# F-matrices in cluster algebras and their applications (団代数におけるF行列とその応用)

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# Contents

# 1 Preface

*Cluster algebra* is a commutative algebra generated by variables called *cluster variables*, discovered by Fomin and Zelevinsky in the early 2000s. The original motivation for the consideration of cluster algebra seems to be an understanding of total positivity and the standard basis of quantum groups. Now, it is known that the structure of cluster variables and the transformations between them, the mutations, appear in various fields of mathematics. Furthermore, the development of cluster algebra theory and its applications is rapid and constant. For example, it is known to have applications in representation theory of algebras [4,5], higher Teichmüller theory [13–15], and number theory (Markov's Diophantine equation) [7,39].

Here, we introduce the special version of cluster algebras. Let $Q$ be a quiver with no loops and no 2-cycles, and $n$ the number of vertices of $Q$. We take a set of $n$ algebraically independent elements $\mathbf{x} = (x_1, \dots, x_n)$ in the rational function field $\mathcal{F}$ of $\mathbb{Q}$ in $n$ variables. We call the pair $(Q, \mathbf{x})$ a *seed*, and $\mathbf{x}$ a *cluster*, and each $x_i$ a *cluster variable*. For $k \in \{1, \dots, n\}$, we define the *mutation* $(Q', \mathbf{x}') = \mu_k(Q, \mathbf{x})$ of a seed $(Q, \mathbf{x})$ in direction $k$ as follows:

- $Q'$ is obtained from $Q$ in the following three steps:
  (1) for each subquiver $i \to k \to j$ in $Q$, we add a new arrow $i \to j$ to $Q$,
  (2) we reverse all arrows connected to $k$ in the quiver obtained by (1),
  (3) we remove all 2-cycles in the quiver obtained by (2).
- $\mathbf{x}' = (x'_1, \dots, x'_n)$ is obtained from $(Q, \mathbf{x})$ in the following way:

$$x'_j = \begin{cases} \dfrac{\prod_{i=1}^{n} x_i^{\max(0, q_{ik})} + \prod_{i=1}^{n} x_i^{\max(0, q_{ki})}}{x_k} & \text{if } j = k, \\ x_j & \text{otherwise.} \end{cases}$$

  where $q_{ik}$ is the number of arrows from $i$ to $k$ in $Q$.

For example, let $Q = 1 \longleftarrow 2 \longleftarrow 3$ and $\mathbf{x} = (x_1, x_2, x_3)$. Then

$$(Q', \mathbf{x}') := \mu_2(Q, \mathbf{x}) = \left( \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd}, \left( x_1, \frac{x_1 + x_3}{x_2}, x_3 \right) \right).$$

Let $\mathcal{X}(Q, \mathbf{x})$ be the set of all cluster variables obtained by mutating $(Q, \mathbf{x})$ in all directions repeatedly. A *cluster algebra* $\mathcal{A}(Q, \mathbf{x})$ associated with a seed $(Q, \mathbf{x})$ is the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}(Q, \mathbf{x})$. In Chapter 2, we define them more generally by using a skew-symmetrizable matrix $B$ instead of $Q$.

A turning point in the study of cluster algebras was the advent in 2006 of the *c-vectors* (*$C$-matrices*), the *g-vectors* (*$G$-matrices*), and the *$F$-polynomials*. The $g$-vectors contain part of the information of the cluster variables, the $c$-vectors contain part of the information of its coefficients, and the $F$-polynomials contain part of the information of both the variables and coefficients, respectively. Fomin and Zelevinsky defined them in [20] from a cluster variable and its coefficients, and showed that the original cluster variables and coefficients can be reconstructed from these vectors (matrices) and polynomials. This led to the recognition of the importance of $c$-, $g$-vectors, $C$-, $G$-matrices, and $F$-polynomials, and the study of these vectors, matrices and polynomials became very active. In particular, there are many applications of $g$-vectors in the representation theory of algebras because of the discovery of a correspondence between $g$-vectors with certain vectors determined by the minimal projective presentations of modules over algebras. For example, Schroll and Trefinger solved the first $\tau$-Brauer-Thrall conjecture by using $g$-vectors and $c$-vectors in [41]. The $F$-polynomials have also shown some development in recent years, for example, methods for constructing the Alexander and Jones polynomials, which are polynomial invariants of the knot, have been devised (see [32], for example).

In 2007, Fu and Keller defined the $f$-vectors[1] in [22] as the maximal exponent of each indeterminate in $F$-polynomials. More precisely, these vectors were given by [20, Conjecture 7.17] implicitly, and they were given the name "$f$-vectors" in [22] to solve this conjecture negatively. The study of $f$-vectors began to develop in 2018 when Fujiwara and the author defined the *$F$-matrix* with the $f$-vectors aligned horizontally in [23].

This thesis summarizes the author's contributions to these studies of $f$-vectors and $F$-matrices, and it is based on the two published papers [23, 27] and the three preprints [21, 25, 26]. In this chapter, we describe the content and the organization of it.

In Chapter 2, we describe the basic properties of cluster algebras. It is based on [15, 18, 20] mainly. In Section 2.1, we define the cluster algebra. We also introduce very important elements and their transformations, the seeds and the mutations. In Sections 2.2 and 2.3, we introduce important families of vectors and polynomials associated with cluster algebras – $d$-vectors, $c$-vectors, $g$-vectors, and $F$-polynomials. Furthermore, we define $D$-matrices, $C$-matrices, and $G$-matrices by using $d$-vectors, $c$-vectors and $g$-vectors, respectively. In Section 2.4, we introduce the cluster complexes for preparation of Chapter 5. They are simplicial complexes which represents the structure formed by seeds and their mutations. In section 2.5, we define cluster algebras from marked surfaces. This is a class of cluster algebras which have geometrical realizations.

In Chapter 3, we define another family of vectors, $f$-vectors. By using them, we also define $F$-matrices. Furthermore, we introduce their important property, the *self-duality*. This chapter is based on [23]. Originally, $f$-vectors and $F$-matrices were defined to represent $F$-polynomials, which tended to explode in number of terms due to mutations, in the form of vectors and matrices. Furthermore, by providing $F$-polynomial alternatives in the form of matrices and vectors, we can investigate $F$-polynomial's properties using transposition operations or matrix products. The self-duality is a property obtained by this approach. This property is described by the following form:

**Theorem 1.0.1** (Theorem 3.2.10). *For any exchange matrix $B$ and $t_0, t \in \mathbb{T}_n$, we have*

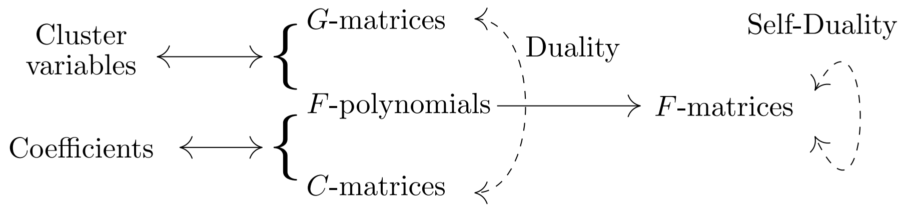
$$(F_t^{B;t_0})^\top = F_{t_0}^{B_t^\top;t}.$$

This equation means the transposition of an $F$-matrix is another $F$-matrix in another cluster algebra. This is an analogue of duality between $C$-matrices and $G$-matrices found by Nakanishi and Zelevinsky [36]:

$$(G_t^{B;t_0})^\top = C_{t_0}^{B_t^\top;t}.$$

The relation between the $C$, $G$ and $F$ matrices is shown in the figure below:

[1] They are different from the *face vectors* of simplicial complexes.




where $A \to B$ implies that $B$ is defined by using $A$, or $B$ is restored from $A$.

In Chapter 4, we deal with the conjecture about the uniqueness of $F$-polynomials associated to $f$-vectors or $F$-matrices. It is based on [25, 27]. We consider the following question;

**Question 1.0.2.** Do the $f$-vectors or the $F$-matrices restore the $F$-polynomials? If it is yes, then how to do it?

Since we define $f$-vectors or $F$-matrices with the intention of "vectorizing" or "matrix-sizing" of $F$-polynomials, this question is natural. In particular, the following Question is considered:

**Question 1.0.3.**

(1) Does a non-zero $f$-vector determine cluster variables?

(2) Does an $F$-matrices determine a cluster?

There is a counterexample to Question 1.0.3 (1), however, we have not found a counterexample to Question 1.0.3 (2) yet. We call this conjecture the *uniqueness conjecture for $F$-matrices*. In Chapter 4, we give some answers to Question 1.0.3 in the case of certain classes of cluster algebras:

**Theorem 1.0.4** (Corollary 4.1.6, Theorem 4.4.1)**.**

*(1) For a cluster algebra of finite type or rank 2, the answer to Question 1.0.3 (1) is yes.*

*(2) For a cluster algebra from a marked surface, of finite type, or of rank 2, the answer to Question 1.0.3 (2) is yes.*

Furthermore, for Question 1.0.2, we obtain the *restoration formula* of cluster algebras of rank 2. This is a formula giving an $F$-polynomial from an $f$-vector. It is introduced in Section 4.8.

In Chapter 4, we give proofs of these theorems. In particular, in the case of cluster algebras arising from marked surfaces, we solve the problem by using coincidence of $f$-vectors with intersection vectors, whose entries are the intersection numbers of two arcs. For cluster algebras of finite type or rank 2, we solve the problem by using coincidence of $f$-vectors with $d$-vectors of cluster variables.

In Chapters 5 and 6, we focus on applications of $f$-vectors or $F$-matrices.

In Chapter 5, we generalize the *compatibility degree* of cluster complexes by using $f$-vectors. It is based on [21]. The classical type of compatibility degree is introduced by Fomin and Zelevinsky in [19]. Originally, this is a function $(\cdot \parallel \cdot)_{\mathrm{cl}}$ on pairs in the set $\Phi_{\geq -1}$ of positive roots and negative simple roots in a finite root system $\Phi$:

$$(\cdot \parallel \cdot)_{\mathrm{cl}} \colon \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}.$$

The *generalized associahedron* $\Delta(\Phi)$ of $\Phi$ is given by using the compatibility degree in the following way: the vertex set is $\Phi_{\geq -1}$, and the simplex set is

$$\{C \subset \Phi_{\geq -1} \mid (\alpha \parallel \beta) = 0 \text{ for all } \alpha, \beta \in C\}$$

In [18], Fomin and Zelevinsky proved this simplicial complex corresponds to a cluster complex. In particular, they give a bijection between $\Phi_{\geq -1}$ and the set of cluster variables. By this identification, we regard the compatibility degree as a function on pairs of cluster variables in a cluster algebra of finite type. Attempts to extend it to the general cluster algebra have been made in the past, and it have been known a generalization by using $d$-vectors [9, 10]. In this paper, it is denoted by $(\cdot \parallel \cdot)_d$.

As mentioned above, in the case of finite type or rank 2, $f$-vectors coincide with $d$-vectors. These two vectors are different in general cases, but the coincidence in rank 2 or finite type implies that $f$-vectors are similar to $d$-vectors. Due to this similarity, we consider constructing the compatibility degree $(\cdot \parallel \cdot)$ by using $f$-vectors instead of $d$-vectors. In Chapter 5, the first main result is the following:

**Theorem 1.0.5** (Theorem 5.3.10)**.** *We fix any finite root system $\Phi$ and its induced cluster algebra $\mathcal{A}$. For any cluster variable $x, x'$, we have*

$$(x \parallel x')_{\mathrm{cl}} = (x \parallel x').$$

Moreover, we also consider the following question:

**Question 1.0.6.** Is $(\cdot \parallel \cdot)$ a "good" generalization of $(\cdot \parallel \cdot)_{\mathrm{cl}}$?

For example, the classical one has the following properties:

(0) for any $x, x'$, there exists a cluster $\mathbf{x}$ containing $x$ and $x'$ if and only if $(x \parallel x')_{\mathrm{cl}} = (x' \parallel x)_{\mathrm{cl}} = 0$,

(1) for any $x, x'$, there exists a set $X$ of cluster variables such that $X \cup x$ and $X \cup x'$ are both clusters, if and only if $(x \parallel x')_{\mathrm{cl}} = (x' \parallel x)_{\mathrm{cl}} = 1$,

(2) we have $(x[\alpha] \parallel x[\beta])_{\mathrm{cl}} = (x[\beta^\vee] \parallel x[\alpha^\vee])_{\mathrm{cl}}$ for every $\alpha, \beta \in \Phi_{\geq -1}$.
In particular, if $\Phi$ is simply-laced, then $(x[\alpha] \parallel x[\beta])_{\mathrm{cl}} = (x[\beta] \parallel x[\alpha])_{\mathrm{cl}}$,

(3) if $(x[\alpha] \parallel x[\beta])_{\mathrm{cl}} = 0$, then $(x[\beta] \parallel x[\alpha])_{\mathrm{cl}} = 0$,

(4) if $\alpha$ and $\beta$ belong to $\Phi(J)_{\geq -1}$ for some proper subset $J \subset I$, then their compatibility degree with respect to the root subsystem $\Phi(J)$ is equal to $(x[\alpha] \parallel x[\beta])_{\mathrm{cl}}$,

where $\alpha^\vee$ is the coroot of $\alpha$, and $x[\alpha]$ is a cluster variable corresponding to the root $\alpha$ by the canonical bijection between $\Phi_{\geq -1}$ and cluster variables. It is desirable that these properties be inherited to the generalized version. We have the following theorem:

**Theorem 1.0.7** (Proposition 5.3.14, Theorems 5.3.17, 5.3.21)**.** *The compatibility degree $(x \parallel x')$ satisfies analogies of (0), (2)–(4), and "only if" part of (1).*

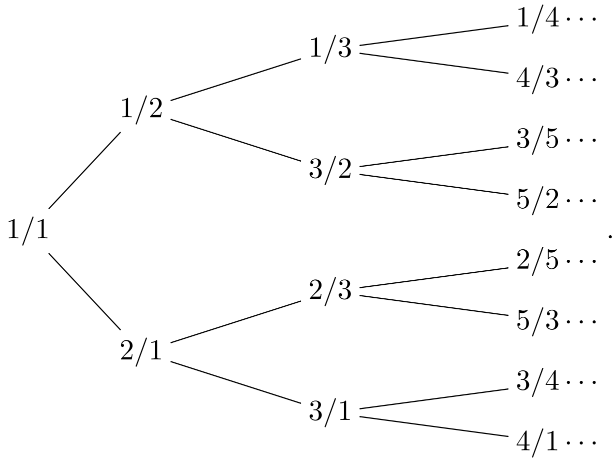
**Theorem 1.0.8** (Theorems 5.3.23, 5.3.24, 5.3.25)**.** *The compatibility degree $(x \parallel x')$ satisfies analogy of "if" part of (1) when the cluster algebra is one of the following:*

- *finite type,*
- *rank 2,*
- *acyclic type,*

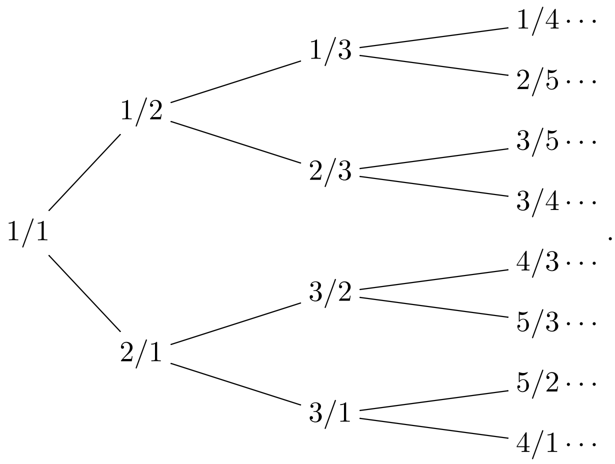
- *from a marked surface,*
- *from a weighted projective line.*

In Chapter 5, we also prove Theorem 1.0.7 and the case of finite type or rank 2 in Theorem 1.0.8.

In Chapter 6, we introduce a new application of $F$-matrices to number theory. We consider two well-known trees, the *Calkin-Wilf tree* and the *Stern-Brocot tree*. It is based on [26]. These trees are both full binary trees, and both trees have a set consisting of all positive irreducible fractions as the vertex set. The Calkin-Wilf tree is the following tree:




On the other hand, the Stern-Brocot tree is the following tree:




For the definition of these trees, see Chapter 6. These trees are very similar, and there are some studies about relation of them. In this paper, we find a new relation of them in context of cluster algebra theory. As for these two trees, there has been known the relation between the Stern-Brocot tree and a certain cluster algebra. For example, [33] gives an explicit description of $c$-vectors and $g$-vectors of a cluster algebra arising from an one-punctured torus. On the other hand, it has not been pointed out that relation between the Calkin-Wilf tree and cluster algebras. In Chapter 6, we give the Calkin-Wilf tree the structure of initial seed mutation introduced in Chapter 3, and we reveal that relation between the Calkin-Wilf tree and the Stern-Brocot tree is a specialization of $F$-matrices duality in Chapter 3.

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# 2 Preliminaries

In this chapter, we will discuss the basics of cluster algebra for use in the following chapters. In Section 2.1, we define cluster patterns and cluster algebras. We define the $d$-vectors in Section 2.2 and the $c$-, $g$-vectors and the $F$-polynomials in Section 2.3. We define in Section 2.4 the cluster complexes needed for the discussion in Chapter 5, and we define in Section 2.5 the cluster structure arising from marked surfaces discussed in the first half of Chapter 4 and in Chapter 6.

## 2.1 Cluster algebras

We start by recalling definitions of seed mutations and cluster patterns according to [20]. A *semifield* $\mathbb{P}$ is an abelian multiplicative group equipped with an addition $\oplus$ which is distributive over the multiplication. We particularly make use of the following two semifields.

Let $\mathbb{Q}_{\mathrm{sf}}(u_1, \dots, u_\ell)$ be the set of rational functions in $u_1, \dots, u_\ell$ which have subtraction-free expressions. Then, $\mathbb{Q}_{\mathrm{sf}}(u_1, \dots, u_\ell)$ is a semifield by the usual multiplication and addition. We call it the *universal semifield* of $u_1, \dots, u_\ell$ ([20, Definition 2.1]).

Let $\mathrm{Trop}(u_1, \dots, u_\ell)$ be the abelian multiplicative group freely generated by the elements $u_1, \dots, u_\ell$. Then, $\mathrm{Trop}(u_1, \dots, u_\ell)$ is a semifield by the following addition:

$$\prod_{j=1}^{\ell} u_j^{a_j} \oplus \prod_{j=1}^{\ell} u_j^{b_j} = \prod_{j=1}^{\ell} u_j^{\min(a_j, b_j)}. \tag{2.1.1}$$

We call it the *tropical semifield* of $u_1, \dots, u_\ell$ ([20, Definition 2.2]). For any semifield $\mathbb{P}$ and $p_1, \dots, p_\ell \in \mathbb{P}$, there exists a unique semifield homomorphism $\pi$ such that

$$\begin{aligned} \pi : \mathbb{Q}_{\mathrm{sf}}(y_1, \dots, y_\ell) &\longrightarrow \mathbb{P} \\ y_i &\longmapsto p_i. \end{aligned} \tag{2.1.2}$$

For $F(y_1, \dots, y_\ell) \in \mathbb{Q}_{\mathrm{sf}}(y_1, \dots, y_\ell)$, we denote

$$F|_{\mathbb{P}}(p_1, \dots, p_\ell) := \pi(F(y_1, \dots, y_\ell)). \tag{2.1.3}$$

and we call it the *evaluation* of $F$ at $p_1, \dots, p_\ell$. We fix a positive integer $n$ and a semifield $\mathbb{P}$. Let $\mathbb{ZP}$ be the group ring of $\mathbb{P}$ as a multiplicative group. Since $\mathbb{ZP}$ is a domain ([17, Section 5]), its total quotient ring is a field $\mathbb{Q}(\mathbb{P})$. Let $\mathcal{F}$ be the field of the rational functions in $n$ indeterminates with coefficients in $\mathbb{Q}(\mathbb{P})$.

A *labeled seed with coefficients in* $\mathbb{P}$ is a triplet $(\mathbf{x}, \mathbf{y}, B)$, where

- $\mathbf{x} = (x_1, \dots, x_n)$ is an $n$-tuple of elements of $\mathcal{F}$ forming a free generating set of $\mathcal{F}$.
- $\mathbf{y} = (y_1, \dots, y_n)$ is an $n$-tuple of elements of $\mathbb{P}$.
- $B = (b_{ij})$ is an $n \times n$ integer matrix which is *skew-symmetrizable*, that is, there exists a positive integer diagonal matrix $S$ such that $SB$ is skew-symmetric. Also, we call $S$ a *skew-symmetrizer* of $B$.

We say that $\mathbf{x}$ is a *cluster* and refer to $x_i, y_i$ and $B$ as the *cluster variables* (or $x$-variables), the *coefficients* (or $y$-variables) and the *exchange matrix*, respectively.

Throughout the paper, for an integer $b$, we use the notation $[b]_+ = \max(b, 0)$. We note that

$$b = [b]_+ - [-b]_+. \tag{2.1.4}$$

Let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed with coefficients in $\mathbb{P}$, and let $k \in \{1, \dots, n\}$. The *seed mutation $\mu_k$ in direction $k$* transforms $(\mathbf{x}, \mathbf{y}, B)$ into another labeled seed $\mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B')$ defined as follows:

- The entries of $B' = (b'_{ij})$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+ \, b_{kj} + b_{ik} \, [-b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{2.1.5}$$

- The coefficients $\mathbf{y}' = (y'_1, \dots, y'_n)$ are given by

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{otherwise.} \end{cases} \tag{2.1.6}$$

- The cluster variables $\mathbf{x}' = (x'_1, \dots, x'_n)$ are given by

$$x'_j = \begin{cases} \dfrac{y_k \prod\limits_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod\limits_{i=1}^{n} x_i^{[-b_{ik}]_+}}{(y_k \oplus 1) x_k} & \text{if } j = k, \\ x_j & \text{otherwise.} \end{cases} \tag{2.1.7}$$

We note that (2.1.6) can be also expressed as follows:

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[-b_{kj}]_+} ({y_k}^{-1} \oplus 1)^{-b_{kj}} & \text{otherwise.} \end{cases} \tag{2.1.8}$$

Let $\mathbb{T}_n$ be the *$n$-regular tree* whose edges are labeled by the numbers $1, \dots, n$ such that the $n$ edges emanating from each vertex have different labels. We write $t \overset{k}{\text{———}} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by $k$. We fix an arbitrary vertex $t_0 \in \mathbb{T}_n$, which is called the *rooted vertex*.

A *cluster pattern with coefficients in* $\mathbb{P}$ is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ with coefficients in $\mathbb{P}$ to every vertex $t \in \mathbb{T}_n$ such that the labeled seeds $\Sigma_t$ and $\Sigma_{t'}$ assigned to the endpoints of any edge $t \overset{k}{\text{———}} t'$ are obtained from each other by the seed mutation in direction $k$. The elements of $\Sigma_t$ are denoted as follows:

$$\mathbf{x}_t = (x_{1;t}, \dots, x_{n;t}), \ \mathbf{y}_t = (y_{1;t}, \dots, y_{n;t}), \ B_t = (b_{ij;t}). \tag{2.1.9}$$

In particular, at $t_0$, we denote

$$\mathbf{x} = \mathbf{x}_{t_0} = (x_1, \dots, x_n), \ \mathbf{y} = \mathbf{y}_{t_0} = (y_1, \dots, y_n), \ B = B_{t_0} = (b_{ij}). \tag{2.1.10}$$

**Definition 2.1.1.** A *cluster algebra* $\mathcal{A}$ associated with a cluster pattern $v \mapsto \Sigma_v$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X} = \{x_{i;t}\}_{1 \le i \le n, t \in \mathbb{T}_n}$.

The degree $n$ of the regular tree $\mathbb{T}_n$ is called the *rank* of $\mathcal{A}$, and $\mathcal{F}$ is the *ambient field* of $\mathcal{A}$. We also denote by $\mathcal{A}(B)$ a cluster algebra with the initial matrix $B$.

**Example 2.1.2.** We give an example for mutations in the case of $A_2$. Let $n = 2$, and we consider a tree $\mathbb{T}_2$ whose edges are labeled as follows:

$$\cdots \overset{1}{\text{---}} t_0 \overset{2}{\text{---}} t_1 \overset{1}{\text{---}} t_2 \overset{2}{\text{---}} t_3 \overset{1}{\text{---}} t_4 \overset{2}{\text{---}} t_5 \overset{1}{\text{---}} \cdots . \tag{2.1.11}$$

We set $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ as the initial exchange matrix at $t_0$. Then, coefficients and cluster variables are given by Table 2.1 [20, Example 2.10].

| $t$ | $\mathbf{y}_t$ | | $\mathbf{x}_t$ | |
|---|---|---|---|---|
| 0 | $y_1$ | $y_2$ | $x_1$ | $x_2$ |
| 1 | $y_1(y_2 \oplus 1)$ | $\dfrac{1}{y_2}$ | $x_1$ | $\dfrac{x_1y_2+1}{(y_2 \oplus 1)x_2}$ |
| 2 | $\dfrac{1}{y_1(y_2 \oplus 1)}$ | $\dfrac{y_1y_2 \oplus y_1 \oplus 1}{y_2}$ | $\dfrac{x_1y_1y_2+y_1+x_2}{(y_1y_2 \oplus y_1 \oplus 1)x_1x_2}$ | $\dfrac{x_1y_2+1}{(y_2 \oplus 1)x_2}$ |
| 3 | $\dfrac{y_1 \oplus 1}{y_1y_2}$ | $\dfrac{y_2}{y_1y_2 \oplus y_1 \oplus 1}$ | $\dfrac{x_1y_1y_2+y_1+x_2}{(y_1y_2 \oplus y_1 \oplus 1)x_1x_2}$ | $\dfrac{y_1+x_2}{x_1(y_1 \oplus 1)}$ |
| 4 | $\dfrac{y_1y_2}{y_1 \oplus 1}$ | $\dfrac{1}{y_1}$ | $x_2$ | $\dfrac{y_1+x_2}{x_1(y_1 \oplus 1)}$ |
| 5 | $y_2$ | $y_1$ | $x_2$ | $x_1$ |

Table 2.1: Coefficients and cluster variables in type $A_2$

Therefore, we have

$$\mathcal{A}(B) = \mathbb{ZP}\left[x_1, x_2, \frac{x_1y_2+1}{(y_2 \oplus 1)x_2}, \frac{x_1y_1y_2+y_1+x_2}{(y_1y_2 \oplus y_1 \oplus 1)x_1x_2}, \frac{y_1+x_2}{x_1(y_1 \oplus 1)}\right].$$

Next, to define the class of cluster algebras of finite type, we define the non-labeled seeds according to [20]. For a cluster pattern $v \mapsto \Sigma_v$, we introduce the following equivalence relations of labeled seeds: We say that

$$\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t), \quad \mathbf{x}_t = (x_{1;t}, \ldots, x_{n;t}), \quad \mathbf{y}_t = (y_{1;t}, \ldots, y_{n;t}), \quad B_t = (b_{ij;t})$$

and

$$\Sigma_s = (\mathbf{x}_s, \mathbf{y}_s, B_s), \quad \mathbf{x}_s = (x_{1;s}, \ldots, x_{n;s}), \quad \mathbf{y}_s = (y_{1;s}, \ldots, y_{n;s}), \quad B_s = (b_{ij;s})$$

are equivalent if there exists a permutation $\sigma$ of indices $1, \ldots, n$ such that

$$x_{i;s} = x_{\sigma(i);t}, \quad y_{j;s} = y_{\sigma(j);t}, \quad b_{ij;s} = b_{\sigma(i),\sigma(j);t}$$

for all $i$ and $j$. We denote by $[\Sigma]$ the equivalent classes represented by a labeled seed $\Sigma$ and call it the *non-labeled seed*. We define the *non-labeled clusters* (resp., *non-labeled coefficients*) as clusters (resp., coefficients) of non-labeled seeds.

**Definition 2.1.3.** The *exchange graph* of a cluster algebra is the regular connected graph whose vertices are the non-labeled seeds of the cluster pattern and whose edges connect the non-labeled seeds related by a single mutation.

Using the exchange graph, we define cluster algebras of finite type.

**Definition 2.1.4.** A cluster algebra $\mathcal{A}$ is of *finite type* if the exchange graph of $\mathcal{A}$ is a finite graph.

## 2.2 Laurent phenomenon and $d$-vectors

Let $\mathcal{A}$ be a cluster algebra. By the *Laurent phenomenon* [20, Theorem 3.5], every cluster variable $x_{j;t} \in \mathcal{A}$ can be uniquely written as

$$x_{j;t} = \frac{N_{j;t}(x_1, \dots, x_n)}{x_1^{d_{1j;t}} \cdots x_n^{d_{nj;t}}}, \quad d_{kj;t} \in \mathbb{Z}, \tag{2.2.1}$$

where $N_{j;t}(x_1, \dots, x_n)$ is a polynomial with coefficients in $\mathbb{ZP}$ which is not divisible by any initial cluster variable $x_j \in \mathbf{x}$.

**Definition 2.2.1.** We define the *$d$-vector* $\mathbf{d}_{j;t}$ as the degree vector of $x_{j;t}$, that is,

$$\mathbf{d}_{j;t}^{B;t_0} = \mathbf{d}_{j;t} = \begin{bmatrix} d_{1j;t} \\ \vdots \\ d_{nj;t} \end{bmatrix}, \tag{2.2.2}$$

in (2.2.1). We define the *$D$-matrix* $D_t^{B;t_0}$ as

$$D_t^{B;t_0} := (\mathbf{d}_{1;t}, \dots, \mathbf{d}_{n;t}). \tag{2.2.3}$$

**Remark 2.2.2.** We remark that $\mathbf{d}_{j;t}$ is independent of the choice of the coefficient system (see [20, Section 7]). Thus, we can also regard $d$-vectors as vectors associated with vertices of $\mathbb{T}_n$. They are also given by the following recursion: For any $j \in \{1, \dots, n\}$,

$$\mathbf{d}_{j;t_0} = -\mathbf{e}_j,$$

and for any $t \overset{k}{\text{———}} t'$,

$$\mathbf{d}_{j;t'} = \begin{cases} \mathbf{d}_{j;t} & \text{if } j \neq k; \\ -\mathbf{d}_{k;t} + \max\left(\sum_{i=1}^{n} [b_{ik;t}]_+ \mathbf{d}_{i;t},\ + \sum_{i=1}^{n} [-b_{ik;t}]_+ \mathbf{d}_{i;t}\right) & \text{if } j = k, \end{cases} \tag{2.2.4}$$

where $\mathbf{e}_j$ is the $j$th canonical basis.

## 2.3 $c, g$-vectors, $F$-polynomials and Separation formulas

First, we define the $c$-vectors, the $g$-vectors and the $F$-polynomials according to [20]. We introduce the principal coefficients to define them.

**Definition 2.3.1.** We say that a cluster pattern $v \mapsto \Sigma_v$ or a cluster algebra $\mathcal{A}$ of rank $n$ has the *principal coefficients* at the rooted vertex $t_0$ if $\mathbb{P} = \mathrm{Trop}(y_1, \dots, y_n)$ and $\mathbf{y}_{t_0} = (y_1, \dots, y_n)$. In this case, we denote $\mathcal{A} = \mathcal{A}_\bullet(B)$.

First, we define the $c$-vectors. For $\mathbf{b} = (b_1, \dots, b_n)^\top$, we use the notation $[\mathbf{b}]_+ = ([b_1]_+, \dots, [b_n]_+)^\top$, where $\top$ means transposition.

**Definition 2.3.2.** Let $\mathcal{A}_\bullet(B)$ be a cluster algebra with principal coefficients at $t_0$. We define the *$c$-vector* $\mathbf{c}_{j;t}$ as the vector consisting of degree of $y_i$ in $y_{j;t}$, that is, if $y_{j;t} = y_1^{c_{1j;t}} \cdots y_n^{c_{nj;t}}$, then

$$\mathbf{c}_{j;t}^{B;t_0} = \mathbf{c}_{j;t} = \begin{bmatrix} c_{1j;t} \\ \vdots \\ c_{nj;t} \end{bmatrix}. \tag{2.3.1}$$

We define the *$C$-matrix* $C_t^{B;t_0}$ as

$$C_t^{B;t_0} := (\mathbf{c}_{1;t}, \dots, \mathbf{c}_{n;t}). \tag{2.3.2}$$

Furthermore, $c$-vectors are the same as those defined by the following recursion: For any $j \in \{1, \dots, n\}$,

$$\mathbf{c}_{j;t_0} = \mathbf{e}_j \quad \text{(canonical basis)},$$

and for any $t \overset{k}{\text{——}} t'$,

$$\mathbf{c}_{j;t'} = \begin{cases} -\mathbf{c}_{j;t} & \text{if } j = k; \\ \mathbf{c}_{j;t} + [b_{kj;t}]_+ \ \mathbf{c}_{k;t} + b_{kj;t}[-\mathbf{c}_{k;t}]_+ & \text{if } j \neq k \end{cases} \tag{2.3.3}$$

(see [20]). Since the recursion formula only depends on exchange matrices, we can regard $c$-vectors as vectors associated with vertices of $\mathbb{T}_n$. In this way, we remark that $c$-vectors are independent of the choice of the coefficient system.

Next, we define the $g$-vectors. We can regard cluster variables in cluster algebras with principal coefficients as homogeneous Laurent polynomials:

**Theorem 2.3.3** ([20, Proposition 6.1])**.** *Let $\mathcal{A}_\bullet(B)$ be a cluster algebra with principal coefficients at $t_0$. Each cluster variables $x_{i;t}$ is a homogeneous Laurent polynomial in $x_1, \dots, x_n, y_1, \dots, y_n$ by the following $\mathbb{Z}^n$-grading:*

$$\deg x_i = \mathbf{e}_i, \quad \deg y_i = -\mathbf{b}_i, \tag{2.3.4}$$

*where $\mathbf{e}_i$ is the ith canonical basis of $\mathbb{Z}^n$ and $\mathbf{b}_i$ is the ith column vector of $B$.*

We denote by $\begin{bmatrix} g_{1j;t} \\ \vdots \\ g_{nj;t} \end{bmatrix}$ the $\mathbb{Z}^n$-grading of $x_{j;t}$.

**Definition 2.3.4.** Let $\mathcal{A}_\bullet(B)$ be a cluster algebra with principal coefficients at $t_0$. We define the *$g$-vector* $\mathbf{g}_{j;t}$ as the degree of the homogeneous Laurent polynomial $x_{j;t}$ by $\mathbb{Z}^n$-grading (2.3.4), that is,

$$\mathbf{g}_{j;t}^{B;t_0} = \mathbf{g}_{j;t} = \begin{bmatrix} g_{1j;t} \\ \vdots \\ g_{nj;t} \end{bmatrix}. \tag{2.3.5}$$

We define the *$G$-matrix* $G_t^{B;t_0}$ as

$$G_t^{B;t_0} := (\mathbf{g}_{1;t}, \dots, \mathbf{g}_{n;t}). \tag{2.3.6}$$

Furthermore, $g$-vectors are the same as those defined by the following recursion: For any $j \in \{1, \dots, n\}$,

$$\mathbf{g}_{j;t_0} = \mathbf{e}_j \quad \text{(canonical basis)},$$

and for any $t \overset{k}{\text{——}} t'$,

$$\mathbf{g}_{j;t'} = \begin{cases} \mathbf{g}_{j;t} & \text{if } j \neq k; \\ -\mathbf{g}_{k;t} + \sum\limits_{i=1}^{n} [b_{ik;t}]_+ \mathbf{g}_{i;t} - \sum\limits_{i=1}^{n} [c_{ik;t}]_+ \mathbf{b}_j & \text{if } j = k \end{cases} \tag{2.3.7}$$

(see [20]). Since the recursion formula only depends on exchange matrices, we can regard $g$-vectors as vectors associated with vertices of $\mathbb{T}_n$. In this way, $g$-vectors are independent of the choice of the coefficient system.

We extend the $g$-vectors from cluster variables to cluster monomials. Let $\mathbf{x}_t$ be a cluster. We consider $\mathbf{x}_t^{\mathbf{v}} = x_{1;t}^{v_1} x_{2;t}^{v_2} \cdots x_{n;t}^{v_n}$, where $v_i \in \mathbb{Z}_{\geq 0}$. We call it a *cluster monomial.* Then, we define the $g$-vector $\mathbf{g}(\mathbf{x}_t^{\mathbf{v}})$ of a cluster monomial $\mathbf{x}_t^{\mathbf{v}}$ as $v_1 \mathbf{g}_{1;t} + \cdots + v_n \mathbf{g}_{n;t}$.

Next, we define the $F$-polynomials and the $f$-vectors.

**Definition 2.3.5.** Let $\mathcal{A}_\bullet(B)$ be a cluster algebra with principal coefficients at $t_0$. We define the *$F$-polynomial* $F_{i;t}^{B;t_0}(\mathbf{y})$ as

$$F_{i;t}^{B;t_0}(\mathbf{y}) = x_{i;t}(x_1,\dots,x_n;y_1,\dots,y_n)|_{x_1=\cdots=x_n=1}, \tag{2.3.8}$$

where $x_{i;t}(x_1,\dots,x_n;y_1,\dots,y_n)$ means the expression of $x_{i;t}$ by $x_1,\dots,x_n,y_1,\dots,y_n$. The fact that seemingly rational functions $F_{j;t}^{B;t_0}(\mathbf{y})$ are polynomials follows from the strongly Laurent phenomenon of cluster variables ([20, Proposition 3.6]).

Furthermore, $F$-polynomials are the same as those defined by the following recursion: For any $j \in \{1,\dots,n\}$,

$$F_{j;t_0}^{B;t_0} = 1$$

and for any $t \overset{k}{\text{———}} t'$ ,

$$F_{j;t'}^{B;t_0}(\mathbf{y}) = \begin{cases} F_{j;t}^{B;t_0}(\mathbf{y}) & \text{if } j \neq k; \\ \dfrac{\prod_{i=1}^{n} y_i^{[c_{ik}]_+} \prod_{i=1}^{n} \left(F_{i;t}^{B;t_0}(\mathbf{y})\right)^{[b_{ik}]_+} + \prod_{i=1}^{n} y_i^{[-c_{ik}]_+} \prod_{i=1}^{n} \left(F_{i;t}^{B;t_0}(\mathbf{y})\right)^{[-b_{ik}]_+}}{F_{k;t}^{B;t_0}(\mathbf{y})} & \text{if } j = k. \end{cases} \tag{2.3.9}$$

Exchange matrices, $c$-vectors, $g$-vectors and $F$-polynomials can restore the cluster variables and the coefficients:

**Proposition 2.3.6** ([20, Proposition 3.13, Corollary 6.3])**.** *Let $\{\Sigma_t\}_{t\in\mathbb{T}_n}$ be a cluster pattern with coefficients in $\mathbb{P}$ with the initial seed* (2.1.10)*. Then, for any $t \in \mathbb{T}_n$ and $j \in \{1,\dots,n\}$, we have*

$$x_{j;t} = \left(\prod_{k=1}^{n} x_k^{g_{kj;t}^{B;t_0}}\right) \frac{F_{j;t}^{B;t_0}|_{\mathcal{F}}(\hat{y}_1,\dots,\hat{y}_n)}{F_{j;t}^{B;t_0}|_{\mathbb{P}}(y_1,\dots,y_n)}, \tag{2.3.10}$$

$$y_{j;t} = \prod_{k=1}^{n} y_k^{c_{kj;t}^{B;t_0}} \prod_{k=1}^{n} (F_{k;t}^{B;t_0}|_{\mathbb{P}}(y_1,\dots,y_n))^{b_{kj;t}}, \tag{2.3.11}$$

*where*

$$\hat{y}_i = y_i \prod_{j=1}^{n} x_j^{b_{ji}}, \tag{2.3.12}$$

*and $g_{ij;t}^{B;t_0}$ and $c_{ij;t}^{B;t_0}$ are the $(i,j)$ entry of $G_t^{B;t_0}$ and $C_t^{B;t_0}$, respectively. Also, the rational function $F_{j;t}^{B;t_0}|_{\mathcal{F}}(\hat{y}_1,\dots,\hat{y}_n)$ is the element of $\mathcal{F}$ obtained by substituting $\hat{y}_i$ for $y_i$ in $F_{j;t}^{B;t_0}(y_1,\dots,y_n)$.*

We call (2.3.10) and (2.3.11) the *separation formulas.*

**Example 2.3.7.** Let $\mathcal{A}(B)$ be the cluster algebra given in Example 2.1.2. In particular, we take one with principal coefficients at $t_0$, that is, we consider $\mathcal{A}_\bullet(B)$. Then, clusters and coefficient tuples are given by Table 2.2 and $F$-polynomials and $C,G$-matrices are given by Table 2.3.

