Renormalized values and desingularized values of the multiple zeta function
(多重ゼータ関数の繰込み値と特異点解消値)

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Abstract

It is known that there are infinitely many singularities of multiple zeta functions and almost all negative integer points are located in their singularities. This causes an indeterminacy of the special values there. It is a fundamental problem to give a nice definition of special values of multiple zeta functions at non-positive integers. As one approach, Guo and Zhang ([GZ]) used the renormalization procedure which arises from the Hopf algebraic approach to perturbative quantum field theory by Connes and Kreimer to introduce the renormalized values as a special values of multiple zeta functions. Other type of renormalized values were introduced by Manchon and Paycha ([MP]) and Ebrahimi-Fard, Manchon and Singer ([EMS16], [EMS17]).

While, Furusho, Komori, Matsumoto and Tsumura ([FKMT17a]) proposed the desingularization method to resolve all singularities of multiple zeta functions, and by using this method, they introduced the desingularized multiple zeta functions, which can be analytically continued to the whole space as entire functions. The desingularized values are defined to be the special values of desingularized multiple zeta functions at integer points. They gave explicit formulae of these special values in terms of Bernoulli numbers. The aim of this thesis is to give a concrete relationships among desingularized values and various renormalized values.

In Chapter 1, We recall the definition of multiple zeta functions and the desingularized multiple zeta functions introduced by Furusho, Komori, Matsumoto and Tsumura, and we explain various properties of the desingularized multiple zeta functions. In Chapter 2, we consider the renormalized values introduced by Ebrahimi-Fard, Manchon and Singer, and the relationship between desingularized values in [FKMT17a] and renormalized values in [EMS17]. In §2.1, we review the definition of the Hopf algebra $\mathcal{H}_0$, which is used to define these renormalized values in §2.2. In §2.3, we give explicit formulae for the coproduct Δ_0 of the Hopf algebra $\mathcal{H}_0$, which are used to prove the recurrence formulae among renormalized values in §2.4. In §2.5, by using these recurrence formulae, we prove an equivalence between desingularized values and renormalized ones and by using this equivalence, we give the explicit formula of the renormalized values in terms of Bernoulli numbers. In Chapter 3, we consider functional relations of desingularized multiple zeta functions. In §3.1, we prove the product formulae of desingularized multiple zeta functions at non-positive integer points. In §3.2 and §3.3, we prove functional relations of desingularized multiple zeta functions as a generalization of that product formulae at non-positive integer points in two different ways. Chapter 4 is on a problem posed by Singer which is on a comparison between the renormalized values of shuffle type and of harmonic type. We settle the problem by giving a universal presentation of the renormalized values introduced by Ebrahimi-Fard, Manchon and Singer as finite linear combinations of any renormalized values of harmonic type.

This doctor thesis is based on three papers [Ko19], [Ko20a], [Ko20b], and current research. In precise, the section Chapter 2 is based on [Ko19], and the sections §3.1 and §3.2 are based on [Ko20a], and the section §3.3 is based on [Ko20b], and the section Chapter 4 is the ongoing research.

Contents

Chapter 0

Introduction

In §0.1, we recall the definitions of multiple zeta values, multiple zeta functions and their various properties and we explain the renormalization method of multiple zeta functions and desingularization method of ones. In §0.2, we briefly describe our main results in the thesis (cf. [Ko19], [Ko20a], [Ko20b]).

0.1 Multiple zeta values and multiple zeta functions

Multiple zeta values (MZVs for short) are real numbers defined by

$$\zeta(k_1,\dots,k_r) := \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \tag{0.1.1}$$

for $k_1,\dots,k_{r-1}\in\mathbb{N}$ and $k_r\in\mathbb{N}_{\geq 2}$ (the condition $k_r\in\mathbb{N}_{\geq 2}$ is required for the convergence of the above series). When $r=1$, the equation (0.1.1) recovers Riemann zeta values $\zeta(k_1)$. It is said that, in 1776, Euler ([Eu]) firstly introduced MZVs for $r=1,2$. More than 200 years later after Euler, these values reappeared in Ecalle's paper [Ec] in 1981. In 1990s, MZVs is also focused by Hoffman ([H92]) and by Zagier ([Za]). It is known that MZVs satisfy various relations ([HS], [IKZ], [Ka], [O] etc.). Especially, the double shuffle relations are one of the most important relations among MZVs, which are obtained as combinations of harmonic products and shuffle products. Harmonic product comes from the decomposition of the summation of definition (0.1.1). While, MZVs have an iterated integral representation, and shuffle product is arisen from their integral representations. The integral expressions enable us to regard it as periods of certain motives ([DG], [Go] and [T]). MZVs appear in calculations of the Kontsevich invariant in knot theory ([CDM] and [LM]). MZVs are also related to mathematical physics in [BK95] and [BK97]. They are explained in [Zh16].

Multiple zeta functions (MZFs for short) are several variables complex analytic functions with $s_1,\dots,s_r\in\mathbb{C}$ in a certain region of convergence defined by

$$\zeta(s_1,\dots,s_r) := \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{s_1}\cdots m_r^{s_r}},$$

which recovers the Riemann zeta function when $r=1$ and evaluates to MZVs at positive integer points. In the early 2000s, Zhao ([Zh00]) and Akiyama, Egami and Tanigawa ([AET]) independently showed that MZFs can be meromorphically continued to $\mathbb{C}^r$. It is shown that almost all non-positive integer points are located in the above singularities, in particular the special values of MZFs are indeterminate in all cases except for $\zeta(-k)$ with $k \in \mathbb{N}_0$ and $\zeta(-k_1, -k_2)$ with $k_1, k_2 \in \mathbb{N}_0$ and $k_1 + k_2$ odd. It is regarded to be a fundamental issue to find a nice definition of the special values "$\zeta(-k_1, \dots, -k_r)$" for $k_1, \dots, k_r \geq 0$ of MZFs at negative integer points.

Connes and Kreimer ([CK]) started a Hopf algebraic approach to the renormalization procedure in perturbative quantum field theory. A fundamental tool in their work is the algebraic Birkhoff decomposition (cf. Theorem 2.2.1). By applying this decomposition to a certain Hopf algebra related to the harmonic product of MZVs, Guo and Zhang ([GZ]) introduced the renormalized values which satisfy the harmonic-type product formulae. Later, by Manchon and Paycha ([MP]) and by Ebrahimi-Fard, Manchon and Singer ([EMS16]), different renormalized values which satisfy the harmonic-type product were introduced. Ebrahimi-Fard, Manchon and Singer ([EMS17]) also introduced another renormalized value (cf. Definition 2.2.4) which satisfies the "shuffle-type" product (see Proposition 2.2.6 for detail) by using a certain Hopf algebra (cf. (2.1.3)) related to the $\mathbb{Q}$-algebra.

On the other hand, in order to resolve all infinitely many singularities of MZFs, Furusho, Komori, Matsumoto and Tsumura ([FKMT17a]) introduced the desingularized multiple zeta functions $\zeta_r^{\mathrm{des}}(s_1, \dots, s_r)$ (cf. Definition 1.2.1). They showed that these functions can be analytically continued to $\mathbb{C}^r$ (see Proposition 1.2.2) and can be represented by a finite linear combination of MZFs (see Proposition 1.2.3). In addition, they showed that desingularized values $\zeta_r^{\mathrm{des}}(-k_1, \dots, -k_r)$ for $k_1, \dots, k_r \geq 0$ (cf. Definition 1.2.7) are explicitly given by Bernoulli numbers (see Proposition 1.2.8).

0.2 Special values of MZFs at non-positive integer points

In this section, we explain our main results in this thesis. We denote the renormalized values in [EMS17] by $\zeta_{\mathrm{EMS}}(-k_1, \dots, -k_r)$, and we define the generating function $Z_{\mathrm{EMS}}(t_1, \dots, t_r)$ of the renormalized values $\zeta_{\mathrm{EMS}}(-k_1, \dots, -k_r)$ and the generating function $Z_{\mathrm{FKMT}}(t_1, \dots, t_r)$ of the desingularized values $\zeta_r^{\mathrm{des}}(-k_1, \dots, -k_r)$:

$$Z_{\mathrm{EMS}}(t_1, \dots, t_r) := \sum_{k_1, \dots, k_r = 0}^{\infty} \frac{(-t_1)^{k_1} \cdots (-t_r)^{k_r}}{k_1! \cdots k_r!} \zeta_{\mathrm{EMS}}(-k_1, \dots, -k_r), \tag{0.2.1}$$

$$Z_{\mathrm{FKMT}}(t_1, \dots, t_r) := \sum_{k_1, \dots, k_r = 0}^{\infty} \frac{(-t_1)^{k_1} \cdots (-t_r)^{k_r}}{k_1! \cdots k_r!} \zeta_r^{\mathrm{des}}(-k_1, \dots, -k_r). \tag{0.2.2}$$

Then their relationship is given as follows:

Theorem 0.2.1 (Theorem 2.5.1, [Ko19, Theorem 3.5])**.** *For $n \in \mathbb{N}$, we have*

$$Z_{\mathrm{EMS}}(t_1, \dots, t_n) = \prod_{i=1}^{n} \frac{1 - e^{-t_i - \cdots - t_n}}{t_i + \cdots + t_n} \cdot Z_{\mathrm{FKMT}}(-t_1, \dots, -t_n).$$

This theorem says that the desingularized values $\zeta_r^{\mathrm{des}}(-k_1, \dots, -k_r)$ and the renormalized values $\zeta_{\mathrm{EMS}}(-k_1, \dots, -k_r)$ are equivalent, that is, they can be represented by finite linear combinations of each other. Because the desingularized values are explicitly calculated in [FKMT17a], as a corollary of the above theorem, we obtain the following:

Corollary 0.2.2 (Corollary 2.5.5, [Ko19, Corollary 3.9])**.** *For $n \in \mathbb{N}$, we have*

$$Z_{\text{EMS}}(t_1, \ldots, t_n) = \prod_{i=1}^{n} \frac{(t_i + \cdots + t_n) - (e^{t_i + \cdots + t_n} - 1)}{(t_i + \cdots + t_n)(e^{t_i + \cdots + t_n} - 1)}.$$

In terms of $\zeta_{\text{EMS}}(-k_1, \ldots, -k_n)$, the above equation is reformulated to

$$\zeta_{\text{EMS}}(-k_1, \ldots, -k_n) = (-1)^{k_1 + \cdots + k_n} \sum_{\substack{\nu_{1i} + \cdots + \nu_{ii} = k_i \\ 1 \le i \le n}} \prod_{i=1}^{n} \frac{k_i!}{\prod_{j=i}^{n} \nu_{ij}!} \frac{B_{\nu_{ii} + \cdots + \nu_{in} + 1}}{\nu_{ii} + \cdots + \nu_{in} + 1}$$

for $k_1, \ldots, k_n \geq 0$. Here, the Bernoulli number B_n $(n \geq 0)$ is defined by

$$\frac{x}{e^x - 1} := \sum_{n \geq 0} \frac{B_n}{n!} x^n. \tag{0.2.3}$$

We note that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$.

By using the equivalence (Theorem 0.2.1) between the desingularized values $\zeta_r^{\text{des}}(-k_1, \ldots, -k_r)$ and the renormalized values $\zeta_{\text{EMS}}(-k_1, \ldots, -k_r)$ and by using a “shuffle-type” product of renormalized values, we see that the desingularized values satisfy a “shuffle-type” product which is the same as the one of the renormalized values:

Theorem 0.2.3 (Theorem 3.1.3, [Ko20a, Theorem 3.3])**.** *For $p, q \in \mathbb{N}$ and $k_1, \ldots, k_p$, $l_1, \ldots, l_q \in \mathbb{N}_0$, we have*

$$\begin{aligned} &\zeta_p^{\text{des}}(-k_1, \ldots, -k_p)\zeta_q^{\text{des}}(-l_1, \ldots, -l_q) \\ &= \sum_{\substack{i_b + j_b = l_b \\ i_b, j_b \geq 0 \\ 1 \leq b \leq q}} \prod_{a=1}^{q} (-1)^{i_a} \binom{l_a}{i_a} \zeta_{p+q}^{\text{des}}(-k_1, \ldots, -k_{p-1}, -k_p - i_1 - \cdots - i_q, -j_1, \ldots, -j_q). \end{aligned}$$

Especially, when $p = q = 1$, the above equation yields the following:

$$\zeta_2^{\text{des}}(-k, -l) = \sum_{\substack{i+j=l \\ i,j \geq 0}} \binom{l}{i} \zeta_1^{\text{des}}(-k - i)\zeta_1^{\text{des}}(-j)$$

for $k, l \geq 0$. While, in [FKMT17b], the following proposition was shown:

Proposition 0.2.4 ([FKMT17b, Proposition 4.3])**.** *For $s \in \mathbb{C}$ and $N \in \mathbb{N}_0$, we have*

$$\zeta_2^{\text{des}}(s, -N) = \sum_{\substack{i+j=N \\ i,j \geq 0}} \binom{N}{i} \zeta_1^{\text{des}}(s - i)\zeta_1^{\text{des}}(-j).$$

We generalize the above proposition to the following:

Theorem 0.2.5 (Theorem 3.2.8, [Ko20a, Proposition 4.8])**.** *For $s_1, \ldots, s_{r-1} \in \mathbb{C}$ and $k \in \mathbb{N}_0$, we have*

$$\zeta_r^{\text{des}}(s_1, \ldots, s_{r-1}, -k) = \sum_{\substack{i+j=k \\ i,j \geq 0}} \binom{k}{i} \zeta_{r-1}^{\text{des}}(s_1, \ldots, s_{r-2}, s_{r-1} - i)\zeta_1^{\text{des}}(-j).$$

As a generalization of Theorem 0.2.3 and Theorem 0.2.5, we obtain the following functional relations of desingularized MZFs.

Theorem 0.2.6 (Theorem 3.3.7, [Ko20b, Theorem 2.7])**.** *For* $s_1, \dots, s_p \in \mathbb{C}$ *and* $l_1, \dots, l_q \in \mathbb{N}_0$*, we have*

$$\begin{aligned}&\zeta_p^{\mathrm{des}}(s_1, \dots, s_p)\zeta_q^{\mathrm{des}}(-l_1, \dots, -l_q)\\ &= \sum_{\substack{i_b+j_b=l_b \\ i_b,j_b\geq 0 \\ 1\leq b\leq q}} \prod_{a=1}^{q} (-1)^{i_a}\binom{l_a}{i_a}\zeta_{p+q}^{\mathrm{des}}(s_1, \dots, s_{p-1}, s_p - i_1 - \cdots - i_q, -j_1, \dots, -j_q).\end{aligned}$$

In our last chapter, we will treat other type of renormalized values, that is, renormalized values of harmonic type (cf. Definition 4.1.5) and consider the following problem posed by Singer in the end of [S].

Problem 0.2.7 (Problem 4.1.6)**.** Which renormalized value of harmonic type has an explicit relationship with the renormalized values $\zeta_{\mathrm{EMS}}(-k_1, \dots, -k_r)$ (defined in Definition 2.2.4)?

The following theorem settle the above problem.

Theorem 0.2.8 (Theorem 4.2.7)**.** *For* $r \geq 1$*, we have*

$$Z_{\mathrm{EMS}}(t_1, \dots, t_r) = \sum_{i=1}^{r} \sum_{\sigma\in\mathcal{P}(r,i)} Z_*\left(u_{\sigma^{-1}(1)}, \dots, u_{\sigma^{-1}(i)}\right). \tag{0.2.4}$$

Here, $Z_{\mathrm{EMS}}(t_1, \dots, t_r)$ *is defined by* (0.2.1) *and* $Z_*(t_1, \dots, t_r)$ *is defined in Definition 4.2.4, and for* $r, i \in \mathbb{N}$ *with* $i \leq r$*, the set* $\mathcal{P}(r, i)$ *(see §4 for detail) is defined by*

$$\mathcal{P}(r, i) := \{\sigma : \{1, \dots, r\} \twoheadrightarrow \{1, \dots, i\}\},$$

and, for $\sigma \in \mathcal{P}(r, i)$*, the symbol* $u_{\sigma^{-1}(k)}$ *is defined by*

$$u_{\sigma^{-1}(k)} := \sum_{n\in\sigma^{-1}(k)} u_n$$

for $u_i := t_i + \cdots + t_r$ $(1 \leq i \leq r)$*.*

We denote the renormalized values introduced in [GZ] and [MP] by

$$\zeta_{\mathrm{GZ}}(-k_1, \dots, -k_r) \quad \text{and} \quad \zeta_{\mathrm{MP}}(-k_1, \dots, -k_r),$$

for $k_1, \dots, k_r \in \mathbb{Z}_{\leq 0}$, and define their generating functions by

$$Z_{\mathrm{GZ}}(t_1, \dots, t_r) := \sum_{k_1,\dots,k_r=0}^{\infty} \frac{(-t_1)^{k_1}\cdots(-t_r)^{k_r}}{k_1!\cdots k_r!}\zeta_{\mathrm{GZ}}(-k_1, \dots, -k_r),$$

$$Z_{\mathrm{MP}}(t_1, \dots, t_r) := \sum_{k_1,\dots,k_r=0}^{\infty} \frac{(-t_1)^{k_1}\cdots(-t_r)^{k_r}}{k_1!\cdots k_r!}\zeta_{\mathrm{MP}}(-k_1, \dots, -k_r).$$

Corollary 0.2.9. *The equation* (0.2.4) *holds for* $Z_* = Z_{\mathrm{GZ}}$ *and* Z_{MP}*. Hence, the renormalized values* $\zeta_{\mathrm{EMS}}(-k_1, \dots, -k_r)$ *can be represented by a finite linear combination of either* $\zeta_{\mathrm{GZ}}(-k_1, \dots, -k_r)$ *or* $\zeta_{\mathrm{GZ}}(-k_1, \dots, -k_r)$*.*

Chapter 1

Multiple zeta functions and desingularization

In this section, we review the definition of multiple zeta functions in §1.1, and the definition of the desingularized multiple zeta functions introduced in [FKMT17a] and we also explain some properties of the desingularized MZFs in §1.2.

