi j/J))$, and $J \in \mathbb{N}$, and

$$q \approx \left(q(z_{i,l})\right)_{-M \leq i,l \leq M-1} \in \mathbb{R}^{(2M)^2}, \tag{7.54}$$

where $z_{i,l} := \left(\frac{(2i+1)R}{2M}, \frac{(2l+1)R}{2M}\right)$, and $M \in \mathbb{N}$ is a number of the division of $[0,R]$, and $[-R,R]^2$ is a square with some $R > 0$, in which support Q of the function q is included, and

$$\mathcal{F}_{B,n} \approx \frac{k^2}{4\pi}\left(\mathrm{e}^{-ik\hat{x}_j\cdot z_{i,l}} q(z_{h,i})\mathrm{e}^{ikz_{i,l}\cdot\theta_n}\right)_{j=1,\ldots,J,\ -M\leq i,l\leq M-1} \in \mathbb{R}^{J\times(2M)^2}. \tag{7.55}$$

Here, we always fix discretization parameters as $J = 20$, $M = 6$, $R = 3$, $N = 15$, and consider true functions as the characteristic function

$$q_j^{true}(x) := \begin{cases} 1 & \text{for } x \in B_j \\ 0 & \text{for } x \notin B_j \end{cases}, \tag{7.56}$$

where the support B_j of the true function is considered as the following two types.

$$B_1 := \left\{(x_1,x_2) : x_1^2 + x_2^2 < 1.5\right\}, \tag{7.57}$$

$$B_2 := \left\{(x_1,x_2) : \begin{array}{c} (x_1+1.5)^2 + (x_2+1.5)^2 < (1.0)^2 \ or \\ 1 < x_1 < 2,\ -2 < x_2 < 2 \ or \\ -2 < x_1 < 2,\ -2.0 < x_2 < -1.0 \end{array}\right\}. \tag{7.58}$$

In Figure 10, the blue closed curve is the boundary ∂B_j of the support B_j of the true function q_j^{true}, and the green brightness indicates the value of the true function on each cell divided into $(2M)^2 = 144$ in the sampling domain $[-R,R]^2 = [-3,3]^2$. Here, we always employ the initial guess q_0 as

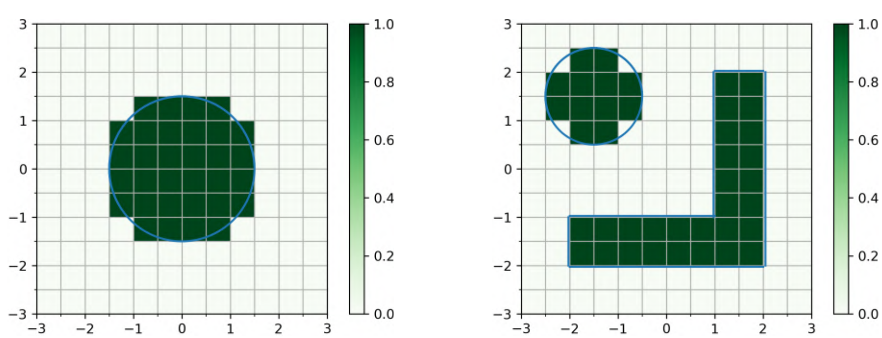
$$q_0 \equiv 0. \tag{7.59}$$

Figure 11 shows the reconstruction by the Kalman filter (KF) and the Full data Tikhonov (FT) discussed in (7.44)–(7.46) and (7.23), respectively. The first and second column correspond to visualization of the state q in the case when four measurements $\{u_B^\infty(\cdot,\theta_n)\}_{n=1}^4$ and full (fifteen) measurements $\{u_B^\infty(\cdot,\theta_n)\}_{n=1}^{N=15}$ are given, respectively, for different methods KF and FT, and for two different shapes B_1 and B_2. The wavenumber and the regularization parameter are fixed as $k = 5$ and $\alpha = 1$, respectively. The third column corresponds to the graph of the Mean Square Error (MSE) defined by

$$e_n := \left\|q^{true} - q_n\right\|^2, \tag{7.60}$$

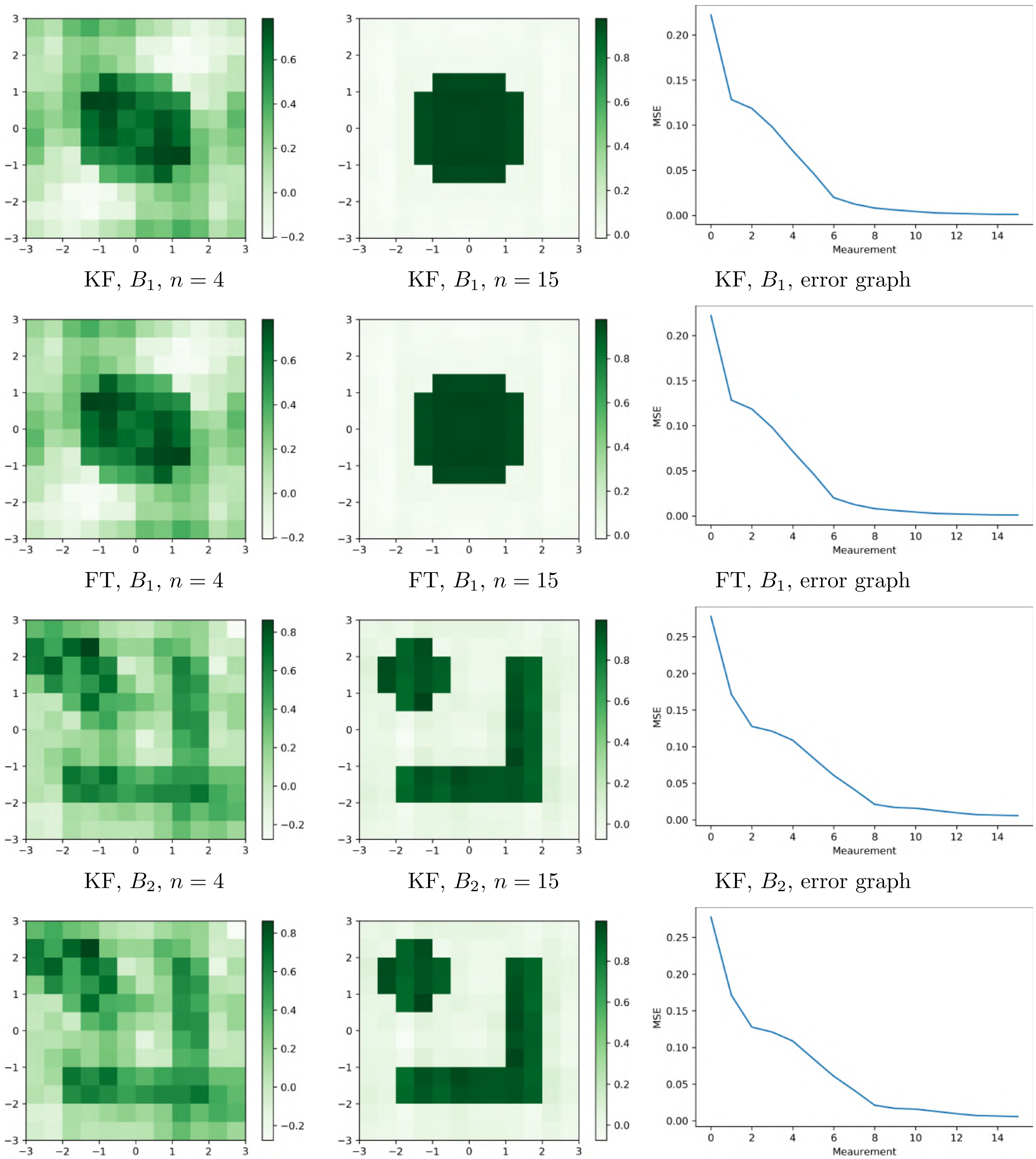
where q_n is associated with nth state reconstructed by some method. The horizontal axis is with respect to number of given measurements, and the vertical axis is the value of MSE. Motivated by Theorem 7.4, we can observe that in Figure 11, KF and FT are also numerically equivalent.

Figure 12 shows the reconstruction by the Kalman filter (KF) for two different wave numbers $k = 3$ and $k = 0.5$, and for two different shape B_1 and B_2. The first and second columns correspond to visualization of the final state given full measurement for different regularization parameters $\alpha = 1$ and $1e{-}8$. The third column corresponds to graphs of MSE, which have four evaluation with respect to $\alpha = 1, 1e-2, 1e-8$, and $\alpha = 0$. We can observe that the error graph converges as $\alpha \to 0$ to the red curve, which corresponds to the error of the least square solution. It agrees with theoretical viewpoints of the least square solution (see Section 7.2). The case of $k = 0.5$ is severely ill-posed because red curve in the case of $\alpha = 0$ does not converge to zero even if the number of measurements increases. This is because the rank of the full far field mapping $\vec{\mathcal{F}}_B := \begin{pmatrix} \mathcal{F}_{B,1} \\ \vdots \\ \mathcal{F}_{B,N} \end{pmatrix}$ degenerates when the wave number k decreases. Figure 13 shows its degeneracy. The horizontal axis is with respect to wave numbers, and the vertical axis is the number of the rank of full far field mappings $\vec{\mathcal{F}}_B$.



q_1^{true} q_2^{true}

Figure 10: true functions



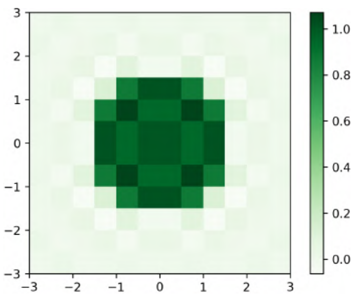
KF, B_1, $n = 4$ | KF, B_1, $n = 15$ | KF, B_1, error graph

FT, B_1, $n = 4$ | FT, B_1, $n = 15$ | FT, B_1, error graph

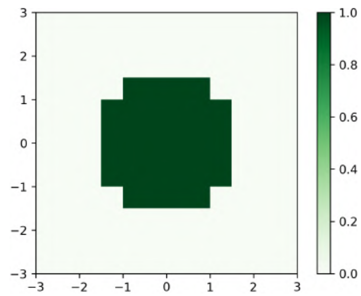
KF, B_2, $n = 4$ | KF, B_2, $n = 15$ | KF, B_2, error graph