Proposition 2.3.6 implies that if $x_{i;t} = x_{j;t'}$ in $\mathcal{A}_\bullet(B)$, then for any cluster algebra $\mathcal{A}(B)$ having the same exchange matrix as $\mathcal{A}_\bullet(B)$ at $t_0$, the $i$th cluster variable associated with $t \in \mathbb{T}_n$ is the same as the $j$th one associated with $t' \in \mathbb{T}_n$. More generally, the following fact is known:

| $t$ | $\mathbf{y}_t$ | | $\mathbf{x}_t$ | |
|---|---|---|---|---|
| 0 | $y_1$ | $y_2$ | $x_1$ | $x_2$ |
| 1 | $y_1$ | $\frac{1}{y_2}$ | $x_1$ | $\frac{x_1y_2+1}{x_2}$ |
| 2 | $\frac{1}{y_1}$ | $\frac{1}{y_2}$ | $\frac{x_1y_1y_2+y_1+x_2}{x_1x_2}$ | $\frac{x_1y_2+1}{x_2}$ |
| 3 | $\frac{1}{y_1y_2}$ | $y_2$ | $\frac{x_1y_1y_2+y_1+x_2}{x_1x_2}$ | $\frac{y_1+x_2}{x_1}$ |
| 4 | $y_1y_2$ | $\frac{1}{y_1}$ | $x_2$ | $\frac{y_1+x_2}{x_1}$ |
| 5 | $y_2$ | $y_1$ | $x_2$ | $x_1$ |

Table 2.2: Coefficients and cluster variables in type $A_2$

**Proposition 2.3.8** ([9, Prposition 6.1 (i)]). *Let $\mathcal{A}_1(B)$ and $\mathcal{A}_2(B)$ be cluster algebras having the same exchange matrix at $t_0$. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be coefficients of $\mathcal{A}_1(B)$ and $\mathcal{A}_2(B)$, respectively. Denoted by $(\mathbf{x}_t(k), \mathbf{y}_t(k), B_t(k))$, the seed of $\mathcal{A}_k(B)$ at $t \in \mathbb{T}_n$, $k = 1, 2$. Then, $x_{i;t}(1) = x_{j;t'}(1)$ if and only if $x_{i;t}(2) = x_{j;t'}(2)$, where $t, t' \in \mathbb{T}_n$ and $i, j \in \{1, 2, \cdots, n\}$.*

**Remark 2.3.9.** By recursions (2.1.7), (2.3.7) ,(2.3.9) and Proposition 2.3.8, for any cluster algebra $\mathcal{A}(B)$, if $x_{i;t} = x_{j;t'}$ then we have $\mathbf{g}_{i;t} = \mathbf{g}_{j;t'}$, $F^{B;t_0}_{i;t}(\mathbf{y}) = F^{B;t_0}_{j;t'}(\mathbf{y})$ (we remark that these vectors and polynomials depend only on $\mathbb{T}_n$ and index $i \in \{1, \ldots, n\}$). Therefore, we can say that $\mathbf{g}_{i;t}$, $F^{B;t_0}_{i;t}(\mathbf{y})$ are a $g$-vector and an $F$-polynomial associated with $x_{i;t}$, respectively.

## 2.4 Cluster complexes

Let us introduce cluster complexes which were defined in [18].

**Definition 2.4.1.** Let $\mathcal{A}(B)$ be a cluster algebra. We define a cluster complex $\Delta(\mathcal{A}(B))$ as a simplicial complex whose simplexes are subsets of cluster variables which is contained in a cluster.

**Example 2.4.2.** We consider the cluster algebra in Example 2.3.7. We give a cluster complex corresponding to this cluster algebra in Figure 2.1.

By Proposition 2.3.8, a cluster complex depends only on $B$ and does not depend on coefficients $\mathbb{P}$.

## 2.5 Cluster structures from marked surfaces

In this section, we introduce cluster structure from marked surfaces and their triangulations along [15].

### 2.5.1 Tagged arcs and tagged triangulations

Let $S$ be a connected compact oriented Riemann surface with (possibly empty) boundary and $M$ a non-empty finite set of marked points on $S$ with at least one marked point on each boundary component. We call the pair $(S, M)$ a *marked surface*. Any marked point

| $t$ | $F_{1;t}^{B;t_0}(\mathbf{y})$ | $F_{2;t}^{B;t_0}(\mathbf{y})$ | $C_t^{B;t_0}$ | $G_t^{B;t_0}$ |
|---|---|---|---|---|
| 0 | $1$ | $1$ | $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ | $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ |
| 1 | $1$ | $y_2+1$ | $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ |
| 2 | $y_1y_2+y_1+1$ | $y_2+1$ | $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ |
| 3 | $y_1y_2+y_1+1$ | $y_1+1$ | $\begin{bmatrix}-1 & 0\\-1 & 1\end{bmatrix}$ | $\begin{bmatrix}-1 & -1\\0 & 1\end{bmatrix}$ |
| 4 | $1$ | $y_1+1$ | $\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & -1\\1 & 1\end{bmatrix}$ |
| 5 | $1$ | $1$ | $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ |

Table 2.3: $F$-polynomials, $C, G$-matrices in type $A_2$

in the interior of $S$ is called a *puncture*. For technical reasons, throughout the paper we assume $(S, M)$ is not a monogon with at most one puncture, a digon without punctures, a triangle without punctures, or a sphere with at most three punctures (cf. [15]).

**Definition 2.5.1.** A *tagged arc* is a curve in $S$, considered up to isotopy, whose endpoints are in $M$ and each end is tagged in one of two ways, *plain* or *notched*, such that the following conditions are satisfied:

- it does not intersect itself except at its endpoints;
- it is disjoint from $M$ and from the boundary of $S$ except at its endpoints;
- it does not cut out a monogon with at most one puncture or a digon without punctures;
- its endpoint lying on the boundary of $S$ is tagged plain;
- both ends of a loop are tagged in the same way,

where a *loop* is a tagged arc with two identical endpoints.

In this paper, we represent tagged arcs as follows:


We call a tagged arc $\delta$

- a *plain arc* if its both ends are tagged plain;
- a 1-*notched arc* if an end of $\delta$ is tagged plain and the other end is tagged notched;
- a 2-*notched arc* if its both ends are tagged notched.

Figure 2.1: Cluster complex of $A_2$ type

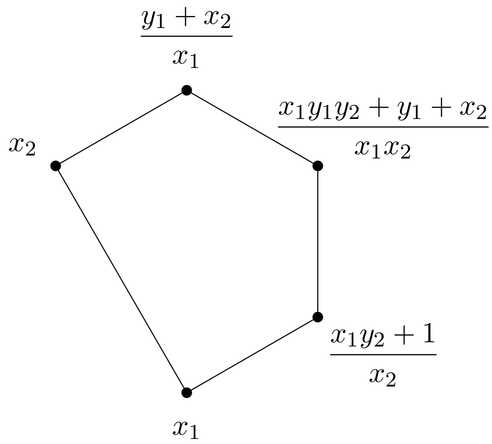



Figure 2.2: Pairs $(\delta, \epsilon)$ of conjugate arcs


We denote by $\overline{\delta}$ the plain arc corresponding to a tagged arc $\delta$ of $(S, M)$. For tagged arcs $\delta$ and $\epsilon$ such that $\overline{\delta} = \overline{\epsilon}$, if exactly one of them is a 1-notched arc, then the pair $(\delta, \epsilon)$ is called a *pair of conjugate arcs* (see Figure 2.2).

For tagged arcs $\delta$ and $\epsilon$ of $(S, M)$, the *intersection number of* $\delta$ *and* $\epsilon$ was defined in [37, Definition 3.3] as follows: We assume that $\delta$ and $\epsilon$ intersect transversally in a minimum number of points in $S \setminus M$. Then we define the intersection number $\mathsf{Int}(\delta, \epsilon) = A + B + C$ [1],where

- $A$ is the number of intersection points of $\delta$ and $\epsilon$ in $S \setminus M$;
- $B$ is the number of pairs of an end of $\delta$ and an end of $\epsilon$ that are incident to a common puncture such that their tags are different;
- $C = 0$ unless $\delta$ and $\epsilon$ form a pair of conjugate arcs, in which case $C = -1$.

Tagged arcs $\delta$ and $\epsilon$ are called *compatible* if $\mathsf{Int}(\delta, \epsilon) = 0$. A *tagged triangulation* is a maximal set of pairwise compatible tagged arcs.

For a tagged arc $\delta$ and a puncture $p$ of $(S, M)$, we define that $\delta^{(p)}$ is the tagged arc obtained from $\delta$ by changing its tags at $p$. If $\delta$ is not incident to $p$, then $\delta^{(p)} = \delta$. By definition, we have $\mathsf{Int}(\delta^{(p)}, \epsilon^{(p)}) = \mathsf{Int}(\delta, \epsilon)$ for any tagged arcs $\delta, \epsilon$ and puncture $p$ of $(S, M)$. Therefore, to consider intersection vectors with respect to a tagged triangulation $T$ of $(S, M)$, by changing tags, we can assume that $T$ satisfies the following condition:

($\diamondsuit$) The tagged triangulation $T$ consists of plain arcs and 1-notched arcs, with at most one 1-notched arc incident to each puncture.

[1] Note that this definition is slightly different from the "intersection number" $(\delta \,|\epsilon)$ defined in [15, Definition 8.4]. The intersection numbers in this paper coincide with entries of $f$-vectors in cluster algebras, and ones in [15, Definition 8.4] coincide with entries of $d$-vectors (see a paragraph right after Theorem 4.1.4 and [15]). They are the same if $(S, M)$ has no puncture.

### 2.5.2 Correspondence between tagged triangulations and clusters

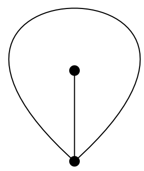
For a tagged triangulation $T$ satisfying ($\Diamond$), we construct an exchange matrix $B_T$ whose vertices are arcs of $T$.

**Definition 2.5.2.** Let $T$ be a tagged triangulation. We obtain an *ideal triangulation* $T^\circ$ associated with $T$ from $T$ in the following way:

(1) we replace all conjugate arcs with self-folded triangles (see Figure 2.3).

(2) we replace all notched tags with plain tags.

Figure 2.3: Self-folded triangle



**Definition 2.5.3.** We associate to each tagged triangulation $T$ the (generalized) *signed adjacency matrix* $B = B_T$ in the following way: The rows and columns of $B_T$ are labeled by the arcs in $T^\circ$. For notational convenience, we arbitrarily label these arcs by the numbers $1, \dots, n$, so that the rows and columns of $B_T$ are numbered from 1 to $n$ as customary, with the understanding that this numbering of rows and columns is temporary rather than intrinsic. For an arc (labeled) $i$, let $\pi_{T^\circ}(i)$ denote (the label of) the arc defined as follows: if there is a self-folded ideal triangle in $T^\circ$ folded along $i$ (see Figure 2.3), then $\pi_{T^\circ}(i)$ is its remaining side (the enclosing loop); if there is no such triangle, set $\pi_{T^\circ}(i) = i$.

For each ideal triangle $\Delta$ in $T^\circ$ which is not self-folded, define the $n \times n$ integer matrix $B^\Delta = (b^\Delta_{ij})$ by setting

$$b^\Delta_{ij} = \begin{cases} 1 & \text{if } \Delta \text{ has sides labeled } \pi_{T^\circ}(i) \text{ and } \pi_{T^\circ}(j), \\ & \quad \text{with } \pi_{T^\circ}(j) \text{ following } \pi_{T^\circ}(i) \text{ in the clockwise order;} \\ -1 & \text{if the same holds, with the counter-clockwise order;} \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.1}$$

The matrix $B = B_T = (b_{ij})$ is then defined by

$$B = \sum_\Delta B^\Delta \,,$$

the sum over all ideal triangles $\Delta$ in $T^\circ$ which are not self-folded. The $n \times n$ matrix $B$ is skew-symmetric, and all its entries $b_{ij}$ are equal to 0, 1, $-1$, 2, or $-2$.

Then we have a cluster algebra $\mathcal{A}(T) := \mathcal{A}(B_T)$ for any tagged triangulation $T$ of $(S, M)$. This cluster algebra has the following properties.

**Theorem 2.5.4** ([15, Theorem 7.11], [16, Theorem 6.1])**.** *Let $T$ be a tagged triangulation of $(S, M)$.*

*(1) If $(S, M)$ is not 1-punctured closed surface, the tagged arcs $\delta$ of $(S, M)$ correspond bijectively with the cluster variables $x_\delta$ in $\mathcal{A}(T)$. This induces that the tagged triangulations $T'$ of $(S, M)$ correspond bijectively with the clusters $\mathbf{x}_{T'}$ in $\mathcal{A}(T)$.*

*(2) If $(S, M)$ is 1-punctured closed surface, the plain arcs $\delta$ of $(S, M)$ correspond bijectively with the cluster variables $x_\delta$ in $\mathcal{A}(T)$. This induces that the tagged triangulations $T'$ which consist of plain arcs $\delta$ of $(S, M)$ correspond bijectively with the clusters $\mathbf{x}_{T'}$ in $\mathcal{A}(T)$.*

# 3 $f$-vectors, $F$-matrices and their dualities

This chapter is based on joint work with Shogo Fujiwara [23].

We describe the duality of $F$-matrices, which are degree matrices of $F$-polynomials. Using $F$-polynomials, we define the $f$-vectors and the $F$-matrices.

**Definition 3.0.1** (Definition 3.1.3). Let $\mathcal{A}_\bullet(B)$ be a cluster algebra with the principal coefficients at $t_0$. We denote by $f_{ij;t}$ the maximal degree of $y_i$ in $F_{j;t}^{B;t_0}(\mathbf{y})$. Then, we define the *$f$-vector* $\mathbf{f}_{j;t}$ as

$$\mathbf{f}_{j;t}^{B;t_0} = \mathbf{f}_{j;t} = \begin{bmatrix} f_{1j;t} \\ \vdots \\ f_{nj;t} \end{bmatrix}. \tag{3.0.1}$$

We define the *$F$-matrix* $F_t^{B;t_0}$ as

$$F_t^{B;t_0} := (\mathbf{f}_{1;t}, \dots, \mathbf{f}_{n;t}). \tag{3.0.2}$$

As above, $f$-vectors are defined as the maximal degree vector of $F$-polynomials. The most noteworthy property of the $F$-matrix is the following one:

**Theorem 3.0.2** (Theorem 3.2.10). *For any exchange matrix $B$ and $t_0, t \in \mathbb{T}_n$, we have*

$$(F_t^{B;t_0})^\top = F_{t_0}^{B_t^\top;t}. \tag{3.0.3}$$

A similar duality was found by Nakanishi and Zelevinsky [36] between the $G$- and $C$-matrices (3.2.23), and its analogy with the $F$-matrix is this theorem. This duality is an important property that will appear in all subsequent chapters. In the rest of the chapter, we will prove this duality. In the process of deriving this theorem, we introduce an initial-seed mutation, which is a dual transformation with the usual mutation.

## 3.1 Final-seed mutations and $F$-matrices

### 3.1.1 Final-seed mutations without sign-coherence of $C$-matrices

We use the following notations [36]. Let $J_\ell$ denote the $n \times n$ diagonal matrix obtained from the identity matrix $I_n$ by replacing the $(\ell, \ell)$ entry with $-1$. For a $n \times n$ matrix $B = (b_{ij})$, let $[B]_+$ be the matrix obtained from $B$ by replacing every entry $b_{ij}$ with $[b_{ij}]_+$. Also, let $B^{k\bullet}$ be the matrix obtained from $B$ by replacing all entries outside of the $k$th row with zeros. Similarly, let $B^{\bullet k}$ be the matrix replacing all entries outside of the $k$th column. Note that the maps $B \mapsto [B]_+$ and $B \mapsto B^{k\bullet}$ commute with each other, and the same is true for $B \mapsto [B]_+$ and $B \mapsto B^{\bullet k}$, so that the notations $[B]_+^{k\bullet}$ and $[B]_+^{\bullet k}$ make sense. Also, we have $AB^{\bullet k} = (AB)^{\bullet k}$ and $A^{k\bullet}B = (AB)^{k\bullet}$.

Let $B$ be any initial exchange matrix at $t_0$. Then, the families of $n \times n$ integer matrices $\{C_t^{B;t_0}\}_{t \in \mathbb{T}_n}$ and $\{G_t^{B;t_0}\}_{t \in \mathbb{T}_n}$ satisfies the following recursions by (2.3.3) and (2.3.7):

(i) We set the initial condition,

$$C_{t_0}^{B;t_0} = I_n, \tag{3.1.1}$$

and for any edge $t \overset{\ell}{\text{———}} t'$ in $\mathbb{T}_n$ and $\varepsilon \in \{\pm 1\}$, we set the recurrence relation,

$$C_{t'}^{B;t_0} = C_t^{B;t_0}(J_\ell + [\varepsilon B_t]_+^{\ell\bullet}) + [-\varepsilon C_t^{B;t_0}]_+^{\bullet\ell} B_t. \tag{3.1.2}$$

(In (3.1.2), it does not make any difference whichever we choose $\varepsilon = 1$ or $\varepsilon = -1$. See Remark 3.1.1.)

(ii) We set the initial condition,

$$G_{t_0}^{B;t_0} = I_n, \tag{3.1.3}$$

and for any edge $t \overset{\ell}{\text{———}} t'$ in $\mathbb{T}_n$ and $\varepsilon \in \{\pm 1\}$, we set the recurrence relation,

$$G_{t'}^{B;t_0} = G_t^{B;t_0}(J_\ell + [\varepsilon B_t]_+^{\bullet\ell}) - B[\varepsilon C_t^{B;t_0}]_+^{\bullet\ell}. \tag{3.1.4}$$

**Remark 3.1.1.** Because of (2.1.4), the right hand side of (3.1.2) does not depend on $\varepsilon$. Meanwhile, the right hand side of (3.1.4) does not depend on $\varepsilon$ due to the following equality ([20, (6.14)]):

$$G_t^{B;t_0} B_t = B C_t^{B;t_0}. \tag{3.1.5}$$

We have two different expressions for the recursion (3.1.2) and (3.1.4) because they will be useful in different situations.

Recall the recursions of $F$-polynomials, explained in the previous chapter. We repeat this recursion: We set the initial condition,

$$F_{j;t_0}^{B;t_0}(\mathbf{y}) = 1 \qquad (j = 1, \dots, n), \tag{3.1.6}$$

and for any edge $t \overset{\ell}{\text{———}} t'$ in $\mathbb{T}_n$, we set the recurrence relation,

$$F_{j;t'}^{B;t_0}(\mathbf{y}) = \begin{cases} F_{\ell;t}^{B;t_0}(\mathbf{y})^{-1} \left( \prod_{i=1}^{n} y_i^{[c_{i\ell;t}^{B;t_0}]_+} \prod_{i=1}^{n} F_{i;t}^{B;t_0}(\mathbf{y})^{[b_{i\ell;t}]_+} \right. \\ \qquad \left. + \prod_{i=1}^{n} y_i^{[-c_{i\ell;t}^{B;t_0}]_+} \prod_{i=1}^{n} F_{i;t}^{B;t_0}(\mathbf{y})^{[-b_{i\ell;t}]_+} \right) & \text{if } j = \ell, \\ F_{j,t}^{B;t_0}(\mathbf{y}) & \text{if } j \neq \ell, \end{cases} \tag{3.1.7}$$

In this paper, we refer to the recurrence relations (“mutations”) (3.1.2), (3.1.4) and (3.1.7) as the *final-seed mutations* in contrast to the initial-seed mutations appearing later. Abusing of notation, we denote $C_{t'}^{B;t_0} = \mu_\ell(C_t^{B;t_0}), G_{t'}^{B;t_0} = \mu_\ell(G_t^{B;t_0})$, and $F_{j:t'}^{B;t_0}(\mathbf{y}) = \mu_\ell(F_{j;t}^{B;t_0}(\mathbf{y}))$.

The following fact is well-known:

**Proposition 3.1.2** ([20, (5.5),(2.13)])**.**

*(1) For any $j \in \{1, \dots, n\}$ and $t \in \mathbb{T}_n$, we have*

$$F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1,\dots,y_n)}(y_1, \dots, y_n) = 1. \tag{3.1.8}$$

(2) *Let $\{\Sigma_t\}_{t\in\mathbb{T}_n}$ be a cluster pattern which has principal coefficients at $t_0$. Then, for any $j \in \{1,\dots,n\}$ and $t \in \mathbb{T}_n$, we have*

$$y_{j;t} = \prod_{k=1}^{n} y_k^{c_{kj;t}^{B;t_0}}. \tag{3.1.9}$$

*Proof.* Because of (3.1.6) and (3.1.7), by using the tropicalization

$$\begin{aligned} \pi : \mathbb{Q}_{\mathrm{sf}}(y_1,\dots,y_n) &\longrightarrow \mathrm{Trop}(y_1,\dots,y_n) \\ y_i &\longmapsto y_i, \end{aligned}$$

we have (3.1.8). Moreover, setting $\mathbb{P} = \mathrm{Trop}(y_1,\dots,y_n)$ in (2.3.11), we obtain (3.1.9) by (3.1.8). □

We introduce another family of matrices, which are the "degree matrices" of $F$-polynomials.

**Definition 3.1.3.** Let $B$ be any initial exchange matrix at $t_0$. For $i \in \{1,\dots,n\}$ and $t \in \mathbb{T}_n$, let $f_{1i;t}^{B;t_0},\dots,f_{ni;t}^{B;t_0}$ be the maximal degrees of $y_1,\dots,y_n$ in the $i$th $F$-polynomial $F_{i;t}^{B;t_0}(y_1,\dots,y_n)$, respectively. We call the non-negative integer vector $\mathbf{f}_{i;t}^{B;t_0} = \begin{bmatrix} f_{1i;t}^{B;t_0} \\ \vdots \\ f_{ni;t}^{B;t_0} \end{bmatrix}$ the *$f$-vector* at $t$. We also call the non-negative integer $n\times n$ matrix $F_t^{B;t_0}$ with columns $\mathbf{f}_{1;t}^{B;t_0},\dots,\mathbf{f}_{n;t}^{B;t_0}$ the *$F$-matrix* at $t$.

**Remark 3.1.4.** By Remark 2.3.9 and definition of $f$-vectors, we can say that $\mathbf{f}_{i;t}$ is the $f$-vector associated with $x_{i;t}$.

To avoid confusion the notation of $F$-matrices and $F$-polynomials, when we write $F$-polynomials, we always write it with arguments.

We have the following description of $F$-matrices: Consider a semifield homomorphism

$$\begin{aligned} \pi : \mathbb{Q}_{\mathrm{sf}}(y_1,\dots,y_n) &\longrightarrow \mathrm{Trop}(y_1^{-1},\dots,y_n^{-1}) \\ y_i &\longmapsto y_i. \end{aligned}$$

Then, we have

$$\pi(F_{\ell;t}^{B;t_0}(y_1,\dots,y_n)) = F_{\ell;t}^{B;t_0}|_{\mathrm{Trop}(y_1^{-1},\dots,y_n^{-1})}(y_1,\dots,y_n) = \prod_{i=1}^{n}(y_i^{-1})^{-f_{i\ell;t}^{B;t_0}} = \prod_{i=1}^{n} y_i^{f_{i\ell;t}^{B;t_0}}. \tag{3.1.10}$$

Moreover, $F$-matrices are uniquely determined by the following recurrence relations:

**Proposition 3.1.5** ([23, Proposition 2.7])**.** *Let $B$ be any initial exchange matrix at $t_0$. Then, $F$-matrices have the following recurrence: The initial condition is*

$$F_{t_0}^{B;t_0} = O_n, \tag{3.1.11}$$

*where $O_n$ is the zero matrix. For any edge $t \overset{\ell}{\text{———}} t'$ in $\mathbb{T}_n$, we have the recurrence relation,*

$$F_{t'}^{B;t_0} = F_t^{B;t_0}J_\ell + \max([C_t^{B;t_0}]_+^{\bullet\ell} + F_t^{B;t_0}[B_t]_+^{\bullet\ell}, [-C_t^{B;t_0}]_+^{\bullet\ell} + F_t^{B;t_0}[-B_t]_+^{\bullet\ell}). \tag{3.1.12}$$

*Proof.* Clearly, (3.1.11) is obtained by (3.1.6). Moreover, applying (3.1.10) to (3.1.7), we have (3.1.12). □

We call the recurrence relation (3.1.12) the *final-seed mutation* for $F$-matrices. As with $C$-, $G$-matrices and $F$-polynomials, we denote $F_{t'}^{B;t_0} = \mu_\ell(F_t^{B;t_0})$.

Under the exchange of the initial exchange matrices $B$ and $-B$ at $t_0$, we have the simple relations between $C$-, $G$- and $F$-matrices.

**Theorem 3.1.6** ([23, Theorem 2.8])**.** *We have the following relations:*

$$C_t^{-B;t_0} = C_t^{B;t_0} + F_t^{B;t_0} B_t, \tag{3.1.13}$$

$$G_t^{-B;t_0} = G_t^{B;t_0} + B F_t^{B;t_0}, \tag{3.1.14}$$

$$F_t^{-B;t_0} = F_t^{B;t_0}. \tag{3.1.15}$$

*Proof.* Let $\{\Sigma_t^B = (\mathbf{x}_t, \mathbf{y}_t, B_t)\}_{t\in\mathbb{T}_n}$ be a cluster pattern with coefficients in any semifield $\mathbb{P}$ with the initial seed $(\mathbf{x}, \mathbf{y}, B)$. Also, let $\{\Sigma_t^{-B} = (\mathbf{x}'_t, \mathbf{y}'_t, B'_t)\}_{t\in\mathbb{T}_n}$ be a cluster pattern with coefficients in $\mathbb{P}$ with the initial seed $(\mathbf{x}, \mathbf{y}^{-1}, -B)$. Then, by the definition of the mutation (2.1.5), (2.1.6) and (2.1.7), we have

$$\mathbf{x}'_t = \mathbf{x}_t, \quad \mathbf{y}'_t = \mathbf{y}_t^{-1}, \quad B'_t = -B_t \tag{3.1.16}$$

([20, Proof of Proposition 5.3]). We also note that for the initial seed $\Sigma_{t_0}^{-B} = (\mathbf{x}', \mathbf{y}', B') = (\mathbf{x}, \mathbf{y}^{-1}, -B)$, we have

$$\hat{y'_i} := y'_i \prod_{j=1}^{n} {x'_j}^{b'_{ji}} = y_i^{-1} \prod_{j=1}^{n} {x_j}^{-b_{ji}} = \hat{y}_i^{-1}. \tag{3.1.17}$$

Now we set $\mathbb{P} = \mathrm{Trop}(y_1^{-1}, \dots, y_n^{-1})$ and apply the separation formulas (2.3.10) and (2.3.11) to (3.1.16). Then, we obtain

$$\left(\prod_{k=1}^{n} x_k^{g_{kj;t}^{-B;t_0}}\right) F_{j;t}^{-B;t_0}|_{\mathcal{F}}(\hat{y}_1^{-1}, \dots, \hat{y}_n^{-1}) = \left(\prod_{k=1}^{n} x_k^{g_{kj;t}^{B;t_0}}\right) \frac{F_{j;t}^{B;t_0}|_{\mathcal{F}}(\hat{y}_1, \dots, \hat{y}_n)}{F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1^{-1},\dots,y_n^{-1})}(y_1, \dots, y_n)}, \tag{3.1.18}$$

$$\left(\prod_{k=1}^{n} (y_k^{-1})^{c_{kj;t}^{-B;t_0}}\right)^{-1} = \prod_{k=1}^{n} y_k^{c_{kj;t}^{B;t_0}} \prod_{k=1}^{n} (F_{k;t}^{B;t_0}|_{\mathrm{Trop}(y_1^{-1},\dots,y_n^{-1})}(y_1, \dots, y_n))^{b_{kj;t}}, \tag{3.1.19}$$

by (3.1.8) and (3.1.9). Applying (3.1.10) to (3.1.19), we obtain

$$\mathbf{c}_{j;t}^{-B;t_0} = \mathbf{c}_{j;t}^{B;t_0} + \sum_{i=1}^{n} b_{ij;t} \mathbf{f}_{i;t}^{B;t_0} \ , \tag{3.1.20}$$

thus we have (3.1.13). To show (3.1.14) from (3.1.18), let us set the $\mathbb{Z}^n$-gradings in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1, \dots, y_n]$ as follows ([20, (6.1), (6.2)]):

$$\deg(x_i) = \mathbf{e}_i, \quad \deg(y_i) = -\mathbf{b}_i, \tag{3.1.21}$$

where $\mathbf{e}_i$ is the $i$th column vector of $I_n$ and $\mathbf{b}_i$ is the $i$th column vector of $B$. Then, we have

$$\deg(\hat{y}_i) = 0. \tag{3.1.22}$$

Hence comparing the $\mathbb{Z}^n$-gradings of both sides of (3.1.18), we obtain

$$\mathbf{g}_{j;t}^{-B;t_0} = \mathbf{g}_{j;t}^{B;t_0} + \sum_{i=1}^{n} f_{ij;t}^{B;t_0} \mathbf{b}_i \ . \tag{3.1.23}$$

Therefore, we have (3.1.14). Let us prove (3.1.15). Substituting $x_i = 1$ $(i = 1, \ldots, n)$ for (3.1.18), we have

$$F_{j;t}^{-B;t_0}(y_1^{-1}, \ldots, y_n^{-1}) = \frac{F_{j;t}^{B;t_0}(y_1, \ldots, y_n)}{F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1^{-1}, \ldots, y_n^{-1})}(y_1, \ldots, y_n)} \tag{3.1.24}$$

([20](5.6)). Under the exchange of $B$ and $-B$ in (3.1.24), we also have

$$F_{j;t}^{B;t_0}(y_1, \ldots, y_n) = \frac{F_{j;t}^{-B;t_0}(y_1^{-1}, \ldots, y_n^{-1})}{F_{j;t}^{-B;t_0}|_{\mathrm{Trop}(y_1, \ldots, y_n)}(y_1^{-1}, \ldots, y_n^{-1})}. \tag{3.1.25}$$

Hence combining (3.1.24) with (3.1.25), we obtain

$$F_{j;t}^{-B;t_0}|_{\mathrm{Trop}(y_1, \ldots, y_n)}(y_1^{-1}, \ldots, y_n^{-1}) F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1^{-1}, \ldots, y_n^{-1})}(y_1, \ldots, y_n) = 1. \tag{3.1.26}$$

Comparing exponents of $y_i$ of both sides of (3.1.26), we have

$$-\mathbf{f}_{j;t}^{-B;t_0} + \mathbf{f}_{j;t}^{B;t_0} = \mathbf{0}, \tag{3.1.27}$$

thus we obtain (3.1.15). □

Thanks to the relation (3.1.13), we have the following alternating expression of the final-seed mutations of $F$-matrices:

**Proposition 3.1.7** ([23, Proposition 2.9])**.** *Let $\varepsilon \in \{\pm 1\}$. For any edge $t \overset{\ell}{\text{———}} t'$ in $\mathbb{T}_n$, the matrices $F_t^{B;t_0}$ and $F_{t'}^{B;t_0}$ are related by*

$$F_{t'}^{B;t_0} = F_t^{B;t_0}(J_\ell + [-\varepsilon B_t]_+^{\bullet \ell}) + [-\varepsilon C_t^{B;t_0}]_+^{\bullet \ell} + [\varepsilon C_t^{-B;t_0}]_+^{\bullet \ell}. \tag{3.1.28}$$

*Proof.* Firstly, we prove the case of $\varepsilon = 1$ in (3.1.28). By (2.1.4), (3.1.12) and (3.1.13), we have

$$\begin{aligned} F_{t'}^{B;t_0} &= F_t^{B;t_0}(J_\ell + [-B_t]_+^{\bullet \ell}) + \max([C_t^{B;t_0}]_+^{\bullet \ell} + F_t^{B;t_0} B_t^{\bullet \ell}, [-C_t^{B;t_0}]_+^{\bullet \ell}) \\ &= F_t^{B;t_0}(J_\ell + [-B_t]_+^{\bullet \ell}) + \max([C_t^{B;t_0}]_+^{\bullet \ell} + (C_t^{-B;t_0})^{\bullet \ell} - (C_t^{B;t_0})^{\bullet \ell}, [-C_t^{B;t_0}]_+^{\bullet \ell}) \\ &= F_t^{B;t_0}(J_\ell + [-B_t]_+^{\bullet \ell}) + \max([-C_t^{B;t_0}]_+^{\bullet \ell} + (C_t^{-B;t_0})^{\bullet \ell}, [-C_t^{B;t_0}]_+^{\bullet \ell}) \\ &= F_t^{B;t_0}(J_\ell + [-B_t]_+^{\bullet \ell}) + [-C_t^{B;t_0}]_+^{\bullet \ell} + [C_t^{-B;t_0}]_+^{\bullet \ell} \end{aligned}$$

as desired. Secondly, we prove the case of $\varepsilon = -1$ in (3.1.28). In the same way as $\varepsilon = 1$, we have

$$\begin{aligned} F_{t'}^{B;t_0} &= F_t^{B;t_0}(J_\ell + [B_t]_+^{\bullet \ell}) + \max([C_t^{B;t_0}]_+^{\bullet \ell}, [-C_t^{B;t_0}]_+^{\bullet \ell} - F_t^{B;t_0} B_t^{\bullet \ell}) \\ &= F_t^{B;t_0}(J_\ell + [B_t]_+^{\bullet \ell}) + \max([C_t^{B;t_0}]_+^{\bullet \ell}, [C_t^{B;t_0}]_+^{\bullet \ell} - (C_t^{-B;t_0})^{\bullet \ell}) \\ &= F_t^{B;t_0}(J_\ell + [B_t]_+^{\bullet \ell}) + [C_t^{B;t_0}]_+^{\bullet \ell} + [-C_t^{-B;t_0}]_+^{\bullet \ell} \end{aligned}$$

as desired. □

### 3.1.2 Final-seed mutations with sign-coherence of $C$-matrices

In this subsection, we reduce the final-seed mutation formulas by applying the sign-coherence of $C$-matrices.

**Definition 3.1.8.** Let $A$ be an (not necessarily square) integer matrix. We say that $A$ is *column sign-coherent* (resp. *row sign-coherent*) if for any column (resp. row) of $A$, its entries are either all non-negative, or all nonpositive, and not all zero.

When $A$ is column sign-coherent (resp. row sign-coherent), we can define its $\ell$th *column sign* $\varepsilon_{\bullet\ell}(A)$ (resp. *row sign* $\varepsilon_{\ell\bullet}(A)$) as the sign of nonzero entries of the $\ell$th column (resp. row) of $A$. We have the following fundamental and nontrivial result:

**Theorem 3.1.9.** [24, Corollary 5.5] *For any initial exchange matrix $B$, every $C$-matrix $C_t^{B;t_0}$ $(t \in \mathbb{T}_n)$ is column sign-coherent.*

The column signs of a $C$-matrix $C_t^{B;t_0}$ are called the *tropical signs* due to (3.1.9). Using them, the following reduced expression of the final-seed mutations of $C$- and $G$-matrices are obtained:

**Proposition 3.1.10** ([36, Proposition 1.3])**.** *For any edge $t \overset{\ell}{\text{---}} t'$ in $\mathbb{T}_n$, we have*

$$C_{t'}^{B;t_0} = C_t^{B;t_0}(J_\ell + [\varepsilon_{\bullet\ell}(C_t^{B;t_0})B_t]_+^{\ell\bullet}), \tag{3.1.29}$$

$$G_{t'}^{B;t_0} = G_t^{B;t_0}(J_\ell + [-\varepsilon_{\bullet\ell}(C_t^{B;t_0})B_t]_+^{\bullet\ell}). \tag{3.1.30}$$

They are obtained from (3.1.2) and (3.1.4) by setting $\varepsilon = \varepsilon_{\bullet\ell}(C_t^{B;t_0})$ and $\varepsilon = -\varepsilon_{\bullet\ell}(C_t^{B;t_0})$, respectively.

The following fact is shown by [20]:

**Proposition 3.1.11** ([20, Proposition 5.6])**.** *For any initial exchange matrix $B$, the following are equivalent:*

*(i) The sign-coherence of $C$-matrices holds.*

*(ii) Every polynomial $F_{\ell;t}^{B;t_0}(\mathbf{y})$ has constant term 1.*

*(iii) Every polynomial $F_{\ell;t}^{B;t_0}(\mathbf{y})$ has a unique monomial of maximal degree. Furthermore, this monomial has coefficient 1, and it is divisible by all the other occurring monomials.*

In proposition 3.1.11, the equivalence of (ii) and (iii) is proved by [20, Proposition 5.3] (see [20, Conjectures 5.4 and 5.5]).

**Remark 3.1.12.** In the definition of the (column) sign-coherence in [20], the nonzero vector property of column vectors are not assumed. However, this property can be easily recovered, since $\det C_t^{B;t_0} = \pm 1$ due to (3.1.29).

By Theorem 3.1.9 and Proposition 3.1.11, we have the following description of $f$-vectors:

**Corollary 3.1.13** ([23, Corollary 2.15])**.** *An $f$-vector $\mathbf{f}_{i;t}^{B;t_0}$ is the exponent vector of the unique monomial with maximal degree of $F_{i;t}^{B;t_0}(\mathbf{y})$. In other words, the unique monomial with maximal degree of $F_{i;t}^{B;t_0}(\mathbf{y})$ is given by $y_1^{f_{1i;t}^{B;t_0}} \dots y_n^{f_{ni;t}^{B;t_0}}$.*

Now, let us give the reduced expression of the final-seed mutations of $F$-matrices by using the tropical signs.

