Cluster categories of formal dg algebras and hereditary Calabi-Yau categories

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Contents

Preface

The main goal of representation theory of algebras is to understand the structure of categories associated to a given algebra. Besides the most fundamental one, namely the category $\mathrm{mod}\,\Lambda$ of modules over an algebra Λ, various triangulated categories are extensively studied, for example the derived category $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$, the singularity category $\mathcal{D}_{\mathrm{sg}}(\Lambda)$, and the cluster category $\mathcal{C}(\Lambda)$. The subject of this thesis is cluster tilting theory in these triangulated categories, especially in cluster categories arising from differential graded (dg for short) algebras.

Cluster tilting theory emerged in the beginning of this century in the contexts of categorification of cluster algebras [FZ], [BMRRT] and higher dimensional Auslander-Reiten theory [Iy1], which has lead to fruitful connections between various areas of mathematics, including commutative or non-commutative algebraic geometry, Lie theory, and singularity theory. A central role is played by the concept of cluster tilting objects which is a categorical counterpart of clusters in a cluster algebra, and at the same time a natural domain for higher dimensional Auslander-Reiten theory. Among such things, Calabi-Yau (CY for short) triangulated categories equipped with cluster tilting objects have been of particular interest.

One of the fundamental examples of such CY triangulated categories with cluster tilting objects are given by Buan–Marsh–Reineke–Reiten–Todorov [BMRRT]. Their *cluster category* is defined for a hereditary algebra H as the orbit category

$$\mathcal{D}^b(\mathrm{mod}\,H)/\tau^{-1}[1].$$

It is a 2-CY triangulated category [Ke2] equipped with a 2-cluster tilting object.

A far-reaching generalization of constructing such triangulated categories is given as Amiot's cluster categories [Am], which is based on the formalism of differential graded (dg) algebras [Ke3]. For a homologically smooth dg algebra Π, its *cluster category* is

$$\mathcal{C}(\Pi) = \mathrm{per}\,\Pi/\mathcal{D}^b(\Pi),$$

the Verdier quotient of the perfect derived category $\mathrm{per}\,\Pi$ by the thick subcategory $\mathcal{D}^b(\Pi)$ consisting of DG modules of finite dimensional total cohomology. The fundamental result due to Amiot and its generalization by Guo [Am, Gu] states that if Π is a $(d+1)$-CY dg algebra, then $\mathcal{C}(\Pi)$ is a d-CY triangulated category and $\Pi \in \mathcal{C}(\Pi)$ is a d-cluster tilting object.

It is these cluster categories we are going to study here.

This thesis consists of two parts, which are based on following two papers by the author, respectively.

[Han1] N. Hanihara, Cluster categories of formal DG algebras and singularity categories, arXiv:2003.7858.

[Han2] ———, Morita theorem for hereditary Calabi-Yau categories, arXiv:2010.14736.

We first explain the background on the theory of cluster categories, and then describe some main results presented in this thesis.

1 Background

In this section we collect preliminary results upon which our discussion is build. Throughout we fix a base field k and denote by $D = \mathrm{Hom}_k(-, k)$ the duality functor on the category of k-vector spaces. We start with the following fundamental notion.

Definition 1.1. Let d be an integer, and let $\mathcal{T}$ be a k-linear, Hom-finite triangulated category. We say that $\mathcal{T}$ is *d-Calabi-Yau* if there are functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) \simeq D\,\mathrm{Hom}_{\mathcal{T}}(Y, X[d])$$

for all $X, Y \in \mathcal{T}$.

CY triangulated categories are ubiquitous in mathematics. The fundamental examples are, as the terminology suggests, the derived categories of Calabi-Yau varieties, which are essential in the homological mirror symmetry conjecture of Kontsevich. They are also classical in representation theory, arising as the singularity categories of commutative Gorenstein rings [Au, Yo, LW], in particular as the homotopy categories of matrix factorizations [E, O].

Recently another class of CY categories, namely the *cluster categories* were invented, motivated by the theory of cluster algebras initiated by Fomin and Zelevinsky [FZ]. The purpose was to categorify the combinatorial structure of cluster algebras. There are several descriptions of a cluster category, namely (the triangulated hull of) the orbit category of the derived category, and the Verdier quotient of the derived category of a dg algebra.

The first definition, as an orbit category, of a cluster category due to Buan–Marsh–Reineke–Reiten–Todorov [BMRRT] is given for a finite acyclic quiver Q by

$$\mathcal{C}(Q) = \mathcal{D}^b(\operatorname{mod} kQ)/\tau^{-1}[1],$$

the orbit category of the derived category of the path algebra kQ by the autoequivalence $\tau^{-1}[1]$, where τ is the Auslander-Reiten translation in $\mathcal{D}^b(\operatorname{mod} kQ)$. Thanks to the fact that kQ is hereditary, that is, the global dimension of kQ is at most 1, the orbit category is canonically triangulated [Ke2]. Moreover $\mathcal{C}(Q)$ is 2-CY and $kQ \in \mathcal{C}(Q)$ is a 2-cluster tilting object in the following sense.

Definition 1.2 ([Iy1, KR1, IYo])**.** Let $d \geq 1$ be an integer. A subcategory $\mathcal{U} \subset \mathcal{T}$ of a triangulated category $\mathcal{T}$ is *d-cluster tilting* if it is functorially finite and satisfies

$$\begin{aligned}\mathcal{U} &= \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(U, X[i]) = 0 \text{ for all } 0 < i < d \text{ and } U \in \mathcal{U}\} \\ &= \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X, U[i]) = 0 \text{ for all } 0 < i < d \text{ and } U \in \mathcal{U}\}.\end{aligned}$$

An object $U \in \mathcal{T}$ is *d-cluster tilting* if its additive closure $\operatorname{add} U$ is a d-cluster tilting subcategory.

It is natural to generalize the cluster category of quivers to those of non-hereditary algebras A and to higher CY dimension d. A natural candidate is the orbit category $\mathcal{D}^b(\operatorname{mod} A)/\nu_d$ by the autoequivalence $\nu_d = - \otimes^{\mathbf{L}}_A DA[-d]$, but the major obstacle is that the orbit category is not necessarily triangulated when A is not hereditary. The generalized cluster category [Am] is defined as the triangulated hull [Ke2] of the orbit category, using a dg enhancement. It turns out (Theorem 1.8) that it can be described in terms of CY dg algebras. We will give this presentation in Definition 1.7, which is the second description of cluster categories.

For a dg algebra Λ we denote by $\operatorname{per} \Lambda$ the perfect derived category, that is, the smallest full triangulated subcategory of the derived category $\mathcal{D}(\Lambda)$ of Λ containing Λ and closed under direct summands. When Λ is an ordinary algebra then $\operatorname{per} \Lambda$ is nothing but the homotopy category of finitely generated projective Λ-modules. Also we denote by Λ^e the enveloping algebra $\Lambda^{\mathrm{op}} \otimes_k \Lambda$ of a dg algebra Λ.

Definition 1.3 ([Gi])**.** Let Π be a dg algebra.

(1) Π is *homologically smooth* if it is perfect as a bimodule, that is, $\Pi \in \operatorname{per} \Pi^e$.

(2) Π is $(d+1)$-*Calabi-Yau* if it is homologically smooth and there is an isomorphism

$$\operatorname{RHom}_{\Pi^e}(\Pi, \Pi^e)[d+1] \simeq \Pi$$

in the derived category $\mathcal{D}(\Pi^e)$ of bimodules.

Suppose that Π is a homologically smooth dg algebra. Slightly generalizing Amiot's definition for CY dg algebras, the following notion of cluster categories for arbitrary homologically smooth dg algebras makes sense. Note that these cluster categories for certain non-CY dg algebras will be important in our later discussion.

Definition 1.4 ([Am]). Let Π be a homologically smooth dg algebra. We have $\mathcal{D}^b(\Pi) \subset \operatorname{per}\Pi$, and call the Verdier quotient

$$\mathcal{C}(\Pi) := \operatorname{per}\Pi/\mathcal{D}^b(\Pi)$$

the *cluster category* of Π.

The following result due to Amiot, and its generalization by Guo, based on the works of Keller, is the fundamental result on cluster categories.

Theorem 1.5 ([Am, Gu]). *Let Π be a $(d+1)$-Calabi-Yau dg algebra such that $H^i\Pi = 0$ for $i > 0$ and $H^0\Pi$ is finite dimensional. Then its cluster category $\mathcal{C}(\Pi)$ is d-CY and $\Pi \in \mathcal{C}(\Pi)$ is a d-cluster tilting object.*

Let us explain how this recovers the previous definition as an orbit category. First of all we need to give examples of CY dg algebras, which is already highly non-trivial. For a dg algebra Λ and a dg (Λ, Λ)-bimodule M, the *derived tensor algebra* of M over Λ is the tensor algebra of a bimodule cofibrant resolution P of M, that is, the complex

$$T^{\mathrm{L}}_{\Lambda}(M) = \Lambda \oplus P \oplus (P \otimes_{\Lambda} P) \oplus (P \otimes_{\Lambda} P \otimes_{\Lambda} P) \oplus \cdots$$

with the natural multiplication. It does not depend on the choice of P up to quasi-isomorphism.

Theorem 1.6 ([Ke6]). *Let d be an integer and Λ a homologically smooth dg algebra. Then the derived tensor algebra*

$$\mathbf{\Pi}_{d+1}(\Lambda) := T^{\mathrm{L}}_{\Lambda}(\operatorname{RHom}_{\Lambda^e}(\Lambda.\Lambda^e)[d])$$

is a $(d+1)$-CY dg algebra. We call this $\mathbf{\Pi}_{d+1}(\Lambda)$ the $(d+1)$-derived preprojective algebra *or the* $(d+1)$-CY completion *of Λ.*

Applying the construction in Theorem 1.5 to the CY completion of a finite dimensional algebra, one obtains its cluster category, which is a d-CY triangulated category with a d-cluster tilting object.

Definition 1.7 ([Am]). Let A be a finite dimensional algebra which is ν_d-finite. The *d-cluster category* $\mathcal{C}_d(A)$ of A is

$$\mathcal{C}_d(A) := \mathcal{C}(\mathbf{\Pi}_{d+1}(A)),$$

the cluster category of the $(d+1)$-CY completion of A.

This definition recovers the previous one by [BMRRT] as an orbit category as follows.

Theorem 1.8 ([Ke2, Am]). *Let $\Pi = \mathbf{\Pi}_{d+1}(A)$ be the $(d+1)$-CY completion of A. Then the functor $-\otimes^{\mathrm{L}}_A \Pi \colon \mathcal{D}^b(\operatorname{mod} A) \to \operatorname{per}\Pi$ induces a fully faithful functor*

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_d \hookrightarrow \mathcal{C}(\Pi),$$

and $\mathcal{C}(\Pi)$ is the triangulated hull of the orbit category.

2 Cluster categories of formal dg algebras and singularity categories

The aim of the first part of this thesis is to provide a new class of CY dg algebras and study its cluster category. It turns out that these cluster categories give systematic examples of cluster categories arising from some roots of the Auslander–Reiten translations.

Let us briefly describe some of the results in this first part, see Section 1.1 for a detailed account. Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a graded algebra. Similarly to Definition 1.3, we have the notion of CY property for (ordinary, non-dg) algebras (see Definition 1.4.1). The fundamental examples are (commutative) polynomial rings. CY algebras can be considered as their non-commutative analogue, and play an important role in non-commutative algebraic geometry, see [AS] and also the references in Section 1.1.1.

Given a graded algebra R, one can view it as a dg algebra with trivial differentials, which we denote by R^{dg}. Such dg algebras are called *formal*. The following observation gives a direct relationship between the CY properties for a graded non-dg algebra and for dg algebras. We refer to Theroem 1.5.2 for a precise statement.

Theorem 2.1 (Theorem 1.5.2)**.** *Let R be a graded $(d+1)$-CY algebra of a-invariant a. Then R^{dg} is sign twisted $(d+a+1)$-CY.*

Although our dg algebra R^{dg} is not CY in general, it is homologically smooth so that one can define the cluster category $\mathcal{C}(R^{\mathrm{dg}})$, and is close enough to being CY so that $R \in \mathcal{C}(R^{\mathrm{dg}})$ is a cluster tilting object. Our goal is to study this cluster category $\mathcal{C}(R^{\mathrm{dg}})$.

Suppose from now on that our CY algebra R is negatively graded, that is, $R_i = 0$ for $i > 0$, and R_0 is finite dimensional. This implies that each R_i is finite dimensional. Define the finite dimensional algebra A by

$$A = \begin{pmatrix} R_0 & 0 & \cdots & 0 \\ R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \end{pmatrix}.$$

Our second result is the relationship between the cluster category $\mathcal{C}(R^{\mathrm{dg}})$ and the derived category of A. We denote by ν the Serre functor $-\otimes^{\mathrm{L}}_A DA$ of $\mathcal{D}^b(\mathrm{mod}\, A)$ and $\nu_d = \nu \circ [-d]$.

Theorem 2.2 (Corollary 1.6.2)**.** *Let R be a negatively graded $(d+1)$-CY algebra of a-invariant a such that R_0 is finite dimensional.*

(1) *The d-AR translation ν_d on $\mathcal{D}^b(\mathrm{mod}\, A)$ has an a-th root $\nu_d^{1/a}$.*

(2) *There exists a fully faithful functor*

$$\mathcal{D}^b(\mathrm{mod}\, A)/\nu_d^{-1/a}[1] \hookrightarrow \mathcal{C}(R^{\mathrm{dg}}),$$

whose image generates $\mathcal{C}(R^{\mathrm{dg}})$ as a thick subcategory.

The existence of a square root of the AR translation appears in [KMV] for generalized Kronecker quivers using the symmetry of the quiver. The first assertion above reduces to this case by a suitable choice of R. Our observation gives a systematic construction of such roots of AR translations, even for higher AR translations.

Concerning the second assertion, this is an analogue of a description of cluster categories as orbit categories. Notice that it has a peculiar feature of involving a root of ν_d. Since we can formally write $\nu_d^{-1/a}[1] = (\nu_d^{-1} \circ [a])^{1/a} = \nu_{d+a}^{-1/a}$, our orbit category is

$$\mathcal{D}^b(\mathrm{mod}\, A)/\nu_d^{-1/a}[1] = \mathcal{D}^b(\mathrm{mod}\, A)/\nu_{d+a}^{-1/a},$$

thus it can be viewed as a "$\mathbb{Z}/a\mathbb{Z}$-quotient" of a usual cluster category.

Our next result is a description of the cluster category $\mathcal{C}(R^{\mathrm{dg}})$ as a singularity category. Let a finite dimensional algebra A be as above, and define the (A, A)-bimodule U and the trivial extension algebra B as follows.

$$U = U(R) = \begin{pmatrix} R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \\ R_{-a} & R_{-(a-1)} & \cdots & R_{-1} \end{pmatrix}, \qquad B = A \oplus U$$

It turns out that this finite dimensional algebra B is d-Iwanaga-Gorenstein (Proposition 1.6.3) in the sense that $\mathrm{inj.dim}\, B \leq d$ on each side. It is an important task to construct Iwanaga-Gorenstein algebras whose singularity categories are CY and have cluster tilting objects, see Section 1.1.2. The following result gives a rich source of such Gorenstein algebras.

Theorem 2.3 (Theorem 1.6.4)**.** *There exists a triangle equivalence*

$$\mathcal{C}(R^{\mathrm{dg}}) \simeq \mathcal{D}_{\mathrm{sg}}(B).$$

In particular, $\mathcal{D}_{\mathrm{sg}}(B)$ is a twisted $(d+a)$-CY category with a $(d+a)$-cluster tilting object.

As a very special case of $a = 1$ we obtain the following equivalence between a cluster category and a singularity category.

Corollary 2.4 (Corollary 1.9.8)**.** *Let A be a d-representation infinite algebra, $U = \mathrm{Ext}^d_A(DA, A)$ an (A, A)-bimodule, and $B = A \oplus U$. Then there exists a triangle equivalence $\mathcal{C}_{d+1}(A) \simeq \mathcal{D}_{\mathrm{sg}}(B)$.*

This gives a partial generalization of a result by Buan–Iyama–Reiten–Scott in the following way. In [BIRSc], they associated for each finite acyclic quiver Q and an element w in the Coxeter group W of Q a finite dimensional 1-Iwanaga-Gorenstein algebra Π_w whose singularity category is 2-CY and has a 2-cluster tilting object. Moreover when Q is non-Dynkin and w is the square of a Coxeter element c of W, then they gave an equivalence [BIRSc, Theorem III.3.4]

$$\mathcal{D}_{\mathrm{sg}}(\Pi_{c^2}) \simeq \mathcal{C}_2(kQ).$$

Now Π_{c^2} is nothing but the trivial extension algebra $kQ \oplus \mathrm{Ext}^1_{kQ}(D(kQ), kQ)$. Our result above is therefore a generalization of this equivalence to higher d.

3 Morita theorem for hereditary Calabi-Yau categories

The aim of the second part of this thesis is to give a Morita-type result for CY categories, which is a certain converse to the construction in the first part. Most classically, Morita theory gives a characterization of module categories among abelian categories in terms of projective generators [Ga]. Similarly, its version for triangulated categories specifies derived categories in terms of tilting objects [Ke1]. We investigate its analogue for CY triangulated categories, which attempts to characterize cluster categories in terms of cluster tilting objects.

In general it is quite difficult to reconstruct the triangulated category from the endomorphism algebra of a cluster tilting object, and the only known results on such Morita-type theorems are essentially the following two, due to Keller–Reiten and Keller–Murfet–Van den Bergh.

Theorem 3.1. *Let $\mathcal{T}$ be an algebraic d-CY triangulated category with a d-cluster tilting object T.*

1. *[KR2] Suppose $d = 2$ and $\mathrm{End}_{\mathcal{T}}(T) = kQ$ for some acyclic quiver Q. Then there exists a triangle equivalence $\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\, kQ)/\tau^{-1}[1]$.*
2. *[KMV] Suppose $d = 3$ and $\mathrm{End}_{\mathcal{T}}(T) = k$. Put $\dim_k \mathrm{Hom}_{\mathcal{T}}(T, T[-1]) = m$. Then there exists a triangle equivalence $\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\, kQ_m)/\tau^{-1/2}[1]$ for the generalized Kronecker quiver $Q_m\colon \circ \Longrightarrow \circ$ with m arrows, and a naturally defined square root $\tau^{1/2}$ of the AR translation.*

The main result of this second part is the following which encompasses both of the above cases.

Theorem 3.2 (=2.4.13)**.** *Let $d \geq 2$ and let $\mathcal{T}$ be an algebraic d-CY triangulated category with a d-cluster tilting object T. Assume that $\mathrm{End}_{\mathcal{T}}(T \oplus T[-1] \oplus \cdots \oplus T[-(d-2)]) = kQ$ for a non-Dynkin quiver Q. Then there exists a triangle equivalence*

$$\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\, kQ)/\tau^{-1/(d-1)}[1]$$

for a naturally defined $(d-1)$-st root $\tau^{1/(d-1)}$ of the AR translation.

We refer to Section 2.1 for the definition of $\tau^{-1/(d-1)}$. When $d = 2$ this gives Keller–Reiten's theorem 3.1(1) for non-Dynkin quivers. Also when $d = 3$ and $\operatorname{End}_{\mathcal{T}}(T) = k$, the algebra $\operatorname{End}_{\mathcal{T}}(T \oplus T[-1])$ in 3.2 is a generalized Kronecker quiver in 3.1(2), thus our result gives Keller–Murfet–Van den Bergh's theorem for $m \geq 2$.

Although the assumption that $\operatorname{End}_{\mathcal{T}}(T \oplus T[-1] \oplus \cdots \oplus T[-(d-2)])$ is hereditary looks restrictive, it in fact follows from hereditaryness of a smaller algebra, which allows us to relax the assumption in the main theorem 3.2.

Theorem 3.3 (cf. 2.1.3)**.** *Let $\mathcal{T}$ be a d-CY triangulated category and $T \in \mathcal{T}$ a d-cluster tilting object. Suppose that* $\operatorname{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(d-3)])$ *is hereditary. Then so is* $\operatorname{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(d-3)] \oplus T[-(d-2)])$.

Moreover we explicitly describe the quiver of $H = \operatorname{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(d-2)])$ in terms of AR $(d+2)$-angles. Combining the above two results together with this description of H, we obtain the following consequence as a very special case. The assertion (1) can be seen a 3-CY version of 3.1(1) as well as a generalization of 3.1(2). Also (2) gives a 4-CY version of Keller–Murfet–Van den Bergh's theorem 3.1(2).

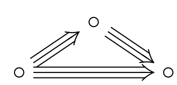
Corollary 3.4. *Let $\mathcal{T}$ be an algebraic d-CY triangulated category with a d-cluster tilting object T.*

(1) *Suppose $d = 3$ and $\operatorname{End}_{\mathcal{T}}(T) = kQ$ for some acyclic quiver Q. Then $\operatorname{End}_{\mathcal{T}}(T \oplus T[-1]) = k\widetilde{Q}$ for some acyclic quiver $\widetilde{Q}$, and there exists a triangle equivalence*

$$\mathcal{T} \simeq \mathcal{D}^b(\operatorname{mod} k\widetilde{Q})/\tau^{-1/2}[1],$$

provided each connected component of $\widetilde{Q}$ is non-Dynkin.

(2) *Suppose $d = 4$ and $\operatorname{End}_{\mathcal{T}}(T) = k$. Put $\dim_k \operatorname{Hom}_{\mathcal{T}}(T, T[-1]) = m$. Then there exists a triangle equivalence $\mathcal{T} \simeq \mathcal{D}^b(\operatorname{mod} kQ'_m)/\tau^{-1/3}[1]$ for the "generalized $\widetilde{A_2}$ quiver" Q'_m below with m arrows, and a naturally defined third root $\tau^{1/3}$ of the AR translation.*



Finally we give examples arising from Calabi-Yau reductions of a higher cluster category of a finite dimensional algebra (Section 2.7) and of the singularity category of an invariant subring (Section 2.8).

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Part 1

Cluster categories of formal dg algebras and singularity categories

1.1 Introduction

This part is based on [Han1]. The aim is to give some descriptions of Amiot's cluster category in Definition 1.4 and Theorem 1.5 for a certain class of dg algebras, namely *formal* dg algebras. Recall that a dg algebra is formal if it is isomorphic to its cohomology in the homotopy category of dg algebras. We therefore start our discussion with a graded (non-dg) algebra R and view it as a dg algebra R^{dg} with trivial differentials.

1.1.1 Cluster categories and orbit categories

Our first observation is that we can obtain a class of CY dg algebras from certain graded (non-dg) algebras. Recall the distinct notion of CY algebra for graded non-dg algebras; a graded algebra R over a field k is *bimodule* $(d+1)$*-CY of* a*-invariant* a if it is homologically smooth and there is an isomorphism

$$\mathrm{RHom}_{R^e}(R, R^e)(a)[d+1] \simeq R$$

in the derived category $\mathcal{D}(\mathrm{Mod}^{\mathbb{Z}} R^e)$ of graded bimodules (this should not be confused with the derived category $\mathcal{D}((R^{\mathrm{dg}})^e)$ of the dg algebra $(R^{\mathrm{dg}})^e$). Here (1) is the degree shift functor on the graded modules, while $[1]$ is the suspension in the derived category. Such algebras arise naturally and are studied extensively in representation theory and commutative or non-commutative algebraic geometry [AS, YZ, Boc1, KS, IR, BS, BSW, MM, AIR, V, RR].

It is well-known that among CY algebras, those of a-invariant 1 are fundamental in the sense that it is the higher preprojective algebra [IO] of its degree 0 part [Ke6, MM, HIO, AIR]. Although our results are already non-trivial for $a = 1$, we study CY algebras of arbitrary a-invariant, which exhibits some additional symmetries.

Let R be a CY algebra. We view it as a dg algebra with vanishing differentials, which we denote by R^{dg}, and study its properties. Note that the gradings on R and on R^{dg} are of different nature; the first one is 'algebraic' while the second one is 'cohomological' (see [Ye, Section 3.1, 15.1]). Such homological properties of dg algebras have been investigated for example in [HM, MGYC]. The following observation shows a relationship between the CY properties of R and R^{dg}. In particular, we obtain from a graded CY algebra a dg algebra which is always very close to being CY, and often in fact CY. We refer to Theorem 1.5.2 for a precise statement. Here we do not need any additional assumptions on R such as (R0) etc below.

Proposition 1.1.1 (Theorem 1.5.2)**.** *Let R be a graded bimodule $(d+1)$-CY algebra of a-invariant a. Then R^{dg} is sign twisted bimodule $(d+a+1)$-CY.*

For a dg algebra Λ satisfying $\operatorname{per}\Lambda \supset \mathcal{D}^b(\Lambda)$, we set

$$\mathcal{C}(\Lambda) := \operatorname{per}\Lambda/\mathcal{D}^b(\Lambda)$$

and call it, by abuse of language, the *cluster category* of Λ. If Λ is a CY dg algebra (e.g. the derived preprojective algebra [Ke6] of a finite dimensional algebra, and the Ginzburg dg algebra [Gi] associated to a quiver with potential) then $\mathcal{C}(\Lambda)$ is the usual cluster category introduced in [Am]. Although our dg algebra R^{dg} is not CY in general, it is close enough to CY so that we can define the cluster category $\mathcal{C}(R^{\mathrm{dg}})$, which gives rise to a cluster tilting object. To understand this category we first study the categories arising from the graded algebra R, and then compare with those arising from R^{dg}.

We now assume the following on the CY algebra R.

(R0) R is negatively graded.

(R1) Each R_i is finite dimensional.

We note that the condition (R0) can be replaced by positive grading up to Theorem 1.1.2 below, but *negative* grading will be essential in the later discussion.

Let $\operatorname{per} R$ be the perfect derived category of R, that is, the thick subcategory of $\mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} R)$ generated by the finitely generated graded projective modules. Also let $\mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}} R)$ be the bounded derived category of graded R-modules of finite length. We set

$$\operatorname{qper}^{\mathbb{Z}} R := \operatorname{per}^{\mathbb{Z}} R/\mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}} R).$$

When R is Noetherian (or more generally graded coherent), we have $\operatorname{per}^{\mathbb{Z}} R = \mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}} R)$, the bounded derived category of finitely presented graded R-modules. Then the Verdier quotient $\operatorname{qper}^{\mathbb{Z}} R$ is nothing but the derived category of the Serre quotient $\operatorname{qgr} R = \operatorname{mod}^{\mathbb{Z}} R/ \operatorname{fl}^{\mathbb{Z}} R$, which is regarded as the category of coherent sheaves over the non-commutative projective scheme [AZ] and plays an essential role in non-commutative algebraic geometry. Our category $\operatorname{qper}^{\mathbb{Z}} R$ is thus a generalization of the derived category $\mathcal{D}^b(\operatorname{qgr} R)$.

Our first main result is the existence of a natural cluster tilting subcategory in $\operatorname{qper}^{\mathbb{Z}} R$, which is of independent interest. More importantly, we prove that the construction of $\operatorname{qper}^{\mathbb{Z}} R$ as the Verdier quotient $\operatorname{per}^{\mathbb{Z}} R/\mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}} R)$ lies on the context of Iyama–Yang's formulation [IYa1] of Amiot's cluster category, see Theorem 1.2.6 and Theorem 1.4.4, which consequently yields a cluster tilting subcategory.

Theorem 1.1.2 (Theorem 1.4.4(3))**.** *Let R be a graded bimodule $(d+1)$-CY algebra of a-invariant a satisfying (R0) and (R1). Then the subcategory*

$$\operatorname{add}\{R(-i)[i] \mid i \in \mathbb{Z}\} \subset \operatorname{qper}^{\mathbb{Z}} R$$

is a $(d+a)$-cluster tilting subcategory.

For example, by setting R to be the polynomial ring with standard positive grading, we deduce that the derived category of coherent sheaves over the projective space $\mathbb{P}^d$ has a $(2d+1)$-cluster tilting subcategory $\operatorname{add}\{\mathcal{O}(i)[i] \mid i \in \mathbb{Z}\}$ (cf. [HIO]).

Now we compare the derived categories of the graded algebra R and that of the dg algebra R^{dg}. An important step is to consider the *total module* (see Section 1.5), which gives a dg functor

$$\operatorname{Tot}\colon \mathcal{C}(\operatorname{Mod}^{\mathbb{Z}} R) \to \mathcal{C}_{\mathrm{dg}}(R^{\mathrm{dg}})$$

on the dg categories of complexes of graded R-modules and of dg R^{dg}-modules, and in turn induces a functor on the derived categories. We deduce the following result as a consequence of Theorem 1.1.2 above.

Corollary 1.1.3 (Theorem 1.6.1)**.** *The functor* Tot *induces a fully faithful functor*

$$\operatorname{qper}^{\mathbb{Z}} R/(-1)[1] \to \mathcal{C}(R^{\mathrm{dg}})$$

whose image generates $\mathcal{C}(R^{\mathrm{dg}})$ as a thick subcategory.

This is a cluster category analogue of the result in [KY1, Theorem 1.3] for the perfect derived category. Note that this gives a reasonable description of the cluster category since on $\mathcal{C}(R^{\mathrm{dg}})$ the degree shift and the suspension is identified, and more accessible in the sense that derived categories are sometimes explicitly described.

Now we apply Minamoto–Mori's equivalence [MM] (see Proposition 1.4.6); there exists a triangle equivalence $\operatorname{qper}^{\mathbb{Z}} R \simeq \mathcal{D}^b(\operatorname{mod} A)$ for the finite dimensional algebra

$$A = A(R) = \begin{pmatrix} R_0 & 0 & \cdots & 0 \\ R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \end{pmatrix}. \tag{1.1.1}$$

This algebra is *d-representation infinite*, which is fundamental in higher dimensional Auslander-Reiten theory [HIO] and non-commutative algebraic geometry [Mi, MM]. By the derived equivalence above we deduce that the autoequivalence $\nu_d = - \otimes^L_A DA[-d]$ of $\mathcal{D}^b(\mathrm{mod}\, A)$ has an a-th root $\nu_d^{1/a}$ (see (1.4.1)). Then we can rewrite Corollary 1.1.3 as a fully faithful functor

$$\mathcal{D}^b(\mathrm{mod}\, A)/\nu_d^{-1/a}[1] \hookrightarrow \mathcal{C}(R^{\mathrm{dg}}) \tag{1.1.2}$$

whose image generates $\mathcal{C}(R^{\mathrm{dg}})$ as a thick subcategory. Note that we can formally write $\nu_d^{-1/a}[1]$ as $\nu_{d+a}^{-1/a}$, thus $\mathcal{C}(R^{\mathrm{dg}})$ can be regarded as a '$\mathbb{Z}/a\mathbb{Z}$-quotient' of the $(d+a)$-cluster category of A in the sense that it is obtained from an a-th root of the automorphism ν_{d+a}.

The existence of a square root of the AR translation appears in [KMV] and was important in their structure theorem for certain CY categories [KMV, Theorem 1.4]. Our result (1.1.2) is an interpretation and a generalization of a situation of their theorem. We discuss in examples (see Example 1.4.10 and 1.6.6) how our results specialize to their setting.

1.1.2 Cluster categories and singularity categories

We further describe the cluster category as a singularity category. Recall that the *singularity category* $\mathcal{D}_{\mathrm{sg}}(\Lambda)$ of a Noetherian ring Λ is the Verdier quotient $\mathcal{D}^b(\mathrm{mod}\,\Lambda)/\mathrm{per}\,\Lambda$, which is widely studied in representation theory and algebraic geometry. If Λ is *Iwanaga-Gorenstein* in the sense that the free module Λ has finite injective dimension on left and right, then $\mathcal{D}_{\mathrm{sg}}(\Lambda)$ is canonically equivalent to the stable category $\underline{\mathrm{CM}}\,\Lambda$ of Cohen-Macaulay modules [Bu]. In the context of cluster tilting theory, Iwanaga-Gorenstein algebras which are stably CY and admit cluster tilting objects, together with the relationship between the cluster categories, have been of particular interest [GLS, Iy1, KR1, IYo, Am, BIRSc, AIRT, ART, KMV, IO, AO4, AIR, TV].

Let a finite dimensional algebra $A = A(R)$ as in (1.1.1) and an (A, A)-bimodule U be

$$U = U(R) = \begin{pmatrix} R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \\ R_{-a} & R_{-(a-1)} & \cdots & R_{-1} \end{pmatrix}.$$

This is a 'relative' d-APR tilting of A. We have a trivial extension algebra

$$B = B(R) = A \oplus U,$$

which turns out to be d-Iwanaga-Gorenstein (Proposition 1.6.3). Our second main result is a description of the cluster category $\mathcal{C}(R^{\mathrm{dg}})$ of a dg algebra in terms of a finite dimensional Iwanaga-Gorenstein algebra B.

Theorem 1.1.4 (Theorem 1.6.4)**.** *There exists a triangle equivalence*

$$\mathcal{C}(R^{\mathrm{dg}}) \simeq \mathcal{D}_{\mathrm{sg}}(B)$$

In particular, $\mathcal{D}_{\mathrm{sg}}(B)$ *is a twisted* $(d+a)$*-CY category with a* $(d+a)$*-cluster tilting object.*

To build the equivalence above, we need a general result on dg orbit categories (Theorem 1.1.5 below). Let us explain the connection, for simplicity, in the case $a = 1$.

In this case, R is bimodule $(d+1)$-CY of a-invariant 1, thus it is the $(d+1)$-preprojective algebra of its degree 0 part, which is A in (1.1.1). Then R^{dg} is the *derived* $(d+2)$*-preprojective algebra* (or the $(d+2)$*-CY completion*)

$$\mathbf{\Pi}_{d+2}(A) = T^L_A\,\mathrm{RHom}_A(DA, A)[d+1]$$

in the sense of [Ke6], thus its cluster category $\mathcal{C}(R^{\mathrm{dg}})$ is the $(d+1)$-cluster category $\mathcal{C}_{d+1}(A)$ of A. On the other hand, we have another description of this cluster category $\mathcal{C}_{d+1}(A)$ as a certain singularity category; setting

$$C = A \oplus DA[-d-2],$$

there exists an equivalence

$$\mathcal{C}(R^{\mathrm{dg}}) = \operatorname{per} R^{\mathrm{dg}}/\mathcal{D}^b(R^{\mathrm{dg}}) \simeq \operatorname{thick}_{\mathcal{D}(C)} A/\operatorname{per} C$$

by the relative Koszul dual [Am]. Therefore the equivalence we need is one between the singularity categories

$$\operatorname{thick}_{\mathcal{D}(C)} A/\operatorname{per} C \simeq \operatorname{thick}_{\mathcal{D}(B)} A/\operatorname{per} B = \mathcal{D}_{\mathrm{sg}}(B).$$

Note that they are precisely Keller's description of triangulated hulls [Ke2], and their equivalence is a consequence of a general equivalence of triangulated hulls, which is our third main result.

Let $\mathcal{A}$ be a pretriangulated dg category, and let F, G be dg endofunctors on $\mathcal{A}$ inducing mutually inverse equivalences on $H^0\mathcal{A}$. We then have dg orbit categories $\mathcal{A}/F$ and $\mathcal{A}/G$, whose perfect derived categories give triangulated hulls of $H^0\mathcal{A}/H^0F \simeq H^0\mathcal{A}/H^0G$. Our result shows that these triangulated hulls are equivalent.

Theorem 1.1.5 (Theorem 1.7.1)**.** *Suppose there exists a natural transformation $G \circ F \to 1_{\mathcal{A}}$ inducing a natural isomorphism on $H^0\mathcal{A}$. Then the dg orbit categories $\mathcal{A}/F$ and $\mathcal{A}/G$ are quasi-equivalent. In particular, the triangulated hulls* $\operatorname{per}(\mathcal{A}/F)$ *and* $\operatorname{per}(\mathcal{A}/G)$ *are equivalent.*

We obtain the singular equivalence of B and C by applying this general result to $\mathcal{A} = \mathcal{C}^b(\operatorname{proj} A)$, $F = - \otimes_A p(\mathrm{RHom}_A(U[1], A))$, and $G = - \otimes_A p(U[1])$ (Corollary 1.7.6), where $p(-)$ means a bimodule projective resolution.

As one of applications and examples of our main results, we give a realization of certain higher cluster categories as singularity categories.

Theorem 1.1.6 (Theorem 1.9.1)**.** *Any m-cluster category of a d-representation infinite algebra with $m > d$ is a singularity category of a d-Iwanaga-Gorenstein algebra.*

For example, any (higher) cluster category of a non-Dynkin quiver is the singularity category of a 1-Iwanaga-Gorenstein algebra. Moreover, we can explicitly describe the Iwanaga-Gorenstein algebra, see Theorem 1.9.1 and Proposition 1.9.4. This should be compared with the results in [HJ], where they give a description of higher cluster categories of 1-representation finite algebras (or of Dynkin types) in terms of singularity categories of self-injective algebras, using a combinatorial method.

We also give systematic examples for the case R is a polynomial ring (Section 1.10), and consider examples arising from dimer models (Section 1.11).

1.2 Preliminaries

We recall some basic concepts on certain structures in triangulated categories. At the end of this section we state Iyama–Yang's result (Theorem 1.2.6) which gives a general framework for the construction of 'cluster-like' categories.

Let us start with the following fundamental notion.

Definition 1.2.1. An object or a subcategory $\mathcal{M}$ in a triangulated category $\mathcal{T}$ is *silting* if $\operatorname{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[>0]) = 0$ and $\operatorname{thick} \mathcal{M} = \mathcal{T}$.

The standard example of a silting object is $\Lambda \in \operatorname{per} \Lambda$ for a negative dg algebra Λ, that is, a dg algebra with $H^{>0}\Lambda = 0$. The same holds for negative dg categories.

Let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of a triangulated category $\mathcal{T}$. We set

$$\mathcal{C} * \mathcal{D} = \{X \in \mathcal{T} \mid \text{there is a triangle } C \to X \to D \to C[1] \text{ for some } C \in \mathcal{C},\ D \in \mathcal{D}\}.$$

By the octahedral axiom, the operation $*$ is associative. One obtains a co-t-structure (or weight structure) [Bon, Pa] from a silting subcategory, which is given as follows.

Proposition-Definition 1.2.2 (See [AI, Proposition 2.17])**.** *Let $\mathcal{T}$ be an idempotent complete triangulated category with a silting subcategory $\mathcal{M}$. Set*

$$t_{\geq 0} = \bigcup_{l \geq 0} \mathcal{M}[-l] * \cdots * \mathcal{M}[-1] * \mathcal{M},$$
$$t_{\leq 0} = \bigcup_{l \geq 0} \mathcal{M} * \mathcal{M}[1] * \cdots * \mathcal{M}[l]$$

Then $(t_{\geq 0}, t_{\leq 0})$ is a co-t-structure. We call it the co-t-structure associated to $\mathcal{M}$*.*

In what follows, we will simply write $t_{\geq 0} = \cdots * \mathcal{M}$, $t_{\leq 0} = \mathcal{M} * \cdots$ and so on. It is important for us that some co-t-structures and t-structures are related.

Definition 1.2.3 ([Bon])**.** Let $\mathcal{M} \subset \mathcal{T}$ be a silting subcategory and $(t_{\geq 0}, t_{\leq 0})$ the associated co-t-structure. Let $t = (t^{\leq 0}, t^{\geq 0})$ be a t-structure in $\mathcal{T}$.

(1) We say t is *right adjacent to* $\mathcal{M}$ if $t_{\leq 0} = t^{\leq 0}$.

(2) We say t is *left adjacent to* $\mathcal{M}$ if $t_{\geq 0} = t^{\geq 0}$.

For example, if Λ be a negative dg algebra which is homologically smooth such that each cohomology is finite dimensional, then the standard t-structure on $\operatorname{per} \Lambda$ is right adjacent to a silting object $\Lambda \in \operatorname{per} \Lambda$. It follows that its image under the duality $\operatorname{RHom}_\Lambda(-, \Lambda)\colon \operatorname{per} \Lambda \leftrightarrow \operatorname{per} \Lambda^{\mathrm{op}}$ is left adjacent to a silting object $\Lambda \in \operatorname{per} \Lambda^{\mathrm{op}}$.

Now let us recall the notion of (relative) Serre functors.

Definition 1.2.4. Let $\mathcal{T}$ be a k-linear Hom-finite triangulated category and $\mathcal{T}^{\mathrm{fd}} \subset \mathcal{T}$ a thick subcategory.

(1) An autoequivalence $S\colon \mathcal{T} \to \mathcal{T}$ is a *Serre functor* on $\mathcal{T}$ if there is a functorial isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(X, Y) \simeq D \operatorname{Hom}_{\mathcal{T}}(Y, SX)$$

for all $X, Y \in \mathcal{T}$.

(2) An autoequivalence $S\colon \mathcal{T} \to \mathcal{T}$ is a *relative Serre functor* for $\mathcal{T}^{\mathrm{fd}} \subset \mathcal{T}$ if it restricts to the autoequivalence on $\mathcal{T}^{\mathrm{fd}}$, and the above functorial isomorphism holds for all $X \in \mathcal{T}^{\mathrm{fd}}$ and $Y \in \mathcal{T}$.

(3) We say that $(\mathcal{T}, \mathcal{T}^{\mathrm{fd}}, S, \mathcal{M})$ is a *relative Serre quadruple* if S is a relative Serre functor for $\mathcal{T}^{\mathrm{fd}} \subset \mathcal{T}$ and $\mathcal{M}$ is a silting subcategory of $\mathcal{T}$.

We have one more notion to recall.

Definition 1.2.5. A k-linear category $\mathcal{C}$ is a *dualizing variety* if $D = \operatorname{Hom}_k(-, k)$ induces a duality $\operatorname{mod} \mathcal{C} \leftrightarrow \operatorname{mod} \mathcal{C}^{\mathrm{op}}$ between the category of finitely presented $\mathcal{C}$-modules.

For example, the category $\operatorname{proj} \Lambda$ (resp. $\operatorname{proj}^{\mathbb{Z}} \Lambda$) of finitely generated (graded) projective modules over a finite dimensional algebra Λ is a dualizing variety.

We are now ready to state the following generalized formulation of Amiot's cluster category.

Theorem 1.2.6 ([IYa1, IYa2])**.** *Let $(\mathcal{T}, \mathcal{T}^{\mathrm{fd}}, S, \mathcal{M})$ be a relative Serre quadruple such that $\mathcal{M}$ is a dualizing variety, and $(t_{\geq 0}, t_{\leq 0})$ be the co-t-structure associated to $\mathcal{M}$.*

(1) *The silting subcategory $\mathcal{M}$ has a right adjacent t-structure with $t_{\leq 0}^{\perp} \subset \mathcal{T}^{\mathrm{fd}}$ if and only if it has a left adjacent t-structure with ${}^{\perp}t_{\geq 0} \subset \mathcal{T}^{\mathrm{fd}}$.*

Suppose in what follows the equivalent conditions above are satisfied.

(2) *The quotient functor* $\pi\colon \mathcal{T} \to \mathcal{T}/\mathcal{T}^{\mathrm{fd}}$ *induces bijections* $\mathrm{Hom}_{\mathcal{T}}(X,Y) \to \mathrm{Hom}_{\mathcal{T}/\mathcal{T}^{\mathrm{fd}}}(X,Y)$ *for all* $X \in t_{\leq 0}$ *and* $Y \in St_{\geq 2}$. *In particular the composition* $t_{\leq 0} \cap St_{\geq 2} \subset \mathcal{T} \to \mathcal{T}/\mathcal{T}^{\mathrm{fd}}$ *is an additive equivalence.*

(3) *Let* $n \geq 1$ *and assume* $\mathcal{M}$ *is stable under* $S_{n+1} = S \circ [-n-1]$. *Then* $\pi(\mathcal{M}) \subset \mathcal{T}/\mathcal{T}^{\mathrm{fd}}$ *is an n-cluster tilting subcategory.*

Proof. (1) is [IYa1, Theorem 4.10]. For (2), adapt the proof of [IYa1, Proposition 5.9]. We include a proof of (3) since it requires a modification from [IYa1, Theorem 5.8]. Since $\mathcal{M}$ is stable under S_{n+1}, so is $t_{\geq i}$ for each $i \in \mathbb{Z}$, thus we have $St_{\geq 2} = S_{n+1}t_{\geq 1-n} = t_{\geq 1-n} = \cdots * \mathcal{M}[n-1]$. Therefore we deduce that the fundamental domain is $\mathcal{M} * \cdots * \mathcal{M}[n-1]$, hence the result. □

If Λ is a bimodule $(n+1)$-CY negative dg algebra with finite dimensional $H^0\Lambda$, then $(\mathrm{per}\,\Lambda, \mathcal{D}^b(\Lambda), [n+1], \Lambda)$ is a relative Serre quadruple. One can apply the above theorem and recover the original results of Amiot and Guo.

1.3 t-structure in $\mathrm{per}^{\mathbb{Z}}R$

We will be interested in a negatively graded CY algebra with an a-invariant. Before that we will place ourselves in a slightly general setting. Let $R = \bigoplus_{i \leq 0} R_i$ be a negatively graded algebra. We assume the following on R.

(R1) Each R_i is finite dimensional.

(R2) Any finite length R-module has finite projective dimension.

The condition (R2) is clearly satisfied if R is homologically smooth. The aim of this section is to show that there is a t-structure in $\mathrm{per}^{\mathbb{Z}}R$ in this setting.