FT, B_2, $n = 4$ | FT, B_2, $n = 15$ | FT, B_2, error graph

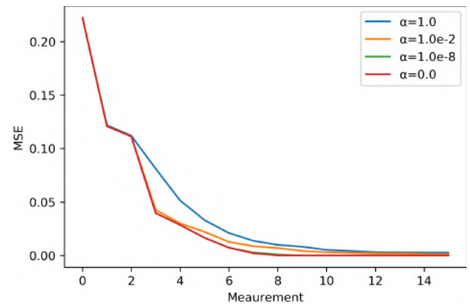
Figure 11: the comparison of KF and FT, $k = 5$, $\alpha = 1$



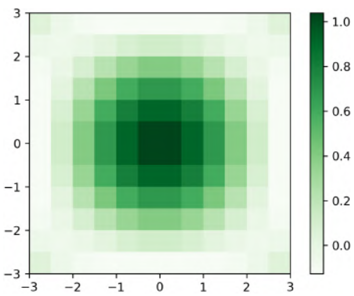
B_1, $k = 3$, $\alpha = 1$, $n = 15$



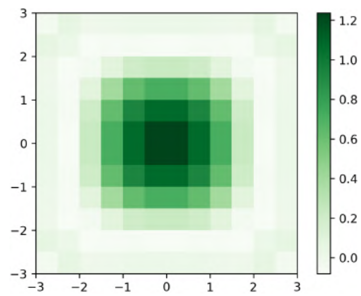
B_1, $k = 3$, $\alpha = 1\mathrm{e}-8$, $n = 15$



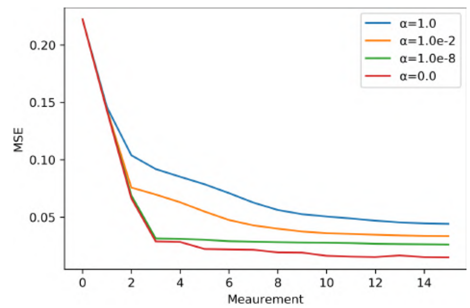
B_1, $k = 3$, error graph



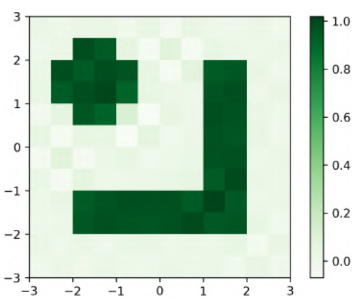
B_1, $k = 0.5$, $\alpha = 1$, $n = 15$



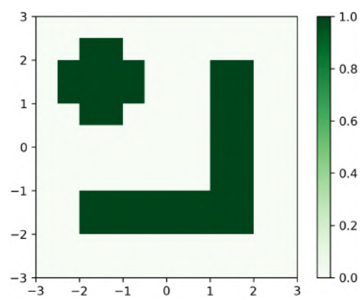
B_1, $k = 0.5$, $\alpha = 1\mathrm{e}-8$, $n = 15$



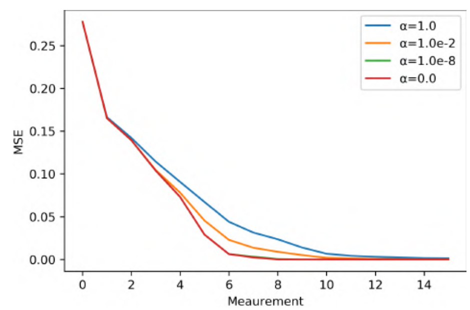
B_1, $k = 0.5$, error graph



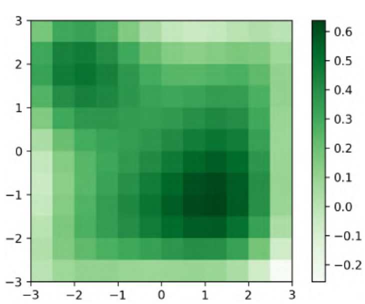
B_2, $k = 3$, $\alpha = 1$, $n = 15$



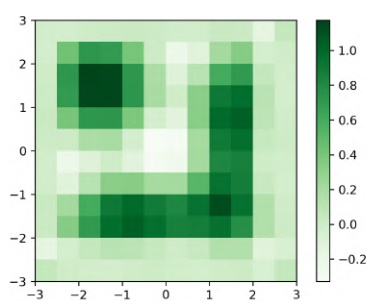
B_2, $k = 3$, $\alpha = 1\mathrm{e}-8$, $n = 15$



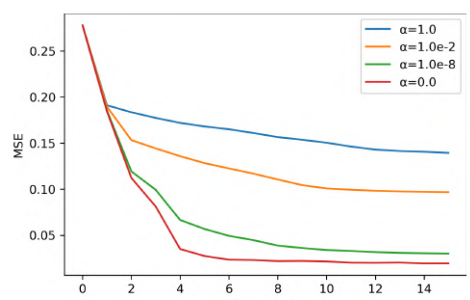
B_2, $k = 3$, error graph



B_2, $k = 0.5$, $\alpha = 1$, $n = 15$



B_2, $k = 0.5$, $\alpha = 1\mathrm{e}-8$, $n = 15$



B_2, $k = 0.5$, error graph

Figure 12: KF reconstruction for different k and α

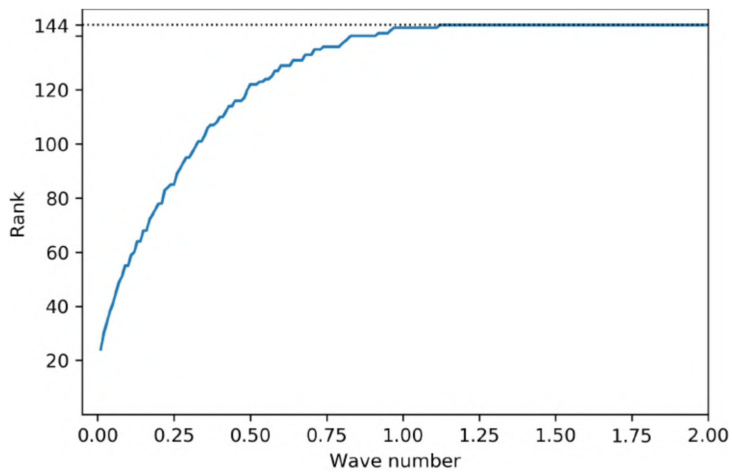


Figure 13: the graph of the rank of $\vec{\mathcal{F}}_B$

8 Inverse medium scattering problems with Kalman filter techniques II. Nonlinear case

8.1 Introduction

Let $k > 0$ be the wave number, and let $\theta \in \mathbb{S}^1$ be incident direction. We denote the incident field $u^{inc}(\cdot, \theta)$ with the direction θ by the plane wave of the form

$$u^{inc}(x, \theta) := \mathrm{e}^{ikx\cdot\theta},\ x \in \mathbb{R}^2. \tag{8.1}$$

Let Q be a bounded open set and let its exterior $\mathbb{R}^2 \setminus \overline{Q}$ be connected. Let $q \in L^{\infty}(\mathbb{R}^d)$ be real valued with a compact support such that $Q = \mathrm{supp}\, q$. Then, the direct scattering problem is to determine the total field $u = u^{sca} + u^{inc}$ such that

$$\Delta u + k^2(1+q)u = 0 \text{ in } \mathbb{R}^2, \tag{8.2}$$

$$\lim_{r\to\infty} \sqrt{r}\biggl(\frac{\partial u^{sca}}{\partial r} - iku^{sca}\biggr) = 0, \tag{8.3}$$

where $r = |x|$. The *Sommerfeld radiation condition* (8.3) holds uniformly in all directions $\hat{x} := \frac{x}{|x|}$. Furthermore, the problem (8.2)–(8.3) is equivalent to the *Lippmann-Schwinger integral equation*

$$u(x, \theta) = u^{inc}(x) + k^2 \int_Q q(y)u(y)\Phi(x, y)dy, \tag{8.4}$$

where $\Phi(x, y)$ denotes the fundamental solution to Helmholtz equation in $\mathbb{R}^2$, that is,

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x-y|),\ x \neq y, \tag{8.5}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order one. It is well known that there exists a unique solution u^{sca} of the problem (8.2)–(8.3), and it has the following asymptotic behaviour (see e.g., Chapter 8 of [18]),

$$u^{sca}(x) = \frac{\mathrm{e}^{ikr}}{\sqrt{r}}\Bigl\{u^{\infty}(\hat{x}, \theta) + O\bigl(1/r\bigr)\Bigr\},\ r \to \infty,\quad \hat{x} := \frac{x}{|x|}. \tag{8.6}$$

The function u^{∞} is called the *far field pattern* of u^{sca}, and it has the form

$$u^{\infty}(\hat{x}, \theta) = \frac{k^2}{4\pi} \int_Q \mathrm{e}^{-ik\hat{x}\cdot y} u(y, \theta) q(y) dy =: \mathcal{F}q(\hat{x}, \theta), \tag{8.7}$$

where the far field mapping $\mathcal{F} : L^2(Q) \to L^2(\mathbb{S}^1 \times \mathbb{S}^1)$ is defined in the second equality. For further details of these direct scattering problems, we refer to Chapter 8 of [18]. The inverse scattering problem we consider here is to reconstruct the function q from the far field pattern $u^{\infty}(\hat{x}, \theta_l)$ for all $\hat{x} \in \mathbb{S}^1$, several incident directions $\{\theta_l\}_{l=1}^{N} \subset \mathbb{S}^1$ with some $N \in \mathbb{N}$, and one fixed wave number $k > 0$.

The equation (8.7) is nonlinear, that is, the far field mappings $\mathcal{F}$ is nonlinear because the function $u(y, \theta)$ depends on q. The well known method to solve the nonlinear problem is the *Newton Method* (see e.g., [5, 18, 51, 52, 67, 78]), which is a classical method to construct an iterative solution based on the first-order linearization. A natural approach applying the Newton method to our situation is to put all available measurements $\{u_l^{\infty}\}_{l=1}^{N}$ and all far field mappings $\{\mathcal{F}_l\}_{l=1}^{N}$, where the index l corresponds to the incident direction θ_l, into one long vector $\vec{u}^{\infty}$ and $\vec{\mathcal{F}}$, respectively, and to apply the *regularized Newton method* to the big system $\vec{u}^{\infty} = \vec{\mathcal{F}}q$, that is, in each iteration step we apply the linear regularization method to linearized system of $\vec{u}^{\infty} = \vec{\mathcal{F}}q$ at the current state. We shall call this way the *Full data Tikhonov Newton*. However, this is computationally expensive because the more available measurement there are, the bigger system we have to construct.