1.1 Multiple zeta functions and their meromorphic continuations

Multiple zeta functions (MZFs for short) are several variables complex analytic functions defined by

$$\zeta(s_1,\ldots,s_r) := \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{s_1}\cdots m_r^{s_r}}, \tag{1.1.1}$$

which give MZVs at positive integer points. When $r=1$, the equation (1.1.1) is nothing but the Riemann zeta function $\zeta(s_1)$. This functions converge absolutely in the region

$$\left\{(s_1,\ldots,s_r)\in\mathbb{C}^r \mid \Re(s_{r-k+1}+\cdots+s_r)>k,\ 1\le k\le r\right\}.$$

In 2001, Akiyama, Egami and Tanigawa proved the following:

Theorem 1.1.1 ([AET, Theorem 1])**.** *MZFs can be meromorphically continued to $\mathbb{C}^r$, and the set of all singularities of MZFs is explicitly given by*

$$\begin{aligned} &s_r = 1,\\ &s_{r-1}+s_r = 2,1,0,-2,-4,\ldots,\\ &s_{r-k+1}+\cdots+s_r = k-n \quad (3\le k\le r,\ n\in\mathbb{N}_0). \end{aligned} \tag{1.1.2}$$

We review other meromorphic continuation of MZFs by Matsumoto ([Mat]). His idea is based on the Mellin-Barnes integral formula:

$$\Gamma(s)(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s+z)\Gamma(-z)\lambda^z dz,$$

where $\Re(s) < c < 0$ and the path of integration is the vertical line $\Re(z) = c$ (see [WW] for detail). By this integral formula, he obtains the following formula ([Mat, (3.7)]):

$$\begin{aligned}&\zeta_r(s_1, \ldots, s_r)\\&= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z)\zeta(-z)dz,\end{aligned} \tag{1.1.3}$$

for $\Re(s_j) > 1$ $(1 \leq j \leq r)$, $-\Re(s_r) < c < 0$ and the path of integration is the same as the one of the Mellin-Barnes integral formula. By this formula, it is shown in [Mat] that MZFs $\zeta_r(s_1, \ldots, s_r)$ can be meromorphically continued to $\mathbb{C}^r$ by induction on r.

Remark 1.1.2. By using (1.1.3), in §3.2 and §3.3, we consider functional relations of the desingularized multiple zeta functions introduced in the next subsection §1.2.

Remark 1.1.3. By (1.1.2), we see that almost all non-positive integer points are located in the above singularities, so the special values of MZFs are indeterminate in all cases except for $\zeta(-k)$ for $k \in \mathbb{N}_0$ and $\zeta(-k_1, -k_2)$ for $k_1, k_2 \in \mathbb{N}_0$ and $k_1 + k_2$ odd. It is regarded to be a fundamental problem to give a nice definition of "$\zeta(-k_1, \ldots, -k_r)$" for $k_1, \ldots, k_r \in \mathbb{N}_0$. Regarding this, several approaches have been proposed (for instance [EMS16], [EMS17], [FKMT17a], [GZ], [MP]).

1.2 Desingularized MZFs

In this subsection, we review the definition of desingularized MZF and the two properties: that the desingularized MZF can be analytically continued to $\mathbb{C}^r$ as an entire function (Proposition 1.2.2), and that it can be represented by a finite "linear" combination of MZFs (Proposition 1.2.3). We also explain some of its properties which are used in our later sections.

We consider the following generating function[1] $\tilde{\mathfrak{H}}_r(t_1, \ldots, t_r; c) \in \mathbb{C}[[t_1, \ldots, t_r]]$ (cf. [FKMT17a, Definition 1.9]):

$$\begin{aligned}\tilde{\mathfrak{H}}_r(t_1, \ldots, t_r; c) &:= \prod_{j=1}^{r} \left(\frac{1}{\exp\left(\sum_{k=j}^{r} t_k\right) - 1} - \frac{c}{\exp\left(c\sum_{k=j}^{r} t_k\right) - 1} \right)\\&= \prod_{j=1}^{r} \left(\sum_{m=1}^{\infty} (1 - c^m) B_m \frac{\left(\sum_{k=j}^{r} t_k\right)^{m-1}}{m!} \right)\end{aligned}$$

for $c \in \mathbb{R}$.

Definition 1.2.1 ([FKMT17a, Definition 3.1])**.** For non-integral complex numbers $s_1, \ldots, s_r$, *desingularized MZF* $\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)$ is defined by

$$\begin{aligned}&\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)\\&:= \lim_{\substack{c \to 1\\ c \in \mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^r} \prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k} - 1)\Gamma(s_k)} \int_{\mathcal{C}^r} \tilde{\mathfrak{H}}_r(t_1, \ldots, t_r; c) \prod_{k=1}^{r} t_k^{s_k - 1} dt_k.\end{aligned} \tag{1.2.1}$$

Here $\mathcal{C}$ is the path consisting of the positive real axis (top side), a circle around the origin of radius ε (sufficiently small), and the positive real axis (bottom side).

[1] It is denoted by $\tilde{\mathfrak{H}}_n((t_j); (1); c)$ in [FKMT17a].

One of the remarkable properties of desingularized MZF is that it is an entire function, i.e., the equation (1.2.1) is well-defined as an analytic function by the following proposition.

Proposition 1.2.2 ([FKMT17a, Theorem 3.4])**.** *The function $\zeta_r^{\rm des}(s_1,\dots,s_r)$ can be analytically continued to $\mathbb{C}^r$ as an entire function in $(s_1,\dots,s_r)\in\mathbb{C}^r$ by the following integral expression:*

$$\zeta_r^{\rm des}(s_1,\dots,s_r)=\prod_{k=1}^r\frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)}$$
$$\cdot\int_{\mathcal{C}^n}\prod_{j=1}^r\lim_{\substack{c\to1\\ c\in\mathbb{R}\setminus\{1\}}}\frac{1}{1-c}\left(\frac{1}{\exp\left(\sum_{k=j}^r t_k\right)-1}-\frac{c}{\exp\left(c\sum_{k=j}^r t_k\right)-1}\right)\prod_{k=1}^r t_k^{s_k-1}dt_k.$$

For indeterminates u_j and v_j $(1\le j\le r)$, we set

$$\mathcal{G}_r(u_1,\dots,u_r;v_1,\dots,v_r):=\prod_{j=1}^r\left\{1-(u_jv_j+\cdots+u_rv_r)(v_j^{-1}-v_{j-1}^{-1})\right\}\tag{1.2.2}$$

with the convention $v_0^{-1}:=0$, and we define the set of integers $\{a^r_{\boldsymbol{l},\boldsymbol{m}}\}$ by

$$\mathcal{G}_r(u_1,\dots,u_r;v_1,\dots,v_r)=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^r\\ |\boldsymbol{m}|=0}}a^r_{\boldsymbol{l},\boldsymbol{m}}\prod_{j=1}^r u_j^{l_j}v_j^{m_j}.\tag{1.2.3}$$

Here, $|\boldsymbol{m}|:=m_1+\cdots+m_r$.
Another remarkable property of desingularized MZF is that the function is given by a finite "linear" combination of shifted MZFs, i.e.,

Proposition 1.2.3 ([FKMT17a, Theorem 3.8])**.** *For $s_1,\dots,s_r\in\mathbb{C}$, we have the following equality between meromorphic functions of the complex variables $(s_1,\dots,s_r)$:*

$$\zeta_r^{\rm des}(s_1,\dots,s_r)=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^r\\ |\boldsymbol{m}|=0}}a^r_{\boldsymbol{l},\boldsymbol{m}}\left(\prod_{j=1}^r(s_j)_{l_j}\right)\zeta(s_1+m_1,\dots,s_r+m_r).\tag{1.2.4}$$

Here, $(s)_k$ is the Pochhammer symbol, that is, for $k\in\mathbb{N}$ and $s\in\mathbb{C}$ $(s)_0:=1$ and $(s)_k:=s(s+1)\cdots(s+k-1)$.

We give some example of the above proposition:

Example 1.2.4. When $r=1$, we have

$$\zeta_1^{\rm des}(s)=(1-s)\zeta(s).$$

Here, $\zeta(s)$ is the Riemann zeta function, this function has a simple pole at $s=1$. Hence, $\zeta_1^{\rm des}(s)$ is entire on $\mathbb{C}$. When $r=2$, we have

$$\begin{aligned}\zeta_2^{\rm des}(s_1,s_2)=&(s_1-1)(s_2-1)\zeta(s_1,s_2)+s_2(s_2+1-s_1)\zeta(s_1-1,s_2+1)\\&-s_2(s_2+1)\zeta(s_1-2,s_2+2).\end{aligned}$$

This summation causes cancellations of all singularities of MZFs in each terms. Hence, $\zeta_2^{\rm des}(s_1,s_2)$ is entire on $\mathbb{C}^2$.

By the above example, we can determine the special values of $\zeta_2^{\mathrm{des}}(s_1, s_2)$ at all integer points. Actually, in [FKMT17b], the following propositions are proven

Proposition 1.2.5 ([FKMT17b, Proposition 4.3])**.** *For $s \in \mathbb{C}$ and $N \in \mathbb{N}_0$, we have*

$$\zeta_2^{\mathrm{des}}(s, -N) = \sum_{k=0}^{N} \binom{N}{k} (k+1)(s-N+k-1)\zeta(s-N+k)\zeta(-k). \tag{1.2.5}$$

Proposition 1.2.6 ([FKMT17b, Proposition 4.5])**.** *For $s \in \mathbb{C}$ and $N \in \mathbb{N}_0$, we have*

$$\begin{aligned}
&\zeta_2^{\mathrm{des}}(-N, s) \\
&= \frac{(s-N-3)(s-N-2)}{(N+3)(N+2)}\zeta(s-N-1) \\
&+ \sum_{k=0}^{N+1} \frac{(ks+N-k+2)(s-N+k-1)}{N+2}\binom{N+2}{k}\zeta(s-N+k)\zeta(-k) \\
&- (N+1)(s-1)\zeta(s)\zeta(-N) + s(s+1+N)\zeta(s+1)\zeta(-N-1) + (s-N-1)\zeta(s-N).
\end{aligned}$$

In §3.2, §3.3, we will give a generalization of the above equation (1.2.5) by two different methods.

We consider the special values of desingularized MZFs at non-positive integer points.

Definition 1.2.7. For $k_1, \ldots, k_r \in \mathbb{N}_0$, the *desingularized value* $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r) \in \mathbb{C}$ is defined to be the special value of desingularized MZF $\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)$ at $(s_1, \ldots, s_r) = (-k_1, \ldots, -k_r)$.

The generating function $Z_{\mathrm{FKMT}}(t_1, \ldots, t_r)$ of $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r)$ is explicitly calculated as follows.

Proposition 1.2.8 ([FKMT17a, Theorem 3.7])**.** *We have*

$$Z_{\mathrm{FKMT}}(t_1, \ldots, t_r) = \prod_{i=1}^{r} \frac{(1-t_i-\cdots-t_r)e^{t_i+\cdots+t_r}-1}{(e^{t_i+\cdots+t_r}-1)^2}.$$

In terms of $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r)$ for $k_1, \ldots, k_r \in \mathbb{N}_0$, the above equation is reformulated to

$$\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r) = (-1)^{k_1+\cdots+k_r} \sum_{\substack{\nu_{1i}+\cdots+\nu_{ii}=k_i \\ 1\le i\le r}} \prod_{i=1}^{r} \frac{k_i!}{\prod_{j=i}^{r}\nu_{ij}!} B_{\nu_{ii}+\cdots+\nu_{ir}+1}.$$

By the above proposition we have the following recurrence formula:

Corollary 1.2.9.

$$Z_{\mathrm{FKMT}}(t_1, \ldots, t_r) = Z_{\mathrm{FKMT}}(t_2, \ldots, t_r) \cdot Z_{\mathrm{FKMT}}(t_1+\cdots+t_r) \quad (r \in \mathbb{N}). \tag{1.2.6}$$

In terms of $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r)$, the equation (1.2.6) is reformulated to

$$\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r) = \sum_{\substack{i_2+j_2=k_2 \\ \vdots \\ i_r+j_r=k_r}} \prod_{a=2}^{r} \binom{k_a}{i_a} \zeta_{r-1}^{\mathrm{des}}(-i_2, \ldots, -i_r)\zeta_1^{\mathrm{des}}(-k_1-j_2-\cdots-j_r)$$

for $k_1, \ldots, k_r \in \mathbb{N}_0$.

In §3.1, we consider the product formulae of desingularized values $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r)$ based on the equivalence between this desingularized values and the renormalized values in Definition 2.2.4.

Chapter 2

Renormalization

In this section, we recall the renormalization procedure to define renormalized values which is introduced by Ebrahimi-Fard, Manchon and Singer ([EMS17]). In §2.1, we start by recalling their framework of a Hopf algebra with the coproduct Δ_0 generated by words and in §2.2 we explain the algebraic Birkhoff decomposition à la Connes and Kreimer which is required to define renormalized values. In §2.3 we show an explicit formula in Proposition 2.3.3 to calculate the reduced coproduct $\tilde{\Delta}_0$ of the coproduct Δ_0. In §2.4, we prove a recurrence formula among renormalized values $\zeta_{\text{EMS}}(-k_1,\dots,-k_n)$ of MZFs in Proposition 2.4.3 to get explicit formulae of $\zeta_{\text{EMS}}(-k_1,\dots,-k_n)$. In §2.5, we prove an equivalence between desingularized values introduced in §1.2 and renormalized ones introduced in §2.2.

2.1 Algebraic frameworks

We follow the conventions of [EMS17]. Let $X_0 := \{j,d,y\}$ be the set of three elements j, d and y. Let W_0 be the associative monoid, with the empty word $\mathbf{1}$ as a unit, generated by X_0 with the rule $jd = dj = \mathbf{1}$ (hence, we sometimes regard d as j^{-1}). Any element $w \in W_0$ can be uniquely represented by

$$w = j^{k_1} y \cdots j^{k_n}$$

for $k_1,\cdots,k_n \in \mathbb{Z}$. An element of W_0 is called a *word*. Put $Y_0 := W_0 y \cup \{\mathbf{1}\}$ and we call an element of Y_0 *admissible*. We denote the $\mathbb{Q}$-linear space $\mathcal{A}_0$ generated by W_0 by $\mathcal{A}_0 := \langle W_0\rangle_{\mathbb{Q}}$. The linear space $\mathcal{A}_0$ is naturally equipped with a structure of a non-commutative algebra. We equip this $\mathcal{A}_0$ with a new product $\sqcup\!\sqcup_0 : \mathcal{A}_0 \otimes \mathcal{A}_0 \to \mathcal{A}_0$ which is a $\mathbb{Q}$-linear map recursively defined by

$$\begin{aligned}
\mathbf{1} \sqcup\!\sqcup_0 w &:= w \sqcup\!\sqcup_0 \mathbf{1} := w \quad (w \in W_0),\\
yu \sqcup\!\sqcup_0 v &:= u \sqcup\!\sqcup_0 yv := y(u \ \sqcup\!\sqcup_0 \ v) \quad (u,v \in W_0),\\
ju \sqcup\!\sqcup_0 jv &:= j(u \sqcup\!\sqcup_0 jv) + j(ju \sqcup\!\sqcup_0 v) \quad (u,v \in W_0),\\
du \sqcup\!\sqcup_0 dv &:= d(u \sqcup\!\sqcup_0 dv) - u \sqcup\!\sqcup_0 d^2 v \quad (u,v \in W_0).
\end{aligned}$$

Then $(\mathcal{A}_0, \sqcup\!\sqcup_0)$ forms a unitary, nonassociative, noncommutative $\mathbb{Q}$-algebra. We put $\mathcal{A}_0' := \langle Y_0\rangle_{\mathbb{Q}}$ to be a linear subspace of the linear space $\mathcal{A}_0$ spanned by Y_0. Then $(\mathcal{A}_0', \sqcup\!\sqcup_0)$ is a subalgebra of $(\mathcal{A}_0, \sqcup\!\sqcup_0)$. We define

$$\mathcal{L}' := \langle j^k\{d(u \ \sqcup\!\sqcup_0 \ v) - du \ \sqcup\!\sqcup_0 \ v - u \ \sqcup\!\sqcup_0 \ dv\} \mid k \in \mathbb{Z},\ u,v \in W_0 y \ \rangle_{(\mathcal{A}_0', \sqcup\!\sqcup_0)},$$

that is, to be the two-sided ideal of $(\mathcal{A}'_0, ш_0)$ algebraically generated by the above elements. We define the quotient algebra

$$\mathcal{B}'_0 := \mathcal{A}'_0/\mathcal{L}'. \tag{2.1.1}$$

We consider the map

$$\zeta_t^{ш} : \mathcal{B}'_0 \to \mathbb{Q}[[t]] \tag{2.1.2}$$

by $\zeta_t^{ш}(\mathbf{1}) := 1$ and for $k_1, \ldots, k_n \in \mathbb{Z}$,

$$\zeta_t^{ш}(j^{k_n}y \cdots j^{k_1}y) := \mathrm{Li}_{k_1,\cdots,k_n}(t).$$

Here $\mathrm{Li}_{k_1,\cdots,k_n}(t)$ is the *multiple polylogarithm* defined by

$$\mathrm{Li}_{k_1,\cdots,k_n}(t) := \sum_{0<m_1<\cdots<m_n} \frac{t^{m_n}}{m_1^{k_1}\cdots m_n^{k_n}}.$$

Lemma 2.1.1. *The map $\zeta_t^{ш}$ is well-defined and forms an algebra homomorphism.*

The first half of the claim of Lemma 2.1.1 is proved in the same way as [EMS17, Proposition 3.5] and the latter half of the claim of Lemma 2.1.1 is proved in [EMS17, Lemma 3.6].