**Proposition 3.1.14** ([23, Proposition 2.16]). *For any edge* $t \overset{\ell}{-\!\!-} t'$ *in* $\mathbb{T}_n$*, we have*

$$\begin{aligned} F_{t'}^{B;t_0} &= F_t^{B;t_0}(J_\ell + [\varepsilon_{\bullet\ell}(C_t^{-B;t_0})B_t]_+^{\bullet\ell}) + [\varepsilon_{\bullet\ell}(C_t^{-B;t_0})C_t^{B;t_0}]_+^{\bullet\ell} \\ &= F_t^{B;t_0}(J_\ell + [-\varepsilon_{\bullet\ell}(C_t^{B;t_0})B_t]_+^{\bullet\ell}) + [\varepsilon_{\bullet\ell}(C_t^{B;t_0})C_t^{-B;t_0}]_+^{\bullet\ell}. \end{aligned} \tag{3.1.31}$$

*Proof.* Substituting $\varepsilon = -\varepsilon_{\bullet\ell}(C_t^{-B;t_0})$ or $\varepsilon = \varepsilon_{\bullet\ell}(C_t^{B;t_0})$ for (3.1.28), we obtain (3.1.31). □

## 3.2 Initial-seed mutations and $F$-matrices

### 3.2.1 Initial-seed mutations of functions $\mathcal{Y}$ and $\mathcal{X}$

We introduce the concept of the *initial-seed mutations* which appears in [20, 36, 39] in the following way. Let $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ be the universal semifield with formal variables $\mathbf{y} = (y_1, \dots, y_n)$ in Section 2.1. Let $\{\Sigma_t\}_{t \in \mathbb{T}_n}$ be the cluster pattern with coefficients in $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ where the initial coefficients $\mathbf{y}_{t_0}$ are taken as the above formal variables $\mathbf{y}$. Then, recursively applying the mutations (2.1.6) from the initial coefficients, $y_{i;t}$ are written as a rational function of $\mathbf{y}$:

$$y_{i;t} = \mathcal{Y}_{i;t}^{B;t_0}(\mathbf{y}) \in \mathbb{Q}_{\mathrm{sf}}(\mathbf{y}). \tag{3.2.1}$$

Similarly, recursively applying the mutations (2.1.7), $x_{i;t} \in \mathcal{F}$ are written as a rational function of the initial cluster variables $\mathbf{x}_{t_0} = \mathbf{x}$ with coefficients in $\mathbb{Q}(\mathbb{Q}_{\mathrm{sf}}(\mathbf{y}))$:

$$x_{i;t} = \mathcal{X}_{i;t}^{B;t_0}(\mathbf{x}) \in \mathbb{Q}(\mathbb{Q}_{\mathrm{sf}}(\mathbf{y}))(\mathbf{x}). \tag{3.2.2}$$

Then, for any cluster pattern $\{\Sigma_t\}_{t \in \mathbb{T}_n}$ with coefficients in $\mathbb{P}$, we recover $x_{i;t}$ and $y_{i;t}$ by the specialisation $\pi : \mathbb{Q}_{\mathrm{sf}}(\mathbf{y}) \to \mathbb{P}$ with $y_i$ setting to be the initial coefficients of $\Sigma_{t_0}$. Let $t_1 \in \mathbb{T}_n$ be the vertex with $t_0 \overset{k}{-\!\!-} t_1$ and let $B_1 = \mu_k(B)$. Then, the rational functions $\mathcal{Y}_{i;t}^{B;t_0}(\mathbf{y})$ and $\mathcal{Y}_{i;t}^{B_1;t_1}(\mathbf{y})$ are related by

$$\mathcal{Y}_{i;t}^{B_1;t_1}(\mathbf{y}) = \rho_k(\mathcal{Y}_{i;t}^{B;t_0}(\mathbf{y})), \tag{3.2.3}$$

where $\rho_k$ is the semifield automorphism of $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ defined by

$$\begin{aligned} \rho_k : \mathbb{Q}_{\mathrm{sf}}(\mathbf{y}) &\to \mathbb{Q}_{\mathrm{sf}}(\mathbf{y}) \\ y_j &\mapsto \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]_+}(y_k \oplus 1)^{-b_{kj}} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2.4}$$

Similarly, the rational functions $\mathcal{X}_{i;t}^{B;t_0}(\mathbf{x})$ and $\mathcal{X}_{i;t}^{B_1;t_1}(\mathbf{x})$ are related by

$$\mathcal{X}_{i;t}^{B_1;t_1}(\mathbf{x}) = \rho_k(\mathcal{X}_{i;t}^{B;t_0}(\mathbf{x})), \tag{3.2.5}$$

where $\rho_k$ is the field automorphism of $\mathbb{Q}(\mathbb{Q}_{\mathrm{sf}}(\mathbf{y}))(\mathbf{x})$ defined by

$$\begin{aligned} \rho_k : \mathbb{Q}(\mathbb{Q}_{\mathrm{sf}}(\mathbf{y}))(\mathbf{x}) &\to \mathbb{Q}(\mathbb{Q}_{\mathrm{sf}}(\mathbf{y}))(\mathbf{x}) \\ y_j &\mapsto \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]_+}(y_k \oplus 1)^{-b_{kj}} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2.6}$$

$$x_j \mapsto \begin{cases} \dfrac{y_k \prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+}}{(y_k \oplus 1)x_k} & \text{if } j = k, \\ x_j & \text{otherwise.} \end{cases} \tag{3.2.7}$$

We call them the *initial-seed mutations of the functions* $\mathcal{Y}$ *and* $\mathcal{X}$.

### 3.2.2 Initial-seed mutations without sign-coherence of $C$-matrices

We use the notations in Section 3.1 continuously. By Proposition 2.3.6, we have

$$\mathcal{X}_{j;t}^{B;t_0}(\mathbf{x}) = \left(\prod_{k=1}^{n} x_k^{g_{kj;t}^{B;t_0}}\right) \frac{F_{j;t}^{B;t_0}(\hat{y}_1,\dots,\hat{y}_n)}{F_{j;t}^{B;t_0}(y_1,\dots,y_n)}, \tag{3.2.8}$$

$$\mathcal{Y}_{j;t}^{B;t_0}(\mathbf{y}) = \prod_{k=1}^{n} y_k^{c_{kj;t}^{B;t_0}} \prod_{k=1}^{n} (F_{k;t}^{B;t_0}(y_1,\dots,y_n))^{b_{kj;t}}. \tag{3.2.9}$$

As with the final-seed mutation, we will define *the initial-seed mutations in direction $k$ of $C$-matrices* (resp. *$G$-matrices, $F$-polynomials, $F$-matrices*) as transformations from $C_t^{B;t_0}$ to $C_t^{B_1;t_1}$ (resp. from $G_t^{B;t_0}$ to $G_t^{B_1;t_1}$, from $F_{j;t}^{B;t_0}$ to $F_{j;t}^{B_1;t_1}$, from $F_t^{B;t_0}$ to $F_t^{B_1;t_1}$). We will deduce the initial-seed mutations of $C$-, $G$-matrices, $F$-polynomials and $F$-matrices. In order to describe these initial-seed mutations, we introduce the *$H$-matrices* according to [20].

**Definition 3.2.1.** Let $B$ be any initial exchange matrix at $t_0$. Then, for any $t$, the $(i,j)$ entry of $H_t^{B;t_0} = (h_{ij;t}^{B;t_0})$ is given by

$$u^{h_{ij;t}^{B;t_0}} = F_{j;t}^{B;t_0}|_{\mathrm{Trop}(u)}(u^{[-b_{i1}]_+},\dots,u^{-1},\dots,u^{[-b_{in}]_+}) \ (u^{-1} \text{ in the } i\text{th position}). \tag{3.2.10}$$

The matrix $H_t^{B;t_0}$ is called the *$H$-matrix* at $t$.

The following fact holds ([20, Proof of Proposition 6.8]).

**Lemma 3.2.2** ([23, Lemma 3.2]). *We have the following equality:*

$$y_k'^{\,h_{kj;t}^{B;t_0}} = F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1',\dots,y_n')}(y_1,\dots,y_n), \tag{3.2.11}$$

*where $(y_1',\dots,y_n')$ are the coefficients at $t_1$ connected with $t_0$ by an edge labeled $k$ in $\mathbb{T}_n$.*

*Proof.* Consider the cluster pattern with coefficients in $\mathrm{Trop}(y_1',\dots,y_n')$. Let $\mathbf{y} = (y_1,\dots,y_n)$ be the coefficients at $t_0$. Then, $\mathbf{y}$ and $\mathbf{y}'$ have the following relation:

$$y_i = \begin{cases} y_k'^{\,-1} & \text{if } i = k, \\ y_i' y_k'^{\,[-b_{ki}]_+} & \text{if } i \neq k. \end{cases} \tag{3.2.12}$$

Therefore, for any $j \in \{1,\dots,n\}$, we have

$$\begin{aligned} F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1',\dots,y_n')}(y_1,\dots,y_n) &= F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1',\dots,y_n')}(y_1' y_k'^{\,[-b_{k1}]_+},\dots,y_k'^{\,-1},\dots,y_n' y_k'^{\,[-b_{kn}]_+}) \\ &\overset{(3.1.8)}{=} F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_k')}(y_k'^{\,[-b_{k1}]_+},\dots,y_k'^{\,-1},\dots,y_k'^{\,[-b_{kn}]_+}) \\ &= y_k'^{\,h_{kj;t}^{B;t_0}}. \end{aligned}$$

□

The initial-seed mutations of $C$- and $G$-matrices are given as follows, where the latter was given in [20, Proposition 6.8]:

**Proposition 3.2.3** ([23, Proposition 3.3]). *Let $t_0 \overset{k}{\text{---}} t_1$ in $\mathbb{T}_n$, $\mu_k(B) = B_1$ and $\varepsilon \in \{\pm 1\}$. Then, for any $t$, we have*

$$C_t^{B_1;t_1} = (J_k + [-\varepsilon B]_+^{k\bullet}) C_t^{B;t_0} + H_t(\varepsilon)^{k\bullet} B_t, \tag{3.2.13}$$

$$G_t^{B_1;t_1} = (J_k + [\varepsilon B]_+^{\bullet k}) G_t^{B;t_0} - B H_t(\varepsilon)^{k\bullet}, \tag{3.2.14}$$

*where $H_t(+) = H_t^{B;t_0}, H_t(-) = H_t^{B_1;t_1}$.*

*Proof.* We denote $C_t^{B;t_0} = (c_{ij;t})$, $C_t^{B_1;t_1} = (c'_{ij;t})$, $H_t^{B;t_0} = (h_{ij;t})$, and $H_t^{B_1;t_1} = (h'_{ij;t})$. The equation (3.2.14) is just [20, Proposition 6.8], rewritten in matrix form.

Let us show (3.2.13). Consider the same cluster pattern as in the proof of Lemma 3.2.2. Then, applying (2.3.11) and (3.1.9) to any coefficient $y_{j;t}$, we have

$$\prod_{i=1}^{n} {y'_i}^{c'_{ij;t}} = \prod_{i=1}^{n} {y_i}^{c_{ij;t}} \prod_{i=1}^{n} F_{i;t}^{B;t_0}|_{\mathrm{Trop}(y'_1,\dots,y'_n)}(y_1,\dots,y_n)^{b_{ij;t}}. \tag{3.2.15}$$

Substituting (3.2.12) for (3.2.15) and using (3.2.11), we have

$$\prod_{i=1}^{n} {y'_i}^{c'_{ij;t}} = \left( \prod_{i\neq k} {y'_i}^{c_{ij;t}} {y'_k}^{[-b_{ki}]_+ c_{ij;t}} \right) {y'_k}^{-c_{kj;t}} \prod_{i=1}^{n} {y'_k}^{h_{ki;t}b_{ij;t}}. \tag{3.2.16}$$

Comparing exponents of $y'_i$ of the both sides of (3.2.16), we obtain

$$c'_{ij;t} = \begin{cases} -c_{kj;t} + \sum_{\ell=1}^{n} [-b_{k\ell}]_+ c_{\ell j;t} + \sum_{\ell=1}^{n} h_{k\ell;t} b_{\ell j;t} & \text{if } i = k, \\ c_{ij;t} & \text{if } i \neq k. \end{cases} \tag{3.2.17}$$

Also, by interchanging $t_0$ and $t_1$, we get

$$c'_{ij;t} = \begin{cases} -c_{kj;t} + \sum_{\ell=1}^{n} [b_{k\ell}]_+ c_{\ell j;t} + \sum_{\ell=1}^{n} h'_{k\ell;t} b_{\ell j;t} & \text{if } i = k, \\ c_{ij;t} & \text{if } i \neq k. \end{cases} \tag{3.2.18}$$

Thus we have (3.2.13). □

The initial-seed mutations of $F$-polynomials were given in [20, Proof of Proposition 6.8] as follows:

**Proposition 3.2.4** ([20, (6.21)]). *Let $t_0 \overset{k}{\text{—}} t_1$ in $\mathbb{T}_n$ and $\mu_k(B) = B_1$. Then, for any $j \in \{1,\dots,n\}$ and $t \in \mathbb{T}_n$, the polynomials $F_{j;t}^{B;t_0}(\mathbf{y})$ and $F_{j;t}^{B_1;t_1}(\mathbf{y})$ are related by*

$$\begin{aligned} F_{j;t}^{B_1;t_1}(y_1,\dots,y_n) &= (1+y_k)^{g_{kj}^{B;t_0}} {y_k}^{-h_{kj}^{B;t_0}} \\ &\quad \times F_{j;t}^{B;t_0}(y_1 {y_k}^{[-b_{k1}]_+}(y_k+1)^{b_{k1}},\dots,{y_k}^{-1},\dots,y_n {y_k}^{[-b_{kn}]_+}(y_k+1)^{b_{kn}}), \end{aligned} \tag{3.2.19}$$

*where ${y_k}^{-1}$ is in the $k$th position.*

The initial-seed mutation of the $F$-matrices are deduced from Proposition 3.2.4 as follows:

**Proposition 3.2.5** ([23, Proposition 3.5]). *Let $t_0 \overset{k}{\text{—}} t_1$ in $\mathbb{T}_n$, $\mu_k(B) = B_1$ and $\varepsilon \in \{\pm 1\}$. Then for any $t$, the matrices $F_t^{B;t_0}$ and $F_t^{B_1;t_1}$ are related by*

$$\begin{aligned} F_t^{B_1;t_1} &= (J_k + [\varepsilon B]_+^{k\bullet}) F_t^{B;t_0} + (\varepsilon G_t^{B;t_0})^{k\bullet} - H_t^{-B;t_0}(\varepsilon)^{k\bullet} - H_t^{B;t_0}(\varepsilon)^{k\bullet} && (3.2.20) \\ &= (J_k + [-\varepsilon B]_+^{k\bullet}) F_t^{B;t_0} + (\varepsilon G_t^{-B;t_0})^{k\bullet} - H_t^{-B;t_0}(\varepsilon)^{k\bullet} - H_t^{B;t_0}(\varepsilon)^{k\bullet}. && (3.2.21) \end{aligned}$$

*Proof.* Let us show (3.2.20) in case of $\varepsilon = 1$. Substituting (3.2.19) for $y_i = y'_i$ and evaluating (3.2.19) at $\mathrm{Trop}(y_1'^{-1}, \dots, y_n'^{-1})$, we have

$$\begin{aligned}
&F_{j;t}^{B_1;t_1}|_{\mathrm{Trop}(y_1'^{-1},\dots,y_n'^{-1})}(y'_1,\dots,y'_n)\\
&= (1\oplus y'_k)^{g_{kj}^{B;t_0}} y_k'^{-h_{kj}^{B;t_0}}\\
&\qquad \times F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1'^{-1},\dots,y_n'^{-1})}(y'_1 y_k'^{[-b_{k1}]_+}(y'_k\oplus 1)^{b_{k1}},\dots,y_k'^{-1},\dots,y'_n y_k'^{[-b_{kn}]_+}(y'_k\oplus 1)^{b_{kn}}),\\
&\overset{(2.1.8)}{=} (1\oplus y'_k)^{g_{kj}^{B;t_0}} y_k'^{-h_{kj}^{B;t_0}}\\
&\qquad \times F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1'^{-1},\dots,y_n'^{-1})}(y'_1 y_k'^{[b_{k1}]_+}(y_k'^{-1}\oplus 1)^{b_{k1}},\dots,y_k'^{-1},\dots,y'_n y_k'^{[b_{kn}]_+}(y_k'^{-1}\oplus 1)^{b_{kn}})\\
&= y_k'^{g_{kj}^{B;t_0}} y_k'^{-h_{kj}^{B;t_0}} F_{j;t}^{B;t_0}|_{\mathrm{Trop}(y_1'^{-1},\dots,y_n'^{-1})}(y'_1 y_k'^{[b_{k1}]_+},\dots,y_k'^{-1},\dots,y'_n y_k'^{[b_{kn}]_+}),\\
&\overset{(3.1.25)}{=} y_k'^{g_{kj}^{B;t_0}} y_k'^{-h_{kj}^{B;t_0}} \frac{F_{j;t}^{-B;t_0}|_{\mathrm{Trop}(y_1'^{-1},\dots,y_n'^{-1})}(y_1'^{-1} y_k'^{-[b_{k1}]_+},\dots,y'_k,\dots,y_n'^{-1} y_k'^{-[b_{kn}]_+})}{F_{j;t}^{-B;t_0}|_{\mathrm{Trop}(y_1,\dots,y_n)}(y_1^{-1},\dots,y_n^{-1})|_{y_i\mapsto y'_i y_k'^{[b_{ki}]_+},\, y_k\mapsto y_k'^{-1}}}\\
&= y_k'^{g_{kj}^{B;t_0}} y_k'^{-h_{kj}^{B;t_0}} y_k'^{-h_{kj}^{-B;t_0}} \prod_{i\neq k}(y_i'^{f_{ij}} y_k'^{[b_{ki}]_+ f_{ij}}) y_k'^{-f_{kj}}.
\end{aligned}$$

Comparing the exponent of both sides, we have

$$f_{ij;t}^{B_1;t_1} = \begin{cases} g_{kj;t}^{B;t_0} - h_{kj;t}^{B;t_0} - h_{kj;t}^{-B;t_0} + \sum_{i=1}^{n}[b_{ki}]_+ f_{ij;t}^{B;t_0} - f_{kj;t}^{B;t_0} & \text{if } i = k,\\ f_{ij;t}^{B;t_0} & \text{if } i \neq k. \end{cases} \tag{3.2.22}$$

Hence we obtain the desired equality (3.2.20). Also replacing $B$ with $-B$ in (3.2.20) and applying (3.1.15) to it, we get (3.2.21). □

### 3.2.3 Initial-seed mutations with sign-coherence of $C$-matrices

In this subsection, we reduce the initial-seed mutation formulas by applying the sign-coherence of the $C$-matrices. Let us introduce a duality between $C$-matrices and $G$-matrices, which is a result in [36], and give the reduced form of the initial-seed mutations of $C$- and $G$-matrices.

Under the sign-coherence of $C$-matrices (Theorem 3.1.9), we have the following result:

**Proposition 3.2.6** ([23, Proposition 3.6]).

(1) *For any exchange matrix $B$ and $t_0, t \in \mathbb{T}_n$, we have*

$$(G_t^{B;t_0})^\top = C_{t_0}^{B_t^\top;t}. \tag{3.2.23}$$

(2) *Let $t_0 \overset{k}{\text{---}} t_1$ in $\mathbb{T}_n$, $\mu_k(B) = B_1$ and $\varepsilon \in \{\pm 1\}$. Then, we have*

$$\begin{aligned} C_t^{B_1;t_1} &= (J_k + [-\varepsilon B]_+^{k\bullet})C_t^{B;t_0} - [-\varepsilon G_t^{B;t_0}]_+^{k\bullet} B_t, &(3.2.24)\\ G_t^{B_1;t_1} &= (J_k + [\varepsilon B]_+^{\bullet k})G_t^{B;t_0} + B[-\varepsilon G_t^{B;t_0}]_+^{k\bullet}. &(3.2.25)\end{aligned}$$

(3) *We have the reduced forms of the initial-seed mutations as follows:*

$$\begin{aligned} C_t^{B_1;t_1} &= (J_k + [-\varepsilon_{k\bullet}(G_t^{B;t_0})B]_+^{k\bullet})C_t^{B;t_0}, &(3.2.26)\\ G_t^{B_1;t_1} &= (J_k + [\varepsilon_{k\bullet}(G_t^{B;t_0})B]_+^{\bullet k})G_t^{B;t_0}. &(3.2.27)\end{aligned}$$

*Proof.* Equalities (3.2.23) and (3.2.25) are the results in [36, (1.13), (4.1)]. Also, (3.2.26) is obtained by combining (3.2.23) with [36, (1.16), (2.7)]. Using (3.2.23), we have (3.2.24) by (3.1.4). We note that $G$-matrices have the row sign-coherence by (3.2.23). By substituting $\varepsilon = \varepsilon_{k\bullet}(G_t^{B;t_0})$ for (3.2.25), we obtain (3.2.27). □

We have the following theorem by the above discussion:

**Theorem 3.2.7** ([24, Corollary 5.11])**.** *For any initial exchange matrix $B$, every $G$-matrix $G_t^{B;t_0}$ $(t \in \mathbb{T}_n)$ is row sign-coherent.*

Through the duality (3.2.23), we can find out the dual equalities between the unreduced form of the final-seed and initial-seed mutations, (3.1.4) and (3.2.24), (3.1.2) and (3.2.25), respectively. Similarly, the reduced form of the final-seed and initial-seed mutations (3.1.29) and (3.2.27), (3.1.30) and (3.2.26) are dual equalities, respectively.

Using Proposition 3.2.6, we prove the conjecture [20, Conjecture 6.10], which is the relation between $H$-matrices and $G$-matrices as follows:

**Theorem 3.2.8** ([23, Theorem 3.8])**.** *For any $t \in \mathbb{T}_n$, we have the following relation:*

$$H_t^{B;t_0} = -[-G_t^{B;t_0}]_+. \tag{3.2.28}$$

*Proof.* We assume the sign-coherence of $C$-matrices. Then, comparing (3.2.14) with (3.2.25) and setting $\varepsilon = 1$, we get

$$B[-G_t^{B;t_0}]_+^{k\bullet} = -B(H_t^{B;t_0})^{k\bullet}. \tag{3.2.29}$$

Since $k$ is arbitrary, we have

$$B[-G_t^{B;t_0}]_+ = -BH_t^{B;t_0}. \tag{3.2.30}$$

If $B$ have no zero column vector, then choosing $i$ which satisfies $b_{ik} \neq 0$, we have $[-g_{kj;t}]_+ = -h_{kj;t}$ and thus we have (3.2.28) as desired. We prove the case that $B$ have $m(\neq 0)$ zero column vectors. Permuting labels of $n$-regular tree $\mathbb{T}_n$, we can assume

$$B = \begin{bmatrix} B' & O \\ O & O \end{bmatrix},$$

where $B'$ is $(n-m)\times(n-m)$ matrix without zero column vector. Under this assumption, for $t_0 \overset{i_1}{\text{———}} \cdots \overset{i_s}{\text{———}} t$ in $\mathbb{T}_n$, we have

$$\mu_{i_s} \cdots \mu_{i_2}\mu_{i_1}(G_{t_0}^{B;t_0})|_{n-m} = \begin{bmatrix} \mu_{i_{\ell'}} \cdots \mu_{i'_2}\mu_{i'_1}(G_{t_0}^{B';t_0}) \\ O \end{bmatrix}, \tag{3.2.31}$$

$$\mu_{i_s} \cdots \mu_{i_2}\mu_{i_1}(H_{t_0}^{B;t_0})|_{n-m} = \begin{bmatrix} \mu_{i'_\ell} \cdots \mu_{i'_2}\mu_{i'_1}(H_{t_0}^{B';t_0}) \\ O \end{bmatrix}, \tag{3.2.32}$$

where $(i'_1, \dots, i'_\ell)$ is a sequence which is obtained by removing $n-m+1, \dots, n$ from $(i_1, \dots, i_s)$, and $|_{n-m}$ means taking the left $n \times (n-m)$ submatrix. Also about the other diagonal entries of $B$, we have the similar equalities. Thus we have

$$\begin{aligned} G_t^{B;t_0} &= G_{t'}^{B';t_0} \oplus G_{t_1}^{(0);t_0} \oplus \cdots \oplus G_{t_m}^{(0);t_0}, \\ H_t^{B;t_0} &= H_{t'}^{B';t_0} \oplus H_{t_1}^{(0);t_0} \oplus \cdots \oplus H_{t_m}^{(0);t_0}, \end{aligned}$$

where $t'$ is a vertex of $\mathbb{T}_{m-n}$ which satisfies $\Sigma_{t'} = \mu_{i'_\ell} \cdots \mu_{i'_1}(\Sigma_{t_0})$ and

$$t_j = \begin{cases} t_0 & \text{if the number of } n-m+j \text{ in } (i_1, \dots, i_s) \text{ is even,} \\ t'_0 & \text{if the number of } n-m+j \text{ in } (i_1, \dots, i_s) \text{ is odd,} \end{cases}$$

in $\mathbb{T}_1$: $t_0 \overset{j}{\text{——}} t_0'$ for any $j \in \{1, \dots, m\}$. Explicitly, we have

$$G_t^{B;t_0} = \begin{bmatrix} G_{t'}^{B';t_0} & & & O \\ & (-1)^{N_1} & & \\ & & \ddots & \\ O & & & (-1)^{N_m} \end{bmatrix},$$

$$H_t^{B;t_0} = \begin{bmatrix} H_{t'}^{B';t_0} & & & O \\ & -[(-1)^{N_1+1}]_+ & & \\ & & \ddots & \\ O & & & -[(-1)^{N_m+1}]_+ \end{bmatrix},$$

where $N_j$ is the number of $n-m+j$ in $(i_1, \dots, i_s)$. Since $B'$ has no zero column vectors and

$$\begin{bmatrix} B'[-G_{t'}^{B';t_0}]_+ & O \\ O & O \end{bmatrix} = \begin{bmatrix} -B'H_{t'}^{B';t_0} & O \\ O & O \end{bmatrix}$$

holds by (3.2.30), we have $[-G_{t'}^{B';t_0}]_+ = -H_{t'}^{B';t_0}$. By direct calculation, we also have $[-G_{t_j}^{(0);t_0}]_+ = -H_{t_j}^{(0);t_0}$ for all $j$. Therefore, we have (3.2.28) as desired. □

Using it, let us give a reduced expression of the initial-seed mutations of $F$-matrices.

**Proposition 3.2.9** ([23, Proposition 3.9])**.**

*(1) Let* $t_0 \overset{k}{\text{—}} t_1$ *in* $\mathbb{T}_n$, $\mu_k(B) = B_1$ *and* $\varepsilon \in \{\pm 1\}$. *Then, we have*

$$F_t^{B_1;t_1} = (J_k + [\varepsilon B]_+^{k\bullet})F_t^{B;t_0} + [-\varepsilon G_t^{-B;t_0}]_+^{k\bullet} + [\varepsilon G_t^{B;t_0}]_+^{k\bullet}. \tag{3.2.33}$$

*(2) We have a reduced form of the initial-seed mutations as follows:*

$$\begin{aligned} F_t^{B_1;t_1} &= (J_k + [\varepsilon_k(G_t^{-B;t_0})B]_+^{k\bullet})F_t^{B;t_0} + [\varepsilon_k(G_t^{-B;t_0})G_t^{B;t_0}]_+^{k\bullet} \\ &= (J_k + [-\varepsilon_k(G_t^{B;t_0})B]_+^{k\bullet})F_t^{B;t_0} + [\varepsilon_k(G_t^{B;t_0})G_t^{-B;t_0}]_+^{k\bullet}. \end{aligned} \tag{3.2.34}$$

*Proof.* (1) Thanks to Proposition 3.2.8, we can substitute $H_t^{B;t_0}(\varepsilon)^{k\bullet} = -[-\varepsilon G_t^{B;t_0}]_+^{k\bullet}$ and $H_t^{-B;t_0}(\varepsilon)^{k\bullet} = -[-\varepsilon G_t^{-B;t_0}]_+^{k\bullet}$ for (3.2.20). Then, we have

$$\begin{aligned} F_t^{B_1;t_1} &= (J_k + [\varepsilon B]_+^{k\bullet})F_t^{B;t_0} + (\varepsilon G_t^{B;t_0})^{k\bullet} + [-\varepsilon G_t^{-B;t_0}]_+^{k\bullet} + [-\varepsilon G_t^{B;t_0}]_+^{k\bullet} \\ &= (J_k + [\varepsilon B]_+^{k\bullet})F_t^{B;t_0} + [-\varepsilon G_t^{-B;t_0}]_+^{k\bullet} + [\varepsilon G_t^{B;t_0}]_+^{k\bullet}, \end{aligned}$$

as desired.
(2) Substituting $\varepsilon = \varepsilon_k(G_t^{-B;t_0})$ or $\varepsilon = -\varepsilon_k(G_t^{B;t_0})$ for (3.2.33), we obtain (3.2.34). □

Like the duality between the final-seed and initial-seed mutations of $C$- and $G$-matrices, (3.1.28) and (3.2.33), (3.1.31) and (3.2.34) are dual equalities, respectively. We show the self-duality of $F$-matrices, which is analogous to the duality (3.2.23) between $C$- and $G$-matrices.

**Theorem 3.2.10** ([23, Theorem 3.10])**.** *For any exchange matrix* $B$ *and* $t_0, t \in \mathbb{T}_n$, *we have*

$$(F_t^{B;t_0})^\top = F_{t_0}^{B_t^\top;t}. \tag{3.2.35}$$

*Proof.* We prove (3.2.35) by the induction on the distance between $t$ and $t_0$ in $\mathbb{T}_n$. When $t = t_0$, we have $(F_t^{B;t_0})^\top = O = F_{t_0}^{B_t^\top;t}$ as desired. We show that if (3.2.35) holds for some $t \in \mathbb{T}_n$, then it also holds for $t' \in \mathbb{T}_n$ such that $t \overset{\ell}{\text{---}} t'$. By the inductive assumption, (3.1.31), Proposition 3.2.9 and (3.2.23), we have

$$\begin{aligned}
(F_{t'}^{B;t_0})^\top &= (J_\ell + [\varepsilon_\ell(G_{t_0}^{-B_t^\top;t})B_t^\top]_+^{\ell\bullet})(F_t^{B;t_0})^\top + [\varepsilon_\ell(G_{t_0}^{-B_t^\top;t})G_{t_0}^{B_t^\top;t}]_+^{\ell\bullet} \\
&= (J_\ell + [\varepsilon_\ell(G_{t_0}^{-B_t^\top;t})B_t^\top]_+^{\ell\bullet})F_{t_0}^{B_t^\top;t} + [\varepsilon_\ell(G_{t_0}^{-B_t^\top;t})G_{t_0}^{B_t^\top;t}]_+^{\ell\bullet} \\
&= F_{t_0}^{B_{t'}^\top;t'}
\end{aligned}$$

as desired. ☐

### 3.2.4 Examples

We introduce an example for the final-seed and initial-seed mutations in the case of $A_2$. Let $n = 2$, and consider a tree $\mathbb{T}_2$ whose edges are labeled as follows:

$$\cdots \overset{1}{\text{---}} t_0 \overset{2}{\text{---}} t_1 \overset{1}{\text{---}} t_2 \overset{2}{\text{---}} t_3 \overset{1}{\text{---}} t_4 \overset{2}{\text{---}} t_5 \overset{1}{\text{---}} \cdots .$$

We set $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ as the initial exchange matrix at $t_0$. Then, the coefficients, the cluster variables, $C$-, $G$- and $F$-matrices are given by Table 1 and Table 2 [20, Example 2.10].

| $t$ | $\mathcal{Y}_t^{B;t_0}$ | | $\mathcal{X}_t^{B;t_0}$ | |
|---|---|---|---|---|
| 0 | $y_1$ | $y_2$ | $x_1$ | $x_2$ |
| 1 | $y_1(y_2 \oplus 1)$ | $\dfrac{1}{y_2}$ | $x_1$ | $\dfrac{x_1y_2+1}{(y_2\oplus 1)x_2}$ |
| 2 | $\dfrac{1}{y_1(y_2\oplus 1)}$ | $\dfrac{y_1y_2\oplus y_1\oplus 1}{y_2}$ | $\dfrac{x_1y_1y_2+y_1+x_2}{(y_1y_2\oplus y_1\oplus 1)x_1x_2}$ | $\dfrac{x_1y_2+1}{(y_2\oplus 1)x_2}$ |
| 3 | $\dfrac{y_1\oplus 1}{y_1y_2}$ | $\dfrac{y_2}{y_1y_2\oplus y_1\oplus 1}$ | $\dfrac{x_1y_1y_2+y_1+x_2}{(y_1y_2\oplus y_1\oplus 1)x_1x_2}$ | $\dfrac{y_1+x_2}{x_1(y_1\oplus 1)}$ |
| 4 | $\dfrac{y_1y_2}{y_1\oplus 1}$ | $\dfrac{1}{y_1}$ | $x_2$ | $\dfrac{y_1+x_2}{x_1(y_1\oplus 1)}$ |
| 5 | $y_2$ | $y_1$ | $x_2$ | $x_1$ |

Table 3.1: Coefficients and cluster variables in type $A_2$

We show the expressions of the coefficients and the cluster variables at $t_0$ in Type $A_2$ in Table 3, and its counterpart $C$-, $G$- and $F$-matrices in Table 4.

Comparing Table 2 with Table 4, we can see the duality of $C$-, $G$- and $F$-matrices in (3.2.23) and (3.2.35).

| $t$ | $C_t^{B;t_0}$ | $G_t^{B;t_0}$ | $F_t^{B;t_0}$ |
|---|---|---|---|
| 0 | $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ | $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ | $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ |
| 1 | $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ |
| 2 | $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$ |
| 3 | $\begin{bmatrix}-1 & 0\\-1 & 1\end{bmatrix}$ | $\begin{bmatrix}-1 & -1\\0 & 1\end{bmatrix}$ | $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$ |
| 4 | $\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & -1\\1 & 1\end{bmatrix}$ | $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ |
| 5 | $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ |

Table 3.2: $C$-, $G$- and $F$-matrices in type $A_2$

| $t$ | $\mathcal{Y}_{t_0}^{B_t^\top;t}$ | | $\mathcal{X}_{t_0}^{B_t^\top;t}$ | |
|---|---|---|---|---|
| 0 | $y_1$ | $y_2$ | $x_1$ | $x_2$ |
| 1 | $y_1(y_2 \oplus 1)$ | $\dfrac{1}{y_2}$ | $x_1$ | $\dfrac{y_2x_1+1}{(y_2\oplus 1)x_2}$ |
| 2 | $\dfrac{y_1y_2\oplus y_2\oplus 1}{y_1}$ | $\dfrac{1}{y_2(y_1\oplus 1)}$ | $\dfrac{y_1x_2+1}{(y_1\oplus 1)x_1}$ | $\dfrac{y_1y_2x_2+y_2+x_1}{(y_1y_2\oplus y_2\oplus 1)x_1x_2}$ |
| 3 | $\dfrac{y_1\oplus 1}{y_1y_2}$ | $\dfrac{y_2}{y_1y_2\oplus y_1\oplus 1}$ | $\dfrac{y_1y_2x_1+y_1+x_2}{(y_1y_2\oplus y_1\oplus 1)x_1x_2}$ | $\dfrac{y_1+x_2}{(y_1\oplus 1)x_1}$ |
| 4 | $\dfrac{1}{y_2}$ | $\dfrac{y_1y_2}{y_2\oplus 1}$ | $\dfrac{y_2+x_1}{(y_2\oplus 1)x_2}$ | $x_1$ |
| 5 | $y_2$ | $y_1$ | $x_2$ | $x_1$ |

Table 3.3: Expressions of coefficients and cluster variables at $t_0$ in type $A_2$

| $t$ | $C_{t_0}^{B_t^\top;t}$ | $G_{t_0}^{B_t^\top;t}$ | $F_{t_0}^{B_t^\top;t}$ |
|---|---|---|---|
| 0 | $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ | $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ | $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ |
| 1 | $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ |
| 2 | $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ | $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ |
| 3 | $\begin{bmatrix}-1 & 0\\-1 & 1\end{bmatrix}$ | $\begin{bmatrix}-1 & -1\\0 & 1\end{bmatrix}$ | $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$ |
| 4 | $\begin{bmatrix}0 & 1\\-1 & 1\end{bmatrix}$ | $\begin{bmatrix}1 & 1\\-1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$ |
| 5 | $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ |

Table 3.4: $C$-, $G$- and $F$-matrices in type $A_2$ (moving the initial vertex)

# 4 Uniqueness conjecture for $F$-matrices

In this chapter, Sections 4.1–4.3 is based on joint work with Toshiya yurikusa [27], and Sections 4.4–4.8 is based on [25].

As mentioned in Chapter 3, the motivation for introducing the $f$-vector and the $F$-matrix is the study of the properties of the $F$-polynomial. However, when the $f$-vector and $F$-matrix are defined from the $F$-polynomial, information other than the main requirements are removed. Therefore, we must consider whether this vector or matrix is intrinsic to the $F$-polynomial. Here, the desired result is that in one cluster algebra, the $f$-vector uniquely defines the $F$-polynomial. Consider the next problem, which is a bit more demanding:

**Question 4.0.1.** Which is a cluster algebra $\mathcal{A}(B)$ satisfying the following condition: $\mathbf{f}_{i;t}^{B;t_0} = \mathbf{f}_{j;s}^{B;t_0} \neq 0$ implies that $x_{i;t}$ and $x_{j;s}$ are the same cluster variable?

If $\mathcal{A}(B)$ satisfies the condition in Question 4.0.1, we say that initial seed $(\mathbf{x}, \mathbf{y}, B)$ (or initial triangulation $(S, M)$) *detects* cluster variables by $f$-vectors. There exist cluster algebras not satisfying the condition (see Remark 4.2.3). Therefore, the goal is to give a classification theorem. We obtain the answer to the question when a cluster algebra is of marked surface type, finite type, or rank 2. When a cluster algebra is marked surface type, we have the following:

**Theorem 4.0.2** (Corollary 4.1.10)**.** *Let $\mathcal{A}(B)$ be a cluster algebra arising from $(S, M)$. We denote by $g$ and $p$ the genus of $S$ and the cardinaly of $M$, respectively.*

*(1) If $S$ is not closed, then there is at least one tagged triangulation of $(S, M)$ detecting cluster variables by $f$-vectors.*

*(2) If $S$ is closed, then there is at least one tagged triangulation of $(S, M)$ detecting cluster variables by $f$-vectors if and only if $(S, M)$ is a 1-punctured closed surface or the following inequality holds:*

$$p \geq \begin{cases} 10 & \text{if } g = 2, \\ \dfrac{7 + \sqrt{1 + 48g}}{2} & \text{if } g \neq 2, \end{cases}$$

*(3) All tagged triangulation of $(S, M)$ detect cluster variables by $f$-vectors if and only if $(S, M)$ is one of the followings:*

- *a 1-punctured closed surface;*
- *a marked surface with no punctures;*
- *a marked surface of genus 0 with exactly 1 boundary component and at most 2 punctures;*
- *a marked surface of genus 0 with exactly 2 boundary components and a 1 puncture.*

When a cluster algebra is of finite type or rank 2, we have the following:

**Theorem 4.0.3** (Propositions 4.6.6, 4.7.4)**.** *Let $\mathcal{A}(B)$ be a cluster algebra of finite type or rank 2. Then, the initial seed detects cluster variables by $f$-vectors.*

Let us consider another question. The $f$-vector does not uniquely determine the cluster variables, but does the $F$-matrix uniquely determine the clusters? We have not find a counterexample of this problem yet. That is, the following are conjectured:

**Conjecture 4.0.4** ([27, Conjecture 4.4])**.** *In a cluster algebra $\mathcal{A}(B)$, for $t, s \in \mathbb{T}_n$, $F_t^{B;t_0} = F_s^{B;t_0}$ implies that $\mathbf{x}_t$ and $\mathbf{x}_s$ are the same non-labeled cluster.*

Note that satisfying the conditions in the Question 4.0.1 does not necessarily implies that the conditions of the Conjecture 4.0.4 are satisfied because Conjecture 4.0.4 is meant to be uniquely determined even when the cluster contains initial cluster variables.