In this section, we propose the reconstruction scheme based on the Kalman filter (see the original paper [50]). The Kalman filter is the linear estimation for the unknown state by the update of the state and its norm using the sequential measurements observed over time. The contributions of this paper are followings.

(A) We propose the reconstruction algorithm, which is equivalent to the Full Tikhonov data Newton (see (8.39)–(8.43)).

(B) We also propose the reconstruction algorithm based on the *Extended Kalman Filter* (see (8.53)–(8.57)).

The advantages of using Kalman Filter over the Newton approach is that we can estimate the unknown function q every time to observe the far field pattern u_l^{∞} with one incident direction θ_l without waiting all available measurements $\{u_l^{\infty}\}_{l=1}^{N}$. Furthermore, we do not need to construct the big system using all measurements, which reduce the computational cost. (A) is derived from the Kalman filter which has been discussed in the first part of our works (see Section 4 in [27]), and we call the reconstruction scheme of (A) the *Kalman filter Newton*. (B) is the different approach from (A). The Extended Kalman filter (see e.g., [31, 32, 48]) is the nonlinear version of the Kalman filter. For every time to observe one measurement, the state is updated by applying the linear Kalman filter to linearized problem at the current state. The figure 14 provides an illustration for the differences of (A) and (B) in the way to use measurements.

This section is organized as follows. In Section 8.2, we recall the Fréchet derivative of the far field mapping $\mathcal{F}$ and its properties. In Section 8.3, we consider the linearization of nonlinear inverse problem, and study the error of the linearized solution. In Section 8.4, we propose two reconstruction algorithms of the Full data Tikhonov Newton and the Kalman filter Newton, and show that they are equivalent. In Section 8.5, we propose the reconstruction algorithm of the iterative Extended Kalman filter. Finally in Section 8.6, we give numerical examples to demonstrate our algorithms.

8.2 Fréchet derivative of the far field mapping

The approach for solving the nonlinear equation (8.7) often requires the linearization by the Fréchet derivative. In this section, we briefly recall the Fréchet derivative of the far field mapping and its properties. The following argument is a brief summary of Section 11.3 of [18].

We denote the far field mappings associated with the incident angle $\theta \in \mathbb{S}^1$ by

$$\mathcal{F}_\theta q(\hat{x}) := \mathcal{F}q(\hat{x}, \theta) = \frac{k^2}{4\pi} \int_Q \mathrm{e}^{-ik\hat{x}\cdot y} u(y, \theta) q(y) dy, \ \hat{x} \in \mathbb{S}^1, \tag{8.8}$$

where the total field $u = u_q(\cdot, \theta)$ is given by the solving the integral equation of (8.4). First, we review the following lemma described in Theorem 11.6 of [18].

Lemma 8.1. *The nonlinear operator $\mathcal{F}_\theta$ is Fréche differentiable, and its derivative $\mathcal{F}'_\theta[q]$ at q is given by*

$$\mathcal{F}'_\theta[q]m = v^\infty, \tag{8.9}$$

where v^∞ is the far field pattern of the radiating solution

$$\Delta v + k^2(1+q)v = -k^2 m u_q(\cdot, \theta) \text{ in } \mathbb{R}^2. \tag{8.10}$$

We observe the integral form of the linear operator $\mathcal{F}'_\theta[q]$. The far field pattern $v^\infty = v^\infty(\cdot, \theta)$ is of the form

$$v^\infty(\hat{x}, \theta) = \frac{k^2}{4\pi} \int_Q \mathrm{e}^{-ik\hat{x}\cdot y} \left[m(y) u_q(y, \theta) + q(y) v(y, \theta)\right] dy. \tag{8.11}$$

Here, we denote the fundamental solution for $-\Delta - k^2(1+q)$ by $\Phi_q(x,y)$, which is of the form

$$\Phi_q(x,y) = \Phi(x,y) + w(x,y),\ x \neq y, \tag{8.12}$$

where $w = w(\cdot, y)$ is the unique solution of the following integral equation

$$w(x,y) = k^2 \int_Q \Phi(x,z) q(z) \left(w(z,y) + \Phi(z,y) \right) dz,\ x \in \mathbb{R}^2. \tag{8.13}$$

By using the fundamental solution Φ_q, the radiating solution $v = v(\cdot, \theta)$ can be of the form

$$v(x,\theta) = k^2 \int_Q \Phi_q(x,y) m(y) u_q(y,\theta) dy,\ x \in \mathbb{R}^2. \tag{8.14}$$

By combining (8.11) and (8.14), and using the Fubini's theorem, we conclude that

$$\mathcal{F}'_\theta[q]m(\hat{x}) = \frac{k^2}{4\pi} \int_Q K_q(\hat{x},y) u(y,\theta) m(y) dy,\ \hat{x} \in \mathbb{S}^1, \tag{8.15}$$

where the function K_q is defined by

$$K_q(\hat{x},y) := \mathrm{e}^{-ik\hat{x}\cdot y} + k^2 \int_Q \mathrm{e}^{-ik\hat{x}\cdot z} q(z) u_q(y,\theta) \Phi_q(z,y) dz. \tag{8.16}$$

Finally in this section, we also review the following properites of the derivative $\mathcal{F}'[q]$ of the mapping $\mathcal{F} : L^2(Q) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ described in Theorem 11.7 of [18].

Lemma 8.2. *For piecewise continuous q, the operator $\mathcal{F}'[q] : L^2(Q) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ is injective.*

8.3 Linearized problems

In this section, we consider the linearization of the nonlinear inverse problem in the general functional analytic situation, and study the error of the linearized solution. Let X and Y be Hilbert spaces over complex variables $\mathbb{C}$ which correspond to the state space $L^2(Q)$ of the inhomogeneous medium function q, and the observation space $L^2(\mathbb{S}^1)$ of the far field pattern u^∞, respectively. Let $A : X \to Y$ be a nonlinear observation operator which corresponds to the far field mapping $\mathcal{F}$.

For give $f \in Y$, we seek the solution $\varphi \in X$ such that

$$A(\varphi) = f. \tag{8.17}$$

We assume that we have an initial guess $\varphi_0 \in X$, which is a starting point of the algorithm, and is usually determined by a priori information of the true solution φ^{true} of (8.17). We also assume that the nonlinear mapping A is Fréchet differentiable at φ_0, which implies that

$$A(\varphi) = A(\varphi_0) + A'[\varphi_0](\varphi - \varphi_0) + r(\varphi - \varphi_0), \tag{8.18}$$

where the linear bounded operator $A'[\varphi_0] : X \to Y$ is the Fréchet derivative of the nonlinear mapping A at φ_0, and $r : X \to Y$ is some mapping corresponding to the remainder term such that $r(h) = o(h)$ as $\|h\| \to 0$. In the case to seek the solution φ close to the initial guess φ_0, we can omit the remainder term r because its influence is small. Then, we have the following linearized problem of (8.17).

$$A'[\varphi_0](\varphi - \varphi_0) = f - A(\varphi_0). \tag{8.19}$$

Although the problem became linear, the equation (8.19) is ill-posed because the Fréchet derivative $A'[\varphi_0]$ of A is not generally invertible. Then, the regularization method must be applied. Here, we briefly recall the Tikhonov regularization method in the following (see e.g., Chapter 4 of [18] and Chapter 3 of [78]).

Lemma 8.3. *Let X and Y be Hilbert space and let $H : X \to Y$ be a compact linear operator from X to Y. Then, followings holds.*

(i) *For $\alpha > 0$, the operator $(\alpha I + H^*H)$ is bounded invertible, and*

$$x_\alpha := (\alpha I + H^*H)^{-1}H^*y, \tag{8.20}$$

is a unique regularized solution of the problem $Hx = y$ given $y \in Y$, that is, $x_\alpha \in X$ is the unique solution of the problem

$$\alpha x + H^*Hx = H^*y. \tag{8.21}$$

(ii) *The solution x_α defined by (8.20) is the unique solution of the minimization problem*

$$\alpha \|x_\alpha\|_X^2 + \|y - Hx_\alpha\|_Y^2 = \min_{x\in X} \left\{\alpha \|x\|_X^2 + \|y - Hx\|_Y^2\right\}. \tag{8.22}$$

(iii) *If $y \in \mathrm{R}(H)$, then there exists $C = C_y$ such that*

$$\|x_\alpha\| \leq C, \ \alpha > 0, \tag{8.23}$$

and if $y \notin \mathrm{R}(H)$, then $\|x_\alpha\|_X \to \infty$ as $\alpha \to 0$.

By applying the above Lemma as $H = A'[\varphi_0]$ and $y = f - A(\varphi_0)$, we have the regularized solution φ_α of (8.19)

$$\varphi_\alpha := \varphi_0 + \left(\alpha I + A'[\varphi_0]^* A'[\varphi_0]\right)^{-1} A'[\varphi_0]^* \left(f - A(\varphi_0)\right), \tag{8.24}$$

where $\alpha > 0$ is a regularization parameter, which is appropriately chosen. Furthermore, we have iterative algorithm for $n \in \mathbb{N}_0$

$$\varphi_{n+1} = \varphi_n + \left(\alpha_n I + A'[\varphi_n]^* A'[\varphi_n]\right)^{-1} A'[\varphi_n]^* \left(f - A(\varphi_n)\right). \tag{8.25}$$

This is known as the *regularized Newton method* (see e.g., [18, 78]). So far, many type of the Newton method have been studied, for example, the *regularized Gauss–Newton method* (see e.g.,[4]) and the *Quasi–Newton method* (see e.g., [79]), and for any other, we refer to [42, 51, 84, 96]. We remark that the regularization parameter $\alpha_n > 0$ in (8.25) is chosen dependently on each iteration step $n \in \mathbb{N}$. For example in [4], the regularization parameter α_n is chosen by

$$\alpha_{n+1} \leq \alpha_n \leq \eta \alpha_{n+1}, \quad \text{and} \quad \lim_{n\to\infty} \alpha_n = 0, \tag{8.26}$$

for some constant $\eta > 1$.