Remark 2.1.2. The restriction of the shuffle product $ш_0$ to admissible words at positive arguments corresponds the usual shuffle product $ш$ as is proved in [EMS17, Lemma 3.7]. Let $\mathcal{C} := \mathbb{Q} \oplus j\mathbb{Q}\langle j, y\rangle y$ and $\mathcal{D} := \mathbb{Q} \oplus \mathbb{Q}\langle x_0, x_1\rangle x_1$. Then the two algebras $(\mathcal{C}, ш_0)$ and $(\mathcal{D}, ш)$ become isomorphic under the linear map $\Phi : (\mathcal{D}, ш) \to (\mathcal{C}, ш_0)$ by $\Phi(\mathbf{1}) := \mathbf{1}$ and for $k_1, \ldots, k_n \in \mathbb{N}$,

$$\Phi(x_0^{k_1-1}x_1 \cdots x_0^{k_n-1}x_1) := j^{k_1}y \cdots j^{k_n}y.$$

Let $L := \{d, y\}$ be the set of two elements d and y. Let L^* be the free monoid of L with empty word $\mathbf{1}$ as a unit. This L^* forms a submonoid of W_0. Put $Y := L^*y \cup \{\mathbf{1}\} \subset Y_0$. So all elements of Y are admissible. The *weight* $\mathrm{wt}(w)$ of a word $w \in L^*$ means the number of letters appearing in w and the *depth* $\mathrm{dp}(w)$ of a word $w \in L^*$ is given by the number of y appearing in w. We denote the free unitary, associative, noncommutative $\mathbb{Q}$-algebra of L by $\mathbb{Q}\langle L\rangle$. Then $(\mathbb{Q}\langle L\rangle, ш_0)$ forms a unitary, nonassociative, noncommutative $\mathbb{Q}$-subalgebra of $\mathcal{A}_0$. The algebra $\mathbb{Q}\langle L\rangle$ also forms a counital, cocommutative coalgebra (see [EMS17, §3.3.5]). We define

$$\mathcal{T}_- := \langle \{wd \mid w \in L^*\}\rangle_{\mathbb{Q}},$$

that is, to be the linear subspace of $\mathbb{Q}\langle L\rangle$ linearly generated by words ending in d. We define

$$\mathcal{L}_- := \langle d^k\{d(u \ ш_0 \ v) - du \ ш_0 \ v - u \ ш_0 \ dv\} \mid k \in \mathbb{N}_0,\ u, v \in Y \ \rangle_{(\mathbb{Q}\langle L\rangle, ш_0)},$$

that is, to be the two-sided ideal of $(\mathbb{Q}\langle L\rangle, ш_0)$ algebraically generated by the above elements. We consider the $\mathbb{Q}$-linear subspace

$$\mathcal{S}_- := \mathcal{T}_- + \mathcal{L}_-$$

of $\mathbb{Q}\langle L\rangle$ generated by $\mathcal{L}_-$ and $\mathcal{T}_-$. This $\mathcal{S}_-$ also forms a two-sided ideal.We put the quotient

$$\mathcal{H}_0 := \mathbb{Q}\langle L\rangle/\mathcal{S}_-. \tag{2.1.3}$$

We note that $\mathcal{H}_0$ is embedding in $\mathcal{B}'_0$. Then $\mathcal{H}_0$ forms a connected, filtered, commutative and cocommutative Hopf algebra (cf. [EMS17, §3.3.6]), whose product is equal to $ш_0$ and whose coproduct is given by

$$\Delta_0(w) := \sum_{\substack{S\subset[n] \\ S:\text{admissible}}} w_S \otimes w_{\overline{S}},$$

for $w \in Y \setminus \{\mathbf{1}\}(\subset \mathcal{H}_0)$. In the summation, S may be empty. we put $n := \mathrm{wt}(w)$, $[n] := \{1, \ldots, n\}$ and $\overline{S} := [n] \setminus S$. For $w := x_1 \cdots x_n$ $(x_i \in L^*,\ i = 1, \ldots, n)$ and $S := \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq n$, we define $w_S := x_{i_1} \cdots x_{i_k}$. We call the set S *admissible* if both $w_S, w_{\overline{S}} \in Y$. See [EMS17, §3.3.8] for combinatorial method using polygons to compute $\Delta_0(w)$. We define the $\mathbb{Q}$-linear map $\tilde{\Delta}_0 : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0$ by

$$\tilde{\Delta}_0(w) := \Delta_0(w) - 1 \otimes w - w \otimes 1 \quad (w \in Y), \tag{2.1.4}$$

and we call $\tilde{\Delta}_0$ the *reduced coproduct*.

2.2 The algebraic Birkhoff decomposition and renormalized values

We explain the algebraic Birkhoff decomposition. This decomposition is a fundamental tool in a work of Connes and Kreimer [CK] on their Hopf algebraic approach to renormalization of perturbative quantum field theory. This decomposition is necessary to define renormalized values.

Based on [Man], we recall the algebraic Birkhoff decomposition. We denote the product and the unit of $\mathbb{Q}$-algebra $\mathcal{A}$ by $m_{\mathcal{A}}$ and $u_{\mathcal{A}}$. For a Hopf algebra $\mathcal{H}$ over $\mathbb{Q}$, we mean $\Delta_{\mathcal{H}}$, $\varepsilon_{\mathcal{H}}$ and $S_{\mathcal{H}}$ to be its coproduct, its counit and its antipode respectively. In this paper, we often use Sweedler's notation:

$$\tilde{\Delta}_0(w) := \sum_{(w)} w' \otimes w''. \tag{2.2.1}$$

Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{Q}$, $\mathcal{A}$ be a $\mathbb{Q}$-algebra and $\mathcal{L}(\mathcal{H}, \mathcal{A})$ be the set of $\mathbb{Q}$-linear maps from $\mathcal{H}$ to $\mathcal{A}$. We define the *convolution* $\phi * \psi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ by

$$\phi * \psi := m_{\mathcal{A}} \circ (\phi \otimes \psi) \circ \Delta_{\mathcal{H}}$$

for $\mathbb{Q}$-linear maps ϕ and $\psi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$. Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{Q}$ and $\mathcal{A}$ be a $\mathbb{Q}$-algebra. The subset

$$G(\mathcal{H}, \mathcal{A}) := \{\phi \in \mathcal{L}(\mathcal{H}, \mathcal{A}) \mid \phi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{A}}\}$$

endowed with the above convolution product $*$ forms a group. The unit is given by a map $e = u_{\mathcal{A}} \circ \varepsilon_{\mathcal{H}}$.

Let $\mathcal{H}$ be a connected filtered Hopf algebra over $\mathbb{Q}$, that is, $\mathcal{H}$ has a filtration of $\mathbb{Q}$-linear subspace:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n \subset \bigcup_{n\in\mathbb{N}_0} \mathcal{H}^n = \mathcal{H}$$

with $\mathcal{H}^0 = \mathbb{Q}$ and with the conditions: $\mathcal{H}^m\mathcal{H}^n \subset \mathcal{H}^{m+n}$ and $S_{\mathcal{H}}(\mathcal{H}^n) \subset \mathcal{H}^n$ and $\Delta_{\mathcal{H}}(\mathcal{H}^n) \subset \sum_{p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q$ for $m, n \in \mathbb{N}_0$.

Put $\mathbb{K} := \mathbb{Q}$ or $\mathbb{C}$. Let $\mathcal{A} := \mathbb{K}[\frac{1}{z}, z]] := \mathbb{K}[[z]][\frac{1}{z}]$ be the algebra consisting of all Laurent series. And we decompose it as $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ where $\mathcal{A}_- := \frac{1}{z}\mathbb{K}[\frac{1}{z}]$ and $\mathcal{A}_+ := \mathbb{K}[[z]]$. We define a projection $\pi : \mathcal{A} \to \mathcal{A}_-$ by

$$\pi\left(\sum_{n=-k}^{\infty} a_n z^n\right) := \sum_{n=-k}^{-1} a_n z^n,$$

with $a_n \in \mathbb{K}$ and $k \in \mathbb{Z}$. Here we use the convention the sum over empty set is zero.

The following theorem is the fundamental tool of Connes and Kreimer ([CK]) in the renormalization procedure of perturbative quantum field theory.

Theorem 2.2.1 ([CK], [EMS17], [Man]: **algebraic Birkhoff decomposition**)**.** *For $\phi \in G(\mathcal{H}, \mathcal{A})$, there are unique linear maps $\phi_+ : \mathcal{H} \to \mathcal{A}_+$ and $\phi_- : \mathcal{H} \to \mathbb{K} \oplus \mathcal{A}_-$ with $\phi_-(\mathbf{1}) = 1 \in \mathbb{K}$ such that*

$$\phi = \phi_-^{-1} * \phi_+.$$

Moreover the maps ϕ_- and ϕ_+ are algebra homomorphisms if ϕ is an algebra homomorphism.

Remark 2.2.2. By the above theorem, the maps ϕ_- and ϕ_+ are recursively given by

$$\phi_-(x) = -\pi\left(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x'')\right), \quad \phi_+(x) = (\mathrm{Id} - \pi)\left(\phi(x) + \sum_{(x)} \phi_-(x')\phi(x'')\right),$$

for $x \in \operatorname{Ker} \varepsilon_{\mathcal{H}}$.

We define the $\mathbb{Q}$-linear map $\phi : \mathcal{H}_0 \to \mathcal{A}$ by $\phi(\mathbf{1}) := 1$ and for $k_1, \ldots, k_n \in \mathbb{N}_0$,

$$d^{k_1}y \cdots d^{k_n}y \mapsto \phi(d^{k_1}y \cdots d^{k_n}y)(z) := \partial_z^{k_1}\left(x\partial_z^{k_2}\right) \cdots \left(x\partial_z^{k_n}\right)(x(z))$$

where $x := x(z) := \frac{e^z}{1-e^z} \in \mathcal{A}$ and ∂_z is the derivative by z.

Proposition 2.2.3 ([EMS17, §4.2])**.** *The $\mathbb{Q}$-linear map $\phi : \mathcal{H}_0 \to \mathcal{A}$ is well-defined and forms an algebra homomorphism. Moreover, the following diagram is commutative:*

$$\begin{tikzcd}
(\mathcal{H}_0, ш_0) \arrow[r, "\zeta_t^{ш}"] \arrow[dr, "\phi"'] & (\mathbb{Q}[[t]], \cdot) \arrow[d, "t \mapsto e^z"] \\
& (\mathcal{A}, \cdot)
\end{tikzcd}$$

where $\zeta_t^{ш}$ is the map in (2.1.2) *(we mention that $\mathcal{H}_0$ is embedding in $\mathcal{B}_0'$).*

Because the map ϕ is algebraic by the above proposition, we obtain the algebraic map:

$$\phi_+ : \mathcal{H}_0 \to \mathcal{A}_+ \tag{2.2.2}$$

which is an algebra homomorphism by Theorem 2.2.1.

Definition 2.2.4 ([EMS17, §4.2])**.** The *renormalized value* [1] $\zeta_{\text{EMS}}(-k_1, \ldots, -k_n)$ is defined by

$$\zeta_{\text{EMS}}(-k_1, \ldots, -k_n) := \lim_{z \to 0} \phi_+(d^{k_n}y \cdots d^{k_1}y)(z)$$

for $k_1, \ldots, k_n \in \mathbb{N}_0$.

[1] If we follow the notations of [EMS17], it should be denoted by $\zeta_+(-k_n, \ldots, -k_1)$.

It is remarkable that the renormalized values coincide with special values of the meromorphic continuation of MZFs at non-positive arguments which do not locate at their singularities.

Proposition 2.2.5 ([EMS17, Theorem 4.3])**.** *For $k_1 \in \mathbb{N}_0$, we have*

$$\zeta_{\text{EMS}}(-k_1) = \zeta(-k_1)$$

and for $k_1, k_2 \in \mathbb{N}_0$ with $k_1 + k_2$ odd, we have

$$\zeta_{\text{EMS}}(-k_1, -k_2) = \zeta(-k_1, -k_2).$$

We recall that, as seen in (1.1.2), $\zeta(s_1, \cdots, s_n)$ is always irregular at $(s_1, \cdots, s_n) = (-k_1, \cdots, -k_n) \in \mathbb{Z}^n_{<0}$ for $n \geq 3$.

Another remarkable property of the renormalized values is that a certain shuffle relation hold for them. Because $ш_0$ is the product of $\mathcal{H}_0$ and $\phi_+ : \mathcal{H}_0 \to \mathbb{Q}[[z]]$ is a unital algebra homomorphism by Theorem 2.2.1, we obtain the following proposition:

Proposition 2.2.6 ([EMS17, §4.2] : **shuffle relation**)**.** *For $w, w' \in Y$, we have*

$$\phi_+(w \,ш_0\, w') = \phi_+(w)\phi_+(w').$$

Here are examples in lower depth:

Example 2.2.7. For $a, b, c \in \mathbb{N}_0$, we have

$$\zeta_{\text{EMS}}(-a) \cdot \zeta_{\text{EMS}}(-b) = \begin{cases} \displaystyle\sum_{k=0}^{a} (-1)^k \binom{a}{k} \zeta_{\text{EMS}}(-b-k, -a+k) & \text{if } b \geq 1, \\ \zeta_{\text{EMS}}(-a, 0) & \text{if } b = 0, \end{cases}$$

$$\zeta_{\text{EMS}}(-a) \cdot \zeta_{\text{EMS}}(-b, -c) = \begin{cases} \displaystyle\sum_{k=0}^{c} (-1)^k \binom{c}{k} \zeta_{\text{EMS}}(-b, -c-k, -a+k) & \text{if } c \geq 1, \\ \displaystyle\sum_{k=0}^{c} (-1)^k \binom{c}{k} \zeta_{\text{EMS}}(-b-k, -a+k, 0) & \text{if } b \geq 1,\ c = 0, \\ \zeta_{\text{EMS}}(-a, 0, 0) & \text{if } b = c = 0. \end{cases}$$

For our comparison, we remind below the usual shuffle relation for positive arguments. For $a, b \in \mathbb{N}_{>1}$,

$$\zeta(a) \cdot \zeta(b) = \sum_{k=0}^{a-1} \binom{b-1+k}{k} \zeta(a-k, b+k) + \sum_{k=0}^{b-1} \binom{a-1+k}{k} \zeta(b-k, a+k),$$

and for $a, c \in \mathbb{N}_{>1}$ and $b \in \mathbb{N}$,

$$\begin{aligned} \zeta(a) \cdot \zeta(b, c) = & \sum_{k=0}^{a-1} \sum_{i=0}^{a-k-1} \binom{c-1+k}{k} \binom{b-1+i}{i} \zeta(a-k-i, b+i, c+k) \\ & + \sum_{k=0}^{a-1} \sum_{j=0}^{b-1} \binom{c-1+k}{k} \binom{a-k-1+j}{j} \zeta(b-j, a-k+j, c+k) \\ & + \sum_{k=0}^{c-1} \binom{a-1+k}{k} \zeta(b, c-k, a+k). \end{aligned}$$

2.3 An explicit formula for the reduced coproduct $\tilde{\Delta}_0$

We show an explicit formula (Proposition 2.3.3) to calculate the reduced coproduct $\tilde{\Delta}_0$ in this subsection. This proposition is important to prove the recurrence formula of $\zeta_{\mathrm{EMS}}(-k_1, \ldots, -k_n)$ in §2.4.

We consider the bilinear map $f : \mathbb{Q}\langle L\rangle \times \mathbb{Q}\langle L\rangle^{\otimes 2} \to \mathbb{Q}\langle L\rangle^{\otimes 2}$ defined by

$$\begin{aligned} f(\mathbf{1}, w \otimes w') &:= w \otimes w', \\ f(d, w \otimes w') &:= dw \otimes w' + w \otimes dw', \\ f(y, w \otimes w') &:= yw \otimes w' + w \otimes yw', \end{aligned}$$

and inductively

$$f(xx_0, w \otimes w') := f\left(x, f(x_0, w \otimes w')\right),$$

for $w, w' \in \mathbb{Q}\langle L\rangle$, $x_0 \in L$ and $x \in L^*$. Then the following lemma holds:

Lemma 2.3.1. *There is a map $\overline{f} : \mathbb{Q}\langle L\rangle \times \mathcal{H}_0^{\otimes 2} \to \mathcal{H}_0^{\otimes 2}$ which makes the following diagram commutative:*

$$\begin{array}{ccc} \mathbb{Q}\langle L\rangle \otimes \mathbb{Q}\langle L\rangle & \xrightarrow{f(x,\cdot)} & \mathbb{Q}\langle L\rangle \otimes \mathbb{Q}\langle L\rangle \\ \pi \big\downarrow & & \big\downarrow \pi \\ \mathcal{H}_0 \otimes \mathcal{H}_0 & \xrightarrow{\overline{f}(x,\cdot)} & \mathcal{H}_0 \otimes \mathcal{H}_0 \end{array}$$

where $x \in \mathbb{Q}\langle L\rangle$ and $\pi : \mathbb{Q}\langle L\rangle^{\otimes 2} \to \mathcal{H}_0^{\otimes 2}$ is the natural projection.

Proof. It is sufficient to prove $f(x, \ker \pi) \subset \ker \pi$ for $x \in L^*$. Here $\ker \pi = \mathbb{Q}\langle L\rangle \otimes \mathcal{S}_- + \mathcal{S}_- \otimes \mathbb{Q}\langle L\rangle$. We show this by induction on $\mathrm{wt}(x)$. Let $x_0 = d$ or y and put $v \in \mathcal{S}_-$. If $v \in \mathcal{T}_-$, it is clear that $x_0 v \in \mathcal{T}_- \subset \mathcal{S}_-$. If $v \in \mathcal{L}_-$, for $x_0 = d$ it is easy to see that $dv \in \mathcal{L}_- \subset \mathcal{S}_-$ by the definition of $\mathcal{L}_-$. Because $\mathcal{L}_-$ is a two-sided ideal of $(\mathbb{Q}\langle L\rangle, ш_0)$, we have $y ш_0 v \in L_-$ for $x_0 = y$. By the definition of $ш_0$, we get

$$y ш_0 v = y(1 ш_0 v) = yv \in \mathcal{L}_- \subset \mathcal{S}_-.$$

Because $\mathcal{S}_-$ is $\mathcal{L}_- + \mathcal{T}_-$, for $v \in \mathcal{S}_-$ and $x_0 = d$ or y, we have $x_0 v \in \mathcal{S}_-$.