Theorem 1.3.1. *Let R be a negatively graded algebra satisfying (R1) and (R2). Set*

$$t^{\leq 0} = \{X \in \mathrm{per}^{\mathbb{Z}}R \mid H^i(X) \in \mathrm{Mod}^{\leq -i}R \text{ for all } i \in \mathbb{Z}\},$$
$$t^{\geq 0} = \{X \in \mathrm{per}^{\mathbb{Z}}R \mid H^i(X) \in \mathrm{Mod}^{\geq -i}R \text{ for all } i \in \mathbb{Z}\}.$$

Then $(t^{\leq 0}, t^{\geq 0})$ *is a t-structure in* $\mathrm{per}^{\mathbb{Z}}R$.

Remark 1.3.2. Under the assumption that R is Noetherian (or more generally graded coherent), there is a version of this for $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}}R)$ without the 'smoothness' condtion (R2), which is in practice far more general than Theorem 1.3.1 above. We give this general result in Appendix 1.B.

In the remainder of this section we simply write $\mathcal{D}$ for $\mathcal{D}(\mathrm{Mod}^{\mathbb{Z}}R)$. Recall from [AI, Definition 4.1] that a subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ with arbitrary (set-indexed) coproducts is *silting* if it forms a compact set of generators such that $\mathrm{Hom}_{\mathcal{T}}(A, B[>0]) = 0$ for all $A, B \in \mathcal{S}$. Note that this is a modified version of Definition 1.2.1.

We start our discussion with the following observation.

Proposition 1.3.3. *The subcategory* $\mathcal{M} = \mathrm{add}\{R(-i)[i] \mid i \in \mathbb{Z}\} \subset \mathcal{D}$ *is silting.*

Proof. Clearly $\mathcal{M}$ is a compact set of generators for $\mathcal{D}$. It remains to show that $\mathrm{Hom}_{\mathcal{D}}(R, R(-i)[i][j])$ vanishes for each $i \in \mathbb{Z}$ and $j > 0$. We only have to consider the case $i = -j < 0$, in which case $\mathrm{Hom}_{\mathcal{D}}(R, R(-i)[i][j]) = \mathrm{Hom}_R^{\mathbb{Z}}(R, R(j)) = 0$ since R is negatively graded. □

We deduce by [Ke1, Theorem 4.3] that $\mathcal{D}$ is triangle equivalent to the derived category $\mathcal{D}(\mathcal{A})$ of a negative dg category $\mathcal{A}$. Then we can consider the standard t-structure $(\mathcal{D}^{\leq 0}_{\mathcal{M}}, \mathcal{D}^{\geq 0}_{\mathcal{M}})$ associated to $\mathcal{M}$, which is given by

$$\mathcal{D}^{\leq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(M, X[i]) = 0 \text{ for all } M \in \mathcal{M} \text{ and } i > 0.\},$$
$$\mathcal{D}^{\geq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(M, X[i]) = 0 \text{ for all } M \in \mathcal{M} \text{ and } i < 0.\}.$$

Now we use the following computation.

Lemma 1.3.4. *We have*

$$\mathcal{D}^{\leq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid H^i(X) \in \mathrm{Mod}^{\leq -i} R \textit{ for all } i \in \mathbb{Z}\},$$
$$\mathcal{D}^{\geq 0}_{\mathcal{M}} = \{X \in \mathcal{D} \mid H^i(X) \in \mathrm{Mod}^{\geq -i} R \textit{ for all } i \in \mathbb{Z}\}.$$

Proof. Since $\mathcal{D}^{\leq 0}_{\mathcal{M}} = \mathcal{M}[<0]^{\perp}$, we have

$$\begin{aligned}\mathcal{D}^{\leq 0}_{\mathcal{M}} &= \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(R(i)[-i][-j], X) = 0 \text{ for all } i \in \mathbb{Z} \text{ and } j > 0\}\\ &= \{X \in \mathcal{D} \mid H^{i+j}(X)_{-i} = 0 \text{ for all } i \in \mathbb{Z} \text{ and } j > 0\}\\ &= \{X \in \mathcal{D} \mid H^{i}(X)_{-i+j} = 0 \text{ for all } i \in \mathbb{Z} \text{ and } j > 0\},\end{aligned}$$

thus the first assertion. By $\mathcal{D}^{\geq 0}_{\mathcal{M}} = \mathcal{M}[>0]^{\perp}$, we similarly have the second equation. □

We need one lemma to ensure that the above t-structure in $\mathcal{D}$ restricts to the small derived category.

Lemma 1.3.5. *Consider the truncation triangle $X' \to X \to X'' \to X'[1]$ for $X \in \mathcal{D}$ with $X' \in \mathcal{D}^{\leq 0}_{\mathcal{M}}$, $X'' \in \mathcal{D}^{\geq 1}_{\mathcal{M}}$, and let $i \in \mathbb{Z}$.*

(1) *The triangle induces a short exact sequence*

$$0 \longrightarrow H^i X' \longrightarrow H^i X \longrightarrow H^i X'' \longrightarrow 0.$$

(2) *The above exact sequence is isomorphic to the truncation*

$$0 \longrightarrow (H^i X)_{\leq -i} \longrightarrow H^i X \longrightarrow (H^i X)_{> -i} \longrightarrow 0$$

of $H^i X \in \mathrm{Mod}^{\mathbb{Z}} R$ with respect to the grading.

Proof. (1) It is enough to show that the connecting homomorphism $H^i X'' \to H^{i+1} X'$ is 0 for each $i \in \mathbb{Z}$. Since $X' \in \mathcal{D}^{\leq 0}_{\mathcal{M}}$ and $X'' \in \mathcal{D}^{\geq 1}_{\mathcal{M}}$, we have $H^i X'' \in \mathrm{Mod}^{\geq i+1} R$ and $H^{i+1} X \in \mathrm{Mod}^{\leq -i-1} R$, hence the assertion. (2) Similarly, we have $H^i X' \in \mathrm{Mod}^{\leq -i} R$ and $H^i X'' \in \mathrm{Mod}^{\geq -i+1} R$, thus the exact sequence in (1) has to be the truncation of $H^i X$ with respect to the grading. □

We are now ready to prove the main result.

Proof of Theorem 1.3.1. We show that the above t-structure in $\mathcal{D}$ restricts to $\mathrm{per}^{\mathbb{Z}} R$. Let $X \in \mathrm{per}^{\mathbb{Z}} R$. We have to show that its truncation X', X'' in Lemma 1.3.5 are perfect. We may replace X by a bounded complex of finitely generated graded projective R-modules. Then we have that $H^i X = 0$ for almost all i and that each $H^i X \in \mathrm{Mod}^{\mathbb{Z}} R$ is bounded above. Moreover by assumption (R1), each vector space $(H^i X)_j$ is finite dimensional. Then by Lemma 1.3.5(1) the cohomology $H^i X''$ is 0 for almost all i and by Lemma 1.3.5(2) that each $H^i X'' \in \mathrm{Mod}^{\mathbb{Z}} R$ is bounded below. Therefore $H^i X''$ lies in $\mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R)$, hence in $\mathrm{per}^{\mathbb{Z}} R$ by (R2). We conclude that the remaining term X' is also perfect. □

1.4 Cluster tilting in $\mathrm{qper}^{\mathbb{Z}} R$ and the a-th root of the AR translation

1.4.1 Cluster tilting

Let us first recall the notion of *(twisted) Calabi-Yau algebras*, in the graded case, which is of our central interest. For a graded automorphism α of a graded ring Λ, we denote by $(-)_\alpha$ the twist automorphism on $\mathrm{Mod}^{\mathbb{Z}} \Lambda$.

Definition 1.4.1. A graded algebra R is *bimodule twisted n-Calabi-Yau of a-invariant a* if it satisfies the following conditions.

- R is homologically smooth, that is, $R \in \operatorname{per}^{\mathbb{Z}} R^e$.
- There exists a graded automorphism α of R such that $\operatorname{RHom}_{R^e}(R, R^e)(a)[n] \simeq {}_\alpha R_1$ in $\mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} R^e)$.

We refer to α as the *Nakayama automorphism*, which is uniquely determined up to inner automorphism. We say that R is *Calabi-Yau* if α is inner.

Let R be a negatively graded bimodule twisted $(d+1)$-CY algebra of a-invariant a with Nakayama automorphism α. We moreover assume that each R_i is finite dimensional over k.

Let us first collect some basic facts on the derived categories $\mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}} R)$, $\operatorname{per}^{\mathbb{Z}} R$, and $\operatorname{qper}^{\mathbb{Z}} R$.

Proposition 1.4.2. (1) *$\mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}} R) \subset \operatorname{per}^{\mathbb{Z}} R$ has a relative Serre functor $(-)_\alpha(a)[d+1]$.*

(2) *$\operatorname{qper}^{\mathbb{Z}} R$ has a Serre functor $(-)_\alpha(a)[d]$.*

Proof. We include a sketch of the proof for the convenience of the reader. (1) follows from [Ke4, Lemma 4.1]. To prove (2), we apply [Am, Section 1]. For this we construct for each $X, Y \in \operatorname{per}^{\mathbb{Z}} R$ a local $\mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}} R)$-envelope of X relative to Y in the sense of [Am, Definition 1.2]. Take a projective resolution $P \to Y$ and pick an integer n such that each term of P is generated in degree $\geq n$. Consider the exact sequence $0 \to X_{<n} \to X \to X_{\geq n} \to 0$ in $\mathcal{C}^b(\operatorname{Mod}^{\mathbb{Z}} R)$ obtained by the truncation with respect to the grading on X. Then $X_{\geq n} \in \mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}} R)$ and it yields a triangle in $\operatorname{per}^{\mathbb{Z}} R$. Since $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{Mod}^{\mathbb{Z}} R)}(Y, X_{<n}) = \operatorname{Hom}_{\mathcal{K}^b(\operatorname{Mod}^{\mathbb{Z}} R)}(P, X_{<n}) = 0$, we see that $X \to X_{\geq n}$ gives a local envelope. □

We need the following reformulation of Proposition 1.3.3.

Proposition 1.4.3. *$\mathcal{M} = \operatorname{add}\{R(-i)[i] \mid i \in \mathbb{Z}\}$ is a silting subcategory of $\operatorname{per}^{\mathbb{Z}} R$.*

Proof. We have seen in Proposition 1.3.3 that $\mathcal{M}$ has no positive self-extensions. Also we clearly have $\operatorname{thick} \mathcal{M} = \operatorname{per}^{\mathbb{Z}} R$. □

Therefore we obtain a relative Serre quadruple $(\operatorname{per}^{\mathbb{Z}} R, \mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}} R), (-)_\alpha(a)[d+1], \mathcal{M})$. The first main result of this part is that this lies on a context of Theorem 1.2.6.

Theorem 1.4.4. *Let R be a negatively graded bimodule twisted $(d+1)$-CY algebra of a-invariant a such that each R_i is finite dimensional.*

(1) *$\mathcal{M}$ is a dualizing variety with left and right adjacent t-structures.*

(2) *The quotient functor $\pi\colon \operatorname{per}^{\mathbb{Z}} R \to \operatorname{qper}^{\mathbb{Z}} R$ induces bijections $\operatorname{Hom}_{\operatorname{per}^{\mathbb{Z}} R}(X, Y) \to \operatorname{Hom}_{\operatorname{qper}^{\mathbb{Z}} R}(X, Y)$ for each $X \in \mathcal{M} * \cdots$ and $Y \in \cdots * \mathcal{M}[d+a-1]$. In particular, π has a fundamental domain $\mathcal{M} * \cdots * \mathcal{M}[d+a-1]$.*

(3) *$\pi(\mathcal{M}) = \operatorname{add}\{R(-i)[i] \mid i \in \mathbb{Z}\} \subset \operatorname{qper}^{\mathbb{Z}} R$ is a $(d+a)$-cluster tilting subcategory.*

In the proof below, we write $\mathcal{D} = \operatorname{per}^{\mathbb{Z}} R$ and $\mathcal{D}^{\mathrm{fd}} = \mathcal{D}^b(\operatorname{fl}^{\mathbb{Z}} R)$.

Proof. (1) Since $\mathcal{M} = \operatorname{add}\{R(-i)[i] \mid i \in \mathbb{Z}\}$, we have $\mathcal{M} \simeq \operatorname{proj}^{\mathbb{Z}} \Lambda$ with $\Lambda = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(R, R(-i)[i]) = R_0$, hence $\mathcal{M}$ is a dualizing variety.

We next show that the silting subcategory $\mathcal{M} \subset \mathcal{D}$ has left and right adjacent t-structures. By Theorem 1.2.6(1), it suffices to show the existence of the right adjacent t-structure with $t_{\leq 0}^{\perp} \subset \mathcal{D}^{\mathrm{fd}}$. Set $t^{\leq 0} = t_{\leq 0}$ and $t^{\geq 0} = (t^{\leq -1})^{\perp}$, where $t^{\leq n} = t^{\leq 0}[-n]$ and so on. Note that $t^{\leq 0} = \mathcal{M}[<0]^{\perp}$ and $t^{\geq 0} = (t_{\leq -1})^{\perp} = \mathcal{M}[>0]^{\perp}$. Then as in Lemma 1.3.4 we have

$$t^{\leq 0} = \{X \in \mathcal{D} \mid H^i(X) \in \operatorname{Mod}^{\leq -i} R \text{ for all } i \in \mathbb{Z}\},$$
$$t^{\geq 0} = \{X \in \mathcal{D} \mid H^i(X) \in \operatorname{Mod}^{\geq -i} R \text{ for all } i \in \mathbb{Z}\}.$$

Now the assertion that $(t^{\leq 0}, t^{\geq 0})$ is a t-structure is precisely what we showed in Theorem 1.3.1, and clearly $t^{\geq 0} \subset \mathcal{D}^{\mathrm{fd}}$.

(2)(3) Let $S = (-)_\alpha(a)[d+1]$ be the relative Serre functor for $\mathcal{D}^{\mathrm{fd}} \subset \mathcal{D}$. Then $S_{d+a+1} = (-)_\alpha(a)[-a]$ preserves $\mathcal{M}$. Therefore we have $St_{\geq 2} = S_{d+a+1}t_{\geq -d-a+1} = t_{\geq -d-a+1}$, hence (2) by Theorem 1.2.6(2), and (3) by Theorem 1.2.6(3). ☐

1.4.2 Tilting and the a-th root of the AR-translation

In this subsection we note the result due to Minamoto–Mori [MM], and give a finite dimensional algebra A which will play a crucial role in the sequel. Before that let us recall the following notion.

Definition 1.4.5 ([HIO])**.** A finite dimensional algebra Λ is *d-representation infinite* if $\operatorname{gl.dim} \Lambda \leq d$ and

$$\nu_d^{-i} \Lambda \in \operatorname{mod} \Lambda$$

holds for all $i \geq 0$, where ν_d is the autoequivalence $- \otimes_\Lambda^L D\Lambda[-d]$ on $\mathcal{D}^b(\operatorname{mod} \Lambda)$.

Proposition 1.4.6 ([MM, Theorem 4.12])**.** *Let R be a negatively graded bimodule twisted $(d+1)$-CY algebra of a-invariant a such that each R_i is finite dimensional.*

(1) $T = \bigoplus_{l=0}^{a-1} R(l)$ *is a tilting object in* $\operatorname{qper}^{\mathbb{Z}} R$.

(2) $A = \operatorname{End}_{\operatorname{qper}^{\mathbb{Z}} R}(T)$ *is d-representation infinite.*

Therefore there exists a triangle equivalence $\operatorname{qper}^{\mathbb{Z}} R \simeq \mathcal{D}^b(\operatorname{mod} A)$.

Proof. We will include a proof using Theorem 1.4.4 in Appendix 1.C. Here we note a complementary discussion to [MM]. Let $\operatorname{QGr} R$ be the Serre quotient of $\operatorname{Mod}^{\mathbb{Z}} R$ by the torsion R-modules, where a graded R-module is torsion if any of its element is annihilated by $R_{\leq n}$ for some $n \leq 0$. By [MM] there is a triangle equivalence $\mathcal{D}(\operatorname{QGr} R) \simeq \mathcal{D}(\operatorname{Mod} A)$ of big derived categories. Consider its restriction to the thick subcategories of compact objects. By [BV, MM], $\mathcal{D}(\operatorname{QGr} R)$ is compactly generated by T, so the compact objects are $\operatorname{thick} T$ [N], thus $\operatorname{qper}^{\mathbb{Z}} R$. Similarly the compact objects in $\mathcal{D}(\operatorname{Mod} A)$ are $\mathcal{D}^b(\operatorname{mod} A)$. ☐

We are now in the position to state the following important consequence. Suppose in what follows that $(-)_\alpha \simeq 1$ on $\operatorname{qper}^{\mathbb{Z}} R$, for example, that R is CY. Let A be the d-representation-infinite algebra given in Proposition 1.4.6, and let F be the autoequivalence on $\mathcal{D}^b(\operatorname{mod} A)$ making the diagram below commutative.

$$\begin{array}{ccc} \operatorname{qper}^{\mathbb{Z}} R & \xrightarrow{\simeq} & \mathcal{D}^b(\operatorname{mod} A) \\ {\scriptstyle (1)}\downarrow & & \downarrow{\scriptstyle F} \\ \operatorname{qper}^{\mathbb{Z}} R & \xrightarrow{\simeq} & \mathcal{D}^b(\operatorname{mod} A) \end{array} \tag{1.4.1}$$

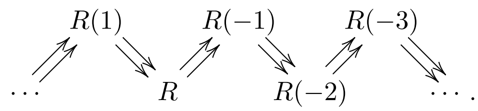
Corollary 1.4.7. *We have $F^a = \nu_d$ as autoequivalences of $\mathcal{D}^b(\operatorname{mod} A)$.*

Proof. Comparing the Serre functors on $\operatorname{qper}^{\mathbb{Z}} R \simeq \mathcal{D}^b(\operatorname{mod} A)$, the autoequivalences (a) on $\operatorname{qper}^{\mathbb{Z}} R$ and ν_d on $\mathcal{D}^b(\operatorname{mod} A)$ are compatible, hence we obtain the desired result. ☐

We can therefore regard F as an a-th root of the d-AR translation ν_d, and denote $F =: \nu_d^{1/a}$, and also $F^{-1} =: \nu_d^{-1/a}$.

Let us give some easy examples of an a-th root of the AR translation.

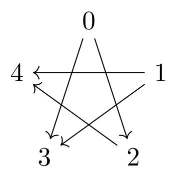
Example 1.4.8. Let $R = k[x, y]$ with $\deg x = \deg y = -1$, so R is 2-CY of a-invariant 2. Applying Proposition 1.4.6, we have a well-known equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{mod} A)$ with A the Kronecker algebra. The AR-quiver of this category looks



By diagram (1.4.1) above, this shows that $\nu_1^{-1/2}$ on $\mathcal{D}^b(\operatorname{mod} A)$ acts by 'moving one place to the right'.

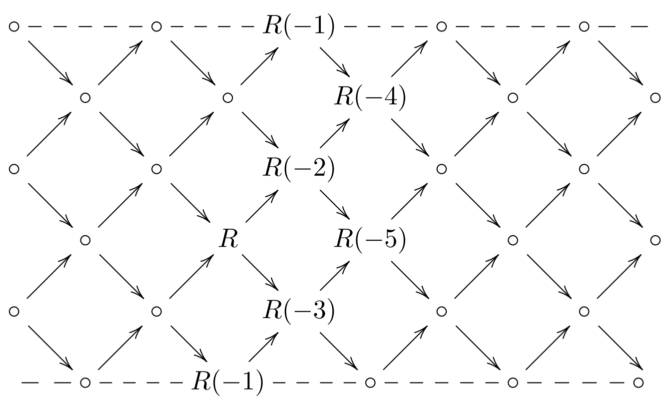
We next look at a higher root of the AR translation.

Example 1.4.9. Let $R = k[x, y]$ with $\deg x = -2$ and $\deg y = -3$, so R is 2-CY of a-invariant 5. By Proposition 1.4.6, there is a triangle equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{mod} A)$, where A is the path algebra over k of the following quiver of type $\widetilde{A_4}$:



,

with the vertex i corresponding to the summand $R(-i)$. By the triangle equivalence, we see that the AR-quiver of the triangulated category $\mathcal{D}^b(\operatorname{qgr} R)$ has the following connected component.



,

where the horizontal ends are identified. We see that $\nu_1^{-1/5} = (-1)$ acts on this component by 'moving one place down'.

We refer to Section 1.10 for more general examples for polynomial rings.

The existence of a square root of the AR translation appears in [KMV] for generalized Kronecker quivers. We show in the following example how to recover their context.

Example 1.4.10. Let $m \geq 2$ and set

$$R = k\langle x_1, \ldots, x_m \rangle / (x_1^2 + \cdots + x_m^2), \quad \deg x_i = -1.$$

This is a (non-Noetherian) Artin-Schelter regular algebra of dimension 2 (see [Z]), thus is twisted CY ([YZ, Proposition 4.5], [RR, Theorem 5.15]). Then it is not difficult to deduce that the complex

$$0 \longrightarrow R \otimes R(2) \xrightarrow{d_2} \bigoplus_{i=1}^{m} R \otimes R(1) \xrightarrow{d_1} R \otimes R \longrightarrow 0$$

with maps

$$d_1((1 \otimes 1)_i) = x_i \otimes 1 - 1 \otimes x_i$$
$$d_2(1 \otimes 1) = \sum_{i=1}^{m} (x_i \otimes 1 + 1 \otimes x_i),$$

together with the multiplication map $R \otimes R \to R$ gives a bimodule projective resolution of R. Applying $\operatorname{Hom}_{R^e}(-, R^e)$ to this complex shows that R is graded bimodule twisted 2-CY of a-invariant 2 with Nakayama automorphism $\sigma \colon x_i \mapsto -x_i$.

By Proposition 1.4.6, we have a derived equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{mod} A)$ with $A = \begin{pmatrix} R_0 & 0 \\ R_{-1} & R_0 \end{pmatrix}$, which is the path algebra of the m-Kronecker quiver $Q_m = (\bullet \xrightarrow{m} \bullet)$. Now the twist automorphism $(-)_\sigma$ is isomorphic to the identity functor on $\operatorname{Mod}^{\mathbb{Z}} R$ so we have an autoequivalence $\nu_1^{-1/2}$ on $\mathcal{D}^b(\operatorname{mod} kQ_m)$. The AR quiver of the derived categories has a connected component

$$\begin{array}{ccccccccccc} & & R(1) & & & & R(-1) & & & & R(-3) & & \\ & \overset{m}{\nearrow} & & \overset{m}{\searrow} & & \overset{m}{\nearrow} & & \overset{m}{\searrow} & & \overset{m}{\nearrow} & & \overset{m}{\searrow} & \\ \cdots & & & & R & & & & R(-2) & & & & \cdots . \end{array}$$

We see that $\nu_1^{-1/2}$ acts by 'moving one place right'; compare Example 1.4.8.

1.5 CY algebras as dg algebras

We will consider a graded algebra R as a dg algebra with the same underlying graded ring and the vanishing differential. We write R^{dg} when considering R as a dg algebra.

We first collect some sign conventions which is heavily used in this section. Throughout this section we denote by $|x|$ the degree of a homogeneous element x in a graded vector space.

Convention 1.5.1. Let Λ and Γ be dg algebras.

(1) Let X be a dg *right* Λ-module. Then its shift $X[1]$ has the same right Λ-action as X.

(2) Let X be a dg *left* Λ-module. Then its shift $X[1]$ has a left Λ-action $a \cdot x = (-1)^{|a|} ax$ for $a \in \Lambda$ and $x \in X[1]$.

(3) There is an isomorphism $\mathcal{H}om_\Lambda(\Lambda[-l], \Lambda) \simeq \Lambda[l]$ of dg left Λ-modules by $f \mapsto (-1)^{l|f|} f(1)$.

(4) We identify $(\Lambda^e)^{\mathrm{op}} \simeq \Lambda^e$ via $x \otimes y \leftrightarrow (-1)^{|x||y|} y \otimes x$.

(5) We identify a dg (Λ, Γ)-bimodule X and a dg $\Lambda^{\mathrm{op}} \otimes_k \Gamma$-module via $\lambda \cdot x \cdot \gamma = (-1)^{|\lambda||x|} x \cdot (\lambda \otimes \gamma)$.

We say that a dg algebra Λ is *twisted bimodule n-CY* if it is homologically smooth and there exists a dg automorphism σ of Λ such that we have an isomorphism $\operatorname{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)[n] \simeq {}_1\Lambda_\sigma$ in $\mathcal{D}(\Lambda^e)$. The aim of this section is to note the following observation. Note that the term 'CY algebra' means precisely as in Definition 1.4.1, and *no* additional conditions are imposed.

Theorem 1.5.2. *Let R be a graded bimodule n-CY algebra of a-invariant a. Then, R^{dg} is twisted bimodule $(n+a)$-CY. Precisely we have the following.*

(1) *If a is odd, then R^{dg} is CY.*

(2) *If a is even, then R^{dg} is homologically smooth and* $\operatorname{RHom}_{(R^{\mathrm{dg}})^e}(R, (R^{\mathrm{dg}})^e)[n+a] \simeq {}_1R_\sigma$ *for the automorphism $\sigma \colon x \mapsto (-1)^{|x|} x$ of R.*

Let us introduce some notations. For a dg algebra Λ, we denote by $\mathrm{C}(\Lambda) = Z^0\mathcal{C}_{\mathrm{dg}}(\Lambda)$ the (abelian) category of dg Λ-modules. Let X be a graded (R,R)-bimodule. We can view it as a dg $(R^{\mathrm{dg}}, R^{\mathrm{dg}})$-bimodule with trivial differentials, hence as an $(R^{\mathrm{dg}})^e$-module, which gives a functor

$$(-)^{\mathrm{dg}}\colon \operatorname{Mod}^{\mathbb{Z}} R^e \longrightarrow \mathrm{C}((R^{\mathrm{dg}})^e).$$

We note the following sign-conventional lemmas. The proofs are left to the reader.

Lemma 1.5.3. *Let $F = R^e(l)$ be a free R^e-module. Then, F^{dg} is isomorphic to the free dg $(R^{\mathrm{dg}})^e$-module $(R^{\mathrm{dg}})^e[l]$. The isomorphism is given by*

$$F^{\mathrm{dg}} \longrightarrow (R^{\mathrm{dg}})^e[l], \quad x \otimes y \mapsto (-1)^{l|x|} x \otimes y.$$

Lemma 1.5.4. *Let $X \in \operatorname{Mod}^{\mathbb{Z}} R^e$ and $l \in \mathbb{Z}$. Then we have an isomorphism $({}_{\sigma^l}X(l))^{\mathrm{dg}} \simeq X^{\mathrm{dg}}[l]$ in $\mathrm{C}((R^{\mathrm{dg}})^e)$.*

Let us now recall the notion of total module of a complex of dg modules. We consider two variations; *total sum* Tot and *total product* $\widehat{\mathrm{Tot}}$. Let $X = (\cdots \to X^{i-1} \xrightarrow{\delta_X^{i-1}} X^i \xrightarrow{\delta_X^i} X^{i+1} \to \cdots)$ be a complex of dg Λ-modules, thus each X^i is a dg Λ-module, each δ_X^i is a morphism of dg Λ-modules, and $\delta_X^{i+1} \circ \delta_X^i = 0$. Then define

$$\operatorname{Tot} X = \bigoplus_{i \in \mathbb{Z}} X^i[-i], \qquad \widehat{\operatorname{Tot}} X = \prod_{i \in \mathbb{Z}} X^i[-i]$$

as graded Λ-modules, and with differentials $\delta_X + \sum_i d_{X^i[-i]}$. Here, δ_X is the differential of the complex X and $d_{X^i[-i]}$ is the differential of the dg module $X^i[-i]$. Then $\operatorname{Tot} X$ and $\widehat{\operatorname{Tot}} X$ are dg Λ-modules.

For a complex $X = (\cdots \to X^{i-1} \to X^i \to X^{i+1} \to \cdots)$ of dg Λ-modules and a dg Λ-module Y, we denote by $\mathcal{H}\mathrm{om}_\Lambda(X,Y)$ the complex

$$\cdots \longrightarrow \mathcal{H}\mathrm{om}_\Lambda(X^{i+1}, Y) \xrightarrow{\cdot\delta_X^i} \mathcal{H}\mathrm{om}_\Lambda(X^i, Y) \xrightarrow{\cdot\delta_X^{i-1}} \mathcal{H}\mathrm{om}_\Lambda(X^{i-1}, Y) \longrightarrow \cdots$$

of dg k-modules, with $\mathcal{H}\mathrm{om}_\Lambda(X^i, Y)$ at degree $-i$.

The following is quite useful for computations.

Lemma 1.5.5. *Let Λ and Γ be dg algebras and X a complex of dg Λ-modules. Then for any dg (Γ,Λ)-bimodule Y, we have an isomorphism $\mathcal{H}\mathrm{om}_\Lambda(\operatorname{Tot} X, Y) \simeq \widehat{\operatorname{Tot}}\, \mathcal{H}\mathrm{om}_\Lambda(X,Y)$ of left dg Γ-modules.*

Proof. It is easily verified that the degree n part of each side is $\prod_{i \in \mathbb{Z}} \mathcal{H}\mathrm{om}_\Lambda(X^i, Y)^{i+n}$, thus two sides coincide as graded vector spaces. Now we check that this identification is compatible with the differentials and the Γ-actions.

On the one hand, the differential on the left-hand-side maps $f \in \mathcal{H}\mathrm{om}_\Lambda(X^i, Y)^{i+n} \subset \mathcal{H}\mathrm{om}_\Lambda(\operatorname{Tot} X, Y)$ to $d_Y f - (-1)^n f d_{\operatorname{Tot} X} = d_Y f - (-1)^n f \delta_X^{i-1} - (-1)^{n+i} f d_{X^i}$.

On the other hand, the differential of the right-hand-side maps $f \in \mathcal{H}\mathrm{om}_\Lambda(X^i, Y)^{i+n} \subset \widehat{\operatorname{Tot}}\, \mathcal{H}\mathrm{om}_\Lambda(X,Y)$ to $f\delta_X^{i-1} + (-1)^{-i} d_{\mathcal{H}\mathrm{om}_\Lambda(X^i,Y)} f = f\delta_X^{i-1} + (-1)^i (d_Y f - (-1)^{i+n} f d_{X^i}) = f\delta_X^{i-1} + (-1)^i d_Y f - (-1)^n f d_{X^i}$.

Then one can check that the map $\mathcal{H}\mathrm{om}_\Lambda(\operatorname{Tot} X, Y) \to \widehat{\operatorname{Tot}}\, \mathcal{H}\mathrm{om}_\Lambda(X,Y)$ given by $f \mapsto (-1)^{i(i+1)/2+in} f$ for $f \in \mathcal{H}\mathrm{om}_\Lambda(X^i, Y)^{i+n}$ is an isomorphism of left dg Γ-modules. ☐

An important step for the proof of Theorem 1.5.2 is the following observation. We denote by $\operatorname{Free}^{\mathbb{Z}} \Lambda$ the category of (not necessarily finitely generated) graded free modules over a graded ring Λ.

Lemma 1.5.6. *Let σ be the automorphism $x \mapsto (-1)^{|x|} x$ on R. The following two functors are naturally isomorphic.*

(a) $F\colon \operatorname{Free}^{\mathbb{Z}} R^e \xrightarrow{\operatorname{Hom}_{R^e}(-,R^e)} \operatorname{Mod}^{\mathbb{Z}} R^e \xrightarrow{{}_\sigma(-)} \operatorname{Mod}^{\mathbb{Z}} R^e$.

(b) $G\colon \operatorname{Free}^{\mathbb{Z}} R^e \xrightarrow{(-)^{\mathrm{dg}}} \mathrm{C}((R^{\mathrm{dg}})^e) \xrightarrow{\mathcal{H}\mathrm{om}_{(R^{\mathrm{dg}})^e}(-,(R^{\mathrm{dg}})^e)} \mathrm{C}((R^{\mathrm{dg}})^e) \xrightarrow{\text{forget}} \operatorname{Mod}^{\mathbb{Z}} R^e$.

Proof. First we define an isomorphism $\varphi_P\colon F(P) \to G(P)$ at the free module $P = R^e(l)$. Clearly we have $F(P) = R \otimes R(-l)$, with the action of R given by $b \cdot (x \otimes y) \cdot a = (-1)^{|b|} xa \otimes by$. On the other hand, using Lemma 1.5.3 we see $G(P) = R \otimes R[-l]$, with the R-action $b \cdot (x \otimes y) \cdot a = (-1)^{|a|(|b|+|y|)+|b|(l+|x|)} xa \otimes by$ for $x, y \in R$. Now we define an isomorphism $F(P) = R \otimes R(-l) \to R \otimes R(-l) = G(P)$ by the formula

$$\varphi_P\colon x \otimes y \mapsto (-1)^{(|x|+l+1)|y|} x \otimes y.$$

We can readily check that this is (R, R)-bilinear.

Next we show that this isomorphism is natural. Let $P \to Q$ be a morphism in $\operatorname{Free}^{\mathbb{Z}} R^e$. We may assume that this is of the form $R^e(l) \to R^e(m)$ with $1 \otimes 1 \mapsto \sum_i u_i \otimes v_i$. Under the functor $(-)^{\mathrm{dg}}$ and the isomorphism in Lemma 1.5.3, it becomes an $(R^{\mathrm{dg}})^e$-linear map $(R^{\mathrm{dg}})^e[l] \to (R^{\mathrm{dg}})^e[m]$ sending $1 \otimes 1$ to $\sum_i (-1)^{m|u_i|} u_i \otimes v_i$. Note that we have $|u_i| + |v_i| - m = -l$. Our task is to show that in the diagram below, the middle square is commutative in $\operatorname{Mod}^{\mathbb{Z}} R^e$.

$$\begin{array}{ccccccc}
{}_\sigma \operatorname{Hom}_{R^e}(R^e(m), R^e) & = & {}_\sigma R^e(-m) & \xrightarrow{\varphi_Q} & (R^{\mathrm{dg}})^e[-m] & = & \mathcal{H}\mathrm{om}_{(R^{\mathrm{dg}})^e}((R^{\mathrm{dg}})^e[m], (R^{\mathrm{dg}})^e) \\
\downarrow & & \downarrow f & & \downarrow g & & \downarrow \\
{}_\sigma \operatorname{Hom}_{R^e}(R^e(l), R^e) & = & {}_\sigma R^e(-l) & \xrightarrow{\varphi_P} & (R^{\mathrm{dg}})^e[-l] & = & \mathcal{H}\mathrm{om}_{(R^{\mathrm{dg}})^e}((R^{\mathrm{dg}})^e[l], (R^{\mathrm{dg}})^e)
\end{array}$$

By our sign conventions the map f is a left $(R^{\mathrm{dg}})^e$-linear map with $1 \otimes 1 \mapsto \sum u_i \otimes v_i$, while g is $1 \otimes 1 \mapsto (-1)^{m(l+1)} \sum_i (-1)^{m|u_i|} u_i \otimes v_i$. Using these we can verify the desired commutativity. ☐

We are now ready to prove the main theorem of this section.

Proof of Theorem 1.5.2. Let $P = (\cdots \to P_1 \to P_0)$ be a free resolution of R in $\mathcal{C}^-(\operatorname{Mod}^{\mathbb{Z}} R^e)$. Then the cohomology of $P^\vee = \operatorname{Hom}_{R^e}(P, R^e)$ is concentrated in degree n, where it is $R(-a)$. Considering P as a complex P^{dg} of dg bimodules as above, the total sum of P^{dg} gives an $(R^{\mathrm{dg}})^e$-cofibrant resolution of the dg $(R^{\mathrm{dg}})^e$-module R. Then $\operatorname{RHom}_{(R^{\mathrm{dg}})^e}(R, (R^{\mathrm{dg}})^e) = \mathcal{H}\mathrm{om}_{(R^{\mathrm{dg}})^e}(\operatorname{Tot} P^{\mathrm{dg}}, (R^{\mathrm{dg}})^e)$, which is isomorphic by Lemma 1.5.5 to the total product of the complex

$$Q = (\mathcal{H}\mathrm{om}_{(R^{\mathrm{dg}})^e}(P_0^{\mathrm{dg}}, (R^{\mathrm{dg}})^e) \to \mathcal{H}\mathrm{om}_{(R^{\mathrm{dg}})^e}(P_1^{\mathrm{dg}}, (R^{\mathrm{dg}})^e) \to \cdots).$$

Now by Lemma 1.5.6 this complex is isomorphic to ${}_\sigma(P^\vee)$ as complexes of graded (R, R)-bimodules. Then it has cohomology only at degree n, where it is isomorphic to ${}_\sigma R(-a)$. Therefore $\widehat{\operatorname{Tot}}\, Q$ is quasi-isomorphic to ${}_\sigma R(-a)^{\mathrm{dg}}[-n]$. We conclude by Lemma 1.5.4 that it is $R[-n-a]$ if a is odd, ${}_\sigma R[-n-a]$ if a is even. ☐

Example 1.5.7. Let $R = k[x_1, \cdots, x_n]$ be the polynomial ring.

(1) Set $\deg x_i = -1$ for all $1 \le i \le n$. Then R is bimodule n-CY of a-invariant n. By Theorem 1.5.2, we see that

- R^{dg} is $2n$-CY if n is odd.
- R^{dg} is twisted $2n$-CY if n is even.

See Example 1.5.8 below for an illustration in $n = 2$ how R^{dg} fails to be CY.

(2) Set $\deg x_i = 1$ for all $1 \le i \le n$. Then R is bimodule n-CY of a-invariant $-n$. By Theorem 1.5.2, we see that

- R^{dg} is 0-CY if n is odd.
- R^{dg} is twisted 0-CY if n is even.

This partially recovers [MGYC, Theorem 6.4], see also [HM, Example 6.1].

(3) Set $\deg x_i = 2$ for all $1 \le i \le n$. Then R is bimodule n-CY of a-invariant $-2n$. By Theorem 1.5.2, we have $\mathrm{RHom}_{(R^{\mathrm{dg}})^e}(R, (R^{\mathrm{dg}})^e)[-n] \simeq {}_1R_\sigma$. Note that the automorphism σ is the identity since R is concentrated in even degrees. Therefore we conclude that R^{dg} is $(-n)$-CY. This partially recovers [MGYC, Theorem 6.2].

We explicitly demonstrate how R^{dg} fails to be CY.

Example 1.5.8. Let $R = k[x, y]$ with $\deg x = \deg y = -1$. Then the graded ring R is bimodule 2-CY of a-invariant 2, and has the Koszul resolution, which we depict in the following way.

$$\begin{array}{ccccc} & \xrightarrow{-y\otimes1+1\otimes y} & R\otimes R(1) & \xrightarrow{x\otimes1-1\otimes x} & \\ R\otimes R(2) & & \oplus & & R\otimes R \\ & \xrightarrow{x\otimes1-1\otimes x} & R\otimes R(1) & \xrightarrow{y\otimes1-1\otimes y} & \end{array},$$

where the values on the arrows show the image of $1 \otimes 1$. Now consider this resolution as a complex of dg modules over $S := (R^{\mathrm{dg}})^e$. Under the isomorphism in Lemma 1.5.3, it becomes

$$\begin{array}{ccccc} & \xrightarrow{y\otimes1+1\otimes y} & S[1] & \xrightarrow{x\otimes1-1\otimes x} & \\ S[2] & & \oplus & & S \\ & \xrightarrow{-x\otimes1-1\otimes x} & S[1] & \xrightarrow{y\otimes1-1\otimes y} & \end{array}.$$

Applying $\mathcal{H}\mathrm{om}_S(-, S)$ we get the complex of left dg S-modules

$$\begin{array}{ccccc} & \xrightarrow{x\otimes1-1\otimes x} & S[-1] & \xrightarrow{-y\otimes1-1\otimes y} & \\ S & & \oplus & & S[-2] \\ & \xrightarrow{y\otimes1-1\otimes y} & S[-1] & \xrightarrow{x\otimes1+1\otimes x} & \end{array},$$

whose total module is $\mathrm{RHom}_S(R, S)$ by Lemma 1.5.5. We therefore see that $\mathrm{RHom}_S(R, S)[4] \simeq {}_1R_\sigma$.

The appearance of the twist automorphism σ suggests that we should twist the multiplication of the CY algebra R in order that the dg algebra R^{dg} to be CY. Let us give an instance where R is twisted CY and R^{dg} is CY.

Example 1.5.9. Let $m \ge 2$ and

$$R = k\langle x_1, \ldots, x_m\rangle / (x_1^2 + \cdots + x_m^2), \quad \deg x_i = -1,$$

If $m = 2$, this is a skew polynomial ring with 2 variables; compare Example 1.5.8 above. For all $m \ge 2$, this algebra is Artin-Schelter regular of dimension 2 (see Example 1.4.10). Recall that the bimodule projective resolution of R is give by the complex

$$0 \longrightarrow R \otimes R(2) \xrightarrow{d_2} \bigoplus_{i=1}^{m} R \otimes R(1) \xrightarrow{d_1} R \otimes R \longrightarrow 0$$

with maps

$$d_1((1 \otimes 1)_i) = x_i \otimes 1 - 1 \otimes x_i$$
$$d_2(1 \otimes 1) = \sum_{i=1}^{m} (x_i \otimes 1 + 1 \otimes x_i).$$

We claim R^{dg} is bimodule 4-CY. For this we follow the computation in Example 1.5.8 above. Set $S = (R^{\mathrm{dg}})^e$. Applying the functor $(-)^{\mathrm{dg}}$ and the isomorphism in Lemma 1.5.3, the above complex becomes

$$0 \longrightarrow S[2] \xrightarrow{d_2} \bigoplus_{i=1}^{m} S[1] \xrightarrow{d_1} S \longrightarrow 0,$$

where the maps are right S-linear morphism such that

$$d_1((1 \otimes 1)_i) = x_i \otimes 1 - 1 \otimes x_i$$
$$d_2(1 \otimes 1) = \sum_{i=1}^{m} (-x_i \otimes 1 + 1 \otimes x_i).$$

Applying $\mathcal{H}\mathrm{om}_S(-, S)$, we get an isomorphic complex, thus we see that $\mathrm{RHom}_S(R, S)[4] \simeq R$ in $\mathcal{D}(S)$.

1.6 Cluster categories, derived orbit categories, and singularity categories

Let R be a CY algebra. We state main results of this part which describes the cluster category of R^{dg} as an orbit category and a singularity category.

1.6.1 Cluster categories and orbit categories

Let R be a negatively graded bimodule $(d+1)$-CY algebra of a-invariant a such that each R_i is finite dimensional. In this subsection, we compare the derived category of R considered as a graded ring, and that of R considered as a dg algebra R^{dg} with vanishing differentials. By Theorem 1.5.2, we know that R^{dg} is twisted bimodule $(d+a+1)$-CY.

Recall the notion of total module from the previous section. As in the previous section, consider the dg functor

$$\mathrm{Tot}\colon \mathcal{C}^b_{\mathrm{dg}}(\mathrm{Mod}^{\mathbb{Z}} R) \to \mathcal{C}_{\mathrm{dg}}(R^{\mathrm{dg}}).$$

Taking the 0-th cohomology, it induces a triangle functor $\mathcal{K}^b(\mathrm{Mod}^{\mathbb{Z}} R) \to \mathcal{K}(R^{\mathrm{dg}})$, which clearly takes acyclic complexes to acyclic dg modules. We therefore obtain a triangle functor

$$\mathrm{Tot}\colon\ \mathrm{per}^{\mathbb{Z}} R \to \mathrm{per}\, R^{\mathrm{dg}}.$$

Note that this restricts to $\mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R) \to \mathcal{D}^b(R^{\mathrm{dg}})$, thus it again induces a triangle functor,

$$\mathrm{Tot}\colon\ \mathrm{qper}^{\mathbb{Z}} R \to \mathcal{C}(R^{\mathrm{dg}}).$$

The following result gives a natural and more concrete description of the cluster category of R^{dg}. Similar type of results for derived or singularity categories are recently obtained in [KY1, Bri].

Theorem 1.6.1. *The functor* $\mathrm{Tot}\colon\ \mathrm{qper}^{\mathbb{Z}} R \to \mathcal{C}(R^{\mathrm{dg}})$ *induces a fully faithful functor*

$$\mathrm{qper}^{\mathbb{Z}} R/(-1)[1] \longrightarrow \mathcal{C}(R^{\mathrm{dg}})$$

whose image generates $\mathcal{C}(R^{\mathrm{dg}})$ *as a thick subcategory. Therefore,* $\mathcal{C}(R^{\mathrm{dg}})$ *is a triangulated hull of the orbit category* $\mathrm{qper}^{\mathbb{Z}} R/(-1)[1]$.

Proof. Note that the cluster tilting subcategory $\mathcal{M} = \mathrm{add}\{R(-i)[i] \mid i \in \mathbb{Z}\} \subset \mathrm{qper}^{\mathbb{Z}} R$ given in Theorem 1.4.4 is mapped to a cluster tilting object $R \in \mathcal{C}(R^{\mathrm{dg}})$, and the functor Tot induces an equivalence $\mathcal{M}/(-1)[1] \xrightarrow{\simeq} \mathrm{add}\, R$. Therefore the assertion follows from covering version of the 'cluster-Beilinson criterion', see [KR2, Lemma 4.5]. □

Let A be the d-representation infinite algebra given in Proposition 1.4.6 as the endomorphism ring of a tilting object in $\mathrm{qper}^{\mathbb{Z}}R$. Explicitly, we have

$$A = \begin{pmatrix} R_0 & 0 & \cdots & 0 \\ R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \end{pmatrix}.$$

Using diagram (1.4.1), we deduce the following.

Corollary 1.6.2. *There exists a fully faithful functor*

$$\mathcal{D}^b(\mathrm{mod}\, A)/\nu_d^{-1/a}[1] \longrightarrow \mathcal{C}(R^{\mathrm{dg}})$$

whose image generates $\mathcal{C}(R^{\mathrm{dg}})$ as a thick subcategory. Therefore, $\mathcal{C}(R^{\mathrm{dg}})$ is a triangulated hull of the orbit category $\mathcal{D}^b(\mathrm{mod}\, A)/\nu_d^{-1/a}[1]$.