Next, we observe the error of the solution φ_α defined by (8.24). Let $f \in Y$ be of the form $f = H(\varphi^{true})$. By substituting φ^{true} for φ in (8.18), we have

$$f - A(\varphi_0) = A'[\varphi_0](\varphi^{true} - \varphi_0) + r(\varphi^{true} - \varphi_0), \tag{8.27}$$

which implies that the error is estimated by

$$\begin{aligned} \left\|\varphi_\alpha - \varphi^{true}\right\| &= \left\|\varphi_0 - \varphi^{true} + R_\alpha \left(f - A(\varphi_0)\right)\right\| \\ &\leq \left\|\left(I - R_\alpha A'[\varphi_0]\right)\left(\varphi_0 - \varphi^{true}\right)\right\| + \|R_\alpha\| \left\|r(\varphi^{true} - \varphi_0)\right\|, \end{aligned} \tag{8.28}$$

where the operator R_α is denoted by $R_\alpha := \left(\alpha I + A'[\varphi_0]^* A'[\varphi_0]\right)^{-1} A'[\varphi_0]^*$. Here, we assume that $A'[\varphi_0]$ is injective, then R_α describes the *regularization scheme*, which satisfies

$$R_\alpha A'[\varphi_0]\varphi \to \varphi, \ \alpha \to 0, \tag{8.29}$$

for all $\varphi \in X$. The first term in (8.28) is the *regularization error*, which arises from the approximation of the inverse operator of $A'[\varphi_0]$ by the regularization scheme R_α. Since R_α is the regularization scheme, the first term converges to zero as $\alpha \to 0$. The second term is the *nonlinearity error*, which arises from the approximation by the linearization. Since we have $\|R_\alpha\| \leq 1/2\sqrt{\alpha}$, the second term diverges as $\alpha \to 0$. Therefore, the regularization error and the nonlinearity error are in the trade-off relationship, and the regularization parameter α has to be chosen such that the total error, which is sum of two errors, is small.

8.4 Full data Newton iteration

The natural approach for solving the equation (8.7) is to put all available measurements $\{u_l^\infty\}_{l=1}^N$ and all far field mappings $\{\mathcal{F}_l\}_{l=1}^N$ where the index l is associated with the incident direction $\theta_l \in \mathbb{S}^1$ into one long vector $\vec{u}^\infty$ and $\vec{\mathcal{F}}$, respectively, and to employ the regularized Newton method (8.25) discussed in the Section 8.3. In order to study the above general situation, let $f_1, ..., f_N \in Y$ be measurements, let $A_1, ..., A_N$ be nonlinear observation operators, and let us consider the problem to determine $\varphi \in X$ such that

$$\vec{A}(\varphi) = \vec{f}, \tag{8.30}$$

where $\vec{f} := \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$, and $\vec{A}(\varphi) := \begin{pmatrix} A_1(\varphi) \\ \vdots \\ A_N(\varphi) \end{pmatrix}$. By applying the regularized Newton method (8.25) to the above system (8.30), we have iterative solution

$$\varphi_{n+1}^{FTN} := \varphi_n^{FTN} + \left(\alpha_n I + \vec{A}'[\varphi_n^{FTN}]^* \vec{A}'[\varphi_n^{FTN}]\right)^{-1} \vec{A}'[\varphi_n^{FTN}]^* \left(\vec{f} - \vec{A}(\varphi_n^{FTN})\right), \tag{8.31}$$

where $\varphi_0^{FTN} := \varphi_0$, and $\vec{A}'[\varphi]$ is denoted by $\vec{A}'[\varphi] = \begin{pmatrix} A_1'[\varphi] \\ \vdots \\ A_N'[\varphi] \end{pmatrix}$, and the regularization parameters $\alpha_n > 0$ is chosen dependently on each iteration step n, like (8.26). We call this the *Full data Tikhonov Newton*. Here, $\vec{A}'[\varphi_0]^*$ is a adjoint operator of $\vec{A}'[\varphi_0]$ with respect to the usual scalar product $\langle \cdot, \cdot \rangle_X$ and the weighted scalar product $\langle \cdot, \cdot \rangle_{Y^N, R^{-1}} := \langle \cdot, R^{-1} \cdot \rangle_{Y^N}$ where $R : Y \to Y$ is the positive definite symmetric invertible operator, which is interpreted as the error covariance matrices of the observation distribution from a statistical viewpoint in the case when Y is the Euclidean space (see e.g., Chapter 5 of [21]). By the same calculation in (3.6) of [27], we have

$$\vec{A}'[\varphi]^* = \left(A_1'[\varphi]^H R^{-1}, ..., A_N'[\varphi]^H R^{-1}\right), \tag{8.32}$$

where $A_n'[\varphi]^H$ is a adjoint operator of $A_n'[\varphi]$ with respect to usual scalar products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$. Then, (8.31) can be of the form

$$\varphi_{n+1}^{FTN} = \varphi_n^{FTN} + \left(\alpha_n I + \sum_{l=1}^{N} A_l'[\varphi_n^{FTN}]^H R^{-1} A_l'[\varphi_n^{FTN}]\right)^{-1} \times \left(\sum_{l=1}^{N} A_l'[\varphi_n^{FTN}]^H R^{-1} \left(f_l - A_l(\varphi_0)\right)\right). \tag{8.33}$$

However, the algorithm (8.31) of the Full data Tikhonov Newton is computationally expensive since the more available measurement there are, the bigger system we have to construct. So, let us consider the alternative approach based on the Kalman filter. The Kalman filter is the linear estimation for the unknown state by the update of the state and its norm using the sequential measurements observed over time. For details of the following derivation, we refer to the first part of our works [27].

We consider the following problem for $l = 1, ..., N$

$$A_l'[\varphi_0]\varphi = f_l - A_l(\varphi_0) + A_l'[\varphi_0]\varphi_0, \tag{8.34}$$

which arises from the linearization of the problem $A_l(\varphi) = f_l$ at the initial guess φ_0. The above problem (8.34) can be applied to the Kalman filter algorithm (see (4.21)–(4.23) in [27]), then we obtain the following algorithm for $l = 1, ..., N$.

$$\varphi_{0,l} := \varphi_{0,l-1} + K_{0,l}\left(f_l - A_l(\varphi_0) + A_l'[\varphi_0]\varphi_0 - A_l'[\varphi_0]\varphi_{0,l-1}\right), \tag{8.35}$$

$$K_{0,l} := B_{0,l-1}A_l'[\varphi_0]^H\left(R + A_l'[\varphi_0]B_{0,l-1}A_l'[\varphi_0]^H\right)^{-1}, \tag{8.36}$$

$$B_{0,l} := \left(I - K_{0,l}A_l'[\varphi_0]^H\right)B_{0,l-1}, \tag{8.37}$$

where $\varphi_{0,0} := \varphi_0$, and $B_{0,0} := \frac{1}{\alpha_0}I$, and some $\alpha_0 > 0$. We denote the final state in the algorithm (8.35) by $\varphi_{1,0} := \varphi_{0,N}$, which is the initial guess of the next iteration step. Next, we consider the following problem

$$A_l'[\varphi_{1,0}]\varphi = f_l - A_l(\varphi_{1,0}) + A_l'[\varphi_{1,0}]\varphi_{1,0}, \tag{8.38}$$

which arises from the linearization of the problem $A_l(\varphi) = f_l$ at $\varphi_{1,0}$. The above problem (8.38) can be applied to the Kalman filter algorithm as well, and we obtain the similar algorithm to (8.35)–(8.37). We can repeat these procedure, then we obtain the following algorithm for $l = 1, ..., N$.

$$\varphi_{n,l}^{KFN} := \varphi_{n,l-1}^{KFN} + K_{n,l}\left(f_l - A_l(\varphi_{n,0}^{KFN}) + A_l'[\varphi_{n,0}^{KFN}]\varphi_{n,0}^{KFN} - A_l'[\varphi_{n,0}^{KFN}]\varphi_{n,l-1}^{KFN}\right), \tag{8.39}$$

$$K_{n,l} := B_{n,l-1}A_l'[\varphi_{n,0}^{KFN}]^H \left(R + A_l'[\varphi_{n,0}^{KFN}]B_{n,l-1}A_l'[\varphi_{n,0}^{KFN}]^H\right)^{-1}, \quad (8.40)$$

$$B_{n,l} := \left(I - K_{n,l}A_l'[\varphi_{n,0}^{KFN}]^H\right) B_{n,l-1}. \quad (8.41)$$

When the iteration time n is raised by one, the final state is renamed as

$$\varphi_{n,0}^{KFN} := \varphi_{n-1,N}^{KFN}, \quad (8.42)$$

and the weight is initialized as

$$B_{n,0} := \frac{1}{\alpha_n}I. \quad (8.43)$$

We call this the *Kalman Filter Newton.* We remark that it has two indexes n and l, where n is associated with the iteration step, and l the Kalman filter step, respectively.

Finally in this section, we show the following equivalent theorem, which is the nonlinear iteration version of Theorem 4.3 in [27].

Theorem 8.4. *For measurements $f_1, ..., f_N$, nonlinear mappings $A_1, ..., A_N$, and the initial guess $\varphi_0 \in X$, the final state of the Kalman filter Newton given by (8.39)–(8.43) is equivalent to the Full data Tikhonov Newton given by (8.33), that is, we have*

$$\varphi_{n,N}^{KFN} = \varphi_{n+1}^{FTN}, \quad (8.44)$$

for all $n \in \mathbb{N}_0$.

Proof. We will prove (8.44) by the induction. By applying Theorem 4.3 of [27] to the linearized problem $A_l'[\varphi_0]\varphi = f_l - A_l(\varphi_0) + A_l'[\varphi_0]\varphi_0$ for $l = 1, ..., N$ with the initial guess φ_0 and the regularization parameter $\alpha_0 > 0$, we have $\varphi_{0,N}^{KFN} = \varphi_1^{FTN}$, which is the case of $n = 0$.

Let us assume that (8.44) in the case of $n-1$ holds, that is, we have $\varphi_{n-1,N}^{KFN}(= \varphi_{n,0}^{KFN}) = \varphi_n^{FTN} =: \varphi_n$. Again, we apply Theorem 4.3 of [27] to the linearized problem $A_l'[\varphi_n]\varphi = f_l - A_l(\varphi_n) + A_l'[\varphi_n]\varphi_n$ for $l = 1, ..., N$ with the initial guess $\varphi_n = \varphi_{n,0}^{KFN} = \varphi_n^{FTN}$ and the regularization parameter $\alpha_n > 0$, then we have $\varphi_{n,N}^{KFN} = \varphi_{n+1}^{FTN}$. Theorem 8.4 has been shown. $\square$

8.5 Iterative Extended Kalman filter

The usual Kalman filter is the linear optimal estimation for solving the linear system. However in realistic applications, most systems are nonlinear, so many studies of the nonlinear estimation have been done. The *Extended*

Kalman filter, which is one of the nonlinear version of the Kalman filter, is to apply the linear Kalman filter to the linearized equation at the current state for every time to observe one measurement. In this section, we introduce the iterative Extended Kalman filter. For further readings of the Extended Kalman filter, we refer to [31, 32, 48], and there also exists other types of the nonlinear Kalman filter such as the Unscented Kalman Filter ([49]) which based on the Monte Carlo sampling without employing the linearization approximation.