Let $w \in L^*$ and $v \in \mathcal{S}_-$. Then $x_0 v \in \mathcal{S}_-$, so we have

$$\begin{aligned} \pi\left(f(x_0, w \otimes v)\right) &= \pi(x_0 w \otimes v + w \otimes x_0 v) \\ &= \pi(x_0 w \otimes v) + \pi(w \otimes x_0 v) \\ &= 0. \end{aligned}$$

Let $w \in L^*$ and $v \in \mathcal{S}_-$. For $x \in L^*$, we get

$$\begin{aligned} \pi\big(f(xx_0, w \otimes v)\big) &= \pi\Big(f\big(x, f(x_0, w \otimes v)\big)\Big) \\ &= \pi\big(f(x, x_0 w \otimes v + w \otimes x_0 v)\big) \\ &= \pi\big(f(x, x_0 w \otimes v)\big) + \pi\big(f(x, w \otimes x_0 v)\big) \\ &= 0, \end{aligned}$$

by our induction assumption. This also applies to the case when $w \in \mathcal{S}_-$ and $v \in L^*$, so the claim holds. ☐

For $x \in L^*$ and $w, w' \in Y$, we simply denote $\overline{f}(x, w \otimes w')$ by $x \bullet (w \otimes w')$ and we define

$$w \otimes_{\mathrm{sym}} w' := w \otimes w' + w' \otimes w \in \mathcal{H}_0 \otimes \mathcal{H}_0.$$

Then, the following equations hold in $\mathcal{H}_0 \otimes \mathcal{H}_0$:

$$d^n \bullet (w \otimes_{\mathrm{sym}} w') = \sum_{i+j=n} \binom{n}{i} d^i w \otimes_{\mathrm{sym}} d^j w', \tag{2.3.1}$$

$$(d^n y) \bullet (w \otimes_{\mathrm{sym}} w') = \sum_{i+j=n} \binom{n}{i} \sum_{\{u,v\}=\{d^i y, d^j\}} uw \otimes_{\mathrm{sym}} vw', \tag{2.3.2}$$

for $n \in \mathbb{N}$, $w,\ w' \in Y$. These equations can be proved inductively on $n \in \mathbb{N}$.

Proposition 2.3.2. *For $w \in Y \setminus \{\mathbf{1}\}$,*

$$\tilde{\Delta}_0(dw) = d \bullet \tilde{\Delta}_0(w), \tag{2.3.3}$$

$$\tilde{\Delta}_0(yw) = y \bullet \tilde{\Delta}_0(w) + y \otimes_{\mathrm{sym}} w. \tag{2.3.4}$$

Proof. Let w be in $Y \setminus \{\mathbf{1}\}$. By the definition of Δ_0 and the equation (2.1.4), we have

$$\begin{aligned}
\tilde{\Delta}_0(dw) &= \Delta_0(dw) - 1 \otimes_{\mathrm{sym}} dw \\
&= \sum_{\substack{S \subset [n+1] \\ S:\text{admissible}}} (dw)_S \otimes (dw)_{\overline{S}} - 1 \otimes_{\mathrm{sym}} dw \\
&= \sum_{\substack{1 \in S \subset [n+1] \\ S:\text{admissible}}} (dw)_S \otimes (dw)_{\overline{S}} + \sum_{\substack{1 \notin S \subset [n+1] \\ S:\text{admissible}}} (dw)_S \otimes (dw)_{\overline{S}} - 1 \otimes_{\mathrm{sym}} dw \\
&= \sum_{\substack{S \subset [n] \\ S:\text{admissible}}} d \cdot w_S \otimes w_{\overline{S}} + \sum_{\substack{S \subset [n] \\ S:\text{admissible}}} w_S \otimes d \cdot w_{\overline{S}} - (d \otimes_{\mathrm{sym}} w + 1 \otimes_{\mathrm{sym}} dw) \\
&= \sum_{\substack{S \subset [n] \\ S:\text{admissible}}} (d \cdot w_S \otimes w_{\overline{S}} + w_S \otimes d \cdot w_{\overline{S}}) - (d \otimes_{\mathrm{sym}} w + 1 \otimes_{\mathrm{sym}} dw) \\
&= d \bullet \left(\sum_{\substack{S \subset [n] \\ S:\text{admissible}}} w_S \otimes w_{\overline{S}} - 1 \otimes_{\mathrm{sym}} w \right) \\
&= d \bullet \tilde{\Delta}_0(w).
\end{aligned}$$

We use $d \otimes_{\mathrm{sym}} w = 0$ in $\mathcal{H}_0 \otimes \mathcal{H}_0$ at the fourth equality. The equation (2.3.4) can be proved in the same way. $\square$

Proposition 2.3.3. *Let $w_m := d^m y$ for $m \in \mathbb{N}_0$. Then for $n \in \mathbb{N}_{\geq 2}$ and $k_1, \ldots, k_n \in \mathbb{N}_0$, we have*

$$\begin{aligned}
\tilde{\Delta}_0(w_{k_1} \cdots w_{k_n}) &= \sum_{i_1+j_1=k_1} \binom{k_1}{i_1} d^{i_1} y \otimes_{\mathrm{sym}} d^{j_1} w_{k_2} \cdots w_{k_n} \\
&+ \sum_{p=2}^{n-1} \sum_{\substack{i_1+j_1=k_1 \\ \vdots \\ i_p+j_p=k_p}} \prod_{a=1}^{p} \binom{k_a}{i_a} \sum_{\substack{\{u_q,\ v_q\}=\{d^{i_q},\ d^{j_q} y\} \\ 1 \leq q \leq p-1}} (u_1 \cdots u_{p-1} d^{i_p} y \otimes_{\mathrm{sym}} v_1 \cdots v_{p-1} d^{j_p} w_{k_{p+1}} \cdots w_{k_n}).
\end{aligned}$$

Here $\{u_q, v_q\} = \{d^{i_q}, d^{j_q}y\}$ *means* $(u_q, v_q) = (d^{i_q}, d^{j_q}y)$ *or* $(d^{j_q}y, d^{i_q})$.

Proof. Because we have

$$\tilde{\Delta}_0(d^a yw) = d^a \bullet \Big(y \otimes_{\mathrm{sym}} w + y \bullet \tilde{\Delta}_0(w)\Big) \quad (a \in \mathbb{N}_0) \tag{2.3.5}$$

by Proposition 2.3.2, we compute

$$\begin{aligned}
&\tilde{\Delta}_0(w_{k_1}w_{k_2}\cdots w_{k_n})\\
&= d^{k_1} \bullet (y \otimes_{\mathrm{sym}} w_{k_2}\cdots w_{k_n}) + (d^{k_1}y) \bullet \tilde{\Delta}_0(w_{k_2}\cdots w_{k_n})\\
&= d^{k_1} \bullet (y \otimes_{\mathrm{sym}} w_{k_2}\cdots w_{k_n}) + (d^{k_1}yd^{k_2}) \bullet (y \otimes_{\mathrm{sym}} w_{k_3}\cdots w_{k_n})\\
&\quad + (d^{k_1}yd^{k_2}y) \bullet \tilde{\Delta}_0(w_{k_3}\cdots w_{k_n}).
\end{aligned}$$

By using the equation (2.3.5) repeatedly, we get

$$\begin{aligned}
&= \sum_{p=1}^{n-1}(d^{k_1}y\cdots yd^{k_p}) \bullet (y \otimes_{\mathrm{sym}} w_{k_{p+1}}\cdots w_{k_n})\\
&\quad + (d^{k_1}y\cdots d^{k_{n-1}}y) \bullet \tilde{\Delta}_0(w_{k_n}).
\end{aligned}$$

Because $\tilde{\Delta}_0(d^a y) = 0$ $(a \in \mathbb{N}_0)$ by the definition of $\tilde{\Delta}_0$, the second term vanishes. Therefore by (2.3.1), we get

$$\begin{aligned}
&\tilde{\Delta}_0(w_{k_1}w_{k_2}\cdots w_{k_n})\\
&= \sum_{p=1}^{n-1}\left(d^{k_1}y\cdots d^{k_{p-1}}y\right) \bullet \left(\sum_{i_p+j_p=k_p}\binom{k_p}{i_p} d^{i_p}y \otimes_{\mathrm{sym}} d^{j_p}w_{k_{p+1}}\cdots w_{k_n}\right).
\end{aligned}$$

And by using (2.3.2) repeatedly, we have

$$\begin{aligned}
&= \sum_{i_1+j_1=k_1}\binom{k_1}{i_1} d^{i_1}y \otimes_{\mathrm{sym}} d^{j_1}w_{k_2}\cdots w_{k_n}\\
&\quad + \sum_{p=2}^{n-1}\sum_{\substack{i_1+j_1=k_1\\ \vdots\\ i_p+j_p=k_p}}\prod_{a=1}^{p}\binom{k_a}{i_a}\sum_{\substack{\{u_q,\ v_q\}=\{d^{i_q},\ d^{j_q}y\}\\ 1\le q\le p-1}}(u_1\cdots u_{p-1}d^{i_p}y \otimes_{\mathrm{sym}} v_1\cdots v_{p-1}d^{j_p}w_{k_{p+1}}\cdots w_{k_n}).
\end{aligned}$$

□

2.4 Recurrence formulas among renormalized values

The goal of this subsection is to prove Proposition 2.4.3 which gives the recurrence formula among renormalized values.

We start with the following key lemma of [EMS17] which is a method to compute recursively the image of ϕ_+ (the equation (2.2.2)).

Lemma 2.4.1 ([EMS17, Corollary 4.4])**.** *For* $w \in Y$ *with* $\mathrm{dp}(w) > 1$, *we have*

$$\phi_+(w) = \frac{1}{2^{\mathrm{dp}(w)}-2}\sum_{(w)}\phi_+(w')\phi_+(w'').$$

Here we use Sweedler's notation (2.2.1).

Proposition 2.4.2. *For $n \in \mathbb{N}_{\geq 2}$ and $k_1, \ldots, k_n \in \mathbb{N}_0$, we have*

$$\zeta_{\text{EMS}}(-k_1, \ldots, -k_n) = \frac{1}{2^{n-1}-1} \left\{ \sum_{i_n+j_n=k_n} \binom{k_n}{i_n} \zeta_{\text{EMS}}(-i_n)\zeta_{\text{EMS}}(-k_1, \ldots, -k_{n-1}-j_n) \right.$$
$$+ \sum_{p=2}^{n-1} \sum_{\substack{i_n+j_n=k_n \\ \vdots \\ i_p+j_p=k_p}} \prod_{a=p}^{n} \binom{k_a}{i_a}$$
$$\left. \times \sum_{\substack{\{\circ_q,\ \diamond_q\}=\{+,\ \boldsymbol{,}\ \} \\ p \leq q \leq n-1}} \zeta_{\text{EMS}}(-i_p \circ_p \cdots \circ_{n-1} -i_n)\zeta_{\text{EMS}}(-k_1, \ldots, -k_{p-1}-j_p \diamond_p \cdots \diamond_{n-1} -j_n) \right\}. \tag{2.4.1}$$

Proof. By Proposition 2.3.3 and Lemma 2.4.1, for $n \in \mathbb{N}_{\geq 2}$ and $k_1, \ldots, k_n \in \mathbb{N}_0$ we get

$$\phi_+(w_{k_n} \cdots w_{k_1}) = \frac{1}{2^{n-1}-1} \left\{ \sum_{i_n+j_n=k_n} \binom{k_n}{i_n} \phi_+(d^{i_n}y)\phi_+(d^{j_n}w_{k_{n-1}} \cdots w_{k_1}) \right.$$
$$\left. + \sum_{p=2}^{n-1} \sum_{\substack{i_n+j_n=k_n \\ \vdots \\ i_p+j_p=k_p}} \prod_{a=p}^{n} \binom{k_a}{i_a} \sum_{\substack{\{u_q,\ v_q\}=\{d^{i_q},\ d^{j_q}y\} \\ p+1 \leq q \leq n}} \phi_+(u_n \cdots u_{p+1}d^{i_p}y)\phi_+(v_n \cdots v_{p+1}d^{j_p}w_{k_{p-1}} \cdots w_{k_1}) \right\},$$

because $\mathrm{dp}(w) = n$. For $p \leq q \leq n-1$, we define

$$(\circ_q, \diamond_q) := \begin{cases} (+, \boldsymbol{,}) & \text{if } (u_{q+1}, v_{q+1}) = (d^{i_{q+1}}, d^{j_{q+1}}y), \\ (\boldsymbol{,}, +) & \text{if } (u_{q+1}, v_{q+1}) = (d^{j_{q+1}}y, d^{i_{q+1}}). \end{cases}$$

Then by the definition of $\zeta_{\text{EMS}}(-k_1, \ldots, -k_n)$, the equation (2.4.1) holds. □

We define the following generating functions in $\mathbb{C}[[x]]$ for $n \in \mathbb{N}_{\geq 2}$ and $k_1, \ldots, k_n \in \mathbb{N}_0$:

$$\mathfrak{h} := \mathfrak{h}(x) := \sum_{k_1=0}^{\infty} \frac{(-x)^{k_1}}{k_1!} \zeta_{\text{EMS}}(-k_1),$$
$$\mathfrak{h}_{k_1,\ldots,k_{n-1}}(x) := \sum_{k_n=0}^{\infty} \frac{(-x)^{k_n}}{k_n!} \zeta_{\text{EMS}}(-k_1, \ldots, -k_n),$$
$$\overline{\mathfrak{h}}_{k_1,\ldots,k_n}(x) := \partial_x^{k_n} \mathfrak{h}_{k_1,\ldots,k_{n-1}}(x).$$

Here for $n \in \mathbb{N}$, we set $\mathfrak{h}_{k_1,\ldots,k_{n-1}}(x) := \mathfrak{h}(x)$.

The equation (2.4.1) looks complicated. But it can be simplified to the following recurrence formula (2.4.2).

Proposition 2.4.3. *For $n \in \mathbb{N}_{\geq 2}$ and $k_1, \ldots, k_n \in \mathbb{N}_0$, we have*

$$\zeta_{\text{EMS}}(-k_1, \ldots, -k_n) = \sum_{i_n + j_n = k_n} \binom{k_n}{i_n} \zeta_{\text{EMS}}(-i_n)\zeta_{\text{EMS}}(-k_1, \ldots, -k_{n-1} - j_n), \tag{2.4.2}$$

and

$$\mathfrak{h}_{k_1,\ldots,k_{n-1}}(x) = (-1)^{k_1+\cdots+k_{n-1}} \left(\mathfrak{h}\partial_x^{k_{n-1}}\right) \cdots \left(\mathfrak{h}\partial_x^{k_1}\right)(\mathfrak{h}). \tag{2.4.3}$$

Proof. We prove (2.4.2) and (2.4.3) by induction on $n \in \mathbb{N}_{\geq 2}$. Let $n = 2$. Then by the equation (2.4.1) of Proposition 2.4.2, the equation (2.4.2) clearly holds. And by the equation (2.4.3) for $n = 2$, we have

$$\begin{aligned}
\mathfrak{h}_{k_1}(x) &= \sum_{k_2=0}^{\infty} \frac{(-x)^{k_2}}{k_2!} \zeta_{\text{EMS}}(-k_1, -k_2) \\
&= \sum_{k_2=0}^{\infty} \frac{(-x)^{k_2}}{k_2!} \sum_{i_2+j_2=k_2} \binom{k_2}{i_2} \zeta_{\text{EMS}}(-i_2)\zeta_{\text{EMS}}(-k_1 - j_2) \\
&= \left\{ \sum_{i_2=0}^{\infty} \frac{(-x)^{i_2}}{i_2!} \zeta_{\text{EMS}}(-i_2) \right\} \left\{ \sum_{j_2=0}^{\infty} \frac{(-x)^{j_2}}{j_2!} \zeta_{\text{EMS}}(-k_1 - j_2) \right\} \\
&= \mathfrak{h} \left\{ (-1)^{k_1} \partial_x^{k_1}(\mathfrak{h}) \right\} \\
&= (-1)^{k_1} \left(\mathfrak{h}\partial_x^{k_1}\right)(\mathfrak{h}).
\end{aligned} \tag{2.4.4}$$

Let $n = n_0 \geq 3$. We assume that (2.4.2) and (2.4.3) hold for $2 \leq n \leq n_0 - 1$. First, we prove the equation (2.4.2). By Lemma 2.4.4 which will be proved later, the second term of the right hand side of the equation (2.4.1) is calculated to be

$$\begin{aligned}
&\sum_{p=2}^{n_0-1} \sum_{\substack{\{\circ_q,\ \diamond_q\}=\{+,\ ,\ \} \\ p \leq q \leq n_0-1}} \left\{ \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0})\zeta_{\text{EMS}}(-k_1, \ldots, -k_{n_0-1} - j_{n_0}) \right\} \\
&= \sum_{p=2}^{n_0-1} 2^{n_0-p} \left\{ \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0})\zeta_{\text{EMS}}(-k_1, \ldots, -k_{n_0-1} - j_{n_0}) \right\} \\
&= (2^{n_0-1} - 2) \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0})\zeta_{\text{EMS}}(-k_1, \ldots, -k_{n_0-1} - j_{n_0}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&(\text{RHS of (2.4.1)}) \\
=& \frac{1}{2^{n_0-1} - 1} \left\{ \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0})\zeta_{\text{EMS}}(-k_1, \ldots, -k_{n_0-1} - j_{n_0}) \right. \\
&\left. + (2^{n_0-1} - 2) \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0})\zeta_{\text{EMS}}(-k_1, \ldots, -k_{n_0-1} - j_{n_0}) \right\} \\
=& \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0})\zeta_{\text{EMS}}(-k_1, \ldots, -k_{n_0-1} - j_{n_0}).
\end{aligned}$$

So we get the equation (2.4.2) for $n \geq 3$.

Secondly, we prove the equation (2.4.3) for $n = n_0 \geq 3$. By using the equation (2.4.2) for $n = n_0$ which we have proved just above, we have

$$\mathfrak{h}_{k_1,\cdots,k_{n_0-1}}(x) = (-1)^{k_{n_0-1}}\mathfrak{h}(x)\partial_x^{k_{n_0-1}}\left(\mathfrak{h}_{k_1,\cdots,k_{n_0-2}}(x)\right)$$

in the same way to case of $n = 2$. By our induction hypotheses,

$$\begin{aligned} &= (-1)^{k_{n_0-1}}\mathfrak{h}(x)\partial_x^{k_{n_0-1}}\left((-1)^{k_1+\cdots+k_{n_0-2}}\left(\mathfrak{h}(x)\partial_x^{k_{n_0-2}}\right)\cdots\left(\mathfrak{h}(x)\partial_x^{k_1}\right)(\mathfrak{h}(x))\right) \\ &= (-1)^{k_1+\cdots+k_{n_0-1}}\left(\mathfrak{h}(x)\partial_x^{k_{n_0-1}}\right)\cdots\left(\mathfrak{h}(x)\partial_x^{k_1}\right)(\mathfrak{h}(x)) \end{aligned}$$

So we get the equation (2.4.3) for $n \geq 3$. □

We prove the following lemma used in the above proof.