About this question, we solved positively when a cluster algebra is of marked surface type, finite type, or rank 2:

**Theorem 4.0.5** (Corollary 4.1.6)**.** *Let $T$ be a tagged triangulation of $(S, M)$. In a cluster algebra $\mathcal{A}(T)$, if tagged triangulations $T'$ and $T''$ of $(S, M)$ satisfy $F_{\mathbf{x}_{T'}} = F_{\mathbf{x}_{T''}}$, then $\mathbf{x}_{T'} = \mathbf{x}_{T''}$.*

**Theorem 4.0.6** (Theorem 4.4.1)**.**

*(1) In a cluster algebra of finite type, for any $t, s \in \mathbb{T}_n$, if $(\mathbf{f}_{1;t}, \dots, \mathbf{f}_{n;t})$ corresponds with $(\mathbf{f}_{1;s}, \dots, \mathbf{f}_{n;s})$ up to order, then $\mathbf{x}_t$ and $\mathbf{x}_s$ are the same non-labeled cluster.*

*(2) In a cluster algebra of rank 2, for $t, s \in \mathbb{T}_2$, if $(\mathbf{f}_{1;t}, \mathbf{f}_{2;t})$ corresponds with $(\mathbf{f}_{1;s}, \mathbf{f}_{2;s})$ up to order, then $\mathbf{x}_t$ and $\mathbf{x}_s$ are the same non-labeled cluster.*

**Remark 4.0.7.** Analogues of Conjecture 4.0.4 for $d$-vectors, $g$-vectors and $c$-vectors have proved already in the case of general cluster algebras. About $d$-vectors, see [8, Theorem 4.22 (i)], and about $c$-vectors and $g$-vectors, see [34, Corollary 4.5, Theorem 4.8].

In this chapter, our goal is to give proofs of the above theorems. We consider the case of marked surface type in Sections 4.1–4.3, and the case of finite type or rank 2 in Sections 4.4–4.7. Moreover, in Section 4.8, we give a way to recover $F$-polynomials from $f$-vectors when $\mathcal{A}(B)$ is of rank 2.

## 4.1 Part 1: Case of Marked surface type

In Sections 4.1–4.3, we consider cluster algebras from marked surfaces. As for definition of cluster algebras from marked surfaces, see Section 2.5. Forthcoming Theorems 4.1.1, 4.1.3, 4.1.4 are proved in Sections 4.2 and 4.3. The number of tagged arcs in a tagged triangulation of $(S, M)$ is constant [15, Theorem 7.9]. Fix a tagged triangulation $T$ of $(S, M)$ with $n$ tagged arcs. For a tagged arc $\delta$ of $(S, M)$, we define

$$\mathsf{Int}(T, \delta) := (\mathsf{Int}(t, \delta))_{t \in T} \in \mathbb{Z}_{\geq 0}^n,$$

called an *intersection vector of $\delta$ with respect to $T$*. For a tagged triangulation $T' = \{\delta_1, \dots, \delta_n\}$ of $(S, M)$, we denote by $\mathsf{Int}(T, T')$ the non-negative integer matrix with columns $\mathsf{Int}(T, \delta_1), \dots, \mathsf{Int}(T, \delta_n)$. We are ready to state the key result of this section.

**Theorem 4.1.1** ([27, Theorem 1.1])**.** *Let $T$ be a tagged triangulation of $(S, M)$. If tagged triangulations $T'$ and $T''$ of $(S, M)$ have $\mathsf{Int}(T, T') = \mathsf{Int}(T, T'')$ up to permutations of columns, then $T' = T''$.*

We consider whether a tagged arc $\delta \notin T$ is determined by its intersection vector with $T$. Note that when $\delta \in T$, then clearly we have $\mathsf{Int}(T, \delta) = 0$, so that arcs in $T$ are clearly not determined by their intersection number with $T$. Thus we study the following property.

**Definition 4.1.2.** For a tagged triangulation $T$ of $(S, M)$, we say that $T$ *detects tagged arcs* if it satisfies the following condition:
• If tagged arcs $\delta$ and $\epsilon$ of $(S, M)$ have a common non-zero intersection vector $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \epsilon)$, then $\delta = \epsilon$.

We give a characterization of this property. In particular, a tagged triangulation does not detect tagged arcs generally.

**Theorem 4.1.3** ([27, Theorem 1.3])**.** *Let $T$ be a tagged triangulation of $(S, M)$. Then $T$ detects tagged arcs if and only if there are no tagged arcs $\delta$ and $\epsilon$ of $T$ connecting two (possibly same) common punctures such that $\overline{\delta} \neq \overline{\epsilon}$.*

Next, we give a complete list of marked surfaces which have tagged triangulations detecting tagged arcs.

**Theorem 4.1.4** ([27, Theorem 1.4])**.**

*(1) If $S$ is not closed, then there is at least one tagged triangulation of $(S, M)$ detecting tagged arcs.*

*(2) If $S$ is closed, then there is at least one tagged triangulation of $(S, M)$ detecting tagged arcs if and only if the inequality*

$$p \geq \begin{cases} 10 & \text{if } g = 2, \\ \dfrac{7 + \sqrt{1 + 48g}}{2} & \text{if } g \neq 2, \end{cases} \tag{4.1.1}$$

*holds where $p$ is the number of punctures of $(S, M)$ and $g$ is the genus of $S$.*[1]

*(3) All tagged triangulation of $(S, M)$ detect tagged arcs if and only if $(S, M)$ is one of the followings:*

- *a marked surface with no punctures;*
- *a marked surface of genus $0$ with exactly $1$ boundary component and at most $2$ punctures;*
- *a marked surface of genus $0$ with exactly $2$ boundary components and a $1$ puncture.*

We apply previous results in this section to a cluster algebra $\mathcal{A}(T)$ associated with a tagged triangulation $T$ by using Theorem 2.5.4. Then each tagged arc $\delta$ of $(S, M)$ gives rise to the cluster variables $x_\delta$ in $\mathcal{A}(T)$.

**Theorem 4.1.5** ([44, Theorem 1.8])**.** *Let $T$ be a tagged triangulation of $(S, M)$. If $(S, M)$ is a $1$-punctured closed surface, for any plain arc $\delta$ of $(S, M)$, we have $f_{x_\delta} = \mathsf{Int}(T, \delta)$. If not, for any tagged arc $\delta$ of $(S, M)$, we have $f_{x_\delta} = \mathsf{Int}(T, \delta)$.*

Thanks to Theorem 4.1.5, we can apply the results in the previous to the theory of cluster algebras.

[1] The lower part of the right hand side of (4.1.1) is known as the *Heawood number*. This number appears in the version of the four-color theorem for higher genus surface [40].

**Corollary 4.1.6** ([27, Corollary 1.5]). *Let $T$ be a tagged triangulation of $(S, M)$. If tagged triangulations $T'$ and $T''$ of $(S, M)$ satisfy $F_{\mathbf{x}_{T'}} = F_{\mathbf{x}_{T''}}$, then $\mathbf{x}_{T'} = \mathbf{x}_{T''}$.*

*Proof.* The assertion follows immediately from Theorems 4.1.1 and 4.1.5. □

**Remark 4.1.7.** In cluster algebras defined from marked surfaces, they are given by $\mathsf{Int}(\cdot,\cdot)$, $(\cdot|\cdot)$ and shear coordinates [12, 16, 38].

| Cluster algebras | $f$-vectors | $d$-vectors | $g$-vectors, $c$-vectors |
|---|---|---|---|
| Marked surfaces | $\mathsf{Int}(\cdot,\cdot)$ | $(\cdot\vert\cdot)$ | shear coordinates |

In particular, also according a footnote of the definition of intersection number in Chapter 2, when $(S, M)$ has no punctures, $f$-vectors coincide with $d$-vectors, and thus Corollary 4.1.6 follows from [8, Theorem 4.22 (i)] in this case.

**Definition 4.1.8.** For a cluster algebra $\mathcal{A}$, we say that $\mathcal{A}$ *detects cluster variables by $f$-vectors* if it satisfies the following condition:
- For non-initial cluster variables $z$ and $z'$ of $\mathcal{A}$, if $f_x = f_{x'}$, then $x = x'$.

**Proposition 4.1.9** ([27, Proposition 4.10]). *Let $T$ be a tagged triangulation of $(S, M)$. Then $T$ detects cluster variables by $f$-vectors if and only if either of the following conditions holds:*

- *$(S, M)$ is a 1-punctured closed surface;*
- *there are no tagged arcs $\delta$ and $\epsilon$ of $T$ connecting two (possibly same) common punctures such that $\overline{\delta} \neq \overline{\epsilon}$.*

*Proof.* If $(S, M)$ is not a 1-punctured closed surface, the assertion follows from Theorems 4.1.3, 2.5.4 (1) and 4.1.5. If $(S, M)$ is a 1-punctured closed surface, there are no 2-notched arcs corresponding to cluster variables by Theorem 2.5.4(2). Therefore, the assertion follows from Corollary 4.2.5 and Theorem 4.1.5. □

**Corollary 4.1.10** ([27, Corollary 4.11]). *Let $\mathcal{A}(B)$ be a cluster algebra arising from $(S, M)$.*

*(1) If $S$ is not closed, then there is at least one tagged triangulation of $(S, M)$ detecting cluster variables by $f$-vectors.*

*(2) If $S$ is closed, then there is at least one tagged triangulation of $(S, M)$ detecting cluster variables by $f$-vectors if and only if $(S, M)$ is a 1-punctured closed surface or the inequality (4.1.1) holds.*

*(3) All tagged triangulation of $(S, M)$ detect cluster variables by $f$-vectors if and only if $(S, M)$ is one of the followings:*
   - *a 1-punctured closed surface;*
   - *a marked surface with no punctures;*
   - *a marked surface of genus 0 with exactly 1 boundary component and at most 2 punctures;*
   - *a marked surface of genus 0 with exactly 2 boundary components and a 1 puncture.*

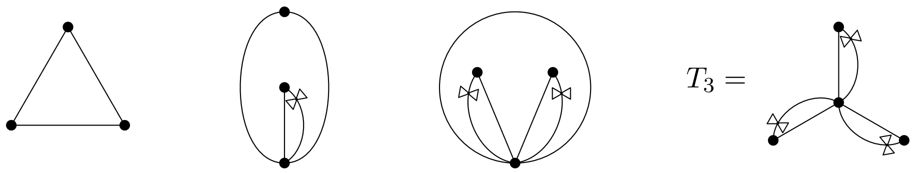
*Proof.* The assertion follows immediately from Theorem 4.1.4 and Proposition 4.1.9. □

## 4.2 Modifications of tagged arcs

### 4.2.1 Puzzle pieces

A key of many proofs in this section is a puzzle piece decomposition of tagged triangulations studied in [15]. We denote by $T_3$ a tagged triangulation satisfying ($\Diamond$) of a 4-punctured sphere consisting of three pairs of conjugate arcs (see the right diagram of Figure 4.1). Any tagged triangulation satisfying ($\Diamond$) which is not $T_3$ is obtained by gluing together a number of puzzle pieces in Figure 4.1 (see [15, Remark 4.2]). We say that a puzzle piece in the first (resp., second, third) diagram from the left on Figure 4.1 is a *triangle piece* (resp., a 1-*puncture piece*, a 2-*puncture piece*).

Figure 4.1: The three puzzle pieces (triangle piece, 1-puncture piece, 2-puncture piece) and the tagged triangulation $T_3$




### 4.2.2 Modifications of tagged arcs

In this subsection, unless otherwise noted, let $T$ be a tagged triangulation of $(S, M)$ satisfying ($\Diamond$). To prove Theorems 4.1.1 and 4.1.3, we define modifications of tagged arcs with respect to $T$.

Let $\delta \notin T$ be a tagged arc of $(S, M)$. First, we change tags of $\delta$ at a puncture $p$ if $\delta$ and a tagged arc of $T$ are tagged notched at $p$, and denote it by $\hat{\delta}$ (see Figure 4.2). Note that a notched arc of $T$ is a 1-notched arc inside a pair of conjugate arcs of $T$ by ($\Diamond$). Second, we construct a deformed curve $\mathsf{M}'_T(\hat{\delta})$ as follows: for $\overline{\delta} \notin T$,

- if $\hat{\delta}$ is a plain arc, $\mathsf{M}'_T(\hat{\delta}) = \hat{\delta}$;
- if $\hat{\delta}$ is a notched arc and is not a loop, $\mathsf{M}'_T(\hat{\delta})$ is obtained from $\hat{\delta}$ by replacing its ends tagged notched as in the left diagram of Figure 4.3;
- if $\hat{\delta}$ is a 2-notched loop and there are both sides of $\hat{\delta}$ in the same puzzle piece divided by $T$, $\mathsf{M}'_T(\hat{\delta})$ is obtained from $\hat{\delta}$ by replacing its ends as in the middle diagram of Figure 4.3;
- otherwise, $\mathsf{M}'_T(\hat{\delta})$ is obtained from $\hat{\delta}$ by replacing its ends as in the right diagram of Figure 4.3;

for $\overline{\delta} \in T$, in particular, $\delta$ is a notched arc since $\delta \notin T$,

- if $\hat{\delta}$ is a 1-notched arc, $\mathsf{M}'_T(\hat{\delta})$ is a 1-punctured loop corresponding to $\hat{\delta}$;
- if $\hat{\delta}$ is a 2-notched arc, $\mathsf{M}'_T(\hat{\delta})$ is a pair of cycles which surround each endpoint of $\hat{\delta}$ and do not include any punctures in their curves (we call this circle a 1-*punctured cycle*).

Finally, we change tags of $\mathsf{M}'_T(\hat{\delta})$ at $p$ again if $\delta$ and a tagged arc of $T$ are tagged notched at $p$. We say that the result is a *modified tagged arc of $\delta$ with respect to $T$*, and denote it by $\mathsf{M}_T(\delta)$ (see Figure 4.2 and Example 4.2.1).

Figure 4.2: From $\delta$ to $\hat{\delta}$ and $\mathsf{M}_T(\delta)$

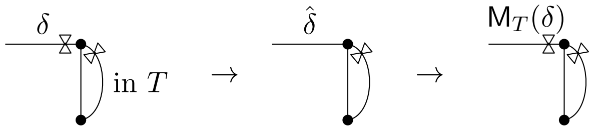
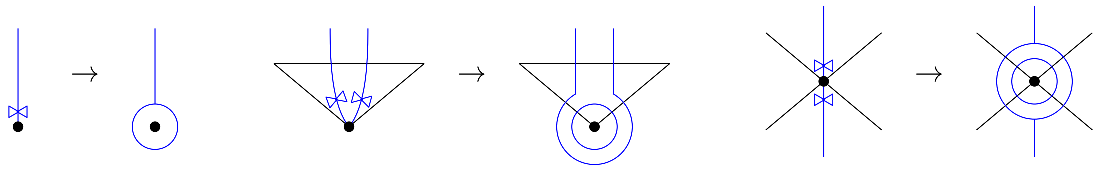
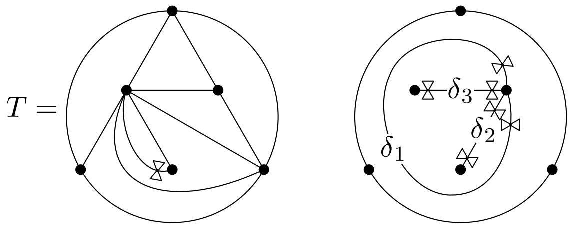


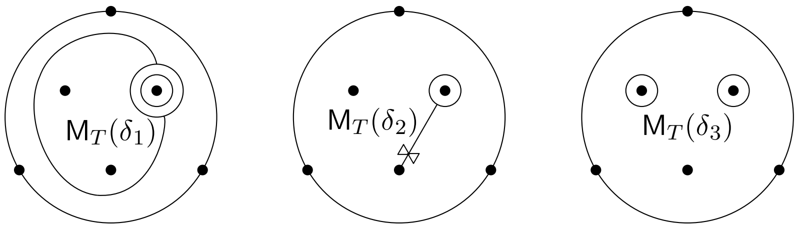

Figure 4.3: Modifications $\mathsf{M}'_T(\hat{\delta})$ of $\hat{\delta}$



**Example 4.2.1.** We consider the following tagged triangulation $T$ and tagged arcs $\delta_1$, $\delta_2$ and $\delta_3$:




Then the corresponding modified tagged arcs $\mathsf{M}_T(\delta_i)$ with respect to $T$ are given as follows:




We can define the intersection number of a modified tagged arc $\mathsf{m}$ and a tagged arc $\delta$ in the same way as of tagged arcs, denote by $\mathsf{Int}(\mathsf{m}, \delta)$. Although the map $\mathsf{M}_T$ may seem strange, it is defined so as to satisfy the following properties.

**Proposition 4.2.2** ([27, Proposition 2.3])**.**

*(1) For a tagged arc $\delta$ of $(S, M)$, we have $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \mathsf{M}_T(\delta))$.*

*(2) The map $\mathsf{M}_T$ restricting to the set*

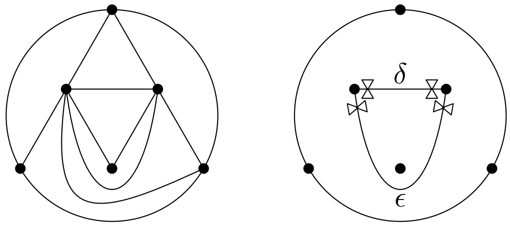
$$A := \{\textit{tagged arcs } \delta \textit{ of } (S, M) \mid \delta \notin T \textit{ and } \mathsf{M}_T(\delta) \textit{ is not a pair of } 1\textit{-punctured cycles}\}$$

*is injective. Moreover, if $\mathsf{M}_T(\delta) = \mathsf{M}_T(\epsilon)$ for $\delta \in A$ and any tagged arc $\epsilon \notin T$, then $\delta = \epsilon$ holds.*

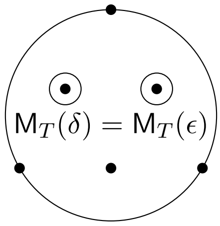
*Proof.* The assertions follow from the definition of intersection numbers and the map $\mathsf{M}_T$. □

**Remark 4.2.3.** For a tagged arc $\delta \notin T \cup A$ of $(S, M)$, $\mathsf{M}(\delta)$ does not always correspond to $\delta$ bijectively. Indeed, we consider the following tagged triangulation $T$ and tagged arcs

$\delta$, $\epsilon$:

$T =$ 


Then the corresponding modified tagged arcs $\mathsf{M}_T(\delta)$ and $\mathsf{M}_T(\epsilon)$ with respect to $T$ are given as follows:




The following theorem is a key of the proofs of Theorems 4.1.1 and 4.1.3.

**Theorem 4.2.4** ([27, Theorem 2.5])**.** *If modified tagged arcs* $\mathsf{m}$ *and* $\mathsf{m}'$ *with respect to* $T$ *have* $\mathsf{Int}(T, \mathsf{m}) = \mathsf{Int}(T, \mathsf{m}')$*, then* $\mathsf{m} = \mathsf{m}'$*.*

This proof is based on a detailed case-by-case analysis and will be omitted here due to space limitations.See [27, Sections 5 and 6].

**Corollary 4.2.5** ([27, Corollary 2.6])**.** *If tagged arcs* $\delta$ *and* $\epsilon$ *in* $A$ *have* $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \epsilon)$*, then* $\delta = \epsilon$*.*

*Proof.* Proposition 4.2.2(1) implies that $\mathsf{Int}(T, \mathsf{M}_T(\delta)) = \mathsf{Int}(T, \mathsf{M}_T(\epsilon))$. By Theorem 4.2.4 and Proposition 4.2.2(2), we have $\delta = \epsilon$. $\square$

These results provide the proofs of Theorems 4.1.1 and 4.1.3.

*Proof of Theorem 4.1.1.* By changing tags, we can assume that $T$ satisfies ($\Diamond$). Let $T' = \{\delta_1, \ldots, \delta_n\}$ and $T'' = \{\epsilon_1, \ldots, \epsilon_n\}$ be tagged triangulations of $(S, M)$ such that $\mathsf{Int}(T, \delta_i) = \mathsf{Int}(T, \epsilon_i)$ for any $i$. We set $V = (v_1 \cdots v_n) = \mathsf{Int}(T, T')$, where $v_i = \mathsf{Int}(T, \delta_i) \in \mathbb{Z}^n_{\geq 0}$. Without loss of generality, we assume that $\delta_i \in A$ for $i \in \{1, \ldots, k\}$ and $\delta_j \notin A$ for $j \in \{k+1, \ldots, n\}$, that is, either $\delta_j, \epsilon_j \in T$ or $\mathsf{M}_T(\delta_j) = \mathsf{M}_T(\epsilon_j)$ is a pair of 1-punctured cycles by Theorem 4.2.4. Corollary 4.2.5 implies that $\delta_i = \epsilon_i$ for $i \in \{1, \ldots, k\}$.

If $T' \neq T''$, then there exist $f, g \in \{k+1, \ldots, n\}$ such that $\mathsf{Int}(\delta_f, \epsilon_g) \neq 0$. Otherwise, it conflicts with the maximality of $T'$. Since $\overline{\delta_f}$ and $\overline{\epsilon_g}$ are contained in $T$, $\delta_f$ and $\epsilon_g$ must have different tags at the common endpoint. Without loss of generality, we assume that $\delta_f$ is contained in $T$ and $\mathsf{M}_T(\delta_g) = \mathsf{M}_T(\epsilon_g)$ is a pair of 1-punctured cycles. Since $\delta_f$ and $\delta_g$ have the common endpoint and $\mathsf{Int}(\delta_f, \delta_g) = 0$, $\delta_f$ is a 1-notched arc of $T$ by ($\Diamond$). Then $\delta_g$ is not a 2-notched arc, thus it is contradictory to the fact that $\mathsf{M}_T(\epsilon_g)$ is a pair of 1-punctured cycles. This finishes the proof. $\square$

*Proof of Theorem 4.1.3.* By changing tags, we can assume that $T$ satisfies ($\Diamond$). First, we prove "if" part. Let $\delta$ and $\epsilon$ be tagged arcs with a common non-zero intersection vector $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \epsilon)$ with respect to $T$. Then $\delta$ and $\epsilon$ are not contained in $T$ by definition of intersection vectors. By Corollary 4.2.5, it suffice to show that if $\mathsf{M}_T(\delta)$ is a pair of 1-punctured cycles, then $\delta = \epsilon$. In this case, $\delta$ and $\epsilon$ are 2-notched arcs such that $\overline{\delta}$ and $\overline{\epsilon}$ are plain arcs of $T$ such that both endpoints of $\overline{\delta}$ correspond to ones of $\overline{\epsilon}$ since $\mathsf{M}_T(\delta) = \mathsf{M}_T(\epsilon)$ by Theorem 4.2.4. Therefore, we have $\delta = \epsilon$ by the assumption.

Second, we prove "only if" part. Suppose that $T$ has a pair of different plain arcs $\gamma$ and $\gamma'$ such that both endpoints of $\gamma$ correspond to ones of $\gamma'$ which are punctures. Let $\delta$ and $\epsilon$ be 2-notched arcs such that $\overline{\delta} = \gamma$ and $\overline{\epsilon} = \gamma'$. Then we have $\delta \neq \epsilon$ and $\mathsf{Int}(T, \delta) = \mathsf{Int}(T, \epsilon)$ which is not zero, that is, $T$ does not detect tagged arcs. □

## 4.3 Proof of Theorem 4.1.4

First of all, we prove Theorem 4.1.4(3).

*Proof of Theorem 4.1.4(3).* It is easy to show that for $(S, M)$ as in Theorem 4.1.4(3), any tagged triangulation of $(S, M)$ detects tagged arcs by Theorem 4.1.3. Conversely, if $(S, M)$ is not one of the above cases, a part of $(S, M)$ must have one of the pairs of plain arcs $\delta$ and $\epsilon$ as in Table 4.1. Then a tagged triangulation $T$ of $(S, M)$ including $\delta$ and $\epsilon$ does not detect tagged arcs by Theorem 4.1.3. □

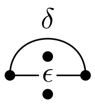
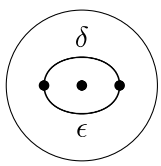
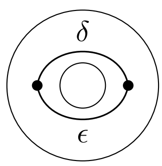
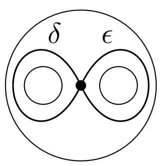
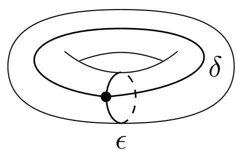
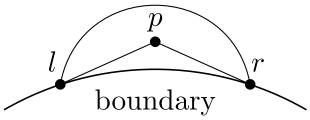
| $g$ | 0 | | | | $\geq 1$ |
|---|---|---|---|---|---|
| $b$ | 0 | 1 | 2 | $\geq 3$ | any |
| $p$ | $\geq 4$ | $\geq 3$ | $\geq 2$ | $\geq 1$ | $\geq 1$ |
| $\delta, \epsilon$ |  |  |  |  |  |

Table 4.1: Tagged arcs $\delta$ and $\epsilon$ connecting two (possibly same) common punctures such that $\overline{\delta} \neq \overline{\epsilon}$, where $g$ is the genus, $b$ is the number of components of the boundary and $p$ is the number of punctures in $(S, M)$

We consider the case that $S$ is not closed. The following lemma is basic.

**Lemma 4.3.1** ([27, Lemma 3.1])**.** *If $S$ is not closed, then there is a tagged triangulation of $(S, M)$ whose any tagged arc is a plain arc with at least one marked point on the boundary of $S$ as its endpoints.*

*Proof.* For a puncture $p$ of $(S, M)$, we can construct triangles with $p$ and two marked points $l$ and $r$ (possibly $l = r$) on the boundary of $S$ as follows:



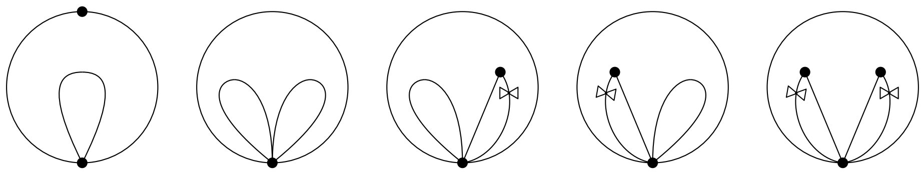

Then, for another puncture $q$ of $(S, M)$, it is easy to construct triangles with $q$, $l$ and $r$ in the same way. We have the set of triangles containing all punctures of $(S, M)$ by the inductive construction. There is a tagged triangulation of $(S, M)$ containing these triangles, thus it is what is desired. □

*Proof of Theorem 4.1.4(1).* The assertion follows from Theorem 4.1.3 and Lemma 4.3.1. □

Next, we consider the case that $S$ is closed. In the rest of this section, let $g$ be the genus of $S$ and $p$ be the number of punctures of $(S, M)$. To prove Theorem 4.1.4(2), we need some preparations.

**Lemma 4.3.2** ([27, Lemma 3.2]). *We assume that $S$ is closed and $g > 0$. If a tagged triangulation $T$ of $(S, M)$ has loops, then $T$ does not detect tagged arcs.*

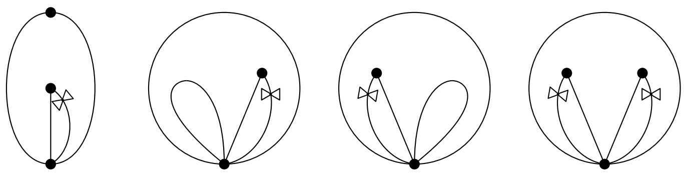
*Proof.* A puzzle piece with loops is one of the followings:



In these puzzle pieces, only the 2-punctured piece does not have a pairs of different plain arcs connecting two (possibly same) common punctures. Therefore, by Theorem 4.1.3, if a tagged triangulation $T$ with loops of $(S, M)$ detects tagged arcs, then $T$ is obtained by gluing two 2-punctured pieces and by changing tags if necessary. This is in conflict with $g > 0$. □

**Lemma 4.3.3** ([27, Lemma 3.3]). *We assume that $S$ is closed and $g > 0$. If a tagged triangulation $T$ of $(S, M)$ satisfies ($\Diamond$) and has 1-notched arcs, then $T$ does not detect tagged arcs.*

*Proof.* A puzzle piece with 1-notched arcs is one of the followings:



In these puzzle pieces, only the 2-punctured piece does not have a pairs of different plain arcs connecting two (possibly same) common punctures. Therefore, the assertion follows in the same way as Lemma 4.3.2. □

**Theorem 4.3.4.** [28, Theorem 1.1] *We assume that $S$ is closed. If $p$ is the minimal integer to satisfy* (4.1.1)*, then there is a tagged triangulation $T$ of $(S, M)$ satisfying the following conditions:*

*(T1) any tagged arc of $T$ is a plain arc;*

*(T2) any triangle of $T$ has three distinct vertices;*

*(T3) the intersection of two distinct triangles of $T$ is either empty, a single vertex, or a single edge.*

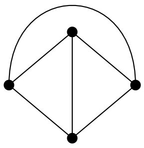
*Conversely, if there is a tagged triangulation of $(S, M)$ satisfying (T1)-(T3), then* (4.1.1) *holds.*

**Proposition 4.3.5** ([27, Proposition 3.5]). *We assume that $S$ is closed and $g > 0$. Then a tagged triangulation $T$ of $(S, M)$ satisfies (T1)-(T3) if and only if $T$ detects tagged arcs.*

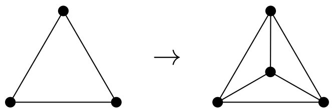
*Proof.* We assume that $T$ satisfies (T1)-(T3) and does not detect tagged arcs. By Theorem 4.1.3, there are tagged arcs $\delta$ and $\epsilon$ of $T$ connecting two common punctures such that $\overline{\delta} \neq \overline{\epsilon}$. Then they are not contained in a single triangle of $T$ by (T2). The intersection of a triangle with $\delta$ and a triangle with $\epsilon$ has two vertices and does not have an edge connecting them. It conflicts with (T3).

Conversely, we assume that $T$ detects tagged arcs. By Lemma 4.3.3, we can also assume that $T$ satisfies (T1). By Lemma 4.3.2, $T$ satisfies (T2). It is easy to show that if the intersection of two distinct triangles of $T$ is either two vertices, three vertices, or two edges, then there are tagged arcs $\delta$ and $\epsilon$ of $T$ connecting two common punctures such that $\overline{\delta} \neq \overline{\epsilon}$. Thus it is a contradiction by Theorem 4.1.3. If the intersection of two distinct triangles of $T$ is three edges, then $(S, M)$ must be a sphere with exactly three punctures, thus it conflicts with our assumption. Therefore, $T$ satisfies (T3). □

*Proof of Theorem 4.1.4(2).* When $g = 0$, we have $p \geq 4$ by our assumption, in which case (4.1.1) holds. We consider the tagged triangulation

$T =$ 

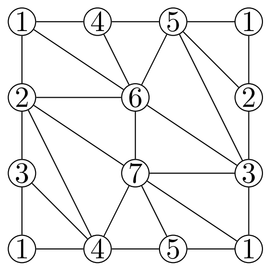
on the 2-dimensional sphere $S$. The tagged triangulation $T$ does not have different plain arcs connecting two common punctures. We add a puncture and arcs to a triangle of $T$ as follows:



Then we have inductively a tagged triangulation without different plain arcs connecting two common punctures for any $p$. By Theorem 4.1.3, it detects tagged arcs.

We assume that $g > 0$. By Theorem 4.3.4 and Proposition 4.3.5, if there is a tagged triangulation of $(S, M)$ detecting tagged arcs, then (4.1.1) holds. Conversely, if $p$ is the minimal integer to satisfy (4.1.1), then there is a tagged triangulation $T$ of $(S, M)$ detecting tagged arcs. In the same way as the case of $g = 0$, we have inductively a tagged triangulation without different plain arcs connecting two common punctures for any $p$ satisfying (4.1.1). By Theorem 4.1.3, it detects tagged arcs. □

**Example 4.3.6.** When $g = 1$, (4.1.1) means that $p \geq 7$. We consider the tagged triangulation

$T =$ 


on the torus $S$ with 7 punctures, where we identify each of two vertical lines and two horizontal lines[2]. Then $T$ does not have different plain arcs connecting two common punctures. Thus $T$ detects tagged arcs by Theorem 4.1.3.

## 4.4 Part 2: Case of finite type or rank 2 type

In Sections 4.4–4.8, we consider cluster algebras of finite type or rank 2.

Our goal in these sections is to prove Conjecture 4.0.4 by showing the following statement:

**Theorem 4.4.1** ([25, Theorem 1.11])**.**

[2]This example is also related to coloring problems of closed surfaces. It is the example that proves that we need at least 7 colors to properly cover a graph on the torus.

(1) *In a cluster algebra of finite type, for any $t, s \in \mathbb{T}_n$, if $(\mathbf{f}_{1;t}, \dots, \mathbf{f}_{n;t})$ corresponds with $(\mathbf{f}_{1;s}, \dots, \mathbf{f}_{n;s})$ up to order, then $\mathbf{x}_t$ and $\mathbf{x}_s$ are the same non-labeled cluster.*

(2) *In a cluster algebra of rank 2, for $t, s \in \mathbb{T}_2$, if $(\mathbf{f}_{1;t}, \mathbf{f}_{2;t})$ corresponds with $(\mathbf{f}_{1;s}, \mathbf{f}_{2;s})$ up to order, then $\mathbf{x}_t$ and $\mathbf{x}_s$ are the same non-labeled cluster.*

the detectivity in Theorem 4.0.3 is shown in the process of proving the above theorem.

**Remark 4.4.2.** In the case of cluster algebras of $A_n$ or $D_n$ type, Theorem 4.4.1 has already been proved by using marked surfaces (Section 4.1–4.3).

The key lemma in these sections is the following one:

**Theorem 4.4.3** ([25, Theorem 1.8])**.** *In a cluster algebra $\mathcal{A}(B)$ of finite type, for any $i \in \{1, \dots, n\}$ and $t \in \mathbb{T}_n$, we have the following relation:*

$$\mathbf{f}_{i;t} = [\mathbf{d}_{i;t}]_+. \tag{4.4.1}$$

It is known that Theorem 4.4.3 holds under the condition that the initial matrix $B$ is bipartite by combining Corollary 10.10 and Proposition 11.1 (1) in [20]. When $B$ is a skew-symmetric matrix, Theorem 4.4.3 has already proved by using 2-Carabi-Yau categories (see [22, Proposition 6.6]). We remove these conditions.

By using Theorem 4.4.3, The consideration of $f$-vectos comes down to the consideration of $d$-vectors.

**Remark 4.4.4.** In the case that $\mathcal{A}(B)$ is of rank 2, we have (4.4.1) by combining Corollary 10.10 and Proposition 11.1 (1) in [20]. If $\mathcal{A}$ is of neither finite type nor rank 2, Theorem 4.4.3 does not hold generally. A counterexample is given by [22, Section 6.4] .

## 4.5 Proof of Theorem 4.4.3

In this section, we will prove Theorem 4.4.3. We say that $B$ is *bipartite* if there is a function $\varepsilon : \{1, \dots, n\} \to \{1, -1\}$ such that for all $i$ and $j$,

$$b'_{ij} > 0 \Rightarrow \begin{cases} \varepsilon(i) = 1, \\ \varepsilon(j) = -1. \end{cases} \tag{4.5.1}$$

.

For an exchange matrix $B$, we define $A(B) = (a_{ij})$ as

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|b'_{ij}| & \text{if } i \neq j. \end{cases}$$

If $A(B)$ is a Cartan matrix, then we say that $B$ is of *finite Cartan type*.

**Remark 4.5.1.** If $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ is of finite type, then the initial matrix $B$ is mutation equivalent to a bipartite matrix $B'$. Furthermore, by permuting their indices appropriately, we can choose $B'$ as one of finite Cartan type (see [18, Theorem 1.8, Theorem 7.1]). If the initial matrix $B$ of $\mathcal{A}$ is mutation equivalent to $B'$ which is finite Cartan $X_n$ type, then there exists a bijection between almost positive roots of $X_n$ type and cluster variables of $\mathcal{A}$ (see [18, Theorem 1.9]).

We start with proving the special case. For any cluster pattern $v \mapsto \Sigma_v$, we fix a seed $\Sigma_s$ such that $B_s$ is bipartite. We define the *source mutation* $\mu_+$ and the *sink mutation* $\mu_-$ as

$$\mu_+ = \prod_{\varepsilon(k)=1} \mu_k, \quad \mu_- = \prod_{\varepsilon(k)=-1} \mu_k, \tag{4.5.2}$$

where $\varepsilon$ is the sign induced by the bipartite matrix $B_s$ (see (4.5.1)). The *bipartite belt* induced by $\Sigma_s$ consists of seeds $\Sigma_t$ satisfying the following condition: there exists a mutation sequence $\mu$ consisting of $\mu_+$ and $\mu_-$ such that $\Sigma_t = \mu(\Sigma_s)$.

**Remark 4.5.2.** Definition of a bipartite belt in this paper is a generalised version of [20, Definition 8.2]. We do not assume that the initial exchange matrix $B$ is bipartite. A bipartite belt in [20] corresponds with that induced by the initial bipartite seed $\Sigma_{t_0}$ in this paper.

**Lemma 4.5.3** ([20, Corollary 10.10]). *In any cluster algebra, if the initial matrix $B$ is bipartite and $\Sigma_t$ belongs to the bipartite belt induced by $\Sigma_{t_0}$, then we have* (4.4.1).

By Remark 4.5.1, if $\mathcal{A}$ is of finite type, then $\mathcal{A}$ has a seed whose exchange matrix is bipartite. We prove the case that the initial matrix $B$ is bipartite.

**Lemma 4.5.4** ([20, Proposition 11.1 (1)]). *In a cluster algebra of finite type, for a bipartite seed $\Sigma_s$, every cluster variable belongs to a seed lying on the bipartite belt induced by $\Sigma_s$.*

**Proposition 4.5.5** ([25, Proposition 2.4]). *We fix a cluster algebra of finite type whose initial matrix $B$ is bipartite. For any $i \in \{1, \dots, n\}$ and $t \in \mathbb{T}_n$, we have* (4.4.1).

*Proof.* It follows from Lemmas 4.5.3 and 4.5.4. □

Let us generalize Proposition 4.5.5 to the case that the initial matrix $B$ is non-bipartite. The next lemma is a generalization of Lemma 4.5.4.