First, let us start with the linearized problem of $A_1(\varphi) = f_1$ at the initial guess φ_0.

$$A_1'[\varphi_0]\varphi = f_1 - A(\varphi_0) + A_1'[\varphi_0]\varphi_0. \tag{8.45}$$

By the same argument in Section 4 of [27] replacing A_1 and f_1 by $A_1'[\varphi_0]$ and $f_1 - H(\varphi_0) + H_1'[\varphi_0]\varphi_0$, respectively, we have the following solution of (8.45).

$$\varphi_1 := \varphi_0 + K_1\left(f_1 - A_1(\varphi_0)\right), \tag{8.46}$$

$$K_1 := B_0 A_1'[\varphi_0]^H \left(R + A_1'[\varphi_0] B_0 A_1'[\varphi_0]^H\right)^{-1}, \tag{8.47}$$

$$B_1 := \left(I - K_1 A_1'[\varphi_0]^H\right) B_0, \tag{8.48}$$

where $B_0 := \frac{1}{\alpha_0} I$ and $\alpha_0 > 0$ is regularization parameter. Next, we consider linearized problem of $A_2(\varphi) = f_2$ at φ_1 defined by (8.46).

$$A_2'[\varphi_1]\varphi = f_2 - A_2(\varphi_1) + A_2'[\varphi_1]\varphi_1, \tag{8.49}$$

Then, by the same argument in Section 4 of [27], we have the solution of (8.49). We can repeat them, then we have the following algorithm.

$$\varphi_l := \varphi_{l-1} + K_l\left(f_l - A_l(\varphi_{l-1})\right), \tag{8.50}$$

$$K_l := B_{l-1} A_l'[\varphi_{l-1}]^H \left(R + A_l'[\varphi_{l-1}] B_{n-1} A_l'[\varphi_{l-1}]^H\right)^{-1}, \tag{8.51}$$

$$B_l := \left(I - K_l A_l'[\varphi_{l-1}]^H\right) B_{l-1}, \tag{8.52}$$

for $l = 1, ..., N$. In order to obtain the iterative algorithm, we discuss (8.45)–(8.52) again as the initial guess is φ_N, and we repeat them. Finally, we obtain the following iterative algorithm for $l = 1, ..., N$.

$$\varphi_{n,l}^{EKF} := \varphi_{n,l-1}^{EKF} + K_{n,l}\left(f_l - A_l(\varphi_{n,l-1}^{EKF})\right), \tag{8.53}$$

$$K_{n,l} := B_{n,l-1} A_l'[\varphi_{n,l-1}^{EKF}]^H \left(R + A_l'[\varphi_{n,l-1}^{EKF}] B_{n,l-1} A_l'[\varphi_{n,l-1}^{EKF}]^H\right)^{-1}, \tag{8.54}$$

$$B_{n,l} := \left(I - K_{n,l} A_l'[\varphi_{n,l-1}^{EKF}]^H\right) B_{n,l-1}. \tag{8.55}$$

When the iteration time n is raised by one, the final state is renamed as

$$\varphi_{n,0}^{EKF} := \varphi_{n-1,N}^{EKF}, \tag{8.56}$$

and the weight is initialized as

$$B_{n,0} := \frac{1}{\alpha_n} I, \tag{8.57}$$

where the regularization parameters $\alpha_n > 0$ is chosen dependently on each iteration step n, like (8.26). We call this the *iteratively Extended Kalman Filter*. We remark that it has two indexes n and l, where n is associated with the iteration step, and l the Kalman filter step, respectively. The figure 14 provide an illustration for the difference of Kalman filter Newton (KFN, left) and iterative Extended Kalman filter (EKF, right). When the state moves horizontally, measurements are used, and when it moves vertically, linearization are done.

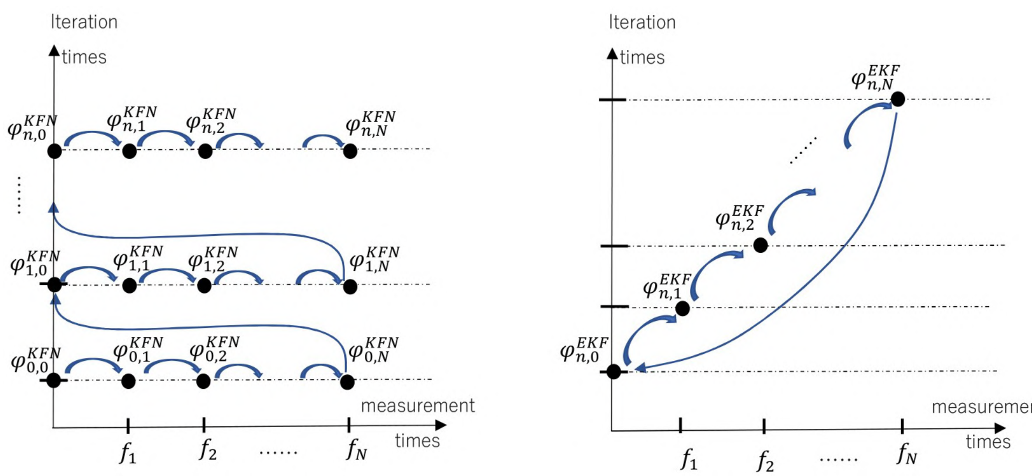


Figure 14: difference of KFN (left) and EKF (right)

8.6 Numerical examples

In this section, we consider numerical studies of the algorithm which have been discussed in above sections. We recall that our inverse scattering problem is to solve the nonlinear integral equation (8.7) with respect to q when the measurements $u_l^\infty := u^\infty(\cdot, \theta_l)$ for $l = 1, ..., N$ are given.

$$\mathcal{F}_l q = u^\infty(\cdot, \theta_l), \tag{8.58}$$

where the operator $\mathcal{F}_l : L^2(Q) \to L^\infty(\mathbb{S}^1)$ is defined by

$$\mathcal{F}_l q(\hat{x}) = \mathcal{F}q(\hat{x}, \theta_l) := \frac{k^2}{4\pi} \int_Q \mathrm{e}^{-ik\hat{x}\cdot y} u(y, \theta_l) q(y) dy, \tag{8.59}$$

where $u = u_q$ is the total field given by solving Lippmann–Schwinger integral equation (8.4). Here, the incident direction is given by $\theta_l := (\cos(2\pi l/N), \sin(2\pi l/N))$ for each $l = 1, ..., N$. The following discretizations are employed.

$$u^\infty(\cdot, \theta) \approx \left(u^\infty(\hat{x}_j, \theta)\right)_{j=1,\ldots,J} \in \mathbb{R}^J, \tag{8.60}$$

where $\hat{x}_j := (\cos(2\pi j/J), \sin(2\pi j/J))$, and $J \in \mathbb{N}$, and

$$q \approx \left(q(z_{i,m})\right)_{-M \le i,m \le M-1} \in \mathbb{R}^{(2M)^2}, \tag{8.61}$$

where $z_{i,m} := \left(\frac{(2i+1)R}{2M}, \frac{(2m+1)R}{2M}\right)$, and $M \in \mathbb{N}$ is a number of division of $[0, R]$, and $[-R, R]^2$ is a square with some $R > 0$, in which support Q of the function q is included. The Fréchet derivative $\mathcal{F}_l'[q]$ of $\mathcal{F}_l$ at q is discretized by

$$\mathcal{F}_l'[q] \approx \frac{k^2}{4\pi} \left(K_q(\hat{x}_j, z_{i,m}) u(z_{i,m}, \theta_l)\right)_{-M \le i,m \le M-1,\ j=1,\ldots,J} \in \mathbb{R}^{J \times (2M)^2}, \tag{8.62}$$

where the function K_q is defined by (8.16).

In this numerical study, we always fix the discretized parameter as $J = 20$, $M = 6$, $R = 3$, $N = 15$, and consider true functions as the characteristic function

$$q_j^{true}(x) := \begin{cases} 1 & \text{for } x \in B_j \\ 0 & \text{for } x \notin B_j \end{cases}, \tag{8.63}$$

where the support B_j of the true function is considered as the following two types.

$$B_1 := \left\{(x_1, x_2) : x_1^2 + x_2^2 < 1.5\right\}, \tag{8.64}$$

$$B_2 := \left\{ (x_1, x_2) : \begin{array}{c} (x_1 + 1.5)^2 + (x_2 + 1.5)^2 < (1.0)^2 \ or \\ 1 < x_1 < 2,\ -2 < x_2 < 2 \ or \\ -2 < x_1 < 2,\ -2.0 < x_2 < -1.0 \end{array} \right\}, \tag{8.65}$$

In Figure 15, the blue closed curve is the boundary ∂B_j of the support B_j of the true function q_j^{true}, and the green brightness indicates values of the true function on each cell divided into $(2M)^2 = 144$ in the sampling domain $[-3, 3]^2$. Here, we always employ the initial guess q_0 as

$$q_0 \equiv 0, \tag{8.66}$$

and employ the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of regularization parameters as

$$\alpha_n := \eta^{-n+1}\alpha_0, \tag{8.67}$$

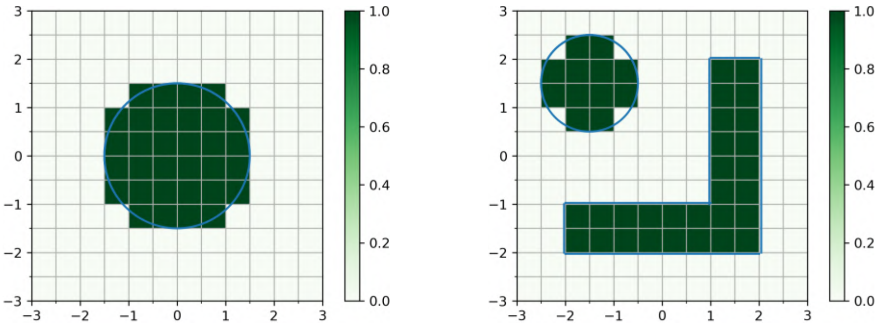
which satisfies the condition (8.26). A positive constant α_0 is the starting parameter and $\eta > 1$ is the decreasing factor. Here, we choose it as $\eta = 2$.