Lemma 2.4.4. *Let $n_0 \geq 3$. We assume that (2.4.3) holds for $n = l$ with $2 \leq l \leq n_0 - 1$. Let $2 \leq p \leq n_0 - 1$ and $\circ_i \in \{+, \,\boldsymbol{,}\,\}$ for $p \leq i \leq n_0 - 1$. Then we have*

$$\begin{aligned} &\sum_{\substack{i_p+j_p=k_p \\ \vdots \\ i_{n_0}+j_{n_0}=k_{n_0}}} \prod_{a=p}^{n_0}\binom{k_a}{i_a}\zeta_{\text{EMS}}(-i_p \circ_p \cdots \circ_{n_0-1} -i_{n_0})\zeta_{\text{EMS}}(-k_1,\ldots,-k_{p-1}-j_p \diamond_p \cdots \diamond_{n_0-1} -j_{n_0}) \\ &\qquad = \sum_{i_{n_0}+j_{n_0}=k_{n_0}}\binom{k_{n_0}}{i_{n_0}}\zeta_{\text{EMS}}(-i_{n_0})\zeta_{\text{EMS}}(-k_1,\ldots,-k_{n_0-1}-j_{n_0}). \end{aligned} \tag{2.4.5}$$

Here $\diamond_i$ is chosen to be with $\{\circ_i,\ \diamond_i\} = \{+, \,\boldsymbol{,}\,\}$ for $p \leq i \leq n_0 - 1$.

Proof. We get

$$\sum_{k_{n_0}=0}^{\infty}\frac{(-x)^{k_{n_0}}}{k_{n_0}!}(\text{RHS of (2.4.5)}) = (-1)^{k_{n_0-1}}\mathfrak{h}\partial_x^{k_{n_0-1}}\left(\mathfrak{h}_{k_1,\ldots,k_{n_0-2}}(x)\right)$$

in the same way to the computations of $\mathfrak{h}_{k_1}(x)$ in (2.4.4). By our induction hypothesis on (2.4.3), for n_0 we obtain

$$= (-1)^{k_1+\cdots+k_{n_0-1}}\left(\mathfrak{h}\partial_x^{k_{n_0-1}}\right)\cdots\left(\mathfrak{h}\partial_x^{k_1}\right)(\mathfrak{h}). \tag{2.4.6}$$

On the other hand, we have

$$\begin{aligned} &\sum_{k_{n_0}=0}^{\infty}\frac{(-x)^{k_{n_0}}}{k_{n_0}!}(\text{LHS of (2.4.5)}) \\ &= \sum_{\substack{i_p+j_p=k_p \\ \vdots \\ i_{n_0-1}+j_{n_0-1}=k_{n_0-1}}} \prod_{a=p}^{n_0-1}\binom{k_a}{i_a}\left\{\sum_{i_{n_0}=0}^{\infty}\frac{(-x)^{i_{n_0}}}{i_{n_0}!}\zeta_{\text{EMS}}(-i_p \circ_p \cdots \circ_{n_0-1} -i_{n_0})\right\} \\ &\quad \times\left\{\sum_{j_{n_0}=0}^{\infty}\frac{(-x)^{j_{n_0}}}{j_{n_0}!}\zeta_{\text{EMS}}(-k_1,\ldots,-k_{p-1}-j_p \diamond_p \cdots \diamond_{n_0-1} -j_{n_0})\right\}. \end{aligned}$$

We also consider the following two cases:

Case i) : When $(\circ_{n_0-1}, \diamond_{n_0-1}) = (\,\boldsymbol{,}\,, +)$, we compute

$$\sum_{k_{n_0}=0}^{\infty} \frac{(-x)^{k_{n_0}}}{k_{n_0}!}(\text{LHS of (2.4.5)})$$

$$= \sum_{\substack{i_p+j_p=k_p \\ \vdots \\ i_{n_0-1}+j_{n_0-1}=k_{n_0-1}}} \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \left\{ \sum_{i_{n_0}=0}^{\infty} \frac{(-x)^{i_{n_0}}}{i_{n_0}!} \zeta_{\text{EMS}}(-i_p \circ_p \cdots \circ_{n_0-2} -i_{n_0-1}, -i_{n_0}) \right\}$$

$$\times \left\{ \sum_{j_{n_0}=0}^{\infty} \frac{(-x)^{j_{n_0}}}{j_{n_0}!} \zeta_{\text{EMS}}(-k_1, \ldots, -k_{p-1} - j_p \diamond_p \cdots \diamond_{n_0-2} -j_{n_0-1} - j_{n_0}) \right\}.$$

Put $m := \begin{cases} p-1 & \text{when } \diamond_i \text{ is } + \text{ for all } i, \\ \max\{l \mid p \leq l \leq n_0-2,\ \diamond_l = \,\boldsymbol{,}\,\} & \text{otherwise.} \end{cases}$

Then we have

$$= \sum_{\substack{i_p+j_p=k_p \\ \vdots \\ i_{n_0-1}+j_{n_0-1}=k_{n_0-1}}} \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \left\{ \sum_{i_{n_0}=0}^{\infty} \frac{(-x)^{i_{n_0}}}{i_{n_0}!} \zeta_{\text{EMS}}(-i_p \circ_p \cdots \circ_{n_0-2} -i_{n_0-1}, -i_{n_0}) \right\}$$

$$\times (-1)^S \partial_x^S \left\{ \sum_{j_{n_0}=0}^{\infty} \frac{(-x)^{j_{n_0}}}{j_{n_0}!} \zeta_{\text{EMS}}(-k_1, \ldots, -k_{p-1} - j_p \diamond_p \cdots \diamond_{m-1} -j_m) \right\}.$$

Here $S := \begin{cases} k_{p-1} + j_p + \cdots + j_{n_0-1} & \text{when } \diamond_i \text{ is } + \text{ for all } i, \\ j_{m+1} + \cdots + j_{n_0-1} & \text{otherwise.} \end{cases}$

$$= \sum_{\substack{i_p+j_p=k_p \\ \vdots \\ i_{n_0-1}+j_{n_0-1}=k_{n_0-1}}} (-1)^S \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \mathfrak{h}_{i_p \circ_p \cdots \circ_{n_0-2}\ i_{n_0-1}}(x) \cdot \overline{\mathfrak{h}}_{k_1, \ldots, k_{p-1}+j_p \diamond_p \cdots \diamond_{n_0-2}\ j_{n_0-1}}(x).$$

Here we use the definitions of $\mathfrak{h}_{k_1,\ldots,k_{n-1}}(x)$ and $\overline{\mathfrak{h}}_{k_1,\ldots,k_{n_0}}(x)$. And by using our induction hypothesis on (2.4.3), we have

$$= \sum_{\substack{i_p+j_p=k_p \\ \vdots \\ i_{n_0-1}+j_{n_0-1}=k_{n_0-1}}} \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \left\{ (-1)^{\sum_{q=p}^{n_0-1} i_q} \left(\mathfrak{h}\partial_x^{i_{n_0-1}}\right) \left(\mathfrak{h}^{\delta_{n_0-2}}\partial_x^{i_{n_0-2}}\right) \cdots \left(\mathfrak{h}^{\delta_p}\partial_x^{i_p}\right)(\mathfrak{h}) \right\}$$

$$\times \left\{ (-1)^{\sum_{q=1}^{p-1} k_q + \sum_{q=p}^{n_0-1} j_q} \partial_x^{j_{n_0-1}} \left(\mathfrak{h}^{1-\delta_{n_0-2}}\partial_x^{j_{n_0-2}}\right) \cdots \left(\mathfrak{h}^{1-\delta_p}\partial_x^{j_p}\right) \left(\mathfrak{h}\partial_x^{k_{p-1}}\right) \cdots \left(\mathfrak{h}\partial_x^{k_1}\right)(\mathfrak{h}) \right\}.$$

Here we put $\delta_i := \begin{cases} 0 & \text{if } \circ_i = +, \\ 1 & \text{if } \circ_i = \boldsymbol{,}, \end{cases}$ for $p \le i \le n_0 - 2$.

$$
\begin{aligned}
=&(-1)^{\sum_{q=1}^{n_0-1} k_q} \mathfrak{h} \sum_{\substack{i_p+j_p=k_p \\ \vdots \\ i_{n_0-1}+j_{n_0-1}=k_{n_0-1}}} \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \left\{ \partial_x^{i_{n_0-1}} \left(\mathfrak{h}^{\delta_{n_0-2}} \partial_x^{i_{n_0-2}}\right) \cdots \left(\mathfrak{h}^{\delta_p}\partial_x^{i_p}\right)(\mathfrak{h}) \right\} \\
&\times \left\{ \partial_x^{j_{n_0-1}} \left(\mathfrak{h}^{1-\delta_{n_0-2}}\partial_x^{j_{n_0-2}}\right)\cdots\left(\mathfrak{h}^{1-\delta_p}\partial_x^{j_p}\right)\left(\mathfrak{h}\partial_x^{k_{p-1}}\right)\cdots\left(\mathfrak{h}\partial_x^{k_1}\right)(\mathfrak{h})\right\} \\
=&(-1)^{\sum_{q=1}^{n_0-1} k_q} \mathfrak{h}\partial_x^{k_{n_0-1}} \Bigg(\mathfrak{h} \sum_{\substack{i_p+j_p=k_p \\ \vdots \\ i_{n_0-2}+j_{n_0-2}=k_{n_0-2}}} \prod_{a=p}^{n_0-2} \binom{k_a}{i_a} \left\{ \partial_x^{i_{n_0-2}} \left(\mathfrak{h}^{\delta_{n_0-3}} \partial_x^{i_{n_0-3}}\right) \cdots \left(\mathfrak{h}^{\delta_p}\partial_x^{i_p}\right)(\mathfrak{h}) \right\} \\
&\times \left\{ \partial_x^{j_{n_0-2}} \left(\mathfrak{h}^{1-\delta_{n_0-3}}\partial_x^{j_{n_0-3}}\right)\cdots\left(\mathfrak{h}^{1-\delta_p}\partial_x^{j_p}\right)\left(\mathfrak{h}\partial_x^{k_{p-1}}\right)\cdots\left(\mathfrak{h}\partial_x^{k_1}\right)(\mathfrak{h})\right\} \Bigg).
\end{aligned}
$$

We use Leibniz rule in last equality. By using this rule repeatedly, we get

$$
=(-1)^{\sum_{q=1}^{n_0-1} k_q} \left(\mathfrak{h}\partial_x^{k_{n_0-1}}\right)\cdots\left(\mathfrak{h}\partial_x^{k_1}\right)(\mathfrak{h}).
$$

This is equal to (2.4.6).

Case ii) : When $(\circ_{n_0-1}, \diamond_{n_0-1}) = (+, \boldsymbol{,})$, it can be proved in the same way to *Case i)*.

$\square$

2.5 An equivalence between desingularized values and renormalized ones

We reveal a close relationship among desingularized values and renormalized ones in Theorem 2.5.1. As a consequence, we get an explicit formula of renormalized values in terms of Bernoulli numbers in Corollary 2.5.5.

Our main theorem in this subsection is the following explicit relationship between the generating function $Z_{\text{FKMT}}(t_1, \ldots, t_n)$ of the desingularized values $\zeta_n^{\text{des}}(-k_1, \ldots, -k_n)$ and the generating function $Z_{\text{EMS}}(t_1, \ldots, t_n)$ of the renormalized values $\zeta_{\text{EMS}}(-k_1, \ldots, -k_n)$.

Theorem 2.5.1. *For $n \in \mathbb{N}$, we have*

$$
Z_{\text{EMS}}(t_1, \ldots, t_n) = \prod_{i=1}^{n} \frac{1 - e^{-t_i - \cdots - t_n}}{t_i + \cdots + t_n} \cdot Z_{\text{FKMT}}(-t_1, \ldots, -t_n). \tag{2.5.1}
$$

Proof. By Proposition 2.4.3 and Lemma 2.4.4 we get

$$\zeta_{\mathrm{EMS}}(-k_1,\dots,-k_n) = \sum_{\substack{i_2+j_2=k_2\\ \vdots \\ i_n+j_n=k_n}} \prod_{a=2}^{n} \binom{k_a}{i_a} \zeta_{\mathrm{EMS}}(-i_2,\dots,-i_n)\zeta_{\mathrm{EMS}}(-k_1-j_2-\cdots-j_n).$$

Here, we use Lemma 2.4.4 for $p=2$ and for all $\circ_q = \boldsymbol{,}$ $(2 \le q \le n)$. It is remarkable that the same recurrence formula holds for $\zeta_n^{\mathrm{des}}(-k_1,\dots,-k_n)$ of (1.2.6). Thus, we get

$$Z_{\mathrm{EMS}}(t_1,\dots,t_n) = Z_{\mathrm{EMS}}(t_2,\dots,t_n)\cdot Z_{\mathrm{EMS}}(t_1+\cdots+t_n) \quad (n\in\mathbb{N}). \tag{2.5.2}$$

Now from [EMS17] Theorem 4.3, $\zeta_{\mathrm{EMS}}(-k_1) = \zeta(-k_1)$ at $k_1 \in \mathbb{N}_0$, so we can write $Z_{\mathrm{EMS}}(x)$ by

$$Z_{\mathrm{EMS}}(x) = \frac{1+x-e^x}{x(e^x-1)}.$$

We get the following equation by $Z_{\mathrm{EMS}}(x)$ and $Z_{\mathrm{FKMT}}(x)$:

$$Z_{\mathrm{EMS}}(x) = \frac{1-e^{-x}}{x} Z_{\mathrm{FKMT}}(-x). \tag{2.5.3}$$

By using (2.5.2), (2.5.3) and (1.2.6), we get (2.5.1). □

By Theorem 2.5.1, we find that desingularized values and renormalized ones are equivalent. Namely, the renormalized values can be given as linear combinations of the desingularized ones.

Example 2.5.2. The desingularized values and the renormalized values are equal at the origin:

$$\zeta_n^{\mathrm{des}}(\underbrace{0,\dots,0}_{n}) = \zeta_{\mathrm{EMS}}(\underbrace{0,\dots,0}_{n}) = B_1^n = \left(-\frac{1}{2}\right)^n$$

Example 2.5.3. For $k_1,k_2,k_3 \in \mathbb{N}_0$, we have

$$\begin{aligned}
\zeta_{\mathrm{EMS}}(-k_1) &= \sum_{\nu_{01}+\nu_{11}=k_1} \binom{k_1}{\nu_{01}} \frac{(-1)^{\nu_{11}}}{\nu_{01}+1} \zeta_1^{\mathrm{des}}(-\nu_{11}),\\
\zeta_{\mathrm{EMS}}(-k_1,-k_2) &= \sum_{\substack{\nu_{01}+\nu_{11}=k_1\\ \nu_{02}+\nu_{12}+\nu_{22}=k_2}} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02}\ \nu_{12}} \frac{1}{\nu_{02}+1} \frac{(-1)^{\nu_{11}+\nu_{22}}}{\nu_{01}+\nu_{12}+1} \zeta_2^{\mathrm{des}}(-\nu_{11},-\nu_{22}),\\
\zeta_{\mathrm{EMS}}(-k_1,-k_2,-k_3) &= \sum_{\substack{\nu_{01}+\nu_{11}=k_1\\ \nu_{02}+\nu_{12}+\nu_{22}=k_2\\ \nu_{03}+\nu_{13}+\nu_{23}+\nu_{33}=k_3}} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02}\ \nu_{12}} \binom{k_3}{\nu_{03}\ \nu_{13}\ \nu_{23}}\\
&\quad \times \frac{1}{\nu_{03}+1} \frac{1}{\nu_{02}+\nu_{13}+1} \frac{(-1)^{\nu_{01}+\nu_{12}+\nu_{23}}}{\nu_{01}+\nu_{12}+\nu_{23}+1} \zeta_3^{\mathrm{des}}(-\nu_{11},-\nu_{22},-\nu_{33}).
\end{aligned}$$

Here $\binom{k_2}{\nu_{02}\ \nu_{12}} := \frac{k_2!}{\nu_{02}!\nu_{12}!(k_2-\nu_{02}-\nu_{12})!}$ and $\binom{k_3}{\nu_{03}\ \nu_{13}\ \nu_{23}} := \frac{k_3!}{\nu_{03}!\nu_{13}!\nu_{23}!(k_3-\nu_{03}-\nu_{13}-\nu_{23})!}$.

On the other hand, desingularized values can be also given as linear combinations of product of renormalized ones and Bernoulli numbers (cf. (0.2.3)):

Example 2.5.4. For $k_1, k_2, k_3 \in \mathbb{N}_0$, we have

$$\zeta_1^{\mathrm{des}}(-k_1) = (-1)^{k_1} \sum_{\nu_{01}+\nu_{11}=k_1} \binom{k_1}{\nu_{01}} B_{\nu_{01}} \zeta_{\mathrm{EMS}}(-\nu_{11}),$$

$$\zeta_2^{\mathrm{des}}(-k_1, -k_2) = (-1)^{k_1+k_2} \sum_{\substack{\nu_{01}+\nu_{11}=k_1 \\ \nu_{02}+\nu_{12}+\nu_{22}=k_2}} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02}\ \nu_{12}} B_{\nu_{02}} B_{\nu_{01}+\nu_{12}} \zeta_{\mathrm{EMS}}(-\nu_{11}, -\nu_{22}),$$

$$\zeta_3^{\mathrm{des}}(-k_1, -k_2, -k_3) = (-1)^{k_1+k_2+k_3} \sum_{\substack{\nu_{01}+\nu_{11}=k_1 \\ \nu_{02}+\nu_{12}+\nu_{22}=k_2 \\ \nu_{03}+\nu_{13}+\nu_{23}+\nu_{33}=k_3}} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02}\ \nu_{12}} \binom{k_3}{\nu_{03}\ \nu_{13}\ \nu_{23}}$$
$$\times B_{\nu_{03}} B_{\nu_{02}+\nu_{13}} B_{\nu_{01}+\nu_{12}+\nu_{23}} \zeta_{\mathrm{EMS}}(-\nu_{11}, -\nu_{22}, -\nu_{33}).$$

By combining Proposition 1.2.8 and Theorem 2.5.1, we obtain the following corollary.