**Lemma 4.5.6** ([27, Lemma 2.5]). *In a cluster algebra of finite type, for a seed $\Sigma_s$, every cluster variable belongs to seeds lying on the bipartite belt induced by $\Sigma_s$.*

*Proof.* Let $\Sigma_t^B$ be a seed and $\Sigma_s^{B'}$ a bipartite seed. By regarding a change of the initial seed from $\Sigma_t^B$ to $\Sigma_s^{B'}$ as a change from the expression of cluster variables and coefficients by $\Sigma_t^B$ to that by $\Sigma_s^{B'}$, the general cases follows from the bipartite cases. □

We introduce a key lemma.

**Lemma 4.5.7** ([39, Theorem 2.2]). *In a cluster algebra $\mathcal{A}(B)$ of finite type, for $t \in \mathbb{T}_n$, we have*

$$D_t^{B;t_0} = (D_{t_0}^{B_t^\top;t})^\top. \tag{4.5.3}$$

**Remark 4.5.8.** In [39, Theorem 2.2], the duality for $D$-matrices is given by

$$D_t^{B;t_0} = (D_{t_0}^{-B_t^\top;t})^\top. \tag{4.5.4}$$

The equation (4.5.3) derives from (4.5.4). In fact, by symmetry of the recursion (2.2.4) of $d$-vectors, we have $D_{t_0}^{-B_t^\top;t} = D_{t_0}^{B_t^\top;t}$.

We are ready to prove Theorem 4.4.3.

*Proof of Theorem 4.4.3.* We fix a bipartite seed $\Sigma$ in $\mathcal{A}(B)$. Note that $\mathcal{A}(B)$ is of finite type if and only if $\mathcal{A}(B^\top)$ is also. Moreover, $B_t^\top$ is bipartite if and only if $B_t$ is bipartite. Therefore, $\mathcal{A}(B^\top)$ is of finite type, and for any $t$ in a bipartite belt induced by $\Sigma$, $B_t^\top$ is bipartite. Thus, we have

$$F_{t_0}^{B_t^\top;t} = \left[D_{t_0}^{B_t^\top;t}\right]_+, \tag{4.5.5}$$

by Proposition 4.5.5 (the operation $[\ ]_+$ on matrices are performed component-wise). Therefore, we have

$$F_t^{B;t_0} = \left[D_t^{B;t_0}\right]_+, \tag{4.5.6}$$

by Proposition 4.5.7 and Theorem 3.2.10. By Lemma 4.5.6, for a cluster variable $x_{j;s}$, there exist $i \in \{1, \dots, n\}$ and a vertex $t$ of the bipartite belt induced by a seed $\Sigma$ such that $x_{j;s} = x_{i;t}$. Thus, $\mathbf{f}_{j;s} = \mathbf{f}_{i;t} = \mathbf{d}_{i;t} = \mathbf{d}_{j;s}$ by (4.5.6), and we have (4.4.1) for any initial vertex $t_0$. □

## 4.6 Proof of Theorem 4.4.1 (1)

In this section, we prove Theorem 4.4.1 (1). We fix any $\mathcal{A}(B)$ of finite type. Through this section, unless otherwise noted, we assume that seeds, cluster variables, clusters, $f$-vectors, $d$-vectors, $F$-matrices, and $D$-matrices are those of $\mathcal{A}(B)$. We start with proving the special case. We say that a vector $\mathbf{b}$ is *positive* (resp. *negative*) if $\mathbf{b} \neq \mathbf{0}$ and all entries of $\mathbf{b}$ is non-negative (resp. non-positive). Due to Theorem 4.4.3, we can use the properties of $d$-vectors to prove Theorem 4.4.1 (1).

**Lemma 4.6.1** ([10, Corollary 3.5])**.** *A cluster variable $x_{i;t}$ is not in the initial cluster if and only if $\mathbf{d}_{i;t}$ is positive.*

By this lemma, we have the following corollary:

**Corollary 4.6.2** ([25, Corollary 3.2])**.** *An f-vector $\mathbf{f}_{i;t}$ is the zero-vector if and only if $x_{i;t}$ is in the initial cluster.*

*Proof.* The "if" part is clear. We prove the "only if" part. By Theorem 4.4.3, $\mathbf{f}_{i;t} = \mathbf{0}$ implies that $\mathbf{d}_{i;t}$ is negative or $\mathbf{0}$. By Lemma 4.6.1, $x_{i;t}$ is in the initial cluster. □

The following propositions and corollary are essential for proving Theorem 4.4.1:

**Proposition 4.6.3** ([20, Theorem 11.1 (2)])**.** *We fix a cluster algebra $\mathcal{A}(B)$ of finite type such that $B$ is bipartite and Cartan finite $X_n$ type. Then d-vectors establish a bijection between cluster variables and the set of all almost positive roots $\Phi_{\geq -1} = \Phi_+ \cup -\Delta$ of $X_n$ Dynkin type, where $\Phi_+$ is the set of all positive roots and $-\Delta$ is the set of negative simple roots.*

Let $\mathcal{D}(B)$ be the set of all $d$-vectors in $\mathcal{A}(B)$.

**Proposition 4.6.4** ([35, Theorem 1.3.3])**.** *We fix a cluster algebra $\mathcal{A}(B)$ of finite type. Then the cardinality $|\mathcal{D}(B)|$ depends only on the Dynkin type $X_n$ of $\mathcal{A}(B)$.*

**Corollary 4.6.5** ([25, Corollary 3.5])**.** *If $\mathbf{d}_{i;t} = \mathbf{d}_{j;s}$ holds, then we have $x_{i;t} = x_{j;s}$.*

*Proof.* Let $B'$ be a bipartite matrix of finite Cartan $X_n$ type which is mutation equivalent to $B$. Then by Proposition 4.6.3 and Proposition 4.6.4, we have

$$|\mathcal{D}(B)| = |\mathcal{D}(B')| = |\Phi_{\geq -1}|. \tag{4.6.1}$$

Let $\mathcal{X}(B)$ be the set of all cluster variables of $\mathcal{A}(B)$. By Remark 4.5.1 and Proposition 4.6.3, we have

$$|\mathcal{X}(B)| = |\mathcal{X}(B')| = |\Phi_{\geq -1}|. \tag{4.6.2}$$

Therefore, we have

$$|\mathcal{D}(B)| = |\mathcal{X}(B)|. \tag{4.6.3}$$

If there exist $d$-vectors $d_{i;t}$ and $d_{j;s}$ such that $d_{i;t} = d_{j;s}$ and $x_{i;t} \neq x_{j;s}$, then we have $|\mathcal{D}(B)| < |\mathcal{X}(B)|$. This conflicts with (4.6.3). □

By Corollary 4.6.2 and Corollary 4.6.5, we have the following proposition:

**Proposition 4.6.6** ([25, Proposition 3.6])**.** *If $\mathbf{f}_{i;t} = \mathbf{f}_{j;s} \neq \mathbf{0}$, then we have $x_{i;t} = x_{j;s}$.*

*Proof.* Let $\mathbf{f}$ be an $f$-vector which is not equal to $\mathbf{0}$. We assume that $\mathbf{f} = \mathbf{f}_{i;t} = \mathbf{f}_{j;s}$. Since all entries of $\mathbf{f}$ are non-negative, and the $f$-vector is not equal to $\mathbf{0}$, we have $\mathbf{f} = \mathbf{d}_{i;t} = \mathbf{d}_{j;s}$ by Theorem 4.4.3 and Lemma 4.6.1. By Proposition 4.6.5, we have $x_{i;t} = x_{j;s}$. ☐

While $d$-vectors can distinguish the initial clusters, $f$-vectors cannot. Thus, we cannot detect the initial cluster variables contained in a cluster by their $f$-vectors directly. However, using the property of $d$-vectors, we can detect them.

**Proposition 4.6.7** ([25, Proposition 3.7])**.** *For a $D$-matrix $D_t^{B;t_0}$, negative column vectors of $D_t^{B;t_0}$ are uniquely determined by positive column vectors of $D_t^{B;t_0}$.*

*Proof.* By (4.5.3), the transposition of a $D$-matrix in a cluster algebra of finite type is another $D$-matrix in a cluster algebra of finite type because $\mathcal{A}(B)$ is of finite type if and only if $\mathcal{A}(B_t^\top)$ is of finite type. Since negative $d$-vectors have the form of $-\mathbf{e}_i$, if the $(i,j)$ entry of $D_t^{B;t_0}$ is $-1$, then entries of the $i$th row and the $j$th column of $D_t^{B;t_0}$ are all 0 except for the $(i,j)$-entry. Since $D_t^{B;t_0}$ do not have the zero column vector by Lemma 4.6.1, if a $D$-matrix has just $m$ positive columns, then we have just $n-m$ indices $i_1,\dots,i_{n-m}$ such that the $i_k (k \in \{1,\dots,n-m\})$th entry of all positive $d$-vectors is 0, and $D_t^{B;t_0}$ has column vectors $-\mathbf{e}_{i_k} (k \in \{1,\dots,n-m\})$. This finishes the proof. ☐

We are ready to prove Theorem 4.4.1 (1).

*Proof of Theorem 4.4.1 (1).* If $\mathbf{f}_{i;t} = \mathbf{f}_{j;s} \neq \mathbf{0}$, then we have $x_{i;t} = x_{j;s}$ by Proposition 4.6.6. We assume that there are $m$ zero-vectors in $(\mathbf{f}_{1;t},\dots,\mathbf{f}_{n;t})$ (or $(\mathbf{f}_{1;s},\dots,\mathbf{f}_{n;s})$). By regarding positive $f$-vectors as $d$-vectors by Theorem 4.4.3, we detect the rest of $d$-vectors in $\mathbf{x}_t$ and $\mathbf{x}_s$ by Proposition 4.6.7. Since positive $d$-vectors in $\mathbf{x}_t$ corresponds with that of $\mathbf{x}_s$, we have $\mathbf{x}_t = \mathbf{x}_s$ by Corollary 4.6.5. ☐

## 4.7 Proof of Theorem 4.4.1 (2)

We prove Theorem 4.4.1 (2). The strategy of this proof is almost the same as Theorem 4.4.1 (1), but we sometimes use the special properties of cluster algebras of rank 2.

For a cluster algebra of rank 2, we may assume that the initial matrix $B$ has the following form without loss of generality:

$$B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}, \quad b,c \in \mathbb{Z}_{\geq 0}, \quad bc \geq 4, \tag{4.7.1}$$

because when $bc \leq 3$, this cluster algebra is of finite type. We name vertices of $\mathbb{T}_2$ by the rule of (2.1.11) and consider a cluster pattern $t_n \mapsto (\mathbf{x}_{t_n}, \mathbf{y}_{t_n}, B_{t_n})$. We abbreviate $\mathbf{x}_{t_n}$ (resp., $\mathbf{y}_{t_n}$, $B_{t_n}, \Sigma_{t_n}$) to $\mathbf{x}_n$ (resp., $\mathbf{y}_n$, $B_n$, $\Sigma_n$). We also abbreviate $d$-vectors, $D$-matrices, $f$-vectors, and $F$-matrices in the same way.

We consider a description of $D$-matrices in the case $n \geq 0$. First, we have

$$D_0^{B;t_0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D_1^{B;t_0} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.7.2}$$

by direct calculation. By [29, (1.13)], if $n > 0$ is even, then we can denote

$$D_n^{B;t_0} = \begin{bmatrix} S_{\frac{n-2}{2}}(u) + S_{\frac{n-4}{2}}(u) & bS_{\frac{n-4}{2}}(u) \\ cS_{\frac{n-2}{2}}(u) & S_{\frac{n-2}{2}}(u) + S_{\frac{n-4}{2}}(u) \end{bmatrix}, \tag{4.7.3}$$

and if $n > 1$ is odd, then we can denote

$$D_n^{B;t_0} = \begin{bmatrix} S_{\frac{n-3}{2}}(u) + S_{\frac{n-5}{2}}(u) & bS_{\frac{n-3}{2}}(u) \\ cS_{\frac{n-3}{2}}(u) & S_{\frac{n-1}{2}}(u) + S_{\frac{n-3}{2}}(u) \end{bmatrix}, \tag{4.7.4}$$

where $u = bc - 2$ and $S_p(u)$ is a (normalized) Chebyshev polynomial of the second kind, that is,

$$S_{-1}(u) = 0, \quad S_0(u) = 1, \quad S_p(u) = uS_{p-1}(t) - S_{p-2}(u)\ (p \in \mathbb{N}). \tag{4.7.5}$$

When $n < 0$, $D_n^{B;t_0}$ is the following matrix:

$$D_n^{B;t_0} = \begin{bmatrix} d_{22;-n}^{-B^\top} & d_{21;-n}^{-B^\top} \\ d_{12;-n}^{-B^\top} & d_{11;-n}^{-B^\top} \end{bmatrix}, \tag{4.7.6}$$

where $d_{ij;-n}^{-B^\top}$ is the $(i,j)$ entry of $D_{-n}^{-B^\top;t_0}$.

We fix any $\mathcal{A}(B)$ of rank 2. Through the rest of this section, unless otherwise noted, we assume that seeds, cluster variables, clusters, $f$-vectors, $d$-vectors, $F$-matrices, and $D$-matrices are those of $\mathcal{A}(B)$. Using the above descriptions, we prove some properties for $d$-vectors.

**Lemma 4.7.1** ([25, Lemma 4.1]). *The initial cluster variables belong to $\Sigma_0$ or $\Sigma_{\pm 1}$. Furthermore, $x_{i;t}$ is not in the initial cluster if and only if $\mathbf{d}_{i;t}$ is positive.*

*Proof.* We prove it in the case $n > 0$. It suffices to show that for any $u \geq 2$ and $p \geq -1$, $S_p(u) \geq 0$ holds and $S_p(u) = 0$ if and only if $p = -1$. The general term of $S_p(u)$ is

$$S_p(u) = \begin{cases} p+1 & \text{if } u = 2; \\ \dfrac{1}{\sqrt{u^2-4}} \left( \left( \dfrac{u+\sqrt{u^2-4}}{2} \right)^{p+1} - \left( \dfrac{u-\sqrt{u^2-4}}{2} \right)^{p+1} \right) & \text{if } u \neq 2. \end{cases} \tag{4.7.7}$$

By direct calculation, we have $S_p(u) \geq 0$. Also, $S_p(u) = 0$ holds if and only if $p = -1$ holds. In the case $n < 0$, we can use the result of the case $n > 0$ by (4.7.6). □

The following corollary is analogous to Corollary 4.6.2:

**Corollary 4.7.2** ([25, Lemma 4.2]). *An $f$-vector $\mathbf{f}_{i;t}$ is the zero-vector if and only if $x_{i;t}$ is in the initial cluster.*

*Proof.* We can prove it in the same way as Corollary 4.6.2: we use Lemma 4.7.1 instead of Lemma 4.6.1. □

The following lemma is analogous to Corollary 4.6.5:

**Lemma 4.7.3** ([25, Lemma 4.3]). *If $\mathbf{d}_{i;t} = \mathbf{d}_{j;s}$, then we have $x_{i;t} = x_{j;s}$.*

*Proof.* Since $d$-vectors are independent of the choice of $\mathbb{P}$, we can assume $\mathbb{P} = \{1\}$. Then using [29, (1.15)] (cf. Section 4.8), we have the expressions of cluster variables induced by $d$-vectors. □

The following proposition is analogous to Corollary 4.6.6:

**Proposition 4.7.4** ([25, Proposition 4.4]). *If $\mathbf{f}_{i;t} = \mathbf{f}_{j;s} \neq \mathbf{0}$, then we have $x_{i;t} = x_{j;s}$.*

*Proof.* We can prove it in the same way as Corollary 4.6.6: we use Corollary 4.7.2 and Lemma 4.7.3 instead of Corollary 4.6.2 and Corollary 4.6.5 respectively. □

The following proposition is analogous to Proposition 4.6.7. Unlike Proposition 4.6.7, we do not need to use the duality for $D$-matrices.

**Proposition 4.7.5** ([25, Proposition 4.5])**.** *For a $D$-matrix $D_n^{B;t_0}$, negative column vectors of $D_n^{B;t_0}$ are uniquely determined by positive column vectors of $D_n^{B;t_0}$.*

*Proof.* When both $d$-vectors in $D_n^{B;t_0}$ are negative vectors, it is clear. Therefore, we can assume that only one $d$-vector is negative. By Lemma 4.7.1, the initial cluster variables only appear in $\Sigma_0$ or $\Sigma_{\pm 1}$. Therefore, if $\mathbf{d}_{1;0} = \mathbf{d}_{1;-1} = -\mathbf{e}_1$ is contained in two $d$-vectors associated with a cluster, then the other is always $\mathbf{d}_{2;-1}$. Similarly, if $\mathbf{d}_{2;0} = \mathbf{d}_{2;1} = -\mathbf{e}_2$ is contained in two $d$-vectors, then the other is always $\mathbf{d}_{1;1}$. By this observation, it suffices to show $\mathbf{d}_{2;-1} \neq \mathbf{d}_{1;1}$. We have $\mathbf{d}_{2;-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{d}_{1;1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by direct calculation. This finishes the proof. □

We are ready to prove Theorem 4.4.1 (2).

*Proof of Theorem 4.4.1 (2).* We can prove it in the same way as Theorem 4.4.1 (1): we use Lemma 4.7.3, Proposition 4.7.4, and Proposition 4.7.5 instead of Corollary 4.6.5, Proposition 4.6.6, and Proposition 4.6.7 respectively. □

## 4.8 Restoration formula of cluster algebras of rank 2

We proved that cluster variables are uniquely determined by their $f$-vectors for cluster algebras of rank 2 in the previous section. In this section, we describe these cluster variables explicitly in the case that coefficients are the principal ones. By this descriptions, we establish a way to restore $F$-polynomials from $f$-vectors. Throughout this section, we assume that $\mathcal{A}(B)$ has the following initial matrix:

$$B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}, \quad b, c \in \mathbb{Z}_{\geq 0}. \tag{4.8.1}$$

We do not assume $bc \geq 4$, thus cluster algebras of finite type and rank 2 ($A_2, B_2, G_2$ Dynkin types) are contained. Unless otherwise noted, we assume that seeds, cluster variables, clusters, $f$-vectors, $d$-vectors, $F$-matrices, and $D$-matrices are those of $\mathcal{A}(B)$.

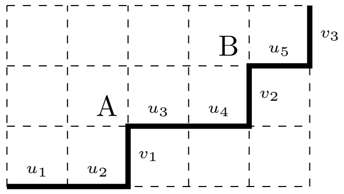
A previous work [29] has given a cluster expansions formula in the case that $\mathbb{P} = \{1\}$. This formula restores the expressions of cluster variables by the initial ones from their $d$-vectors. We start with an explanation of this formula.

We define Dyck Paths and some notations along [29, Section 1]. Let $(a_1, a_2)$ be a pair of non-negative integers. A *Dyck path* of type $a_1 \times a_2$ is a lattice path from $(0,0)$ to $(a_1, a_2)$ and it does not go above the diagonal combining $(0,0)$ with $(a_1, a_2)$. For the Dyck paths of $a_1 \times a_2$ type, there is the *maximal* one $\mathcal{D}^{a_1 \times a_2}$. It is defined by the following property: for any lattice point $A$ on $\mathcal{D}$, there is no lattice points between $A$ and the crosspoint of a vertical line including $A$ and the diagonal combining $(0,0)$ with $(a_1, a_2)$.

For $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$, let $\mathcal{D}_1 = \{u_1, \dots, u_{a_1}\}$ be the set of horizontal edges of $\mathcal{D}$ indexed from left to right, and $\mathcal{D}_2 = \{v_1, \dots, v_{a_2}\}$ be the set of vertical edges of $\mathcal{D}$ indexed from bottom to top.

For any lattice points $A$ and $B$ on $\mathcal{D}$, let $AB$ be the subpath of $\mathcal{D}$ that starts at $A$ and goes along $D$ in the upper right direction until it reaches $B$. If $(a_1, a_2)$ is reached before $B$ is reached, it starts over at $(0,0)$. Now, $(0,0)$ and $(a_1, a_2)$ are considered to be the same point, thus if $A = (a_1, a_2)$, then $AA$ corresponds to the maximal Dyck path. We will call $(AB)_1$ the set of horizontal edges of $AB$ and $(AB)_2$ the set of vertical edges of $AB$. Let $AB^\circ$ be the set of lattice points on the subpath $AB$ except for the endpoints $A$ and $B$.

Figure 4.4: A maximal Dyck path ($(a_1, a_2) = (5, 3)$).




**Example 4.8.1.** We fix $(a_1, a_2) = (5, 3)$, and let $A = (2, 1)$, $B = (4, 2)$. Then

$$(AB)_1 = \{u_3, u_4\},\ (AB)_2 = \{v_2\}, (BA)_1 = \{u_5, u_1, u_2\},\ (BA)_2 = \{v_3, v_1\},$$

and the subpath $AA$ has length 8 (see Figure 4.4).

Next, we define the compatibility in $\mathcal{D}$:

**Definition 4.8.2** ([29, Definition 1.10])**.** For $S_1 \subseteq \mathcal{D}_1$, $S_2 \subseteq \mathcal{D}_2$, we say that the pair $(S_1, S_2)$ is *compatible* if for every $u \in S_1$ and $v \in S_2$, denoting by $E$ the left endpoint of $u$ and $F$ the upper endpoint of $v$, there exists a lattice point $A \in EF^\circ$ such that

$$|(AF)_1| = b|(AF)_2 \cap S_2| \text{ or } |(EA)_2| = c|(EA)_1 \cap S_1|. \tag{4.8.2}$$

We are ready to describe a cluster expansion formula for cluster algebras of rank 2.

**Theorem 4.8.3** ([29, Theorem 1.11])**.** *For every d-vector* $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, *the cluster variable* $x_{\mathbf{d}}$ *corresponding to* $\mathbf{d}$ *is given by the following equation:*

$$x_{\mathbf{d}} = x_1^{-d_1} x_2^{-d_2} \sum_{(S_1, S_2)} x_1^{b|S_2|} x_2^{c|S_1|}, \tag{4.8.3}$$

*where the sum is over all compatible pairs* $(S_1, S_2)$ *in* $\mathcal{D}^{[d_1]_+ \times [d_2]_+}$.

**Remark 4.8.4.** In [29, Theorem 1.11], (4.8.3) is defined for any $(a_1, a_2) \in \mathbb{Z}^2$ and is called a *greedy element.*

We generalize this formula to the principal coefficients version in a way which is analogous to [30]. When a cluster algebra is of rank 2, $g$-vectors are obtained from $d$-vectors:

**Theorem 4.8.5** ([25, Theorem 5.5])**.** *For a g-vector* $\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ *and a d-vector* $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ *of a cluster variable, we have the following equation:*

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} -d_1 \\ cd_1 - d_2 \end{bmatrix}. \tag{4.8.4}$$

*Proof.* This is the spacial case of [20, Theorem 10.12]. □

Using $g$-vectors, we have the following generalization of Theorem 4.8.3:

**Theorem 4.8.6** ([25, Theorem 5.6])**.** *For a d-vector* $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, *the cluster variable* $x_{\mathbf{d}}$ *with the principal coefficients corresponding to* $\mathbf{d}$ *is given by the following equation:*

$$x_{\mathbf{d}} = x_1^{-d_1} x_2^{-d_2} \sum_{(S_1, S_2)} y_1^{[d_1]_+ - |S_1|} y_2^{|S_2|} x_1^{b|S_2|} x_2^{c|S_1|}, \tag{4.8.5}$$

*where the sum is over all compatible pairs* $(S_1, S_2)$ *in* $\mathcal{D}^{[d_1]_+ \times [d_2]_+}$.

*Proof.* When a $d$-vector is the negative, we have (4.8.5) by direct calculation. We assume that a $d$-vector is positive. For any compatible pair $(S_1, S_2) \in \mathcal{D}^{d_1 \times d_2}$, let $a_1(S_1, S_2)$ and $a_2(S_1, S_2)$ be integers satisfying

$$x_{\mathbf{d}} = x_1^{-d_1} x_2^{-d_2} \sum_{(S_1,S_2)} y_1^{a_1(S_1,S_2)} y_2^{a_2(S_1,S_2)} x_1^{b|S_2|} x_2^{c|S_1|}. \tag{4.8.6}$$

Since $x_{\mathbf{d}}$ is homogeneous by the grading (2.3.4), and its degree is $\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} -d_1 \\ cd_1 - d_2 \end{bmatrix}$ by Theorem 4.8.5, the following equation holds for any compatible pair $(S_1, S_2)$:

$$\begin{bmatrix} -d_1 \\ cd_1 - d_2 \end{bmatrix} = - \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + a_1(S_1, S_2) \begin{bmatrix} 0 \\ c \end{bmatrix} + a_2(S_1, S_2) \begin{bmatrix} -b \\ 0 \end{bmatrix} + \begin{bmatrix} b|S_2| \\ c|S_1| \end{bmatrix}. \tag{4.8.7}$$

By solving the equation, we have

$$a_1(S_1, S_2) = d_1 - |S_1|, \quad a_2(S_1, S_2) = |S_2|. \tag{4.8.8}$$

□

By Theorem 4.8.6, definition of $F$-polynomials, and Remark 4.4.4, we have the following restoration formula of $F$-polynomials from $f$-vectors:

**Corollary 4.8.7** ([25, Corollary 5.7])**.** *For a $f$-vector $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, the $F$-polynomial $F_{\mathbf{f}}(\mathbf{y})$ whose maximal degree vector is $\mathbf{f}$ is given by the following formula:*

$$F_{\mathbf{f}}(y_1, y_2) = \sum_{(S_1,S_2)} y_1^{f_1 - |S_1|} y_2^{|S_2|}, \tag{4.8.9}$$

*where the sum is over all compatible pairs $(S_1, S_2)$ in $\mathcal{D}^{f_1 \times f_2}$.*

**Example 4.8.8.** Let $B = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{d} = \mathbf{f} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. If $(S_1, S_2) \in \mathcal{D}^{3 \times 2}$ is compatible, then at least one of the sets $S_1$ and $S_2$ is empty, or $(S_1, S_2)$ is one of pairs in the following list:

$$(\{u_1\}, \{v_2\}), (\{u_2\}, \{v_2\}), (\{u_3\}, \{v_1\}). \tag{4.8.10}$$

Then we have an expression of the cluster variable $x_{\mathbf{d}}$ corresponding to $d$-vector $\mathbf{d}$ in $\mathcal{A}_{\bullet}(B)$ as follows:

$$x_{\mathbf{d}} = \frac{x_1^8 y_1^3 y_2^2 + 2x_1^4 y_1^3 y_2 + y_1^3 + 3x_1^4 x_2 y_1^2 y_2 + 3x_2 y_1^2 + 3x_2^2 y_1 + x_2^3}{x_1^3 x_2^2}. \tag{4.8.11}$$

Also we have the $F$-polynomial $F_{\mathbf{f}}(\mathbf{y})$ corresponding to the $f$-vector $\mathbf{f}$ as follows:

$$F_{\mathbf{f}}(\mathbf{y}) = y_1^3 y_2^2 + y_1^3 y_2 + y_1^3 + y_1^2 y_2 + y_1^2 + y_1 + 1. \tag{4.8.12}$$

# 5 Compatibility degree of cluster complexes

This chapter is based on joint work with Changjian Fu [21]. In this chapter, we consider a generalization of the compatibility degree of cluster complexes. The compatibility degree $(\cdot \parallel \cdot)$ is function on pairs of cluster variables, and originally introduced by Fomin and Zelevinsky. In [18, 19], the compatibility degree is defined in cluster algebra of finite type by using finite root systems. Fomin and Zelevinsky classify cluster algebras of finite type by using this degree. We call it the *classical* compatibility (See Section 5.3.1). The classical compatibility degree has the following properties:

(1) There is a cluster $\mathbf{x}$ containing both $x$ and $x'$ if and only if $(x \parallel x') = (x' \parallel x) = 0$.

(2) There is a mutation $\mu_k$ exchanging $x$ for $x'$ (or $x'$ for $x$) if and only if $(x \parallel x') = (x' \parallel x) = 1$.

After Fomin and Zelevinsky's work, Ceballos and Pilaud found that the classical compatibility degree can be also defined by using $d$-vectors in [10], and Cao and Li generalized it in [9] from cluster algebras of finite type to general cluster algebras.

On the other hand, as we see in Section 4.4, in cluster algebra of finite type, $d$-vectors and $f$-vectors are coincide. Therefore, the classical compatibility degree can be also defined by using $f$-vectors. In this chapter, we introduce a generalization of the classical one from from cluster algebras of finite type to general cluster algebras by using $f$-vectors (Section 5.3.2).

We consider comparing compatibility degree derived from $f$-vectors with that from $d$-vectors. For example, both compatibility degree derived from $f$-vectors and $d$-vectors satisfy the condition (1) above (Theorems 5.3.17 and 5.3.18), however $d$-vector's one does not satisfy the condition (2) (See Examples 5.3.26, 5.3.27). On the other hand, we can find that $f$-vector's one satisfies the condition (2) in many cluster algebras and counterexamples have not been obtained yet (See Theorems 5.3.23, 5.3.24, and 5.3.25).

Moreover, we will see in this chapter that the compatibility degree derived from the $f$-vector inherit various properties of the classical one.

## 5.1 Scattering diagrams and enough $g$-pairs property

As preparation, we introduce Scattering diagrams and enough $g$-pairs property.

### 5.1.1 Scattering diagrams

The scattering diagrams were introduced in [24] to study the canonical basis of cluster algebras. In this paper, we give only a few definitions and properties used in proofs along [9, 31].

**Definition 5.1.1.** We fix a skew-symmetrizable matrix $B$ whose order is $n$, and let $S$ be a skew-symmetrizer of $B$. For the matrix $B$, We call the pair $(\mathbf{v}, W)$ a *wall* associated with $B$, where $\mathbf{v} \in \mathbb{Z}^n_{\geq 0}$ is a non-zero vector, and $W$ is a convex cone spanning $\mathbf{v}^{\perp} := \{\mathbf{m} \in \mathbb{R}^n \mid \mathbf{v}^{\top} S\mathbf{m} = 0\}$. We say that a set $\mathfrak{D}(B)$ of walls associated with $B$ is a *scattering diagram* of $B$, and each connected compartment of $\mathbb{R}^n - \mathfrak{D}(B)$ is a *chamber* of $\mathfrak{D}(B)$.

The open half space $\{\mathbf{m} \in \mathbb{R}^n \mid \mathbf{v}^\top S\mathbf{m} > 0\}$ is the *green side* of $W$, and $\{\mathbf{m} \in \mathbb{R}^n \mid \mathbf{v}^\top S\mathbf{m} < 0\}$ is the *red side* of $W$.

**Definition 5.1.2.** Let $\mathfrak{D}(B)$ be a scattering diagram in $\mathbb{R}^n$. We say that a continued path $\rho\colon [0,1] \to \mathbb{R}^n$ is *finite transverse* at $\mathfrak{D}(B)$ if $\rho$ satisfies the following conditions:

- Neither $\rho(0)$ or $\rho(1)$ is in any wall $W$ of $\mathfrak{D}(B)$.
- The image of $\rho$ crosses each wall $W$ of $\mathfrak{D}(B)$ transversely.
- The image of $\rho$ crosses finitely many walls of $\mathfrak{D}(B)$ and it does not cross the boundaries of walls or intersection of walls spanning two distinct hyperplanes.

We fix a skew-symmetrizable matrix $B$ of order $n$. We take a skew-symmetrizer $S = \mathrm{diag}(s_1, \dots, s_n)$ of $B$ so that $s_1, \dots, s_n$ are relatively prime. We consider

$$R = \mathbb{Q}[x_1^{\pm1}, \dots, x_n^{\pm1}][[y_1, \dots, y_n]],$$

that is, formal power series in the variables $y_1, \dots, y_n$ with coefficients in $\mathbb{Q}[x_1^{\pm1}, \dots, x_n^{\pm1}]$. For $\mathbf{v} = (v_1, \dots, v_n)^\top \in \mathbb{Z}^n_{\geq 0}$, we define the *formal elementary transformation* $E_{\mathbf{v}} \in \mathrm{Aut}(R)$ as

$$E_{\mathbf{v}}(\mathbf{x}^{\mathbf{w}}) = (1 + \mathbf{x}^{B\mathbf{v}}\mathbf{y}^{\mathbf{v}})^{\frac{\mathbf{v}^\top S\mathbf{m}}{\gcd(s_1v_1, \dots, s_nv_n)}}, \quad E_{\mathbf{v}}(\mathbf{y}^{\mathbf{w}'}) = \mathbf{y}^{\mathbf{w}'}$$

with the inverse

$$E_{\mathbf{v}}^{-1}(\mathbf{x}^{\mathbf{w}}) = (1 + \mathbf{x}^{B\mathbf{v}}\mathbf{y}^{\mathbf{v}})^{-\frac{\mathbf{v}^\top S\mathbf{m}}{\gcd(s_1v_1, \dots, s_nv_n)}}, \quad E_{\mathbf{v}}^{-1}(\mathbf{y}^{\mathbf{w}'}) = \mathbf{y}^{\mathbf{w}'}.$$

Let $\rho$ be a finite transverse path of a scattering diagram $\mathfrak{D}(B)$, and we assume that $\rho$ crosses walls of $\mathfrak{D}(B)$ in the following order:

$$(\mathbf{v}_1, W_1), \dots, (\mathbf{v}_n, W_n).$$

We define the *path-ordered product of elementary transformations* of $\rho$ as

$$E_\rho = E_{\mathbf{v}_s}^{\varepsilon_s} \circ \cdots \circ E_{\mathbf{v}_1}^{\varepsilon_1} \in \mathrm{Aut}(R),$$

where $\varepsilon_i = 1$ (resp., $\varepsilon_i = -1$) if $\rho$ crosses $W_i$ from its green side to its red side (resp., from its red side to its green side). A scattering diagram $\mathfrak{D}(B)$ which has finitely many walls is *consistent* if for any finite transverse path $\rho$ whose starting point coincides with terminal point, $E_\rho = 1$ holds. Moreover, it is generalized to scattering diagrams which have infinitely many walls. Let $I$ be a monomial ideal of $\mathbb{Q}[[y_1, \dots, y_n]]$. For each $\mathbf{v} \in \mathbb{Z}_0^n$, $E_{\mathbf{v}}$ and $E_{\mathbf{v}}^{-1}$ induce automorphisms of $R/I$. A finite scattering diagram is *consistent* mod $I$ if the path-ordered product associated to every transverse loop is a trivial automorphism of $R/I$. The *reduction* $\mathfrak{D}/I$ of a scattering diagram $\mathfrak{D}$ is obtained by deleting any wall of the form $(\mathbf{v}, W)$ with $y^n \in I$. A scattering diagram $\mathfrak{D}(B)$ which has infinitely many walls is *consistent* if for any monomial ideal $I$ of $\mathbb{Q}[[y_1, \dots, y_n]]$with finite dimensional quotient, the reduction $\mathfrak{D}/I$ is finite and consistent mod $I$.

**Lemma 5.1.3** ([24, Theorem 1.12])**.** *Let $B$ be a skew-symmetrizable matrix of order $n$. There exists a unique consistent diagram $\mathfrak{D}_0(B)$ satisfying the following conditions:*

- *For any $i \in \{1, \dots, n\}$, $(\mathbf{e}_i, \mathbf{e}_i^\perp)$ are walls of $\mathfrak{D}_0(B)$.*
- *For any else walls $(\mathbf{v}, W)$ of $\mathfrak{D}_0(B)$, $B\mathbf{v} \notin W$ holds.*

We note that since $\mathbf{v} \in \mathbb{Z}_{\geq 0}$, we have $W \cap (\mathbb{R}_{>0})^n = W \cap (\mathbb{R}_{<0})^n = \emptyset$. Therefore, $(\mathbb{R}_{>0})^n$ and $(\mathbb{R}_{<0})^n$ are chambers of $\mathfrak{D}_0(B)$, and we call them *all positive chamber* and *all negative chamber* respectively. If there exists a path which is finitely transverse from all positive chamber to a chamber $\mathcal{C}$, then we call $\mathcal{C}$ a *reachable chamber*.

**Lemma 5.1.4** ([24, Lemma 2.10])**.** *Let $\mathcal{A}(B)$ be a cluster algebra of rank $n$ whose initial matrix is $B$. Each reachable chamber of $\mathfrak{D}_0(B)$ is expressed by the following form:*

$$\mathbb{R}_{>0}\mathbf{g}_1 + \cdots + \mathbb{R}_{>0}\mathbf{g}_n, \tag{5.1.1}$$

*where $G = (\mathbf{g}_1, \ldots, \mathbf{g}_n)$ is one of $G$-matrices of $\mathcal{A}(B)$.*

### 5.1.2 Enough $g$-pairs property

The definition of $g$-pairs was introduced in [9] for cluster algebras with principal coefficients with the aim to study $d$-vectors. By our convention of $g$-vectors and Proposition 2.3.8, it generalizes to cluster algebras with arbitrary coefficients directly.

**Definition 5.1.5.** Let $\mathcal{A}(B)$ be a cluster algebra of rank $n$ with the rooted vertex $t_0$, and $I = \{i_1, \cdots, i_p\}$ be a subset of $\{1, 2, \cdots, n\}$.

(1) We say that a seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ of $\mathcal{A}(B)$ is *connected with* $(\mathbf{x}, \mathbf{y}, B)$ *by an $I$-sequence*, if there exists a composition of mutations $\mu_{k_s} \cdots \mu_{k_2}\mu_{k_1}$, such that $(\mathbf{x}_t, \mathbf{y}_t, B_t) = \mu_{k_s} \cdots \mu_{k_2}\mu_{k_1}(\mathbf{x}, \mathbf{y}, B)$, where $k_1, \ldots, k_s \in I$.

(2) We say that a cluster $\mathbf{x}_t$ of $\mathcal{A}(B)$ is *connected with* $\mathbf{x}$ *by an $I$-sequence*, if there exists a seed containing the cluster $\mathbf{x}_t$ such that this seed is connected with a seed containing the cluster $\mathbf{x}$ by an $I$-sequence.

**Definition 5.1.6.** Let $\mathcal{A}(B)$ be a cluster algebra of rank $n$ with rooted vertex $t_0$, and $I = \{i_1, \ldots, i_m\}$ be a subset of $\{1, \ldots, n\}$. For any two cluster $\mathbf{x}_t$ and $\mathbf{x}'_t$, the pair $(\mathbf{x}_t, \mathbf{x}'_t)$ is called a *$g$-pair* along $I$ if it satisfies the following conditions:

- $\mathbf{x}_{t'}$ is connected with $\mathbf{x}_{t_0}$ by an $I$-sequence,
- for any cluster monomial $\mathbf{x}_t^{\mathbf{v}}$ in $\mathbf{x}_t$, there exists a cluster monomial $\mathbf{x}_{t'}^{\mathbf{v}'}$ with $v'_i = 0$ for $i \notin I$ such that

$$\pi_I(g(\mathbf{x}_t^{\mathbf{v}})) = \pi_I(g(\mathbf{x}_{t'}^{\mathbf{v}'})) \tag{5.1.2}$$

  where $g(\mathbf{x}_t^{\mathbf{v}})$ and $g(\mathbf{x}_{t'}^{\mathbf{v}'})$ are $g$-vectors of the cluster monomials $\mathbf{x}_t^{\mathbf{v}}$ and $\mathbf{x}_{t'}^{\mathbf{v}'}$ and $\pi_I$ is a canonical projection from $\mathbb{R}^n$ to $\mathbb{R}^{|I|}$.