Figure 16 shows the graph of the error of the solution (8.18) linearized at $q_0 \equiv 0$ when the wavenumber $k = 3$ is fixed, and the regularization paramters $\alpha > 0$ are changed, for two different true functions q_1^{true}, q_2^{true}. The blue curve corresponds to the regularization error, the yellow one corresponds to the nonlinearity error, and the green one correponds to total error, which is the sum of two errors. We can observe that it would be good to choose the regularization parameter α around one hundred seventy such that the total error decreased significantly in both cases. From this point of view, we choose the starting parameter α_0 as $\alpha_0 = 175$.

Figures 17, 18 show the reconstruction by the Kalman filter Newton (KFN), and the iterative Extended Kalman filter (EKF) discussed in (8.39)–(8.43), and (8.53)–(8.57), respectively. The first and second column corresponds to visualization of the state in 4th and 15th iteration step, respectively, for different two shapes B_1 and B_2, and for different two wavenumbers $k = 1$ and $k = 3$. The third column corresponds to the graph of the Mean Square Error (MSE) defined by

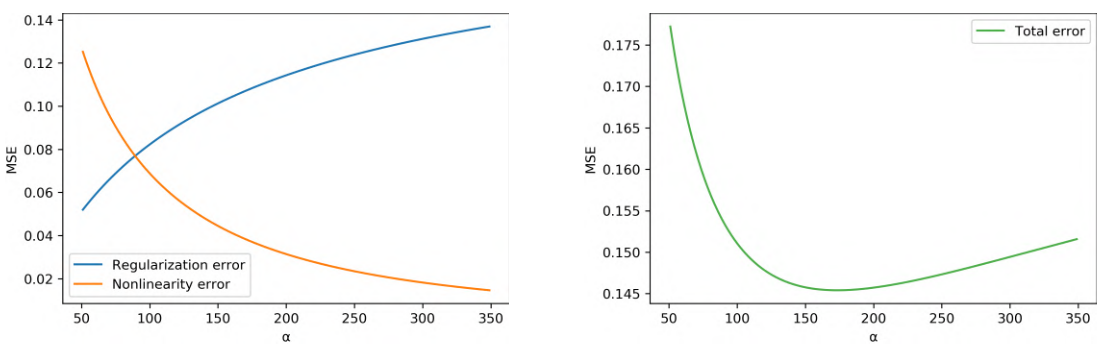
$$e_n := \left\| q^{true} - q_n \right\|^2, \tag{8.68}$$

where q_n is associated with the state of nth iteration step by some reconstruction method. The horizontal axis is with respect to number of iterations, and the vertical axis is the value of MSE. In both cases, the true functions are successfully reconstructed. The iterative Extended Kalman filter requires more calculations of the derivative than the Kalman filter Newton because we have to linearize the nonlinear problem for every time to observe one measurement, but instead, we can observe that in the third column of Figures 17 and 18, the convergence of the Extended Kalman filter to the true function is faster than that of the Kalman filter Newton.

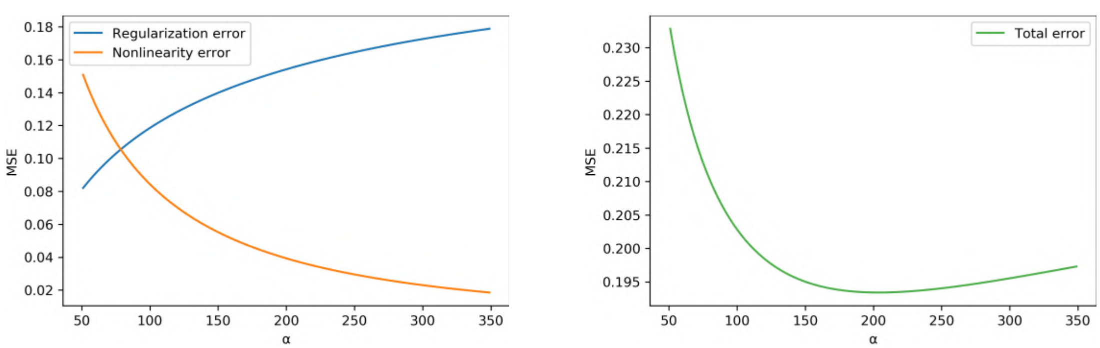


q_1^{true} q_2^{true}

Figure 15: true functions

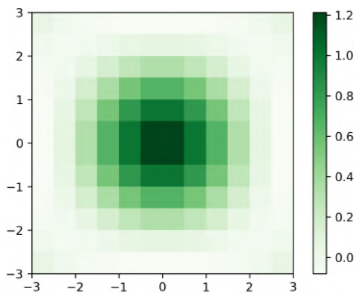


q_1^{true} q_1^{true}

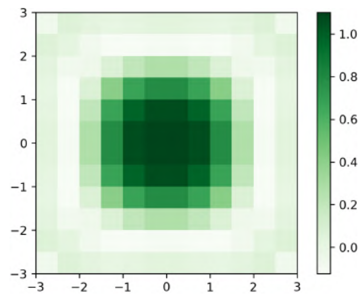


q_2^{true} q_2^{true}

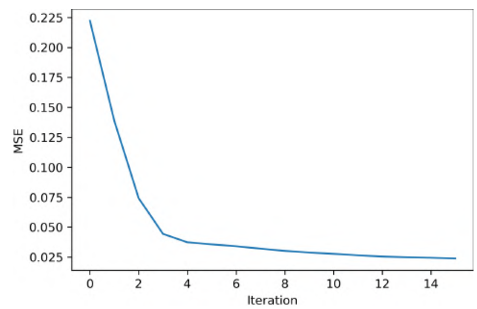
Figure 16: error graphs, $k = 3$, $q_0 = 0$