1.6.2 Cluster categories and singularity categories

We present another description of the cluster category $\mathcal{C}(R^{\mathrm{dg}})$. Set

$$U = \begin{pmatrix} R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \\ R_{-a} & R_{-(a-1)} & \cdots & R_{-1} \end{pmatrix},$$

which is an (A, A)-bimodule and let $B = A \oplus U$ be the trivial extension algebra. Note that $U = \nu_d^{-1/a} A$ in $\mathcal{D}^b(\mathrm{mod}\, A)$. We have the following basic properties.

Proposition 1.6.3. (1) *U is a cotilting bimodule over A.*

(2) *B is a d-Iwanaga-Gorenstein algebra.*

Proof. Since $U = \mathrm{RHom}_{\mathrm{qper}^{\mathbb{Z}}R}(T, T(1))$ for a tilting object T given in Proposition 1.4.6, it is a cotilting bimodule over $A = \mathrm{End}_{\mathrm{qper}^{\mathbb{Z}}R}(T)$. Then the second assertion follows from [MY, Theorem 4.3]. ☐

Now we state the second main result of this part.

Theorem 1.6.4. *There exists a triangle equivalence*

$$\mathcal{C}(R^{\mathrm{dg}}) \simeq \mathcal{D}_{\mathrm{sg}}(B).$$

In particular, $\mathcal{D}_{\mathrm{sg}}(B)$ is a twisted $(d + a)$-CY category with a $(d + a)$-cluster tilting object.

We postpone the proof of this result to Section 1.8 since it requires a general result on dg orbit categories, which we give in the following section.

1.6.3 Examples

Before going on let us give some demonstrations of our results. See Section 1.9, 1.10, 1.11 for systematic examples.

Example 1.6.5. Let us start with an almost trivial example. Let

$$R = k[x], \quad \deg x = -1.$$

This is bimodule 1-CY of a-invariant 1, thus R^{dg} is bimodule 2-CY by Theorem 1.5.2. We have $A = k$ (which is '0-representation infinite') and $U = k$, thus $B = A \oplus U = k[t]/(t^2)$. By Corollary 1.6.2 and Theorem 1.6.4, we have equivalences of triangulated categories

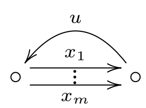
$$\mathcal{D}^b(\operatorname{mod} k)/[1] \simeq \mathcal{C}(R^{\mathrm{dg}}) \simeq \mathcal{D}_{\mathrm{sg}}(k[t]/(t^2)),$$

which are triangulated categories with 1 point. See Example 1.A.5 for a generalization, where the case $\deg x = -n$ for arbitrary $n \geq 1$ is discussed.

Example 1.6.6. This is a continuation of Example 1.4.10 and 1.5.9. Let $m \geq 2$ and set

$$R = k\langle x_1, \dots, x_m \rangle / (x_1^2 + \cdots + x_m^2), \quad \deg x_i = -1,$$

which is twisted 2-CY of a-invariant 2 (Example 1.4.10), and R^{dg} is 4-CY (Example 1.5.9). The 1-representation infinite algebra A is the path algebra of the m-Kronecker quiver $Q_m \colon \bullet \xrightarrow{m} \bullet$, and the autoequivalence ν_1 of $\mathcal{D}^b(\operatorname{mod} kQ_m)$ has a square root (Example 1.4.10). Also it is not difficult to see that the 1-Iwanaga-Gorenstein algebra B is presented by the following quiver with relations.



, $\quad \sum_{i=1}^m (x_i u x_i + x_i u x_i) = 0,\ u x_i u = 0$

By Corollary 1.6.2 and Theorem 1.6.4, there exist triangle equivalences

$$\mathcal{D}^b(\operatorname{mod} kQ_m)/\nu_1^{-1/2}[1] \simeq \mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}(R^{\mathrm{dg}}).$$

Remark 1.6.7. In [KMV, Theorem 1.4], Keller–Murfet–Van den Bergh proved that any algebraic 3-CY triangulated category $\mathcal{T}$ with a 3-cluster tilting object T such that $\operatorname{End}_{\mathcal{T}}(T) = k$ and $\operatorname{Hom}_{\mathcal{T}}(T, T[-1]) = k^m$ is triangle equivalent to $\mathcal{D}^b(\operatorname{mod} kQ_m)/\tau^{-1/2}[1]$, where $\tau^{-1/2}$ is the square root of the AR translation defined in [KMV] using reflection functors.

Now, since $\mathcal{C} = \mathcal{C}(R^{\mathrm{dg}})$ is a 3-CY category with a cluster tilting object R satisfying $\operatorname{End}_{\mathcal{C}}(R) = k$ and $\operatorname{Hom}_{\mathcal{C}}(R, R[-1]) = k^m$, the above equivalent triangulated categories are precisely the 3-CY category in [KMV, Theorem 1.4]. Our result shows that the 3-CY category in [KMV] is also the singularity category of B, and can be realized as the cluster category of the dg algebra R^{dg}. We refer to Example 1.A.6 for a generalization, which covers the situation in [KMV, Remark 3.4.5].

1.7 Quasi-equivalence of dg orbit categories

Let $\mathcal{T}$ be a triangulated category and $F \colon \mathcal{T} \to \mathcal{T}$ an autoequivalence. In order to discuss when the orbit category $\mathcal{T}/F$ is triangulated, a triangulated hull of this orbit category was constructed by Keller [Ke2]. The idea was to take an orbit at the level of enhancement of $\mathcal{T}$.

Let $\mathcal{A}$ be a pretriangulated dg category and F a dg endofunctor on $\mathcal{A}$ inducing an equivalence on $H^0\mathcal{A}$. Then the *dg orbit category* of $\mathcal{A}$ with respect to F, which we denote by $\mathcal{B} = \mathcal{A}/F$, is the dg category with the same objects as $\mathcal{A}$ and with the morphism complex

$$\mathcal{B}(L, M) = \operatorname{colim} \left(\bigoplus_{n \geq 0} \mathcal{A}(F^n L, M) \xrightarrow{F} \bigoplus_{n \geq 0} \mathcal{A}(F^n L, FM) \xrightarrow{F} \bigoplus_{n \geq 0} \mathcal{A}(F^n L, F^2 M) \xrightarrow{F} \cdots \right)$$

for each $L, M \in \mathcal{B}$. It follows that we have $H^0\mathcal{A}/H^0F = H^0\mathcal{B}$, hence $\operatorname{per}\mathcal{B}$ is a triangulated hull of $H^0\mathcal{A}/H^0F$ in the sense that there is a fully faithful functor $H^0\mathcal{A}/H^0F \hookrightarrow \operatorname{per}\mathcal{B}$ whose image generates $\operatorname{per}\mathcal{B}$ as a thick subcategory.

Observe that the above embedding is not dense in general, and thus it is not clear a naive expectation for orbit categories carries over to triangulated hulls. We give an answer to one of such problems.

Let us briefly recall some relevant notions, see [Ke1, Section 7] for details. Let $\mathcal{B}$ and $\mathcal{C}$ be dg categories. A *quasi-functor* $\mathcal{C} \to \mathcal{B}$ is a $(\mathcal{C}, \mathcal{B})$-bimodule X, whose value we denote by $X(B, C)$ for $B \in \mathcal{B}$ and $C \in \mathcal{C}$, such that the dg $\mathcal{B}$-module $X(-, C)$ is isomorphic in $\mathcal{D}(\mathcal{B})$ to a representable dg $\mathcal{B}$-module for each $C \in \mathcal{C}$. A quasi-functor $X : \mathcal{C} \to \mathcal{B}$ is a *quasi-equivalence* if $- \otimes^L_\mathcal{C} X : \mathcal{D}(\mathcal{C}) \to \mathcal{D}(\mathcal{B})$ is a triangle equivalence. In this case the equivalence restricts to $\operatorname{per}\mathcal{C} \xrightarrow{\simeq} \operatorname{per}\mathcal{B}$.

Theorem 1.7.1. *Let $\mathcal{A}$ be a pretriangulated dg category, and let F, G be dg endofunctors on $\mathcal{A}$ such that H^0F and H^0G are mutually inverse equivalences on $H^0\mathcal{A}$. Suppose that there is a morphism $G \circ F \to 1_\mathcal{A}$ of dg functors inducing a natural isomorphism on $H^0\mathcal{A}$. Then the dg orbit categories $\mathcal{B} = \mathcal{A}/F$ and $\mathcal{C} = \mathcal{A}/G$ are quasi-equivalent. In particular, the triangulated hulls* $\operatorname{per}\mathcal{B}$ *and* $\operatorname{per}\mathcal{C}$ *are triangle equivalent.*

Remark 1.7.2. (1) We do not need a natural transformation $F \circ G \to 1_\mathcal{A}$.

(2) The assumption on the existence of a natural transformation $G \circ F \to 1_\mathcal{A}$ is satisfied in the following typical case: Let Λ be a finite dimensional algebra which is homologically smooth and $\mathcal{A} = \mathcal{C}^b(\operatorname{proj}\Lambda)$. Let X a complex of projective (Λ, Λ)-bimodules such that $F = - \otimes_\Lambda X : \mathcal{A} \to \mathcal{A}$ gives an equivalence on $H^0\mathcal{A} = \mathcal{D}^b(\operatorname{mod}\Lambda)$. Letting Y be the bimodule projective resolution of $\operatorname{RHom}_\Lambda(X, \Lambda)$, $G = - \otimes_\Lambda Y : \mathcal{A} \to \mathcal{A}$ gives an inverse of F on $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Moreover, a quasi-isomorphism $Y \otimes_\Lambda X \to \Lambda$ of (Λ, Λ)-bimodule complexes gives a natural transformation $G \circ F \to 1_\mathcal{A}$.

In view of relating the categories $\mathcal{B}$ and $\mathcal{C}$, consider the following direct system of complexes indexed by $\mathbb{N} \times \mathbb{N}$:

$$\begin{array}{ccccccc}
\bigoplus_{n\geq 0}\mathcal{A}(F^nL, M) & \longrightarrow & \bigoplus_{n\geq 0}\mathcal{A}(GF^nL, M) & \longrightarrow & \bigoplus_{n\geq 0}\mathcal{A}(G^2F^nL, M) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{n\geq 0}\mathcal{A}(GF^nL, GM) & \longrightarrow & \bigoplus_{n\geq 0}\mathcal{A}(G^2F^nL, GM) & \longrightarrow & \bigoplus_{n\geq 0}\mathcal{A}(G^3F^nL, GM) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array} \tag{1.7.1}$$

where the vertical transition maps are induced by G, and the horizontal ones by $G^{p+1}F^{1+n}L \to G^pF^nL$.

We first fix $q \geq 0$ and consider the colimit

$$U_q(L, M) := \operatorname*{colim}_{p\geq 0} \bigoplus_{n\geq 0} \mathcal{A}(G^{p+q}F^nL, G^qM)$$

of the $(q+1)$-st row. We can regard $U_q(-, M)$ as a dg $\mathcal{B}$-module for each $M \in \mathcal{A}$ as follows: Let $b \in \mathcal{B}(K, L)$ a morphism represented by $F^mK \to F^rL$ and $u \in U_q(L, M)$ an element represented by $G^{p+q}F^nL \to G^qM$. Enlarging n if necessarily we may assume $r \leq n$. Then define $u \cdot b$ by the composition $G^{p+q}F^{m+n-r}K \xrightarrow{G^{p+q}F^{n-r}b} G^{p+q}F^nL \xrightarrow{u} G^qM$. Since there is a commutative diagram

$$\begin{array}{ccccc}
G^{p+q}F^{m+n-r}K & \xrightarrow{G^{p+q}F^{n-r}b} & G^{p+q}F^nL & \xrightarrow{u} & G^qM \\
\uparrow & & \uparrow & & \| \\
G^{p+q+1}F^{1+m+n-r}K & \xrightarrow{G^{p+q+1}F^{1+n-r}b} & G^{p+q+1}F^{1+n}L & \longrightarrow & G^qM
\end{array}$$

for each $r \leq n$, we see that this action is well-defined.

Let us note the following property of U_0.

Lemma 1.7.3. *The maps* $\mathcal{A}(F^n(-), F^p M) \xrightarrow{G^p} \mathcal{A}(G^p F^n(-), G^p F^p M) \to \mathcal{A}(G^p F^n(-), M)$ *induce a quasi-isomorphism*

$$\mathcal{B}(-, M) = \operatorname{colim}_{p\geq 0} \bigoplus_{n\geq 0} \mathcal{A}(F^n(-), F^p M) \longrightarrow \operatorname{colim}_{p\geq 0} \bigoplus_{n\geq 0} \mathcal{A}(G^p F^n(-), M) = U_0(-, M)$$

of dg $\mathcal{B}$*-modules.*

Proof. The commutative diagram

$$\begin{array}{ccccc}
\mathcal{A}(F^n(-), F^p M) & \xrightarrow{G^p} & \mathcal{A}(G^p F^n(-), G^p F^p M) & \longrightarrow & \mathcal{A}(G^p F^n(-), M) \\
{\scriptstyle F}\downarrow & & & & \downarrow \\
\mathcal{A}(F^{n+1}(-), F^{p+1} M) & \xrightarrow{G^{p+1}} & \mathcal{A}(G^{p+1} F^{n+1}(-), G^{p+1} F^{p+1} M) & \longrightarrow & \mathcal{A}(G^{p+1} F^{n+1}(-), M)
\end{array}$$

shows the existence of the morphism on the colimts.

It is clear that the induced morphism is a quasi-isomorphism since F and G are mutually inverse equivalences on $H^0\mathcal{A}$. □

Note that $U_q(-,-)$ constructed above does not have a $\mathcal{C}$-action. The next step toward relating $\mathcal{B}$ and $\mathcal{C}$ is constructing a dg $(\mathcal{C}, \mathcal{B})$-bimodule which is quasi-isomorphic over $\mathcal{B}$ to U_0.

Lemma 1.7.4. *The vertical maps in (1.7.1) induce a sequence of quasi-isomorphisms*

$$U_0(-, M) \longrightarrow U_1(-, M) \longrightarrow U_2(-, M) \longrightarrow \cdots$$

of dg $\mathcal{B}$*-modules for each* $M \in \mathcal{A}$*.*

Proof. Since the vertical maps in (1.7.1) are quasi-isomorphisms, the induced map $U_q(L, M) \to U_{q+1}(L, M)$ is a quasi-isomorphism for each $q \geq 0$, $L \in \mathcal{B}$. It is easily verified that $U_q(-, M) \to U_{q+1}(-, M)$ is $\mathcal{B}$-linear. □

Now define the dg $\mathcal{B}$-module $U(-, M)$ by the colimit

$$U(-, M) := \operatorname*{colim}_{q\geq 0} U_q(-, M).$$

For each $L \in \mathcal{B}$ we have $U(L, M) = \operatorname{colim}_{p,q\geq 0} \bigoplus_{n\geq 0} \mathcal{A}(G^{p+q} F^n L, G^q M)$.

We observe that $U(L, M)$ has a left $\mathcal{C}$-action as follows: Let $c \in \mathcal{C}(M, N)$ be a morphism presented by $G^m M \to G^r N$ and $u \in U(L, M)$ an element presented by $G^{p+q} F^n L \to G^q M$. Enlarging m and p if necessarily we may assume $q \leq m$ and $r \leq p$. Then define $c \cdot u$ by the composition $G^{p+m} F^n L \xrightarrow{G^{m-q} u} G^m M \xrightarrow{c} G^r N$. It is clear that this is well-defined, and compatible with the right $\mathcal{B}$-action. We have therefore obtained a $(\mathcal{C}, \mathcal{B})$-bimodule U.

As a final preparation, we describe an equivalence between the orbit categories $H^0\mathcal{B}$ and $H^0\mathcal{C}$.

Lemma 1.7.5. *The maps* $H^0\mathcal{A}(F^n L, F^p M) \xrightarrow{G^{n+p}} H^0\mathcal{A}(G^{n+p} F^n L, G^{n+p} F^p M) \xrightarrow{\simeq} H^0\mathcal{A}(G^{n+p} F^n L, G^n M)$ $\xleftarrow{\simeq} H^0\mathcal{A}(G^p L, G^n M) \to H^0\mathcal{C}(L, M)$ *induce an equivalence* $H^0\mathcal{B} \simeq H^0\mathcal{C}$*.*

Now we are ready to prove the main theorem of this section.

Proof of Theorem 1.7.1. We show that the $(\mathcal{C}, \mathcal{B})$-bimodule

$$U(L, M) = \operatorname*{colim}_{p,q\geq 0} \bigoplus_{n\geq 0} \mathcal{A}(G^{p+q} F^n L, G^q M)$$

constructed above gives a quasi-equivalence.

For each $M \in \mathcal{C}$ we have quasi-isomorphisms

$$u_M \colon \mathcal{B}(-, M) \to U_0(-, M) \to U(-, M)$$

of dg $\mathcal{B}$-modules by Lemma 1.7.3 and Lemma 1.7.4, thus U is a quasi-functor.

It remains to show that the induced map

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\mathcal{C}(-, L), \mathcal{C}(-, M)) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(U(-, L), U(-, M))$$

is an isomorphism for each $L, M \in \mathcal{A}$. It suffices to show that the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\mathcal{C}(-,L),\mathcal{C}(-,M)) & \xrightarrow{-\otimes^L_{\mathcal{C}} U} & \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(U(-,L),U(-,M)) & \xrightarrow[u_L]{\simeq} & \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(\mathcal{B}(-,L),U(-,M)) \\ \| & & & & \uparrow u_M \wr \\ H^0\mathcal{C}(L,M) & \xleftarrow{\simeq} & H^0\mathcal{B}(L,M) & = & \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(\mathcal{B}(-,L),\mathcal{B}(-,M)). \end{array}$$

The equivalence $H^0\mathcal{B} \to H^0\mathcal{C}$ given in Lemma 1.7.5 shows that if $f \in H^0\mathcal{B}(L, M)$ is presented by a morphism $F^nL \to F^pM$ in $Z^0\mathcal{A}$, then the corresponding morphism $g \in H^0\mathcal{C}(L, M)$ is presented by $G^pL \to G^nM$ in $Z^0\mathcal{A}$ so that the diagram

$$\begin{array}{ccc} G^{n+p}F^nL & \xrightarrow{G^{n+p}f} & G^{n+p}F^pM \\ \downarrow & & \downarrow \\ G^pL & \xrightarrow{g} & G^nM \end{array} \qquad (1.7.2)$$

is commutative in $H^0\mathcal{A}$.

Let f and g be of the form above. Then the commutativity we want amounts to saying that $u_M \cdot f = g \cdot u_L$ in $H^0U(L, M)$, where we may regard $\cdot$ as the right action of $\mathcal{B}$ (resp. left action of $\mathcal{C}$) on U. Since $u_L \in U(L, L)$ is presented by the identity morphism in $\mathcal{A}$ the element $g \cdot u_L \in U(L, M)$ is presented by $G^{p+n}F^nL \to G^pL \xrightarrow{1} G^pL \xrightarrow{g} G^nM$. Similarly, since $u_M \in U(M, M)$ is presented by the identity morphism in $\mathcal{A}$, the element $u_M \cdot f \in U(L, M)$ is presented by $G^pF^nL \xrightarrow{G^pf} G^pF^pM \to M$, hence by one obtained by applying G^n. Then the commutativity of the diagram (1.7.2) in $H^0\mathcal{A}$ implies that we have $u_M \cdot f = g \cdot u_L$ in $H^0U(L, M)$. □

Let us apply our general result to the setting of finite dimensional algebras. Let Λ be a finite dimensional algebra which is homologically smooth, and X a complex of (Λ, Λ)-bimodules such that $F = - \otimes^L_\Lambda X$ gives an autoequivalence on $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$. We assume that for each $L, M \in \mathcal{D}^b(\mathrm{mod}\,\Lambda)$, we have $\mathrm{Hom}_{\mathcal{D}(\Lambda)}(L, F^iM) = 0$ for almost all $i \in \mathbb{Z}$.

Let Γ be the trivial extension dg algebra of Λ by $X[-1]$, that is, $\Gamma = \Lambda \oplus X[-1]$ as a complex, and the multiplication is given by the bimodule structure on $X[-1]$. Then Keller's theorem [Ke2, Theorem 2] gives the equivalence

$$\mathrm{per}\,\mathcal{B} \xrightarrow{\simeq} \mathrm{thick}_{\mathcal{D}(\Gamma)}\,\Lambda / \mathrm{per}\,\Gamma$$

which is compatible with the natural functors from $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$.

We have the same equivalence arising from the inverse of F; Let Y be a bimodule projective resolution of $\mathrm{RHom}_\Lambda(X, \Lambda)$ and $G = - \otimes_\Lambda Y \colon \mathcal{A} \to \mathcal{A}$, which induces a quasi-inverse to $F = - \otimes_\Lambda X$ on $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$. Let $\mathcal{C} = \mathcal{A}/G$ be the dg orbit category and $\Delta = \Lambda \oplus Y[-1]$ be the trivial extension dg algebra so that we have an equivalence

$$\mathrm{per}\,\mathcal{C} \xrightarrow{\simeq} \mathrm{thick}_{\mathcal{D}(\Delta)}\,\Lambda / \mathrm{per}\,\Delta$$

By the above equivalences and Theorem 1.7.1 we deduce the following consequence.

Corollary 1.7.6. *There exists a triangle equivalence*

$$\operatorname{thick}_{\mathcal{D}(\Gamma)} \Lambda / \operatorname{per} \Gamma \xrightarrow{\simeq} \operatorname{thick}_{\mathcal{D}(\Delta)} \Lambda / \operatorname{per} \Delta$$

which is compatible with the projection functors from $\mathcal{D}^b(\operatorname{mod} \Lambda)$.

Observe that the above result gives a certain singular equivalence of dg algebras Γ and Δ. We end this section by posing the following question.

Question 1.7.7. Is it possible to describe the equivalence in Corollary 1.7.6 directly?

1.8 Proof of Theorem 1.6.4

We now give a proof of a main result Theorem 1.6.4 of this part using the result from the previous section. In fact, the essential part of the proof does not depend on our specific setup, so we first state the intermediate result in Proposition 1.8.1 below, which can also be viewed as a dg version of Theorem 1.6.4.

Let Λ be a finite dimensional algebra which is homologically smooth, X a complex of (Λ, Λ)-bimodules such that $F = - \otimes_\Lambda^L X$ gives an autoequivalence on $\mathcal{D}^b(\operatorname{mod} \Lambda)$. We assume the following on the tilting complex X:

(X1) For each $L, M \in \mathcal{D}^b(\operatorname{mod} \Lambda)$, we have $\operatorname{Hom}_{\mathcal{D}(\Lambda)}(L, F^i M) = 0$ for almost all $i \in \mathbb{Z}$.

(X2) X is concentrated in degree ≤ 0.

Let

$$\Gamma = \Lambda \oplus X[-1], \quad \Sigma = T_\Lambda^L X$$

be the trivial extension dg algebra, and respectively the derived tensor algebra, that is, the tensor algebra of a bimodule projective resolution of X. Then the conditions imply that Σ is a negative dg algebra and its cohomology $H^0\Sigma = T_\Lambda(H^0 X)$ is finite dimensional.

Recall from the introduction that we have denoted by

$$\mathcal{C}(\Pi) = \operatorname{per} \Pi / \mathcal{D}^b(\Pi)$$

and called it the cluster category for any dg algebra Π satisfying $\operatorname{per} \Pi \supset \mathcal{D}^b(\Pi)$. Our general intermediate result is an equivalence between the cluster category of the tensor algebra and the singularity category of the trivial extension algebra.

Proposition 1.8.1. *The dg algebra* Σ *satisfies* $\operatorname{per} \Sigma \supset \mathcal{D}^b(\Sigma)$*, and there exists a triangle equivalence*

$$\mathcal{C}(\Sigma) \simeq \operatorname{thick}_{\mathcal{D}(\Gamma)} \Lambda / \operatorname{per} \Gamma.$$

The first step is to apply Corollary 1.7.6. Let Y be a bimodule projective resolution of $\operatorname{RHom}_\Lambda(X, \Lambda)$ and set

$$\Delta = \Lambda \oplus Y[-1].$$

Then we have a triangle equivalence

$$\operatorname{thick}_{\mathcal{D}(\Gamma)} \Lambda / \operatorname{per} \Gamma \simeq \operatorname{thick}_{\mathcal{D}(\Delta)} \Lambda / \operatorname{per} \Delta. \qquad (1.8.1)$$

We next use the following computation of a dg endomorphism algebra.

Lemma 1.8.2 (See [Am, Lemma 4.13])**.** *There exists an isomorphism* $\operatorname{RHom}_\Delta(\Lambda, \Lambda) \simeq T_\Lambda^L X = \Sigma$ *in the homotopy category of dg algebras, that is, these two dg algebras are related by a zig-zag of quasi-isomorphisms of dg algebras.*

We also need the equivalence by relative Koszul dual.

Lemma 1.8.3. *The functor* $\mathrm{RHom}_\Delta(\Lambda,-)\colon \mathcal{D}(\Delta)\to\mathcal{D}(\Sigma)$ *restricts to equivalences* $\mathrm{thick}_{\mathcal{D}(\Delta)}\Lambda\simeq\mathrm{per}\,\Sigma$ *and* $\mathrm{per}\,\Delta\simeq\mathcal{D}^b(\Sigma)$*. Therefore we have* $\mathrm{per}\,\Sigma\supset\mathcal{D}^b(\Sigma)$ *and an equivalence* $\mathrm{thick}_{\mathcal{D}(\Delta)}\Lambda/\mathrm{per}\,\Delta\xrightarrow{\simeq}\mathcal{C}(\Sigma)$*.*

Proof. The first assertion is clear. We prove second one. Since $\Delta=\mathrm{RHom}_\Lambda(\Delta,Y[-1])$ as (Λ,Δ)-bimodules, we have $\mathrm{RHom}_\Delta(\Lambda,\Delta)=\mathrm{RHom}_\Delta(\Lambda,\mathrm{RHom}_\Lambda(\Delta,Y[-1]))=\mathrm{RHom}_\Lambda(\Lambda,Y[-1])=Y[-1]$, which has finite dimensional total cohomology. Therefore the functor maps $\mathrm{per}\,\Delta$ into $\mathcal{D}^b(\Sigma)$. It remains to show the essential surjectivity. Since Σ is a negative dg algebra, the finite dimensional derived category $\mathcal{D}^b(\Sigma)$ has a bounded t-structure whose heart is equivalent to the category of finite dimensional modules over $H^0\Sigma$, hence it suffices to show that the heart is contained in the image of the functor. Note that $H^0\Sigma=T_\Lambda(H^0X)$ is a finite dimensional graded algebra whose degree 0 part Λ has finite global dimension. Therefore it is sufficient to prove that $D\Lambda$ is in the image. Clearly $D\Lambda=\mathrm{RHom}_\Delta(\Lambda,D\Delta)$, so it remains to show $D\Delta\in\mathrm{per}\,\Delta$. But we have $D\Delta=\mathrm{RHom}_\Lambda(\Delta,D\Lambda)$ and $D\Lambda\in\mathrm{thick}_{\mathcal{D}(\Lambda)}Y[-1]$, hence the assertion. □

Proof of Proposition 1.8.1. It is a consequence of (1.8.1) and Lemma 1.8.3. □

We now return to our setup from Section 1.6. Recall that R is a negatively graded bimodule $(d+1)$-CY algebra of a-invariant a such that each R_i is finite dimensional, and that A is a d-representation infinite algebra in Proposition 1.4.6 given by

$$A=\begin{pmatrix} R_0 & 0 & \cdots & 0\\ R_{-1} & R_0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0\end{pmatrix},$$

whose d-AR-translation ν_d has an a-th root defined by diagram (1.4.1). We have a cotilting bimodule $U=\nu_d^{-1/a}A$ and a d-Iwanaga-Gorenstein algebra $B=A\oplus U$ (see Proposition 1.6.3).

Let us first apply Proposition 1.8.1. Set $\Lambda=A$ and $X=U[1]$. Since U is a preprojective module over a d-representation infinite algebra, it clearly satisfies (X1) and (X2). Then the dg algebra $\Gamma=\Lambda\oplus X[-1]$ concentrates in degree 0, and is nothing but our Iwanaga-Gorenstein algebra $B=A\oplus U$. Moreover, we see that the dg algebra Σ is the tensor algebra

$$S:=T_A(U[1])$$

with trivial differentials. Proposition 1.8.1 for this case gives the following equivalence.

Corollary 1.8.4. *There exists a triangle equivalence*

$$\mathcal{C}(S)\simeq\mathcal{D}_{\mathrm{sg}}(B).$$

To compare the cluster categories of S and R^{dg}, we need another intermediate dg algebra, which is

$$\widetilde{S}:=\mathcal{E}\mathrm{nd}_R(T),\ \text{with}\ T=R\oplus R[-1]\oplus\cdots\oplus R[-(a-1)].$$

Let us first state an easy relationship between R^{dg} and $\widetilde{S}$. We say that dg algebras Π_1 and Π_2 are *dg Morita equivalent* if there is a (Π_1,Π_2)-bimodule X such that $-\otimes^L_{\Pi_1}X$ induces an equivalence $\mathcal{D}(\Pi_1)\xrightarrow{\simeq}\mathcal{D}(\Pi_2)$, or equivalently there exists a compact generator $M\in\mathcal{D}(\Pi_2)$ whose dg endomorphism algebra is quasi-equivalent to Π_1 [Ke1, Theorem 8.2][Ke3, Theorem 3.11].

We immediately have the following lemma.

Lemma 1.8.5. (1) *The dg algebras* R^{dg} *and* $\widetilde{S}$ *are dg Morita equivalent.*

(2) *We have* $\mathrm{per}\,\widetilde{S}\supset\mathcal{D}^b(\widetilde{S})$*.*

(3) *The cluster categories of R^{dg} and $\widetilde{S}$ are equivalent.*

Proof. Since $\widetilde{S}$ is the dg endomorphism ring of a compact generator $T \in \mathcal{D}(R^{\mathrm{dg}})$, we have (1). Then the assertions (2) and (3) follow. □

We next discuss the relationship between S and $\widetilde{S}$.

Lemma 1.8.6. (1) *S is a finite codimensional subalgebra of $\widetilde{S}$.*

(2) *The cluster categories of S and $\widetilde{S}$ are equivalent.*

Proof. (1) This is a consequence of $S = \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathcal{D}(A)}(A, U^i)$ and $\widetilde{S} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}(A)}(A, U^i)$.
(2) Consider the pair of adjoint functors $F = -\otimes^L_S \widetilde{S} \colon \mathcal{D}(S) \to \mathcal{D}(\widetilde{S})$ and $G = \mathrm{res}\colon \mathcal{D}(\widetilde{S}) \to \mathcal{D}(S)$.

Step 1: Restrictions of the adjoint pair. We observe that these functors restrict to the perfect and finite dimensional derived categories. Clearly $F = -\otimes^L_S \widetilde{S}$ restricts to $\mathrm{per}\, S \to \mathrm{per}\, \widetilde{S}$, and $G = \mathrm{res}$ to $\mathcal{D}^b(\widetilde{S}) \to \mathcal{D}^b(S)$. Also since $\widetilde{S}$ is perfect over S by (1) and Lemma 1.8.3, the remaining assertions follow. Therefore the above functors induce an adjoint pair between the Verdier quotients.

Step 2: The unit and counit maps. We show that the unit and counit maps are isomorphisms. Let $X \in \mathcal{C}(S)$ and consider the unit map $u_X \colon X \to X \otimes^L_S \widetilde{S}$. Since this is obtained by applying $X \otimes^L_S -$ to an isomorphism $S \to \widetilde{S}$ in $\mathcal{C}(S)$, it is an isomorphism. Next let $Y \in \mathcal{C}(\widetilde{S})$ and $v_Y \colon FGY \to Y$ the counit. Note that G detects isomorphisms. Indeed v_Y is an isomorphism in $\mathcal{C}(\widetilde{S})$ if and only if, as a map in $\mathrm{per}\, \widetilde{S}$, the cone of v_Y is in $\mathcal{D}^b(\widetilde{S})$. But this property is detected by the restriction functor G. Now the claim $G(v_Y)$ is an isomorphism follows from the fact that the composition $GY \xrightarrow{u_{GY}} GFGY \xrightarrow{G(v_Y)} GY$ equals the identity, which is a general property of an adjoint pair, and the isomorphism of the unit. □

Now, Theorem 1.6.4 is a consequence of the following sequence of equivalences in the upper row.

$$\mathcal{D}_{\mathrm{sg}}(B) \overset{\text{Cor 1.8.4}}{=\!=\!=\!=\!=} \mathcal{C}(S) \overset{\text{Lem 1.8.6}}{=\!=\!=} \mathcal{C}(\widetilde{S}) \overset{\text{Lem 1.8.5}}{=\!=\!=} \mathcal{C}(R^{\mathrm{dg}})$$
$$\mathcal{D}_{\mathrm{sg}}(B) \overset{\text{Cor 1.7.6}}{=\!=} \mathrm{thick}_{\mathcal{D}(C)} A/\mathrm{per}\, C \overset{\text{Lem 1.8.3}}{=\!=} \mathcal{C}(S)$$

Here we have included $C = A \oplus \mathrm{RHom}_A(U[1], A)[-1]$, which is the dg algebra Δ in Proposition 1.8.1 for our specific setup.

We record the following formula for the d-representation infinite algebra A, the cotilting bimodule U, and the d-Iwanaga-Gorenstein algebra B which are determined by R.

$$A = \begin{pmatrix} R_0 & 0 & \cdots & 0 \\ R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \end{pmatrix}, \quad U = \begin{pmatrix} R_{-1} & R_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & R_0 \\ R_{-a} & R_{-(a-1)} & \cdots & R_{-1} \end{pmatrix}, \quad B = A \oplus U. \tag{1.8.2}$$

Let us also record the equivalences we have shown.

$$\mathcal{D}^b(\mathrm{mod}\, A)/\nu_d^{-1/a}[1] \hookrightarrow \mathcal{D}_{\mathrm{sg}}(B) \xrightarrow{\simeq} \mathcal{C}(R^{\mathrm{dg}}) \tag{1.8.3}$$

We end this section with the obvious lemma, which is useful for later computation.

Lemma 1.8.7. *Let J_0 be the Jacobson radical of R_0.*

(1) *The Jacobson radical J_A of A is* $\begin{pmatrix} J_0 & 0 & \cdots & 0 \\ R_{-1} & J_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{-(a-1)} & R_{-(a-2)} & \cdots & J_0 \end{pmatrix}$.

(2) *The Jacobson radical J_B of B is $J_A \oplus U$.*

(3) *We have $J_B/J_B^2 = J_A/J_A^2 \oplus U/(J_A U + U J_A)$, with*

$$U/(J_AU+UJ_A) = \begin{pmatrix} 0 & R_0/J_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_0/J_0 \\ \frac{R_{-a}}{J_0R_{-a}+R_{-a}J_0+\sum_{i+j=a,i,j>0} R_{-i}R_{-j}} & 0 & \cdots & 0 \end{pmatrix}.$$

1.9 Higher cluster categories of higher representation infinite algebras

We give an application of Theorem 1.6.4, which is given by taking the CY algebra R as a (higher) preprojective algebra. We prove that any m-cluster category of a d-representation infinite algebra with $m > d$ is the singularity category of a d-Iwanaga-Gorenstein algebra. Moreover we explicitly describe the quiver and relations of the Iwanaga-Gorenstein algebra for the case $d = 1$, that is, when A is hereditary.

Theorem 1.9.1. *Let A be a d-representation infinite algebra and fix $n \geq 1$. Let $U = \operatorname{Ext}^d_A(DA, A)$ and*

$$B = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix} \oplus \begin{pmatrix} 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \\ U & 0 & \cdots & 0 \end{pmatrix}$$

the trivial extension algebra, where the matrix is $n \times n$. Then B is d-Iwanaga-Gorenstein and there exists a triangle equivalence

$$\mathcal{C}_{d+n}(A) \simeq \mathcal{D}_{\rm sg}(B).$$

Proof. Let $\Pi = T_A U$ be the $(d+1)$-preprojective algebra of A. Give a grading on Π by setting $\deg U = -n$ so that Π is a bimodule $(d+1)$-CY algebra of a-invariant n. Note that $\Pi^{\rm dg}$ is quasi-isomorphic to the derived $(d+n+1)$-preprojective algebra (or the $(d+n+1)$-Calabi-Yau completion) $T^L_A \operatorname{RHom}_A(DA, A)[d+n]$, in the sense of [Ke6]. Therefore the cluster category $\mathcal{C}(\Pi^{\rm dg})$ is nothing but the $(d+n)$-cluster category $\mathcal{C}_{d+n}(A)$ of A. By Theorem 1.6.4, we have a triangle equivalence $\mathcal{C}(\Pi^{\rm dg}) \simeq \mathcal{D}_{\rm sg}(B(\Pi))$ for the d-Iwanaga-Gorenstein algebra $B(\Pi)$ in (1.8.2) for the Calabi-Yau algebra Π, which is precisely the above B in the statement. ☐

Remark 1.9.2. We give a general discussion in Appendix 1.A the effect of 'multiplying gradings' as we did in the above proof. See Corollary 1.A.2 for a description as a derived orbit category, which predicts the above equivalence.

1.9.1 The case $d = 1$.

Let us record the special case $d = 1$, that is, when A is hereditary.

Corollary 1.9.3. *Let Q be a finite connected acyclic non-Dynkin quiver, $A = kQ$ and fix $n \geq 1$. Then we have a triangle equivalence*

$$\mathcal{C}_{n+1}(kQ) \simeq \mathcal{D}_{\rm sg}(B)$$

for the 1-Iwanaga-Gorenstein algebra B in Theorem 1.9.1.

In this case we can explicitly describe the quiver and relations for B.

Proposition 1.9.4. *The 1-Iwanaga-Gorenstein algebra B in Theorem 1.9.1 is presented by the quiver $\widehat{Q}$ with*

(a) *vertices* $Q_0 \times \{1, \dots, n\}$,

(b) *three kinds of arrows*

  (i) $a = a^l \colon (i,l) \to (j,l)$ *for each* $a \colon i \to j$ *in* Q_1 *and* $1 \leq l \leq n$.

  (ii) $v = v_i^l \colon (i, l+1) \to (i,l)$ *for each* $i \in Q_0$ *and* $1 \leq l < n$.

  (iii) $a^* \colon (j,1) \to (i,n)$ *for each* $a \colon i \to j$ *in* Q_1.

(c) *three kinds of relations*

  (i) $a^l v_i^l = v_j^l a^{l+1}$ *for each* $a \colon i \to j$ *in* Q_1 *and* $1 \leq l < n$.

  (ii) $\sum_{s(a)=i} a^* a^1 = \sum_{t(a)=i} a^l a^*$ *for all* $i \in Q_0$.

  (iii) $v_i^{l-1} v_i^l = 0$, $a^* v = 0$, $v a^* = 0$ *if* $n \geq 2$, *and* $a^* b c^* = 0$ *for any composable* $a, b, c \in Q_1$ *if* $n = 1$.

We use a reformulation of a well-known fact on preprojective algebras. We denote by J_Λ the Jacobson radical of a ring Λ.

Lemma 1.9.5. *Let* Q *be a finite acyclic quiver,* $A = kQ$*, and* $U = \mathrm{Ext}^1_A(DA, A)$*. Then there exists a subset* $\{u(a^*) \mid a \in Q_1\}$ *of* U *which gives a basis of* $U/(J_A U + U J_A)$ *as a* (kQ_0, kQ_0)*-bimodule, and such that* $\sum_{a \in Q_1} (a u(a^*) - u(a^*) a) = 0$ *in* U.

Proof. Consider the two presentations $T_A U = k\overline{Q}/(\sum_{a \in Q_1}(aa^* - a^* a))$ of the preprojective algebra Π of Q, where $\overline{Q}$ is the double quiver of Q obtained by adding the opposite arrows $\{a^* \colon j \to i \mid a \colon i \to j \text{ in } Q\}$. Take the elements of U corresponding to $\{a^* \mid a \in Q_1\} \subset \overline{Q}_1$. This is a desired set since we have $U/(J_A U + U J_A) = kQ_1^*$ as (kQ_0, kQ_0)-bimodules by looking at the degree 1 part of J_Π / J_Π^2. □

Proof of Proposition 1.9.4. By Lemma 1.8.7, we have

$$J_B/J_B^2 = \begin{pmatrix} J_A/J_A^2 & 0 & \cdots & 0 \\ 0 & J_A/J_A^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_A/J_A^2 \end{pmatrix} \oplus \begin{pmatrix} 0 & A/J_A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A/J_A \\ U/(J_A U + U J_A) & 0 & \cdots & 0 \end{pmatrix}.$$

Therefore we see that the quiver of B consists of the following.

- n copies $Q^1, \dots, Q^n$ of Q.
- The arrows from Q^{l+1} to Q^l corresponding to the idempotents in A.
- The arrows from Q^1 to Q^n corresponding to the basis of $U/(J_A U + U J_A)$ as (kQ_0, kQ_0)-bimodules.

In view of Lemma 1.9.5, these vertices and arrows are precisely the ones described in (a) and (b), hence the quiver of B is $\widehat{Q}$.

Now we determine the relations. To simplify the discussion below, we give a grading on B in Theorem 1.9.1 by setting the first factor to have degree 0, and the second one to have degree 1. Similarly, give a grading on $\widehat{Q}$ by setting the arrows in (i) to have degree 0, and in (ii), (iii) to have degree 1. Take a subset $\{u(a^*) \mid a \in Q_1\}$ of U given in Lemma 1.9.5 and consider the map $\widehat{Q} \to B$ defined by

- the natural embedding $kQ^l \to A \times \cdots \times A = B_0$ into the l-th factor,
- $v_i^l \mapsto e_i$, where e_i is the corresponding idempotent of A in the $(l+1, l)$-component of B_1,
- $a^* \mapsto u(a^*)$, where $u(a^*) \in U$ is in the $(n, 1)$-component of B_1.

These maps induce a homogeneous homomorphism $\varphi\colon k\widehat{Q} \to B$, which clearly preserves the relations. Denoting by I the ideal generated by the relations, we obtain a homomorphism $k\widehat{Q}/I \to B$. Since it is clearly an isomorphism in degree 0, it is enough to consider the degree 1 part. Let $e_{(i,l)}$ be the idempotent of $k\widehat{Q}/I$ at the vertex (i,l), and set $e_l = \sum_{i\in Q_0} e_{(i,l)}$. We denote their images under φ by the same symbols. It is sufficient to show that φ induces an isomorphism $e_l(k\widehat{Q}/I)e_m \to e_l B e_m$ for each $1 \le l, m \le n$. By the relation (iii), each term is 0 in degree 1 unless $m - l = 1$ or $(l,m) = (n,1)$, so we only have to consider these two cases.

Case 1: The map $e_l(k\widehat{Q}/I)e_{l+1} \to e_l B e_{l+1}$ is an isomorphism for each $1 \le l < n$. By the relation (i), any element in $e_l(k\widehat{Q}/I)e_{l+1}$ can be written as $a\cdot(\sum_{i\in Q_0} v_i^l)$ for some $a \in kQ^l = A$. This observation immediately shows the map is an isomorphism.

Case 2: The map $e_n(k\widehat{Q}/I)e_1 \to e_n B e_1$ is an isomorphism. By the relation (ii), the space $e_n(k\widehat{Q}/I)e_1$ is isomorphic to the degree 1 part of the preprojective algebra of Q, thus to U. On the other hand, the space $e_n B e_1$ is also clearly U. □

We look at the most special case $d = 1$ and $n = 1$.

Example 1.9.6. Let Q be a finite connected acyclic non-Dynkin quiver and kQ its path algebra, which is 1-representation-infinite.

The 1-Iwanaga-Gorenstein algebra $B = kQ \oplus U$ with $U = \tau^{-1}kQ$ is a truncation of the preprojective algebra Π of Q, which is presented by the same quiver as Π and the additional relations 'the elements of U square to zero', as stated in Proposition 1.9.4.

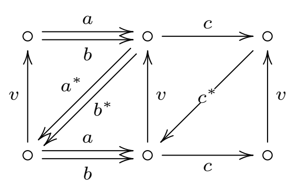
The equivalence $\mathcal{D}_{\rm sg}(B) \simeq \mathcal{C}_2(kQ)$ in Corollary 1.9.3 is given in [BIRSc, ART] as the 2-CY category associated to the square of the Coxeter element in the Coxeter group of Q. Our proof is different from theirs since our equivalence comes from quasi-equivalence of dg orbit categories.

The next example is the case $d = 1$ and $n = 2$.

Example 1.9.7. Let Q be the following quiver

$$\circ \overset{a}{\underset{b}{\rightrightarrows}} \circ \xrightarrow{c} \circ \ ,$$

thus $A = kQ$ is 1-representation infinite. Let $n = 2$, so by Proposition 1.9.4, the 1-Iwanaga-Gorenstein algebra B is presented by the following quiver with relations.



$$av = va,\ bv = vb,\ cv = vc$$
$$a^*a + b^*b = 0,\ aa^* + bb^* = c^*c,\ cc^* = 0$$
$$va^* = 0,\ vb^* = 0,\ vc^* = 0,\quad a^*v = 0,\ b^*v = 0,\ c^*v = 0$$

By Corollary 1.9.3, we have a triangle equivalence $\mathcal{C}_3(kQ) \simeq \mathcal{D}_{\rm sg}(B)$.

1.9.2 The case $n = 1$.

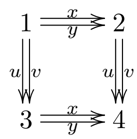
We now turn to another special case of $n = 1$. In this case, the algebra B is a truncation of the $(d+1)$-preprojective algebra of A.