**Theorem 5.1.7** ([9, Theorems 4.8, 5.5])**.** *Let $\mathcal{A}(B)$ be a cluster algebra of rank $n$ with arbitrary coefficients. For any subset $I \subset \{1, \ldots, n\}$ and any cluster $\mathbf{x}_t$, there exists a cluster $\mathbf{x}_{t'}$ such that $(\mathbf{x}_t, \mathbf{x}_{t'})$ is a $g$-pair along $I$. Furthermore, if $(\mathbf{x}_t, \mathbf{x}_s)$ is also a $g$-pair along $I$, then we have $\mathbf{x}_{t'} = \mathbf{x}_s$.*

We often refer to Theorem 5.1.7 the *enough $g$-pairs property* of $\mathcal{A}(B)$.

## 5.2 $d$-vectors versus $f$-vectors

Today, it is known that $d$-vectors and $f$-vectors are different vectors essentially, but it was pointed out that these two classes of vectors have similarities. Fomin and Zelevinsky expected that $d$-vectors of non-initial cluster variables coincide with $f$-vectors of them in any cluster algebras [20, Conjecture 7.17]. A counterexample of it was given by [22, Example 6.7], but it is known that this conjecture is true in cluster algebras of rank 2 and of finite type (Theorem 4.4.3 and Remark 4.4.4). In this section, we show a weaker similarity of these two families of vectors than (4.4.1) in general cluster algebras.

Fomin and Zelevinsky conjectured the following properties about $d$-vectors in [20, Conjecture 7.4], which was proved in [9]:

**Theorem 5.2.1** ([9, Theorem 6.3]). *Let $\mathcal{A}(B)$ be any cluster algebra. The following statements hold:*

*(1) A cluster variable $x_{i;t}$ is not one of the initial cluster variables if and only if $\mathbf{d}_{i;t}$ is a non-negative vector.*

*(2) The $(i,j)$th entry $d^{B_{t'};t'}_{ij;t}$ of $D^{B_{t'};t'}_{t}$ equals the $(k,\ell)$th entry $d^{B_s;s}_{k\ell;s'}$ of $D^{B_s;s}_{s'}$ if $x_{i;t'} = x_{k;s'}$ and $x_{j;t} = x_{\ell:s'}$.*

*(3) There is a cluster containing $x_{i;t}(\neq x_k)$ and $x_k$ if and only if $d_{ki;t} = 0$.*

We prove the similar theorem of it about $f$-vectors:

**Theorem 5.2.2** ([21, Theorem 3.3]). *Let $\mathcal{A}(B)$ be any cluster algebra. The following statements hold:*

*(1) A cluster variable $x_{i;t}$ is not one of the initial cluster variables if and only if $\mathbf{f}_{i;t}$ is a non-zero vector.*

*(2) The $(i,j)$th entry $f^{B_t;t}_{ij;t'}$ of $F^{B_t;t}_{t'}$ equals the $(k,\ell)$th entry $f^{B_s;s}_{k\ell;s'}$ of $F^{B_s;s}_{s'}$ if $x_{i;t} = x_{k;s}$ and $x_{j;t'} = x_{\ell:s'}$.*

*(3) There is a cluster containing $x_{i;t}$ and $x_k$ if and only if $f_{ki;t} = 0$.*

*(4) A cluster $\mathbf{x}_t$ contains $x_k$ if and only if entries of the $k$th row of $F^{B:t_0}_{t}$ are all $0$.*

**Remark 5.2.3.** Because of Proposition 2.3.8, it suffices to show Theorem 5.2.2 in the case of cluster algebras with principal coefficients. In fact, for example, we assume that Theorem 5.2.2 (1) holds in principal coefficient cases and there is a cluster algebra $\mathcal{A}_1(B)$ with coefficients in $\mathbb{P}_1$ which does not satisfy Theorem 5.2.2 (1). Then, there is a non initial cluster variable $x(1)$ whose $f$-vector is zero. Then, for any $i$, we have $x(1) \neq x_i(1)$ in $\mathcal{A}_1(B)$. Since $f$-vectors are independent of coefficients, we have a cluster variable $x$ whose $f$-vector is zero in $\mathcal{A}_\bullet(B)$ such that for any $i$, we have $x \neq x_i$ by Proposition 2.3.8. This is a contradiction. Therefore, we assume $\mathcal{A}(B) = \mathcal{A}_\bullet(B)$ in the proof of Theorem 5.2.2.

The first statement implies there are no cluster variables whose expansions in the initial cluster variables are Laurent monomials except for the initial cluster variables, and this is a generalization of Corollaries 4.6.2 and 4.7.2. The second statement is important for defining the compatibility degree in the next section. We prove Theorem 5.2.2 in the rest of this section.

To prove Theorem 5.2.2 (1), we use the following two lemmas.

**Lemma 5.2.4** ([9, Lemma 7.3]). *Let $\mathcal{A}(B)$ be any cluster algebra, and we fix any cluster variable $x$ of $\mathcal{A}(B)$. If, for all $i \in \{p+1,\dots,n\}$, there exists a cluster containing $x$ and $x_i$, then there exists a cluster containing $x$ and initial cluster variables $x_{p+1},\dots,x_n$.*

**Lemma 5.2.5** ([9, Lemma 6.2]). *Let $\mathcal{A}(B)$ be any cluster algebra. We fix any subset $X$ of a cluster. Then, all seeds which have a cluster containing $X$ form a connected component of exchange graph of $\mathcal{A}(B)$.*

*Proof of Theorem 5.2.2 (1).* "If" part is clear. We prove "Only if "part by proving the contraposition of the statement. Since $\mathbf{f}_{i;t} = 0$ and by (3.1.8), we have $F_{i;t}(\mathbf{y}) = 1$. Then we have $x_{i;t} = \prod_j x_j^{g_{ji}}$ by the separation formula (2.3.10). We note that $\mathbf{d}_{i;t}$ coincides with $-\mathbf{g}_{i;t}$. According to Theorem 5.2.1 (1), we have $\mathbf{g}_{i;t} \in \mathbb{Z}^n_{\leq 0}$ or $\mathbf{g}_{i;t} = \mathbf{e}_l$. We assume $\mathbf{g}_{i;t} \in \mathbb{Z}^n_{\leq 0}$. Without loss of generality, we can assume that

$$\mathbf{g}_{i;t} = (a_1, \cdots, a_k, 0, \cdots, 0)^\top \in \mathbb{Z}^n_{\leq 0},\ a_1, \ldots, a_k < 0.$$

By Theorem 5.2.1 (3) and Lemma 5.2.4, there exists a cluster $\mathbf{x}$ such that $\mathbf{x}$ contains $x_{i;t}$ and initial cluster variables $x_{k+1}, \ldots, x_n$. We set $\mathbf{x} = \{x_{i;t}, z_2, \cdots, z_k, x_{k+1}, \cdots, x_n\}$. By the sign-coherence of $G$-matrices (Theorem 3.2.7), the first $k$ components of $g$-vectors of $z_2, \ldots, z_k$ lie in $\mathbb{Z}_{\leq 0}$. Then $G$-matrix associated with $\mathbf{x}$ is a partition matrix $\begin{bmatrix} G & O \\ X & E_{n-k} \end{bmatrix}$, where each column of $G$ lies in $\mathbb{Z}^k_{\leq 0}$. We consider the cluster algebra $\mathcal{A}'$ by freezing the initial cluster variables $x_{k+1}, \cdots, x_n$ of $\mathcal{A}(B)$. One can show that the $G$-matrix of the cluster $\tilde{\mathbf{x}} = \{x_{i;t}, z_2, \cdots, z_k\}$ of $\mathcal{A}'$ is $G$. We note that columns of $G$ are linearly independent because of $\det G = \pm 1$. Then, according to Lemma 5.1.4, the chamber induced by $G$ coincides with the all negative chamber. Therefore, each column of $G$ is a scalar multiplication of a unit vector and we have $k = 1$. Thus we have $\det G = -1$, $\mathbf{g}_{i;t} = -\mathbf{e}_1$ and $x_{i;t} = 1/x_1$. We note that $\mathbf{d}_{i;t} = \mathbf{e}_1$ and thus there exists a cluster $\mathbf{x}'$ such that $\mathbf{x}' = \{1/x_1, x_2 \ldots, x_n\}$ by Theorem 5.2.1 (1) again. However, by Lemma 5.2.5, clusters containing $\{x_2, \ldots, x_n\}$ are the initial cluster or a cluster which is obtained by mutating the initial cluster in direction 1. Clearly, $\mathbf{x}'$ is not the initial cluster. Since the numerator of $\mathbf{x}'$ does not have any initial cluster variables, the entries of first column and the first row of $B$ are all 0. In this case, $\mu_1(x_1) = (y_1 + 1)/x_1$. Thus $\mathbf{x}' \neq \mu_1(\mathbf{x}_{t_0})$. This is a contradiction. Therefore, we have $\mathbf{g}_{i;t} = -\mathbf{d}_{i;t} = \mathbf{e}_l$. Because of Theorem 5.2.1 (1), we have $x_{i;t} = x_l$. This finishes the proof. □

Next, we prove Theorem 5.2.2 (2). This statement follows from the fact that the $(i, j)$ entry of a $F$-matrix is invariant by mutations in direction $k$ such that $k \neq j$ and by initial-seed mutations in direction $\ell$ such that $\ell \neq i$. We prepare a lemma. This gives an recursion of the $F$-matrices by an initial-seed mutation.

*Proof of Theorem 5.2.2 (2).* Since $x_{i;t} = x_{k;s}$, according to Lemma 5.2.5, there exists a permutation $\sigma$ of indices such that $\sigma(k) = i$ and a vertex $s_0 \in \mathbb{T}_n$ such that the seed $\Sigma_{s_0}$ is the permutation of $\Sigma_t$ by the permutation $\sigma$, that is

$$x_{u;s} = x_{\sigma(u);s_0}, y_{u;s} = y_{\sigma(u);s_0}, b_{uv;s} = b_{\sigma(u),\sigma(v);s_0}$$

for all $u$ and $v$. Moreover, the seed $\Sigma_{s_0}$ is connected with $\Sigma_t$ by a $\{1, \ldots, n\}\backslash\{i\}$-sequence.

By definition of $f$-vectors, for any cluster variable $z$, the $f$-vector of $z$ with respect to the seed $\Sigma_{s_0}$ is the permutation of the $f$-vector of $z$ with respect to the seed $\Sigma_s$ by $\sigma$. In particular, $f^{B_s;s}_{k\ell;s'} = f^{B_{s_0};s_0}_{i\ell;s'}$. On the other hand, since $x_{j;t'} = x_{\ell;s'}$, we have $f^{B_t;t}_{i\ell;s'} = f^{B_t;t}_{ij;t'}$. By Theorem 3.2.9, the initial-seed mutation at $m$ only change the $m$th row of $F$-matrices. Therefore, we have $f^{B_{s_0};s_0}_{i\ell;s'} = f^{B_t;t}_{i\ell;s'}$. Putting all of these together, we obtain

$$f^{B_t;t}_{ij;t'} = f^{B_s;s}_{k\ell;s'}.$$

□

Let us prove Theorem 5.2.2 (3). The following fact is known.

**Lemma 5.2.6** ([9, Lemma 5.2]). *Suppose that $\mathcal{A}(B)$ is an arbitrary cluster algebra of rank $n$, and $(\mathbf{x}_t, \mathbf{x}_{t'})$ is a $g$-pair along $I = \{1, \cdots, n\}\backslash\{k\}$. Let $G_t^{B_{t'};t'} = (g'_{ij})$ and $\mathbf{d}^{B_{t'};t'}_{i;t} = (d'_1, \cdots, d'_n)^\top$ be the $d$-vector of $x_{i;t}$ with respect to $\boldsymbol{x}_{t'}$. We have that*

*(1)* $g'_{ki} > 0$ *if and only if* $d'_k = -1$*, and if only if* $x_{i;t} \in \mathbf{x}_{t'}$ *and* $x_{i;t} = x_{k;t'}$*.*

*(2)* $g'_{ki} = 0$ *if and only if* $d'_k = 0$*, and if only if* $x_{i;t} \in \mathbf{x}_{t'}$ *and* $x_{i;t} \neq x_{k;t'}$*.*

*(3)* $g'_{ki} < 0$ *if and only if* $d'_k > 0$*, and if only if* $x_{i;t} \notin \mathbf{x}_{t'}$*.*

We consider a similar lemma to Lemma 5.2.6 for the $f$-vectors:

**Lemma 5.2.7** ([21, Lemma 3.9])**.** *Suppose that* $\mathcal{A}(B)$ *is an arbitrary cluster algebra of rank* $n$*, and* $(\boldsymbol{x}_t, \boldsymbol{x}_{t'})$ *is a* $g$*-pair along* $I = \{1, \cdots, n\}\backslash\{k\}$*. Let* $\mathbf{f}_{i;t}^{B_{t'};t'} = (f'_1, \cdots, f'_n)^\top$ *be the* $f$*-vector of* $x_{i;t}$ *with respect to* $\mathbf{x}_{t'}$*. We have that*

*(1)* $g'_{ki} \geq 0$ *if and only if* $f'_k = 0$*, and if only if* $x_{i;t} \in \mathbf{x}_{t'}$*.*

*(2)* $g'_{ki} < 0$ *if and only if* $f'_k > 0$*, and if only if* $x_{i;t} \notin \mathbf{x}_{t'}$*.*

*Proof of Lemma 5.2.7.* By Lemma 5.2.6, it suffices to show that $g'_{ki} < 0$ implies $f'_k > 0$. By Lemma 3.2.8, $g'_{ki} < 0$ implies $h_{ki;t}^{B';t'} < 0$. Then, we have $f'_k > 0$ by definition of the $H$-matrix. □

*Proof of Theorem 5.2.2 (3).* We set $I = \{1, \ldots, n\}\backslash\{k\}$. First, we prove "only if" part. Let $\mathbf{x}_{t'}$ be a cluster such that $\mathbf{x}_{t'}$ contains both $x_{i;t}$ and $x_k$. Then, according to Lemma 5.2.5, $\mathbf{x}_{t'}$ is connected with $\mathbf{x}_{t_0}$ by an $I$-sequence. We set $\mathbf{x}_{t'} = \mu(\mathbf{x}_{t_0})$. If we regard $\mathbf{x}_{t'}$ as the initial cluster, then $f_{ki;t}^{B_{t'};t'} = 0$. We can change the initial cluster from $\mathbf{x}_{t'}$ to $\mathbf{x}_{t_0}$ by initial-seed mutation induced by $\mu^{-1}$. Then, by Theorem 5.2.2 (2), we have $f_{ki;t} = f_{ki;t}^{B_{t'};t'} = 0$. Second, we prove "if" part. Let $\mathbf{x}_s$ be a cluster containing the cluster variable $x_{i;t}$. By Lemma 5.1.7, there is a cluster $\mathbf{x}_{s'}$ such that $(\mathbf{x}_s, \mathbf{x}_{s'})$ is a $g$-pair along $I$. Then $x_{k;s'} = x_k$ holds because $\mathbf{x}_{s'}$ is connected with $\mathbf{x}_{t_0}$ by an $I$-sequence. Since $f_{ki;t} = 0$ implies $f_{ki;t}^{B_{s'};s'} = 0$ by Theorem 5.2.2 (2), we have $x_{i;t} \in \mathbf{x}_{s'}$ by Lemma 5.2.7. This finishes the proof. □

*Proof of Theorem 5.2.2 (4).* "Only if" part follows from "Only if" part of Theorem 5.2.2 (3). We prove "if" part. By Theorem 3.2.10, the transposition of $F$-matrix $F_t^{B;t_0}$ is another $F$-matrix $F_{t_0}^{B_t^\top;t}$. By assumption, the $k$th column of $F_{t_0}^{B_t^\top;t}$ is the zero vector. By Theorem 5.2.2 (1), a cluster variable of $\mathcal{A}(B_t^\top)$ associated with this column is an initial cluster variable. Then, by Theorem 5.2.2 (3), there is a $j \in \{1, \ldots, n\}$ such that all entries of the $j$th row of $F_{t_0}^{B_t^\top;t}$ are 0. This implies the $j$th column of $F_t^{B;t_0}$ is the zero vector. Therefore, all entries of $j$th column and $k$th row of $F_t^{B;t_0}$ are all 0. By Theorem 5.2.2 (1), $\mathbf{x}_t$ has at least one initial cluster variable. We show that one of these initial cluster variable is $x_k$. We assume that $x_k \notin \mathbf{x}_t$. Then, $\mathbf{x}_t$ has an initial cluster variable which is not $x_k$. We assume that this cluster variable is $x_{k'}$. Then by Theorem 5.2.2 (3), the $k'$th column of $F_{t_0}^{B_t^\top;t}$ is the zero vector. In the same way as the previous argument, there exists a $j' \in \{1, \ldots, n\}$ such that the $j'$ column of $F_t^{B;t_0}$ is the zero vector. Since cluster variables in a cluster are algebraically independent, we note that $j \neq j'$. Therefore, all entries of the $j, j'$th columns and the $k, k'$th rows of $F_t^{B;t_0}$ are all 0. By Theorem 5.2.2 (1) again, $\mathbf{x}_t$ has at least two initial cluster variables. By assumption, $\mathbf{x}_t$ has a cluster variable which is neither $x_k$ nor $x_{k'}$. By repeating this argument, we have $F_t^{B;t_0} = O$. Therefore by Theorem 5.2.2 (1), $\mathbf{x}_t = \mathbf{x}_{t_0}$. This conflicts with the assumption. □

In particular, Theorem 5.2.2 (2) admits the *compatibility degree* between two cluster variables, and Theorem 5.2.2 (3) implies that the *compatibility property* holds. See Section 5.3.2.

By these theorems, we have the following corollary:

**Corollary 5.2.8** ([21, Corollary 3.13]). *For any cluster algebra $\mathcal{A}(B)$, $f_{ij;t} = 0$ if and only if $d_{ij;t} = 0$ or $-1$.*

*Proof.* It follows from Theorem 5.2.1 (1),(3) and Theorem 5.2.2 (3). □

The property of $d$-vectors corresponding to Theorem 5.2.2 (4) had not been known. However, we obtain it by using Theorem 5.2.2 (4) and Corollary 5.2.8.

**Corollary 5.2.9** ([21, Corollary 3.14]). *For any cluster algebra $\mathcal{A}(B)$, all entries of the $k$th row of $D_t^{B;t_0}$ are all non-positive if and only if $\mathbf{x}_t$ contains $x_k$.*

*Proof.* "If" part follows from Theorem 5.2.1 (3). We show "only if" part. By Corollary 5.2.8, the $k$th row of $F_t^{B;t_0}$ are all 0. By Theorem 5.2.2 (4), $\mathbf{x}_t$ contains $x_k$. This finishes the proof. □

## 5.3 Compatibility degree and its properties

The classical compatibility degree was introduced to define the generalized associahedron. This is a function on the set of pairs of roots, and the generalized associahedra are simplicial complexes whose simplexes are sets consisting of roots such that each classical compatibility degree of a pair of roots is 0. In this section, we generalize it to a function on pairs of cluster variables in a different way from [9] by using $f$-vectors and give some properties of the generalized one, the compatibility degree.

### 5.3.1 Classical compatibility degree and generalized associahedra

In this subsection, we explain the *classical compatibility degree* and the *generalized associahedra* introduced by [19]. Let $\Phi$ be a root system of finite type. We denote by $\Phi_{\geq -1}$ the set of almost positive roots, that is, the union of all negative simple roots and all positive roots. Let $C_\Phi$ be a Cartan matrix corresponding to $\Phi$ and $\Gamma_\Phi$ be a Dynkin graph corresponding to $\Phi$. Denote by $I$ the set of vertices of $\Gamma_\Phi$. Let $I_+$ be a maximal set of vertices such that each two vertices in $I_+$ are not neighbors on $\Gamma_\Phi$, and we also define $I_-$ as $I_- = I - I_+$. We remark that $I_-$ is also the maximal set of vertices such that each two vertices in $I_-$ are not neighbors on $\Gamma_\Phi$ because $\Gamma_\Phi$ is a bipartite graph. The sign function $\varepsilon\colon I \to \{\pm 1\}$ is defined by

$$\varepsilon(i) = \begin{cases} 1 & \text{if } i \in I_+ \\ -1 & \text{if } i \in I_- \end{cases} \tag{5.3.1}$$

Next, we define $t_\pm$ which are compositions of simple reflections as follows:

$$t_+ = \prod_{\varepsilon(i)=1} s_i, \quad t_- = \prod_{\varepsilon(i)=-1} s_i. \tag{5.3.2}$$

We define transformations $\tau_\pm\colon \Phi_{\geq -1} \to \Phi_{\geq -1}$ as follows:

$$\tau_+(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_j,\ \varepsilon(j) = -1; \\ t_+(\alpha) & \text{otherwise,} \end{cases} \tag{5.3.3}$$

$$\tau_-(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_j,\ \varepsilon(j) = 1; \\ t_-(\alpha) & \text{otherwise.} \end{cases} \tag{5.3.4}$$

For $k \in \mathbb{Z}$ and $i \in I$, we abbreviate

$$\alpha(k;i) = (\tau_-\tau_+)^k(-\alpha_i). \tag{5.3.5}$$

In particular, $\alpha(0;i) = -\alpha_i$ for all $i$ and $\alpha(\pm 1;i) = \alpha_i$ for $i \in I_{\mp}$.

Let $h$ be the Coxeter number of $\Phi$ and $w_\circ$ be the longest element of the Weyl group of $\Phi$. Let $i \mapsto i^*$ denote the involution on $I$ defined by $-\alpha_{i^*} := w_\circ(\alpha_i)$. It is known that this involution preserves each of the sets $I_+$ and $I_-$ when $h$ is even, and interchanges them when $h$ is odd.

**Proposition 5.3.1** ([19, Proposition 2.5])**.**

*(1) Suppose $h = 2e$ is even. Then the map $(k,i) \mapsto \alpha(k;i)$ restricts to a bijection*

$$[0,e] \times I \to \Phi_{\geq -1}. \tag{5.3.6}$$

*Furthermore, $\alpha(e+1;i) = -\alpha_{i^*}$ for any $i$.*

*(2) Suppose $h = 2e+1$ is odd. Then the map $(k,i) \mapsto \alpha(k;i)$ restricts to a bijection*

$$([0,e+1] \times I_-) \cup ([0,e] \times I_+) \to \Phi_{\geq -1}. \tag{5.3.7}$$

*Furthermore, $\alpha(e+2;i) = -\alpha_{i^*}$ for $i \in I_-$, and $\alpha(e+1;i) = -\alpha_{i^*}$ for $i \in I_+$.*

By this proposition, we can express any root $\beta \in \Phi_{\geq -1}$ with $\beta = \tau(-\alpha_i)$, where $\tau$ is a composition of $\tau_+$ and $\tau_-$, and $-\alpha_i$ is a negative simple root. We consider a function $(\cdot \parallel \cdot)_{\mathrm{cl}} \colon \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ characterized by the following property: For any negative simple root $-\alpha_i$ and any root $\beta$, we have

$$(-\alpha_i \parallel \beta)_{\mathrm{cl}} = [(\beta : \alpha_i)]_+, \tag{5.3.8}$$

and for any roots $\alpha, \beta$, we have

$$(\alpha \parallel \beta)_{\mathrm{cl}} = (\tau_\varepsilon(\alpha) \parallel \tau_\varepsilon(\beta))_{\mathrm{cl}}, \tag{5.3.9}$$

where$(\beta : \alpha_i)$ is the coefficient integer of $\alpha_i$ in the expansion of $\beta$ in simple roots, and $\varepsilon \in \{\pm 1\}$. This function is well-defined by Proposition 5.3.1. We call it the *classical compatibility degree*. In [19], the classical compatibility degree is called simply the *compatibility degree*, but we adopt this name in imitation of [10] to distinguish between it and forthcoming other degrees. For $\alpha, \beta \in \Phi_{\geq -1}$, we say that $\alpha$ and $\beta$ are *compatible* if $(\alpha \parallel \beta)_{\mathrm{cl}} = (\beta \parallel \alpha)_{\mathrm{cl}} = 0$.

By using the classical compatibility degree, we define the generalized associahedra.

**Definition 5.3.2.** For a root system $\Phi$, we define the *generalized associahedron* $\Delta(\Phi)$ as a simplicial complex whose simplexes are subsets of almost positive roots such that their elements are pairwise compatible.

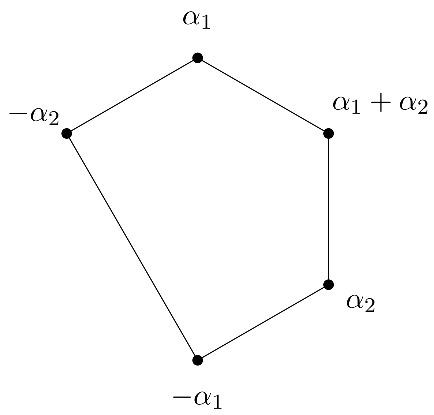
**Example 5.3.3.** We consider the root system of $A_2$ type. We give a generalized associahedron of $A_2$ type in Figure 5.1. We remark that this complex are the same as the cluster complex given in Example 2.4.2. We introduce the correspondence between cluster complexes and generalized associahedra in Theorems 5.3.4 and 5.3.5.

The function $(\cdot \parallel \cdot)_{\mathrm{cl}}$ can be regarded as a function on cluster variables of a cluster algebra in the following way: We fix a root system $\Phi$ and the sign of vertices $I = I_+ \cup I_-$ of $\Gamma_\Phi$. We denote by $B(C_\Phi) = (b_{ij})$ a skew-symmetrizable matrix obtained from the Cartan matrix $C_\Phi = (C_{ij})$ by the following equation:

$$b_{ij} = \begin{cases} 0 & \text{if } i = j, \\ -\varepsilon C_{ij} & \text{if } i \neq j \text{ and } i \in I_\varepsilon. \end{cases}$$

We call $\mathcal{A}(B(C_\Phi))$ a cluster algebra induced by $\Phi$.

Figure 5.1: Generalized associahedron of $A_2$ type




**Theorem 5.3.4** ([18, Theorem 1.9]). *For a root system $\Phi$, there is a unique bijection $\alpha \mapsto x[\alpha]$ from $\Phi_{\geq -1}$ to the set $\mathcal{X}$ of all cluster variables in $\mathcal{A}(B(C_\Phi))$, such that, for any $\alpha = \sum_i a_i\alpha_i \in \Phi_{\geq -1}$, the cluster variable $x[\alpha]$ is expressed in terms of the initial cluster $x_1, \ldots, x_n$ as*

$$x[\alpha] = \frac{P(x_1, \ldots, x_n)}{x_1^{a_1} \cdots x_n^{a_n}}, \tag{5.3.10}$$

*where $P(x_1, \ldots, x_n)$is a polynomial over $\mathbb{ZP}$ with nonzero constant term. Under this bijection, $x[-\alpha_i] = x_i$.*

The bijection in Theorem 5.3.4 is natural in the sense of the following.

**Theorem 5.3.5** ([18, Theorem 1.12]). *Under the bijection of Theorem 5.3.4, the cluster complex $\Delta(\mathcal{A}(B(C_\Phi)))$ is identified with the simplicial complex $\Delta(\Phi)$.*

By Theorems 5.3.4 and 5.3.5, we can identify almost positive roots in $\Phi_{\geq -1}$ with the $d$-vectors of cluster variables of $\mathcal{A}(B(C_\Phi))$. By abusing notation, we use $(\cdot \parallel \cdot)_{\mathrm{cl}}$ as a function on $\mathcal{X} \times \mathcal{X}$. We denote by $\Phi^\vee$ a dual root system of $\Phi$, and for any $\alpha \in \Phi$, we denote by $\alpha^\vee \in \Phi^\vee$ the coroot of $\alpha$. By definition, it is clear that $\mathcal{A}(B(C_{\Phi^\vee}))$ is $\mathcal{A}(B(C_\Phi)^\top)$ or $\mathcal{A}(-B(C_\Phi)^\top)$ (depending on the choice of $I_+$). We remark that the classical compatibility on $\mathcal{X} \times \mathcal{X}$ depends only on root systems, therefore we can assume $\mathcal{A}(B(C_{\Phi^\vee})) = \mathcal{A}(-B(C_\Phi)^\top)$ without loss of generality.

The classical compatibility degree satisfies the following property:

**Proposition 5.3.6** ([19, Proposition 3.3]). *We fix $\Phi$ and an induced cluster algebra $\mathcal{A}(B(C_\Phi))$.*

*(1) We have $(x[\alpha] \parallel x[\beta])_{\mathrm{cl}} = (x[\beta^\vee] \parallel x[\alpha^\vee])_{\mathrm{cl}}$ for every $\alpha, \beta \in \Phi_{\geq -1}$.
In particular, if $\Phi$ is simply-laced, then $(x[\alpha] \parallel x[\beta])_{\mathrm{cl}} = (x[\beta] \parallel x[\alpha])_{\mathrm{cl}}$.*

*(2) If $(x[\alpha] \parallel x[\beta])_{\mathrm{cl}} = 0$, then $(x[\beta] \parallel x[\alpha])_{\mathrm{cl}} = 0$.*

*(3) If $\alpha$ and $\beta$ belong to $\Phi(J)_{\geq -1}$ for some proper subset $J \subset I$ then their compatibility degree with respect to the root subsystem $\Phi(J)$ is equal to $(x[\alpha] \parallel x[\beta])_{\mathrm{cl}}$.*

We call (1) the *duality property*, (2) the *symmetry property*, and (3) the *embedding property* respectively. Moreover, the classical compatibility degree satisfies the following two property, the *compatibility property* and the *exchangeability property*:

**Proposition 5.3.7** ([21, Proposition 4.7])**.** *Let $\Phi$ be a root system and $\mathcal{A}(B(C_\Phi))$ be an induced cluster algebra by $\Phi$. For any set of cluster variables $X$, there exists a cluster $\mathbf{x}$ such that $\mathbf{x}$ contains $X$ if and only if the classical compatibility degrees of any pairs of cluster variables in $X$ are* $0$*.*

*Proof.* It follows from Theorem 5.3.5 immediately. □

**Proposition 5.3.8** ([21, Proposition 4.8])**.** *Let $\Phi$ be a root system and $\mathcal{A}(B(C_\Phi))$ be the induced cluster algebra by $\Phi$. For any $x[\alpha], x[\beta]$, there exists a set $X$ of cluster variables such that $X \cup x[\alpha]$ and $X \cup x[\beta]$ are both clusters, if and only if $(x[\alpha] \parallel x[\beta])_{\mathrm{cl}} = (x[\beta] \parallel x[\alpha])_{\mathrm{cl}} = 1$.*

*Proof.* The exchangeability of almost positive roots is proved by [11, Lemma 2.2] and [18, Corollary 4.4]. The proposition is shown by combining it with Theorem 5.3.5. □

We consider a natural generalization of the classical compatibility degree keeping these properties in the next subsection.

### 5.3.2 Compatibility degree

We introduce the compatibility degree. This is defined by using components of $f$-vectors. In this subsection, we prove that compatibility degree keeps Proposition 5.3.6 and Proposition 5.3.7.

**Definition 5.3.9.** Let $\mathcal{A}(B)$ be a cluster algebra. We define the *compatibility degree* $(\cdot \parallel \cdot)\colon \mathcal{X} \times \mathcal{X} \to \mathbb{Z}_{\geq 0}$ of $\mathcal{A}(B)$ as follows: For any two cluster variables $x$ and $x'$, if $x = x_{i;t}, x' = x_{j;t'}$, then

$$(x \parallel x') = f^{B_t;t}_{ij;t'}. \tag{5.3.11}$$

When we want to emphasize that this function is defined by $f$-vector, we use $(x \parallel x')_f$ as the notation.

We remark that the choice of $i, j, t, t'$ satisfying $x = x_{i;t}$ and $x' = x_{j;t'}$ is not unique, but the compatibility degree is well-defined by Theorem 5.2.2 (2). This function is a generalization of the classical compatibility degree.

**Theorem 5.3.10** ([21, Theorem 4.10])**.** *We fix any root system $\Phi$ and its induced cluster algebra $\mathcal{A}(B(C_\Phi))$. For any cluster variable $x, x'$, we have*

$$(x \parallel x')_{\mathrm{cl}} = (x \parallel x'). \tag{5.3.12}$$

To prove the theorem, we introduce $d$-compatibility degree defined by [9].

**Definition 5.3.11.** Let $\mathcal{A}(B)$ be a cluster algebra. We define the *$d$-compatibility degree* $(\cdot \parallel \cdot)_d\colon \mathcal{X} \times \mathcal{X} \to \mathbb{Z}_{\geq 0}$ of $\mathcal{A}(B)$ as follows: For any two cluster variables $x$ and $x'$, if $x = x_{i;t}, x' = x_{j;t'}$, then

$$(x \parallel x')_d = \left[d^{B_t;t}_{ij;t'}\right]_+. \tag{5.3.13}$$

In [9], the compatibility degree is not defined by $\left[d^{B_t;t}_{ij;t'}\right]_+$ but $d^{B_t;t}_{ij;t'}$. We adopt this definition for simplicity of the notation. The following theorem is essential for Theorem 5.3.10:

**Theorem 5.3.12** ([10, Corollary 3.2]). *We fix any root system $\Phi$ and its induced cluster algebra $\mathcal{A}(B(C_\Phi))$. For any cluster variable $x, x'$ we have*

$$(x \parallel x')_{\mathrm{cl}} = (x \parallel x')_d. \tag{5.3.14}$$

*Proof of Theorem 5.3.10.* Since $\mathcal{A}(B(C_\Phi))$ is of finite type, it follows from Theorem 4.4.3 and Theorem 5.3.12. □

**Remark 5.3.13.** Theorem 5.3.10 can be generalized from the classical compatibility degree to *c-compatiblity degree*, which is a function of the set of pairs of cluster variables of any cluster algebras of finite type, defined by [10, Definition 2.8]. This fact follows from [10, Corollary 3.3] and Theorem 4.4.3.

We will show that the compatibility degree satisfies properties which are analogous to Proposition 5.3.6 and Proposition 5.3.7. First, we consider the following proposition, which is a generalization of Proposition 5.3.6.

**Proposition 5.3.14** ([21, Proposition 4.14]). *We fix any cluster algebra $\mathcal{A}(B)$. For any $x = x_{i;t}$, we denote by $x^\vee$ the $i$th cluster variable in the cluster at $t$ of $\mathcal{A}(-B^\top)$.*

*(1) For any two cluster variables $x, x'$, we have $(x \parallel x') = ((x')^\vee \parallel x^\vee)$. In particular, if $B$ is skew-symmetric, then $(x \parallel x') = (x' \parallel x)$.*

*(2) If $x = x_{i;t}, x' = x_{j;t'}$, we have $(x \parallel x') = s_i^{-1}s_j(x' \parallel x)$, where $s_i$ is the $(i,i)$th entry of skew-symmetrizer $S$ of $B$. In particular, if $(x \parallel x') = 0$, then we have $(x' \parallel x) = 0$.*

*(3) For skew-symmetrizable matrix $B$ and the index set $I$, let $J = \{k_1, \dots k_m\} \subset I$ and $B_J$ be submatrix of $B$ such that $B_J = (b_{k_ik_j})$. For any pair of cluster variables $x, x'$ of $\mathcal{A}(B_J)$, which we regard as a pair of cluster variables of $\mathcal{A}(B)$ by embedding, $(x \parallel x')$ on $\mathcal{A}(B_J)$ equals to $(x \parallel x')$ on $\mathcal{A}(B)$.*

To prove Proposition 5.3.14, we prepare a lemma:

**Lemma 5.3.15** ([21, Lemma 4.16]). *For any exchange matrix $B$ and $t_0, t \in \mathbb{T}_n$, we have*

$$F_t^{B;t_0} = S^{-1}F_t^{-B^\top;t_0}S. \tag{5.3.15}$$

*Proof.* By (3.1.7), [36, (2.7)] and definition of $S$, for any $t$, we have

$$C_t^{B;t_0} = S^{-1}C_t^{-B^\top;t_0}S, \tag{5.3.16}$$
$$B_t = S^{-1}(-B_t^\top)S. \tag{5.3.17}$$

We prove (5.3.15) by induction on distances of $t$ from $t_0$. If $t = t_0$, then (5.3.15) holds clearly because $F_{t_0}^{B;t_0}$ is the zero matrix. When we assume (5.3.15) holds at $t$, we have (5.3.15) holds at $t'$ by substituting (3.1.12) with (5.3.16) and (5.3.17). □

*Proof of Proposition 5.3.14.* First, we prove (1). By Theorem 3.2.10 and (3.1.15), for any $t$ and $t'$, we have

$$F_{t'}^{B_t;t} = \left(F_t^{-B_{t'}^\top;t'}\right)^\top. \tag{5.3.18}$$

Thus, we have

$$f_{ij;t'}^{B_t;t} = f_{ji;t}^{-B_{t'}^\top;t'}. \tag{5.3.19}$$

This implies the first statement of (1). Furthermore, if $B$ is skew-symmetric, then we have $B_{t'} = -B_{t'}^{\top}$. This implies the second statement of (1). Second, we prove (2). By (1) and Lemma 5.3.15, for any $t$ and $t'$, we have

$$F_{t'}^{B_t;t} = \left(F_t^{-B_{t'}^{\top};t'}\right)^{\top} = \left(SF_t^{B_{t'};t'}S^{-1}\right)^{\top} = S^{-1}\left(F_t^{B_{t'};t'}\right)^{\top} S. \tag{5.3.20}$$

Thus, we have

$$f_{ij;t'}^{B_t;t} = s_i^{-1}s_j f_{ji;t}^{B_{t'};t'}. \tag{5.3.21}$$

This implies (2). Finally, we prove (3). Without loss of generality, we can assume $J = \{1,\dots,m\}$. It suffice to show that $F_t^{B_J;t_0}$ equals the $m\times m$ principal submatrix of $F_t^{B;t_0}$ for any $t_0 \in \mathbb{T}_n$ and $t \in \mathbb{T}_m$, where $\mathbb{T}_m$ is $m$-regular graph whose labels of edges are $1,\dots,m$ and which is a connected component of $\mathbb{T}_n$ containing $t_0$. We prove (3) by induction on distances of $t$ from $t_0$. The base case $t = t_0$ is immediate as $F_{t_0}^{B;t_0} = O_n$ and $F_{t_0}^{B_J;t_0} = O_m$. Let $C_t^{B_J;t_0} = (\overline{c}_{ij;t})$ and we abbreviate $F_{i;t}^{B;t_0}(\mathbf{y}) = F_{i;t}$ and $F_{i;t}^{B_J;t_0}(\mathbf{y}) = \overline{F}_{i;t}$. We have the following fact by directly calculation: $B_{Jt}$ equals the $m\times m$ principal matrix $B_t$, $F_{i;t} = 1$ for all $i \in \{m+1,\dots,n\}$, and the left side $m\times n$ submatrix of $C_t^{B;t_0}$ is $\begin{bmatrix} C_t^{B_J;t_0} \\ O \end{bmatrix}$. By these facts and inductive assumption, for $t \overset{\ell}{\text{—}} t'$, we have

$$\begin{aligned}
\overline{F}_{i;t'} &= \overline{F}_{i;t} = F_{i;t} = F_{i;t'} \quad \text{if } i \neq \ell,\\
\overline{F}_{\ell;t'} &= \frac{\displaystyle\prod_{j=1}^{m} y_j^{[\overline{c}_{j\ell;t}]_+} \prod_{i=1}^{m} \overline{F}_{i;t}^{[\overline{b}_{i\ell;t}]_+} + \prod_{j=1}^{m} y_j^{[-\overline{c}_{j\ell;t}]_+} \prod_{i=1}^{m} \overline{F}_{i;t}^{[-\overline{b}_{i\ell;t}]_+}}{\overline{F}_{\ell;t}}\\
&= \frac{\displaystyle\prod_{j=1}^{n} y_j^{[c_{j\ell;t}]_+} \prod_{i=1}^{n} F_{i;t}^{[b_{i\ell;t}]_+} + \prod_{j=1}^{n} y_j^{[c_{j\ell;t}]_+} \prod_{i=1}^{n} F_{i;t}^{[-b_{i\ell;t}]_+}}{F_{\ell;t}}\\
&= F_{\ell;t'}.
\end{aligned}$$

Therefore, $F_t^{B_J;t_0}$ equals the $m\times m$ principal submatrix of $F_t^{B;t_0}$. ☐

**Remark 5.3.16.** We can prove the second statement of (1) by using the first statement of (2) because when $B$ is skew-symmetric, $s_i^{-1}s_j$ is always 1.