Corollary 2.5.5. *For $n \in \mathbb{N}$, we have*

$$Z_{\mathrm{EMS}}(t_1, \ldots, t_n) = \prod_{i=1}^{n} \frac{(t_i + \cdots + t_n) - (e^{t_i+\cdots+t_n} - 1)}{(t_i + \cdots + t_n)(e^{t_i+\cdots+t_n} - 1)}.$$

Therefore the renormalized values are described explicitly in terms of Bernoulli numbers:

Example 2.5.6. For $k_1, k_2, k_3 \in \mathbb{N}_0$, we have

$$\zeta_{\mathrm{EMS}}(-k_1) = \frac{(-1)^{k_1}}{k_1+1} B_{k_1+1},$$

$$\zeta_{\mathrm{EMS}}(-k_1, -k_2) = (-1)^{k_1+k_2} \sum_{\nu_{12}+\nu_{22}=k_2} \binom{k_2}{\nu_{12}} \frac{B_{\nu_{22}+1}}{\nu_{22}+1} \frac{B_{k_1+\nu_{12}+1}}{k_1+\nu_{12}+1},$$

$$\zeta_{\mathrm{EMS}}(-k_1, -k_2, -k_3) = (-1)^{k_1+k_2+k_3} \sum_{\substack{\nu_{12}+\nu_{22}=k_2 \\ \nu_{13}+\nu_{23}+\nu_{33}=k_3}} \binom{k_2}{\nu_{12}} \binom{k_3}{\nu_{13}\ \nu_{23}}$$
$$\times \frac{B_{\nu_{33}+1}}{\nu_{33}+1} \frac{B_{\nu_{22}+\nu_{23}+1}}{\nu_{22}+\nu_{23}+1} \frac{B_{k_1+\nu_{12}+\nu_{13}+1}}{k_1+\nu_{12}+\nu_{13}+1}.$$

Chapter 3

Functional relations of desingularized MZFs

In §3.1, we prove the product formulae of desingularized values at non-positive integer points. In §3.2, by using a combinatorial method, we prove a generalization of the equation (3.1.3) in Theorem 3.2.8 and general "shuffle product" of $\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)$ in Corollary 3.2.10. In §3.3, by using an analytic method, we prove a generalization (Theorem 3.3.7) of Theorem 3.1.3 and Corollary 3.2.10.

3.1 The product formulae at non-positive integer points

In this subsection, we prove the shuffle-type product formulae of desingularized values at non-positive integer points (Theorem 3.1.3).

Lemma 3.1.1. *For $r \in \mathbb{N}$, we have*

$$Z_{\mathrm{FKMT}}(u_1) \cdots Z_{\mathrm{FKMT}}(u_r) = Z_{\mathrm{FKMT}}(u_1 - u_2, u_2 - u_3, \ldots, u_{r-1} - u_r, u_r). \tag{3.1.1}$$

Proof. Let $r \in \mathbb{N}$. Using the equation (1.2.6) repeatedly, we get

$$Z_{\mathrm{FKMT}}(t_1, \ldots, t_r) = \prod_{i=1}^{r} Z_{\mathrm{FKMT}}(t_i + \cdots + t_r).$$

Setting $u_i = t_i + \cdots + t_r$ for $i = 1, \ldots, r$, the equation (3.1.1) follows. $\square$

We obtain the following lemma by direct calculation.

Lemma 3.1.2. *For $r \in \mathbb{N}$, $a_1, \ldots, a_r \in \mathbb{C}$ and $f : \mathbb{N}_0 \to \mathbb{C}$, we have*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(a_1 + \cdots + a_r)^k}{k!} f(k) &= \sum_{k=0}^{\infty} \frac{f(k)}{k!} \sum_{i_1 + \cdots + i_r = k} \frac{k!}{i_1! \cdots i_r!} a_1^{i_1} \cdots a_r^{i_r} \\ &= \sum_{i_1, \ldots, i_r = 0}^{\infty} \frac{a_1^{i_1} \cdots a_r^{i_r}}{i_1! \cdots i_r!} f(i_1 + \cdots + i_r). \end{aligned}$$

Using the above two lemmas, we have the following theorem.

Theorem 3.1.3. *For $p, q \in \mathbb{N}$ and $k_1, \ldots, k_p, l_1, \ldots, l_q \in \mathbb{N}_0$, we have*

$$\begin{aligned}&\zeta_p^{\mathrm{des}}(-k_1, \ldots, -k_p)\zeta_q^{\mathrm{des}}(-l_1, \ldots, -l_q)\\&= \sum_{\substack{i_1+j_1=l_1\\ \vdots \\ i_q+j_q=l_q}} \prod_{a=1}^{q} (-1)^{i_a} \binom{l_a}{i_a} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_{p-1}, -k_p - i_1 - \cdots - i_q, -j_1, \ldots, -j_q).\end{aligned} \tag{3.1.2}$$

Proof. Using the equation (1.2.6) repeatedly, we get

$$\begin{aligned}&Z_{\mathrm{FKMT}}(s_1, \ldots, s_p) Z_{\mathrm{FKMT}}(t_1, \ldots, t_q)\\&= Z_{\mathrm{FKMT}}(s_1 + \cdots + s_p) \cdots Z_{\mathrm{FKMT}}(s_p) Z_{\mathrm{FKMT}}(t_1 + \cdots + t_q) \cdots Z_{\mathrm{FKMT}}(t_q).\end{aligned}$$

By setting $u_i = \begin{cases} s_i + \cdots + s_p & (1 \leq i \leq p), \\ t_{i-p} + \cdots + t_q & (p+1 \leq i \leq p+q), \end{cases}$ and applying the equation (3.1.1) to the above equation, we have

$$\begin{aligned}
&= Z_{\mathrm{FKMT}}(s_1, \ldots, s_{p-1}, s_p - t_1 - \cdots - t_q, t_1, \ldots, t_q)\\
&= \sum_{k_1, \ldots, k_p \geq 0} \frac{(-s_1)^{k_1} \cdots (-s_{p-1})^{k_{p-1}} (-s_p + t_1 + \cdots + t_q)^{k_p}}{k_1! \cdots k_{p-1}! k_p!}\\
&\qquad \cdot \sum_{j_1, \ldots, j_q \geq 0} \frac{(-t_1)^{j_1} \cdots (-t_q)^{j_q}}{j_1! \cdots j_q!} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_p, -j_1, \ldots, -j_q)\\
&= \sum_{\substack{k_1, \ldots, k_{p-1} \geq 0 \\ j_1, \ldots, j_q \geq 0}} \frac{(-s_1)^{k_1} \cdots (-s_{p-1})^{k_{p-1}}}{k_1! \cdots k_{p-1}!} \frac{(-t_1)^{j_1} \cdots (-t_q)^{j_q}}{j_1! \cdots j_q!}\\
&\qquad \cdot \sum_{k_p \geq 0} \frac{(-s_p + t_1 + \cdots + t_q)^{k_p}}{k_p!} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_p, -j_1, \ldots, -j_q).
\end{aligned}$$

Using Lemma 3.1.2, we get

$$\begin{aligned}
&= \sum_{\substack{k_1, \ldots, k_{p-1} \geq 0 \\ j_1, \ldots, j_q \geq 0}} \frac{(-s_1)^{k_1} \cdots (-s_{p-1})^{k_{p-1}}}{k_1! \cdots k_{p-1}!} \frac{(-t_1)^{j_1} \cdots (-t_q)^{j_q}}{j_1! \cdots j_q!}\\
&\qquad \cdot \sum_{k_p, i_1, \ldots, i_q \geq 0} \frac{(-s_p)^{k_p} t_1^{i_1} \cdots t_q^{i_q}}{k_p! i_1! \cdots i_q!} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_p - i_1 - \cdots - i_q, -j_1, \ldots, -j_q)\\
&= \sum_{k_1, \ldots, k_p \geq 0} \frac{(-s_1)^{k_1} \cdots (-s_p)^{k_p}}{k_1! \cdots k_p!}\\
&\qquad \cdot \sum_{\substack{i_1, \ldots, i_q \geq 0 \\ j_1, \ldots, j_q \geq 0}} \frac{t_1^{i_1} \cdots t_q^{i_q}}{i_1! \cdots i_q!} \frac{(-t_1)^{j_1} \cdots (-t_q)^{j_q}}{j_1! \cdots j_q!} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_p - i_1 - \cdots - i_q, -j_1, \ldots, -j_q)\\
&= \sum_{k_1, \ldots, k_p \geq 0} \frac{(-s_1)^{k_1} \cdots (-s_p)^{k_p}}{k_1! \cdots k_p!}\\
&\qquad \cdot \sum_{\substack{i_1, \ldots, i_q \geq 0 \\ j_1, \ldots, j_q \geq 0}} \frac{(-t_1)^{i_1+j_1} \cdots (-t_q)^{i_q+j_q}}{i_1! \cdots i_q! j_1! \cdots j_q!} (-1)^{i_1 + \cdots + i_q} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_p - i_1 - \cdots - i_q, -j_1, \ldots, -j_q)
\end{aligned}$$

$$= \sum_{\substack{k_1,\ldots,k_p\geq 0\\ l_1,\ldots,l_q\geq 0}} \frac{(-s_1)^{k_1}\cdots(-s_p)^{k_p}}{k_1!\cdots k_p!}\frac{(-t_1)^{l_1}\cdots(-t_q)^{l_q}}{l_1!\cdots l_q!}$$

$$\cdot \sum_{\substack{i_1+j_1=l_1\\ \vdots\\ i_q+j_q=l_q}} \prod_{a=1}^{q}\binom{l_a}{i_a}(-1)^{i_a}\zeta^{\mathrm{des}}_{p+q}(-k_1,\ldots,-k_p-i_1-\cdots-i_q,-j_1,\ldots,-j_q).$$

On the other hand, by the definition of $Z_{\mathrm{FKMT}}(t_1,\ldots,t_q)$, we have

$$Z_{\mathrm{FKMT}}(s_1,\ldots,s_p)Z_{\mathrm{FKMT}}(t_1,\ldots,t_q)$$

$$=\left\{\sum_{k_1,\ldots,k_p\geq 0}\frac{(-s_1)^{k_1}\cdots(-s_p)^{k_p}}{k_1!\cdots k_p!}\zeta^{\mathrm{des}}_p(-k_1,\ldots,-k_p)\right\}$$

$$\cdot\left\{\sum_{l_1,\ldots,l_q\geq 0}\frac{(-t_1)^{l_1}\cdots(-t_q)^{l_q}}{l_1!\cdots l_q!}\zeta^{\mathrm{des}}_q(-l_1,\ldots,-l_q)\right\}$$

$$= \sum_{\substack{k_1,\ldots,k_p\geq 0\\ l_1,\ldots,l_q\geq 0}} \frac{(-s_1)^{k_1}\cdots(-s_p)^{k_p}}{k_1!\cdots k_p!}\frac{(-t_1)^{l_1}\cdots(-t_q)^{l_q}}{l_1!\cdots l_q!}\zeta^{\mathrm{des}}_p(-k_1,\ldots,-k_p)\zeta^{\mathrm{des}}_q(-l_1,\ldots,-l_q).$$

Therefore, we obtain the equation (3.1.2). □

Here are examples for $(p,q)=(1,1),\ (1,2)$.

Example 3.1.4. For $a,b,c\in\mathbb{N}_0$, we have

$$\zeta^{\mathrm{des}}_1(-a)\zeta^{\mathrm{des}}_1(-b)=\sum_{i_1+j_1=b}(-1)^{i_1}\binom{b}{i_1}\zeta^{\mathrm{des}}_2(-a-i_1,-j_1),$$

$$\zeta^{\mathrm{des}}_1(-a)\zeta^{\mathrm{des}}_2(-b,-c)=\sum_{\substack{i_1+j_1=b\\ i_2+j_2=c}}(-1)^{i_1+i_2}\binom{b}{i_1}\binom{c}{i_2}\zeta^{\mathrm{des}}_3(-a-i_1-i_2,-j_1,-j_2).$$

Remark 3.1.5. The above recurrence formula (3.1.2) also yields

$$\zeta^{\mathrm{des}}_r(-k_1,\ldots,-k_r)=\sum_{\substack{i+j=k_r\\ i,j\geq 0}}\binom{k_r}{i}\zeta^{\mathrm{des}}_{r-1}(-k_1,\ldots,-k_{r-2},-k_{r-1}-i)\zeta^{\mathrm{des}}_1(-j). \quad (3.1.3)$$

3.2 Combinatorial proof

We prove generalizations of (3.1.3) in Theorem 3.2.8. We assume $r\in\mathbb{N}_{\geq 2}$ in this subsection. We start with the following lemma on the property of the Pochhammer symbol.

Lemma 3.2.1. *For $a,b\in\mathbb{C}$ and $n\in\mathbb{N}_0$, we have*

$$(a+b)_n=\sum_{i+j=n}\binom{n}{i}(a)_i(b)_j.$$

Proof. We prove this claim by the induction on n. When $n = 0$, this claim holds clearly. Let $n \in \mathbb{N}_0$. We have

$$\begin{aligned}
(a+b)_{n+1} &= (a+b)_n(a+b+n) \\
&= \sum_{i+j=n} \binom{n}{i} (a)_i (b)_j (a+b+n) \\
&= \sum_{i+j=n} \binom{n}{i} (a)_i (b)_j (a+i) + \sum_{i+j=n} \binom{n}{i} (a)_i (b)_j (b+j) \\
&= \sum_{i+j=n} \binom{n}{i} (a)_{i+1} (b)_j + \sum_{i+j=n} \binom{n}{i} (a)_i (b)_{j+1} \\
&= \sum_{i+j=n+1} \binom{n}{i-1} (a)_i (b)_j + \sum_{i+j=n+1} \binom{n}{i} (a)_i (b)_j \\
&= \sum_{i+j=n+1} \left\{ \binom{n}{i-1} + \binom{n}{i} \right\} (a)_i (b)_j \\
&= \sum_{i+j=n+1} \binom{n+1}{i} (a)_i (b)_j.
\end{aligned}$$

□

The above lemma is used in the proof of Proposition 3.2.7.
Next, we prove a property of $\mathcal{G}_r((u_j);(v_j))$ defined by the equation (1.2.2).

Proposition 3.2.2. *We have*

$$\begin{aligned}
&\mathcal{G}_r \left(u_1, \ldots, u_r; v_1, \ldots, v_{r-1}, \frac{u_r + z}{u_r} v_{r-1} \right) \\
&\quad = (z+1) \mathcal{G}_{r-1}(u_1, \ldots, u_{r-2}, u_{r-1} + u_r + z; v_1, \ldots, v_{r-1}). \qquad (3.2.1)
\end{aligned}$$

Proof. By the definition of $\mathcal{G}_r((u_j);(v_j))$, we have

$$\begin{aligned}
&\mathcal{G}_r \left(u_1, \ldots, u_r; v_1, \ldots, v_{r-1}, \frac{u_r + z}{u_r} v_{r-1} \right) \\
&= \prod_{j=1}^{r-1} \left\{ 1 - \left(u_j v_j + \cdots + u_{r-1} v_{r-1} + u_r \frac{u_r + z}{u_r} v_{r-1} \right) (v_j^{-1} - v_{j-1}^{-1}) \right\} \\
&\quad \cdot \left\{ 1 - u_r \frac{u_r + z}{u_r} v_{r-1} \left(\left(\frac{u_r + z}{u_r} v_{r-1} \right)^{-1} - v_{r-1}^{-1} \right) \right\} \\
&= \prod_{j=1}^{r-1} \left\{ 1 - (u_j v_j + \cdots + u_{r-1} v_{r-1} + (u_r + z) v_{r-1}) (v_j^{-1} - v_{j-1}^{-1}) \right\} \\
&\quad \cdot \left\{ 1 - (u_r + z) v_{r-1} \left(\frac{u_r}{u_r + z} - 1 \right) v_{r-1}^{-1} \right\}
\end{aligned}$$

$$
\begin{aligned}
=&\prod_{j=1}^{r-1}\left\{1-(u_jv_j+\cdots+u_{r-2}v_{r-2}+(u_{r-1}+u_r+z)v_{r-1})\left(v_j^{-1}-v_{j-1}^{-1}\right)\right\}\\
&\cdot\{1-(u_r-(u_r+z))\}\\
=&(z+1)\mathcal{G}_{r-1}(u_1,\dots,u_{r-2},u_{r-1}+u_r+z;v_1,\dots,v_{r-1}).
\end{aligned}
$$

□

It is easy to prove the following lemma by comparing coefficients $a^r_{\boldsymbol{l},\boldsymbol{m}}$ of the equations (1.2.2) and (1.2.3).

Lemma 3.2.3. *Let $\boldsymbol{l}:=(l_j)\in\mathbb{N}_0^r$ and $\boldsymbol{m}:=(m_j)\in\mathbb{Z}^r$. If $m_r\neq l_r-1,l_r$ or $m_r<0$, then we have*

$$a^r_{\boldsymbol{l},\boldsymbol{m}}=0.$$

For our simplicities, we employ the following symbols:

Notation 3.2.4. Let $s_1,\dots,s_r$ and z be indeterminates. For r-tuple symbol $\boldsymbol{s}:=(s_1,\dots,s_r)$, the symbols $\boldsymbol{s}'$ and $\boldsymbol{s}^-$ are defined by

$$
\begin{aligned}
\boldsymbol{s}'&:=(s_1,\dots,s_{r-2},s_{r-1}+s_r),\\
\boldsymbol{s}^-&:=(s_1,\dots,s_{r-1}),\\
|\boldsymbol{s}|&:=s_1+\cdots+s_r,
\end{aligned}
$$

and we define $\boldsymbol{z}:=(\underbrace{0,\dots,0}_{r-1},z)$.