Corollary 1.9.8. *Let A be a d-representation infinite algebra, $U = \operatorname{Ext}^d_A(DA, A)$, and $B = A \oplus U$. Then B is d-Iwanaga-Gorenstein and there is a triangle equivalence $\mathcal{C}_{d+1}(A) \simeq \mathcal{D}_{\rm sg}(B)$.*

This is a higher dimensional analogue of Example 1.9.6 above. It is predicted in [Iy3] as a generalization of [BIRSc] that $\mathcal{D}_{\rm sg}(B)$ has a $(d+1)$-cluster tilting object. We deduce this from our equivalence with the $(d+1)$-cluster category.

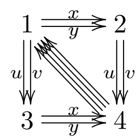
Let us now give an example. See also Example 1.11.2(1) for an example in $d = 2$.

Example 1.9.9. (1) Let A be the tensor product of two path algebras of Kronecker quivers, thus is presented by the following quiver with relations.



$$xu = ux,\ xv = vx,\ yu = uy,\ yv = vy.$$

This is a 2-representation infinite algebra. (See [HIO, Theorem 2.10].) This is also the endomorphism algebra of a tilting bundle $\mathcal{O} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,1)$ over $\mathbb{P}^1 \times \mathbb{P}^1$. The preprojective algebra Π of A is presented by the following quiver with suitable commutativity relations,

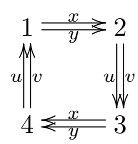


thus so is its truncation B with some additional relations. By Corollary 1.9.8 we have a triangle equivalence $\mathcal{C}_3(A) \simeq \mathcal{D}_{\rm sg}(B)$.

(2) Let A' be the algebra presented by the following quiver with relations.

$$1 \underset{y}{\overset{x}{\rightrightarrows}} 2 \underset{v}{\overset{u}{\rightrightarrows}} 3 \underset{y}{\overset{x}{\rightrightarrows}} 4\ , \quad xuy = yux,\ xvy = yvx.$$

This is obtained from A by taking for example the left mutation at vertex 3, that is, the endomorphism algebra of another tilting bundle $\mathcal{O} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(2,1)$. Then it is easy to see that A' is also 2-representation infinite. Its preprojective algebra Π' is given by the following quiver with commutativity relations,



thus its truncation B' by the same quiver with suitable additional relations. By Corollary 1.9.8 we have a triangle equivalence $\mathcal{C}_3(A') \simeq \mathcal{D}_{\rm sg}(B')$.

(3) The 2-representation infinite algebras A and A' above are derived equivalent, hence their cluster categories are equivalent; $\mathcal{C}_3(A) \simeq \mathcal{C}_3(A')$. Therefore we deduce that all the relevant 3-CY categories are equivalent; $\mathcal{D}_{\rm sg}(B) \simeq \mathcal{C}_3(A) \simeq \mathcal{C}_3(A') \simeq \mathcal{D}_{\rm sg}(B')$.

1.10 Examples: Polynomial rings

In this section, we apply our main results for polynomial rings and give a concrete description of the d-representation-infinite algebra A and the d-Iwanaga-Gorenstein algebra B. (See (1.8.2) for the definition of A and B.)

Let $R = k[x_0, \dots, x_d]$ with $\deg x_i = -a_i < 0$. It is bimodule $(d+1)$-CY algebra with a-invariant $a = \sum_{i=0}^d a_i$. We have the following result on the algebras A and B. For a finite subgroup $G \subset \mathrm{GL}_{d+1}(k)$, which naturally acts on R, the *skew group algebra* $R * G$ is a vector space $R \otimes_k kG$ with multiplication $(a \otimes g)(b \otimes h) = ag(b) \otimes gh$.

Proposition 1.10.1. *Suppose that a is invertible in k, there exists a primitive a-th root of unity $\zeta \in k$, and that $a_0, \dots, a_d$ are relatively prime.*

(1) *The algebra A is presented by the quiver Q with the vertices $\{0,1,\dots,a-1\}$, the arrows $x_i = x_i^l \colon l \to l+a_i$ for each $0 \le i \le d$ and $0 \le l \le a-1$ such that $l+a_i \le a-1$, and with the commutativity relations $x_j^{l+a_i} x_i^l = x_i^{l+a_j} x_j^l$.*

(2) *Let $g = \operatorname{diag}(\zeta^{a_0}, \dots, \zeta^{a_d}) \in \mathrm{SL}_{d+1}(k)$ and G the cyclic subgroup generated by g. Then, the $(d+1)$-preprojective algebra of A is isomorphic to $R * G$.*

(3) *A is d-representation-infinite of type $\widetilde{A}$, in the sense of [HIO].*

(4) *B is presented by the quiver $\widehat{Q}$ obtained by adding to Q the arrows $u = u^l \colon l \to l-1$ for each $1 \le l \le a-1$ and two types of additional relations:*

(i) $x_i^{l-1} u^l = u^{l+a_i} x_i^l$ whenever $1 \le l$ and $l + a_i \le a-1$.

(ii) $u^{l+a_i-1} x_i^{l-1} u^l = 0$ whenever $1 \le l$ and $l + a_i \le a$.

Proof. (1) Note that the category $\operatorname{proj}^{\mathbb{Z}} R$ is presented by the quiver with vertices set $\mathbb{Z}$, arrows $x_i^l \colon l \to l+a_i$, and with the commutativity relations. Since $A \simeq \operatorname{End}_R^{\mathbb{Z}}(R \oplus R(-1) \oplus \cdots R(-(a-1)))$, it is presented by its full subquiver with vertices $\{0,1,\dots,a-1\}$ and with the induced relations.
(2)(3) We follow the construction of d-representation-infinite algebras of type $\widetilde{A}$ [HIO, Section 5].

Let L be the free $\mathbb{Z}$-module with basis $\alpha_1, \dots, \alpha_d$, and set $\alpha_0 := -\alpha_1 - \cdots - \alpha_d$. Let Q be the quiver with vertices $Q_0 = L$ and the set of arrows $Q_1 = \{x_i = x_i^l \colon l \to l + \alpha_i \mid l \in L,\ 0 \le i \le d\}$. Moreover for each $l \in L$ and $0 \le i < j \le d$, define the relation $r_{ij} = r_{ij}^l = x_i x_j - x_j x_i$. Then we have a category $\mathcal{L}$ presented by this quiver and relations. We assign for each point $l = \sum_{i=1}^d l_i \alpha_i \in L$ the integer $m(l) = \sum_{i=1}^d l_i a_i$.

Now let $B \subset L$ the subgroup consisting of points l in L such that $m(l)$ is a multiple of a. The subgroup B has finite index a, and acts on $\mathcal{L}$ by translation. We then have the orbit category $\mathcal{L}/B$, which can naturally be identified with the algebra Π presented by the (finite) quiver Q/B and the induced relations. By [HIO, Lemma 5.3], Π is isomorphic to the skew group algebra $R * G$.

Going back to the original quiver Q, we set

$$C = \{x \colon l \to l' \text{ in } Q_1 \mid m(l) < na \le m(l') \text{ for some } n \in \mathbb{Z}\},$$

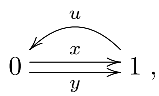
which is a periodic and bounding cut in the sense of [HIO, Definitions 5.4, 5.5], and is stable under B. Then C induces a grading on Q/B, hence on Π by

$$\deg x = \begin{cases} 1 & (x \in C) \\ 0 & (x \notin C) \end{cases}$$

for each $x \in Q_1$. Now by [HIO, Theorem 5.6], Π is the preprojective algebra of its degree 0 part, and we see that it is nothing but A, since they are presented by the quiver $(Q/B) \setminus (C/B)$ and the commutativity relations.
(4) We first compute the quiver of B using Lemma 1.8.7(3). Since R is generated by degree $> -a$, the vector space on the lower left corner of $U/(J_A U + U J_A)$ is 0. Therefore the arrows we have to add are just the ones corresponding to $1 \in R_0/J_0 = k$, and the quiver of B is $\widehat{Q}$. Then there exists a natural homomorphism $k\widehat{Q} \to B$, which preserves the relations, thus induces a homomorphism $\varphi \colon k\widehat{Q}/I \to B$, where I is the ideal generated by the relations. We show that φ is an isomorphism. We may truncate by the idempotents; denote by e_i the idempotent of $k\widehat{Q}/I$ at vertex i, and we show that the induced map $e_j(k\widehat{Q}/I)e_i \to e_j B e_i$ is an isomorphism for each $0 \le i, j \le a-1$. Note that the relations show that each space $e_j(k\widehat{Q}/I)e_i$ has a basis consisting of monomials of degree $-(j-i+1)$ (defined from the grading on R) each of which contains at most one of the u^l's (which have degree 0). It is then clear that φ maps these basis to the basis of $e_j B e_i$. $\square$

Let us first look at the easiest case.

Example 1.10.2. This is a continuation of Example 1.4.8. Let $R = k[x, y]$ with $\deg x = \deg y = -1$, so R is 2-CY of a-invariant 2. By Theorem 1.5.2, R^{dg} is twisted 4-CY. As we have seen in Example 1.4.8, we have an equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{mod} A)$ with A the Kronecker algebra, and its AR translation ν_1 has a square root. On the other hand, the Iwanaga-Gorenstein algebra B is presented by the following quiver with relations:



$$xuy = yux,\ uxu = uyu = 0.$$

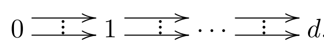
By (1.8.3), there are equivalences

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_1^{-1/2}[1] \simeq \mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}(R^{\mathrm{dg}})$$

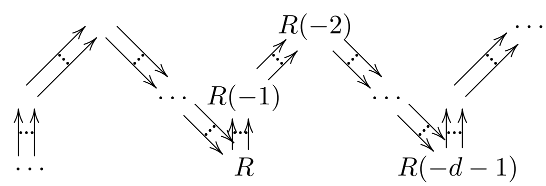
of twisted 3-CY categories; compare the case $m = 2$ in Example 1.5.9. Using the description as an orbit category, we can classify the objects in $\mathcal{C}(R^{\mathrm{dg}})$ or $\mathcal{D}_{\mathrm{sg}}(B)$, which we leave to the reader.

We look at higher dimensional case.

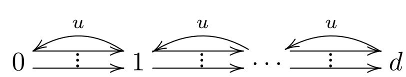
Example 1.10.3. Let $R = k[x_0, x_1, \dots, x_d]$ with $\deg x_0 = \cdots = \deg x_d = -1$. Then R is $(d+1)$-CY of a-invariant $d+1$, thus by Theorem 1.5.2, R^{dg} is sign-twisted $(2d+2)$-CY. It is well-known that $\operatorname{qgr} R$ is equivalent to the category $\operatorname{coh} \mathbb{P}^d$ of coherent sheaves over the projective space $\mathbb{P}^d$. The tilting object in $\mathcal{D}^b(\operatorname{qgr} R)$ given in Proposition 1.4.6 is the tilting bundle $T = \bigoplus_{l=0}^d \mathcal{O}_{\mathbb{P}^d}(l)$ on $\mathbb{P}^d$, whose endomorphism ring A is the d-Beilinson algebra. It is presented by the following quiver with commutativity relations:

$A =$ 

The category $\operatorname{add}\{R(-i) \mid i \in \mathbb{Z}\} = \operatorname{add}\{\mathcal{O}(i) \mid i \in \mathbb{Z}\} = \operatorname{add}\{\nu_d^{-i} A \mid i \in \mathbb{Z}\}$ (which are identified via the equivalence $\mathcal{D}^b(\operatorname{qgr} R) \simeq \mathcal{D}^b(\operatorname{coh} \mathbb{P}^d) \simeq \mathcal{D}^b(\operatorname{mod} A)$) is presented by the following quiver.



The autoequivalence $\nu_d^{-1/(d+1)}$ on $\mathcal{D}^b(\operatorname{mod} A)$ acts on this subcategory by 'moving one place' along the d-fold arrows. On the other hand the Iwanaga-Gorenstein algebra $B = A \oplus U$ is presented by the quiver



and the commutativity relations, and $u^2 = 0$. This is a truncation of $T_A U = A \oplus U \oplus U^2 \oplus \cdots$, which is the endomorphism ring of a tilting object $\pi^* T$ over the total space of the line bundle $\pi \colon \mathcal{O}(-1) \to \mathbb{P}^d$. Applying (1.8.3), we have an embedding and an equivalence

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_d^{-1/(d+1)}[1] \hookrightarrow \mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}(R^{\mathrm{dg}}).$$

1.11 Examples: Jacobian algebras arising from dimer models

A *dimer model* is a finite bipartite graph Γ on a real 2-torus inducing a polygonal cell decomposition. We denote by Γ_0, (resp. Γ_1, Γ_2) the set of vertices (resp. edges, faces) of Γ. It gives rise to a quiver with potential (Q, W) in the sense of [DWZ] in the following way: Let Q denote the dual quiver of Γ, thus the set of vertices Q_0 (resp. arrows Q_1) corresponds bijectively to Γ_2 (resp. Γ_1). By convention, the arrows of Q see white vertices on the right. Then for each vertex v there is a unique cycle c_v consisting of arrows of Q corresponding to the edges of Γ which are adjacent to v. Now define the potential by $W = \sum_{v\colon \text{white}} c_v - \sum_{v\colon \text{black}} c_v$.

We assume that Γ is *consistent* (see [Boc2]) in the sense that there exists a map $\mathrm{R}\colon \Gamma_1 \to \mathbb{R}_{>0}$ such that

- $\sum_{v \in \partial a} \mathrm{R}(a) = 2$ for all $v \in \Gamma_0$, where the sum runs over $a \in \Gamma_1$ adjacent to v.
- $\sum_{a \in \partial f} (1 - \mathrm{R}(a)) = 2$ for all $f \in \Gamma_2$, where the sum runs over $a \in \Gamma_1$ in the boundary of f.

Fix a map

$$d \colon Q_1 = \Gamma_1 \longrightarrow \mathbb{Z} \tag{1.11.1}$$

such that $\sum_{v \in \partial a} d(a)$ is a constant l for all $v \in \Gamma_0$. Such maps are typically given by perfect matchings on Γ. Recall that a *perfect matching* on a graph is a set of its edges such that each vertex is contained in precisely one edge in the set. It is known that the consistency condition ensures the existence of perfect matchings [Bro, Section 2.3]. We can identify a perfect matching P on Γ with a map $d \colon \Gamma_1 \to \{0, 1\}$ such that $\sum_{v \in \partial a} d(a) = 1$ for all $v \in \Gamma_0$ by setting $d(a) = 1$ if and only if $a \in P$. Consequently, any $\mathbb{Z}$-linear combination of perfect matchings gives a function (1.11.1).

Proposition 1.11.1 (See [Bro, Theorem 7.7], [AIR, Proposition 6.1])**.** *Let Γ be a consistent dimer model, and let d be a map (1.11.1) such that $\sum_{v \in \partial a} d(a) = l$ for all $v \in \Gamma_0$. Then d gives a grading on the Jacobian algebra making it into a bimodule 3-CY algebra of a-invariant $-l$.*

Proof. Give a grading on the quiver Q by setting $\deg a = d(a)$ for $a \in Q_1$. Then the potential W is homogeneous of degree l, thus d induces a grading on the Jacobian algebra R. Consider the complex

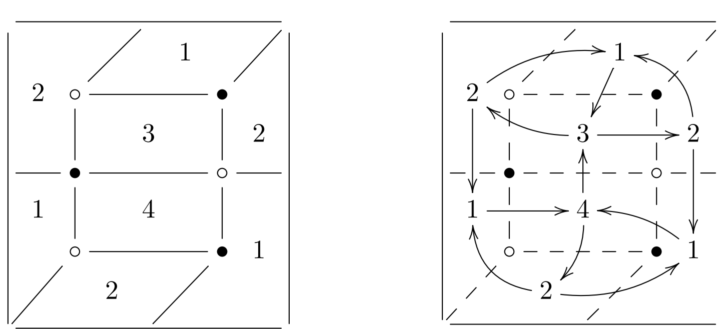
$$\bigoplus_{i \in Q_0} Re_i \otimes e_i R \xrightarrow{d_3} \bigoplus_{a \in Q_1} Re_{s(a)} \otimes e_{t(a)} R \xrightarrow{d_2} \bigoplus_{a \in Q_1} Re_{t(a)} \otimes e_{s(a)} R \xrightarrow{d_1} \bigoplus_{i \in Q_0} Re_i \otimes e_i R$$

with maps

$$\begin{aligned} d_1(e_{t(a)} \otimes e_{s(a)}) &= a \otimes e_{s(a)} - e_{t(a)} \otimes a \\ d_2(e_{s(a)} \otimes e_{t(a)}) &= \sum_{b \in Q_1} p \otimes q \text{ for each cycle } apbq \text{ in } W \\ d_3(e_i \otimes e_i) &= \sum_{t(a) = i} a \otimes e_i - \sum_{s(a) = i} e_i \otimes a. \end{aligned}$$

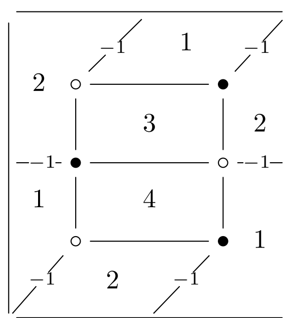
By [Bro, Theorem 7.7], this complex together with the multiplication map $\bigoplus_{i \in Q_0} Re_i \otimes e_i R \to R$ gives a bimodule projective resolution of R making it into a 3-CY algebra. Now, taking degree into account, we deduce that R is graded bimodule 3-CY of a-invariant $-l$. $\square$

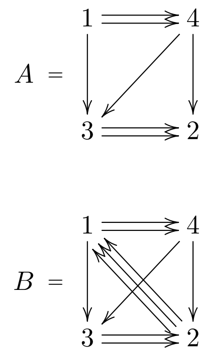
Example 1.11.2. Let Γ be a dimer model as in the left picture below, where the vertical and horizontal ends are identified so that it has 4 faces which are labeled by 1, 2, 3, and 4. It gives a 3-CY algebra R presented by the quiver in the right picture.



Now we consider the grading d in (1.11.1). We discuss two variations.

(1) First we consider the grading below. The labels on the edges show the values under d, and unlabeled ones have degree 0, thus it is a perfect matching. This grading makes R into a bimodule 3-CY algebra of a-invariant 1, thus $A = R_0$ and $B = R_{\leq -1}$ are given by quivers below.

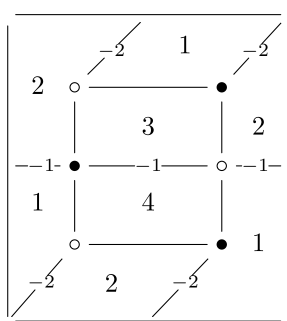


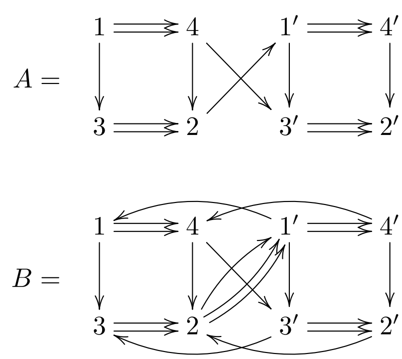


This 2-representation infinite algebra A is the endomorphism ring of a tilting bundle T on the Hirzeburch surface $\Sigma_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ which is the blow-up of $\mathbb{P}^2$ at one point as well. Also, R is the 3-preprojective algebra of A, which is the endomorphism ring of a tilting bundle π^*T on the total space of the canonical bundle $\pi\colon \omega \to \Sigma_1$ over Σ_1 (see [BH]). Now the dg algebra R^{dg} is 4-CY by Theorem 1.5.2, and applying Corollary 1.9.8 we have triangle equivalences

$$\mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}_3(A) = \mathcal{C}(R^{\mathrm{dg}}).$$

(2) We next consider the grading below, again the non-zero degree of each edge is labeled. This is not given by a perfect matching (but by a sum of two perfect matchings), and makes R into a bimodule 3-CY algebra of a-invariant 2. In this case, the 2-representation infinite algebra $A = \begin{pmatrix} R_0 & 0 \\ R_{-1} & R_0 \end{pmatrix}$ and the 2-Iwanaga-Gorenstein algebra $B = A \oplus U$ with $U = \begin{pmatrix} R_{-1} & R_0 \\ R_{-2} & R_{-1} \end{pmatrix}$ is presented as follows.





Applying our results, the dg algebra R^{dg} is sign twisted 5-CY, and there exist an embedding and a triangle equivalence

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_2^{-1/2}[1] \hookrightarrow \mathcal{D}_{\mathrm{sg}}(B) \simeq \mathcal{C}(R^{\mathrm{dg}}).$$

1.A Multiplying gradings

Let R be a graded ring. For a fixed integer $n \geq 1$, define the graded ring ${}^nR$ by

$$({}^nR)_i = \begin{cases} R_{i/n} & \text{if } n \mid i \\ 0 & \text{if } n \nmid i \end{cases}.$$

If R is bimodule $(d+1)$-CY of a-invariant a, then clearly ${}^nR$ is bimodule $(d+1)$-CY of a-invariant na. Although the category $\mathrm{qper}^{\mathbb{Z}n}R$ just splits as a direct product of n copies of $\mathrm{qper}^{\mathbb{Z}}R$ and yields nothing new, the cluster category $\mathcal{C}({}^nR^{\mathrm{dg}})$, being a triangulated hull of $\mathrm{qper}^{\mathbb{Z}n}R/(-1)[1]$, becomes 'connected' by the action of the automorphism $(-1)[1]$, which yields something new.

The aim of this section is to describe the category $\mathcal{C}({}^nR^{\mathrm{dg}})$ in terms of the relevant objects from R. Although this can be regarded as a special case of our main results, we shall obtain a better presentation of orbit categories.

Recall that R is a negatively graded bimodule $(d+1)$-CY algebra of a-invariant a such that each R_i is finite dimensional, and recall the definitions of the d-representation infinite algebra $A = A(R)$, the cotilting bimodule $U = U(R)$, and the d-Iwanaga-Gorenstein algebra $B = B(R)$ from (1.8.2). We have the following description of $\mathcal{C}({}^nR^{\mathrm{dg}})$ in terms of A, which generalizes Theorem 1.6.1 and Corollary 1.6.2.

Theorem 1.A.1. *There exists a fully faithful functor*

$$\mathcal{D}^b(\mathrm{mod}\,A)/\nu_d^{-1/a}[n] = \mathrm{qper}^{\mathbb{Z}}R/(-1)[n] \longrightarrow \mathcal{C}({}^nR^{\mathrm{dg}})$$

whose image generates $\mathcal{C}({}^nR^{\mathrm{dg}})$ as a thick subcategory.

Let the d-representation infinite algebra $\widetilde{A} = A({}^nR)$, the cotilting $(\widetilde{A},\widetilde{A})$-bimodule $\widetilde{U} = U({}^nR)$, and the d-Iwanaga-Gorenstein algebra $\widetilde{B} = B({}^nR)$ be as given in (1.8.2) for ${}^nR$, thus we have

$$\widetilde{A} = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix} = A \times \cdots \times A, \quad \widetilde{U} = \begin{pmatrix} 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \\ U & 0 & \cdots & 0 \end{pmatrix}, \quad \widetilde{B} = \widetilde{A} \oplus \widetilde{U}. \tag{1.A.1}$$

Then we also have the consequence in terms of singularity category of $\widetilde{B}$.

Corollary 1.A.2. *There exists a fully faithful functor*

$$\mathcal{D}^b(\mathrm{mod}\,A)/\nu_d^{-1/a}[n] \longrightarrow \mathcal{D}_{\mathrm{sg}}(\widetilde{B})$$

whose image generates $\mathcal{D}_{\mathrm{sg}}(\widetilde{B})$ as a thick subcategory.

Now we start the proof. The first step is to apply our results in Section 1.6 for the CY algebra ${}^nR$. By Corollary 1.4.7, the d-AR translation ν_d on $\mathcal{D}^b(\mathrm{mod}\,\widetilde{A})$ has an na-th root, and by Theorem 1.6.1 and Corollary 1.6.2, we have an equivalence and an embedding

$$\mathcal{D}^b(\mathrm{mod}\,\widetilde{A})/\nu_d^{-1/na}[1] \simeq \mathrm{qper}^{\mathbb{Z}n}R/(-1)[1] \hookrightarrow \mathcal{C}({}^nR^{\mathrm{dg}}). \tag{1.A.2}$$

We next compare the derived orbit categories of ${}^nR$ (resp. $\widetilde{A}$) and of R (resp. A). Obviously there is a diagram of equivalences and compatible autoequivalences

$$\begin{array}{ccc} (-1) \circlearrowright & \mathrm{qper}^{\mathbb{Z}n}R = & \mathrm{qper}^{\mathbb{Z}}R \times \cdots \times \mathrm{qper}^{\mathbb{Z}}R \\ & \wr\downarrow & \downarrow\wr \\ \nu_d^{-1/na} \circlearrowright & \mathcal{D}^b(\mathrm{mod}\,\widetilde{A}) = & \mathcal{D}^b(\mathrm{mod}\,A) \times \cdots \times \mathcal{D}^b(\mathrm{mod}\,A). \end{array}$$

We describe the action of these autoequivalences on the right-hand-side.

Lemma 1.A.3. (1) *The action of (-1) on $\mathrm{qper}^{\mathbb{Z}n}R$ becomes $(X_1,\dots,X_n) \mapsto (X_n(-1), X_1,\dots,X_{n-1})$ on $\mathrm{qper}^{\mathbb{Z}}R \times \cdots \times \mathrm{qper}^{\mathbb{Z}}R$.*

(2) *The action of* $\nu_d^{-1/na}$ *on* $\mathcal{D}^b(\operatorname{mod}\widetilde{A})$ *is* $(X_1,\dots,X_n)\mapsto(\nu_d^{-1/a}X_n,X_1,\dots,X_{n-1})$ *on* $\mathcal{D}^b(\operatorname{mod}A)\times\cdots\times\mathcal{D}^b(\operatorname{mod}A)$*.*

Proof. We only prove (2), the proof of (1) is similar. By the form (1.A.1) of $\widetilde{U}$, we see that $-\otimes^L_{\widetilde{A}}\widetilde{U}$ maps $(X_1,\dots,X_n)$ to $(X_n\otimes^L_A U,X_1,\dots,X_{n-1})$. □

We next relate the orbit categories arising from R and ${}^nR$.

Lemma 1.A.4. (1) *The functor* $\operatorname{qper}^{\mathbb{Z}}R\to\operatorname{qper}^{\mathbb{Z}n}R$ *given by* $X\mapsto(X,0,\dots,0)$ *induces an equivalence*

$$\operatorname{qper}^{\mathbb{Z}}R/(-1)[n]\xrightarrow{\simeq}\operatorname{qper}^{\mathbb{Z}n}R/(-1)[1].$$

(2) *The functor* $\mathcal{D}^b(\operatorname{mod}A)\to\mathcal{D}^b(\operatorname{mod}\widetilde{A})$ *given by* $X\mapsto(X,0,\dots,0)$ *induces an equivalence*

$$\mathcal{D}^b(\operatorname{mod}A)/\nu_d^{-1/a}[n]\xrightarrow{\simeq}\mathcal{D}^b(\operatorname{mod}\widetilde{A})/\nu_d^{-1/na}[1].$$

Proof. Again, we only prove (2). We first want to show that there is a natural isomorphism

$$\bigoplus_{l\in\mathbb{Z}}\operatorname{Hom}_{\mathcal{D}(A)}(X,(\nu_d^{-1/a}[n])^lY)\longrightarrow\bigoplus_{l\in\mathbb{Z}}\operatorname{Hom}_{\mathcal{D}(\widetilde{A})}(FX,(\nu_d^{-1/na}[1])^lFY)\ ,$$

where $FX=(X,0,\dots,0)$. Since $F(\nu_d^{-1/a}Y[n])=(\nu_d^{-1/na}[1])^nF(Y)$ and $\operatorname{Hom}_{\mathcal{D}(\widetilde{A})}(FX,(\nu_d^{-1/na}[1])^lFY)=0$ unless $n\mid l$ by Lemma 1.A.3, we have a natural bijection.
We next verify that the functor is dense. Note that $(0,\dots,X_i,\dots,0)\simeq((\nu_d^{-1/a}[1])^{-i+1}X_i,0\dots,0)$ in the orbit category $\mathcal{D}^b(\widetilde{A})/\nu_d^{-1/na}[1]$. Therefore $\bigoplus_{i=1}^n(\nu_d^{-1/a}[1])^{-i+1}X_i\in\mathcal{D}^b(A)$ is mapped to $(X_1,\dots,X_n)$. □

We now have our desired results.

Proof of Theorem 1.A.1 and Corollary 1.A.2. We have Theorem 1.A.1 by (1.A.2) and Lemma 1.A.4. Then Corollary 1.A.2 follows by Theorem 1.6.4. □

Let us demonstrate the difference of $\mathcal{C}(R^{\mathrm{dg}})$ and $\mathcal{C}({}^nR^{\mathrm{dg}})$.

Example 1.A.5. This is a generalization of Example 1.6.5, which is still almost trivial. Let

$$R=k[x],\quad \deg x=-1,$$

which is bimodule 1-CY of a-invariant 1. Then we have $A=k$ and $U=k$. Now fix $n\geq1$ and consider the graded algebra ${}^nR$. We have

$$\widetilde{A}=\begin{pmatrix}k&0&\cdots&0\\0&k&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&k\end{pmatrix},\quad \widetilde{U}=\begin{pmatrix}0&k&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&k\\k&0&\cdots&0\end{pmatrix},$$

thus $\widetilde{B}=\widetilde{A}\oplus\widetilde{U}$ is the self-injective Nakayama algebra with n vertices and of Loewy length 2. By Theorem 1.A.1 and Corollary 1.A.2 we have equivalences of triangulated categories

$$\mathcal{D}^b(\operatorname{mod}k)/[n]\simeq\mathcal{C}({}^nR^{\mathrm{dg}})\simeq\mathcal{D}_{\mathrm{sg}}(\widetilde{B}),$$

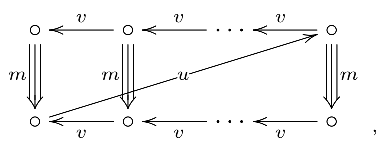
which is the n-cluster category of k.

Example 1.A.6. This is a generalization of Example 1.6.6. As in Example 1.4.10, 1.5.9, and 1.6.6, let

$$R = k\langle x_1, \dots, x_m\rangle \,/ (x_1^2 + \cdots + x_m^2), \quad \deg x_i = -1,$$

which is twisted 2-CY of a-invariant 2, the DG algebra R^{dg} is 4-CY, and the 1-representation infinite algebra A is the path algebra kQ_m of the m-Kronecker quiver.

Now we consider the cluster category $\mathcal{C}({}^nR^{\mathrm{dg}})$. The algebra $\widetilde{A}$ is just the n copies of $A = kQ_m$, and the 1-Iwanaga-Gorenstein algebra $\widetilde{B}$ is presented by the following quiver with relations.



$$x_i v = v x_i \; (1 \le i \le m)$$

$$\textstyle\sum_{i=1}^m x_i u x_i = 0,$$

where we have denoted by $x_1, \dots, x_m$ the m-fold arrows. By Theorem 1.A.1, we obtain triangle equivalences

$$\mathcal{D}^b(\operatorname{mod} kQ_m)/\nu_1^{-1/2}[n] \simeq \mathcal{D}_{\mathrm{sg}}(\widetilde{B}) \simeq \mathcal{C}({}^nR^{\mathrm{dg}}).$$

Similarly to Corollary 1.6.6 and Remark 1.6.7, these are precisely the $(2n+1)$-CY triangulated category in [KMV, Remark 3.4.5].

1.B t-structure in $\mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}} R)$

We give a version of Theorem 1.3.1 for the derived category $\mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}} R)$ for graded coherent rings, as announced in Remark 1.3.2. Let R be a negatively graded, graded coherent ring. Then the category $\operatorname{mod}^{\mathbb{Z}} R$ of finitely presented graded R-modules is abelian. We impose the following technical assumption.

(R3) The ideal $R_{\le i}$ is finitely generated as a right R-module for each $i \le 0$.

Note that this is automatic when R is Noetherian.

Lemma 1.B.1. *Let R be a negatively graded ring satisfying (R3) and let X be a finitely presented graded R-module. Then the truncation $X_{>i}$ is finitely presented for each $i \in \mathbb{Z}$.*

Proof. Let $P_1 \to P_0 \to X \to 0$ be a finite presentation of X and consider its truncation $(-)_{>i}$. Since the $(P_0)_{>i}$ and $(P_1)_{>i}$ are finitely presented by the assumption (R3), so is $X_{>i}$. □

Theorem 1.B.2 (cf. Theorem 1.3.1)**.** *Let R be a negatively graded, graded coherent ring satisfying (R3). Set*

$$t^{\le 0} = \{X \in \mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}} R) \mid H^i(X) \in \operatorname{mod}^{\le -i} R \text{ for all } i \in \mathbb{Z}\},$$
$$t^{\ge 0} = \{X \in \mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}} R) \mid H^i(X) \in \operatorname{mod}^{\ge -i} R \text{ for all } i \in \mathbb{Z}\}.$$

Then $(t^{\le 0}, t^{\ge 0})$ is a t-structure in $\mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}} R)$.

We give two independent proofs. The first one is a short proof using silting theory and DG categories. For the sake of reader who is not familiar with these, we include the second direct proof.

1.B.1 The first proof

Recall that we have a t-structure in the big derived category $\mathcal{D} := \mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} R)$ which is given by

$$\mathcal{D}^{\le 0}_{\mathcal{M}} = \{X \in \mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} R) \mid H^i(X) \in \operatorname{Mod}^{\le -i} R \text{ for all } i \in \mathbb{Z}\},$$
$$\mathcal{D}^{\ge 0}_{\mathcal{M}} = \{X \in \mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} R) \mid H^i(X) \in \operatorname{Mod}^{\ge -i} R \text{ for all } i \in \mathbb{Z}\}.$$

As in Section 1.3, we show that the t-structure $(\mathcal{D}^{\le 0}_{\mathcal{M}}, \mathcal{D}^{\ge 0}_{\mathcal{M}})$ above on $\mathcal{D}$ restricts to that on $\mathcal{D}^b(\operatorname{mod}^{\mathbb{Z}} R)$.

Proof of Theorem 1.B.2. Since R is right graded coherent the small derived category $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$ identifies with the thick subcategory of $\mathcal{D}$ whose cohomology is bounded and each one is finitely presented. Let $X \in \mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$ and consider the truncation triangle $X' \to X \to X'' \to X'[1]$ in $\mathcal{D}$. Since X has bounded cohomology, so do X' and X'' by Lemma 1.3.5(1). Moreover, since each $H^i X$ is finitely presented, so are $H^i X'$ and $H^i X''$ by Lemma 1.3.5(2) and Lemma 1.B.1. Therefore the t-structure in the big derived category restricts to that of the small one, which is precisely $(t^{\leq 0}, t^{\geq 0})$. □

1.B.2 The second proof

We turn to the second direct proof. In this subsection we will use $\mathcal{D}$ for the small derived category $\mathcal{D}^b(\mathrm{mod}^{\mathbb{Z}} R)$. We need several lemmas for the proof. Put, as usual, $t^{\leq n} = t^{\leq 0}[-n]$ and $t^{\geq n} = t^{\geq 0}[-n]$. The first one is obvious.

Proposition 1.B.3. *We have $t^{\leq -1} \subset t^{\leq 0}$ and $t^{\geq 1} \subset t^{\geq 0}$.*

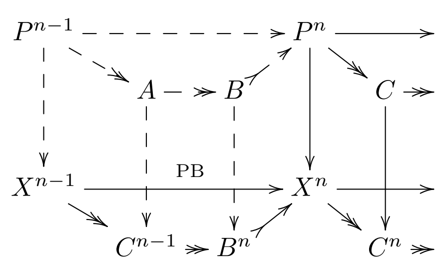
The following easy observations will be useful.

Lemma 1.B.4. *Let $\mathcal{A}$ be an abelian category with enough projectives $\mathcal{P}$. Let $P \in \mathcal{K}^-(\mathcal{P})$, $X \in \mathcal{D}^b(\mathcal{A})$ and suppose that $\mathrm{Hom}_{\mathcal{A}}(P^i, H^i(X)) = 0$ for all $i \in \mathbb{Z}$. Then $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(P, X) = 0$.*

Proof. We may assume by induction on the length of X that $X \in \mathcal{A}$. Then we have $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(P, X) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P, X) \twoheadleftarrow \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(P, X) \subset \mathrm{Hom}_{\mathcal{A}}(P^0, X) = 0$. □

Lemma 1.B.5. *Let $X \in t^{\leq 0}$. Then there exists a projective resolution $P \to X$ such that each term $P^i \in \mathrm{add}\, R(\geq i)$ for each $i \in \mathbb{Z}$.*

Proof. This can be seen by recalling the construction of a projective resolution: Suppose we have constructed such P for degree $\geq n$. As in the diagram below, let $B = \mathrm{Ker}(P^n \to C)$, A the pull-back, and $P^{n-1} \to A$ a surjection from a projective.



Then $B \in \mathrm{mod}^{\leq -n} R$ since it is a subset of P^n and $P^n \in \mathrm{add}\, R(\geq n)$. Also, since there exists an exact sequence $0 \to H^{n-1}(X) \to A \to B \to 0$ and $H^{n-1}(X) \in \mathrm{mod}^{\leq -n+1} R$ by $X \in t^{\leq 0}$, we have $A \in \mathrm{mod}^{\leq -n+1}$. Therefore we can take its projective cover $P^{n-1} \in \mathrm{add}\, R(\geq n-1)$. □

These observations yield the following.

Proposition 1.B.6. *We have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $X \in t^{\leq 0}$ and $Y \in t^{\geq 1}$.*

Proof. Take a projective resolution $P \to X$ in Lemma 1.B.5. On the other hand, we have $H^i(Y) \in \mathrm{mod}^{>-i} R$ for each $i \in \mathbb{Z}$. Therefore we deduce $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(P, Y) = 0$ by Lemma 2.4.3. □

We now give a final observation.

Proposition 1.B.7. *For any $X \in \mathcal{D}$, there exists a triangle $X' \to X \to X'' \to X'[1]$ in $\mathcal{D}$ with $X' \in t^{\leq 0}$ and $X'' \in t^{\geq 1}$.*

Proof. We proceed by induction on $w(X) = \max\{i \in \mathbb{Z} \mid H^i(X) \neq 0\} - \min\{i \in \mathbb{Z} \mid H^i(X) \neq 0\}$.

If $w(X) = 0$, then $X \in (\mathrm{mod}^{\mathbb{Z}} R)[n]$ for some $n \in \mathbb{Z}$. In this case, truncating the graded module X as $0 \to X_{\leq -n} \to X \to X_{>-n} \to 0$ in $\mathrm{mod}^{\mathbb{Z}} R$ yields a desired triangle by Lemma 1.B.1.

If $w(X) > 0$, there exists $n \in \mathbb{Z}$ such that in the truncation $X^{\leq n} \to X \to X^{>n} \to X^{\leq n}[1]$ of X with respect to (the shift of) the standard t-structure $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$, one has $w(Y), w(Z) < w(X)$, where $Y := X^{\leq n}$ and $Z := X^{>n}$. By induction hypothesis, there exist triangles $Y' \to Y \to Y'' \to Y'[1]$ and $Z' \to Z \to Z'' \to Z'[1]$ such that $Y', Z' \in t^{\leq 0}$ and $Y'', Z'' \in t^{\geq 1}$, thus the diagram below.

$$\begin{array}{ccccccc}
Z'[-1] & \longrightarrow & Z[-1] & \longrightarrow & Z''[-1] & \longrightarrow & Z' \\
\vdots\downarrow & & \downarrow & & \vdots\downarrow & & \vdots\downarrow \\
Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & Y'[1] \\
 & & \downarrow & & & & \\
 & & X & & & & \\
 & & \downarrow & & & & \\
Z' & \longrightarrow & Z & \longrightarrow & Z'' & \longrightarrow & Z'[1]
\end{array}$$

We claim that $\mathrm{Hom}_{\mathcal{D}}(Z'[-1], Y'') = 0$. This allows us to complete the morphism $Z[-1] \to Y$ to a morphism of triangles as in the dashed line above, thus the diagram above to a 3×3 diagram of triangles by [BBD, Proposition 1.1.11]. We then have a triangle $X' \to X \to X'' \to X'[1]$ in the third row, which is a desired one since the first and the third column are triangles and $t^{\leq 0}$ and $t^{\geq 0}$ are extension-closed.

We now prove the claim. Observe that any triangle $W' \to W \to W'' \to W'[1]$ in $\mathcal{D}$ with $W' \in t^{\leq 0}$ and $W'' \in t^{\geq 1}$ yields a short exact sequence

$$0 \longrightarrow H^i(W') \longrightarrow H^i(W) \longrightarrow H^i(W'') \longrightarrow 0$$

in $\mathrm{mod}^{\mathbb{Z}} R$ for all $i \in \mathbb{Z}$. Indeed, we have $H^{i-1}(W'') \in \mathrm{mod}^{>-i+1} R$ by $W'' \in t^{\geq 1}$ and $H^i(W') \in \mathrm{mod}^{\leq -i} R$ by $W' \in t^{\leq 0}$, thus the connecting homomorphisms are 0. In particular, $W', W'' \in \mathcal{D}^{\leq n}$ (resp. $\in \mathcal{D}^{\geq n}$) if and only if $W \in \mathcal{D}^{\leq n}$ (resp. $\in \mathcal{D}^{\geq n}$).

Now apply the above argument to $Z' \to Z \to Z'' \to Z'[1]$, which shows $Z' \in \mathcal{D}^{\geq n+1}$, hence $Z'[-1] \in t^{\leq 1} \cap \mathcal{D}^{\geq n+2}$. Therefore $H^i(Z'[-1]) \in \mathrm{mod}^{\leq -n-1} R$ for all $i \in \mathbb{Z}$ since it is 0 for $i \leq n+1$ and is in $\mathrm{mod}^{\leq -i+1} R$ for $i \geq n+2$. Similarly, we have $H^i(Y'') \in \mathrm{mod}^{\geq -n+1} R$ for all i since $Y'' \in \mathcal{D}^{\leq n} \cap t^{\geq 1}$. We therefore obtain the claim. ◻

Now Theorem 1.B.2 is a consequence of the Propositions.

1.C Proof of Proposition 1.4.6

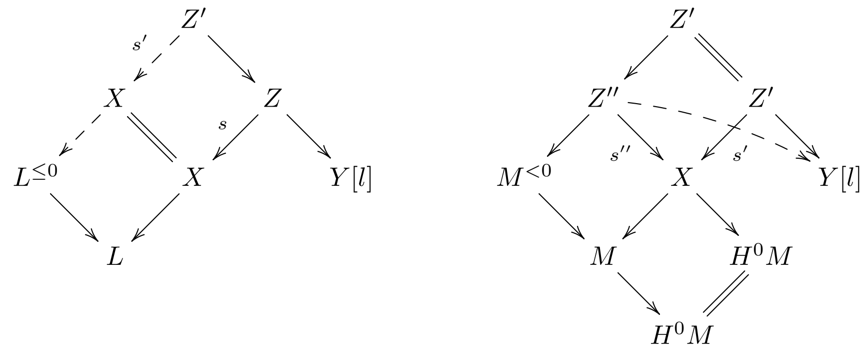
In this section we give a proof of Minamoto–Mori's equivalence [MM] based on Theorem 1.4.4. The main tool is the realization of the Verdier quotient as a subcategory given in Theorem 1.4.4(2).

We need the following computation of morphism in $\mathrm{qper}^{\mathbb{Z}} R$ which is not covered by Theorem 1.4.4(2).

Lemma 1.C.1. *Let $X, Y \in \mathrm{Mod}^{\mathbb{Z}} R$ such that $X, Y \in \mathrm{per}^{\mathbb{Z}} R$ and $\mathrm{Hom}_R^{\mathbb{Z}}(L, Y) = 0$ for all $L \in \mathrm{fl}^{\mathbb{Z}} R$. Then we have* $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(X, Y[<0]) = 0$.

Proof. Let $l < 0$ and let a morphism $X \to Y[l]$ in $\mathrm{qper}^{\mathbb{Z}} R$ be presented by a diagram $X \overset{s}{\leftarrow} Z \to Y[l]$ in $\mathrm{per}\, R$

with cone $s \in \mathcal{D}^b(\mathrm{fl}^{\mathbb{Z}} R)$. We claim that we can replace s by a morphism whose cone lies in $\mathrm{fl}^{\mathbb{Z}} R$.



First complete s to a triangle $Z \xrightarrow{s} X \to L \to Z[1]$ and consider the truncation $L^{\leq 0} \to L \to L^{>0} \to L^{\leq 0}[1]$ with respect to the standart t-structure. Since $\mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}} R}(X, L^{>0}) = 0$ there is a map $X \to L^{\leq 0}$ and as in the left diagram above, the original moprhism equals the morphism $X \xleftarrow{s'} Z' \to Z \to Y[l]$ with cone $s' = L^{\leq 0}$ concentrated in (cohomological) degree ≤ 0.

Next consider the truncation of $M := L^{\leq 0}$ along the standard t-structure: $M^{<0} \to M \to H^0 M \to M^{<0}[1]$. By the octahedral axiom we find a commutative diagram in the above right. Now we have $\mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}} R}(M^{<0}[-1], Y[l]) = 0$, the morphism $Z' \to Y$ factors through Z''. Then the diagram $X \xleftarrow{s''} Z' \to Y[l]$ with cone $s'' = H^0 M \in \mathrm{fl}^{\mathbb{Z}} R$ gives the same morphism in $\mathrm{qper}^{\mathbb{Z}} R$ as the original one, which establishes our claim.