Next, we consider the *compatibility property*, which is a generalization of Proposition 5.3.7.

**Theorem 5.3.17** ([21, Theorem 4.18])**.** *For any cluster algebra $\mathcal{A}(B)$ and any set $X$ of cluster variables, there exists a cluster $\mathbf{x}$ such that $\mathbf{x}$ contains $X$ if and only if the compatibility degrees of any pairs of cluster variables in $X$ are* 0.

*Proof.* It follows from Theorem 5.2.2 (3) and Lemma 5.2.4 immediately. ☐

Let us compare the compatibility degree with the $d$-compatibility degree. It is proved by [9] that $d$-compatibility degree also has the similar property of Theorem 5.3.17.

**Theorem 5.3.18** ([9, Theorem 7.4])**.** *For any cluster algebra $\mathcal{A}(B)$ and any set $X$ of cluster variables, there exists a cluster $\mathbf{x}$ such that $\mathbf{x}$ contains $X$ if and only if the $d$-compatibility degrees of any pairs of cluster variables in $X$ are either* 0 *or* $-1$.

However, the $d$-compatibility degree does not satisfy the similar property of the duality and symmetry properties (Proposition 5.3.14 (1),(2)). Actually, if these properties hold for $d$-vectors, the $D$-matrices must satisfy the following equation when $B$ is skew-symmetric:

$$\left(D_t^{B;t_0}\right)^\top = D_{t_0}^{-B_t^\top;t} = D_{t_0}^{B_t;t}. \tag{5.3.22}$$

However, this equation does not hold generally unlike the $F$-matrices. For the class of cluster algebras arising from the marked surface, [39] gave the complete classification of marked surfaces whose corresponding cluster algebras satisfy (5.3.22).

**Theorem 5.3.19** ([39, Theorem 2.4])**.** *The equation* (5.3.22) *hold for a cluster algebra arising from a marked surface if and only if the marked surface is one of the following.*

*(1) A disk with at most one puncture (finite types A and D).*

*(2) An annulus with no punctures and one or two marked points on each boundary component (affine types $\tilde{A}_{1,1}$, $\tilde{A}_{2,1}$, and $\tilde{A}_{2,2}$).*

*(3) A disk with two punctures and one or two marked points on the boundary component (affine types $\tilde{D}_3$ and $\tilde{D}_4$).*

*(4) A sphere with four punctures and no boundary components.*

*(5) A torus with exactly one marked point (either one puncture or one boundary component containing one marked point).*

According to Theorem 5.3.19, for example, cluster algebras arising from a disk with three punctures and one marked point on the boundary component do not satisfy (5.3.22). Let us see a concrete example.

**Example 5.3.20.** We set $\mathbb{P} = \{1\}$ the trivial semifield and consider a seed $(\mathbf{x}, B)$, where

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7), \quad B = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}.$$

Then $\mathcal{A}(B)$ is a cluster algebra arising from the marked surface in Figure 5.2. Moreover, we set

$$\mathbf{x}' = (x_1, x_2', x_3', x_4', x_5', x_6, x_7') = \mu_7\mu_5\mu_4\mu_3\mu_2(\mathbf{x}).$$

Then, we have $(x_2 \parallel x_7')_d = 2, (x_7'^\vee \parallel x_2^\vee)_d = (x_7' \parallel x_2)_d = 1$. We consider flipping the

Figure 5.2: Marked surface corresponding to $B$

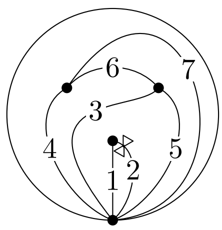
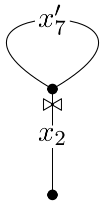


Figure 5.3: Relative position of arc corresponding to $x_2$ and $x'_7$




marked surface in Figure 5.2 at 2,3,4,5,7. The relative position of arc corresponding to $x_2$ and $x'_7$ is as in Figure 5.3.

As mentioned in [15, Example 8.5], considering the intersection number([15, Definition 8.4]), we have $(x_2 \parallel x'_7)_d = 2, (x'_7 \parallel x_2)_d = 1$. This example implies $(\cdot \parallel \cdot)_d$ does not satisfy the similar property of Proposition 5.3.14 (1),(2).

Next, we consider a generalization of Proposition 5.3.8. The following statement is clear:

**Theorem 5.3.21** ([21, Theorem 4.22])**.** *For any cluster algebra $\mathcal{A}(B)$ and any pair of its cluster variables $x, x'$, if there exists a set $X$ of cluster variables such that $X \cup x$ and $X \cup x'$ are both clusters, then $(x \parallel x') = (x' \parallel x) = 1$.*

*Proof.* We take a seed whose cluster is $X \cup x$ as the initial seed and consider a mutation such that it changes cluster from $X \cup x$ to $X \cup x'$. By Lemma 5.2.5, there is a mutation satisfying this condition. The statement is followed by definition of the cluster mutation (2.1.7) and of the $f$-vectors (Definition 3.1.3). □

The converse of Theorem 5.3.21 is still open:

**Conjecture 5.3.22.** *For any cluster algebra $\mathcal{A}(B)$ and any pair of its cluster variables $x, x'$, if $(x \parallel x') = (x' \parallel x) = 1$, then there exists a set $X$ of cluster variables such that $X \cup x$ and $X \cup x'$ are both clusters.*

We call Theorem 5.3.21 and Conjecture 5.3.22 the *exchangeability property.* In the case of finite type, Conjecture 5.3.22 is correct:

**Theorem 5.3.23** ([21, Theorem 4.24])**.** *For any cluster algebra $\mathcal{A}(B)$ of finite type and any pair of its cluster variables $x, x'$, if $(x \parallel x') = (x' \parallel x) = 1$, then there exists a set $X$ of cluster variables such that $X \cup x$ and $X \cup x'$ are both clusters.*

*Proof.* It follows from Proposition 5.3.8 and Theorem 5.3.10. □

Also in the case of rank 2, we can prove Conjecture 5.3.22 by using descriptions of the $F$-matrices.

**Theorem 5.3.24** ([21, Theorem 4.25])**.** *For any cluster algebra $\mathcal{A}$ of rank 2 and any pair of its cluster variables $x, x'$, if $(x \parallel x') = (x' \parallel x) = 1$, then there exists a set $X$ of cluster variables such that $X \cup x$ and $X \cup x'$ are both clusters.*

*Proof.* If $\mathcal{A}$ is of finite type, the result follows from Theorem 5.3.23. We assume that $\mathcal{A}$ is not of finite type. Let $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ be a labeled seed of $\mathcal{A}$ which contains the cluster variable $x$. Without loss of generality, we may assume that $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$ for some $b, c \in \mathbb{Z}_{\geq 0}$ such that $bc \geq 4$.

We name vertices of $\mathbb{T}_2$ by the rule of (2.1.11) and fix a cluster pattern $t_n \mapsto (\mathbf{x}_{t_n}, \mathbf{y}_{t_n}, B_{t_n})$ by assigning the seed $\Sigma$ to the vertex $t_0$. We abbreviate $\mathbf{x}_{t_n}$ (resp., $\mathbf{y}_{t_n}$, $B_{t_n}, \Sigma_{t_n}$) to $\mathbf{x}_n$

(resp., $\mathbf{y}_n$, $B_n$, $\Sigma_n$). We also abbreviate cluster variables, $f$-vectors and $F$-matrices in the same way.

Let us consider the case $x = x_{1;0}$, the case $x = x_{2;0}$ can be proved similarly. In this case, it suffices to prove that $x' \in \{x_{1;2}, x_{1;-1}\}$.

A direct computation show that

$$F_0^{B;t_0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_1^{B;t_0} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.3.23}$$

By descriptions of $D$-matrices of rank 2, that is (4.7.3) and (4.7.4), and Remark 4.4.4, if $n > 0$ is even, then we have

$$F_n^{B;t_0} = \begin{bmatrix} S_{\frac{n-2}{2}}(u) + S_{\frac{n-4}{2}}(u) & bS_{\frac{n-4}{2}}(u) \\ cS_{\frac{n-2}{2}}(u) & S_{\frac{n-2}{2}}(u) + S_{\frac{n-4}{2}}(u) \end{bmatrix}, \tag{5.3.24}$$

and if $n > 1$ is odd, then we have

$$F_n^{B;t_0} = \begin{bmatrix} S_{\frac{n-3}{2}}(u) + S_{\frac{n-5}{2}}(u) & bS_{\frac{n-3}{2}}(u) \\ cS_{\frac{n-3}{2}}(u) & S_{\frac{n-1}{2}}(u) + S_{\frac{n-3}{2}}(u) \end{bmatrix}, \tag{5.3.25}$$

where $u = bc - 2$ and $S_p(u)$ is a (normalized) Chebyshev polynomial of the second kind (See (4.7.5)). When $n < 0$, $F_n^{B;t_0}$ is the following matrix:

$$F_n^{B;t_0} = \begin{bmatrix} f_{22;-n}^{-B^\top} & f_{21;-n}^{-B^\top} \\ f_{12;-n}^{-B^\top} & f_{11;-n}^{-B^\top} \end{bmatrix}, \tag{5.3.26}$$

where $f_{ij;-n}^{-B^\top}$ is the $(i,j)$ entry of $F_{-n}^{-B^\top;t_0}$. For any $p \geq 0$, we have $S_p(u) - S_{p-1}(u) > 0$. Indeed, we have $S_0(u) - S_{-1}(u) = 1 > 0$. Assume that $S_q(u) - S_{q-1}(u) > 0$, then $S_{q+1}(u) - S_q(u) = (u-1)S_q(u) - S_{q-1}(u) > 0$. In particular, for any $p \in \mathbb{Z}_{\geq -1}$, we have

$$\cdots > S_{p+1}(u) > S_p(u) > \cdots > S_0(u) = 1 > S_{-1}(u) = 0. \tag{5.3.27}$$

We claim that $(x_{1;0} \parallel x_{i;n})(x_{i;n} \parallel x_{1;0}) > 1$ for $i \in \{1,2\}$ whenever $n \geq 4$ or $n \leq -3$. As a consequence, $x' \neq x_{i;n}$ for any $n \geq 4$ and $n \leq -3$ and $x' \in \bigcup_{-2 \leq n \leq 3} \mathbf{x}_n$. Recall that according to Proposition 5.3.14 (2), $(x_{1;0} \parallel x_{i;n}) = 0$ if and only if $(x_{i;n} \parallel x_{1;0}) = 0$. For $i = 1$ and $n \geq 4$,

$$\begin{aligned} (x_{1;0} \parallel x_{1;n})(x_{1;n} \parallel x_{1;0}) &= f_{11;n}^{B;t_0}(x_{1;n} \parallel x_{1;0}) \\ &\geq (S_1(u) + S_0(u))(x_{1;n} \parallel x_{1;0}) && \text{by (5.3.24)(5.3.25)} \\ &> 1. && \text{by (5.3.27)} \end{aligned}$$

For $i = 2$ and $n \geq 4$,

$$\begin{aligned} (x_{1;0} \parallel x_{2;n})(x_{2;n} \parallel x_{1;0}) &= f_{12;n}^{B;t_0}(x_{2;n} \parallel x_{1;0}) \\ &\geq b(x_{2;n} \parallel x_{1;0}) && \text{by (5.3.24)(5.3.25)} \\ &= b(x_{1;0}^\vee \parallel x_{2;n}^\vee) && \text{by Propostion 5.3.14(1)} \\ &= bf_{12;n}^{-B^\top;t_0} \\ &\geq bc \geq 4. && \text{by (5.3.24)(5.3.25)} \end{aligned}$$

This completes the proof of the claim for $n \geq 4$. For $n \leq -3$, one can prove the statement by using (4.7.6) similarly and we omit the proof.

According to the cluster pattern, we have

$$x_{2;-2} = x_{2;-3}, x_{1;-2} = x_{1;-1}, x_{2;-1} = x_{2;0}, x_{1;0} = x_{1;1}, x_{2;1} = x_{2;2}, x_{1;2} = x_{1;3}, x_{2;3} = x_{2;4}.$$

By the above claim, we conclude that $x' \neq x_{2;-2}$ and $x' \neq x_{2;3}$. By Theorem 5.3.17, we have

$$(x_{1;0} \parallel x_{1;0}) = (x_{1;0} \parallel x_{2;0}) = (x_{1;0} \parallel x_{2;1}) = 0$$

since there is a cluster contains $x_{1;0}$ and $x_{i;n}$ for $(i, n) \in \{(1, 0), (2, 0), (2, 1)\}$. It follows that $x' \in \{x_{1;-1}, x_{1;2}\}$. This finishes the proof. □

It is known that Conjecture 5.3.22 holds in some classes of cluster algebras:

**Theorem 5.3.25** ([21, Theorems 6.3, 6.6, 6.11]). *Conjecture 5.3.22 holds if $\mathcal{A}(B)$ is*

- *an acyclic cluster algebra of skew-symmetric type,*
- *a cluster algebra arising from weighted projective lines, or*
- *a cluster algebra arising from marked surfaces.*

Since we need to use 2-Carabi-Yau categorification, we omit the proof of this statement. See [21, Sections 5 and 6].

In the case of finite type or rank 2, the $d$-compatibility degree also has the exchangeability property. However, it is not correct in general. Let us see some examples.

**Example 5.3.26.** We set $\mathbb{P} = \{1\}$ the trivial semifield and consider a seed $(\mathbf{x}, B)$, where

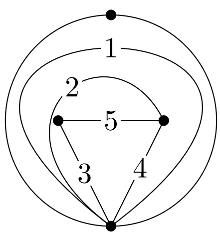
$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5), \quad B = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

This is of $\hat{D}_4$ type. Moreover, we set

$$\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4, x_5) = \mu_4\mu_3\mu_2\mu_1(\mathbf{x}).$$

Then, we have $(x_1 \parallel x'_4)_d = (x'_4 \parallel x_1)_d = 1$ and $(x_1 \parallel x'_4) = (x'_4 \parallel x_1) = 2$. Let us see this fact by using marked surface and their flips. $\mathcal{A}(B)$ is a cluster algebra arising from the marked surface in Figure 5.4. We consider flipping the marked surface in figure 5.4 at

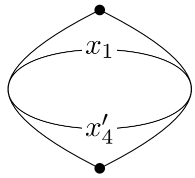
Figure 5.4: Marked surface corresponding to $B$




1,2,3,4. The relative position of arc corresponding to $x_1$ and $x'_4$ is as in Figure 5.5.

Considering the intersection number induced by the $d$-vector ([15, Definition 8.4]), we have $(x_1 \parallel x'_4)_d = (x'_4 \parallel x_1)_d = 1$. On the other hand, considering the intersection number induced by the $f$-vector ([44, Section 1]), we have $(x_1 \parallel x'_4) = (x'_4 \parallel x_1) = 2$. Therefore, by Corollary 5.3.25, $x_1$ and $x'_4$ are not exchangeable. This example implies $(\cdot \parallel \cdot)_d$ does not satisfy the similar property of Conjecture 5.3.22. We remark that $\mathcal{A}(B)$ satisfies the similar property to Proposition 5.3.14 for the $d$-vectors because of Theorem 5.3.19 (3).

Figure 5.5: Relative position of arc corresponding to $x_1$ and $x'_4$




**Example 5.3.27.** We set $\mathbb{P} = \{1\}$ and consider a seed $(\mathbf{x}, B)$, where

$$\mathbf{x} = (x_1.x_2.x_3), \quad B = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

We fix a cluster pattern by assigning $\Sigma_{t_0} = (\mathbf{x}, B)$ to the rooted vertex $t_0$ of $\mathbb{T}_3$. The cluster algebra $\mathcal{A}(B)$ is acyclic (type $\hat{A}_2$). Let

$$t_0 \overset{3}{\text{———}} t_1 \overset{2}{\text{———}} t_2 \overset{1}{\text{———}} t_3$$

be a subgraph of $\mathbb{T}_3$. According to [22, Example 6.7], we have

$$\mathbf{f}_{1,t_3}^{B,t_0} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{d}_{1;t_3}^{B,t_0} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore $(x_3 \parallel x_{1;t_3}) = (x_{1;t_3} \parallel x_3) = 2$ by Proposition 5.3.14 (2). Hence $x_3$ and $x_{1;t_3}$ are not exchangeable by Theorem 5.3.25. On the other hand, a direct computation shows that

$$\mathbf{d}_{3,t_0}^{B_{t_3};t_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence $(x_3 \parallel x_{1;t_3})_d = (x_{1,t_3} \parallel x_3)_d = 1$. In particular, $(\cdot \parallel \cdot)_d$ does not satisfy the similar property of Conjecture 5.3.22.

# 6 Cluster duality between Calkin-Wilf tree and Stern-Brocot tree

In this chapter, we give an application of $f$-vectors and $F$-matrices to number theory. We introduce two cluster structures, which are dual to each other, into the *Calkin-Wilf tree* and the *Stern-Brocot tree*. This chapter is based on [26].

The Calkin-Wilf tree, whose vertex set has a bijection with the set of positive rational numbers $\mathbb{Q}_+$, is introduced by Neil Calkin and Herbert S. Wilf [6] to count all positive rational numbers efficiently. This is a full binary tree whose vertices are positive fractions given by the following way: the root is $\dfrac{1}{1}$, and the generation rule is that a parent $\dfrac{x}{y}$ has the following two children:

$$\begin{tikzcd}[column sep=small] & \dfrac{x}{y} \arrow[dl, dash] \arrow[dr, dash] & \\ \dfrac{x}{x+y} & & \dfrac{x+y}{y} \end{tikzcd}.$$

The first few terms are as follows:

$$\begin{tikzcd}[row sep=tiny] & & & 1/4\cdots \\ & & 1/3 \arrow[ur, dash] \arrow[dr, dash] & \\ & & & 4/3\cdots \\ & 1/2 \arrow[uur, dash] \arrow[ddr, dash] & & \\ & & & 3/5\cdots \\ & & 3/2 \arrow[ur, dash] \arrow[dr, dash] & \\ & & & 5/2\cdots \\ 1/1 \arrow[uuuur, dash] \arrow[ddddr, dash] & & & \\ & & & 2/5\cdots \\ & & 2/3 \arrow[ur, dash] \arrow[dr, dash] & \\ & & & 5/3\cdots \\ & 2/1 \arrow[uur, dash] \arrow[ddr, dash] & & \\ & & & 3/4\cdots \\ & & 3/1 \arrow[ur, dash] \arrow[dr, dash] & \\ & & & 4/1\cdots \end{tikzcd}.$$

We can easily verify that all fractions appearing in the Calkin-Wilf tree is irreducible.

On the other hand, the Stern-Brocot tree[1] is named after Moritz Stern, Achille Brocot, and their researches in 1800's [3, 43]. The vertex set of the Stern-Brocot tree also has a bijection with $\mathbb{Q}_+$. This tree is given as follows: first, we consider the *Farey triple tree*. This is a full binary tree given in the following way: the root is $\left(\dfrac{0}{1}, \dfrac{1}{0}, \dfrac{1}{1}\right)$, and the generation rule is that a parent $\left(\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}\right)$ has the following two children: if the second

[1] It is also called the *Farey tree*.

largest fraction is (i) $\dfrac{a}{b}$, (ii) $\dfrac{c}{d}$, (iii)$\dfrac{d}{e}$, then

(i)

$$\begin{array}{c} \left(\dfrac{a}{b},\dfrac{c}{d},\dfrac{e}{f}\right) \\ \swarrow \qquad \searrow \\ \left(\dfrac{a}{b},\dfrac{a+e}{b+f},\dfrac{e}{f}\right)\left(\dfrac{a}{b},\dfrac{c}{d},\dfrac{a+c}{b+d}\right) \end{array}$$

(ii)

$$\begin{array}{c} \left(\dfrac{a}{b},\dfrac{c}{d},\dfrac{e}{f}\right) \\ \swarrow \qquad \searrow \\ \left(\dfrac{c+e}{d+f},\dfrac{c}{d},\dfrac{e}{f}\right)\left(\dfrac{a}{b},\dfrac{c}{d},\dfrac{a+c}{b+d}\right) \end{array}$$

(iii)

$$\begin{array}{c} \left(\dfrac{a}{b},\dfrac{c}{d},\dfrac{e}{f}\right) \\ \swarrow \qquad \searrow \\ \left(\dfrac{c+e}{d+f},\dfrac{c}{d},\dfrac{e}{f}\right)\left(\dfrac{a}{b},\dfrac{a+e}{b+f},\dfrac{e}{f}\right). \end{array}$$

The Stern-Brocot tree is a full binary tree obtained from the Farey triple tree by replacing each vertex with the second largest fraction of it. The first few terms are as follows:

- 1/1
  - 1/2
    - 1/3
      - 1/4 $\cdots$
      - 2/5 $\cdots$
    - 2/3
      - 3/5 $\cdots$
      - 3/4 $\cdots$
  - 2/1
    - 3/2
      - 4/3 $\cdots$
      - 5/3 $\cdots$
    - 3/1
      - 5/2 $\cdots$
      - 4/1 $\cdots$

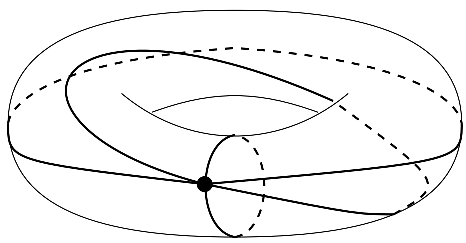
.

Like the Calkin-Wilf tree, all fractions appearing this tree is irreducible.

So far, it is pointed out that there are some relations of these two trees. For example, Backhouse and Ferreira found relation of these two and the *matrix tree* [1, 2, 42]. In this chapter, we introduce a new relation between these trees derived from cluster algebra theory.

Now, we consider a one-punctured torus and their triangulations (See Figure 6.1).

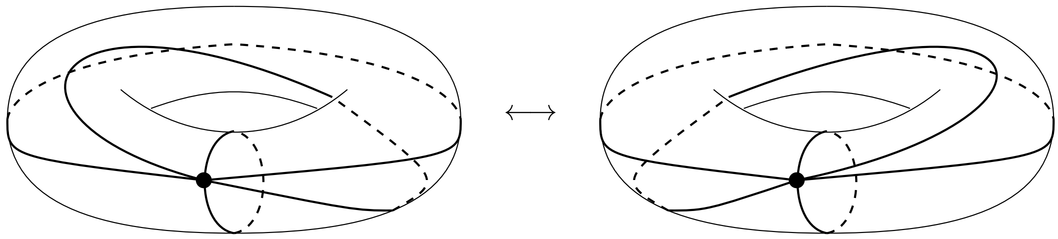
Figure 6.1: Triangulation of one-punctured torus



We fix a triangulation $L = (\ell_1, \ell_2, \ell_3)$. For another triangulation $M$ and an arc $\ell$ included in $M$, we define the *intersection vector* $\mathsf{Int}(L, \ell) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$, where $f_i$ is the intersection number of $\ell$ and $\ell_i$.

Next, we consider the operation called flip, defined in Chapter 2. We can apply flip to a triangulation $M$ and obtain a new triangulation $M'$, and also, we can apply flip to an arc $l$ and obtain a new arc $l'$. See an example in Figure 6.2.

Figure 6.2: Flip of triangulation



There are 3 ways to flip per triangulation. By applying flips to triangulations again and again, we can obtain a full binary tree called the *intersection vector tree*:

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
 & & \begin{bmatrix}2\\1\\4\end{bmatrix}\cdots \\
 & \begin{bmatrix}1\\0\\2\end{bmatrix} \arrow[ur, dash] \arrow[dr, dash] & \\
 & & \begin{bmatrix}2\\0\\3\end{bmatrix}\cdots \\
\begin{bmatrix}0\\0\\1\end{bmatrix} \arrow[uur, dash] \arrow[ddr, dash] & & \\
 & & \begin{bmatrix}1\\2\\4\end{bmatrix}\cdots \\
 & \begin{bmatrix}0\\1\\2\end{bmatrix} \arrow[ur, dash] \arrow[dr, dash] & \\
 & & \begin{bmatrix}0\\2\\3\end{bmatrix}\cdots
\end{tikzcd}.
$$

The first main theorem presents a relation between the intersection vector tree and the Stern-Brocot tree:

**Theorem 6.0.1** (Theorem 6.2.1)**.** *We consider a map*

$$g\colon \mathbb{Z}^3_{\geq 0} \to \mathbb{Q}, \qquad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \mapsto \frac{f_1+1}{f_2+1}.$$

*The Stern-Brocot tree is obtained by replacing each vertex $v$ of the intersection vector tree with $g(v)$.*

Next, we introduce a counterpart of the Calkin-Wilf tree. In contrast to the previous, we fix an arc $\ell$ and consider changing triangulations from $L$ to $L'$ by a flip. In parallel with change of triangulations, we obtain another intersection vector $\mathsf{Int}(L', \ell)$. By doing it repeatedly, we define another tree, the *initial intersection vector tree*:

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
 & & \begin{bmatrix}2\\4\\1\end{bmatrix}\cdots \\
 & \begin{bmatrix}2\\0\\1\end{bmatrix} \arrow[ur, dash, "2"] \arrow[dr, dash, "3"'] & \\
 & & \begin{bmatrix}2\\0\\3\end{bmatrix}\cdots \\
\begin{bmatrix}0\\0\\1\end{bmatrix} \arrow[uur, dash, "1"] \arrow[ddr, dash, "2"'] & & \\
 & & \begin{bmatrix}4\\2\\1\end{bmatrix}\cdots \\
 & \begin{bmatrix}0\\2\\1\end{bmatrix} \arrow[ur, dash, "1"] \arrow[dr, dash, "3"'] & \\
 & & \begin{bmatrix}0\\2\\3\end{bmatrix}\cdots
\end{tikzcd}, \tag{6.0.1}
$$

where numbers on edges are the positions of a exchanged arc in the triangulation. The second main theorem presents a relation between the initial intersection vector tree and the Calkin-Wilf tree:

**Theorem 6.0.2** (Theorem 6.3.3)**.** *We define a map $h$ from vertices of the initial intersection vector tree to $\mathbb{Q}$ inductively as follows: we assign the leftmost vertex* $\begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$ *to* $\dfrac{f_1+1}{f_2+1}=\dfrac{1}{1}$. *Let* $\{a,b,c\}=\{1,2,3\}$. *When* $\mathsf{Int}(L_t,\ell)=\begin{bmatrix}f_{1;t}\\f_{2;t}\\f_{3;t}\end{bmatrix}\mapsto\dfrac{f_{a;t}+1}{f_{b;t}+1}$, *and* $\mathsf{Int}(L_t,\ell)\overset{k}{\text{———}}\mathsf{Int}(L_{t'},\ell)$ *as in* (6.0.1),

- *if* $k=a$, *then we assign* $\mathsf{Int}(L_{t'},\ell)\mapsto\dfrac{f_{c;t}+1}{f_{b;t}+1}$,
- *if* $k=b$, *then we assign* $\mathsf{Int}(L_{t'},\ell)\mapsto\dfrac{f_{a;t}+1}{f_{c;t}+1}$.

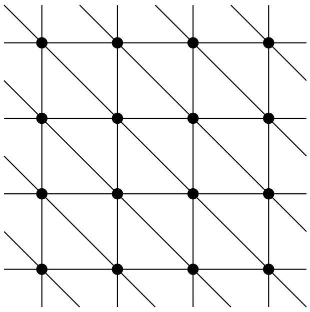
*The Calkin-Wilf tree is obtained by replacing each vertex $v$ of the initial intersection vector tree with $h(v)$.*

In the context of cluster algebra theory, we can regard the relation between Theorem 6.0.1 and Theorem 6.0.2 as a specialization of the $F$-matrix duality (Theorem 3.2.10).

## 6.1 Cluster pattern from one-punctured torus

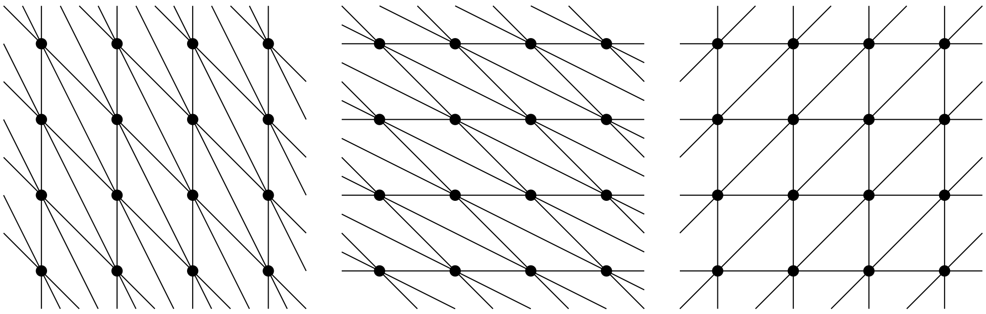
In this section, we introduce the cluster pattern from a one-punctured torus. This is the special case of the cluster structure from marked surfaces introduced by [15]. Let $S$ be a one-punctured torus. We denote by $p$ the puncture of $S$. We consider a triangulation of $S$ by (homotopy equivalence classes of) arcs whose both endpoints are $p$. On the universal covering of $S$, a triangulation of $S$ is given as in Figure 6.3. Clearly, a triangulation of $S$ consists of 3 arcs. Hereinafter, arcs constructing a triangulation is referred to as a *triangulation* simply. In this chapter, we define a triangulation as an ordered set. Let $L=(\ell_1,\ell_2,\ell_3)$ be a triangulation. Due to symmetry, we can assume $\ell_1$ is the horizontal line, $\ell_2$ is the vertical line, and $\ell_3$ is the diagonal line in Figure 6.3 without loss of generality. For

Figure 6.3: Triangulation of one-punctured torus (universal covering)



$k \in \{1, 2, 3\}$, we define a *flip* $\varphi_k(L)$ of $L$ in direction $k$ as the operation that obtains another triangulation from $L$ by exchanging $\ell_k$ for another arc. Figure 6.4 shows triangulations of $S$ flipped from $L$ in directions 1,2,3, respectively. Let $\mathbb{T}_3$ be the 3-*regular tree* whose

Figure 6.4: flipped Triangulation of one-punctured torus



edges are labeled by the numbers 1,2,3 such that the 3 edges emanating from each vertex have different labels. We use the notation $t \overset{k}{\text{———}} t'$ to indicate that vertices $t, t' \in \mathbb{T}_3$ are joined by an edge labeled by $k$. We fix an arbitrary vertex $t_0 \in \mathbb{T}_3$, which is called the *rooted vertex*, and a triangulation $L$. A *cluster pattern* with the initial triangulation $L$ is an assignment of a triangulation $L_t = (\ell_{1;t}, \ell_{2;t}, \ell_{3;t})$ to every vertex $t \in \mathbb{T}_n$ such that $L$ are assigned $t_0$ and triangulations $L_t$ and $L_{t'}$ assigned to the endpoints of any edge $t \overset{k}{\text{———}} t'$ are obtained from each other by a flip in direction $k$. We denote by $P_L \colon t \mapsto L_t$ this assigment. For a cluster pattern $P_L$, when $\ell_{i;t}$ intersects with $\ell_1, \ell_2$ and $\ell_3$ at least $f_{i1}, f_{i2}$ and $f_{i3}$ times on $S \setminus \{p\}$ respectively, we define the *intersection vector* $\mathsf{Int}(L, \ell_{i;t})$ associate with $\ell_{i;t}$ as

$$\mathsf{Int}(L, \ell_{i;t}) = \begin{bmatrix} f_{i1} \\ f_{i2} \\ f_{i3} \end{bmatrix}.$$

We regard the intersection number of the same arc as 0. Furthermore, we define the *intersection matrix* $\mathsf{Int}(L, L_t)$ associate with $L_t$ as

$$\mathsf{Int}(L, L_t) = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}.$$

They are specialization of intersection vectors and intersection matrices in Section 4.1. Therefore, they correspond with $f$-vectors and $F$-polynomials by Theorem 4.1.5.

## 6.2 Intersection vector tree and Stern-Brocot tree

Let $t_1$ be a vertex of $\mathbb{T}_3$ connected with $t_0$ by an edge labeled 3, and $\mathbb{T}_3'$ a full subtree whose vertex set is the union of $t_0$ and all vertices which are reachable to $t_1$ without going through $t_0$:

$$\begin{tikzcd}[row sep=tiny]
 & & & & t_8 \cdots \\
 & & & t_4 \arrow[ur, dash, "1"] \arrow[dr, dash, "3"'] & \\
 & & & & t_9 \cdots \\
 & & t_2 \arrow[uur, dash, "2"] \arrow[ddr, dash, "3"'] & & \\
 & & & & t_{10} \cdots \\
 & & & t_5 \arrow[ur, dash, "1"] \arrow[dr, dash, "2"'] & \\
 & & & & t_{11} \cdots \\
t_0 \arrow[r, dash, "3"] & t_1 \arrow[uuuur, dash, "1"] \arrow[ddddr, dash, "2"'] & & & \\
 & & & & t_{12} \cdots \\
 & & & t_6 \arrow[ur, dash, "2"] \arrow[dr, dash, "3"'] & \\
 & & & & t_{13} \cdots \\
 & & t_3 \arrow[uur, dash, "1"] \arrow[ddr, dash, "3"'] & & \\
 & & & & t_{14} \cdots \\
 & & & t_7 \arrow[ur, dash, "1"] \arrow[dr, dash, "2"'] & \\
 & & & & t_{15} \cdots
\end{tikzcd} . \tag{6.2.1}$$

Let $\mathbb{T}_3''$ be a full subtree of $\mathbb{T}_3'$ whose vertex set consists of all vertices of $\mathbb{T}_3'$ except for $t_0$. We correspond the intersection vectors to vertices of $\mathbb{T}_3''$ as

$$\begin{tikzcd}[row sep=tiny]
 & & & \mathsf{Int}(L, \ell_{1;t_8}) \cdots \\
 & & \mathsf{Int}(L, \ell_{2;t_4}) \arrow[ur, dash, "1"] \arrow[dr, dash, "3"'] & \\
 & & & \mathsf{Int}(L, \ell_{3;t_9}) \cdots \\
 & \mathsf{Int}(L, \ell_{1;t_2}) \arrow[uur, dash, "2"] \arrow[ddr, dash, "3"'] & & \\
 & & & \mathsf{Int}(L, \ell_{1;t_{10}}) \cdots \\
 & & \mathsf{Int}(L, \ell_{3;t_5}) \arrow[ur, dash, "1"] \arrow[dr, dash, "2"'] & \\
 & & & \mathsf{Int}(L, \ell_{2;t_{11}}) \cdots \\
\mathsf{Int}(L, \ell_{3;t_1}) \arrow[uuuur, dash, "1"] \arrow[ddddr, dash, "2"'] & & & \\
 & & & \mathsf{Int}(L, \ell_{2;t_{12}}) \cdots \\
 & & \mathsf{Int}(L, \ell_{1;t_6}) \arrow[ur, dash, "2"] \arrow[dr, dash, "3"'] & \\
 & & & \mathsf{Int}(L, \ell_{3;t_{13}}) \cdots \\
 & \mathsf{Int}(L, \ell_{2;t_3}) \arrow[uur, dash, "1"] \arrow[ddr, dash, "3"'] & & \\
 & & & \mathsf{Int}(L, \ell_{1;t_{14}}) \cdots \\
 & & \mathsf{Int}(L, \ell_{3;t_7}) \arrow[ur, dash, "1"] \arrow[dr, dash, "2"'] & \\
 & & & \mathsf{Int}(L, \ell_{2;t_{15}}) \cdots
\end{tikzcd} . \tag{6.2.2}$$

That is, if an edge labeled $k$ emanates from the left side of $t_i$ in (6.2.1), we assign $\mathsf{Int}(L, \ell_{k;t_i})$ to $t_i$. The first seven vertices of it is as follows:

$$\begin{tikzcd}[row sep=tiny]
 & & \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \\
 & \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \arrow[ur, dash, "2"] \arrow[dr, dash, "3"'] & \\
 & & \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \\
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \arrow[uur, dash, "1"] \arrow[ddr, dash, "2"'] & & \\
 & & \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \\
 & \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \arrow[ur, dash, "1"] \arrow[dr, dash, "3"'] & \\
 & & \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}
\end{tikzcd} .$$

We denote by $\mathrm{Tree}(F)$ this tree, and we call $\mathrm{Tree}(F)$ the *intersection vector tree*. In this section, we prove the following theorem:

**Theorem 6.2.1** ([26, Theorem 3.1])**.** *We consider a map*

$$g\colon \mathbb{Z}^3_{\geq 0} \to \mathbb{Q}, \qquad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \mapsto \frac{f_1+1}{f_2+1}.$$

*The Stern-Brocot tree is obtained by replacing each vertex $v$ of* $\mathrm{Tree}(F)$ *with $g(v)$.*

For $k \in \{1,2,3\}$, we define the *intersection matrix flip* $\Phi_k$ of $\mathsf{Int}(L, L_t)$ in direction $k$ as

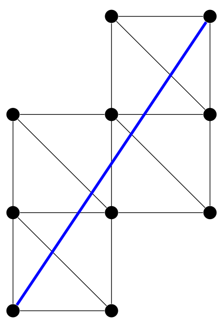
$$\Phi_k(\mathsf{Int}(L, L_t)) = \mathsf{Int}(L, \varphi_k(L_t)). \tag{6.2.3}$$

By regarding punctures on the universal cover of $S$ as lattice points on $\mathbb{R}^2$ with the coordinate axis $\ell_1 = \ell_{1;t_0}$ and $\ell_2 = \ell_{2;t_0}$, we consider the *gradient* of arcs of $L_t$. We denote by $\mathrm{grad}_L(\ell)$ the gradient of $\ell$. We assume $\mathrm{grad}_L(\ell_1) = 0, \mathrm{grad}_L(\ell_2) = \infty, \mathrm{grad}_L(\ell_3) = -1$.