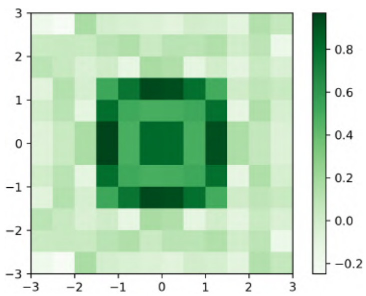
B_1, $k = 1$, $n = 4$



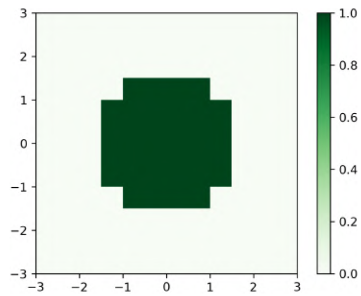
B_1, $k = 1$, $n = 15$



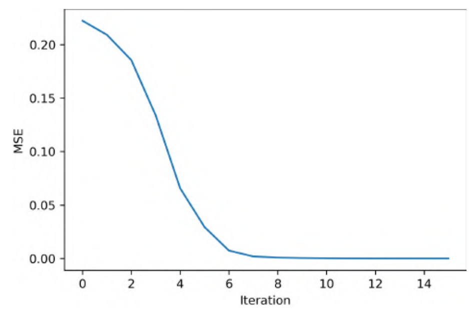
B_1, $k = 1$, error graph



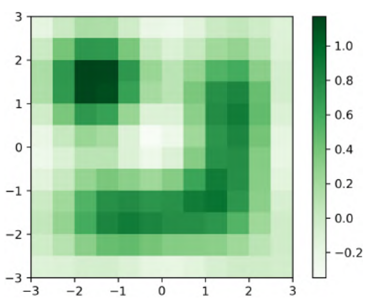
B_1, $k = 3$, $n = 4$



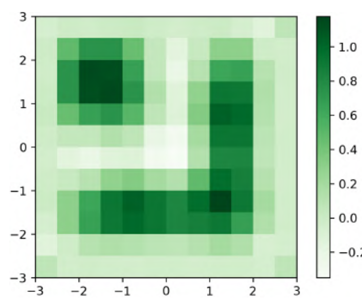
B_1, $k = 3$, $n = 15$



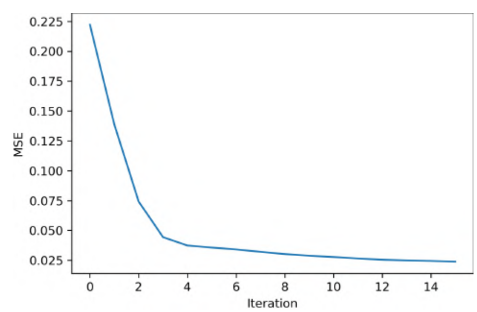
B_1, $k = 3$, error graph



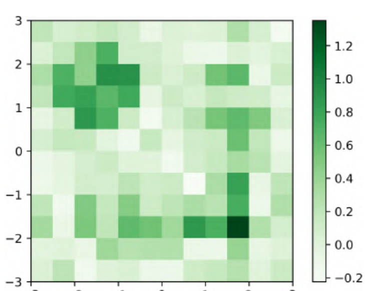
B_2, $k = 1$, $n = 4$



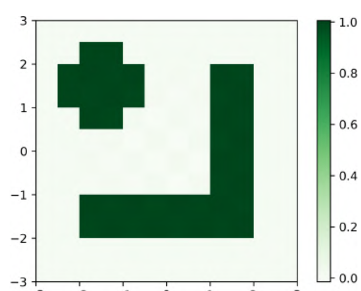
B_2, $k = 1$, $n = 15$



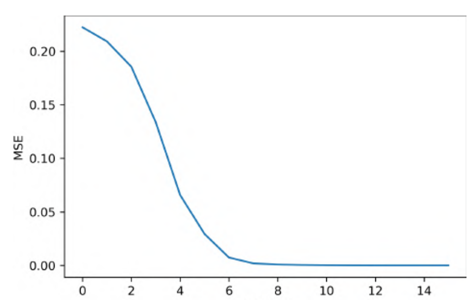
B_2, $k = 1$, error graph



B_2, $k = 3$, $n = 4$

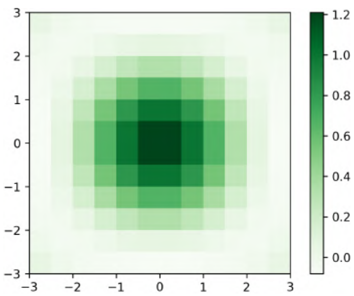


B_2, $k = 3$, $n = 15$

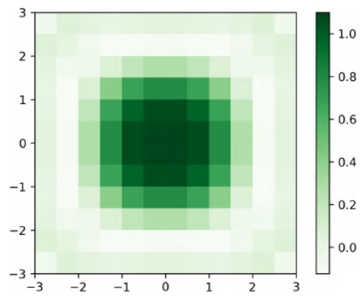


B_2, $k = 3$, error graph

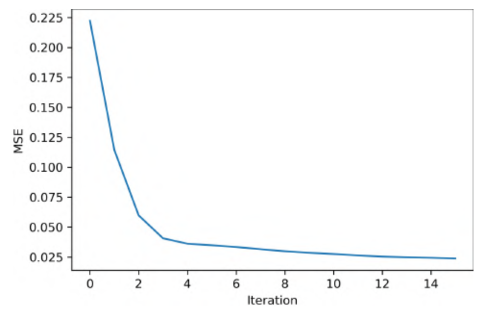
Figure 17: KFN



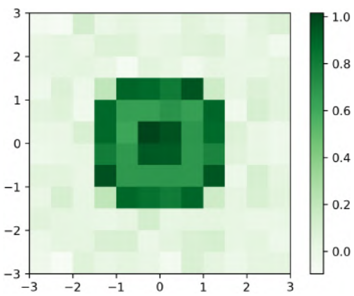
B_1, $k = 1$, $n = 4$



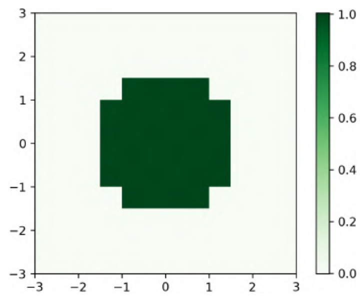
B_1, $k = 1$, $n = 15$



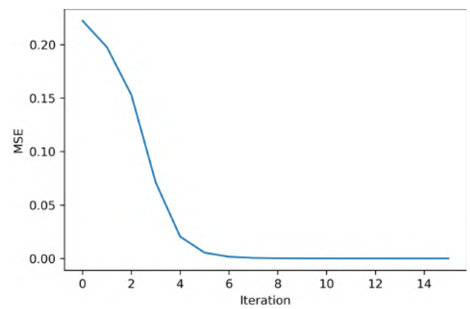
B_1, $k = 1$, error graph



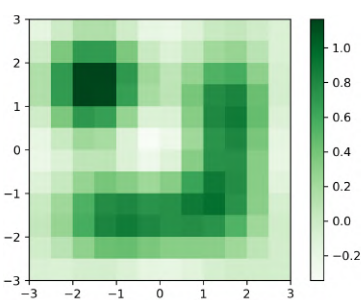
B_1, $k = 3$, $n = 4$



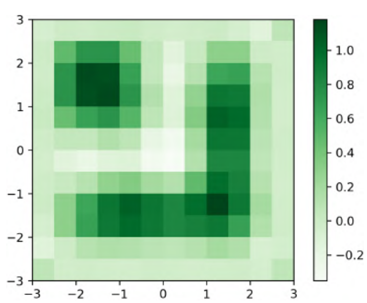
B_1, $k = 3$, $n = 15$



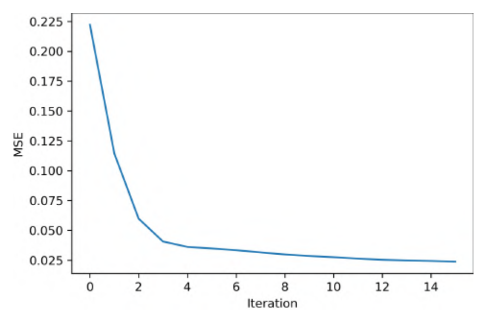
B_1, $k = 3$, error graph



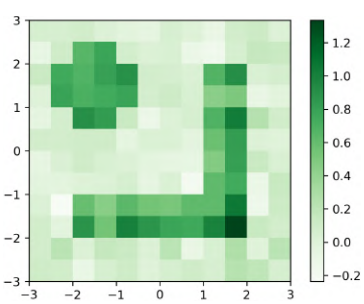
B_2, $k = 1$, $n = 4$



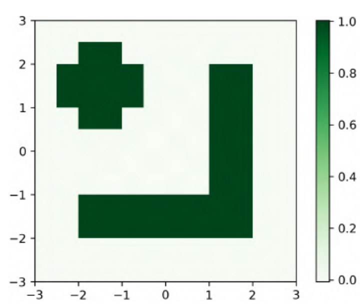
B_2, $k = 1$, $n = 15$



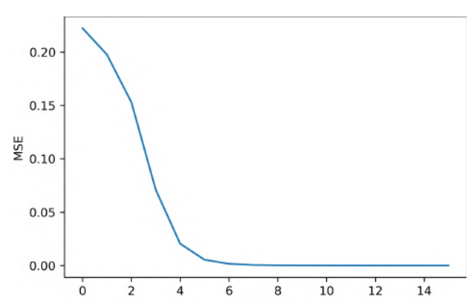
B_2, $k = 1$, error graph



B_2, $k = 3$, $n = 4$



B_2, $k = 3$, $n = 15$



B_2, $k = 3$, error graph

Figure 18: EKF

References

[1] T. Aktosun, V. G. Papanicolau, V. Zisis, *Inverse scattering on the line for a generalized nonlinear Schrödinger equation*, Inverse Probl., **20**, 1267–1280, (2004).

[2] A. Albicker, R. Griesmaier *Monotonicity in inverse obstacle scattering on unbounded domains*, Inverse Probl., **36**, 085014, (2020).

[3] K. A. Anagnostopoulos, A. Charalambopoulos, A. Kleefeld, *The factorization method for the acoustic transmission problem*, Inverse Probl., **29**, 115015. (2013).

[4] A. B. Bakushinsky, *The problem of the convergence of the iteratively regularized Gauss– Newton method*, Comput. Maths. Maths. Phys., **32**, 1353–1359 (1992).

[5] A. B. Bakushinsky, M. Yu. Kokurin, *Iterative Methods for Approximate Solution of Inverse Problems*, Springer, Dordrecht, (2004).

[6] G. Bao, F. Triki. *Error estimates for the recursive linearization of inverse medium problems*, J. Comput. Math., **28**, 725–744, (2010).

[7] O. Bondarenko, A. Kirsch, X. Liu, *The factorization method for inverse acoustic scattering in a layered medium*, Inverse Probl., **29**, 045010, (2013).

[8] A. Bonnet-Ben Dhia, G. Dakhia, C. Hazard, L. Chorfi, *Diffraction by a defect in an open waveguide: a mathematical analysis based on a modal radiation condition*, SIAM J. Appl. Math., **11**, 677–693, (2009).

[9] L. Bourgeois, S. Fliss, *On the identification of defects in a periodic waveguide from far field data*, Inverse Probl., **30**, 095004, (2014).

[10] A. Bukhgeim, *Recovering a potential from Cauchy data in the two-dimensional case*, J. Inverse Ill-Posed Probl., **16**, 19–33, (2008).

[11] F. Cakoni, D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer, Berlin, (2006).

[12] S. N. Chandler-Wilde, *The impedance boundary value problem for the Helmholtz equation in a half-plane*, Math. Meth. Appl. Sci., **20**, 813–840, (1997).

[13] S. N. Chandler-Wilde, R. R. Christopher, *Scattering by rough surfaces: the Dirichlet problem for the Helmholtz equation in a non-locally perturbed half-plane*, Math. Meth. Appl. Sci., **19**, 959–976, (1996).

[14] S. N. Chandler-Wilde, B. Zhang, *A uniqueness result for scattering by infinite rough surfaces*, SIAM J. Appl. Math., **58**, 1774–1790, (1998).

[15] S. N. Chandler-Wilde, B. Zhang, *Electromagnetic scattering by an inhomogeneous conducting or dielectric layer on a perfectly conducting plate*, Proc. R. Soc. Lond., A **454**, 519–542, (1998).

[16] X. Chen, *Computational Methods for Electromagnetic Inverse Scattering*, Wiley-IEEE, Singapore, (2018).

[17] D. Colton, A. Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inverse Probl., **12**, 383–393, (1996).

[18] D. Colton, R. Kress, *Inverse acoustic and electromagnetic scattering theory*, Fourth edition, Springer, New York, (2019).

[19] G. Eskin, *Lectures on linear partial differential equations* , American Mathematical Society, **123**, (2011).

[20] A. Feizmohammadi, L. Oksanen, *An inverse problem for a semi-linear elliptic equation in Riemannian geometries*, J. Differ. Equ., **269**, 4683–4719, (2020).

[21] M. Freitag, R. Potthast, *Synergy of inverse problems and data assimilation techniques*, Large Scale Inverse Problems, (2013).

[22] T. Furuya, *A modification of the factorization method for scatterers with different physical properties*, Math. Meth. Appl. Sci., **42**, 4017–4030, (2019).

[23] T. Furuya, *Scattering by the local perturbation of an open periodic waveguide in the half plane*, J. Math. Anal. Appl., Elsevier, **489**, 124149, (2020).

[24] T. Furuya, T. Daimon, R. Saiin, *The monotonicity method for the inverse crack scattering problem*, to appear in Inverse Probl. Sci. Eng., (2020).

[25] T. Furuya, *The direct and inverse scattering problem for the semilinear Schrödinger equation*, Nonlinear Differ. Equ. Appl., **27**, Paper No. 24, (2020).

[26] T. Furuya, *The factorization and monotonicity method for the defect in an open periodic waveguide*, to appear in J. Inverse Ill-Posed Probl., (2020).

[27] T. Furuya, Potthast, *Inverse medium scattering problems with Kalman filter techniques I. Linear case*, preprint, (2020).

[28] T. Furuya, Potthast, *Inverse medium scattering problems with Kalman filter techniques II. Nonlinear case*, preprint, (2020).

[29] D. Ghosh Roy, L. Couchman, *Inverse Problems and Inverse Scattering of Plane Waves*, Academic Press, New York, (2002).

[30] G. Giorgi, M. Brignone, R. Aramini, M. Piana, *Application of the Inhomogeneous Lippmann– Schwinger Equation to Inverse Scattering Problems*, SIAM J. Appl. Math., **73**, 212–231, (2013).

[31] M. S. Grewal, A. P. Andrews, Kalman Filtering: Theory and Practice with MATLAB, Fourth edition, Wiley-IEEE Press, (2014).