Now let $X \to Y[l]$ be a morphism in $\mathrm{qper}^{\mathbb{Z}} R$ presented by the diagram $X \xleftarrow{s} Z \to Y[l]$ in $\mathrm{per}\, R$ with $L := \mathrm{cone}\, s \in \mathrm{fl}^{\mathbb{Z}} R$. Since $\mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}} R}(L[-1], Y[l]) = 0$ by the assumption on Y, the map $Z \to Y[l]$ factors through s, hence we have $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(X, Y[l]) \twoheadleftarrow \mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}} R}(X, Y[l]) = 0$. □

Now we are ready to give our proof.

Proof of Proposition 1.4.6. (1) We first show the vanishing of extensions, that is, $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T, T[i]) = 0$ for all $i \neq 0$. The case $i < 0$ follows from Lemma 2.2.11. Also, when $i > d$ we have $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T, T[i]) = D\,\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T, T(a)[d-i])$ by Serre duality, thus 0 again by Lemma 2.2.11. Therefore it remains to consider the case $0 \leq i \leq d$. Note that in $\mathrm{per}^{\mathbb{Z}} R$, we have $R(l) = (R(l)[-l])[l] \in \mathcal{M}[l]$, thus $T \in \mathcal{M} * \cdots * \mathcal{M}[a-1]$. Therefore, $T[i]$ lies in the fundamental domain $\mathcal{M} * \cdots * \mathcal{M}[d+a-1]$ for all $0 \leq i \leq d$. This shows $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T, T[i]) = \mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}} R}(T, T[i]) = 0$ for $0 < i \leq d$. We next show that T generates $\mathrm{qper}^{\mathbb{Z}} R$. Consider the minimal projective resolution $0 \to P_{d+1} \to \cdots \to P_0 \to R_0 \to 0$ of the graded R-module R_0. We know that $P_0 = R$. Since R is twisted $(d+1)$-CY of a-invariant a, we see that $P_{d+1} = R(a)$ and $P_i \in \mathrm{add}\{R(l) \mid 0 < l < a\}$ for all $0 < i < d+1$. (See the proofs of [MM, Proposition 4.3] or [AIR, Lemma 3.8].) Therefore $R(a) \in \mathrm{thick}\, T$ in $\mathrm{qper}^{\mathbb{Z}} R$ and we see inductively that T generates $\mathrm{qper}^{\mathbb{Z}} R$.
(2) We deduce by (1) that there exists a triangle equivalence $\mathrm{qper}^{\mathbb{Z}} R \simeq \mathcal{D}^b(\mathrm{mod}\, A)$. Comparing the Serre functor of each category, we have $(-)_\alpha(a)[d] \leftrightarrow \nu := - \otimes_A^L DA$, thus $(-)_\alpha(a) \leftrightarrow \nu_d := - \otimes_A^L DA[-d]$.

We first show $\nu_d^{-i} A \in \mathrm{mod}\, A$ for all $i \geq 0$. For this we have to show $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(A, \nu_d^{-i} A[l]) = 0$ for all $l \neq 0$ and $i \geq 0$. By the triangle equivalence, this is to show $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T, T(-ia)[l]) = 0$ since $R_\alpha \simeq R$, or $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T(ia), T[l]) = 0$. By Lemma 2.2.11 and the Serre duality we may assume $0 < l \leq d$. To prove this we apply Theorem 1.4.4(2). Note that $T(ia) \in \mathcal{M} * \cdots$ for all $i \geq 0$ and $T[l] \in \cdots * \mathcal{M}[d+a-1]$ for all $l \leq d$. By Theorem 1.4.4(2), we deduce $\mathrm{Hom}_{\mathrm{qper}^{\mathbb{Z}} R}(T(ia), T[l]) = \mathrm{Hom}_{\mathrm{per}^{\mathbb{Z}} R}(T(ia), T[l])$, which is zero for $l \neq 0$.

Finally, we prove $\mathrm{gl.dim}\, A \leq d$. Since $\mathrm{gl.dim}\, A$ is certainly finite, it is sufficient to show $\mathrm{Ext}_A^l(DA, A) = 0$ for $l > d$. For any $i > 0$, we have $\mathrm{Ext}_A^{d+i}(DA, A) = \mathrm{Hom}_{\mathcal{D}^b(A)}(\nu A, A[d+i]) = \mathrm{Hom}_{\mathcal{D}^b(A)}(A, \nu_d^{-1} A[i])$, which is 0 by the previous claim. □

1.D On silting-cluster tilting correspondence By Norihiro Hanihara and Osamu Iyama

Throughout this appendix, let d be a positive integer, and let Γ be a bimodule $(d+1)$-CY DG algebra over a field k such that $H^i\Gamma = 0$ for all $i > 0$ and $H^0\Gamma$ is finite dimensional. Amiot [Am] and Guo [Gu] introduced the *cluster category* as

$$\mathcal{C}(\Gamma) := \operatorname{per}\Gamma/\mathcal{D}^b(\Gamma).$$

Then Γ is a silting object in $\operatorname{per}\Gamma$ and a d-cluster tilting object in $\mathcal{C}(\Gamma)$ [Am, Theorem 2.1][Gu, Theorem 2.2]; see Theorem 1.2.6. More generally, the functor $\operatorname{per}\Gamma \to \mathcal{C}(\Gamma)$ sends each silting objet in $\operatorname{per}\Gamma$ to a d-cluster tilting object in $\mathcal{C}(\Gamma)$ [IYa1, Corollary 5.12]. Therefore we have a map

$$\operatorname{silt}\Gamma \longrightarrow d\text{-ctilt}\,\Gamma\ , \tag{1.D.1}$$

where $\operatorname{silt}\Gamma$ (resp. d-ctilt Γ) is the set of isomorphism classes of silting objects in $\operatorname{per}\Gamma$ (resp. d-cluster tilting objects in $\mathcal{C}(\Gamma)$). The aim of this appendix is to discuss the following problem.

Question 1.D.1. How far is the map (1.D.1) being from bijective?

A natural approach to study this question is to consider the full subcategory (called the *fundamental domain*)

$$\mathcal{F} = \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[d-1] \subset \operatorname{per}\Gamma,$$

where $\mathcal{P} = \operatorname{add}\Gamma \subset \operatorname{per}\Gamma$. Then the composition $\mathcal{F} \subset \operatorname{per}\Gamma \to \mathcal{C}(\Gamma)$ is an additive equivalence (see Theorem 1.2.6). In particular the map (1.D.1) restricts to an injection

$$\operatorname{silt}^{\mathcal{F}}\Gamma \hookrightarrow d\text{-ctilt}\,\Gamma\ , \tag{1.D.2}$$

where $\operatorname{silt}^{\mathcal{F}}\Gamma$ is the subset of $\operatorname{silt}\Gamma$ consisting of silting objects contained in $\mathcal{F}$.

Definition 1.D.2. We call Γ *mild* if the map (1.D.2) is bijective.

For $d = 1$ and 2, Γ is always mild [KN] (see also [IYa1, Corollary 5.12]). It was asked in [IYa1, Conjecture 5.14] if Γ is always mild for $d \geq 3$. We will show that this is far from being true. In fact, under the assumption that $H^0\Gamma = k$, we will characterize mildness as follows.

Theorem 1.D.3. *Let Γ be a $(d+1)$-Calabi-Yau DG k-algebra such that $H^i\Gamma = 0$ for $i > 0$ and $H^0\Gamma = k$. Then the following are equivalent.*

(a) Γ is mild.

(b) d-ctilt $\Gamma = \{\Gamma, \ldots, \Gamma[d-1]\}$.

(c) Γ is quasi-isomorphic to $k[x]$ with $\deg x = -d$ and with zero differentials.

(d) $\mathcal{C}(\Gamma)$ is triangle equivalent to $\mathcal{C}_d(k)$, the d-cluster category of k.

Proof. (a)⇔(b) Since $H^0\Gamma = k$, Γ is an indecomposable silting object in $\operatorname{per}\Gamma$, thus $\operatorname{silt}\Gamma = \{\Gamma[i] \mid i \in \mathbb{Z}\}$. Then $\operatorname{silt}^{\mathcal{F}}\Gamma = \{\Gamma[i] \mid 0 \leq i \leq d-1\}$ and the desired equivalence follows.
(b)⇒(c) We first show that Γ is d-periodic in $\mathcal{C}(\Gamma)$, that is, $\Gamma \simeq \Gamma[d]$ in $\mathcal{C}(\Gamma)$. Suppose that d-ctilt $\Gamma = \{\Gamma[i] \mid 0 \leq i \leq d-1\}$. Then the d-cluster tilting object $\Gamma[d] \in \mathcal{C}(\Gamma)$ has to be isomorphic to $\Gamma[i]$ for some $0 \leq i \leq d-1$. The only possible i is 0 since $\operatorname{Hom}_{\mathcal{C}(\Gamma)}(\Gamma, \Gamma[i]) = 0$ for $1 \leq i \leq d-1$. Thus we have $\Gamma \simeq \Gamma[d]$ in $\mathcal{C}(\Gamma)$.
We next compute the cohomology of Γ. By Theorem 1.2.6(2) the functor $\operatorname{per}\Gamma \to \mathcal{C}(\Gamma)$ induces bijections $\operatorname{Hom}_{\operatorname{per}\Gamma}(\Gamma[i], \Gamma) \to \operatorname{Hom}_{\mathcal{C}(\Gamma)}(\Gamma[i], \Gamma)$ for each $i \geq 0$, thus $H^{-i}\Gamma = \operatorname{Hom}_{\mathcal{C}(\Gamma)}(\Gamma[i], \Gamma)$ for $i \geq 0$. By periodicity of Γ, this is k if $i \mid d$, and 0 if $i \not\mid d$ since $\operatorname{Hom}_{\mathcal{C}(\Gamma)}(\Gamma, \Gamma[i]) = 0$ for $0 < i < d$. Now lift an isomorphism

$\Gamma[d] \to \Gamma$ in $\mathcal{C}(\Gamma)$ to a morphism $f\colon \Gamma[d] \to \Gamma$ in $\operatorname{per}\Gamma$, and let $y \in Z^{-d}\Gamma$ give the morphism f in $H^{-d}\Gamma$. Then we obtain a homomorphism $k[x] \to \Gamma$ of DG algebras, taking x to y. Consider the power $y^n \in Z^{nd}\Gamma$ of y. It presents the morphism $\Gamma[nd] \xrightarrow{f[(n-1)d]} \Gamma[(n-1)d] \to \cdots \xrightarrow{f[1]} \Gamma[1] \xrightarrow{f} \Gamma$ in $\operatorname{per}\Gamma$ which is an isomorphism in $\mathcal{C}(\Gamma)$. Therefore y^n is non-zero in $H^{-nd}\Gamma$. We conclude that $k[x] \to \Gamma$ is a quasi-isomorphism.
(c)⇒(d) Since $k[x]$ with $\deg x = -d$ is the derived $(d+1)$-preprojective algebra of k, the assertion follows.
(d)⇒(b) If $\mathcal{C}(\Gamma) \simeq \mathcal{C}_d(k)$ then clearly $d\text{-ctilt}\,\Gamma = \{\Gamma[i] \mid 0 \leq i \leq d-1\}$. □

Example 1.D.4. We regard the polynomial ring $\Gamma = k[x_1, \ldots, x_n]$ as a DG algebra with $\deg x_i = -a_i < 0$ and zero differential. We assume that $a = \sum_{i=1}^{n} a_i$ is odd. Then Γ is an $(n+a)$-CY DG algebra by Theorem 1.5.2. By (c) of the above theorem we immdediately deduce the following.

Corollary 1.D.5. Γ *is mild if and only if* $n = 1$.

Part 2

Morita theorem for hereditary Calabi-Yau categories

2.1 Introduction

This part is based on [Han2]. The subject of this part is Morita theorem for CY categories which attempts to characterize cluster categories in terms of cluster tilting objects. The only known results on such Morita-type theorems are the following two, due to Keller–Reiten and Keller–Murfet–Van den Bergh.

Theorem 2.1.1. *Let $\mathcal{T}$ be an algebraic d-CY triangulated category with a d-cluster tilting object T.*

(1) *[KR2] Suppose* $\operatorname{End}_{\mathcal{T}}(T) = kQ$ *for some acyclic quiver Q, and* $\mathcal{T}(T, T[-i]) = 0$ *for* $0 < i < d-1$. *Then there exists a triangle equivalence* $\mathcal{T} \simeq \mathcal{D}^b(\operatorname{mod} kQ)/\tau^{-1}[d-1]$.

(2) *[KMV] Suppose* $d = 2n+1 \geq 3$, $\operatorname{End}_{\mathcal{T}}(T) = k$, *and* $\mathcal{T}(T, T[-i]) = 0$ *for* $0 < i < n$. *Put* $\dim_k \operatorname{Hom}_{\mathcal{T}}(T, T[-n]) = m$. *Then there exists a triangle equivalence* $\mathcal{T} \simeq \mathcal{D}^b(\operatorname{mod} kQ_m)/\tau^{-1/2}[n]$ *for the m-Kronecker quiver* $Q_m\colon \circ \Longrightarrow \circ$ *with m arrows, and a naturally defined square root $\tau^{1/2}$ of the AR translation.*

This part is devoted to prove the following result which encompasses both of the above two cases, providing a general Morita-type theorem for CY categories arising from representation-*infinite* hereditary algebras. Recall that a hereditary algebra is 1-*representation infinite* if any of its ring indecomposable summand is representation-infinite. Also we denote by J_Λ the Jacobson radical of a ring Λ.

Theorem 2.1.2 (=2.4.13)**.** *Let $d \geq 2$ and let $\mathcal{T}$ be an algebraic d-CY triangulated category with a d-cluster tilting object T. Assume that* $H = \operatorname{End}_{\mathcal{T}}(T \oplus T[-1] \oplus \cdots \oplus T[-(d-2)])$ *is 1-representation infinite and that H/J_H is separable over k. Then there exists a triangle equivalence*

$$\mathcal{T} \simeq \mathcal{D}^b(\operatorname{mod} H)/\tau^{-1/(d-1)}[1]$$

for a naturally defined $(d-1)$-st root $\tau^{1/(d-1)}$ of the AR translation.

We also give a certain converse in Section 2.6. We show that if H has a $(d-1)$-st root of τ, then there exists a projective H-module P such that $P \oplus \tau^{-1/(d-1)}P \oplus \cdots \oplus \tau^{-(d-2)/(d-1)}P \simeq H$, justifying our choice of H.

Our $(d-1)$-st root $\tau^{-1/(d-1)}$ is defined by $- \otimes^{\mathrm{L}}_H U$ for the (H,H)-bimodule $U = \mathcal{T}(X, X[-1])$ where $X = T \oplus \cdots \oplus T[-(d-2)]$. One can show that the $(d-1)$-fold derived tensor product of U is isomorphic to $\operatorname{Ext}^1_H(DH, H)$ (2.4.10), justifying the notation $- \otimes^{\mathrm{L}}_H U = \tau^{-1/(d-1)}$. A square root of the AR translation for generalized Kronecker quivers appears in [KMV]. More general roots of (higher) AR translations for various algebras are studied in [Han1].

The main theorem recovers both of the known results in 2.1.1 for the infinite type. When $d = 2$ this immediately gives Keller–Reiten's recognition theorem 2.1.1(1) for the case Q is of non-Dynkin type. Also when $d = 3$ and $\operatorname{End}_{\mathcal{T}}(T) = k$, then H is the path algebra of a generalized Kronecker quiver, thus we deduce Keller–Murfet–Van den Bergh's theorem 2.1.1(2) for $m \geq 2$. We can also recover the versions for arbitrary d in 2.1.1, despite the seemingly different assumption and conclusion; see the second paragraph after 2.1.4.

Even if H is of Dynkin type we have a partial result. For arbitrary hereditary H there is an additive equivalence (2.4.5)

$$\mathcal{T}/[T[1] \oplus T \oplus \cdots \oplus T[-(d-2)]] \simeq \underline{\operatorname{mod}}\, H,$$

which gives a classification $\operatorname{ind}\mathcal{T} \simeq \operatorname{ind}(\operatorname{mod} H) \sqcup \operatorname{ind}(\operatorname{add} T[1])$ of objects of $\mathcal{T}$, as 2.1.2 implies for non-Dynkin cases. Here $\operatorname{ind}\mathcal{C}$ means the set of isomorphism classes of indecomposable objects in an additive category $\mathcal{C}$.

Although the assumption in 2.1.2 that $H = \mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(d-2)])$ is hereditary looks strong, it in fact follows from the same property for a smaller algebra, in particular, from that of T when $d = 3$. More generally, we have the following sufficient condition for H to be hereditary using vanishing of some negative self-extensions of T.

Theorem 2.1.3. *Let $\mathcal{T}$ be a d-CY triangulated category with a d-cluster tilting object T.*

(1) (=2.2.8) *Suppose $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < d/2$ and $H' = \mathrm{End}_{\mathcal{T}}(T)$ is hereditary. Then we have $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < d-1$, thus $\mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(d-2)])$ is the direct product of $(d-1)$ copies of H', hence is hereditary.*

(2) (=2.2.12) *Suppose $d = 2n+1 \geq 3$, $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < n$, and $\mathrm{End}_{\mathcal{T}}(T)$ is hereditary. Then $\mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary, and $\mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(2n-1)])$ is the direct product of n copies of it, hence is hereditary.*

(3) (=2.2.21) *Suppose $d = 2n+2 \geq 4$, $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < n$, and $\mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary. Then $H = \mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-2n])$ is also hereditary.*

Note that we are allowing $\mathcal{T}(T, T[-n])$ to survive in (2) and (3) while it is supposed to vanish in (1). One can thus view (1) as a 'degenerate' version of (2) and (3). We also explicitly describe the quiver of $\mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(d-2)])$ for (2) and (3) in terms of AR sequences in $\mathrm{add}\, T$, see 2.2.13 and 2.2.22.

Combining 2.1.2 and 2.1.3, together with an interpretation of $(d-1)$-th root of the AR translation for (1) and (2) which we explain below, we obtain the following version of 2.1.2.

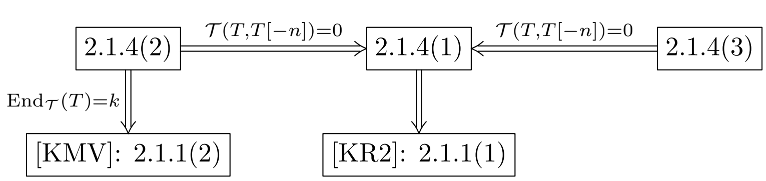
Corollary 2.1.4. *Let $\mathcal{T}$ be an algebraic d-CY triangulated category with a d-cluster tilting object T.*

(1) (=2.4.14) *Suppose $H' = \mathrm{End}_{\mathcal{T}}(T)$ is hereditary and $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < d/2$. Then there exists a triangle equivalence*
$$\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\, H')/\tau^{-1}[d-1]$$
when H' is 1-representation infinite and $H'/J_{H'}$ is separable over k.

(2) (=2.4.15) *Suppose $d = 2n+1 \geq 3$, $\mathrm{End}_{\mathcal{T}}(T)$ is hereditary, and $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < n$. Then $H' = \mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary, and there exists a triangle equivalence*
$$\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\, H')/\tau^{-1/2}[n]$$
when H' is 1-representation infinite and $H'/J_{H'}$ is separable over k.

(3) (=2.4.16) *Suppose $d = 2n+2 \geq 4$, $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < n$, and $\mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary. Then $H = \mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-2n])$ is hereditary, and there exists a triangle equivalence*
$$\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\, H)/\tau^{-1/(2n+1)}[1]$$
when H is 1-representation infinite and H/J_H is separable over k.

The above (1) shows that we in fact need only half of the vanishings of negative extensions in 2.1.1(1). Also (2) reduces to Keller–Murfet–Van den Bergh's theorem 2.1.1(2) when $\mathrm{End}_{\mathcal{T}}(T) = k$ as well as relaxes the assumption on the vanishing of negative extensions in Keller–Reiten's theorem 2.1.1(1). This 2.1.4(2) is thus a common generalization of 2.1.1(1)(2). Let us summarize the implications diagrammatically.



Now we briefly explain how to deduce the corollaries in 2.1.4 from 2.1.2 and the respective results in 2.1.3. It depends on an interpretation of the $(d-1)$-st root $\tau^{1/(d-1)}$. Under the assumption of 2.1.4(1), the algebra $H = \mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(d-2)])$ in 2.1.2 is the direct product of $(d-1)$ copies of $\mathrm{End}_{\mathcal{T}}(T) =: H'$ by 2.1.3(1). It turns out, under the identification $\mathcal{D}^b(\mathrm{mod}\, H) = \mathcal{D}^b(\mathrm{mod}\, H') \times \cdots \times \mathcal{D}^b(\mathrm{mod}\, H')$, that our $(d-1)$-st root is given just by $\tau^{1/(d-1)}\colon (L_1, \ldots, L_{d-1}) \mapsto (L_2, \ldots, L_{d-1}, \tau' L_1)$ using the AR translation τ' of H'. This yields an equivalence of orbit categories (2.5.2)

$$\mathcal{D}^b(\mathrm{mod}\, H)/\tau^{-1/(d-1)}[1] \simeq \mathcal{D}^b(\mathrm{mod}\, H')/\tau'^{-1}[d-1],$$

which gives 2.1.4(1). A similar interpretation of $2n$-th root of H in terms of a square root for H' yields 2.1.4(2). Note that in 2.1.4(3) the $(2n+1)$-st root cannot be made easier in general.

Let us now mention one intermediate result toward proving 2.1.2 which is a general consideration on realizing a triangulated category as a cluster category. It is based on a description of cluster categories in terms of differential graded (dg) algebras [Ke3]. Recall that a dg algebra Π is *homologically smooth* if Π is perfect as a bimodule. In this case we have that $\mathcal{D}^b(\Pi)$, the derived category of dg Π-modules of finite dimensional total cohomology, is contained in the perfect derived category $\mathrm{per}\,\Pi$. For a homologically smooth dg algebra Π, we call the Verdier quotient

$$\mathcal{C}(\Pi) := \mathrm{per}\,\Pi/\mathcal{D}^b(\Pi)$$

the *cluster category* of Π. Such kind of categories was established by Amiot [Am] and generalized in [Gu] for CY dg algebras [Gi, Ke6], giving rise to CY triangulated categories with cluster tilting objects. We refer to [AMY, ART, AIR, AO1, AO2, AO3, BT, FM, Han1, IQ, IO, IYa1, KY2, KY3, KY4, Ke6, Ki, KQ, Pl, Pr, TV] for further studies on this subject. Although the cluster category in the above general sense is not CY nor have cluster tilting objects, those for certain non-CY dg algebras will be important for us.

Let $\mathcal{T}$ be a triangulated category and $X \in \mathcal{T}$ satisfying the following.

(a) $\mathcal{T}$ is Hom-finite over k and $\mathrm{thick}_{\mathcal{T}} X = \mathcal{T}$.

(b) If $\mathcal{T}(X, Y[i]) = 0$ for $i \ll 0$ then $Y = 0$.

(c) There exists an enhancement $\mathcal{A}$ of $\mathcal{T}$ such that the truncated derived endomorphism algebra $\Pi := \mathrm{RHom}_{\mathcal{A}}(X, X)^{\leq 0}$ is homologically smooth.

For example if $X \in \mathcal{T}$ is a d-cluster tilting object for some $d \geq 1$, then the above (b) indeed holds (see 2.3.2). In this situation we can realize our triangulated category $\mathcal{T}$ as a cluster category of Π in (c).

Theorem 2.1.5 (=2.3.3)**.** *Let $\mathcal{T}$ be a triangulated category satisfying (a), (b), and (c) above. Then $\mathcal{T}$ is triangle equivalent to the cluster category of Π.*

Our main result 2.1.2 is obtained as an application of this observation together with a description of $\mathcal{C}(\Pi)$ as an orbit category of a derived category (see 2.3.1) and our method is thus quite different from those in 2.1.1. It should be noted that a similar result on realizing a triangulated category as a cluster category is obtained in [KY4], in the setting where $\mathcal{T}$ is CY and $X \in \mathcal{T}$ is cluster tilting. It would be interesting to investigate the relationship of these results. We refer also to [T] for another result based on a different model.

2.2 Hereditaryness of shifted sum of cluster tilting objects

Let $\mathcal{T}$ be a d-CY triangulated category with a d-cluster tilting object T. While the object T, if $\mathrm{End}_{\mathcal{T}}(T)$ is hereditary, alone can recover the category $\mathcal{T}$ when $d = 2$, the same does not hold in larger dimensions. Our generator for larger d is the shifted sum $X = T \oplus \cdots \oplus T[-(d-2)]$ of T which, if $\mathrm{End}_{\mathcal{T}}(X)$ is hereditary, turns out to be essential to recover the triangulated category $\mathcal{T}$, see 2.4.13.

The aim of this section is to give a sufficient condition for $\mathrm{End}_{\mathcal{T}}(X)$ to be hereditary in terms of much smaller endomorphism algebra under some vanishing of negative self-extensions of T (2.2.12 and 2.2.21). Moreover we explicitly describe the quiver of $\mathrm{End}_{\mathcal{T}}(X)$ in each case (2.2.13 and 2.2.22).

2.2.1 Rigid objects with hereditary endomorphism algebras

Some part of our discussion does not depend on the setup of cluster tilting object in a CY triangulated category. Let $\mathcal{T}$ be a triangulated category and $T \in \mathcal{T}$. For each indecomposable summand T_a of T, define the objects $A_a^{(i)}$ and $S_a^{(i)}$ by the triangles

$$S_a^{(1)} \longrightarrow A_a^{(0)} \xrightarrow{a_0} T_a \xrightarrow{c_0} S_a^{(1)}[1]$$
$$S_a^{(i+1)} \longrightarrow A_a^{(i)} \xrightarrow{a_i} S_a^{(i)} \xrightarrow{\delta_i} S_a^{(i+1)}[1] \qquad \text{for } i \geq 1$$

with a_0 the sink map in $\operatorname{add} T$, and a_i the minimal right $(\operatorname{add} T)$-approximation for each $i \geq 1$. We denote these triangles as a complex below.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A_a^{(2)} & & \xrightarrow{f_2} & & A_a^{(1)} & & \xrightarrow{f_1} & & A_a^{(0)} \\ & \searrow \ \cdots \ \nearrow & & \searrow^{a_2} & S_a^{(2)} & \nearrow & & \searrow^{a_1} & S_a^{(1)} & \nearrow & & \searrow^{a_0} \ T_a \end{array}$$

Let $c_i \colon T_a[-i] \to S_a^{(i+1)}[1]$ be the composite of the connecting morphisms, precisely it is given by

$$c_i \colon T_a[-i] \longrightarrow S_a^{(1)}[-(i-1)] \xrightarrow{\delta_1[-(i-1)]} S_a^{(2)}[-(i-2)] \longrightarrow \cdots \longrightarrow S_a^{(i)} \xrightarrow{\delta_i} S_a^{(i+1)}[1]\ ,$$

and thus satisfies $c_i = \delta_i \circ (c_{i-1}[-1])$.

An important observation is that for each $m \geq 1$ we can determine the quiver of the endomorphism algebra of the shifted sum $T \oplus T[-1] \oplus \cdots \oplus T[-m]$ from the complex above when a smaller algebra $\operatorname{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(m-1)])$ is hereditary.

Proposition 2.2.1. *Let $m \geq 1$ and $T \in \mathcal{T}$ an $(m+1)$-rigid object such that $\operatorname{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(m-1)])$ is hereditary. Then there exists an octahedral*

$$\begin{array}{ccccccc} S_a^{(m)}[-1] & \xrightarrow{-\delta_m[-1]} & S_a^{(m+1)} & \longrightarrow & A_a^{(m)} & \xrightarrow{a_m} & S_a^{(m)} \\ \| & & \downarrow & & \downarrow & & \| \\ S_a^{(m)}[-1] & \longrightarrow & \coprod_{i=0}^{m-1} A_a^{(i)}[-(m-i)] & \xrightarrow{p_m} & T_a[-m] & \xrightarrow{c_{m-1}[-1]} & S_a^{(m)} \\ & & \downarrow & & \downarrow & & \\ & & \bullet & = & \bullet & & \end{array}$$

with p_m the sink map in $\operatorname{add}(T[-1] \oplus \cdots \oplus T[-m])$. Moreover, the triangle

$$S_a^{(m+1)} \longrightarrow \coprod_{i=0}^{m} A_a^{(i)}[-(m-i)] \xrightarrow{q_m} T_a[-m] \xrightarrow{c_m} S_a^{(m+1)}[1]$$

obtained from the above homotopy cartesian square gives the sink map q_m at $T_a[-m]$ in $\operatorname{add}(T \oplus \cdots \oplus T[-m])$.

Proof. Consider the following statements.

(a)$_l$ The morphism $c_{l-1}[-1] \colon T_a[-l] \to S_a^{(l)}$ induces a surjection $\mathcal{T}(T, T_a[-l]) \to \mathcal{T}(T, S_a^{(l)})$.

(b)$_l$ The conclusions of the proposition for $m = l$.

We prove that (b)$_{l-1}$ and (a)$_l$ implies (b)$_l$ when $l \leq m$, and (b)$_l$ gives (a)$_{l+1}$ when $l \leq m-1$. Starting with the sink map

$$S_a^{(1)} \longrightarrow A_a^{(0)} \xrightarrow{q_0 := a_0} T_a \longrightarrow S_a^{(1)}[1]$$

which can be viewed as (b)$_0$, this will prove our result (b)$_m$ by induction.

Suppose first (b)$_l$ and $l \le m-1$. Put $\widetilde{T} = T \oplus \cdots \oplus T[-l]$ and consider the exact sequence

$$\mathcal{T}(\widetilde{T}, T_a[-l-1]) \xrightarrow{c_l[-1]} \mathcal{T}(\widetilde{T}, S_a^{(l+1)}) \longrightarrow \mathcal{T}(\widetilde{T}, \textstyle\coprod_{i=0}^{l} A_a^{(i)}[-(l-i)]) \longrightarrow \mathcal{T}(\widetilde{T}, T_a[-l])$$

obtained from the triangle in (b)$_l$. Since $\operatorname{End}_{\mathcal{T}}(\widetilde{T})$ is hereditary the sink map in $\operatorname{add}\widetilde{T}$ is a monomorphism, which is to say that the last map in the above exact sequence is injective. Therefore the first map, in particular its direct summand $\mathcal{T}(T, T_a[-l-1]) \to \mathcal{T}(T, S_a^{(l+1)})$, is surjective. This shows (a)$_{l+1}$.

Suppose next (b)$_{l-1}$ and (a)$_l$. Shifting the triangle in (b)$_{l-1}$ by $[-1]$ we have the triangle in the second row in the diagram below. By (b)$_{l-1}$ the map $q_l[-1]$ is the sink map in $\operatorname{add}(T[-1]\oplus\cdots\oplus T[-l])$. We compare it with the triangle in the first row. By (a)$_l$, we can lift $a_l\colon A_a^{(l)} \to S_a^{(l)}$ to $T_a[-l]$, which can be completed to a desired octahedral. This gives the first part of (b)$_l$.

$$\begin{array}{ccccccc}
S_a^{(l)}[-1] & \xrightarrow{-\delta_l[-1]} & S_a^{(l+1)} & \longrightarrow & A_a^{(l)} & \xrightarrow{a_l} & S_a^{(l)} \\
\| & & \downarrow & & \downarrow & & \| \\
S_a^{(l)}[-1] & \longrightarrow & \coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)] & \xrightarrow{q_{l-1}[-1]} & T_a[-l] & \xrightarrow{c_{l-1}[-1]} & S_a^{(l)} \\
 & & \downarrow & & \downarrow & & \\
 & & \bullet & = & \bullet & &
\end{array}$$

We have to prove that in the triangle

$$S_a^{(l+1)} \longrightarrow \coprod_{i=0}^{l} A_a^{(i)}[-(l-i)] \xrightarrow{q_l} T_a[-l] \xrightarrow{c_l} S_a^{(l+1)}[1]$$

given by the above octahedral, the middle map is the sink map at $T_a[-l]$ in $\operatorname{add}(T \oplus \cdots \oplus T[-l])$.

We first show that the map is right almost split. Since the map $q_l[-1]$ in second row of the octahedral is the sink map at $T_a[-l]$ in $\operatorname{add}(T[-1] \oplus \cdots \oplus T[-l])$, any radical map $T[-i] \to T_a[-l]$ with $0 < i \le l$ factors through $\coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)]$. It remains to consider $\varphi\colon T \to T_a[-l]$.

$$\begin{array}{ccccc}
 & & T & & \\
 & \swarrow & \downarrow{\scriptstyle\varphi} & \searrow & \\
 & & A_a^{(l)} & \xrightarrow{a_l} & S_a^{(l)} \\
 & & \downarrow & & \| \\
\coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)] & \longrightarrow & T_a[-l] & \longrightarrow & S_a^{(l)}
\end{array}$$

Since $a_l\colon A_a^{(l)} \to S_a^{(l)}$ is a right $(\operatorname{add} T)$-approximation, the morphism $T \xrightarrow{\varphi} T_a[-l] \to S_a^{(l)}$ can be lifted to $A_a^{(l)}$. Then the difference of the two maps in the triangle formed by T, $A_a^{(l)}$, and $T_a[-l]$ vanishes under $T_a[-l] \to S_a^{(l)}$, thus it factors through $\coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)]$. We conclude that φ factors through $\coprod_{i=0}^{l} A_a^{(i)}[-(l-i)]$. This proves that q_l is right almost split.

We next show q_l is right minimal. For this we prove that $S_a^{(l+1)} \xrightarrow{\binom{r_1}{r_2}} \coprod_{i=0}^{l} A_a^{(i)}[-(l-i)] = A_a^{(l)} \oplus \coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)]$ is a radical map. Since a_l is right minimal, the summand r_1 is certainly a radical map. It remains to consider $r_2\colon S_a^{(l+1)} \to \coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)]$. If it is not a radical map, then there is a non-zero direct summand M of $\coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)]$ which is mapped to 0 under the vertical map $\coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)] \to \bullet$ in the octahedral. Then the composite $M \subset \coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)] \xrightarrow{q_{l-1}[-1]} T_a[-l] \to \bullet$ is 0, so

$M \to T_a[-l]$ factors through $A_a^{(l)}$. Now since T is $(m+1)$-rigid we have $\mathcal{T}(M, A_a^{(l)}) = 0$, thus the restriction of $q_{l-1}[-1]\colon \coprod_{i=0}^{l-1} A_a^{(i)}[-(l-i)] \to T_a[-l]$ to M is 0. This contradicts right minimality of q_{l-1}. Therefore r_2 is a radical map. $\square$

Let us note some consequences of this inductive construction of sink maps.

Lemma 2.2.2. *Let $m \geq 1$ and $T \in \mathcal{T}$ an m-rigid object such that $\operatorname{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(m-1)])$ is hereditary.*

(1) *The map $a_i\colon A_a^{(i)} \to S_a^{(i)}$ induces an injection $\mathcal{T}(T, A_a^{(i)}) \hookrightarrow \mathcal{T}(T, S_a^{(i)})$ for each $0 \leq i \leq m-1$, hence an isomorphism for $1 \leq i \leq m-1$.*

(2) *The maps $f_i\colon A_a^{(i)} \to A_a^{(i-1)}$ are 0 for all $1 \leq i \leq m$.*

Proof. Since the assumptions on T for m implies the same for any smaller m, it is enough to prove the assertions for the largest possible i, that is, $i = m-1$ for (1) and $i = m$ for (2). We can apply 2.2.1 for $m-1$, so there is a triangle

$$S_a^{(m)} \longrightarrow \coprod_{i=0}^{m-1} A_a^{(i)}[-(m-1-i)] \longrightarrow T_a[-(m-1)] \longrightarrow S_a^{(m)}[1]\ ,$$

in which the middle map is the sink map in $\operatorname{add} \widetilde{T}$, where $\widetilde{T} = T \oplus \cdots \oplus T[-(m-1)]$. Applying $\mathcal{T}(\widetilde{T}, -)$ we have an exact sequence

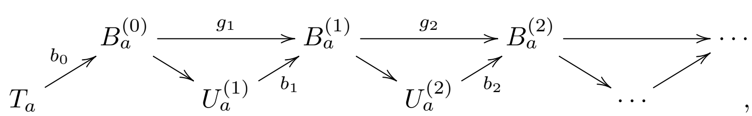
$$\mathcal{T}(\widetilde{T}, T_a[-m]) \longrightarrow \mathcal{T}(\widetilde{T}, S_a^{(m)}) \longrightarrow \mathcal{T}(\widetilde{T}, \textstyle\coprod_{i=0}^{m-1} A_a^{(i)}[-(m-1-i)]) \longrightarrow \mathcal{T}(\widetilde{T}, T_a[-(m-1)])\ .$$

Since $\operatorname{End}_{\mathcal{T}}(\widetilde{T})$ is hereditary, any sink map in $\operatorname{add} \widetilde{T}$ is a monomorphism, thus the last map in the above exact sequence is injective. Then the middle map, in particular its direct summand

$$\mathcal{T}(T, S_a^{(m)}) \longrightarrow \mathcal{T}(T, A_a^{(m-1)})$$

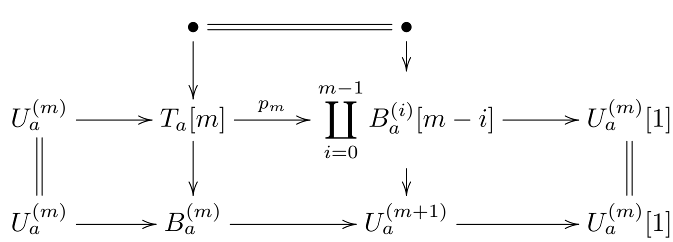
is 0. We conclude that $\mathcal{T}(T, A_a^{(m-1)}) \to \mathcal{T}(T, S_a^{(m-1)})$ is injective, and also $f_m\colon A_a^{(m)} \to A_a^{(m-1)}$ is 0 by substituting $T = A_a^{(m)}$ and considering the image of $a_m \in \mathcal{T}(A_a^{(m)}, S_a^{(m)})$. $\square$

Dually we define the objects $B_a^{(i)} \in \operatorname{add} T$ and $U_a^{(i)} \in \mathcal{T}$ by the sequence of triangles below.



where b_0 the source map in $\operatorname{add} T$, and b_i the minimal left $(\operatorname{add} T)$-approximation for each $i \geq 1$. We state without proof the following dual results.

Proposition 2.2.3. *Let $m \geq 1$ and $T \in \mathcal{T}$ an $(m+1)$-rigid object such that $\operatorname{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[m-1])$ is hereditary. Then there exists an octahedral*



with p_m the source map in $\operatorname{add}(T[1] \oplus \cdots \oplus T[m])$*. Moreover, the triangle*

$$U_a^{(m+1)}[-1] \longrightarrow T_a[m] \xrightarrow{q_m} \coprod_{i=0}^{m} B_a^{(i)}[m-i] \longrightarrow U_a^{(m+1)}$$

obtained from the above homotopy cartesian square gives the source map q_m at $T_a[m]$ in $\operatorname{add}(T \oplus \cdots \oplus T[m])$*.*

Lemma 2.2.4. *Let $m \geq 1$ and $T \in \mathcal{T}$ an m-rigid object such that $\operatorname{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[m-1])$ is hereditary.*

(1) *The map $b_i \colon U_a^{(i)} \to B_a^{(i)}$ induces an injection $\mathcal{T}(B_a^{(i)}, T) \hookrightarrow \mathcal{T}(U_a^{(i)}, T)$ for each $0 \leq i \leq m-1$, hence an isomorphism for $1 \leq i \leq m-1$.*

(2) *The maps $g_i \colon B_a^{(i-1)} \to B_a^{(i)}$ are 0 for all $1 \leq i \leq m$.*

We end this section with a technical lemma.

Lemma 2.2.5. *Let $T \in \mathcal{T}$ be a rigid object with $\operatorname{End}_{\mathcal{T}}(T)$ hereditary. Then the $\operatorname{End}_{\mathcal{T}}(T)$-module $\mathcal{T}(T, U_a^{(1)})$ has no non-zero projective summands.*

Proof. Applying $\mathcal{T}(T, -)$ to the defining triangle of $U_a^{(1)}$ gives an exact sequence

$$\mathcal{T}(T, T_a) \longrightarrow \mathcal{T}(T, B_a^{(0)}) \longrightarrow \mathcal{T}(T, U_a^{(1)}) \longrightarrow \mathcal{T}(T, T_a[1]) = 0 \ .$$

Since $T_a \to B_a^{(0)}$ is a source map, the first map in the above sequence is left minimal, hence there cannot be a projective summand in $\mathcal{T}(T, U_a[1])$. □

2.2.2 AR sequences

A crucial ingredient of the proof of our main results for this section is AR sequences in a cluster tilting subcategory in a triangulated category. Let us first recall its notion and the fundamental existence theorem.

Theorem 2.2.6 ([IYo]). *Let $\mathcal{T}$ be a k-linear,* Hom*-finite, idempotent-complete triangulated category with a Serre functor ν, and let $\mathcal{C}$ be a d-cluster tilting subcategory. Then for any $C \in \mathcal{C}$ there exists a sequence, unique up to isomorphism of complexes,*

$$\begin{array}{ccccccccccccccccc}
 & C_1 & \longrightarrow & C_2 & \to \cdots \to & C_{i-1} & \longrightarrow & C_i & \longrightarrow & C_{i+1} & \to \cdots \to & C_{d-1} & \longrightarrow & C_d & \\
{\scriptstyle f_0}\nearrow & & {\scriptstyle g_1}\searrow \quad \nearrow{\scriptstyle f_1} & & & & {\scriptstyle g_{i-1}}\searrow \quad \nearrow{\scriptstyle f_{i-1}} & & {\scriptstyle g_i}\searrow \quad \nearrow{\scriptstyle f_i} & & & & {\scriptstyle g_{d-1}}\searrow \quad \nearrow{\scriptstyle f_{d-1}} & & \searrow{\scriptstyle g_d} \\
C_0 & & Y_1 & & & & Y_{i-1} & & Y_i & & & & Y_{d-1} & & C_{d+1}
\end{array}$$

with $C_0 = C$ (resp. $C_{d+1} = C$) and all $C_i \in \mathcal{C}$ such that

- *each $Y_{i-1} \xrightarrow{f_{i-1}} C_i \xrightarrow{g_i} Y_i$, $1 \leq i \leq d$ is a part of a triangle $Y_{i-1} \to C_i \to Y_i \to Y_{i-1}[1]$ in $\mathcal{T}$, where we understand $Y_0 = C_0$ and $Y_d = C_{d+1}$,*
- *$f_0 \colon C_0 \to C_1$ is a source map and $g_d \colon C_d \to C_{d+1}$ is a sink map in $\mathcal{C}$,*
- *each $C_i \to C_{i+1}$ is a radical map.*

We call the above sequence an AR $(d+2)$-angle *in $\mathcal{C}$. Moreover, it satisfies the following.*

(1) *For each $1 \leq i \leq d-1$, the morphism f_i is a minimal left $\mathcal{C}$-approximation and g_i is a minimal right $\mathcal{C}$-approximation.*

(2) *$C_{d+1} = \nu_d^{-1} C_0$ for $\nu_d = \nu \circ [-d]$.*

In particular if $\mathcal{T}$ is d-CY the AR $(d+2)$-angles have the same end terms, say C. In this case we call it the *AR $(d+2)$-angle at C*.

Now assume that $\mathcal{T}$ is d-CY and $T \in \mathcal{T}$ is d-cluster tilting such that $\operatorname{End}_{\mathcal{T}}(T)$ is hereditary. We denote by T_a the indecomposable direct summand of T corresponding to the vertex a of the quiver Q of $\operatorname{End}_{\mathcal{T}}(T)$. For each vertex a, define the objects $S_a, U_a \in \mathcal{T}$ by the triangles

$$\coprod_{b\to a} T_b \longrightarrow T_a \longrightarrow S_a \longrightarrow \coprod_{b\to a} T_b$$

$$U_a \longrightarrow T_a \longrightarrow \coprod_{a\to b} T_b \longrightarrow U_a[1],$$

where the sum $\coprod_{b\to a}$ (resp. $\coprod_{a\to b}$) runs over all the arrows ending (resp. starting) at a, giving a sink map (resp. source map) at T_a in $\operatorname{add} T$. Note that $S_a[-1] = S_a^{(1)}$ and $U_a[1] = U_a^{(1)}$ in Section 2.2.1.

We note an easy observation on these sink and source maps.

Lemma 2.2.7. *Suppose $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < n$. Then for each $0 < i \le n$, there are exact sequences*

$$\mathcal{T}(T, \textstyle\coprod_{b\to a} T_b[-i]) \longrightarrow \mathcal{T}(T, T_a[-i]) \longrightarrow \mathcal{T}(T, S_a[-i]) \longrightarrow 0$$

$$\mathcal{T}(\textstyle\coprod_{a\to b} T_b[i], T) \longrightarrow \mathcal{T}(T_a[i], T) \longrightarrow \mathcal{T}(U_a[i], T) \longrightarrow 0.$$

In particular, $\mathcal{T}(T, S_a[-i]) = 0$ and $\mathcal{T}(U_a[i], T) = 0$ for each vertex a and $0 < i < n$.