**Example 6.2.2.** We consider an arc $\ell$ in Figure 6.5. Then we have

$$\mathrm{grad}_L(\ell) = \frac{3}{2}, \quad \text{and} \quad \mathsf{Int}(L, \ell) = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Figure 6.5: Arc $\ell$



**Definition 6.2.3.** For $q \in \mathbb{Q} \cup \{\infty\}$, if $n(q), d(q) \in \mathbb{Z}$ satisfy the following conditions, we say that $\dfrac{n(q)}{d(q)}$ is the *reduced expression* of $q$:

- $q = \dfrac{n(q)}{d(q)}$,
- $\gcd(n(q), d(q)) = 1$,
- $d(q) \geq 0$.

Moreover, for a fraction $\dfrac{a}{b}$, if there exists $q \in \mathbb{Q} \cup \{\infty\}$ such that $\dfrac{a}{b}$ is the reduced expression of $q$, then we say that $\dfrac{a}{b}$ is *irreducible*.

This expression is determined uniquely. In particular, $\dfrac{0}{1}, \dfrac{1}{0}$ are reduced expressions of $0, \infty$ respectively.

**Lemma 6.2.4** ([26, Lemma 3.4])**.** *Let $M = (m_1, m_2, m_3)$ a triangulation and $\{i, j, k\} = \{1, 2, 3\}$. The following two conditions are equivalent:*

*(1) Either of the following two inequalities holds:*

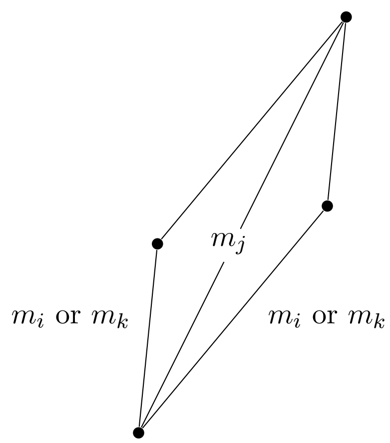
$$\mathrm{grad}_L(m_i) < \mathrm{grad}_L(m_j) < \mathrm{grad}_L(m_k) \quad \textit{or} \quad \mathrm{grad}_L(m_k) < \mathrm{grad}_L(m_j) < \mathrm{grad}_L(m_i). \tag{6.2.4}$$

*(2) We assume that $\frac{a}{b}$ and $\frac{c}{d}$ are irreducible fractions. If $\mathrm{grad}_L(m_i) = \frac{a}{b}$ and $\mathrm{grad}_L(m_k) = \frac{c}{d}$, then $\mathrm{grad}_L(m_j) = \frac{a+c}{b+d}$.*

*In particular, for any triangulation $M$, there exists $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{Z}_{\geq 0}$ such that $\frac{a}{b}$ and $\frac{c}{d}$ are irreducible and $\{\mathrm{grad}_L(m_1), \mathrm{grad}_L(m_2), \mathrm{grad}_L(m_3)\} = \left\{\frac{a}{b}, \frac{c}{d}, \frac{a+c}{b+d}\right\}$.*

*Proof.* We prove that (1) implies (2). We note that $b, d \geq 0$ and therefore $(b, a)$ and $(d, c)$ are in the first or the forth quadrant. Since $M$ is a triangulation, $m_i, m_j, m_k$ is as in Figure 6.6 on the universal covering of $S$. When the coordinate of a point shared by 3 arcs is $(0, 0)$, then the coordinate of the other endpoint of $m_i$ is $(b, a)$, and that of $m_k$ is $(d, c)$. Therefore, that of $m_j$ is $(b + d, a + c)$ and $\mathrm{grad}_L(m_j) = \frac{a+c}{b+d}$. It is clear that (2) implies (1). □

Figure 6.6: Triangulation under the assumption (6.2.4)




**Remark 6.2.5.** We note that if (2) in Lamma 6.2.4 holds, then $\mathrm{grad}_L(m_j) = \frac{a+c}{b+d}$ is irreducible. Indeed, it is shown in the following way: we assume that $\frac{a+c}{b+d}$ is not irreducible. Then $m_j$ passes through a lattice point in the section between $(0, 0)$ and $(a + c, b + d)$. Since all lattice points are a point on $S$, $m_i$ intersects with $m_j$ at non-lattice points. This conflicts that $M$ is a triangulation. Moreover, if $\{\mathrm{grad}_L(m_1), \mathrm{grad}_L(m_2), \mathrm{grad}_L(m_3)\} = \left\{\frac{a}{b}, \frac{c}{d}, \frac{c-a}{d-b}\right\}$ and $\frac{a}{b}, \frac{c}{d}$ are irreducible, then the reduced expression of $\frac{c-a}{d-b}$ is $\frac{c-a}{d-b}$ or $\frac{a-c}{b-d}$. This fact is proved in the same way as the above.

By using the above lemma, we obtain a property for magnitude relation of the gradients of triangulation.

**Lemma 6.2.6** ([26, Lemma 3.6]). *Let $t \in \mathbb{T}_3$, $L_t = (\ell_{1;t}, \ell_{2;t}, \ell_{3;t})$ a triangulation and $\{i, j, k\} = \{1, 2, 3\}$. We assume that*

$$\mathrm{grad}_L(\ell_{i;t}) < \mathrm{grad}_L(\ell_{j;t}) < \mathrm{grad}_L(\ell_{k;t}). \tag{6.2.5}$$

*(1) Let $L_{t'} = \{\ell_{i;t}, \ell_{j;t'}, \ell_{k;t}\}$ be a triangulation of $S$ obtained from $L_t$ by a flip in direction $j$. Then we have*

$$\mathrm{grad}_L(\ell_{j;t'}) < \mathrm{grad}_L(\ell_{i;t}) < \mathrm{grad}_L(\ell_{k;t}) \quad \textit{or} \quad \mathrm{grad}_L(\ell_{i;t}) < \mathrm{grad}_L(\ell_{k;t}) < \mathrm{grad}_L(\ell_{j;t'}),$$

*(2) Let $L_{t''} = \{\ell_{i;t''}, \ell_{j;t}, \ell_{k;t}\}$ be a triangulation of $S$ obtained from $L_t$ by a flip in direction $i$. Then we have*

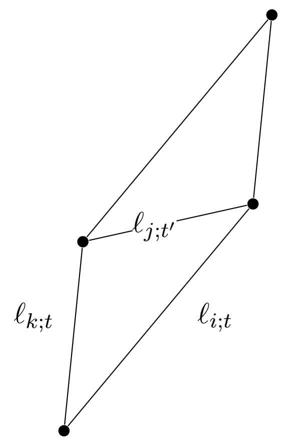
$$\mathrm{grad}_L(\ell_{j;t}) < \mathrm{grad}_L(\ell_{i;t''}) < \mathrm{grad}_L(\ell_{k;t})$$

*(3) Let $L_{t'''} = \{\ell_{i;t}, \ell_{j;t}, \ell_{k;t'''}\}$ be a triangulation of $S$ obtained from $L_t$ by a flip in direction $k$. Then we have*

$$\mathrm{grad}_L(\ell_{i;t}) < \mathrm{grad}_L(\ell_{k;t'''}) < \mathrm{grad}_L(\ell_{j;t}).$$

*Proof.* We prove (1). By flipping the triangulation in Figure 6.6 in direction $j$, we have a triangulation in Figure 6.7. If $\mathrm{grad}_L(\ell_{i;t}) = \dfrac{a}{b}$ and $\mathrm{grad}_L(\ell_{k;t}) = \dfrac{c}{d}$, then $\mathrm{grad}_L(\ell_{j;t}) =$

Figure 6.7: Flipped triangulation




$\dfrac{c-a}{d-b}$. If the reduced expression of $\mathrm{grad}_L(\ell_{j;t})$ is $\dfrac{c-a}{d-b}$, then by Lemma 6.2.4 and (6.2.5), we have $\mathrm{grad}_L(\ell_{i;t}) < \mathrm{grad}_L(\ell_{k;t}) < \mathrm{grad}_L(\ell_{j;t'})$. On the other hand, if the reduced expression is $\dfrac{a-c}{b-d}$, then we have $\mathrm{grad}_L(\ell_{j;t'}) < \mathrm{grad}_L(\ell_{i;t}) < \mathrm{grad}_L(\ell_{k;t})$. The case (2),(3) is also proved in the same way. □

We note that if $t \in \mathbb{T}_3''$, then the gradient of arcs of $L_t$ are 0 or more by Lemma 6.2.6. Let us consider relation between the gradient and the intersection vector of an arc $\ell$ of $L_t$. The following fact is useful:

**Lemma 6.2.7** ([26, Lemma 3.7]). *Let $\ell \in L_t$ be an edge satisfying $t \in \mathbb{T}_3''$ and $\mathsf{Int}(L, \ell) \neq 0$. Then,* $\mathrm{grad}_L(\ell) = \dfrac{a}{b}$ *holds and* $\dfrac{a}{b}$ *is irreducible if and only if* $\mathsf{Int}(L, \ell) = \begin{bmatrix} a-1 \\ b-1 \\ a+b-1 \end{bmatrix}$ *holds.*

**Remark 6.2.8.** By Lemma 6.2.7, for $\ell \in L_t$ satisfying $t \in \mathbb{T}_3''$ and $\mathsf{Int}(L, \ell_t) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq 0$, $\dfrac{a+1}{b+1}$ and $\dfrac{b+1}{a+1}$ are irreducible.

We define the *non-middle gradient flip* as a flip that removes an arc whose gradient is the smallest or the largest in the three arcs and adds another arc. We are ready to prove the first main theorem.

*Proof of Theorem 6.2.1.* Let $t \in \mathbb{T}_3$. For $L_t = (\ell_{1;t}, \ell_{2;t}, \ell_{3;t})$, we consider the triple

$$\mathrm{grad}_L(L_t) = (\mathrm{grad}_L(\ell_{1;t}), \mathrm{grad}_L(\ell_{2;t}), \mathrm{grad}_L(\ell_{3;t})),$$

where all entries are irreducible. According to Lemma 6.2.7, a restriction of $g$ to $\{\mathsf{Int}(L, \ell_{i;t})\}_{t \in \mathbb{T}_3'', i \in \{1,2,3\}} \setminus \{0\}$ is given by $\mathsf{Int}(L, \ell) \mapsto$ (the reduced expression of) $\mathrm{grad}_L(\ell)$. Therefore, it suffices to show that a tree

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & \mathrm{grad}_L(\ell_{2;t_4}) \arrow[r, dash, "1"] \arrow[dr, dash, "3"'] & \mathrm{grad}_L(\ell_{1;t_8}) \cdots \\
 & \mathrm{grad}_L(\ell_{1;t_2}) \arrow[ur, dash, "2"] \arrow[dr, dash, "3"'] & & \mathrm{grad}_L(\ell_{3;t_9}) \cdots \\
 & & \mathrm{grad}_L(\ell_{3;t_5}) \arrow[r, dash, "1"] \arrow[dr, dash, "2"'] & \mathrm{grad}_L(\ell_{1;t_{10}}) \cdots \\
\mathrm{grad}_L(\ell_{3;t_1}) \arrow[uur, dash, "1"] \arrow[ddr, dash, "2"'] & & & \mathrm{grad}_L(\ell_{2;t_{11}}) \cdots \\
 & & \mathrm{grad}_L(\ell_{1;t_6}) \arrow[r, dash, "2"] \arrow[dr, dash, "3"'] & \mathrm{grad}_L(\ell_{2;t_{12}}) \cdots \\
 & \mathrm{grad}_L(\ell_{2;t_3}) \arrow[ur, dash, "1"] \arrow[dr, dash, "3"'] & & \mathrm{grad}_L(\ell_{3;t_{13}}) \cdots \\
 & & \mathrm{grad}_L(\ell_{3;t_7}) \arrow[r, dash, "1"] \arrow[dr, dash, "2"'] & \mathrm{grad}_L(\ell_{1;t_{14}}) \cdots \\
 & & & \mathrm{grad}_L(\ell_{2;t_{15}}) \cdots
\end{tikzcd} \quad . \tag{6.2.6}$$

is the Stern-Brocot tree. Flips in direction 1 and 2 at $t_1$ are non-middle gradient flips, and we find that flips from left to right in (6.2.1) are all non-middle gradient flips inductively by Lemma 6.2.6. Furthermore, $\mathrm{grad}_L(\ell_{k;t})$ lying in (6.2.6) is the second largest number in $\mathrm{grad}_L(L_t)$. Therefore, it suffice to show that a tree

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & \mathrm{grad}_L(L_{t_4}) \arrow[r, dash, "1"] \arrow[dr, dash, "3"'] & \mathrm{grad}_L(L_{t_8}) \cdots \\
 & \mathrm{grad}_L(L_{t_2}) \arrow[ur, dash, "2"] \arrow[dr, dash, "3"'] & & \mathrm{grad}_L(L_{t_9}) \cdots \\
 & & \mathrm{grad}_L(L_{t_5}) \arrow[r, dash, "1"] \arrow[dr, dash, "2"'] & \mathrm{grad}_L(L_{t_{10}}) \cdots \\
\mathrm{grad}_L(L_{t_1}) \arrow[uur, dash, "1"] \arrow[ddr, dash, "2"'] & & & \mathrm{grad}_L(L_{t_{11}}) \cdots \\
 & & \mathrm{grad}_L(L_{t_6}) \arrow[r, dash, "2"] \arrow[dr, dash, "3"'] & \mathrm{grad}_L(L_{t_{12}}) \cdots \\
 & \mathrm{grad}_L(L_{t_3}) \arrow[ur, dash, "1"] \arrow[dr, dash, "3"'] & & \mathrm{grad}_L(L_{t_{13}}) \cdots \\
 & & \mathrm{grad}_L(L_{t_7}) \arrow[r, dash, "1"] \arrow[dr, dash, "2"'] & \mathrm{grad}_L(L_{t_{14}}) \cdots \\
 & & & \mathrm{grad}_L(L_{t_{15}}) \cdots
\end{tikzcd} \quad . \tag{6.2.7}$$

is the Farey triple tree. We have $\mathrm{grad}_L(L_{t_1}) = \left(\dfrac{0}{1}, \dfrac{1}{0}, \dfrac{1}{1}\right)$. We assume $\mathrm{grad}_L(L_t) = \left(\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}\right)$. Because of Lemma 6.2.6 and Lemma 6.2.4, if $\dfrac{a}{b}$ is the smallest or largest in those three, then there exists an edge labeled by 1 on the right of $\mathrm{grad}_L(L_t)$ in (6.2.7), and we have $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) \mapsto \mathrm{grad}_L(L_{t'}) = \left(\frac{c+e}{d+f}, \frac{c}{d}, \frac{e}{f}\right)$ by a flip in direction 1 (we note that all of $\frac{c+e}{d+f}, \frac{c}{d}, \frac{e}{f}$ are irreducible again by Remark 6.2.5). Similarly, if $\frac{c}{d}$ or $\frac{e}{f}$ is the smallest or largest in those three, we have the desired triple. Therefore, (6.2.7) corresponds with the Farey triple tree, and this finishes the proof. $\square$

For the sake of discussion in Section 6.3, we give the explicit description of intersection matrices. First, we consider intersection matrices not containing zero vectors.

**Corollary 6.2.9** ([26, Corollary 3.9])**.** *Let $t \in \mathbb{T}_3''$ and $L_t = (\ell_{1;t}, \ell_{2;t}, \ell_{3;t})$ a triangulation. For all $i \in \{1,2,3\}$, if $\mathsf{Int}(L, \ell_{i;t}) \neq 0$, then $\mathsf{Int}(L, L_t) = (f_{ij})$ satisfies just one of the followings:*

*(i)* $$\begin{bmatrix} f_{11} & f_{12} & f_{11}+f_{12}+1 \\ f_{21} & f_{22} & f_{21}+f_{22}+1 \\ f_{11}+f_{21}+1 & f_{12}+f_{22}+1 & f_{11}+f_{12}+f_{21}+f_{22}+3 \end{bmatrix}$$

*(ii)* $$\begin{bmatrix} f_{12}+f_{13}+1 & f_{12} & f_{13} \\ f_{22}+f_{23}+1 & f_{22} & f_{23} \\ f_{12}+f_{13}+f_{22}+f_{23}+3 & f_{12}+f_{22}+1 & f_{13}+f_{23}+1 \end{bmatrix}$$

*(iii)* $$\begin{bmatrix} f_{11} & f_{11}+f_{13}+1 & f_{13} \\ f_{21} & f_{21}+f_{23}+1 & f_{23} \\ f_{11}+f_{21}+1 & f_{11}+f_{21}+f_{13}+f_{23}+3 & f_{13}+f_{23}+1 \end{bmatrix}$$

*Proof.* It follows from Lemmas 6.2.7 and 6.2.4. $\square$

The following is the key lemma in Section 6.3.

**Lemma 6.2.10** ([26, Lemma 3.10])**.** *We fix $t \in \mathbb{T}_3''$. The intersection matrix $\mathsf{Int}(L, L_t) = (f_{ij})$ satisfies just one of the followings:*

*(i)* $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*(ii)* $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & f_{22} & f_{22}+1 \\ 0 & f_{22}+1 & f_{22}+2 \end{bmatrix}$$

*(iii)* $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & f_{23}+1 & f_{23} \\ 0 & f_{23}+2 & f_{23}+1 \end{bmatrix}$$

*(iv)* $$\begin{bmatrix} f_{11} & 0 & f_{11}+1 \\ 0 & 0 & 0 \\ f_{11}+1 & 0 & f_{11}+2 \end{bmatrix}$$

*(v)* $$\begin{bmatrix} f_{13}+1 & 0 & f_{13} \\ 0 & 0 & 0 \\ f_{13}+2 & 0 & f_{13}+1 \end{bmatrix}$$

*(vi)* $\begin{bmatrix} f_{11} & f_{12} & f_{11}+f_{12}+1 \\ f_{21} & f_{22} & f_{21}+f_{22}+1 \\ f_{11}+f_{21}+1 & f_{12}+f_{22}+1 & f_{11}+f_{12}+f_{21}+f_{22}+3 \end{bmatrix}$

*(vii)* $\begin{bmatrix} f_{12}+f_{13}+1 & f_{12} & f_{13} \\ f_{22}+f_{23}+1 & f_{22} & f_{23} \\ f_{12}+f_{13}+f_{22}+f_{23}+3 & f_{12}+f_{22}+1 & f_{13}+f_{23}+1 \end{bmatrix}$

*(viii)* $\begin{bmatrix} f_{11} & f_{11}+f_{13}+1 & f_{13} \\ f_{21} & f_{21}+f_{23}+1 & f_{23} \\ f_{11}+f_{21}+1 & f_{11}+f_{21}+f_{13}+f_{23}+3 & f_{13}+f_{23}+1 \end{bmatrix}$

*Moreover, we have the following diagram:*

$$\begin{tikzcd}
 & (\mathrm{v}) \arrow[r, shift left, "3"] \arrow[d, "2"] & (\mathrm{iv}) \arrow[l, shift left, "1"] \arrow[dl, "2"] & \\
(\mathrm{i}) \arrow[ur, "1"] \arrow[dr, "2"'] & (\mathrm{viii}) \arrow[r, shift left, "1"] \arrow[rr, bend left, shift left, "3"] & (\mathrm{vii}) \arrow[l, shift left, "2"] \arrow[r, shift left, "3"] & (\mathrm{vi})\ , \arrow[l, shift left, "1"] \arrow[ll, bend right, "2"'] \\
 & (\mathrm{iii}) \arrow[ur, "1"] \arrow[r, shift left, "3"] & (\mathrm{ii}) \arrow[l, shift left, "2"] \arrow[u, "1"'] &
\end{tikzcd}$$

*where* $(\mathrm{n}) \xrightarrow{k} (\mathrm{m})$ *implies that* $\mathsf{Int}(L_t, L)$ *satisfying* (m) *is obtained from* (n) *by* $\Phi_k$*.*

*Proof.* When $t = t_1$, $\mathsf{Int}(L, L_t)$ satisfies (i). Since each intersection matrix flip is an involution, it suffices to consider flips from left to right on (6.2.2), that is, the diagram in the lemma. First, we prove a part of the diagram between (vi)–(viii). We note that an arc of $S$ which is flipped by a flip in the above diagram between (vi)–(viii) has the second largest gradient in the flipped triangulation because of Lemmas 6.2.7, 6.2.6. Therefore, it follows from Corollary 6.2.9. Next, we prove a part of the diagram between (ii) and (iii). We will show the following statement inductively: when $\mathsf{Int}(L, L_t)$ satisfies (ii) or (iii), $\mathrm{grad}_L(\ell_{1;t}) = 0$ and $(\mathrm{iii}) \underset{2}{\overset{3}{\rightleftarrows}} (\mathrm{ii})$ holds. If $\mathsf{Int}(L, L_t) = \Phi_2(\mathsf{Int}(L, L_{t_1}))$, $\mathsf{Int}(L, L_t)$ satisfies (iii) and $\mathrm{grad}_L(\ell_{1;t}) = 0$. Under the assumption, when $\mathsf{Int}(L, L_t)$ satisfies (ii), we have $\mathrm{grad}_L(\ell_{1;t}) < \mathrm{grad}_L(\ell_{3;t}) < \mathrm{grad}_L(\ell_{2;t})$ by Lemma 6.2.7. Therefore, by Lemmas 6.2.6, 6.2.4 and 6.2.7, we have $(\mathrm{ii}) \xrightarrow{2} (\mathrm{iii})$ . We denote by $\mathsf{Int}(L, L_{t'}) = \Phi_2(\mathsf{Int}(L, L_t))$. Since $\ell_{1;t} = \ell_{1;t'}$, we have $\mathrm{grad}_L(\ell_{1;t'}) = 0$. When $\mathsf{Int}(L, L_{t'})$ satisfies (iii), it follows from the same argument as the above. Moreover, $(\mathrm{iii}) \xrightarrow{1} (\mathrm{iv})$ and $(\mathrm{ii}) \xrightarrow{1} (\mathrm{iv})$ follows from the fact that $\mathrm{grad}_L(\ell_{1;t}) = 0$ and Lemmas 6.2.6, 6.2.4, and 6.2.7. The diagram between (iv),(v),(viii) is proved in the same way as the above. Finally, we have $(\mathrm{i}) \xrightarrow{2} (\mathrm{iii})$ and $(\mathrm{i}) \xrightarrow{1} (\mathrm{v})$ because $\mathsf{Int}(L, L_t)$ satisfies (i) if and only if $\mathsf{Int}(L, L_t) = \mathsf{Int}(L, L_{t_1})$. $\square$

## 6.3 Initial intersection vector tree and Calkin-Wilf tree

In contrast to the previous section, We correspond the intersection vectors to vertices of a subtree $\mathbb{T}''_3$ of (6.2.1) as

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & \mathsf{Int}(L_{t_8},\ell_3)\cdots \\
 & & \mathsf{Int}(L_{t_4},\ell_3) \arrow[ur, dash, "1"] \arrow[dr, dash, "3"'] & \\
 & & & \mathsf{Int}(L_{t_9},\ell_3)\cdots \\
 & \mathsf{Int}(L_{t_2},\ell_3) \arrow[uur, dash, "2"] \arrow[ddr, dash, "3"'] & & \\
 & & & \mathsf{Int}(L_{t_{10}},\ell_3)\cdots \\
 & & \mathsf{Int}(L_{t_5},\ell_3) \arrow[ur, dash, "1"] \arrow[dr, dash, "2"'] & \\
 & & & F^{(}(L_{t_{11}},\ell_3)\cdots \\
\mathsf{Int}(L_{t_1},\ell_3) \arrow[uuuur, dash, "1"] \arrow[ddddr, dash, "2"'] & & & \\
 & & & \mathsf{Int}(L_{t_{12}},\ell_3)\cdots \\
 & & \mathsf{Int}(L_{t_6},\ell_3) \arrow[ur, dash, "2"] \arrow[dr, dash, "3"'] & \\
 & & & \mathsf{Int}(L_{t_{13}},\ell_3)\cdots \\
 & \mathsf{Int}(L_{t_3},\ell_3) \arrow[uur, dash, "1"] \arrow[ddr, dash, "3"'] & & \\
 & & & \mathsf{Int}(L_{t_{14}},\ell_3)\cdots \\
 & & \mathsf{Int}(L_{t_7},\ell_3) \arrow[ur, dash, "1"] \arrow[dr, dash, "2"'] & \\
 & & & \mathsf{Int}(L_{t_{15}},\ell_3)\cdots
\end{tikzcd}. \tag{6.3.1}$$

That is, we assign $\mathsf{Int}(L_{t_i},\ell_3)$ to $t_i$. The first seven vertices of it is as follows:

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & \begin{bmatrix}2\\4\\1\end{bmatrix} \\
 & \begin{bmatrix}2\\0\\1\end{bmatrix} \arrow[ur, dash, "2"] \arrow[dr, dash, "3"'] & \\
 & & \begin{bmatrix}2\\0\\3\end{bmatrix} \\
\begin{bmatrix}0\\0\\1\end{bmatrix} \arrow[uur, dash, "1"] \arrow[ddr, dash, "2"'] & & \\
 & & \begin{bmatrix}4\\2\\1\end{bmatrix} \\
 & \begin{bmatrix}0\\2\\1\end{bmatrix} \arrow[ur, dash, "1"] \arrow[dr, dash, "3"'] & \\
 & & \begin{bmatrix}0\\2\\3\end{bmatrix}
\end{tikzcd}.$$

We denote by $\mathrm{Tree}(F^{\dagger})$ and we call it the *initial intersection vector tree.* We make a preparation for describing the main theorem of this section. In this section, we regard $L_t$ as the initial triangulation. We define the *initial intersection matrix flip* $\Psi_k$ of $\mathsf{Int}(L_t, L)$ in direction $k$ as

$$\Psi_k(\mathsf{Int}(L_t, L)) = \mathsf{Int}(\varphi_k(L_t), L). \tag{6.3.2}$$

The following proposition is clear:

**Proposition 6.3.1** ([26, Proposition 4.1])**.** *We have*

$$\mathsf{Int}(L_t, L) = (\mathsf{Int}(L, L_t))^{\top},$$

*Proof.* It follows from Theorem 4.1.5 and Theorem 3.2.35. ☐

By using this duality, we have the following property:

**Proposition 6.3.2** ([26, Proposition 4.2])**.** *For* $t \overset{k}{-\!\!-\!\!-} t' \in \mathbb{T}''_3$*, let*

$$\mathsf{Int}(L_t,\ell_3) = \begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix} \neq 0 \quad \textit{and} \quad \mathsf{Int}(L_{t'},\ell_3) = \begin{bmatrix}f'_1\\f'_2\\f'_3\end{bmatrix} \neq 0.$$

*(1) If $i \neq k$, then we have $f_i = f'_i$.*

*(2) The integer $f_k$ is not maximal in $\{f_1, f_2, f_3\}$ if and only if $f'_k$ is maximal in $\{f'_1, f'_2, f'_3\}$.*

*Proof.* The statement (1) follows from Proposition 6.3.1. We prove "if" part of (2). Since if $f'_k = 0$, then this is not the maximal in $\{f'_1, f'_2, f'_3\}$ clearly, we assume that $f'_k \neq 0$. If $f'_k$ is the maximal in $\{f'_1, f'_2, f'_3\}$, the gradient of an arc corresponding to $k$th row of $\mathsf{Int}(L, L_{t'})$ is the second largest in three arcs because of Proposition 6.3.1 and Lemmas 6.2.7, 6.2.4. By Lemma 6.2.6, the gradient of an arc corresponding to $k$th row of $\mathsf{Int}(L, L_t)$ is not the second largest in three arcs. By Lemma Proposition 6.3.1 and Lemmas 6.2.7, 6.2.4 again, $f_k$ is not maximal in $\{f_1, f_2, f_3\}$. We prove "only if" part of (2). We assume that $f_k = 0$. Then by Proposition 6.3.1 and Lemma 6.2.6, $\mathsf{Int}(L, L_t) = 0$ and the gradient of an arc corresponding to $k$th row of $\mathsf{Int}(L, L_t)$ is 0 or $\infty$. Thus by Lemma 6.2.6, the gradient of an arc corresponding to $k$th row of $\mathsf{Int}(L, L_{t'})$ is the second largest in three arcs. By Proposition 6.3.1 and Lemmas 6.2.7, 6.2.4, $f'_k$ is the maximal in $\{f'_1, f'_2, f'_3\}$. In the case of $f_k \neq 0$, it is proved by considering the inverse of "if" part with $f'_k \neq 0$. □

By Proposition 6.3.2, if $\mathsf{Int}(L_t, \ell_3)$ lies on the right endpoint of an edge labeled by $k$ in the tree of (6.3.1), then the $k$th element of $\mathsf{Int}(L_t, \ell_3)$ is the maximal in those three. In the rest of this section, we prove the following theorem:

**Theorem 6.3.3** ([26, Theorem 4.3])**.** *We define a map*

$$h\colon \{\mathsf{Int}(L_t, \ell_3)\}_{t \in T''_3} \to \mathbb{Q}$$

*inductively as follows: we assign*

$$\mathsf{Int}(L_{t_1}, \ell_3) \mapsto \frac{f_{13;t_1} + 1}{f_{23;t_1} + 1} = \frac{1}{1}.$$

*Let $\{a, b, c\} = \{1, 2, 3\}$. When $\mathsf{Int}(L_t, \ell_3) \mapsto \dfrac{f_{a3;t} + 1}{f_{b3;t} + 1}$, and $\mathsf{Int}(L_t, \ell_3) \overset{k}{\text{---}} \mathsf{Int}(L_{t'}, \ell_3)$ as in (6.3.1),*

- *if $k = a$, then we assign $\mathsf{Int}(L_{t'}, \ell_3) \mapsto \dfrac{f_{c3;t} + 1}{f_{b3;t} + 1}$,*
- *if $k = b$, then we assign $\mathsf{Int}(L_{t'}, \ell_3) \mapsto \dfrac{f_{a3;t} + 1}{f_{c3;t} + 1}$.*

*The Calkin-Wilf tree is obtained by replacing each vertex $v$ of* $\mathrm{Tree}(F^\dagger)$ *with $h(v)$.*

In the rest of this chapter, we prove Theorem 6.3.3. The following lemma is duality of Lemma 6.2.10:

**Lemma 6.3.4** ([26, Lemma 4.4])**.** *We fix $t \in \mathbb{T}''_3$. The intersection matrix $\mathsf{Int}(L_t, L) = (f_{ij})$ satisfies just one of the following:*

*(i)* $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

*(ii)* $\begin{bmatrix} 0 & 0 & 0 \\ 0 & f_{22} & f_{22}+1 \\ 0 & f_{22}+1 & f_{22}+2 \end{bmatrix}$

*(iii)* $\begin{bmatrix} 0 & 0 & 0 \\ 0 & f_{32}+1 & f_{32}+2 \\ 0 & f_{32} & f_{32}+1 \end{bmatrix}$

*(iv)* $\begin{bmatrix} f_{11} & 0 & f_{11}+1 \\ 0 & 0 & 0 \\ f_{11}+1 & 0 & f_{11}+2 \end{bmatrix}$

*(v)* $\begin{bmatrix} f_{31}+1 & 0 & f_{31}+2 \\ 0 & 0 & 0 \\ f_{31} & 0 & f_{31}+1 \end{bmatrix}$

*(vi)* $\begin{bmatrix} f_{11} & f_{12} & f_{11}+f_{12}+1 \\ f_{21} & f_{22} & f_{21}+f_{22}+1 \\ f_{11}+f_{21}+1 & f_{12}+f_{22}+1 & f_{11}+f_{12}+f_{21}+f_{22}+3 \end{bmatrix}$

*(vii)* $\begin{bmatrix} f_{21}+f_{31}+1 & f_{22}+f_{32}+1 & f_{21}+f_{31}+f_{22}+f_{32}+3 \\ f_{21} & f_{22} & f_{21}+f_{22}+1 \\ f_{31} & f_{32} & f_{31}+f_{32}+1 \end{bmatrix}$

*(viii)* $\begin{bmatrix} f_{11} & f_{12} & f_{11}+f_{12}+1 \\ f_{11}+f_{31}+1 & f_{12}+f_{32}+1 & f_{11}+f_{12}+f_{31}+f_{32}+3 \\ f_{31} & f_{32} & f_{31}+f_{32}+1 \end{bmatrix}.$

*Moreover, we have the following diagram:*

$$\begin{tikzcd}
& \mathrm{(v)} \arrow[r, shift left, "3"] \arrow[d, "2"] & \mathrm{(iv)} \arrow[l, shift left, "1"] \arrow[dl, "2"] & \\
\mathrm{(i)} \arrow[ur, "1"] \arrow[dr, "2"'] & \mathrm{(viii)} \arrow[r, shift left, "1"] \arrow[rr, bend left, "3"] & \mathrm{(vii)} \arrow[l, shift left, "2"] \arrow[r, shift left, "3"] & \mathrm{(vi)}\ , \arrow[l, shift left, "1"] \arrow[ll, bend right, "2"'] \\
& \mathrm{(iii)} \arrow[r, shift left, "3"] \arrow[ur, "1"] & \mathrm{(ii)} \arrow[l, shift left, "2"] \arrow[u, "1"'] &
\end{tikzcd}$$

*where* (n) $\xrightarrow{k}$ (m) *implies that* $\mathsf{Int}(L_t, L)$ *satisfying* (m) *is obtained from* (n) *by* $\Psi_k$.

*Proof.* It follows from Lemma 6.2.10 and Proposition 6.3.1. □

Let us prove the main theorem in this section. For a fraction $q$, we denote by $q_n, q_d$ the numerator and denominator of $q$, respectively.

*Proof of Theorem 6.3.3.* It suffice to show the following: for any (n) in (i)–(viii) in the diagram of Lemma 6.3.4 and $\mathsf{Int}(L_t, \ell_3)$ satisfying (n), if

$$h(\mathsf{Int}(L_t, \ell_3))_n = x \quad h(\mathsf{Int}(L_t, \ell_3))_d = y,$$

then we have

$$\begin{aligned} h(\Psi_a(\mathsf{Int}(L_t), \ell_3))_n &= x + y, \quad h(\Psi_a(\mathsf{Int}(L_t), \ell_3))_d = y, \\ h(\Psi_b(\mathsf{Int}(L_t), \ell_3))_n &= x, \quad h(\Psi_b(\mathsf{Int}(L_t), \ell_3))_d = x + y. \end{aligned}$$

In the case that $\mathsf{Int}(L_t, \ell_3)$ satisfies (i), we can check them by direct calculation. We prove the case that $\mathsf{Int}(L_t, \ell_3)$ satisfies (iii). Then we have

$$h(\mathsf{Int}(L_t, \ell_3)) = \frac{f_{13;t_1} + 1}{f_{33;t} + 1} = \frac{1}{f_{33;t} + 1}$$

by inductive argument. By Lemma 6.3.4 and definition of $h$, for $t \overset{3}{\text{———}} t'$,

$$\begin{aligned} h(\mathsf{Int}(L_t, \ell_3)) &= \frac{f_{13;t_1} + 1}{f_{33;t} + 1} = \frac{1}{f_{33;t} + 1} = \frac{1}{f_{32;t} + 2}, \\ h(\mathsf{Int}(L_{t'}, \ell_3)) &= \frac{1}{f_{23;t} + 1} = \frac{1}{f_{32;t} + 3} = \frac{1}{(f_{32;t} + 2) + 1}. \end{aligned}$$

On the other hand, for $t \overset{1}{\text{———}} t''$, we have

$$h(\mathsf{Int}(L_{t''}, \ell_3)) = \frac{f_{23;t} + 1}{f_{33;t} + 1} = \frac{f_{32:t} + 3}{f_{32;t} + 2} = \frac{(f_{32:t} + 2) + 1}{f_{32;t} + 2}.$$

Therefore, in (iii), $\mathsf{Int}(L_t, \ell_3)$ satisfies the desired condition. Next, we prove the case that $\mathsf{Int}(L_t, \ell_3)$ satisfies (ii). We have

$$h(\mathsf{Int}(L_t, \ell_3)) = \frac{f_{13;t_1} + 1}{f_{23;t} + 1} = \frac{1}{f_{23;t} + 1}$$

by inductive argument. By Lemma 6.3.4 and definition of $h$, for $t \overset{2}{\text{———}} t'$ and $t \overset{1}{\text{———}} t''$, we have

$$\begin{aligned} h(\mathsf{Int}(L_t, \ell_3)) &= \frac{1}{f_{23;t} + 1} = \frac{1}{f_{22;t} + 2}, \\ h(\mathsf{Int}(L_{t'}, \ell_3)) &= \frac{1}{f_{33;t} + 1} = \frac{1}{f_{22;t} + 3} = \frac{1}{(f_{22;t} + 2) + 1}, \\ h(\mathsf{Int}(L_{t''}, \ell_3)) &= \frac{f_{33;t} + 1}{f_{23;t} + 1} = \frac{f_{22:t} + 3}{f_{22;t} + 2} = \frac{(f_{22:t} + 2) + 1}{f_{22;t} + 2}. \end{aligned}$$

Therefore, in (ii), $\mathsf{Int}(L_t, \ell_3)$ satisfies the desired condition. By symmetry, we can also prove the case that $\mathsf{Int}(L_t, \ell_3)$ satisfies (iv) or (v). Next, we prove the case that $\mathsf{Int}(L_t, \ell_3)$ satisfies (vi). First, we assume that

$$h(\mathsf{Int}(L_t, \ell_3)) = \frac{f_{13;t} + 1}{f_{23;t} + 1}.$$

By Lemma 6.3.4 and definition of $h$, for $t \overset{1}{\text{——}} t'$ and $t \overset{2}{\text{——}} t''$, we have

$$\begin{aligned}
h(\mathsf{Int}(L_t, \ell_3)) &= \frac{f_{13;t}+1}{f_{23;t}+1} = \frac{f_{11;t}+f_{12;t}+2}{f_{21;t}+f_{22;t}+2}, \\
h(\mathsf{Int}(L_{t'}, \ell_3)) &= \frac{f_{33;t}+1}{f_{23;t}+1} = \frac{f_{11;t}+f_{12;t}+f_{21;t}+f_{22;t}+4}{f_{21;t}+f_{22;t}+2} \\
&= \frac{(f_{11;t}+f_{12;t}+2)+(f_{21;t}+f_{22;t}+2)}{f_{21;t}+f_{22;t}+2}, \\
h(\mathsf{Int}(L_{t''}, \ell_3)) &= \frac{f_{13;t}+1}{f_{33;t}+1} = \frac{f_{11;t}+f_{12;t}+2}{f_{11;t}+f_{12;t}+f_{21;t}+f_{22;t}+4} \\
&= \frac{f_{11;t}+f_{12;t}+2}{(f_{11;t}+f_{12;t}+2)+(f_{21;t}+f_{22;t}+2)}.
\end{aligned}$$

Second, in the case that

$$h(\mathsf{Int}(L_t, \ell_3)) = \frac{f_{23;t}+1}{f_{13;t}+1},$$

we can prove in the same way as the first case. Therefore, in (vi), $\mathsf{Int}(L_t, \ell_3)$ satisfies the desired condition. By symmetry, we can also prove the case that $\mathsf{Int}(L_t, \ell_3)$ satisfies (vii) or (viii). $\square$

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