[32] M. S. Grewal, A. P. Andrews, *Application of Kalman filtering in aerospace 1960 to the present*, IEEE Control System Magazine, 69–78, (2010).

[33] R. Griesmaier, B. Harrach, *Monotonicity in inverse medium scattering on unbounded domains*, SIAM J. Appl. Math., **78**, 2533–2557, (2018).

[34] C. W. Groetsch, *Inverse Problems in the Mathematical Sciences*, Springer, (1993).

[35] M. Hanke, *Taste of Inverse Problems: Basic Theory and Examples*, SIAM–Society for Industrial and Applied Mathematics, (2017).

[36] M. Harju, V. Serov, *Three-dimensional Direct and Inverse Scattering for the Schrödinger Equation with a General Nonlinearity*, Operator Theory: Advances and Applications, **236**, 257–273, (2014).

[37] B. Harrach, V. Pohjola, M. Salo, *Dimension bounds in monotonicity methods for the Helmholtz equation*, SIAM J. Appl. Math., **51**, 2995–3019, (2019).

[38] B. Harrach, V. Pohjola, M. Salo, *Monotonicity and local uniqueness for the Helmholtz equation*, Anal. PDE, **12**, 1741–1771, (2019).

[39] B. Harrach, M. Ullrich, *Local uniqueness for an inverse boundary value problem with partial data*, Proc. Amer. Math. Soc., **145**, 1087–1095, (2017).

[40] B. Harrach, M. Ullrich, *Monotonicity based shape reconstruction in electrical impedance tomography*, SIAM J. Math. Anal., **45**, 3382–3403, (2013).

[41] C. Hazard, *On the absence of trapped modes in locally perturbed open waveguides*, IMA J. Appl. Math., **80**, 1049–1062, (2015).

[42] T. Hohage, *On the numerical solution of a three-dimensional inverse medium scattering problem*, Inverse Probl., **17**, 1743–1763, (2001).

[43] N. Honda, G. Nakamura, R. Potthast, M. Sini, *The no-response approach and its relation to non-iterative methods in inverse scattering*, Annali di Matematica Pura ed Applicata, **187**, 7–37, (2008).

[44] M. Ikehata, *Reconstruction of the shape of an obstacle from the scattering amplitude at a fixed frequency*, Inverse Probl., **14**, 949–954, (1998).

[45] V. Isakov, *Inverse Problems for Partial Differential Equations*, 2nd ed, Springer, Berlinm, (2006).

[46] V. Isakov, A. I. Nachman, *Global uniqueness for a two-dimensional semilinear elliptic inverse problem*, Trans. Amer. Math. Soc., **347**, 3375–3390, (1995).

[47] E. Jalade, *Inverse problem for a nonlinear Helmholtz equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **21**, 517–531, (2004).

[48] A. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic Press, (1970).

[49] S. Julier, J. Uhlmann, *New extension of the Kalman filter to nonlinear systems*, Proceedings of SPIE, (1997).

[50] R. E. Kalman, *A New Approach to Linear Filtering and Prediction Problems*, J. Basic Eng., Trans., ASME, 35–45, (1960).

[51] B. Kaltenbacher, A. Neubauer, O. Scherzer, *Iterative Regularization Methods for Nonlinear Ill-Posed Problems*, Berlin, De Gruyter, (2008).

[52] A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, Second edition, Springer, Berlin, (2011).

[53] A. Kirsch, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Probl., **14**, 1489–1512, (1998).

[54] A. Kirsch, *Remarks on the Born Approximation and the Factorization Method*, Appl. Anal., **96**, 70–84, (2017).

[55] A. Kirsch, *Scattering by a periodic tube in $\mathbb{R}^3$: part i. The limiting absorption principle*, Inverse Probl., **35**, 104004, (2019).

[56] A. Kirsch, *Scattering by a periodic tube in $\mathbb{R}^3$: part ii. A radiation condition*, Inverse Probl., **35**, 104005, (2019).

[57] A. Kirsch, *The factorization method for a class of inverse elliptic problems*, Math. Nachr., **278**, 258–277, (2004).

[58] A. Kirsch, N. Grinberg, *The Factorization Method for Inverse Problems*, Oxford University Press, Oxford, (2008).

[59] A. Kirsch, N. Grinberg, *The factorization method for obstacles with a priori separated sound-soft and sound-hard parts*, Math. Comput. Simulation, **66**, 267–279, (2004).

[60] A. Kirsch, A. Kleefeld, *The factorization method for a conductive boundary condition*, J. Integral Equations Appl., **24**, 575–601, (2012).

[61] A. Kirsch, A. Lechleiter, *A radiation condition arising from the limiting absorption principle for a closed full- or half-waveguide problem*, Math. Meth. Appl. Sci., **41**, 3955–3975, (2018).

[62] A. Kirsch, A. Lechleiter, *The limiting absorption principle and a radiation condition for the scattering by a periodic layer*, SIAM J. Math. Anal., **50**, 2536–2565, (2018).

[63] A. Kirsch, X. Liu, *A modification of the factorization method for the classical acoustic inverse scattering problems*, Inverse Probl., **30**, 035013, (2014).

[64] A. Kirsch, X. Liu, *Direct and inverse acoustic scattering by a mixed-type scatterer*, Inverse Probl., **29**, 065005, (2013).

[65] A. Kirsch, X. Liu, *The factorization method for inverse acoustic scattering by a penetrable anisotropic obstacle*, Math. Meth. Appl. Sci., **37**, 1159–1170, (2014).

[66] A. Kirsch, S. Ritter, *A linear sampling method for inverse scattering from an open arc*, Inverse Probl., **16**, 89–105, (2000).

[67] R. Kress, *Inverse scattering from an open arc*, Math. Meth. Appl. Sci., **18**, 267–293, (1995).

[68] R. Kress, *Linear Integral Equations*, Third edition, Springer, Berlin, (2014).

[69] E. Lakshtanov, A. Lechleiter, *Difference factorizations and monotonicity in inverse medium scattering for contrasts with fixed sign on the boundary*, SIAM J. Math. Anal., **48**, 3688–3707, (2016).

[70] M. Lassas, T. Liimatainen, Y.-H. Lin, M. Salo, *Inverse problems for elliptic equations with power type nonlinearities*, to appear in J. Math. Pures Appl., (2020).

[71] M. Lassas, T. Liimatainen, Y.-H. Lin, M. Salo, *Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations*, Prepreint arXiv:1905.02764, (2019).

[72] A. Lechleiter, *The factorization method is independent of transmission eigenvalues*, Inverse Probl. Imaging, **3**, 123–138, (2009).

[73] A. Lechleiter, *The Floquet-Bloch transform and scattering from locally perturbed periodic surfaces*, J. Math. Anal. Appl., **446**, 605–627, (2017).

[74] X. Liu, *The factorization method for scatterers with different physical properties*, Discrete Contin. Dyn. Syst. Ser., S **8**, 563–577, (2015).

[75] X. Liu, B. Zhang, G. Hu, *Uniqueness in the inverse scattering problem in a piecewise homogeneous medium*, Inverse Probl., **26**, 015002, (2010).

[76] W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, (2000).

[77] A. I. Nachman, *Reconstructions From Boundary Measurements*, Ann. Math., **128**, 531–576, (1988).

[78] G. Nakamura, R. Potthast, *Inverse Modeling*, IOP Publishing, Bristolerl, (2015).

[79] J. Nocedal, S. J. Wright, *Numerical Optimization*, New York, Springer, (2006).

[80] A. Nosich, *Radiation conditions, limiting absorption principle, and general relations in open waveguide scattering*, J. Electromagn. Waves and Applications, **8**, 329–353, (1994).

[81] R. G. Novikov, *Multidimensional inverse spectral problems for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Func. Anal. and its Appl., **22**, 263–272, (1989).

[82] L. Päivärinta, M. Salo, G. Uhlmann, *Inverse scattering for the magnetic Schrödinger operator*, J. Funct. Analysis, **259**, 1771–1798, (2010).

[83] R. Pike, P. Sabatier, *Scattering. Scattering and Inverse Scattering in Pure and Applied Science*, Academic Press, New York, (2002).

[84] R. Potthast, *On the convergence of a new Newton–type method in inverse scattering*, Inverse Probl., **17**, 1419–1434, (2001).

[85] R. Potthast, *Point-sources and Multipoles in Inverse Scattering*, Chapman & Hall, London, (2001).

[86] R. Potthast, *Stability estimates and reconstructions in inverse acoustic scattering using singular sources*, J. Comput. Appl. Math., **114**, 247–274, (2000).

[87] A. G. Ramm, *Recovery of the potential from fixed-energy scattering data*, Inverse Probl., **4**, 877–886, (1988).

[88] V. Serov, *Inverse fixed energy scattering problem for the generalized nonlinear Schrödinger operator*, Inverse Probl., **28**, 025002, (2012).

[89] V. Serov, M. Harju, G. Fotopoulosc, *Direct and inverse scattering for nonlinear Schrödinger equation in 2D*, J. Math. Phys., **53**, 123522, (2012).

[90] V. Serov, M. Harju, *A uniqueness theorem and reconstruction of singularities for a two-dimensional nonlinear Schrödinger equation*, Nonlinearity, **21**, 1323–1337, (2008).

[91] J. Sylvester, G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. Math., **125** 153–169, (1987).

[92] G. Uhlmann, *Electrical impedance tomography and Calderon's problem*, Inverse Probl., **25**, 123011, (2009).

[93] M. Watanabe, *Time-dependent method for non-linear Schrödinger equations in inverse scattering problems*, J. Math. Anal. Appl., **459**, 932–944, (2018).

[94] R. Weder, *Lp–Lp Estimates for the Schrödinger Equation on the Line and Inverse Scattering for the Nonlinear Schrödinger Equation with a Potential*, J. Funct. Analysis, **170**, 37–68, (2000).

[95] R. Weder, *Inverse Scattering for the Nonlinear Schrödinger Equation II. Reconstruction of the Potential and the Nonlinearity in the Multidimensional Case*, Proc. Amer. Math. Soc., **129**, 3637–3645, (2001).

[96] C. Xiao, Y. Deng, *A new Newton-Landweber iteration for nonlinear inverse problems*, J. Appl. Math. Comput. Sci., **36**, 489–505, (2011).

[97] J. Yang, B. Zhang, H. Zhang, *The factorization method for reconstructing a penetrable obstacle with unknown buried objects*, SIAM J. Appl. Math., **73**, 617–635, (2013).