Proof. We only prove the statement for S_a. By the defining triangle for S_a and vanishing of negative self-extensions of T, we immediately have the exact sequence for $1 < i \le n$. When $i = 1$, consider the exact sequence

$$\mathcal{T}(T, \textstyle\coprod_{b\to a} T_b[-1]) \longrightarrow \mathcal{T}(T, T_a[-1]) \longrightarrow \mathcal{T}(T, S_a[-1]) \longrightarrow \mathcal{T}(T, \coprod_{b\to a} T_b) \longrightarrow \mathcal{T}(T, T_a)\ ,$$

in which the last map is injective since it is the sink map in $\operatorname{add} T$ and $\operatorname{End}_{\mathcal{T}}(T)$ is hereditary. This proves our assertion. □

Let us also mention that vanishing of negative extensions up to a half of d automatically yields the vanishing for the other half, which allows us to weaken the vanishing assumption in Keller–Reiten's theorem 2.1.1(1).

Proposition 2.2.8. *Suppose $\mathcal{T}(T, T[-i]) = 0$ for $0 < i \le (d-1)/2$. Then $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < d-1$.*

Proof. Consider the AR $(d+2)$-angle at each indecomposable direct summand T_a of T.

$$\begin{array}{ccccccccccc} & & \coprod_{a\to b} T_b & \longrightarrow & T_a^{(d-2)} & \longrightarrow \cdots \longrightarrow & T_a^{(1)} & \longrightarrow & \coprod_{b\to a} T_b & & \\ & \nearrow & & \searrow \quad \nearrow & & \searrow \nearrow \quad \searrow \nearrow & & \searrow \quad \nearrow & & \searrow & \\ T_a & & & U_a[1] & & & & S_a[-1] & & & T_a \end{array}$$

By the assertion for S_a in 2.2.7, the middle terms $T_a^{(i)}$ are 0 for $1 \le i \le (d-1)/2$. Also from that for U_a, we have $T_a^{(d-1-i)} = 0$ for $1 \le i \le (d-1)/2$. Therefore all the middle terms $T_a^{(1)}, \dots, T_a^{(d-2)}$ are 0, hence $\mathcal{T}(T, S_a[-i]) = 0$ for $0 < i < d-1$. Now we prove $\mathcal{T}(T, T_a[-i]) = 0$ for $0 < i < d-1$ by induction on the vertices of Q, precisely, induction on the maximal length of path ending at the vertex a. If a is a source, then $T_a = S_a$ and we are done. Applying $\mathcal{T}(T, -)$ to the rightmost triangle, we have an exact sequence

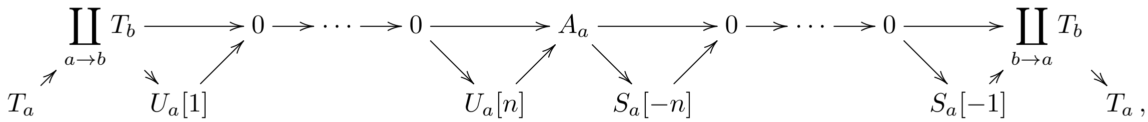
$$\mathcal{T}(T, \textstyle\coprod_{b\to a} T_b[-i]) \longrightarrow \mathcal{T}(T, T_a[-i]) \longrightarrow \mathcal{T}(T, S_a[-i])\ ,$$

in which the right term is 0 by the former claim, and the left term is 0 by the induction hypothesis. We therefore conclude that $\mathcal{T}(T, T_a[-i]) = 0$. □

2.2.3 Odd CY triangulated category with a cluster titling object

We apply our general observations of Section 2.2.1 to the setting of a CY triangulated category and a cluster tilting object. Let $n \geq 1$ and let $\mathcal{T}$ be a $(2n+1)$-CY triangulated category with a $(2n+1)$-cluster tilting object T such that $\mathrm{End}_{\mathcal{T}}(T)$ is hereditary, and $\mathrm{Hom}_{\mathcal{T}}(T, T[-i]) = 0$ for $0 < i < n$. Our proof shows that we can detect the structure of $H = \mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(2n-1)])$ from the AR $(2n+3)$-angles in $\mathrm{add}\, T$. The following observation is therefore fundamental.

Proposition 2.2.9. *The AR $(2n+3)$-angle at T_a is of the following form for some $A_a \in \mathrm{add}\, T$.*



with all the omitted middle terms 0.

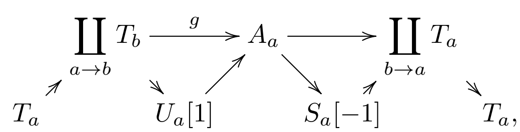
Proof. We know that the sink map in $\mathrm{add}\, T$ at T_a is $\coprod_{b \to a} T_b \to T_a$, so we have the rightmost triangle. Then we have $\mathcal{T}(T, S_a[-i]) = 0$ for $0 < i < n$ by 2.2.7, which gives the triangles on the right half. We dually get the triangles on the left half. □

We will refer to these AR $(2n+3)$-angles as (AR).

The symmetry of AR sequences has the following consequences. Compare the first one with 2.2.2(1).

Lemma 2.2.10. *The map $A_a \to S_a[-n]$ in the middle triangle in (AR) induces an isomorphism $\mathcal{T}(T, A_a) \simeq \mathcal{T}(T, S_a[-n])$. In particular, the $\mathrm{End}_{\mathcal{T}}(T)$-module $\mathcal{T}(T, S_a[-n])$ is projective.*

Proof. The map is always surjective since $A_a \to S_a[-n]$ is a right $(\mathrm{add}\, T)$-approximation. If $n > 1$, we see $\mathcal{T}(T, U_a[n]) = 0$ by the leftmost triangle in (AR), which gives injectivity. Now assume $n = 1$. In this case (AR) has the form



in which we have $g = 0$ by 2.2.4(2) for $m = 1$. Then applying $\mathcal{T}(T, -)$ to $g = 0$ yields a 0-map $\mathcal{T}(T, \coprod_{a \to b} T_b) \to \mathcal{T}(T, U_a[1]) \to \mathcal{T}(T, A_a)$, with the first map being surjective. Therefore, the second map is 0, hence applying $\mathcal{T}(T, -)$ to the middle triangle yields an isomorphism $\mathcal{T}(T, A_a) \xrightarrow{\simeq} \mathcal{T}(T, S_a[-1])$. □

Lemma 2.2.11. *We have $\mathcal{T}(T, T[-j]) = 0$ for $n < j < 2n$.*

Proof. We first show $\mathcal{T}(T, S_a[-j]) = 0$ for all vertices a and $n < j < 2n$. Put $j = n + i$ so that $0 < i < n$. Applying $\mathcal{T}(T, -)$ to the middle triangle in (AR) we have an exact sequence

$$\mathcal{T}(T, A_a[-i]) \longrightarrow \mathcal{T}(T, S_a[-n-i]) \longrightarrow \mathcal{T}(T, U_a[n-i+1]) \ ,$$

but $\mathcal{T}(T, A_a[-i]) = 0$ since $A_a \in \mathrm{add}\, T$, and the leftmost triangle yields $\mathcal{T}(T, U_a[n-i+1]) = 0$, hence $\mathcal{T}(T, S_a[-n-i]) = 0$.

Applying $\mathcal{T}(T, -)$ to the rightmost triangle, we have an exact sequence

$$\mathcal{T}(T, \textstyle\coprod_{b \to a} T_b[-j]) \longrightarrow \mathcal{T}(T, T_a[-j]) \longrightarrow \mathcal{T}(T, S_a[-j]) \ .$$

We therefore see by induction on the vertices of the quiver of $\mathrm{End}_{\mathcal{T}}(T)$ that $\mathcal{T}(T, T_a[-j]) = 0$ (cf. proof of 2.2.8). □

Now we are ready to prove the first main result of this section.

Theorem 2.2.12. *Let $n \geq 1$ and $\mathcal{T}$ be a $(2n+1)$-CY triangulated category with a $(2n+1)$-cluster tilting object T. Suppose that $\mathrm{End}_{\mathcal{T}}(T)$ is hereditary and $\mathrm{Hom}_{\mathcal{T}}(T, T[-i]) = 0$ for $0 < i < n$.*

(1) *The algebra $H' = \mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary.*

(2) *The algebra $H = \mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-(2n-1)])$ is a direct product of n copies of H', thus is hereditary.*

Proof. (2) follows from (1) and 2.2.11, so we prove (1). For this we show that any sink map in $\mathrm{proj}\, H' = \mathrm{add}(T \oplus T[-n])$ is a monomorphism. This is clear for the sink maps at objects in $\mathrm{add}\, T$ since there are no non-zero morphisms from $\mathrm{add}\, T[-n]$ to $\mathrm{add}\, T$, so sink maps in $\mathrm{add}\, T$ give sink maps in $\mathrm{add}(T \oplus T[-n])$. We consider the sink map at $T_a[-n]$. Applying 2.2.1 for $m = n$ there exists a commutative diagram of triangles

$$\begin{array}{ccccccc} S_a[-n-1] & \longrightarrow & U_a[n] & \longrightarrow & A_a & \longrightarrow & S_a[-n] \\ \| & & \downarrow & & \downarrow & & \| \\ S_a[-n-1] & \longrightarrow & \coprod_{b \to a} T_b[-n] & \longrightarrow & T_a[-n] & \longrightarrow & S_a[-n] \end{array} \qquad (2.2.1)$$

giving a triangle

$$U_a[n] \longrightarrow A_a \oplus \coprod_{b \to a} T_b[-n] \longrightarrow T_a[-n] \longrightarrow U_a[n+1] \ ,$$

in which the middle map is the sink map in $\mathrm{add}(T \oplus \cdots \oplus T[-n])$. Since it has terms in $\mathrm{add}(T \oplus T[-n])$ it is the one in $\mathrm{add}(T \oplus T[-n])$. Applying $\mathcal{T}(T \oplus T[-n], -)$ gives an exact sequence

$$\mathcal{T}(T \oplus T[-n], U_a[n]) \longrightarrow \mathcal{T}(T \oplus T[-n], A_a \oplus \textstyle\coprod_{b \to a} T_b[-n]) \longrightarrow \mathcal{T}(T \oplus T[-n], T_a[-n]) \ ,$$

in which we want to show that the first map is 0. It is clear for the summand $\mathcal{T}(T[-n], -)$ since the above sequence reduces to

$$\mathcal{T}(T[-n], U_a[n]) \longrightarrow \mathcal{T}(T[-n], \textstyle\coprod_{b \to a} T_b[-n]) \longrightarrow \mathcal{T}(T[-n], T_a[-n]) \ ,$$

and the second map, being the sink map in $\mathrm{add}\, T[-n]$, is a monomorphism. Also we see that $\mathcal{T}(T, U_a[n]) \to \mathcal{T}(T, A_a)$ is 0 by the sequence

$$\mathcal{T}(T, U_a[n]) \longrightarrow \mathcal{T}(T, A_a) \longrightarrow \mathcal{T}(T, S_a[-n]) \ ,$$

in which the second map is injective (in fact an isomorphism) by 2.2.10. It remains to show that the map

$$\mathcal{T}(T, U_a[n]) \longrightarrow \mathcal{T}(T, \textstyle\coprod_{b \to a} T_b[-n])$$

is 0. If $n > 1$ then $\mathcal{T}(T, U_a[n]) = 0$ and we have the assertion. Now assume $n = 1$. We prove the following statements by induction on the vertices of Q, which will complete the proof by (i).

(i) The map $\mathcal{T}(T, U_a[1]) \to \mathcal{T}(T, \coprod_{b \to a} T_b[-1])$ is 0.

(ii) The $\mathrm{End}_{\mathcal{T}}(T)$-module $\mathcal{T}(T, T_a[-1])$ is projective.

If a is a source, then we have $\coprod_{b \to a} T_b = 0$ thus (i), and $T_a = S_a$ thus (ii) by 2.2.10. Suppose now that a is a general vertex of Q. By 2.2.5, the $\mathrm{End}_{\mathcal{T}}(T)$-module $\mathcal{T}(T, U_a[1])$ has no projective summands while $\mathcal{T}(T, \coprod_{b \to a} T_b[-1])$ is projective by induction hypothesis. Therefore the map in (i) has to be 0. Then by the leftmost commutative square in (2.2.1), we have that $\mathcal{T}(T, S_a[-2]) \to \mathcal{T}(T, \coprod_{b \to a} T_b[-1])$ is 0, hence applying $\mathcal{T}(T, -)$ to the second row gives a short exact sequence below by 2.2.7.

$$0 \longrightarrow \mathcal{T}(T, \textstyle\coprod_{b \to a} T_b[-1]) \longrightarrow \mathcal{T}(T, T_a[-1]) \longrightarrow \mathcal{T}(T, S_a[-1]) \longrightarrow 0$$

This has projective end terms by induction hypothesis and 2.2.10 respectively, so we conclude that the middle term $\mathcal{T}(T, T_a[-1])$ is also projective. □

We conclude our discussion by explicitly describing the quiver $\widetilde{Q}$ of $\operatorname{End}_{\mathcal{T}}(T \oplus T[-n])$. Suppose that $\operatorname{End}_{\mathcal{T}}(T) = kQ$ for an acyclic quiver Q. Note first that T and $T[-n]$ have no common direct summand by $\mathcal{T}(T[-n], T) = 0$, thus $\widetilde{Q}$ have two copies of Q as a subquiver. We need to investigate the arrows from the subquiver for T to the one for $T[-n]$. We identify the vertices of $\widetilde{Q}$ and the corresponding indecomposable summands of $T \oplus T[-n]$.

Proposition 2.2.13. *The following are equal.*

(a) The number of arrows from T_b to $T_a[-n]$.

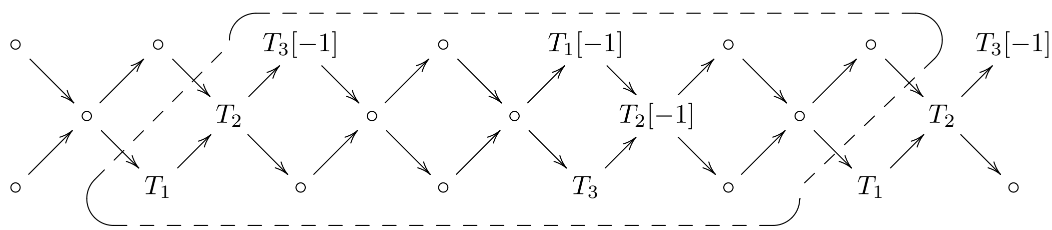
(b) The number of direct summands T_b in A_a.

(c) The number of arrows from T_a to $T_b[-n]$.

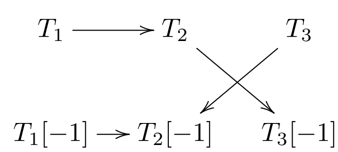
(d) The number of direct summands T_a in A_b.

Proof. By 2.2.1 the map $A_a \oplus \coprod_{b \to a} T_b[-n] \to T_a[-n]$ is the sink map in $\operatorname{add}(T \oplus T[-n])$. This gives (a)=(b), and similarly (c)=(d). Dually, 2.2.3 shows that the map $T_a[n] \to A_a \oplus \coprod_{a \to b} T_b[n]$ is the source map in $\operatorname{add}(T \oplus T[n])$, which shows (c)=(b). □

Example 2.2.14. Let Q be the quiver of linearly oriented type A_3 and $\mathcal{T} = \mathcal{C}_3(kQ)$ the 3-cluster category of Q. We have its AR quiver as below, with a fundamental domain inside the dotted line. It has a 3-cluster tilting object $T = T_1 \oplus T_2 \oplus T_3$ which is obtained by mutating the initial cluster tilting object $kQ \in \mathcal{C}_3(kQ)$ at the sink.

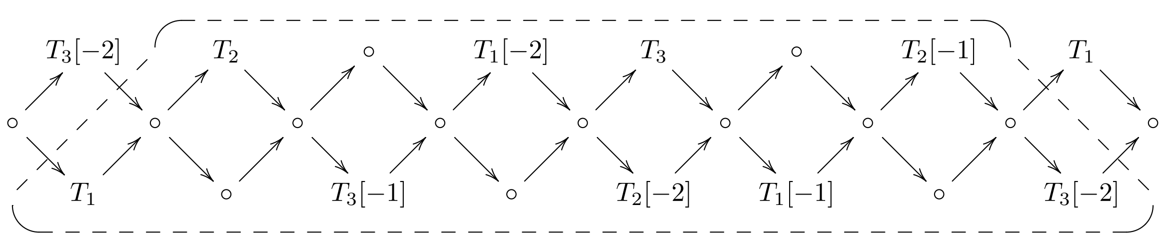


We see that $\operatorname{End}_{\mathcal{T}}(T) = kA_2 \times k$, and $\operatorname{End}_{\mathcal{T}}(T \oplus T[-1])$ is presented by the quiver below; a direct product of 2 path algebras of type A_3 with different orientations.

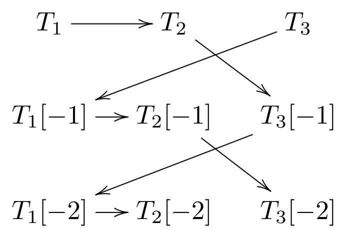


We end this subsection with the following non-example, which shows that 2.2.12 fails for 4-CY case, that is, even if $T \in \mathcal{T}$ is a 4-cluster tilting such that $\operatorname{End}_{\mathcal{T}}(T)$ is hereditary, the shifted sum $T \oplus T[-1] \oplus T[-2]$ does not necessarily have hereditary endomorphism ring.

Example 2.2.15. Let Q be a quiver of type A_3, and $\mathcal{T} = \mathcal{C}_4(kQ)$ the 4-cluster category of Q. Consider the 4-cluster tilting object $T = T_1 \oplus T_2 \oplus T_3$ which is obtained by mutating $T_1 \oplus T_2 \oplus T_3[-1]$ at $T_3[-1]$.



The algebra $H = \mathrm{End}_{\mathcal{T}}(T \oplus T[-1] \oplus T[-2])$ is presented by the quiver



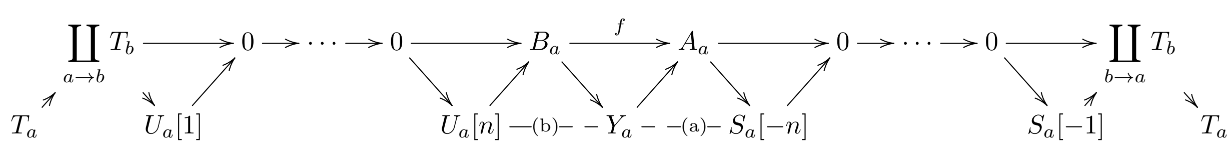
with relations "any composite of arrows equals 0". It follows that H has global dimension 4.

2.2.4 Even CY triangulated category with a cluster tilting object

As we have just seen in 2.2.15 above, hereditaryness of T does not imply that of $T\oplus T[-1]\oplus\cdots\oplus T[-(d-2)]$ when $d \geq 4$. Nevertheless we prove in this section that in dimension 4, the endomorphism algebra of $T \oplus T[-1] \oplus T[-2]$ is hereditary as soon as that of $T \oplus T[-1]$ is (2.2.21). As before, we work in a higher dimensional setting with vanishing of some negative extensions.

Let $n \geq 1$, and let $\mathcal{T}$ be a $(2n+2)$-CY triangulated category with a $(2n+2)$-cluster tilting object T such that $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < n$, and $\mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary. In this case $\mathrm{End}_{\mathcal{T}}(T)$ is also hereditary, whose quiver we denote by Q. We denote by T_a the indecomposable direct summand of T corresponding to the vertex a of Q. As in the previous subsection, our starting point is a computation of AR sequences. Recall the definition of S_a and U_a from Section 2.2.2.

Proposition 2.2.16. *The AR $(2n+4)$-angle at T_a is of the form below for some $A_a, B_a \in \mathrm{add}\, T$ and $Y_a \in \mathcal{T}$.*



Moreover the middle map f is 0.

Proof. We draw the AR sequence from the right. Clearly we have the rightmost triangle. By 2.2.7 we get the triangles on the right half up to $S_a[-n]$. Dually we can draw the left triangles to obtain the AR $(2n+2)$-angle above. Now, since $\mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary we can apply 2.2.2(2) for $m = n + 1$, which shows $f = 0$. □

As before we will refer to these AR $(2n + 4)$-angles as (AR). Similarly to the previous subsection, the symmetry of (AR) gives the following.

Lemma 2.2.17. *We have $\mathcal{T}(T, T_a[-j]) = 0$ for $n + 1 < j < 2n$.*

Proof. Put $j = n + i$ so that $1 < i < n$. By the triangle (a) in (AR) we have an exact sequence

$$0 = \mathcal{T}(T, A_a[-i]) \longrightarrow \mathcal{T}(T, S_a[-j]) \longrightarrow \mathcal{T}(T, Y_a[-i+1]) \longrightarrow \mathcal{T}(T, A_a[-i+1]) = 0 ,$$

in which the two end terms are 0 by vanishing of small negative self-extensions of T. Also by the triangle (b) in (AR) we have

$$0 = \mathcal{T}(T, B_a[-i+1]) \longrightarrow \mathcal{T}(T, Y_a[-i+1]) \longrightarrow \mathcal{T}(T, U_a[n+2-i]) = 0 ,$$

where the leftmost term is 0 by vanishing of small negative self-extensions of T, and so is the rightmost term by the leftmost triangle in (AR). We deduce by these exact sequences that $\mathcal{T}(T, S_a[-j]) = \mathcal{T}(T, Y_a[-i+1]) = 0$.

Now by the exact sequence

$$\mathcal{T}(T, \textstyle\coprod_{b\to a} T_b[-j]) \longrightarrow \mathcal{T}(T, T_a[-j]) \longrightarrow \mathcal{T}(T, S_a[-j])\ ,$$

we see inductively that $\mathcal{T}(T, T_a[-j]) = 0$. □

We need some more technical observations.

Lemma 2.2.18. *We have* $\mathcal{T}(T, Y_a[-n+1]) = \begin{cases} \mathcal{T}(T, B_a) & (n=1) \\ 0 & (n>1) \end{cases}$.

Proof. This is easily shown by applying $\mathcal{T}(T, -)$ to the triangle (b) in (AR). □

Lemma 2.2.19. *The triangle (a) in (AR) yields a short exact sequence*

$$0 \longrightarrow \mathcal{T}(T, A_a[-n]) \longrightarrow \mathcal{T}(T, S_a[-2n]) \longrightarrow \mathcal{T}(T, Y_a[-n+1]) \longrightarrow 0\ .$$

Consequently the $\mathrm{End}_{\mathcal{T}}(T)$*-module* $\mathcal{T}(T, S_a[-2n])$ *is projective.*

Proof. We apply $\mathcal{T}(T, -)$ to the triangle (a) in (AR). For any $n \geq 1$ it yields an exact sequence

$$\mathcal{T}(T, Y_a[-n]) \xrightarrow{g} \mathcal{T}(T, A_a[-n]) \longrightarrow \mathcal{T}(T, S_a[-2n]) \longrightarrow \mathcal{T}(T, Y_a[-n+1]) \longrightarrow \mathcal{T}(T, A_a[-n+1]) \xrightarrow{g'} \mathcal{T}(T, S_a[-2n+1])$$

We first show right exactness. By 2.2.18 we only have to consider $n = 1$. Then g' is an isomorphism by 2.2.10, which gives the result.

We next show that the map g is 0, which yields left exactness. Applying $\mathcal{T}(T, -)$ to the middle map $0 = f \colon B_a \to Y_a \to A_a$ in (AR), we have a zero map $\mathcal{T}(T, B_a[-n]) \xrightarrow{h} \mathcal{T}(T, Y_a[-n]) \xrightarrow{g} \mathcal{T}(T, A_a[-n])$. Now, applying $\mathcal{T}(T, -)$ to the triangle (b) in (AR) gives an exact sequence

$$\mathcal{T}(T, B_a[-n]) \xrightarrow{h} \mathcal{T}(T, Y_a[-n]) \longrightarrow \mathcal{T}(T, U_a[1]) \xrightarrow{k} \mathcal{T}(T, B_a[-n+1]) \xrightarrow{k'} \mathcal{T}(T, Y_a[-n+1])\ ,$$

in which we have $k = 0$. Indeed, if $n > 1$ then $\mathcal{T}(T, B_a[-n+1]) = 0$, and if $n = 1$ then k' is an isomorphism by 2.2.18. It follows that $\mathcal{T}(T, U_a[1]) = \mathrm{Coker}\, h$, so g factors through $\mathcal{T}(T, U_a[1])$. But as $\mathcal{T}(T, U_a[1])$ has no non-zero projective summand by 2.2.5 while $\mathcal{T}(T, A_a[-n])$ is projective since $\mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary, this shows $g = 0$ as desired.

Finally, the $\mathrm{End}_{\mathcal{T}}(T)$-module $\mathcal{T}(T, Y_a[-n+1])$ is projective by 2.2.18 and so is $\mathcal{T}(T, A_a[-n])$ since $\mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary, thus the remaining assertion follows. □

As a final preparation we need an observation on the behavior of sink maps.

Lemma 2.2.20. *Let* $n + 1 \leq j \leq 2n$*. Then the sink map at* $T_a[-j]$ *in* $\mathrm{add}(T \oplus \cdots \oplus T[-2n])$ *is obtained from the one at* $T[-n-1]$ *by shifting by* $[-(j-n-1)]$*.*

Proof. Applying 2.2.3 for $m = n+1$, the middle map in the triangle

$$U_a[n] \longrightarrow B_a \oplus A_a[-1] \oplus \coprod_{b\to a} T_b[-n-1] \longrightarrow T_a[-n-1] \longrightarrow U_a[n+1]$$

gives the sink map at $T_a[-n-1]$ in $\mathrm{add}(T \oplus \cdots \oplus T[-n-1])$. In view of rigidity of T, this is also the sink map in $\mathrm{add}(T \oplus \cdots \oplus T[-2n])$. We have to show that in the shifted triangle

$$U_a[2n+1-j] \longrightarrow B_a[-(j-n-1)] \oplus A_a[-(j-n)] \oplus \coprod_{b\to a} T_b[-j] \xrightarrow{v} T_a[-j] \longrightarrow U_a[2n+2-j]\ ,$$

the middle map is the sink map in $\mathrm{add}(T \oplus \cdots \oplus T[-2n])$. Since T is rigid and $\mathcal{T}(T, T[-i]) = 0$ for $i \in \{0, \ldots, 2n\} \setminus \{0, n, n+1, 2n\}$ by 2.2.17, this assertion is clear for $j < 2n$. It remains to verify $j = 2n$. It is enough to show $\mathcal{T}(T, v)$ is surjective, but this is clear by $\mathcal{T}(T, U_a[2]) = 0$. $\square$

Now we are ready to prove the 4-CY version, or more generally the even CY version, which is the second main result of this section.

Theorem 2.2.21. *Let $n \geq 1$ and let $\mathcal{T}$ be a $(2n+2)$-CY triangulated category with a $(2n+2)$-cluster tilting object T. Suppose that $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < n$, and $\mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary. Then $H = \mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-2n])$ is hereditary.*

Proof. We show that for each $a \in Q_0$ and $0 \leq j \leq 2n$ the sink map at $T_a[-j]$ in $\mathrm{add}(T \oplus \cdots \oplus T[-2n])$ is a monomorphism.

By our assumptions this is clear for $0 \leq j \leq n$.

Let $n+1 \leq j < 2n$, in particular $n > 1$. By 2.2.1 and 2.2.20, the map v in the triangle

$$U_a[n] \xrightarrow{u} B_a \oplus A_a[-1] \oplus \coprod_{b \to a} T_b[-n-1] \xrightarrow{v} T_a[-n-1] \longrightarrow U_a[n+1]$$

gives the sink map at $T_a[-n-1]$ in $\mathrm{add}(T \oplus \cdots \oplus T[-2n])$, and the same holds for the sequence shifted by $[-(j-n-1)]$. We want to show $\mathcal{T}(T \oplus \cdots \oplus T[-2n], v[-(j-n-1)])$ is injective, or equivalently $\mathcal{T}(T \oplus \cdots \oplus T[-2n], u[-(j-n-1)]) = 0$. By vanishing of some negative and positive self-extension of T, it is enough to consider $j = n+1$, thus is reduced to showing $\mathcal{T}(T \oplus T[-1] \oplus T[-n-1], u) = 0$. This follows from $\mathcal{T}(T \oplus T[-1] \oplus T[-n-1], U_a[n]) = 0$, where we used $n > 1$ for $\mathcal{T}(T, U_a[n]) = 0$.

Finally consider $j = 2n$. By 2.2.1 and 2.2.20 we have the commutative diagram

$$\begin{array}{ccccccc}
Y_a[-n] & \longrightarrow & U_a[1] & \longrightarrow & B_a[-n+1] & \longrightarrow & Y_a[-n+1] \\
\| & & \downarrow & & \downarrow & & \| \\
Y_a[-n] & \longrightarrow & A_a[-n] \oplus \coprod_{b \to a} T_b[-2n] & \longrightarrow & T_a[-2n] & \longrightarrow & Y_a[-n+1]
\end{array} \tag{2.2.2}$$

giving the sink map v at $T_a[-2n]$

$$U_a[1] \xrightarrow{u} B_a[-n+1] \oplus A_a[-n] \oplus \coprod_{b \to a} T_b[-2n] \xrightarrow{v} T_a[-2n] \longrightarrow U_a[2] \ ,$$

in which we want to show $\mathcal{T}(T \oplus \cdots \oplus T[-2n], v)$ is injective. Since $\mathcal{T}(T[-1] \oplus \cdots \oplus T[-2n], U_a[1]) = 0$ by the leftmost triangle in (AR), it remains to prove that $\mathcal{T}(T, u) = 0$. We prove by induction on the vertices of Q the following statements, which will complete the proof.

(i) The map $\mathcal{T}(T, U_a[1]) \to \mathcal{T}(T, B_a[-n+1] \oplus A_a[-n] \oplus \coprod_{b \to a} T_b[-2n])$ is 0.

(ii) The $\mathrm{End}_{\mathcal{T}}(T)$-module $\mathcal{T}(T, T_a[-2n])$ is projective.

If a is a source, then $\mathcal{T}(T, B_a[-n+1] \oplus A_a[-n])$ is projective while $\mathcal{T}(T, U_a[1])$ has no non-zero projective summand by 2.2.5, so we have (i). Also in this case we have $T_a = S_a$, so (ii) by 2.2.19. Now suppose that a is a general vertex of Q. Then the $\mathrm{End}_{\mathcal{T}}(T)$-module $\mathcal{T}(T, B_a[-n+1] \oplus A_a[-n] \oplus \coprod_{b \to a} T_b[-2n])$ is projective by induction hypothesis, and $\mathcal{T}(T, U_a[1])$ does not have a non-zero projective summand by 2.2.5, which gives (i). Then the leftmost commutative square in (2.2.2) shows that the map $\mathcal{T}(T, Y_a[-n]) \to \mathcal{T}(T, A_a[-n] \oplus \coprod_{b \to a} T_b[-2n])$ is 0, hence we obtain an exact sequence

$$0 \longrightarrow \mathcal{T}(T, A_a[-n] \oplus \textstyle\coprod_{b \to a} T_b[-2n]) \longrightarrow \mathcal{T}(T, T_a[-2n]) \longrightarrow \mathcal{T}(T, Y_a[-n+1]) \ .$$

Since the rightmost term is projective by 2.2.18, and so is the leftmost term by induction hypothesis, we deduce that the middle term is also projective, which completes the induction step, hence gives our result. $\square$

Finally let us describe the quiver of our hereditary algebra $H = \mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-2n])$. As in 2.2.13 we can compute the quiver $\widetilde{Q}$ of H from the AR $(2n+4)$-angles. Since T is $(2n+2)$-rigid the quiver $\widetilde{Q}$ has $(2n+1)$ copies of Q as a subquiver. We investigate the arrows from $T_a[-i]$ to $T_b[-j]$. By 2.2.17 we know that there is an arrow $T_a[-i] \to T_b[-j]$ in Q only if $j - i \in \{0, n, n+1, 2n\}$.

Proposition 2.2.22. (1) *Let $0 \leq i \leq n$ and $0 \leq i' \leq n-1$. The following are equal.*

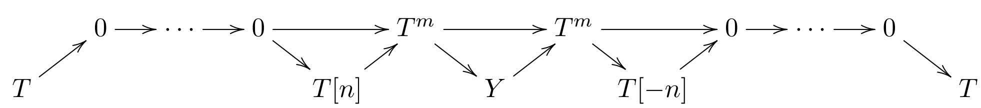
- (a) *The number of arrows from $T_a[-i]$ to $T_b[-i-n]$.*
- (b) *The number of summands T_a in A_b.*
- (c) *The number of summands T_b in B_a.*
- (d) *The number of arrows from $T_b[-i']$ to $T_a[-i'-n-1]$.*

(2) *There is an arrow T_a to $T_b[-2n]$ only if $n = 1$, in which case the number of arrows is given in (d) above.*

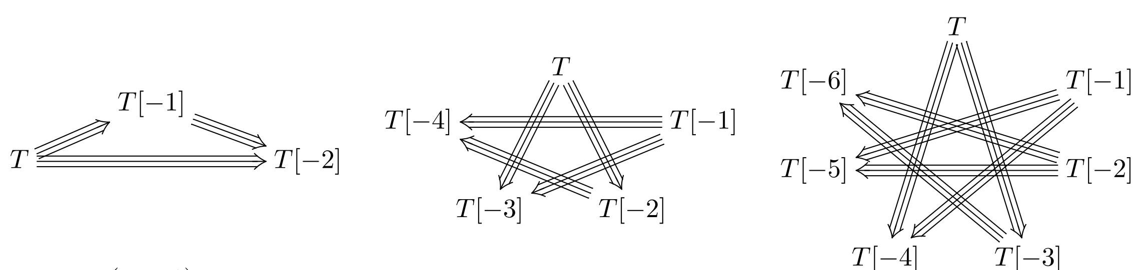
Proof. (1) By 2.2.1 the values of (a) are equal for $i = 0$ and $i = 1$, and in view of 2.2.20 they are equal to the ones for all the other i. Therefore it is enough to consider the case $i = 0$. Similarly by 2.2.20 it suffices to assume $i' = 0$. Applying 2.2.1 to T_b and $m = n$ gives (a)=(b), and 2.2.3 to T_a and $m = n$ gives (a)=(c). Also, applying 2.2.1 to T_a and $m = n+1$ gives (d)=(c).
(2) This follows from 2.2.20. □

We give several examples of the quiver of H for the case $\mathrm{End}_{\mathcal{T}}(T) = k$.

Example 2.2.23. Suppose that $\mathrm{End}_{\mathcal{T}}(T) = k$ and put $m = \dim_k \mathcal{T}(T, T[-n])$. Then the AR $(2n+4)$-angle is of the form



for some $Y \in \mathcal{T}$. We see that for $n = 1, 2, 3$, the algebra H is presented by the quiver below with m-fold arrows between the vertices.



$(n = 1)$ $(n = 2)$ $(n = 3)$

2.3 Realizing triangulated categories as cluster categories

Recall that the *cluster category* of a homologically smooth dg algebra Π is the Verdier quotient

$$\mathcal{C}(\Pi) := \mathrm{per}\,\Pi / \mathcal{D}^b(\Pi).$$

We think about how a triangulated category can be realized as a cluster category in the above sense, which plays an important role in the proof of main results in this part.

Let us note a preliminary result on cluster categories of a special class of dg algebras, namely (derived) tensor algebras. This gives a description the cluster category in terms of the triangulated hull of a derived category. Let A be a finite dimensional algebra of finite global dimension, and X a two-sided tilting complex over A, thus X is a complex of (A,A)-bimodules such that $F = - \otimes^{\mathrm{L}}_A X \colon \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{D}^b(\operatorname{mod} A)$ gives an autoequivalence. We impose the following two conditions on X.

- For each $L, M \in \mathcal{D}^b(\operatorname{mod} A)$ we have $\operatorname{Hom}_{\mathcal{D}(A)}(L, F^i M) = 0$ for all but finitely many $i \in \mathbb{Z}$.
- X is concentrated in (cohomological) degree ≤ 0.

Replacing X by a projective resolution over A^e, we put $\Pi = T_A X$, the tensor algebra of X over A.

Proposition 2.3.1 ([Ke2, Section 7], [Am, 4.13], see also [Han1, Section 8])**.** *The dg algebra* Π *is homologically smooth, and its cluster category* $\mathcal{C}(\Pi)$ *is equivalent to the canonical triangulated hull of* $\mathcal{D}^b(\operatorname{mod} A)/-\otimes^{\mathrm{L}}_A X$.

Now we turn to the main subject of this section. Let $\mathcal{T}$ be a k-linear, Hom-finite triangulated category with an generator X, that is, $\operatorname{thick} X = \mathcal{T}$. Assuming that $\mathcal{T}$ is algebraic and taking a derived endomorphism algebra Γ of X, we have $\mathcal{T} = \operatorname{per}\Gamma$, and we consider the following condition on the dg algebra Γ.

(a) Each cohomology of Γ is finite dimensional.

(b) For each $Y \in \operatorname{per}\Gamma$, $H^i Y = 0$ for $i \ll 0$ implies $Y = 0$.

(c) The truncation $\Pi := \Gamma^{\leq 0}$ is homologically smooth.

The following example is our principle one for the condition (b).

Lemma 2.3.2. *If* $e\Gamma \in \operatorname{per}\Gamma$ *is* d*-cluster tilting for some idempotent* $e \in \Gamma$ *and* $d \geq 1$*, then (b) above hold.*

Proof. Suppose $e\Gamma \in \operatorname{per}\Gamma$ is d-cluster tilting and $\operatorname{Hom}_{\mathcal{D}(\Gamma)}(\Gamma, Y[\ll 0]) = 0$. Then some d-successive extensions from $e\Gamma$ to Y vanish, so there exists $l \in \mathbb{Z}$ such that $\operatorname{Hom}_{\operatorname{per}\Gamma}(e\Gamma, Y[l][i]) = 0$ for $0 \leq i < d$. The vanishing for $0 < i < d$ shows $Y[l] \in \operatorname{add} e\Gamma$ since $e\Gamma \in \operatorname{per}\Gamma$ is d-cluster tilting, and the vanishing for $i = 0$ shows $Y[l] = 0$. □

We prove that our triangulated category $\mathcal{T} = \operatorname{per}\Gamma$ arises as the cluster category; in fact it is the cluster category of Π which is very simply described as above.

Theorem 2.3.3. *Let* Γ *be a dg algebra satisfying the above (a), (b), and (c). Then the functor* $- \otimes^{\mathrm{L}}_{\Pi} \Gamma$ *induces an equivalence*

$$\mathcal{C}(\Pi) = \operatorname{per}\Pi/\mathcal{D}^b(\Pi) \xrightarrow{\simeq} \operatorname{per}\Gamma\ .$$

We will temporarily have to work in big triangulated categories. Let $\mathcal{D}$ be a triangulated category with arbitrary (set-indexed) coproducts. Recall that the *homotopy colimit* of a sequence

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots$$

in $\mathcal{D}$ is an object M which fits into a triangle

$$\coprod_{i\geq 0} M_i \xrightarrow{f} \coprod_{i\geq 0} M_i \longrightarrow M \longrightarrow \coprod_{i\geq 0} M_i[1]$$

with f having components $M_i \xrightarrow{\binom{1}{-f_i}} M_i \oplus M_{i+1} \hookrightarrow \coprod_{i\geq 0} M_i$. Thus M is uniquely determined up to (non-unique) isomorphism, which is denoted by $\operatorname{hocolim}_{i\geq 0} M_i$.

We will use some easy computations on homotopy colimits.

Lemma 2.3.4. *Let* Λ *be an arbitrary negative dg algebra.*

(1) *We have* $M = \operatorname{hocolim}_{p\geq 0} M^{\leq p}$ *for all* $M \in \mathcal{D}(\Lambda)$.

(2) *For each* $L \in \operatorname{per}\Lambda$ *and a sequence* $M_0 \to M_1 \to \cdots$ *in* $\mathcal{D}(\Lambda)$, *there exists a natural isomorphism*

$$\operatorname*{colim}_{p\geq 0} \operatorname{Hom}_{\mathcal{D}(\Lambda)}(L, M_p) \xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{D}(\Lambda)}(L, \operatorname{hocolim}_{p\geq 0} M).$$

(3) *For each* $L \in \operatorname{per}\Lambda$, $M \in \mathcal{D}(\Lambda)$, *and* $N \in \mathcal{D}(\Lambda^e)$, *there is a natural isomorphism*

$$\operatorname*{colim}_{p\geq 0} \operatorname{Hom}_{\mathcal{D}(\Lambda)}(L, M \otimes^{\mathrm{L}}_{\Lambda} N^{\leq p}) \xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{D}(\Lambda)}(L, M \otimes^{\mathrm{L}}_{\Lambda} N).$$

Proof. (1) is easy and (2) is well-known [N]. Then (3) follows since $M \otimes^{\mathrm{L}}_{\Lambda} N = \operatorname{hocolim}_{p\geq 0}(M \otimes^{\mathrm{L}}_{\Lambda} N^{\leq p})$. $\square$

The following observation is crucial.

Lemma 2.3.5. *We have* $L \otimes^{\mathrm{L}}_{\Pi} \Gamma = 0$ *for any* $L \in \mathcal{D}^b(\Pi)$.

Proof. Consider the triangle

$$\Pi \longrightarrow \Gamma \longrightarrow W \longrightarrow \Pi[1]$$

in $\mathcal{D}(\Pi^e)$ obtained from the inclusion $\Pi \to \Gamma$. Since Π is homologically smooth we have $L \in \operatorname{per}\Pi$, hence $M := L \otimes^{\mathrm{L}}_{\Pi} \Gamma \in \operatorname{per}\Gamma$. We show that M is bounded below, and the claim follows from our assumption (b). Applying $L \otimes^{\mathrm{L}}_{\Pi} -$ to the above triangle yields a triangle $L \to M \to L \otimes^{\mathrm{L}}_{\Pi} W \to L[1]$, in which $L \in \mathcal{D}^b(\Pi)$ is bounded below, and $L \otimes^{\mathrm{L}}_{\Pi} W \in \operatorname{thick}_{\mathcal{D}(\Pi)} W$ is also bounded below. Therefore the middle term M is bounded below. $\square$

Now we are ready to prove our general structure theorem.

Proof of 2.3.3. First note that 2.3.5 gives rise to a functor we want: The tensor product $- \otimes^{\mathrm{L}}_{\Pi} \Gamma$ indeed induces a functor $\mathcal{C}(\Pi) \to \operatorname{per}\Gamma$. We have to show that the map $\operatorname{Hom}_{\mathcal{C}(\Pi)}(L, M) \to \operatorname{Hom}_{\mathcal{D}(\Gamma)}(L \otimes^{\mathrm{L}}_{\Pi} \Gamma, M \otimes^{\mathrm{L}}_{\Pi} \Gamma)$ is bijective for each $L, M \in \operatorname{per}\Pi$. Note that the right-hand-side is isomorphic to $\operatorname{Hom}_{\mathcal{D}(\Pi)}(L, M \otimes^{\mathrm{L}}_{\Pi} \Gamma)$ by adjunction, and then the map in question is induced by applying $\operatorname{Hom}_{\mathcal{D}(\Pi)}(L, -)$ to the natural map $M = M \otimes^{\mathrm{L}}_{\Pi} \Pi \to M \otimes^{\mathrm{L}}_{\Pi} \Gamma$, that is, we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{D}(\Pi)}(L, M) & \longrightarrow & \operatorname{Hom}_{\mathcal{D}(\Pi)}(L, M \otimes^{\mathrm{L}}_{\Pi} \Gamma) \\ \downarrow & & \| \\ \operatorname{Hom}_{\mathcal{C}(\Pi)}(L, M) & \longrightarrow & \operatorname{Hom}_{\mathcal{D}(\Gamma)}(L \otimes^{\mathrm{L}}_{\Pi} \Gamma, M \otimes^{\mathrm{L}}_{\Pi} \Gamma). \end{array}$$

We first show the injectivity. Let $L \to M$ be a morphism in $\operatorname{per}\Pi$ such that the composite $L \to M \to M \otimes^{\mathrm{L}}_{\Pi} \Gamma$ is 0. Put $W = \operatorname{cone}(\Pi \to \Gamma)$ as in 2.3.5. Then it factors through $M \otimes^{\mathrm{L}}_{\Pi} W[-1]$ as in the diagram below, thus through $M \otimes^{\mathrm{L}}_{\Pi} (W[-1])^{\leq p}$ for some $p \geq 0$ by 2.3.4(3). Now this is in $\mathcal{D}^b(\Pi)$ since $M \in \operatorname{per}\Pi$ and $(W[-1])^{\leq p} \in \mathcal{D}^b(\Pi)$. Therefore $L \to M$ is 0 in $\mathcal{C}(\Pi)$.

$$\begin{array}{ccccccc} & \swarrow\!\!- - - & L & & & & \\ & & \downarrow & & & & \\ M \otimes^{\mathrm{L}}_{\Pi} W[-1] & \longrightarrow & M & \longrightarrow & M \otimes^{\mathrm{L}}_{\Pi} \Gamma & \longrightarrow & M \otimes^{\mathrm{L}}_{\Pi} W \end{array}$$

We next show the surjectivity. Let $f \colon L \to M \otimes^{\mathrm{L}}_{\Pi} \Gamma$ be a morphism in $\mathcal{D}(\Pi)$. It factors through $M \otimes^{\mathrm{L}}_{\Pi} \Gamma^{\leq p}$ for some $p \geq 0$ by 2.3.4(3). Then the canonical map $s \colon M \to M \otimes^{\mathrm{L}}_{\Pi} \Gamma^{\leq p}$ has finite dimensional mapping cone

$M \otimes^{\mathbf{L}}_{\Pi} W^{\leq p}$, so we have the diagram $L \xrightarrow{g} M \otimes^{\mathbf{L}}_{\Pi} \Gamma^{\leq p} \xleftarrow{s} M$ in $\operatorname{per}\Pi$ which presents a morphism in $\mathcal{C}(\Pi)$. We verify that this is mapped to the original morphism in $\mathcal{D}(\Gamma)$.

$$\begin{array}{ccccc} L & \xrightarrow{\ g\ } & M \otimes^{\mathbf{L}}_{\Pi} \Gamma^{\leq p} & \xleftarrow{\ s\ } & M \\ \downarrow & & \downarrow & \overset{t}{\searrow} & \downarrow \\ L \otimes^{\mathbf{L}}_{\Pi} \Gamma & \longrightarrow & M \otimes^{\mathbf{L}}_{\Pi} \Gamma^{\leq p} \otimes^{\mathbf{L}}_{\Pi} \Gamma & \xleftarrow{\ \simeq\ } & M \otimes^{\mathbf{L}}_{\Pi} \Gamma \end{array}$$

Applying $- \otimes^{\mathbf{L}}_{\Pi} \Gamma$ to $s^{-1}g\colon L \to M$ in $\mathcal{C}(\Pi)$ yields the second row in the diagram above, which is the morphism $s^{-1}g \otimes 1\colon L \otimes^{\mathbf{L}}_{\Pi} \Gamma \to M \otimes^{\mathbf{L}}_{\Pi} \Gamma$ in $\operatorname{per}\Gamma$. Under the adjunction it becomes $L \xrightarrow{g} M \otimes^{\mathbf{L}}_{\Pi} \Gamma^{\leq p} \to M \otimes^{\mathbf{L}}_{\Pi} \Gamma^{\leq p} \otimes^{\mathbf{L}}_{\Pi} \Gamma \xleftarrow{\simeq} M \otimes^{\mathbf{L}}_{\Pi} \Gamma$ in $\mathcal{D}(\Pi)$. On the other hand, the original morphism $f\colon L \to M \otimes^{\mathbf{L}}_{\Pi} \Gamma$ is $L \xrightarrow{g} M \otimes^{\mathbf{L}}_{\Pi} \Gamma^{\geq p} \xrightarrow{t} M \otimes^{\mathbf{L}}_{\Pi} \Gamma$. Therefore it remains to show that the lower triangle in the right square in the above diagram is commutative. Clearly the upper triangle is commutative, so the difference of two maps $M \otimes^{\mathbf{L}}_{\Pi} \Gamma^{\geq p} \to M \otimes^{\mathbf{L}}_{\Pi} \Gamma$ factors through $\operatorname{cone} s = M \otimes^{\mathbf{L}}_{\Pi} W^{\leq p} \in \mathcal{D}^b(\Pi)$. But there is no non-zero map in $\mathcal{D}(\Pi)$ from $X \in \mathcal{D}^b(\Pi)$ to $Y \in \mathcal{D}(\Gamma)$. Indeed, we have $\operatorname{Hom}_{\mathcal{D}(\Pi)}(X, Y) = \operatorname{Hom}_{\mathcal{D}(\Gamma)}(X \otimes^{\mathbf{L}}_{\Pi} \Gamma, Y)$, which is 0 by 2.3.5. This completes the proof of surjectivity.