Lemma 3.2.5. *For the functions $f:\mathbb{Z}^r\to\mathbb{C}$ and $g:\mathbb{N}_0^{r+1}\to\mathbb{C}$ with*

$$\#\{\boldsymbol{n}\in\mathbb{Z}^r\mid f(\boldsymbol{n})\neq0\}<\infty \text{ and } \#\{\boldsymbol{a}\in\mathbb{N}_0^{r+1}\mid g(\boldsymbol{a})\neq0\}<\infty,$$

we have

$$\sum_{\substack{\boldsymbol{n}=(n_j)\in\mathbb{Z}^r\\|\boldsymbol{n}|=0}}f(\boldsymbol{n})=\sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}\sum_{\substack{p+q=m_{r-1}\\p,q\in\mathbb{Z}}}f(\boldsymbol{m}^-,p,q), \tag{3.2.2}$$

$$\sum_{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r}g(\boldsymbol{l}',l_{r-1},l_r)=\sum_{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1}}\sum_{\substack{p+q=k_{r-1}\\p,q\in\mathbb{N}_0}}g(\boldsymbol{k},p,q). \tag{3.2.3}$$

Proof. We only prove the equation (3.2.2), because the proof of the equation (3.2.3) can be done in the same way to that of the equation (3.2.2). We have

$$
\begin{aligned}
\sum_{\substack{\boldsymbol{n}=(n_j)\in\mathbb{Z}^r\\|\boldsymbol{n}|=0}}f(\boldsymbol{n})&=\sum_{n_1,\dots,n_{r-2},n_{r-1}\in\mathbb{Z}}f(n_1,\dots,n_{r-2},n_{r-1},-n_1-\cdots-n_{r-2}-n_{r-1})\\
&=\sum_{m_1,\dots,m_{r-2}\in\mathbb{Z}}\sum_{n_{r-1}\in\mathbb{Z}}f(m_1,\dots,m_{r-2},n_{r-1},-m_1-\cdots-m_{r-2}-n_{r-1}).
\end{aligned}
$$

When we put $m_{r-1}:=-m_1-\cdots-m_{r-2}$, then m_{r-1} can run over all integers. So we get

$$=\sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}\sum_{n_{r-1}\in\mathbb{Z}}f(m_1,\dots,m_{r-2},n_{r-1},m_{r-1}-n_{r-1}).$$

When we put $p := n_{r-1}$ and $q := m_{r-1} - n_{r-1}$, then p and q run over all integers with $p + q = m_{r-1}$. So we obtain

$$= \sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \sum_{\substack{p+q=m_{r-1}\\ p,q\in\mathbb{Z}}} f(\boldsymbol{m}^-, p, q).$$

□

Using Lemma 3.2.3 and Lemma 3.2.5, we get the following corollary.

Corollary 3.2.6. *For* $\boldsymbol{l} := (l_1, \ldots, l_r) \in \mathbb{N}_0^r$ *and* $\boldsymbol{m} := (m_1, \ldots, m_{r-1}) \in \mathbb{Z}^{r-1}$ *with* $|\boldsymbol{m}| = 0$*, we have*

$$\begin{aligned} a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)} &= \binom{l_{r-1}+l_r}{l_{r-1}} a^{r-1}_{\boldsymbol{l'},\boldsymbol{m}} \\ &= -a^r_{(\boldsymbol{l}^-,l_r+1),(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}. \end{aligned} \tag{3.2.4}$$

Proof. Let $r \in \mathbb{N}$. By the equation (3.2.1) and the equation (1.2.3) (the definition of the coefficient $a^r_{\boldsymbol{l},\boldsymbol{m}}$ of the function $\mathcal{G}_r$), we have

$$\text{L.H.S. of (3.2.1)} = \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^r\\ |\boldsymbol{n}|=0}} a^r_{\boldsymbol{l},\boldsymbol{n}} \left(\prod_{j=1}^{r} u_j^{l_j}\right) \left(\prod_{j=1}^{r-1} v_j^{n_j}\right) \left(\frac{u_r+z}{u_r} v_{r-1}\right)^{n_r}.$$

By using the equation (3.2.2) of Lemma 3.2.5, we have

$$\begin{aligned} &= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \sum_{\substack{p+q=m_{r-1}\\ p,q\in\mathbb{Z}}} a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)} \left(\prod_{j=1}^{r} u_j^{l_j}\right) \left(\prod_{j=1}^{r-2} v_j^{m_j}\right) v_{r-1}^{p} \left(\frac{u_r+z}{u_r} v_{r-1}\right)^{q} \\ &= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \sum_{\substack{p+q=m_{r-1}\\ p,q\in\mathbb{Z}}} a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)} \left(\prod_{j=1}^{r} u_j^{l_j}\right) \left(\frac{u_r+z}{u_r}\right)^{q} \left(\prod_{j=1}^{r-1} v_j^{m_j}\right). \end{aligned}$$

By Lemma 3.2.3, we get $a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)} = 0$ for $q \neq l_r - 1, l_r$. So we have

$$\begin{aligned} = \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \Bigg\{ & a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)} \left(\prod_{j=1}^{r-1} u_j^{l_j}\right) u_r(u_r+z)^{l_r-1} \left(\prod_{j=1}^{r-1} v_j^{m_j}\right) \\ & + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} \left(\prod_{j=1}^{r-1} u_j^{l_j}\right) (u_r+z)^{l_r} \left(\prod_{j=1}^{r-1} v_j^{m_j}\right) \Bigg\} \end{aligned}$$

$$= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \left\{ a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)} \left(\prod_{j=1}^{r-1} u_j^{l_j}\right)(-z)(u_r+z)^{l_r-1}\left(\prod_{j=1}^{r-1} v_j^{m_j}\right) \right.$$

$$+ a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)} \left(\prod_{j=1}^{r-1} u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1} v_j^{m_j}\right)$$

$$\left. + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} \left(\prod_{j=1}^{r-1} u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1} v_j^{m_j}\right) \right\}.$$

By Lemma 3.2.3, we get $a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}+1,-1)} = 0$ (i.e. the case of $l_r = 0$). By replacing $l_r - 1$ with l_r, we have

$$= -z \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \left\{ a^r_{(\boldsymbol{l}^-,l_r+1),(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} \left(\prod_{j=1}^{r-1} u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1} v_j^{m_j}\right)\right\}$$

$$+ \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \left\{ \left(a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)} \right) \right.$$

$$\left. \cdot \left(\prod_{j=1}^{r-1} u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1} v_j^{m_j}\right)\right\}. \tag{3.2.5}$$

On the other hand, we have

R.H.S. of (3.2.1)

$$= (z+1) \sum_{\substack{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1} \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} a^{r-1}_{\boldsymbol{k},\boldsymbol{m}} \left(\prod_{j=1}^{r-2} u_j^{k_j}\right)(u_{r-1}+u_r+z)^{k_{r-1}}\left(\prod_{j=1}^{r-1} v_j^{m_j}\right)$$

$$= (z+1) \sum_{\substack{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1} \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} a^{r-1}_{\boldsymbol{k},\boldsymbol{m}} \left(\prod_{j=1}^{r-2} u_j^{k_j}\right) \sum_{\substack{p+q=k_{r-1} \\ p,q\in\mathbb{N}_0}} \binom{k_{r-1}}{p} u_{r-1}^p (u_r+z)^q \left(\prod_{j=1}^{r-1} v_j^{m_j}\right).$$

By using the equation (3.2.3) of Lemma 3.2.5, we have

$$= (z+1) \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(n_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \binom{l_{r-1}+l_r}{l_{r-1}} a^{r-1}_{\boldsymbol{l}',\boldsymbol{m}} \left(\prod_{j=1}^{r-1} u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1} v_j^{m_j}\right). \tag{3.2.6}$$

We compare the coefficients of (3.2.5) and (3.2.6), then we obtain (3.2.4). □

By tracing proof of Corollary 3.2.6 inversely, we get the following proposition.

Proposition 3.2.7. *For $s_1, \dots, s_r, z \in \mathbb{C}$,*

$$\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^r \\ |\boldsymbol{n}|=0}} a^r_{\boldsymbol{l},\boldsymbol{n}} \left(\prod_{j=1}^{r}(s_j)_{l_j}\right) \frac{\Gamma(s_r+n_r+z)\Gamma(-z)}{\Gamma(s_r+n_r)} \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{n}'+\boldsymbol{z}')$$

$$= (1+z)\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta^{\mathrm{des}}_{r-1}(\boldsymbol{s}'+\boldsymbol{z}') \tag{3.2.7}$$

holds except for singularities.

Proof. Let $s_1, \dots, s_r, z \in \mathbb{C}$. Using Corollary 3.2.6, we have

$$\begin{aligned}
&(z+1)\sum_{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r} \binom{l_{r-1}+l_r}{l_{r-1}} a^{r-1}_{\boldsymbol{l}',\boldsymbol{m}} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r} \\
&= -z \sum_{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r} \left\{ a^r_{(\boldsymbol{l}^-,l_r+1),(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r} \right\} \\
&+ \sum_{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r} \left\{ \left(a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\right) \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r} \right\}.
\end{aligned} \tag{3.2.8}$$

By multiplying the function $\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')$ and taking summation over $\boldsymbol{m}\in\mathbb{Z}^{r-1}$ with $|\boldsymbol{m}|=0$, we have

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} (\text{R.H.S. of (3.2.8)}) \cdot \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \\
&= -z \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \left\{ a^r_{(\boldsymbol{l}^-,l_r+1),(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \right\} \\
&\quad + \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \left\{ \left(a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\right) \right. \\
&\qquad\qquad \left. \cdot \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \right\} \\
&= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \left\{ a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(-z)(s_r+z)_{l_r-1}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \right. \\
&\qquad + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \\
&\qquad \left. + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \right\}
\end{aligned}$$

$$= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \left\{ a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+l_r-1)(s_r+z)_{l_r-1}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \right.$$

$$\left. + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \right\}.$$

By Lemma 3.2.3, we get $a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)} = 0$ for $q \neq l_r - 1, l_r$. So we have

$$= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \sum_{\substack{p+q=m_{r-1} \\ p,q\in\mathbb{Z}}} a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)} \left(\prod_{j=1}^{r}(s_j)_{l_j}\right) \frac{(s_r+z)_q}{(s_r)_q}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}').$$

By using the equation (3.2.2) of Lemma 3.2.5, we have

$$= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^{r} \\ |\boldsymbol{n}|=0}} a^r_{\boldsymbol{l},\boldsymbol{n}} \left(\prod_{j=1}^{r}(s_j)_{l_j}\right) \frac{(s_r+z)_{n_r}}{(s_r)_{n_r}}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{n}'+\boldsymbol{z}'). \tag{3.2.9}$$

We have $\Gamma(s+n) = (s)_n\Gamma(s)$ for $s\in\mathbb{C}$ and $n\in\mathbb{N}_0$, by the relation $\Gamma(s+1)=s\Gamma(s)$. By multiplying the equation (3.2.9) with $\Gamma(s_r+z)\Gamma(-z)/\Gamma(s_r)$, we obtain

$$\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \cdot (\text{the equation (3.2.9)})$$

$$= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^{r} \\ |\boldsymbol{n}|=0}} a^r_{\boldsymbol{l},\boldsymbol{n}} \left(\prod_{j=1}^{r}(s_j)_{l_j}\right) \frac{\Gamma(s_r+n_r+z)\Gamma(-z)}{\Gamma(s_r+n_r)}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{n}'+\boldsymbol{z}'). \tag{3.2.10}$$

On the other hand, we have

$$\sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} (\text{L.H.S. of (3.2.8)}) \cdot \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')$$

$$= (z+1) \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \binom{l_{r-1}+l_r}{l_{r-1}} a^{r-1}_{\boldsymbol{l}',\boldsymbol{m}} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}').$$

By using the equation (3.2.3) of Lemma 3.2.5, we have

$$= (z+1) \sum_{\substack{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1} \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} \sum_{\substack{p+q=k_{r-1} \\ p,q\in\mathbb{N}_0}} \binom{k_{r-1}}{p} a^{r-1}_{\boldsymbol{k},\boldsymbol{m}}$$

$$\cdot \left(\prod_{j=1}^{r-2}(s_j)_{k_j}\right)(s_{r-1})_p(s_r+z)_q\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}').$$

Using Lemma 3.2.1, we have

$$= (z+1) \sum_{\substack{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1} \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1} \\ |\boldsymbol{m}|=0}} a^{r-1}_{\boldsymbol{k},\boldsymbol{m}} \left(\prod_{j=1}^{r-2} (s_j)_{k_j}\right) (s_{r-1}+s_r+z)_{k_r-1} \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}'). \tag{3.2.11}$$

By multiplying the equation (3.2.11) with $\Gamma(s_r+z)\Gamma(-z)/\Gamma(s_r)$ and by the equation (1.2.4) of the desingularized function $\zeta_r^{\mathrm{des}}(\boldsymbol{s})$, we obtain

$$\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \cdot (\text{the equation (3.2.11)}) = (1+z)\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}'+\boldsymbol{z}'). \tag{3.2.12}$$

So we obtain the equation (3.2.7), by combining the equations (3.2.10) and (3.2.12) because we have (3.2.9) = (3.2.11). □

Theorem 3.2.8. *For $s_1, \ldots, s_{r-1} \in \mathbb{C}$ and $k \in \mathbb{N}_0$, we have*

$$\zeta_r^{\mathrm{des}}(s_1, \ldots, s_{r-1}, -k) = \sum_{\substack{i+j=k \\ i,j\geq 0}} \binom{k}{i} \zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-2}, s_{r-1}-i)\zeta_1^{\mathrm{des}}(-j). \tag{3.2.13}$$

Proof. Let $\boldsymbol{s} := (s_1, \ldots, s_r) \in \mathbb{C}^r$. We recall the equation (1.1.3):

$$\zeta_r(\boldsymbol{s}) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{z}')\zeta(-z)dz,$$

for $\Re(s_j) > 1$ $(1 \leq j \leq r)$, $-\Re(s_r) < c < 0$ and the path of integration is the vertical line $\Re(z) = c$. By this formula and the definition of $\zeta_r^{\mathrm{des}}(\boldsymbol{s})$, we have

$$\begin{aligned}
\zeta_r^{\mathrm{des}}(\boldsymbol{s}) = & \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^{r} \\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^{r} \\ |\boldsymbol{n}|=0}} a^{r}_{\boldsymbol{l},\boldsymbol{n}} \left(\prod_{j=1}^{r} (s_j)_{l_j}\right) \zeta(\boldsymbol{s}+\boldsymbol{n}). \\
= & \frac{1}{2\pi i}\int_{(c)} \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^{r} \\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^{r} \\ |\boldsymbol{n}|=0}} a^{r}_{\boldsymbol{l},\boldsymbol{n}} \left(\prod_{j=1}^{r} (s_j)_{l_j}\right) \frac{\Gamma(s_r+n_r+z)\Gamma(-z)}{\Gamma(s_r+n_r)} \\
& \cdot \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{n}'+\boldsymbol{z}')\zeta(-z)dz.
\end{aligned}$$

Using Proposition 3.2.7, we get

$$= \frac{1}{2\pi i}\int_{(c)} (1+z)\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}'+\boldsymbol{z}')\zeta(-z)dz.$$

By Proposition 1.2.3, we have the formula $\zeta_1^{\mathrm{des}}(s) = (1-s)\zeta(s)$, so we obtain

$$= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}'+\boldsymbol{z}')\zeta_1^{\mathrm{des}}(-z)dz.$$

For $M \in \mathbb{N}$ and sufficiently small $\varepsilon > 0$, we set $\mathcal{D} := \{z \in \mathbb{C} \mid c < \Re(z) < M - \varepsilon\}$. For $z \in \mathcal{D}$, we have $\Re(s_r + z) > 0$ by $-\Re(s_r) < c < 0$. So singularities of the above integrand, which lie on $\mathcal{D}$, are only at $z = 0, 1, 2, \ldots, M-1$. By using the residue theorem, we get

$$\begin{aligned} &= \sum_{j=0}^{M-1} \operatorname{Res}\left[\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}' + \boldsymbol{z}')\zeta_1^{\mathrm{des}}(-z), z = j\right] \\ &\quad + \frac{1}{2\pi i}\int_{(M-\varepsilon)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}' + \boldsymbol{z}')\zeta_1^{\mathrm{des}}(-z)dz \end{aligned}$$

(By the same arguments as in [Mat], the second term above converge). By using the fact that the residue of gamma function $\Gamma(s)$ at $s = -j$ is $\frac{(-1)^n}{n!}$, we get

$$\begin{aligned} &= \sum_{j=0}^{M-1} \binom{-s_r}{j} \zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + j)\zeta_1^{\mathrm{des}}(-j) \\ &\quad + \frac{1}{2\pi i \Gamma(s_r)}\int_{(M-\varepsilon)} \Gamma(s_r+z)\Gamma(-z)\zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}' + \boldsymbol{z}')\zeta_1^{\mathrm{des}}(-z)dz. \end{aligned}$$

Setting $s_r = -k$ and $M = k+1$ for $k \in \mathbb{N}_0$, we obtain

$$\zeta_r^{\mathrm{des}}(s_1, \ldots, s_{r-1}, -k) = \sum_{j=0}^{k} \binom{k}{j} \zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-2}, s_{r-1} - k + j)\zeta_1^{\mathrm{des}}(-j),$$

because $1/\Gamma(-k) = 0$ for $k \in \mathbb{N}_0$. □

Remark 3.2.9. In case of $r = 2$, the above theorem recovers the equation

$$\zeta_2^{\mathrm{des}}(s, -N) = \sum_{i+j=N} \binom{N}{i} \zeta_1^{\mathrm{des}}(s-i)\zeta_1^{\mathrm{des}}(-j),$$

which is equivalent to (1.2.5) in the paper [FKMT17b, Proposition 4.3].

By Theorem 3.2.8, we obtain the following corollary.

Corollary 3.2.10. *For $s_1, \ldots, s_{r-1} \in \mathbb{C}$ and $l \in \mathbb{N}_0$, we have*

$$\zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-1})\zeta_1^{\mathrm{des}}(-l) = \sum_{i+j=l} (-1)^i \binom{l}{i} \zeta_r^{\mathrm{des}}(s_1, \ldots, s_{r-2}, s_{r-1} - i, -j). \tag{3.2.14}$$

Proof. We prove this claim by induction on l. It is clear that the case of $l = 0$ follows from the case of $k = 0$ of Theorem 3.2.8. By putting $k = l_0$ (≥ 1) in the equation (3.2.13), we get

$$\begin{aligned} &\zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-1})\zeta_1^{\mathrm{des}}(-l_0) \\ &\qquad = \zeta_r^{\mathrm{des}}(s_1, \ldots, s_{r-1}, -l_0) - \sum_{j=0}^{l_0-1} \binom{l_0}{j} \zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-2}, s_{r-1} - l_0 + j)\zeta_1^{\mathrm{des}}(-j). \end{aligned}$$

In the second term of the right hand side of this equation, we obtain the equation (3.2.14) of $l = l_0$ by using our induction hypothesis (i.e. the equation (3.2.14) in the case of $0 \leq l \leq l_0 - 1$). □

3.3 Analytic proof

As a generalization of Corollary 3.2.10 in §3.2, we give an analytic proof of shuffle-type product formulae between $\zeta_p^{\rm des}(s_1,\ldots,s_p)$ and $\zeta_q^{\rm des}(-l_1,\ldots,-l_q)$ in Theorem 3.3.7. We assume $r\in\mathbb{N}_{\geq 2}$ in this section. In [FKMT17a], the *multiple zeta-function of the generalized Euler-Zagier type* is defined by

$$\zeta_r(s_1,\ldots,s_r;\gamma_1,\ldots,\gamma_r):=\sum_{m_1,\ldots,m_r\geq 1}\prod_{k=1}^{r}(\gamma_1 m_1+\cdots+\gamma_k m_k)^{-s_k},$$

for $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ with the condition $\Re(\gamma_j)>0\quad(1\leq j\leq r)$. This series absolutely converges in the region

$$\{(s_1,\ldots,s_r)\in\mathbb{C}^r\mid \Re(s_{r-k+1}+\cdots+s_r)>k\ (1\leq k\leq r)\}. \tag{3.3.1}$$

In [Mat], it is proved that the function $\zeta_r(s_1,\ldots,s_r;\gamma_1,\ldots,\gamma_r)$ can be meromorphically continued to $\mathbb{C}^r$. For simplicity, we sometimes denote it by $\zeta_r((s_j);(\gamma_j))$.