Finally we see that the functor $\mathcal{C}(\Pi) \to \operatorname{per}\Gamma$ is clearly dense. □

2.4 Proof of main theorems

We apply the result from previous section to prove our main result 2.1.2 of this part which is an explicit structure theorem for CY categories with cluster tilting objects. It will be proved in 2.4.13 in this section.

Let us give a sketch of the proof. Let $\mathcal{T}$ be an algebraic d-CY triangulated category with $d \geq 2$. Suppose that there exists a d-cluster tilting object $T \in \mathcal{T}$. Put

$$X = T \oplus T[-1] \oplus \cdots \oplus T[-(d-2)].$$

Since X has a d-cluster tilting object T as a direct summand, we can apply 2.3.3. For this we need to know its truncated derived endomorphism algebra Π. It turns out that its cohomology which is given by

$$S = \coprod_{i \geq 0} \mathcal{T}(X, X[-i]),$$

is intrinsically formal (2.4.11, 2.4.12), that is, any dg algebra with cohomology S is formal, thus Π is nothing but S regarded as a dg algebra with trivial differentials. We are therefore reduced to study the algebra S. For this we consider the functor

$$F = \mathcal{T}(X, -)\colon \mathcal{T} \longrightarrow \operatorname{mod} H$$

to the category of modules over $H = \operatorname{End}_{\mathcal{T}}(T)$ and compare $\mathcal{T}$ with it. An important observation is that any object in $\mathcal{T}$ has a 2-term resolution by objects from $\operatorname{add} X$ (2.4.1), by which we show that this functor is close enough to being an equivalence; in fact it is full (2.4.4) and induces an equivalence between stable categories (2.4.5). Using this functor F, we prove in 2.4.9 that there is an isomorphism

$$S = T_H \mathcal{T}(X, X[-1])$$

of graded algebras, where the right-hand-side is the tensor algebra of an (H, H)-bimodule $\mathcal{T}(X, X[-1])$. In view of 2.3.1, this description of S as a tensor algebra allows us to write the cluster category $\mathcal{C}(\Pi)$ as an orbit category of $\mathcal{D}^b(\operatorname{mod} H)$.

2.4.1 Proof of 2.1.2

The first and an important step toward 2.4.13 is the observation that any objects in $\mathcal{T}$ has a 2-term resolution by $X = T \oplus \cdots \oplus T[-(d-2)]$. For subcategories $\mathcal{U}, \mathcal{V} \subset \mathcal{T}$, we denote by $\mathcal{U} * \mathcal{V}$ the full subcategory of $\mathcal{T}$ formed by $A \in \mathcal{T}$ such that there exists a triangle $U \to A \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. This operation $*$ is associative by the octahedral axiom. For objects $U, V \in \mathcal{T}$, we will simply write $U * V$ for $\operatorname{add} U * \operatorname{add} V$. Also we denote by $\mathcal{U} \vee \mathcal{V}$ the smallest additive subcategory containing $\mathcal{U}$ and $\mathcal{V}$, and similarly $U \vee V$ for $\operatorname{add} U \vee \operatorname{add} V$.

Proposition 2.4.1. *Let $\mathcal{T}$ be a triangulated category and $T \in \mathcal{T}$ a d-rigid object. Put*

$$X = T \oplus \cdots \oplus T[-(d-3)] \oplus T[-(d-2)],$$
$$Y = T \oplus \cdots \oplus T[-(d-3)],$$

and suppose that $\operatorname{End}_{\mathcal{T}}(Y)$ is hereditary. Then we have

$$T[-(d-2)] * \cdots * T * T[1] = X * X[1].$$

In particular, if $T \in \mathcal{T}$ is d-cluster tilting, then $\mathcal{T} = X * X[1]$.

Proof. The inclusion $\supset$ is easy. Indeed, we have $X * X[1] \subset (T[-(d-2)] * \cdots * T) * (T[-(d-3)] * \cdots * T[1])$, which equals $T[-(d-2)] * \cdots * T[1]$ by d-rigidity of T. We prove the converse inclusion by induction on d. The case $d = 2$ is clear (where we understand $Y = 0$ for $d = 2$), so let $d \geq 3$.

Let $A \in T[-(d-2)] * \cdots * T * T[1]$. Then there is a triangle

$$T_0 \longrightarrow B \longrightarrow A \longrightarrow T_0[1]$$

in $\mathcal{T}$ with $T_0 \in \operatorname{add} T$ and $B \in T[-(d-2)] * \cdots * T$. By induction hypothesis applied to a $(d-1)$-rigid object $T[-1]$, we see that $B \in Y[-1] * Y$, and the same is true for any direct summand of B since $T[-(d-2)] * \cdots * T$ is closed under direct summands ([IYo, 2.1]).

Now write $B = B' \oplus T_1$ with $T_1 \in \operatorname{add} T$ and $\operatorname{add} B' \cap \operatorname{add} T = 0$. We know that $B' \in Y[-1] * Y$. We claim that there exists a triangle

$$Y_0[-1] \longrightarrow B' \longrightarrow Y_1 \longrightarrow Y_0$$

with $Y_0, Y_1 \in \operatorname{add} Y$ which induces a surjection $\mathcal{T}(Y, Y_0[-1]) \twoheadrightarrow \mathcal{T}(Y, B')$. Since we can discuss summandwise, it is enough to consider each indecomposable direct summand B_0 of B'. If $B_0 \in \operatorname{add} Y$, then we have $B_0 \in \operatorname{add} Y[-1]$ since $B_0 \notin \operatorname{add} T$, so we can take $Y_0[-1] = B_0$ and $Y_1 = 0$, which gives a desired triangle. If $B_0 \notin \operatorname{add} Y$, we show that any triangle as above has the desired surjectivity. Indeed, since $\operatorname{End}_{\mathcal{T}}(Y)$ is hereditary, the morphism $Y_1 \to Y_0$, which becomes under the equivalence $\mathcal{T}(Y, -)\colon \operatorname{add} Y \to \operatorname{proj} \operatorname{End}_{\mathcal{T}}(Y)$ the morphism between projective $\operatorname{End}_{\mathcal{T}}(Y)$-modules, is isomorphic to the direct sum of $Y_1' \to Y_0$ inducing an injection $\mathcal{T}(Y, Y_1') \hookrightarrow \mathcal{T}(Y, Y_0)$, and $Y_1'' \to 0$. This forces $Y_1'' \in \operatorname{add} B_0$, so Y_1'' has to be 0 since B_0 is an indecomposable $\notin \operatorname{add} Y$. It follows that $\mathcal{T}(Y, Y_1) \to \mathcal{T}(Y, Y_0)$ is injective, hence $\mathcal{T}(Y, Y_0[-1]) \to \mathcal{T}(Y, B_0)$ is surjective, which finishes the proof of the claim.

By the triangle $T_0 \to B \to A \to T_0[1]$ with $B = B' \oplus T_1$, we can form an octahedral on the left below.

$$\begin{array}{ccccccc}
 & & A'[-1] & = & A'[-1] & & \\
 & & \downarrow & & \downarrow & & \\
\bullet & \longrightarrow & T_0 & \longrightarrow & T_1 & \longrightarrow & \bullet \\
\| & & \downarrow & & \downarrow & & \| \\
\bullet & \longrightarrow & B' & \longrightarrow & A & \longrightarrow & \bullet \\
 & & \downarrow & & \downarrow & & \\
 & & A' & = & A' & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & T_0 & = & T_0 & & \\
 & & \downarrow & & \downarrow & & \\
Y_1[-1] & \to & Y_0[-1] & \longrightarrow & B' & \longrightarrow & Y_1 \\
\| & & \downarrow & & \downarrow & & \| \\
Y_1[-1] & \longrightarrow & C & \longrightarrow & A' & \longrightarrow & Y_1 \\
 & & \downarrow & & \downarrow & & \\
 & & T_0[1] & = & T_0[1] & &
\end{array}$$

On the other hand, taking a triangle for B' in the claim above, the morphism $T_0 \to B'$ can be lifted to $T_0 \to Y_0[-1]$ since $T_0 \in \operatorname{add} Y$. This gives another octahedral as in the right diagram above.

We can now reach the conclusion using these diagrams. Looking at the second octahedral, we have $C \in Y[-1] * T[1]$ by the left vertical triangle, so using the lower horizontal one, we see $A' \in Y[-1] * T[1] * Y$, which equals $Y[-1] * X[1]$ by $T[1] * Y = T[1] \vee Y = \operatorname{add} X[1]$. Now we move to the left octahedral. By the right vertical triangle, we see $A \in T * Y[-1] * X[1]$, which is $X * X[1]$ by $T * Y[-1] = T \vee Y[-1] = \operatorname{add} X$. □

Now we place ourselves in the setup as in 2.4.13: $\mathcal{T}$ is a d-CY triangulated category and $T \in \mathcal{T}$ is d-cluster tilting. Put $X = T \oplus \cdots \oplus T[-(d-2)]$ and $H = \operatorname{End}_{\mathcal{T}}(X)$, which we assume to be hereditary. For the moment we do not need that H is 1-representation infinite. Let us note a complementary observation on the 2-term resolution, although it will not be used later.

Remark 2.4.2. Let $A \in \mathcal{T}$ and let $X_0 \to A$ be any right $(\operatorname{add} X)$-approximation. Then its mapping cocone is in $\operatorname{add} X$.

Proof. We may assume that the approximation is minimal and A is indecomposable. If $A \in \operatorname{add} T[1]$, then the minimal right $(\operatorname{add} X)$-approximation is $0 \to A$, thus its mapping cocone is $A[-1] \in \operatorname{add} X$. If $A \in \operatorname{add} X$, then the minimal approximation is the identity, thus its mapping cocone is 0. Finally suppose that $A \notin \operatorname{add}(X \oplus T[1])$. Then by 2.4.7(1) below, in any triangle $X_1 \to X_0 \xrightarrow{f} A \to X_1[1]$ in 2.4.1, the map f is a right $(\operatorname{add} X)$-approximation. In particular, there exists a right $(\operatorname{add} X)$-approximation of A whose mapping cocone is in $\operatorname{add} X$. It follows that the minimal right $(\operatorname{add} X)$-approximation has the same property. □

Now we consider the functor

$$F = \mathcal{T}(X, -) \colon \mathcal{T} \longrightarrow \operatorname{mod} H \ .$$

Let us give some easy observations which we will often use.

Lemma 2.4.3. (1) *FX and $FX[1]$ are projective H-modules.*

(2) *$FX[d]$ and $FX[d-1]$ are injective H-modules.*

(3) *For $A \in \mathcal{T}$, we have $FA = 0$ if and only if $A \in \operatorname{add} T[1]$.*

(4) *We have a commutative diagram of equivalences*

$$\begin{array}{ccc} \operatorname{add} X & \xrightarrow{F} & \operatorname{proj} H \\ {\scriptstyle [d]}\downarrow & & \downarrow{\scriptstyle -\otimes_H DH} \\ \operatorname{add} X[d] & \xrightarrow{F} & \operatorname{inj} H. \end{array}$$

Proof. We have $FX = H \in \operatorname{proj} H$. Also all the direct summands of $X[1]$ except $T[1]$ are in $\operatorname{add} X$, and $FT[1] = 0$ since T is d-rigid, so $FX[1] \in \operatorname{proj} H$. This proves (1). Using Serre duality we similarly have (2). We see (3) since $T \in \mathcal{T}$ is d-cluster tilting. Finally (4) is clear. □

We next discuss more essential properties of the functor F.

Lemma 2.4.4. *The functor $F = \mathcal{T}(X, -) \colon \mathcal{T} \to \operatorname{mod} H$ is full.*

Proof. We have to show that for each $A, B \in \mathcal{T}$, the functor F induces surjections $F_{A,B} \colon \mathcal{T}(A, B) \to \operatorname{Hom}_H(FA, FB)$. First $F_{A,B}$ is clearly bijective if $A \in \operatorname{add} X$. Note also that it is surjective if $A \in \operatorname{add} X[1]$. Indeed, all the direct summands of $X[1]$ except $T[1]$ lies in $\operatorname{add} X$, so we may assume $A = T[1]$. But we have $FT[1] = 0$ by 2.4.3(3), hence $F_{T[1],B}$ is surjective.

Now we prove $F_{A,B}$ is surjective for all $A \in \mathcal{T}$. By 2.4.1, we have a triangle

$$X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow X_1[1]$$

with $X_0, X_1 \in \operatorname{add} X$. Applying $\mathcal{T}(-, B)$ yields the exact sequence in the first row of the diagram below. On the other hand, applying $F = \mathcal{T}(X, -)$ gives an exact sequence $FX_1 \to FX_0 \to FA \xrightarrow{u} FX_1[1] \xrightarrow{v} FX_0[1]$ in $\operatorname{mod} H$. Now since $FX_0[1]$ is projective by 2.4.3(1) the morphism v is a split epimorphism to its image, hence applying $\operatorname{Hom}_H(-, FB)$ yields a complex in the second row, which is acyclic at the middle term.

$$
\begin{array}{ccccccccc}
\mathcal{T}(X_0[1], B) & \longrightarrow & \mathcal{T}(X_1[1], B) & \longrightarrow & \mathcal{T}(A, B) & \longrightarrow & \mathcal{T}(X_0, B) & \longrightarrow & \mathcal{T}(X_1, B) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \wr & & \downarrow \wr \\
{}_H(FX_0[1], FB) & \longrightarrow & {}_H(FX_1[1], FB) & \longrightarrow & {}_H(FA, FB) & \longrightarrow & {}_H(FX_0, FB) & \longrightarrow & {}_H(FX_1, FB)
\end{array}
$$

Now the right two vertical maps are isomorphisms and the left two are surjective by the starting remark. It easily follows that the middle map is surjective. □

Moreover this functor gives equivalences when passing to ideal quotients.

Proposition 2.4.5. (1) *The functor $F = \mathcal{T}(X, -) \colon \mathcal{T} \to \operatorname{mod} H$ induces stable equivalences*

$$\underline{F} \colon \mathcal{T}/[X \oplus T[1]] \xrightarrow{\simeq} \underline{\operatorname{mod}}\, H \quad \text{and} \quad \overline{F} \colon \mathcal{T}/[T[1] \oplus X[d]] \xrightarrow{\simeq} \overline{\operatorname{mod}}\, H.$$

(2) *We have an isomorphism of functors $F \circ [d-1] \simeq \tau \circ F \colon \mathcal{T} \to \operatorname{mod} H$. Consequently there exists a commutative diagram of equivalences*

$$
\begin{array}{ccc}
\mathcal{T}/[X \oplus T[1]] & \xrightarrow{\underline{F}} & \underline{\operatorname{mod}}\, H \\
{\scriptstyle [d-1]} \downarrow & & \downarrow {\scriptstyle \tau} \\
\mathcal{T}/[T[1] \oplus X[d]] & \xrightarrow{\overline{F}} & \overline{\operatorname{mod}}\, H.
\end{array}
$$

We first prove (1).

Proof of 2.4.5(1). Since $FX = H$, $FT[1] = 0$, and $FX[d] = DH$ (see 2.4.3), the functor F induces the functors $\underline{F}$ and $\overline{F}$ on stable categories.

We only prove that $\underline{F}$ is an equivalence. The statement for $\overline{F}$ is proved dually. We immediately see that this is full by 2.4.4. We show that this is faithful. Let $f \colon A \to B$ be a morphism in $\mathcal{T}$ such that Ff in $\operatorname{mod} H$ factors through a projective H-module. Taking a right $(\operatorname{add} X)$-approximation $X' \to B$, the morphism Ff factors through a surjection $FX' \twoheadrightarrow FB$ as in the left diagram below.

$$
\begin{array}{ccc}
 & & FA \\
 & \swarrow & \downarrow {\scriptstyle Ff} \\
FX' & \twoheadrightarrow & FB,
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
X_1 & \longrightarrow & X_0 & \longrightarrow & A & \longrightarrow & X_1[1] \\
 & & & \swarrow & \downarrow {\scriptstyle f} & \swarrow & \\
 & & X' & \longrightarrow & B & &
\end{array}
$$

Since F is full, the lift $FA \to FX'$ comes from a morphism $A \to X'$ in $\mathcal{T}$. Now take a triangle $X_1 \to X_0 \to A \to X_1[1]$ in 2.4.1 and consider the right diagram above. Since the triangle formed by A, X', and B becomes commutative under $F = \mathcal{T}(X, -)$, the maps $f \colon A \to B$ and $A \to X' \to B$ coincide when precomposing $X_0 \to A$. Therefore the difference of these two maps factors through $X_1[1]$, and f, being the sum of $A \to X' \to B$ and $A \to X_1[1] \to B$, factors through an object in $\operatorname{add}(X \oplus X[1]) = \operatorname{add}(X \oplus T[1])$. This proves faithfulness.

Finally we show that $\underline{F}$ is dense. Let $M \in \operatorname{mod} H$ and consider the projective resolution $0 \to P_1 \to P_0 \to M \to 0$. Let $X_1 \to X_0$ be a morphism in $\operatorname{add} X \subset \mathcal{T}$ corresponding to $P_1 \to P_0$ under the equivalence

$\operatorname{proj} H \simeq \operatorname{add} X$, and complete it to a triangle $X_1 \to X_0 \to A \to FX_1[1]$. Applying F we have an exact sequence below, with M (resp. P) the cokernel of $FX_1 \to FX_0$ (resp. $FX_0 \to FA$).

$$\begin{array}{ccccccccc}
FX_1 & \longrightarrow & FX_0 & & \longrightarrow & FA & & \longrightarrow & FX_1[1] \\
\| & & \| & \searrow & & \nearrow \quad \searrow & & \nearrow & \\
\| & & \| & & M & & P & & \\
\| & & \| & \nearrow & & & & & \\
P_1 & \longrightarrow & P_0 & & & & & &
\end{array}$$

Note that P is a submodule a projective H-module $FX_1[1]$ (2.4.3(1)), hence is projective. Then the short exact sequence $0 \to M \to FA \to P \to 0$ splits, thus $FA \simeq M$ in $\underline{\operatorname{mod}}\, H$. $\square$

This yields a certain desirable behavior of objects under F.

Lemma 2.4.6. *Let $A \in \mathcal{T}$ be an indecomposable object.*

(1) *If $A \not\simeq T[1]$ then FA is indecomposable.*

(2) *$FA \in \operatorname{mod} H$ is projective if and only if $A \in \operatorname{add}(X \oplus T[1])$.*

(3) *$FA \in \operatorname{mod} H$ is injective if and only if $A \in \operatorname{add}(T[1] \oplus X[d])$.*

Proof. Immediate by 2.4.5(1). $\square$

We next discuss a relationship between triangles in $\mathcal{T}$ and short exact sequences in $\operatorname{mod} H$.

Lemma 2.4.7. *Let $A \in \mathcal{T}$ be an indecomposable object and let*

$$X_1 \xrightarrow{f} X_0 \longrightarrow A \longrightarrow X_1[1]$$

be a triangle with $X_0, X_1 \in \operatorname{add} X$. Suppose that $A \notin \operatorname{add}(X \oplus T[1])$.

(1) *The functor $F = \mathcal{T}(X, -)$ induces a short exact sequence*

$$0 \longrightarrow FX_1 \longrightarrow FX_0 \longrightarrow FA \longrightarrow 0\ .$$

(2) *The functor $G = \mathcal{T}(X, -[d])$ induces a short exact sequence*

$$0 \longrightarrow GA[-1] \longrightarrow GX_1 \longrightarrow GX_0 \longrightarrow 0\ .$$

Proof. We only prove (1). We have an exact sequence

$$FA[-1] \longrightarrow FX_1 \xrightarrow{u} FX_0 \longrightarrow FA \xrightarrow{v} FX_1[1]\ .$$

Since H is hereditary, each of the morphisms u and v is a split epimorphism to its image by 2.4.3(1).

We first show that u is injective. Note that f is right minimal. Indeed, if f is not right minimal, then X_1 and $A[-1]$ share a direct summand, which is impossible by our assumption $A[-1] \notin \operatorname{add} X$. It follows that $u = Ff$ is also right minimal, hence injective.

We next show that $v = 0$. Since $A \notin \operatorname{add}(X \oplus T[1])$, we see FA is indecomposable non-projective H-module by 2.4.6(1)(2), thus v has to be 0. $\square$

Now we can prove the second part of 2.4.5.

Proof of 2.4.5(2). Let $A, B \in \mathcal{T}$ without a direct summand in $\operatorname{add}(X \oplus T[1])$, and let $f\colon A \to B$ be a morphism in $\mathcal{T}$. We compute the AR translation of $Ff\colon FA \to FB$ in $\operatorname{mod} H$. Let $X_1 \to X_0 \to A \to X_1[1]$ and $Y_1 \to Y_0 \to B \to Y_1[1]$ be triangles with $X_0, X_1, Y_0, Y_1 \in \operatorname{add} X$ in 2.4.1. By 2.4.7(1), f can be lifted to a morphism of triangles

$$\begin{array}{ccccccc} A[-1] & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & A \\ \downarrow{\scriptstyle f[-1]} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ B[-1] & \longrightarrow & Y_1 & \longrightarrow & Y_0 & \longrightarrow & B. \end{array}$$

Applying $F = \mathcal{T}(X, -)$ it induces a commutative diagram of short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}(X, X_1) & \longrightarrow & \mathcal{T}(X, X_0) & \longrightarrow & \mathcal{T}(X, A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Ff} & & \\ 0 & \longrightarrow & \mathcal{T}(X, Y_1) & \longrightarrow & \mathcal{T}(X, Y_0) & \longrightarrow & \mathcal{T}(X, B) & \longrightarrow & 0 \end{array}$$

again by 2.4.7(1). In view of 2.4.3(4), applying $- \otimes_H DH$ to these sequences gives complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}(X, A[d-1]) & \longrightarrow & \mathcal{T}(X, X_1[d]) & \longrightarrow & \mathcal{T}(X, X_0[d]) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}(X, B[d-1]) & \longrightarrow & \mathcal{T}(X, Y_1[d]) & \longrightarrow & \mathcal{T}(X, Y_0[d]) & \longrightarrow & 0, \end{array}$$

which are exact by 2.4.7(2). This shows $\mathcal{T}(X, A[d-1]) \simeq \tau\mathcal{T}(X, A)$ and $\mathcal{T}(X, B[d-1]) \simeq \tau\mathcal{T}(X, B)$, and $Ff[d-1] = \tau(Ff)$ under these isomorphisms, that is, $F \circ [d-1] \simeq \tau \circ F$ as functors. ☐

From now on we assume that H is 1-representation infinite. The reason we (have to) assume this is the following nice behavior of objects under suspension of $\mathcal{T}$, which allows us to use 2.4.7 quite freely.

Lemma 2.4.8. *The objects $\{T[-l] \mid l \in \mathbb{Z}\}$ do not mutually share a direct summand, that is,* $\operatorname{add} T[-i] \cap \operatorname{add} T[-j] = 0$ *whenever $i \neq j$. In particular, $T[-l]$ does not share a direct summand with $X \oplus T[1]$ whenever $l \geq d-1$.*

Proof. By 2.4.5(2), the functor F takes $T[\leq 0]$ to preprojective H-modules, $T[1]$ to 0, and $T[\geq 2]$ to preinjective H-modules. Therefore the H-modules $FT[-l]$ mutually do not share a summand for all $l \in \mathbb{Z}$ since H is 1-representation infinite. Then the statements follow from 2.4.6. ☐

Now consider the algebra

$$S = \coprod_{i \geq 0} \mathcal{T}(X, X[-i]).$$

Also let $U := \mathcal{T}(X, X[-1])$, which we view as a bimodule over H.

Proposition 2.4.9. *We have an isomorphism $T_H U \simeq S$ of graded algebras.*

Proof. We have to show that $U \otimes_H U \otimes_H \cdots \otimes_H U \simeq \mathcal{T}(X, X[-l])$ for all $l \geq 0$. By induction it is enough to show that the natural map

$$\mathcal{T}(X[-1], X[-l]) \otimes_H \mathcal{T}(X, X[-1]) \longrightarrow \mathcal{T}(X, X[-l])$$

is an isomorphism for all $l \geq 1$. One can check that its dual is isomorphic to the composite of the following natural maps.

$$\begin{aligned} D(\mathcal{T}(X[-1],X[-l])\otimes_H \mathcal{T}(X,X[-1])) &= \mathrm{Hom}_H(\mathcal{T}(X[-1],X[-l]),D\mathcal{T}(X,X[-1])) \\ &= \mathrm{Hom}_H(\mathcal{T}(X[-1],X[-l]),\mathcal{T}(X[-1],X[d])) \\ &= \mathrm{Hom}_H(\mathcal{T}(X,X[-l+1]),\mathcal{T}(X,X[d+1])) \\ &\xleftarrow{F} \mathcal{T}(X[-l+1],X[d+1]) = \mathcal{T}(X[-l],X[d]) \\ &= D\mathcal{T}(X,X[-l]) \end{aligned}$$

Therefore it remains to show that $F = \mathcal{T}(X,-)$ induces bijections

$$\mathcal{T}(T[-l],X[d+1]) \longrightarrow \mathrm{Hom}_H(FT[-l],FX[d+1])$$

for all $l \geq 0$. By 2.4.4, we know that this is surjective. We claim that this is injective. When $0 \leq l \leq d-2$ then $T[-l] \in \operatorname{add} X$ and the assertion is clear. We assume $l \geq d-1$, so that $T[-l]$ do not share a direct summand with $X \oplus T[1]$ by 2.4.8 and we can apply 2.4.7. Let $f\colon T[-l] \to X[d+1]$ be a morphism which is 0 under F. Taking a triangle $X_1 \to X_0 \to T[-l] \to X_1[1]$ in 2.4.1, we see that f is mapped to 0 under the last map in the exact sequence below.

$$\begin{array}{ccccccc} \mathcal{T}(X_0,X[d]) & \xrightarrow{a} & \mathcal{T}(X_1,X[d]) & \longrightarrow & \mathcal{T}(T[-l],X[d+1]) & \longrightarrow & \mathcal{T}(X_0,X[d+1]) \\ & & & & f & \longmapsto & 0 \end{array}$$

Also the first map a is dual to $\mathcal{T}(X,X_1) \to \mathcal{T}(X,X_0)$, which is injective by 2.4.7(1). We conclude that a is surjective, hence $f = 0$ as desired. □

We next give the following property of the bimodule U, which shows that the bimodule U gives a $(d-1)$-st root of the AR translation.

Proposition 2.4.10. *We have isomorphisms*

$$U \otimes^{\mathrm{L}}_H \cdots \otimes^{\mathrm{L}}_H U \simeq U \otimes_H \cdots \otimes_H U \simeq \mathrm{Ext}^1_H(DH,H)$$

in the derived category of (H,H)-bimodules, where the tensor factor is $(d-1)$-times. Therefore the functor $-\otimes^{\mathrm{L}}_H U$ gives an autoequivalence on $\mathcal{D}^b(\operatorname{mod} H)$ whose $(d-1)$-st power is the AR translation.

Proof. Since U is a preprojective H-module by 2.4.5(2) we have the first isomorphism. We prove the second one. By 2.4.9 we have to show $\mathcal{T}(X,X[-d+1]) \simeq \mathrm{Ext}^1_H(DH,H)$. It follows from the dual of 2.4.5(2) that $\mathcal{T}(X,X[-d+1]) \simeq \tau^{-1}H = \mathrm{Ext}^1_H(DH,H)$ in $\operatorname{mod} H$. Naturality of this isomorphism shows that it is compatible with left H-actions, that is, $\mathcal{T}(X,X[-d+1]) \simeq \mathrm{Ext}^1_H(DH,H)$ as (H,H)-bimodules. □

Now assume that the algebra H/J_H is separable over k, which is the case if H is the path algebra of an acyclic quiver.

Lemma 2.4.11. *The graded algebra S is homologically smooth of dimension ≤ 2, that is,* $\operatorname{proj.dim}_{S^e} S \leq 2$.

Proof. Since H is 1-representation infinite and U is a preprojective module such that $-\otimes^{\mathrm{L}}_H U$ gives an autoequivalence on $\mathcal{D}^b(\operatorname{mod} H)$, we see that the derived tensor algebra $T^{\mathrm{L}}_H U := T_H \Psi$, where $\Psi \to U$ is a bimodule projective resolution, has its cohomology concentrated in degree 0, where it is S. This shows that there is a triangle

$$S \otimes^{\mathrm{L}}_H U \otimes^{\mathrm{L}}_H S \longrightarrow S \otimes^{\mathrm{L}}_H S \longrightarrow S \longrightarrow$$

in $\mathcal{D}(\operatorname{Mod} S^e)$. Applying $\mathrm{RHom}_{S^e}(-,S^e)$ we obtain a triangle

$$S^e \otimes^{\mathrm{L}}_{H^e} \mathrm{RHom}_{H^e}(H,H^e)[1] \longrightarrow S^e \otimes^{\mathrm{L}}_{H^e} \mathrm{RHom}_{H^e}(U,H^e)[1] \longrightarrow \mathrm{RHom}_{S^e}(S,S^e)[2] \longrightarrow \ ,$$

cf. [Ke6, 4.8]. Now the first term is concentrated in degree ≤ 0 by $\operatorname{proj.dim}_{H^e} H \leq 1$, and so is the second term since $\operatorname{RHom}_{H^e}(U, H^e)[1] = \operatorname{RHom}_{H^{\mathrm{op}}\otimes H}(U, \operatorname{RHom}_k(DH, H)) = \operatorname{RHom}_{H^{\mathrm{op}}}(U \otimes^{\mathrm{L}}_H DH, H)$, which equals $(U \otimes^{\mathrm{L}}_H U^{\otimes^{\mathrm{L}}_H -(d-1)})^{-1} = U^{\otimes^{\mathrm{L}}_H (d-2)}$ by 2.4.10. We conclude that the third term also lies in degree ≤ 0, hence $\operatorname{proj.dim}_{S^e} S \leq 2$. □

Recall that a dg algebra is *formal* if it is isomorphic to its cohomology in the homotopy category of dg categories. A graded algebra Λ is *intrinsically formal* if any dg algebra with cohomology Λ is formal. We use the following criterion for intrinsic formality using Hochschild cohomology. We denote by [1] the degree shift of graded vector spaces. For a graded bimodule M over Λ, the shifted bimodule $M[1]$ has Λ-actions $a \cdot x \cdot b = (-1)^{\deg a} axb$.

Proposition 2.4.12 ([Ka], see also [ST, 4.7][RW, 1.7])**.** *Let Λ be a graded k-algebra such that $\operatorname{Ext}^i_{\Lambda^e}(\Lambda, \Lambda[2-i]) = 0$ for all $i > 2$. Then Λ is intrinsically formal. In particular if $\operatorname{proj.dim}_{\Lambda^e} \Lambda \leq 2$ then Λ is intrinsically formal.*

We are now ready to prove our main theorem of this part which gives a Morita-type result for cluster categories arising from hereditary algebras.

Theorem 2.4.13. *Let $\mathcal{T}$ be an algebraic d-CY triangulated category with a d-cluster tilting object T. We put $X = T \oplus T[-1] \oplus \cdots \oplus T[-(d-2)]$. Suppose that $H = \operatorname{End}_{\mathcal{T}}(X)$ is 1-representation infinite and H/J_H is separable over k. Set $U = \mathcal{T}(X, X[-1])$, which we view as an (H, H)-bimodule.*

(1) *There exists an isomorphism $U^{\otimes^{\mathrm{L}}_H (d-1)} \simeq \operatorname{Ext}^1_H(DH, H)$ of (H, H)-bimodules.*

(2) *There exists a triangle equivalence $\mathcal{T} \simeq \mathcal{D}^b(\operatorname{mod} H)/ - \otimes^{\mathrm{L}}_H U[1]$.*

Therefore we can write $\mathcal{T} \simeq \mathcal{D}^b(\operatorname{mod} H)/\tau^{-1/(d-1)}[1]$ for $\tau^{-1/(d-1)} := - \otimes^{\mathrm{L}}_H U$.

Proof. We have seen (1) in 2.4.10, so we prove (2). Let Γ be a derived endomorphism ring of X in some dg enhancement of $\mathcal{T}$ and set $\Pi = \Gamma^{\leq 0}$. By 2.4.9 we know the cohomology of Π is isomorphic to the tensor algebra $S = T_H U$. By 2.4.11 and 2.4.12, it is intrinsically formal, thus Π is quasi-isomorphic to S viewed as a dg algebra with $\deg U = -1$ and zero differential; $\Pi = T_H(U[1])$. It follows from the first triangle in the proof of 2.4.11 that Π is homologically smooth. Now, X has a d-cluster tilting object T as a direct summand so we can apply 2.3.3, and therefore there exists a triangle equivalence $\mathcal{T} = \operatorname{per} \Gamma \simeq \mathcal{C}(\Pi)$. By 2.3.1 the cluster category $\mathcal{C}(\Pi)$ is equivalent to $\mathcal{D}^b(\operatorname{mod} H)/ - \otimes^{\mathrm{L}}_H U[1]$. □

2.4.2 Corollaries

Some of the $(d-1)$-st root $\tau^{1/(d-1)}$ stated 2.4.13 arises in a somewhat absurd way, for example, H is a direct product of $(d-1)$ copies of an algebra H', and $\tau^{1/(d-1)}$ on $\mathcal{D}^b(\operatorname{mod} H) = \mathcal{D}^b(\operatorname{mod} H') \times \cdots \times \mathcal{D}^b(\operatorname{mod} H')$ is given by $(L_1, \ldots, L_{d-1}) \mapsto (L_2, \ldots, L_{d-1}, \tau' L_1)$ using the AR translation τ' for H'. The proof of corollaries consists of such an interpretation of $\tau^{1/(d-1)}$. Together with an observation in 2.5.2 below allows us to rewrite the orbit category $\mathcal{D}^b(\operatorname{mod} H)/\tau^{-1/(d-1)}[1]$ in terms of $\mathcal{D}^b(\operatorname{mod} H')$ and ${\tau'}^{-1}$.

Now we look at some consequences of our main theorem 2.4.13. When $d = 2$, this immediately reduces to Keller–Reiten's recognition theorem for non-Dynkin quivers. More generally, using an interpretation of the $(d-1)$-st root $\tau^{-1/(d-1)}$ as above, we can recover its generalization to higher dimension.

Corollary 2.4.14 (cf. [KR2])**.** *Let $\mathcal{T}$ be an algebraic d-CY triangulated category with a d-cluster tilting object T. Suppose that $H' = \operatorname{End}_{\mathcal{T}}(T)$ is 1-representation infinite, $H'/J_{H'}$ is separable over k, and that $\operatorname{Hom}_{\mathcal{T}}(T, T[-i]) = 0$ for $0 < i < d/2$. Then there exists a triangle equivalence $\mathcal{T} \simeq \mathcal{D}^b(\operatorname{mod} H')/\tau^{-1}[d-1]$.*

Proof. By 2.2.8 we have $\mathcal{T}(T, T[-i]) = 0$ for all $1 \leq i \leq d-2$. Then we have the following forms of $H = \operatorname{End}_{\mathcal{T}}(X)$ and the bimodule $U = \mathcal{T}(X, X[-1])$:

$$H = H' \times \cdots \times H', \qquad U = \begin{pmatrix} 0 & H' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H' \\ V & 0 & \cdots & 0 \end{pmatrix},$$

with $V = \mathcal{T}(T, T[-(d-1)])$. Note that by the above descriptions we have $U^{\otimes_H (d-1)} = V \times \cdots \times V$, hence using 2.4.10, we see $V \simeq \mathrm{Ext}^1_{H'}(DH', H')$ as (H', H')-bimodules. Now since $H = H' \times \cdots \times H'$ is 1-representation infinite and H/J_H is separable over k, we can apply 2.4.13, so our triangulated category $\mathcal{T}$ is triangle equivalent to $\mathcal{D}^b(\mathrm{mod}\, H)/-\otimes^{\mathrm{L}}_H U[1]$. By 2.5.2 this is triangle equivalent to $\mathcal{D}^b(\mathrm{mod}\, H')/-\otimes^{\mathrm{L}}_{H'} V[d-1]$, and we deduce by $V \simeq \mathrm{Ext}^1_{H'}(DH', H')$ that this is precisely the d-cluster category of H'. □

The next case $d = 3$ generalizes a theorem of Keller–Murfet–Van den Bergh as well as giving a 3-CY version of Keller–Reiten's theorem above. Again with a suitable vanishing conditions and an interpretation of $(d-1)$-st root, we have the following structure theorem for CY categories of dimension $d = 2n+1$. In this case the $2n$-th root $\tau^{1/2n}$ can generally be reduced to a square root.

Corollary 2.4.15. *Let $\mathcal{T}$ be an algebraic $(2n+1)$-CY triangulated category with a $(2n+1)$-cluster tilting object T such that $\mathrm{End}_{\mathcal{T}}(T)$ is hereditary, and $\mathcal{T}(T, T[-i]) = 0$ for $0 < i < n$. Then, the algebra $H' = \mathrm{End}_{\mathcal{T}}(T \oplus T[-n])$ is hereditary, and there exists a triangle equivalence $\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\, H')/\tau^{-1/2}[n]$ if H' is 1-representation infinite and $H'/J_{H'}$ is separable over k.*

Proof. Recall that $X = T \oplus \cdots \oplus T[-(2n-1)])$ and put $Y = T \oplus T[-n]$. Then by 2.2.12 the algebra $H' = \mathrm{End}_{\mathcal{T}}(Y)$ is hereditary, and by 2.2.11 the algebra $H = \mathrm{End}_{\mathcal{T}}(X)$ and the bimodule $U = \mathcal{T}(X, X[-1])$ has the following form along the decomposition $X = Y \oplus \cdots \oplus Y[-(n-1)]$.

$$H = H' \times \cdots \times H', \qquad U = \begin{pmatrix} 0 & H' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H' \\ V & 0 & \cdots & 0 \end{pmatrix},$$

where the product has n factors, the matrix is $n \times n$, and $V = \mathcal{T}(Y, Y[-n])$. Now the above descriptions show $U^{\otimes_H (2n)} = V^{\otimes_{H'} 2} \times \cdots \times V^{\otimes_{H'} 2}$, hence by 2.4.10 we deduce $V \otimes_{H'} V \simeq \mathrm{Ext}^1_{H'}(DH', H')$ as (H', H')-bimodules so that $-\otimes^{\mathrm{L}}_{H'} V$ can be regarded as a square root $\tau^{-1/2}$ of the AR translation on $\mathcal{D}^b(\mathrm{mod}\, H')$. When H' is 1-representation infinite and $H'/J_{H'}$ is separable over k, so is H, so we can apply 2.4.13 to obtain a triangle equivalence $\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\, H)/-\otimes^{\mathrm{L}}_H U[1]$. By 2.5.2 this is equivalent to $\mathcal{D}^b(\mathrm{mod}\, H')/-\otimes^{\mathrm{L}}_{H'} V[n]$. □

The final case is $d = 4$. Yet again, we state this more generally for even dimensional CY categories.

Corollary 2.4.16. *Let $\mathcal{T}$ be an algebraic $(2n+2)$-CY triangulated category with a $(2n+2)$-cluster tilting object T such that $\mathrm{Hom}_{\mathcal{T}}(T, T[-i]) = 0$ for $0 < i < n$ and $\mathrm{End}_{\mathcal{T}}(T) = k \times \cdots \times k$. Then the algebra $H = \mathrm{End}_{\mathcal{T}}(T \oplus \cdots \oplus T[-2n])$ is hereditary and there exists a triangle equivalence $\mathcal{T} \simeq \mathcal{D}^b(\mathrm{mod}\, H)/\tau^{-1/(2n+1)}[1]$.*

We remark that there is no analogue of interpreting the $(d-1)$-st root in this situation; in general $\tau^{1/(d-1)}$ cannot be made easier. This is because H is in general connected in this case, see 2.2.23.

2.5 Adjusting orbits

We observe that certain interpretation of $(d-1)$-st root of the AR translation as in Section 2.4.2 gives an equivalence of orbit categories, leading to the proof of corollaries. The aim of this section is to prove such equivalences in the level of dg enhancements. The content here is in this way a refinement of [Han1, Appendix A].

Let $\mathcal{A}$ be a dg category and let $F\colon \mathcal{A} \to \mathcal{A}$ be a dg functor. Then we can define the dg orbit category $\mathcal{A}/F$ [Ke2]; it has the same objects as $\mathcal{A}$ and the morphism complex

$$\mathcal{A}/F(L, M) = \mathrm{colim}\left(\coprod_{k \geq 0} \mathcal{A}(F^k L, M) \longrightarrow \coprod_{k \geq 0} \mathcal{A}(F^k L, FM) \longrightarrow \cdots \right).$$

Lemma 2.5.1. *Let $\mathcal{A}$ be a dg category and $\mathcal{B} = \mathcal{A} \times \cdots \times \mathcal{A}$ the n-fold product of $\mathcal{A}$. Suppose that $F\colon \mathcal{A} \to \mathcal{A}$ and $G\colon \mathcal{B} \to \mathcal{B}$ are dg functors such that $G^n = F \times \cdots \times F$ on $\mathcal{B}$ and $\mathcal{B}(G^i(L,0,\dots,0), G^j(M,0,\dots,0)) = 0$ unless $n \mid i-j$. Then the functor $\mathcal{A} \to \mathcal{B}$ given by $L \mapsto (L,0,\dots,0)$ induces a fully faithful functor $\mathcal{A}/F \to \mathcal{B}/G$.*

Proof. Writing $\widetilde{L} = (L,0,\dots,0)$, we have to show $\mathcal{B}/G(\widetilde{L},\widetilde{M}) = \mathcal{A}/F(L,M)$ for each $L, M \in \mathcal{A}$. The left-hand-side is

$$\operatorname{colim}\left(\coprod_{k\geq 0} \mathcal{B}(G^k\widetilde{L},\widetilde{M}) \xrightarrow{G} \coprod_{k\geq 0} \mathcal{B}(G^k\widetilde{L},G\widetilde{M}) \xrightarrow{G} \coprod_{k\geq 0} \mathcal{B}(G^k\widetilde{L},G^2\widetilde{M}) \longrightarrow \cdots \right),$$

which equals

$$\operatorname{colim}\left(\coprod_{k\geq 0} \mathcal{B}(G^k\widetilde{L},\widetilde{M}) \xrightarrow{G^n} \coprod_{k\geq 0} \mathcal{B}(G^k\widetilde{L},G^n\widetilde{M}) \xrightarrow{G^n} \coprod_{k\geq 0} \mathcal{B}(G^k\widetilde{L},G^{2n}\widetilde{M}) \longrightarrow \cdots \right)$$

by cofinality. Moreover, by vanishing of $\mathcal{B}(G^i\widetilde{L},G^j\widetilde{M})$ for $n \nmid i-j$, this is equal to

$$\operatorname{colim}\left(\coprod_{k\geq 0} \mathcal{B}(G^{nk}\widetilde{L},\widetilde{M}) \xrightarrow{G^n} \coprod_{k\geq 0} \mathcal{B}(G^{nk}\widetilde{L},G^n\widetilde{M}) \xrightarrow{G^n} \coprod_{k\geq 0} \mathcal{B}(G^{nk}\widetilde{L},G^{2n}\widetilde{M}) \longrightarrow \cdots \right),$$

hence to

$$\operatorname{colim}\left(\coprod_{k\geq 0} \mathcal{A}(F^kL,M) \xrightarrow{F} \coprod_{k\geq 0} \mathcal{A}(F^kL,FM) \xrightarrow{F} \coprod_{k\geq 0} \mathcal{A}(F^kL,F^2M) \longrightarrow \cdots \right),$$

which is nothing but $\mathcal{A}/F(L,M)$. □

Let us note the following consequence in which form we use.

Proposition 2.5.2. *Let $\mathcal{A}$ be a pretriangulated dg category and $F\colon \mathcal{A} \to \mathcal{A}$ a dg functor inducing an equivalence on $H^0\mathcal{A}$. Let $\mathcal{B} = \mathcal{A} \times \cdots \times \mathcal{A}$ the n-fold product and define $G\colon \mathcal{B} \to \mathcal{B}$ by $(L_1,\dots,L_n) \mapsto (FL_n, L_1,\dots,L_{n-1})$. Then $L \mapsto (L,0,\dots,0)$ gives a quasi-equivalence $\mathcal{A}/F[n] \to \mathcal{B}/G[1]$.*

Proof. Since $(G[1])^n = F[n] \times \cdots \times F[n]$, we can apply 2.5.1 so that we have a fully faithful functor $\mathcal{A}/F[n] \to \mathcal{B}/G[1]$. It remains to show that the induced functor $H^0(\mathcal{A}/F[n]) \to H^0(\mathcal{B}/G[1])$ is dense. This follows from the fact that for each $L \in \mathcal{A}$ and $1 \leq i \leq n$, the object $L[-i+1] \in H^0(\mathcal{A}/F[n])$ is mapped to $(L[-i+1],0,\dots,0)$, which is isomorphic in $H^0(\mathcal{B}/G[1])$ to $(G[1])^{i-1}(L[-i+1],0,\dots,0) = (0,\dots,L,\dots,0)$ with L at the i-th factor. □

Applying 2.5.2 to finite dimensional algebras, we can realize the $(d+n)$-cluster category of a finite dimensional algebra A of global dimension $\leq d$ as the canonical triangulated hull of the orbit category of n-fold product algebra.

Example 2.5.3. Let A be a finite dimensional algebra of global dimension $\leq d$, and $\theta_d \to \operatorname{RHom}_A(DA,A)[d]$ the bimodule projective resolution. Consider the pretriangulated dg category and its dg endofunctor given by

$$\mathcal{A} = \mathcal{C}^b(\operatorname{proj} A), \quad F = -\otimes_A \theta_d\colon \mathcal{A} \to \mathcal{A}.$$

Then, letting $\mathbf{\Pi}_{d+n+1}(A) := T_A(\theta_d[n])$ the $(d+n+1)$-CY completion of A, we have an equivalence

$$\operatorname{per}(\mathcal{A}/F[n]) \xrightarrow{\simeq} \mathcal{C}(\mathbf{\Pi}_{d+n+1}(A)),$$

cf. 2.3.1. Let us now consider the dg orbit category $\mathcal{B}/G[1]$ in 2.5.2. Putting $B = A \times \cdots \times A$, we have

$$\mathcal{B} = \mathcal{A} \times \cdots \times \mathcal{A} \simeq \mathcal{C}^b(\operatorname{proj} B), \quad G = - \otimes_B U \colon \mathcal{B} \to \mathcal{B} \text{ with } U = \begin{pmatrix} 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \\ \theta_d & 0 & \cdots & 0 \end{pmatrix}.$$

Then, define $\mathbf{S} = T_B(U[1])$ so that 2.3.1 gives an equivalence

$$\operatorname{per}(\mathcal{B}/G[1]) \xrightarrow{\simeq} \mathcal{C}(\mathbf{S}).$$

We conclude by 2.5.2 that there exists an equivalence

$$\mathcal{C}(\mathbf{S}) \xrightarrow{\simeq} \mathcal{C}(\mathbf{\Pi}_{d+n+1}).$$

2.6 Combinatorial roots of τ

We give a description of quivers whose derived categories have some roots of the AR translation. Observing that a root of τ in the derived category a quiver Q gives rise to a root of τ as an automorphism of the infinite translation quiver $\mathbb{Z}Q$, we give a necessarily and sufficient condition for Q to have such a root on $\mathbb{Z}Q$.

Throughout this section let Q be a finite acyclic quiver. Note that we do not assume it is connected. Suppose that there exists an autoequivalence F of $\mathcal{D}^b(\operatorname{mod} kQ)$ such that $F^l \simeq \tau^{-1}$ for some $l \geq 1$. We first translate this categorical autoequivalence into a combinatorial one on the infinite translation quiver $\mathbb{Z}Q$ [ASS, Hap]. We say that two quivers are *derived equivalent* if their path algebras are derived equivalent. This is the case if and only if their infinite translation quivers are isomorphic.

Lemma 2.6.1. *Let F be a triangle autoequivalence of $\mathcal{D}^b(\operatorname{mod} kQ)$ such that $F^l \simeq \tau^{-1}$. Then F induces an automorphism F' of the translation quiver $\mathbb{Z}Q$ such that $(F')^l = \tau^{-1}$.*

Proof. The proof depends on the description of AR components of $\mathcal{D}^b(\operatorname{mod} kQ)$ (see [Hap, I.5.6]). Divide the components of Q into the derived equivalence classes $Q_1 \sqcup \cdots \sqcup Q_n$. Then F acts on each derived category $\mathcal{D}^b(\operatorname{mod} kQ_i)$ so we may assume that the components of Q are mutually derived equivalent. Suppose first that each component of Q is Dynkin. Then the AR quiver of $\mathcal{D}^b(\operatorname{mod} kQ)$ is $\mathbb{Z}Q$, thus F gives a desired autoequivalence. We now suppose that Q is non-Dynkin and decompose it into the components $Q_1 \sqcup \cdots \sqcup Q_n$. Then the AR quiver of $\mathcal{D}^b(\operatorname{mod} kQ)$ contains the components $\mathcal{C}_p(Q_i) \simeq \mathbb{Z}Q_i$ each of which is characterized as the component containing $kQ_i[p] \in \mathcal{D}^b(\operatorname{mod} kQ)$. In view of the shape of the AR quiver of $\mathcal{D}^b(\operatorname{mod} kQ)$, the triangle autoequivalence F induces an automorphism of the translation quiver $\bigcup_{p \in \mathbb{Z}, 1 \leq i \leq n} \mathcal{C}_p(Q_i)$. Then there exists a permutation σ on $\{1, \ldots, n\}$ and $p_i \in \mathbb{Z}$ such that the component $\mathcal{C}_0(Q_i)$ is mapped to $\mathcal{C}_{p_i}(Q_{\sigma(i)})$. Let G be the triangle automorphism of $\mathcal{D}^b(\operatorname{mod} kQ) = \mathcal{D}^b(\operatorname{mod} kQ_1) \times \cdots \times \mathcal{D}^b(\operatorname{mod} kQ_n)$ given by $(L_1, \ldots, L_n) \mapsto (L_1[-p_{\sigma^{-1}(1)}], \ldots, L_n[-p_{\sigma^{-1}(n)}])$. Then the composite $F' := G \circ F$ preserves $\mathcal{C}_0(Q_1) \sqcup \cdots \sqcup \mathcal{C}_0(Q_n) \simeq \mathbb{Z}Q$, whose l-th power equals τ^{-1}. $\square$

Recall that a *section* of $\mathbb{Z}Q$ [ASS, VIII.1.2], (also [ABS, 1.7]) is a full subquiver Σ of $\mathbb{Z}Q$ such that

- each τ-orbit of $\mathbb{Z}Q$ intersects Σ exactly once.
- If $x \to y$ is an arrow in $\mathbb{Z}Q$ and $x \in \Sigma$, then $y \in \Sigma$ or $\tau y \in \Sigma$.

If Σ is a section of $\mathbb{Z}Q$ then there is an isomorphism $\mathbb{Z}\Sigma \simeq \mathbb{Z}Q$ of translation quivers [ASS, VIII.1.6]. The following main result in this section shows that we can take a section as an orbit under the root of τ.

Theorem 2.6.2. *Let Q be a finite acyclic quiver and F an automorphism of $\mathbb{Z}Q$ such that $F^l = \tau^{-1}$. Then there exists a full subquiver T of $\mathbb{Z}Q$ such that the full subquiver consisting of $T \cup FT \cup \cdots \cup F^{l-1}T$ forms a section of $\mathbb{Z}Q$. Moreover, for any points $s, t \in T$ and $a > 0$, there is no arrow from $F^a s$ to t.*

Consequently, we can characterize the quivers whose infinite translation quiver $\mathbb{Z}Q$ has an l-th root of τ.

Corollary 2.6.3. *There exists an l-th root of τ^{-1} on $\mathbb{Z}Q$ if and only if Q is derived equivalent to the quiver Q' satisfying the following (a), (b) and (c).*

(a) Q' has l copies $T^{(0)}, T^{(1)}, \dots, T^{(l-1)}$ of a quiver T as a full subquiver.

We denote by F the permutation of the vertices of Q' taking $T^{(i)}$ to $T^{(i+1)}$, where $T^{(l)} := T^{(0)}$.

(b) There are additional arrows $x \to y$ in Q' with $x \in T^{(i)}$, $y \in T^{(j)}$ only if $i < j$.

(c) F extends to an automorphism of the underlying graph of Q'.

For the proof let us introduce the notion of "a section with respect to a root of τ".

Definition 2.6.4. Let F be an automorphism of $\mathbb{Z}Q$ such that $F^l = \tau^{-1}$. An *F-section* of $\mathbb{Z}Q$ is a full subquiver T of $\mathbb{Z}Q$ satisfying the following.

(a) Each F-orbit of $\mathbb{Z}Q$ intersects T exactly once.

(b) If $t \to x$ is an arrow in $\mathbb{Z}Q$ and $t \in T$ then $x \in T \cup FT \cup \cdots \cup F^{l-1}T$ or $\tau x \in T$.

Clearly the condition (b) is equivalent to its dual, which have the following form.

(b') If $x \to t$ is an arrow in $\mathbb{Z}Q$ and $t \in T$ then $x \in T$ or $\tau^{-1}x \in T \cup FT \cup \cdots \cup F^{l-1}T$.

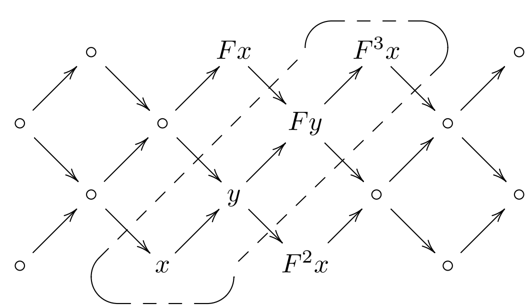
The following observation shows that an F-section gives a desired (τ-)section of $\mathbb{Z}Q$.

Lemma 2.6.5. *Let T be an F-section of $\mathbb{Z}Q$. Then $\Sigma = T \cup FT \cup \cdots \cup F^{l-1}T$ is a section of $\mathbb{Z}Q$.*

Proof. It is easily seen that Σ forms a complete set of representatives of the τ-orbits of $\mathbb{Z}Q$. Let $x \in \Sigma$ and $x \to y$ an arrow in $\mathbb{Z}Q$. We have to show $y \in \Sigma$ or $\tau y \in \Sigma$, that is, $y \in \bigcup_{i=0}^{2l-1} F^iT$. Since $x \in \Sigma$ we have $x \in F^aT$ for some $0 \le a \le l-1$. Then $y \in F^a\Sigma$ or $\tau y \in F^aT$ since T is an F-section. It follows that $y \in \bigcup_{i=a}^{a+l} F^iT \subset \bigcup_{i=0}^{2l-1} F^iT$. $\square$

Construction 2.6.6. We construct an F-section as a subquiver of the standard section Q of $\mathbb{Z}Q$. Let T be the subquiver of Q such that each vertex x of Q can be written uniquely as $x = F^at$ for some $t \in T$ and $a \ge 0$. Thus T is obtained by dividing the vertices of Q into F-orbits, and taking from each orbit the vertices which are written as the smallest powers of F.

Example 2.6.7. Let Q be the quiver of linearly oriented type A_4. Then the infinite translation quiver $\mathbb{Z}Q$ has a square root of τ^{-1}, indeed, consider the automorphism F which "turns up side down", moving slightly to the right.



The standard section Q which is depicted in the dotted line consists of two F-orbits; letting x its source and y the adjacent vertex, the other two vertices are Fy and F^3x. Then the construction says $T = \{x, y\}$.

Proposition 2.6.8. *The quiver T given in 2.6.6 is an F-section of $\mathbb{Z}Q$.*

To prove this we need some observations on the arrows in Q with sources or targets in T.

Lemma 2.6.9. *Let $x \to t$ be an arrow in Q with $t \in T$. Then $x \in T$.*

Proof. Since $x \in Q$ we can write $x = F^a s$ for some $s \in T$ and $a \geq 0$. Then there is an arrow $s \to F^{-a}t$ with $s \in Q$. Since Q is a section we have $F^{-a}t \in Q$ or $F^{-a-l}t = \tau F^{-a}t \in Q$, but by our construction of T, $F^p t \in Q$ for some $p \in \mathbb{Z}$ forces $p \geq 0$, thus we must have $a = 0$, hence $x = s \in T$. □

Lemma 2.6.10. *Let $t \to x$ be an arrow in Q with $t \in T$. Then $x \in T \cup FT \cup \cdots \cup F^{l-1}T$.*

Proof. Since $x \in Q$ we can write $x = F^a s$ with $s \in T$ and $a \geq 0$. We have to show $a \leq l-1$. Consider the arrow $F^{-a}t \to s$. Since $s \in Q$ and Q is a section, we have $F^{-a}t \in Q$ or $\tau^{-1}F^{-a}t \in Q$. If $F^{-a}t \in Q$ then $F^{-a}t \in T$ by 2.6.9, hence $a = 0$. If $\tau^{-1}F^{-a}t$, which equals $F^{l-a}t$, is in Q, we must have $l - a \geq 0$ by the construction of T. It remains to exclude $a = l$. In this case, we have that both s and $x = F^a s = \tau^{-1}s$ lies in Q, which is absurd. □

Proof of 2.6.8. It is clear from the construction that each F-orbits in $\mathbb{Z}Q$ intersects T exactly once. Let $t \to x$ be an arrow in $\mathbb{Z}Q$ with $t \in T$. Since $t \in Q$ and Q is a section, we have $x \in Q$ or $\tau x \in Q$. If $x \in Q$, then $x \in T \cup FT \cup \cdots \cup F^{l-1}T$ by 2.6.10 . If $\tau x \in Q$, then $\tau x \in T$ by 2.6.9. □

Now we are ready to prove the main results of this section.

Proof of 2.6.2. The first assertion follows from 2.6.8 and 2.6.5, so we prove the second one. Let $s, t \in T$ and $F^a s \to t$ an arrow in $\mathbb{Z}Q$. We have to show $a \leq 0$. Since T is an F-section we have $F^a s \in T$ or $\tau^{-1}F^a s \in T \cup FT \cup \cdots \cup F^{l-1}T$. If $F^a s \in T$ then $a = 0$ since T intersects each F-orbit only once. Similarly if $\tau^{-1}F^a s \in T \cup FT \cup \cdots \cup F^{l-1}T$ then comparing the exponent of F, we must have $0 \leq a + l \leq l-1$, thus $a \leq -1$. □

For a pair x, y in a quiver we denote by $\{x \to y\}$ (resp. $\{x - y\}$) the set of arrows from x to y (resp. unoriented edges between x and y).

Proof of 2.6.3. We first show the "only if" part. Suppose $\mathbb{Z}Q$ has an l-th root F of τ^{-1}. Then by 2.6.2 there exists a subquiver T of Q such that $Q' := T \cup FT \cup \cdots \cup F^{l-1}T$ is a section of $\mathbb{Z}Q$. We claim that this Q' has the desired properties. Letting $T^{(i)}$ be the full subquiver of Q' consisting of the vertices from F^iT we have (a). Also the second assertion in 2.6.2 shows (b). Now we turn to (c). For each point $t \in T$ we write $t^{(i)}$ the corresponding point in $T^{(i)}$. We have to show $\sharp\{s^{(i)} - t^{(j)}\} = \sharp\{s^{(i+1)} - t^{(j+1)}\}$, where the superscripts are read modulo l. We may assume $i < j$, and the assertion is clear if $j < l-1$. If $j = l-1$, then we have $\sharp\{s^{(i)} - t^{(l-1)}\} = \sharp\{F^i s - F^{l-1}t\} = \sharp\{F^{i+1}s - F^l t\} = \sharp\{t - F^{i+1}s\}$ since $F^l = \tau^{-1}$, and the last term equals $\sharp\{t - s^{(i+1)}\}$.

We next show the "if" part. We may assume $Q = Q'$, satisfying (a), (b), and (c). Define the automorphism on the set of vertices of $\mathbb{Z}Q$ by taking $T^{(i)}$ to $T^{(i+1)}$ for $0 \leq i < l-1$, and $T^{(l-1)}$ to $\tau^{-1}T^{(0)}$, via F. It is easily seen that this extends to the automorphism of $\mathbb{Z}Q$ whose l-th power is τ^{-1}. □

We end this section with a complete list of connected Dynkin quivers whose infinite translation quivers have roots of τ.

Example 2.6.11. Let Q be a connected Dynkin quiver and $l \geq 2$. Then $\mathbb{Z}Q$ *has an l-th root of τ if and only if $l = 2$ and Q is of even type A*. Indeed, suppose that $\mathbb{Z}Q$ has an l-th root of τ. We use that the underlying graph of Q has a free action of $\mathbb{Z}/l\mathbb{Z}$ (2.6.3(c)). We can exclude type D and E since it has only one trivalent node. Then Q has to be of type A, which has exactly two univalent nodes. This forces $l = 2$ and thus the number of vertices to be even. Conversely if Q is of even type A, then the automorphism of $\mathbb{Z}Q$ which turns up-side-down as in 2.6.7 gives a square root of τ.

2.7 Application: Calabi-Yau reduction of cluster categories

Let us start with a reduction process, called *Calabi-Yau reduction*, of triangulated categories, which yields a smaller CY triangulated category from a given one.

Theorem 2.7.1 ([IYo, Section 4])**.** *Let $\mathcal{T}$ be a d-CY triangulated category and $\mathcal{P} \subset \mathcal{T}$ a functorially finite d-rigid subcategory. Put $\mathcal{Z} = \{X \in \mathcal{T} \mid \mathcal{T}(P, X[i]) = 0$ for all $P \in \mathcal{P}$ and $0 < i < d\}$.*

(1) *The additive quotient $\mathcal{U} := \mathcal{Z}/[\mathcal{P}]$ has a natural structure of a triangulated category, which is d-CY.*

(2) *The projection $\mathcal{Z} \to \mathcal{U}$ induces a bijection between the set of d-cluster tilting subcategories of $\mathcal{T}$ containing $\mathcal{P}$, and the set of d-cluster tilting subcategories of $\mathcal{U}$.*

(3) *The projection $\mathcal{Z} \to \mathcal{U}$ preserves AR $(d+2)$-angles, that is, if $\mathcal{C} \subset \mathcal{T}$ be a d-cluster tilting subcategory containing $\mathcal{P}$, then for each indecomposable $C \in \mathcal{Z} \setminus \mathcal{P}$ the image of the AR $(d+2)$-angle at C in $\mathcal{C}$ is the AR $(d+2)$-angle at C in $\mathcal{U}$.*

We apply this CY reduction to an important class of CY triangulated categories with cluster tilting objects, namely the cluster categories of finite dimensional algebras. Let A be a finite dimensional algebra which is ν_d*-finite* [Iy4, Am], that is, we have $\operatorname{gl.dim} A \leq d$ and $\operatorname{Hom}_{\mathcal{D}(A)}(A, \nu_d^{-i} A) = 0$ for almost all $i \in \mathbb{Z}$ where $\nu = - \otimes^{\mathbf{L}}_A DA$ and $\nu_d = \nu \circ [-d]$. Then the *d-cluster category* $\mathcal{C}_d(A)$ of A is the cluster category of the $(d+1)$-CY completion $\mathbf{\Pi}_{d+1}(A)$ of A, which is also the triangulated hull of the orbit category $\mathcal{D}^b(\operatorname{mod} A)/\nu_d$;

$$\mathcal{D}^b(\operatorname{mod} A)/\nu_d \hookrightarrow \mathcal{C}_d(A) = \mathcal{C}(\mathbf{\Pi}_{d+1}(A)).$$

It is a Hom-finite d-CY triangulated category with a d-cluster titling obejct $A \in \mathcal{C}_d(A)$ [Am, Gu], whose endomorphism algebra is the $(d+1)$-preprojective algebra $\Pi_{d+1}(A)$ [IO] of A.

Recall that an idempotent e is *stratifying* if $A \to A/(e)$ the induced functor $\mathcal{D}^b(\operatorname{mod} A/(e)) \to \mathcal{D}^b(\operatorname{mod} A)$ on the derived categories is fully faithful. It is known [IYa1, Ke6] that the CY reduction of the cluster category $\mathcal{C}_d(A)$ with respect to a stratifying idempotent $e \in A$ is the cluster category $\mathcal{C}_d(A/(e))$. In this section, we apply our main result to give a description of the CY reduction by a *non*-stratifying idempotent, and in particular observe that it is not necessarily the cluster category of the quotient algebra.

Applying 2.4.15 to a CY reduction of the 3-cluster category gives the following result.

Proposition 2.7.2. *Let A be a finite dimensional algebra which is ν_3-finite and Π its 4-preprojective algebra. Suppose $e \in A$ is idempotent such that $\Pi/(e)$ is hereditary. Consider the CY reduction $\mathcal{U}$ of $\mathcal{C}_3(A)$ with respect to eA.*

(1) *The algebra $H = \operatorname{End}_{\mathcal{U}}(A \oplus A[-1])$ is hereditary.*

(2) *When H is 1-representation infinite and H/J_H is separable over k, there is triangle equivalence $\mathcal{T} \simeq \mathcal{D}^b(\operatorname{mod} H)/\tau^{-1/2}[1]$.*

In the rest of this subsection let A be a finite dimensional algebra of global dimension ≤ 2, and $\mathcal{C}$ its 3-cluster category. Then the 4-preprojective algebra $\mathcal{C}(A, A)$ of A is just A. Let us describe the hereditary algebra H explicitly in this case. For this we give a general description of AR 5-angles at A in $\mathcal{C}$. We start with an easy computation.

Lemma 2.7.3. *We have $\mathcal{C}(A, A[-1]) = \operatorname{Ext}^2_A(DA, A)$.*

Proof. This is easily seen by $\mathcal{C}(A, A[-1]) = \coprod_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}(A)}(A, \nu_3^{-i} A[-1])$, in which only $i = 1$ survives. $\square$

Let P be an indecomposable direct summand of A and S the corresponding simple A-module. We let

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0$$
$$0 \longrightarrow S \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow 0$$

the minimal projective and injective resolutions of S, thus $P_0 = P$ and $I^0 = \nu P_0$. Clearly the AR 5-angle in $\operatorname{add} A \subset \mathcal{C}$ at P is of the form

$$\begin{array}{ccccccccc} & & \nu^{-1}I^1 & \longrightarrow & Q & \longrightarrow & P_1 & & \\ & \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow & \\ \nu^{-1}I^0 & & & \bullet & & X & & & P_0 \end{array}$$

for some $Q \in \operatorname{add} A$ and $X \in \mathcal{C}$. In fact we can also determine Q.

Lemma 2.7.4. *We have* $Q = P_2 \oplus \nu^{-1}I^2$.

Proof. Since $Q \to X$ is a minimal right $(\operatorname{add} A)$-approximation in $\mathcal{C}$, the morphism $\mathcal{C}(A, Q) \to \mathcal{C}(A, X)$ is a projective cover, thus we have to determine the A-module $\mathcal{C}(A, X)$. By the first triangle we have an exact sequence

$$\mathcal{C}(A, P_1[-1]) \longrightarrow \mathcal{C}(A, P_0[-1]) \longrightarrow \mathcal{C}(A, X) \longrightarrow \mathcal{C}(A, P_1) \longrightarrow \mathcal{C}(A, P_0) .$$

The rightmost map is isomorphic to $P_1 \to P_0$, thus has projective kernel P_2. On the other hand, the leftmost map is isomorphic by 2.7.3 to $\operatorname{Ext}^2_A(DA, P_1) \to \operatorname{Ext}^2_A(DA, P_0)$, thus its cokernel is $\operatorname{Ext}^2_A(DA, S)$ since $\operatorname{Ext}^2_A(DA, -)$ is right exact by $\operatorname{gl.dim} A \leq 2$. Therefore we obtain a split short exact sequence

$$0 \longrightarrow \operatorname{Ext}^2_A(DA, S) \longrightarrow \mathcal{C}(A, X) \longrightarrow P_2 \longrightarrow 0 .$$

Now by the minimal injective resolution of S we have a projective presentation

$$\nu^{-1}I^1 \longrightarrow \nu^{-1}I^2 \longrightarrow \operatorname{Ext}^2_A(DA, S) \longrightarrow 0 ,$$

which gives the projective cover $\nu^{-1}I^2$ of $\operatorname{Ext}^2_A(DA, S)$. We conclude that the projective cover of $\mathcal{C}(A, X) = P_2 \oplus \operatorname{Ext}^2_A(DA, S)$ is $P_2 \oplus \nu^{-1}I^2$. □

Now we describe the algebra H. Suppose A is given by a quiver with relations; $A = kQ/I$ and let R be a minimal set of relations generating I. Recall that the minimal projective and injective resolutions of simple modules are controlled by the arrows and the relations ([BIRSm, Section 3]), precisely, we have exact sequences

$$0 \longrightarrow \coprod_{\rho\colon c \to a} e_c A \longrightarrow \coprod_{\alpha\colon b \to a} e_b A \longrightarrow e_a A \longrightarrow S_a \longrightarrow 0 ,$$

where the first (resp. second) sum runs over the relations ρ in R (resp. paths α) ending at a, and dually

$$0 \longrightarrow S_a \longrightarrow D(Ae_a) \longrightarrow \coprod_{\alpha\colon a \to b} D(Ae_b) \longrightarrow \coprod_{\rho\colon a \to c} D(Ae_c) \longrightarrow 0 .$$

We therefore obtain the following explicit description of H.

Theorem 2.7.5. *Let $A = kQ/(R)$ be a finite dimensional algebra of global dimension ≤ 2 given by quiver Q and a minimal set of relations R. Let $e \in A$ be an idempotent such that $A/(e)$ is hereditary, and consider the CY reduction $\mathcal{U}$ of the 3-cluster category of A with respect to eA.*

(1) *The algebra $H = \operatorname{End}_{\mathcal{U}}(A \oplus A[-1])$ is the path algebra of the quiver $\widetilde{Q}$ obtained as follows.*

- *(i) Consider two copies of the quiver $\underline{Q}$ obtained from Q by deleting the vertices corresponding to e. For each vertex $a \in \underline{Q}$ we denote by a' the corresponding vertex in the other copy.*
- *(ii) For each relation $a \to b$ in R with $a, b \in \underline{Q}$, add arrows $a \to b'$ and $b \to a'$.*

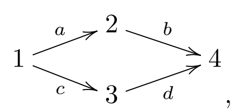
(2) *If each connected component of* $\widetilde{Q}$ *is non-Dynkin, then there exists a triangle equivalence*

$$\mathcal{U} \simeq \mathcal{D}^b(\operatorname{mod} k\widetilde{Q})/\tau^{-1/2}[1].$$

Proof. The triangle equivalence is given in 2.7.2 so we only have to prove (1). Clearly the quiver of H has two copies of Q as a subquiver, and by 2.2.13 the arrows between these copies are given by the middle terms of the AR 5-angles in $\mathcal{U}$. By 2.7.1(3) they are the image of the ones in $\mathcal{C}$, whose middle terms are as in 2.7.4. We then deduce the result by the remark on resolutions of simple modules. □

The following example shows that a CY reduction of a usual 3-cluster category of a finite dimensional algebra is *not* a usual 3-cluster category; it involves a square root of the AR translation.

Example 2.7.6. Let A be the algebra presented by the following quiver with relations, which has global dimension 2.

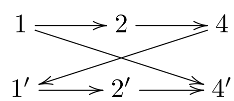


$dc = 0.$

Let $e = e_3$ be the idempotent corresponding to the vertex 3. Then the quotient $A/(e_3)$ is hereditary. Letting $\mathcal{U}$ be the CY reduction of the 3-cluster category $\mathcal{C}_3(A)$ with respect to e_3A, we have a triangle equivalence

$$\mathcal{U} \simeq \mathcal{D}^b(\operatorname{mod} H)/\tau^{-1/2}[1],$$

where H is the path algebra of the following quiver of type $\widetilde{A_5}$.



Note that $\mathcal{D}^b(\operatorname{mod} H)/\tau^{-1/2}[1]$ has infinitely many indecomposables, so it cannot be equivalent to the cluster category of the quotient algebra $A/(e_3)$.

2.8 Application: Singularity categories of truncated skew group algebras

Throughout this section let k be an algebraically closed field of characteristic 0, $S = k[x_0, \ldots, x_d]$ the polynomial ring, and G a finite subgroup of $\mathrm{SL}_{d+1}(k)$, which acts naturally on S. Let $R = S^G$ the invariant ring and $S * G$ the skew group algebra, that is, the vector space $S \otimes_k kG$ with multiplication $(s \otimes g)(t \otimes h) = sg(t) \otimes gh$ for $s, t \in S$ and $g, h \in G$. The following result gives examples of CY triangulated categories with cluster tilting objects.

Theorem 2.8.1 ([IYo, 10.1], [AIR, 2.3])**.** *Let* e *be an idempotent of* $\Gamma = S * G$ *such that* $\Gamma/(e)$ *is finite dimensional. Put* $\Lambda = e\Gamma e$.

(1) *The algebra* Λ *is a symmetric* R*-order, thus the singularity category* $\underline{\mathrm{CM}}\,\Lambda$ *is* d*-CY.*

(2) *The* Λ*-module* $T = \Gamma e$ *is Cohen-Macaulay and is* d*-cluster tilting in* $\mathrm{CM}\,\Lambda$*.*

(3) *We have* $\underline{\mathrm{End}}_\Lambda(T) = \Gamma/(e)$*.*

Recall that AR sequences in $\operatorname{add}\Gamma e \subset \underline{\mathrm{CM}}\,\Lambda$ can be obtained from the Koszul complex [Iy2]. Let V be the vector space with basis $\{x_0, \ldots, x_d\}$, so that we have the Koszul complex of S:

$$0 \longrightarrow S \otimes_k \textstyle\bigwedge^{d+1} V \longrightarrow S \otimes_k \bigwedge^{d} V \longrightarrow \cdots \longrightarrow S \otimes_k V \longrightarrow S \longrightarrow k \longrightarrow 0\,.$$

Let $M = eS \in \operatorname{mod} R$ be the direct summand of S corresponding to the idempotent $e \in \Gamma = \operatorname{End}_R(S)$. Applying $\operatorname{Hom}_R(M, -)$ yields an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M,S) \otimes_k \bigwedge^{d+1} V \longrightarrow \cdots \longrightarrow \operatorname{Hom}_R(M,S) \otimes_k V \longrightarrow \operatorname{Hom}_R(M,S) \ ,$$

in $\operatorname{CM}\Lambda$, which is a direct sum of d-almost split sequences and d-fundamental sequences in $\operatorname{add}\Gamma e$. Passing these to the stable category $\underline{\operatorname{CM}}\,\Lambda$ gives the AR $(d+2)$-angles.

In what follows we use a standard notation for elements in $\operatorname{GL}_n(k)$: for $a, a_1, \ldots, a_n \in \mathbb{Z}$, we write $1/a(a_1, \ldots, a_n)$ for $\operatorname{diag}(\zeta^{a_1}, \ldots, \zeta^{a_n}) \in \operatorname{GL}_n(k)$ for a fixed primitive a-th root of unity ζ.

Lemma 2.8.2. *Let $G \subset \operatorname{SL}_{d+1}(k)$ is the cyclic subgroup generated by $1/n(a_0, \ldots, a_d)$.*

(1) *The McKay quiver Q of G has vertices $\mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$ and arrows $x_i \colon j \to j + a_i$.*

(2) *The skew group algebra Γ is presented by Q with commutativity relations.*

We view Q as an $\{x_0, \ldots, x_d\}$-colored quiver. Let $e \in \Gamma$ be an idempotent and let I be the corresponding subset of the vertices of Q.

(3) $\Gamma/(e)$ *is presented by the quiver $Q \setminus I$ obtained from Q by removing the vertices in I.*

(4) $\Gamma/(e)$ *is finite dimensional hereditary if and only if there are no cycles consisting of arrows of a single color, nor composable arrows of different colors in $Q \setminus I$.*

Proof. (1) is obvious, (2) is well-known (see e.g. [CMT, 2.8][BSW, 4.1]), and (3) is then clear. We verify (4). If $Q \setminus I$ has no cycles of a single color and no composable arrows of different colors, then there are no relations on $Q \setminus I$ since Γ has only commutativity relations. Also the assumption implies that $Q \setminus I$ is acyclic. Therefore $\Gamma/(e)$ is finite dimensional and hereditary. Conversely, if $Q \setminus I$ has a cycle of a single color, then $\Gamma/(e)$ is infinite dimensional. If $Q \setminus I$ has composable arrows of different colors, then it has a relation. Indeed, let $a \xrightarrow{x_i} b \xrightarrow{x_j} c$ be a subquiver of $Q \setminus I$ with $i \neq j$. Then Q has a subquiver of the form

$$\begin{array}{ccccc} & & b & & \\ & \overset{x_i}{\nearrow} & & \overset{x_j}{\searrow} & \\ a & & & & c \\ & \underset{x_j}{\searrow} & & \underset{x_i}{\nearrow} & \\ & & b' & & \end{array} .$$

If $b' \in Q \setminus I$ then $Q \setminus I$ has a commutativity relation, and if $b' \notin Q \setminus I$ then $Q \setminus I$ has a zero relation. $\square$

Let $G = \langle 1/n(a_0, \ldots, a_d) \rangle \subset \operatorname{SL}_{d+1}(k)$ and its McKay quiver Q as in 2.8.2 above. We naturally regard $a_i \in \mathbb{Z}/n\mathbb{Z}$. Let M_j be the indecomposable direct summand of S corresponding to the vertices j of Q. Then the construction of almost split sequences and a computation of exterior powers show that it is given for M_j by the exact sequence

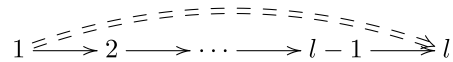
$$M_j \longrightarrow \coprod_{0 \le i_1 < \ldots < i_d \le d} M_{j - a_{i_1} - \cdots - a_{i_d}} \longrightarrow \cdots \longrightarrow \coprod_{0 \le i_1 < i_2 \le d} M_{j - a_{i_1} - a_{i_2}} \longrightarrow \coprod_{0 \le i \le d} M_{j - a_i} \longrightarrow M_j \ .$$

Now let $e \in \Gamma$ is an idempotent as in 2.8.1, I the corresponding subset of the vertives of Q, and M the corresponding direct summand of S. Putting $N_j = \operatorname{Hom}_R(M, M_j) \in \operatorname{CM}\Lambda$, the AR sequence at N_j in $\operatorname{add}\Gamma e \subset \underline{\operatorname{CM}}\,\Lambda$ is

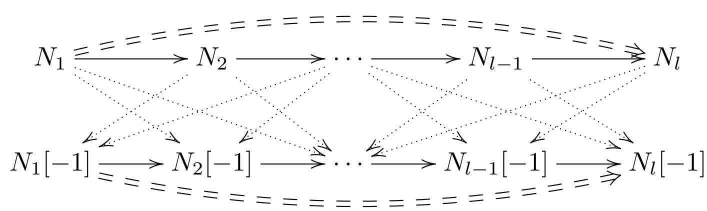
$$N_j \longrightarrow \coprod_{\substack{0 \le i_1 < \ldots < i_d \le d \\ j - a_{i_1} - \cdots - a_{i_d} \notin I}} N_{j - a_{i_1} - \cdots - a_{i_d}} \longrightarrow \cdots \longrightarrow \coprod_{\substack{0 \le i_1 < i_2 \le d \\ j - a_{i_1} - a_{i_2} \notin I}} N_{j - a_{i_1} - a_{i_2}} \longrightarrow \coprod_{\substack{0 \le i \le d \\ j - a_i \notin I}} N_{j - a_i} \longrightarrow N_j \ .$$

Our first example is 3-CY category with a 3-cluster tilting object with hereditary endomorphism ring, so that we can apply 2.4.15.

Example 2.8.3. Let $S = k[x, y, z, w]$ and G the cyclic subgroup of $\mathrm{SL}_4(k)$ generated by $1/n(a_0, a_1, a_2, a_3)$ with $0 \le a_i \le n-1$. Assume that $a_0 = 1$, and $a_1, a_2, a_3 > 1$. Moreover pick an integer $l > 0$ such that $l-1 \le a_1, a_2, a_3 \le n-l$ and let $e := e_0 + e_{l+1} + \cdots + e_{n-1} = 1 - (e_1 + \cdots + e_l)$. Putting $M = M_0 \oplus M_{l+1} \oplus \cdots \oplus M_{n-1}$ we have $\Lambda = e\Gamma e = \mathrm{End}_R(M)$, and by 2.8.1 the stable category $\underline{\mathrm{CM}}\,\Lambda$ is 3-CY and $N = N_1 \oplus \cdots \oplus N_l \in \underline{\mathrm{CM}}\,\Lambda$ with $N_i := \mathrm{Hom}_R(M, M_i)$ is 3-cluster tilting. By 2.8.1(4) we see that $\underline{\mathrm{End}}_\Lambda(N) = \Gamma/(e)$ is hereditary whose quiver as below, where the number of arrows $1 \to l$ is the number of a_1, a_2, a_3 which equals $l-1$.



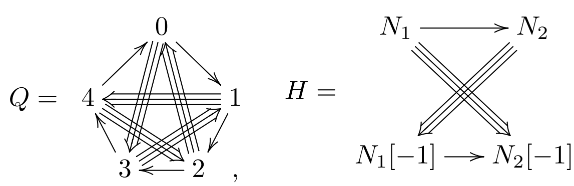
The middle term of the AR 5-angle at N_j is $\coprod_{0 \le i_1 < i_2 \le 3} N_{j - a_{i_1} - a_{i_2}}$, thus equals $N_l^{m_{jl}}$ with m_{jl} the number of pairs (a_{i_1}, a_{i_2}) such that $a_{i_1} + a_{i_2} = j - l$. Note that $m_{jl} = m_{lj}$ by 2.2.13. By 2.2.12 the algebra $H = \underline{\mathrm{End}}_\Lambda(N \oplus N[-1])$ is hereditary, which by 2.2.13 is presented by the quiver which looks as below, with m_{jl} arrows from N_j to $N_l[-1]$.



We obtain by 2.4.15 a triangle equivalence

$$\underline{\mathrm{CM}}\,\Lambda \xrightarrow{\simeq} \mathcal{D}^b(\mathrm{mod}\, H)/\tau^{-1/2}[1] \, .$$

Let us list a specific example of the McKay quiver Q, and the hereditary algebra $H = \underline{\mathrm{End}}_\Lambda(N \oplus N[-1])$ for $G = \langle 1/5(1, 3, 3, 3) \rangle$ and $l = 2$.



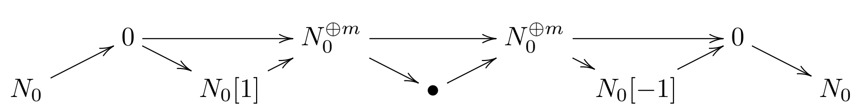
The next example is a 4-CY category $\underline{\mathrm{CM}}\,\Lambda$ with a semisimple 4-cluster tilting object N. In this case the algebra $\underline{\mathrm{End}}_\Lambda(N \oplus N[-1])$ is clearly hereditary, so we can apply 2.4.16.

Example 2.8.4. Let $G \subset \mathrm{SL}_5(k)$ be the cyclic subgroup generated by $1/n(a_0, a_1, a_2, a_3, a_4)$ with $1 \le a_i \le n-1$.

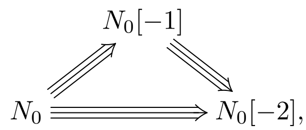
(1) Let e_0 a primitive idempotent of $\Gamma = S * G$, $e = 1 - e_0$, and M_0 (resp. M) the R-module summand of S corresponding to e_0 (resp. e). Then the almost split sequence at M_0 in $\mathrm{CM}\, R$ is given by

$$M_0 \longrightarrow \coprod_{0 \le i \le 4} M_{a_i} \longrightarrow \coprod_{0 \le i_1 < i_2 \le 4} M_{a_{i_1} + a_{i_2}} \longrightarrow \coprod_{0 \le i_1 < i_2 \le 4} M_{-a_{i_1} - a_{i_2}} \longrightarrow \coprod_{0 \le i \le 4} M_{-a_i} \longrightarrow M_0 \, .$$

Therefore letting m be the number of pairs (a_{i_1}, a_{i_2}) with $0 \le i_1 < i_2 \le 4$ such that $a_{i_1} + a_{i_2} = n$, the AR 6-angle in $\underline{\mathrm{CM}}\,\Lambda$ at $N_0 = \mathrm{Hom}_R(M, M_0)$, where $\Lambda = e\Gamma e$, is

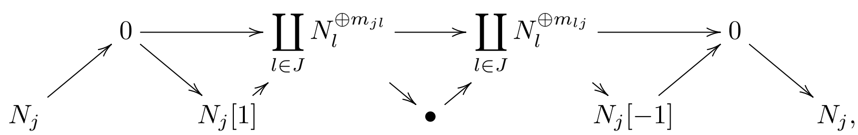


It follows from 2.2.22 that $H = \underline{\mathrm{End}}_\Lambda(N_0 \oplus N_0[-1] \oplus N_0[-2])$ is the path algebra of the following quiver of type $\widetilde{A_2}$ with m-fold arrows, and from 2.4.16 that there exists a triangle equivalence below.



$$\underline{\mathrm{CM}}\,\Lambda \simeq \mathcal{D}^b(\mathrm{mod}\, H)/\tau^{-1/3}[1].$$

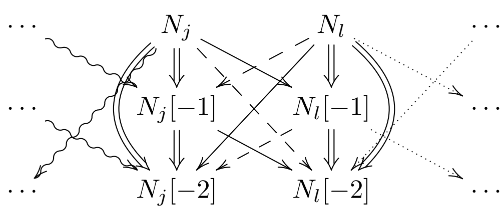
(2) More generally, let I_0 be a subset of $\mathbb{Z}/n\mathbb{Z}$ such that for each $j, j' \in I_0$ we have $j - j' \neq a_i$ for any $0 \leq i \leq 4$. Then letting e be the idempotent of Γ corresponding to $J := \mathbb{Z}/n\mathbb{Z} \setminus I_0$, the algebra $\Gamma/(e)$ is semisimple. Also the AR 6-angle in $\mathrm{add}\,\Gamma e \subset \underline{\mathrm{CM}}\,\Lambda$ is



where m_{jl} is the number of pairs (a_{i_1}, a_{i_2}) with $0 \leq i_1 < i_2 \leq 4$ such that $a_{i_1} + a_{i_2} = l - j$ in $\mathbb{Z}/n\mathbb{Z}$. By 2.2.22 the algebra $H = \underline{\mathrm{End}}_\Lambda(N \oplus N[-1] \oplus N[-2])$ is the path algebra of the quiver given as follows.

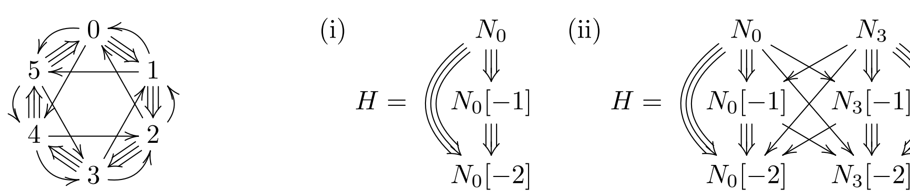
- vertices are $J \times \{0, -1, -2\}$,
- m_{jl} arrows from $(j, 0)$ to $(l, -1)$ and from $(j, -1)$ to $(l, -2)$, and m_{lj} arrows from $(j, 0)$ to $(l, -2)$.

The quiver of H looks as in the following picture, where the same kind of arrows indicates that there are same number of arrows. By 2.4.16 we have a triangle equivalence below.



$$\underline{\mathrm{CM}}\,\Lambda \simeq \mathcal{D}^b(\mathrm{mod}\, H)/\tau^{-1/3}[1]$$

We end this thesis with some specific examples for $G = \langle 1/6(1,1,1,4,5) \rangle$ and (i) $I_0 = \{0\}$, (ii) $I_0 = \{0, 3\}$, which describe the McKay quiver of G and the hereditary algebra $H = \underline{\mathrm{End}}_\Lambda(N \oplus N[-1] \oplus N[-2])$ in each case.



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