Lemma 3.3.1. *For $s_1,\ldots,s_r\in\mathbb{C}$, we have*

$$\zeta_r^{\rm des}(s_1,\ldots,s_r)=\lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}}\frac{1}{(1-c)^r}\sum_{\delta_1,\ldots,\delta_r\in\{0,1\}}(-c)^{\delta_1+\cdots+\delta_r}\zeta_r(s_1,\ldots,s_r;c^{\delta_1},\ldots,c^{\delta_r}). \tag{3.3.2}$$

Proof. Let $c>0$ such that $|c-1|$ is sufficiently small. We assume $(s_1,\ldots,s_r)\in\mathbb{C}^r$ satisfies

$$\Re(s_{r-k+1}+\cdots+s_r)>k\quad(1\leq k\leq r).$$

Then, we have

$$\begin{aligned}&\lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}}\frac{1}{(1-c)^r}\sum_{\delta_1,\ldots,\delta_r\in\{0,1\}}(-c)^{\delta_1+\cdots+\delta_r}\zeta_r(s_1,\ldots,s_r;c^{\delta_1},\ldots,c^{\delta_r})\\ &=\lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}}\frac{1}{(1-c)^r}\sum_{\delta_1,\ldots,\delta_r\in\{0,1\}}(-c)^{\delta_1+\cdots+\delta_r}\sum_{m_1,\ldots,m_r\geq 1}\prod_{k=1}^{r}\left(c^{\delta_1}m_1+\cdots+c^{\delta_k}m_k\right)^{-s_k}.\end{aligned}$$

Because we have

$$m^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-tm}t^{s-1}dt$$

by using the Mellin transformation, we get

$$\begin{aligned}&\lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}}\frac{1}{(1-c)^r}\sum_{\delta_1,\ldots,\delta_r\in\{0,1\}}(-c)^{\delta_1+\cdots+\delta_r}\zeta_r(s_1,\ldots,s_r;c^{\delta_1},\ldots,c^{\delta_r})\\ &\quad=\lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}}\frac{1}{(1-c)^r}\sum_{\delta_1,\ldots,\delta_r\in\{0,1\}}(-c)^{\delta_1+\cdots+\delta_r}\\ &\qquad\qquad\cdot\sum_{m_1,\ldots,m_r\geq 1}\prod_{k=1}^{r}\left\{\frac{1}{\Gamma(s_k)}\int_0^\infty e^{-t_k\sum_{j=1}^{k}c^{\delta_j}m_j}t_k^{s_k-1}dt_k\right\}\\ &\quad=\lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}}\frac{1}{(1-c)^r}\sum_{\delta_1,\ldots,\delta_r\in\{0,1\}}(-c)^{\delta_1+\cdots+\delta_r}\sum_{m_1,\ldots,m_r\geq 1}\\ &\qquad\qquad\cdot\prod_{k=1}^{r}\frac{1}{\Gamma(s_k)}\int_{(0,\infty)^r}\prod_{n=1}^{r}e^{-t_n\sum_{j=1}^{n}c^{\delta_j}m_j}\prod_{l=1}^{r}t_l^{s_l-1}dt_l.\end{aligned}$$

By using $\prod_{n=1}^{r} e^{-t_n \sum_{j=1}^{n} c^{\delta_j} m_j} = \prod_{j=1}^{r} e^{-m_j c^{\delta_j} \sum_{n=j}^{r} t_n}$, we have

$$
\begin{aligned}
= \lim_{\substack{c \to 1 \\ c \in \mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^r} \sum_{\delta_1,\ldots,\delta_r \in \{0,1\}} (-c)^{\delta_1+\cdots+\delta_r} \sum_{m_1,\ldots,m_r \geq 1} \\
\cdot \prod_{k=1}^{r} \frac{1}{\Gamma(s_k)} \int_{(0,\infty)^r} \prod_{j=1}^{r} e^{-m_j c^{\delta_j} \sum_{n=j}^{r} t_n} \prod_{l=1}^{r} t_l^{s_l - 1} dt_l.
\end{aligned}
$$

Because $\zeta((s_j);(c^{\delta_j}))$ converges absolutely, the integral $\int_{(0,\infty)^r}$ and the sum $\sum_{m_1,\ldots,m_r \geq 1}$ can be interchanged. So we have

$$
\begin{aligned}
&\lim_{\substack{c \to 1 \\ c \in \mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^r} \sum_{\delta_1,\ldots,\delta_r \in \{0,1\}} (-c)^{\delta_1+\cdots+\delta_r} \zeta_r(s_1,\ldots,s_r; c^{\delta_1},\ldots,c^{\delta_r}) \\
&= \lim_{\substack{c \to 1 \\ c \in \mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^r} \prod_{k=1}^{r} \frac{1}{\Gamma(s_k)} \\
&\quad \cdot \int_{(0,\infty)^r} \sum_{\delta_1,\ldots,\delta_r \in \{0,1\}} (-c)^{\delta_1+\cdots+\delta_r} \sum_{m_1,\ldots,m_r \geq 1} \prod_{j=1}^{r} e^{-m_j c^{\delta_j} \sum_{n=j}^{r} t_n} \prod_{l=1}^{r} t_l^{s_l-1} dt_l \\
&= \lim_{\substack{c \to 1 \\ c \in \mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^r} \prod_{k=1}^{r} \frac{1}{\Gamma(s_k)} \\
&\quad \cdot \int_{(0,\infty)^r} \prod_{j=1}^{r} \left\{ \sum_{\delta_j \in \{0,1\}} (-c)^{\delta_j} \sum_{m_j \geq 1} e^{-m_j c^{\delta_j} \sum_{n=j}^{r} t_n} \right\} \prod_{l=1}^{r} t_l^{s_l-1} dt_l.
\end{aligned}
$$

By using the definition of $\tilde{\mathfrak{H}}_r$ and the following formula

$$
\frac{1}{e^y - 1} - \frac{c}{e^{cy} - 1} = \sum_{m \geq 1} e^{-my} - c \sum_{m \geq 1} e^{-mcy} = \sum_{\delta \in \{0,1\}} (-c)^{\delta} \sum_{m \geq 1} e^{-mc^{\delta} y},
$$

for $y > 0$, we get

$$
\begin{aligned}
&\lim_{\substack{c \to 1 \\ c \in \mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^r} \sum_{\delta_1,\ldots,\delta_r \in \{0,1\}} (-c)^{\delta_1+\cdots+\delta_r} \zeta_r(s_1,\ldots,s_r; c^{\delta_1},\ldots,c^{\delta_r}) \\
&= \lim_{\substack{c \to 1 \\ c \in \mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^r} \prod_{k=1}^{r} \frac{1}{\Gamma(s_k)} \int_{(0,\infty)^r} \tilde{\mathfrak{H}}_r(t_1,\ldots,t_r; c) \prod_{l=1}^{r} t_l^{s_l-1} dt_l \\
&= \zeta_r^{\mathrm{des}}(s_1,\ldots,s_r).
\end{aligned}
$$

Therefore, we get the claim for $(s_1,\ldots,s_r) \in \mathbb{C}^r$ with

$$
\Re(s_{r-k+1} + \cdots + s_r) > k \quad (1 \leq k \leq r).
$$

Because $\zeta_r^{\mathrm{des}}(s_1,\ldots,s_r)$ and $\zeta_r((s_j);(\gamma_j))$ are meromorphic on $\mathbb{C}^r$ and the limit of meromorphic functions is also meromorphic, the equation (3.3.2) holds for $(s_1,\ldots,s_r) \in \mathbb{C}^r$. $\square$

Lemma 3.3.2. *Let $\gamma_1, \ldots, \gamma_r > 0$, $s_1, \ldots, s_r \in \mathbb{C}$ with $\Re(s_j) > 1$ $(1 \leq j \leq r)$. Put $1 \leq t \leq r-1$ and take $a_{t+1}, \ldots, a_r \in \mathbb{R}$ with $-\Re(s_k) < a_k < 0$ $(t+1 \leq k \leq r)$. Then, we have*

$$\begin{aligned}\zeta_r((s_j);(\gamma_j)) = \left(\frac{1}{2\pi i}\right)^{r-t} \int_{(a_{t+1})\times\cdots\times(a_r)} \prod_{k=t+1}^{r} \frac{\Gamma(s_k+z_k)\Gamma(-z_k)}{\Gamma(s_k)} \\ \cdot \zeta_t\left(s_1, \ldots, s_{t-1}, s_t + \sum_{j=t+1}^{r}(s_j+z_j); \gamma_1, \ldots, \gamma_t\right) \\ \cdot \zeta_{r-t}(-z_{t+1}, \ldots, -z_r; \gamma_{t+1}, \ldots, \gamma_r) \prod_{l=t+1}^{r} dz_l. \qquad (3.3.3)\end{aligned}$$

Here, the symbol (a_k) is the path of integration on the vertical line $\Re(z_k) = a_k$ from $a_k - i\infty$ to $a_k + i\infty$, for $t+1 \leq k \leq r$.

Proof. Consider Mellin-Barnes integral formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(a)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \lambda^z dz,$$

where $\lambda, s \in \mathbb{C}$, $\lambda \neq 0$, $|\arg \lambda| < \pi$, $\Re(s) > 0$, $-\Re(s) < a < 0$.

For $m_1, \ldots, m_r \geq 1$, by putting $\lambda = \frac{\gamma_{t+1}m_{t+1}+\cdots+\gamma_j m_j}{\gamma_1 m_1+\cdots+\gamma_t m_t}$ and $s = s_j$ and $a = a_j$ for $j = t+1, \ldots, r$ $(1 \leq t \leq r-1)$, we have

$$\left(\frac{\gamma_1 m_1 + \cdots + \gamma_j m_j}{\gamma_1 m_1 + \cdots + \gamma_t m_t}\right)^{-s_j} = \frac{1}{2\pi i}\int_{(a_j)} \frac{\Gamma(s_j+z_j)\Gamma(-z_j)}{\Gamma(s_j)} \left(\frac{\gamma_{t+1}m_{t+1} + \cdots + \gamma_j m_j}{\gamma_1 m_1 + \cdots + \gamma_t m_t}\right)^{z_j} dz_j.$$

So we get

$$\begin{aligned}&(\gamma_1 m_1 + \cdots + \gamma_j m_j)^{-s_j} \\ &= \frac{1}{2\pi i}\int_{(a_j)} \frac{\Gamma(s_j+z_j)\Gamma(-z_j)}{\Gamma(s_j)} (\gamma_1 m_1 + \cdots + \gamma_t m_t)^{-s_j-z_j} (\gamma_{t+1}m_{t+1} + \cdots + \gamma_j m_j)^{z_j} dz_j.\end{aligned}$$

Taking product over $j = t+1, \ldots, r$ and taking summation over $m_{t+1}, \ldots, m_r \geq 1$, we have

$$\begin{aligned}&\sum_{m_{t+1},\ldots,m_r \geq 1} \prod_{j=t+1}^{r} (\gamma_1 m_1 + \cdots + \gamma_j m_j)^{-s_j} \\ &= \left(\frac{1}{2\pi i}\right)^{r-t} \sum_{m_{t+1},\ldots,m_r \geq 1} \int_{(a_{t+1})\times\cdots\times(a_r)} \prod_{k=t+1}^{r} \frac{\Gamma(s_k+z_k)\Gamma(-z_k)}{\Gamma(s_k)} \\ &\cdot (\gamma_1 m_1 + \cdots + \gamma_t m_t)^{-\sum_{j=t+1}^{r}(s_j+z_j)} \prod_{j=t+1}^{r} (\gamma_{t+1}m_{t+1} + \cdots + \gamma_j m_j)^{z_j} \prod_{l=t+1}^{r} dz_l. \qquad (3.3.4)\end{aligned}$$

By multiplying Equation (3.3.4) by $\prod_{j=1}^{t}(\gamma_1 m_1 + \cdots + \gamma_j m_j)^{-s_j}$ and taking summation over $m_1, \ldots, m_t \geq 1$, we see that the left hand side becomes $\zeta_r((s_j);(\gamma_j))$. The series $\zeta_r((s_j);(\gamma_j))$ converges absolutely in the region (3.3.1) and we have $\Re(s_j) > 1$ $(1 \leq j \leq r)$, so we get the equation (3.3.3). □

We set $(-z_1,\ldots,-z_t) := \left(s_1,\ldots,s_{t-1},s_t+\sum_{j=t+1}^r (s_j+z_j)\right)$.

Lemma 3.3.3. *Let $c\in\mathbb{R}\backslash\{1\}$ satisfying that $|c-1|$ is sufficiently small. Let $s_1,\ldots,s_r\in\mathbb{C}$ with $\Re(s_k)>1$ and let $a_j\in\mathbb{R}$ with $-\Re(s_j)<a_j<-1$ $(t+1\le j\le r)$. Then, the integral*

$$\int_{(a_{t+1})\times\cdots\times(a_r)} \left\{\prod_{k=t+1}^r \frac{\Gamma(s_k+z_k)\Gamma(-z_k)}{\Gamma(s_k)}\right\} \frac{1}{(1-c)^r} \sum_{\delta_1,\ldots,\delta_r\in\{0,1\}} \cdot(-c)^{\delta_1+\cdots+\delta_r}\zeta_t\left(-z_1,\ldots,-z_t;c^{\delta_1},\ldots,c^{\delta_t}\right)\cdot\zeta_{r-t}(-z_{t+1},\ldots,-z_r;c^{\delta_{t+1}},\ldots,c^{\delta_r})\prod_{l=t+1}^r dz_l. \tag{3.3.5}$$

converges uniformly.

Proof. We have

$$\begin{aligned}
&\prod_{j=1}^t\left\{\frac{1}{\exp\left(\sum_{k=j}^t u_k\right)-1}-\frac{c}{\exp\left(c\sum_{k=j}^t u_k\right)-1}\right\}\\
&\qquad\cdot\prod_{j=t+1}^r\left\{\frac{1}{\exp\left(\sum_{k=j}^r u_k\right)-1}-\frac{c}{\exp\left(c\sum_{k=j}^r u_k\right)-1}\right\}\\
&=\prod_{j=1}^t\left\{\sum_{\delta_j\in\{0,1\}}\frac{(-c)^{\delta_j}}{\exp\left(c^{\delta_j}\sum_{k=j}^t u_k\right)-1}\right\}\cdot\prod_{j=t+1}^r\left\{\sum_{\delta_j\in\{0,1\}}\frac{(-c)^{\delta_j}}{\exp\left(c^{\delta_j}\sum_{k=j}^r u_k\right)-1}\right\}\\
&=\sum_{\delta_1,\ldots,\delta_r\in\{0,1\}}(-c)^{\delta_1+\cdots+\delta_r}\left\{\prod_{j=1}^t\frac{1}{\exp\left(c^{\delta_j}\sum_{k=j}^t u_k\right)-1}\right\}\left\{\prod_{j=t+1}^r\frac{1}{\exp\left(c^{\delta_j}\sum_{k=j}^r u_k\right)-1}\right\}.
\end{aligned}$$

By using this and the following integral expression of $\zeta_r((s_j);(\gamma_j))$

$$\zeta_r((s_j);(\gamma_j))=\prod_{k=1}^r\frac{1}{\Gamma(s_k)}\int_{(0,\infty)^r}\prod_{j=1}^r\frac{1}{\exp\left(\gamma_j\sum_{k=j}^r u_k\right)-1}\prod_{l=1}^r u_l^{s_l-1}du_l,$$

we have

$$\begin{aligned}
&\frac{1}{(1-c)^r}\sum_{\delta_1,\ldots,\delta_r\in\{0,1\}}(-c)^{\delta_1+\cdots+\delta_r}\zeta_t\left(-z_1,\ldots,-z_t;c^{\delta_1},\ldots,c^{\delta_t}\right)\\
&\qquad\qquad\cdot\zeta_{r-t}(-z_{t+1},\ldots,-z_r;c^{\delta_{t+1}},\ldots,c^{\delta_r})\\
&=\frac{1}{(1-c)^r}\prod_{k=1}^r\frac{1}{\Gamma(-z_k)}\int_{(0,\infty)^r}\prod_{j=1}^t\left\{\frac{1}{\exp\left(\sum_{k=j}^t u_k\right)-1}-\frac{c}{\exp\left(c\sum_{k=j}^t u_k\right)-1}\right\}\\
&\qquad\cdot\prod_{j=t+1}^r\left\{\frac{1}{\exp\left(\sum_{k=j}^r u_k\right)-1}-\frac{c}{\exp\left(c\sum_{k=j}^r u_k\right)-1}\right\}\prod_{l=1}^r u_l^{-z_l-1}du_l.
\end{aligned}$$

By [FKMT17a, Lemma 3.6], for $c \in \mathbb{R} \setminus \{1\}$ such that $|c-1|$ is sufficiently small, we have a constant $A > 0$ independent of c such that

$$\left|\frac{1}{c-1}\right|\left|\frac{1}{e^y-1}-\frac{c}{e^{cy}-1}\right| < Ae^{-y/2}$$

holds for any $y > 0$. Therefore, we get

$$\begin{aligned}
&\Bigg|\frac{1}{(1-c)^r}\sum_{\delta_1,\dots,\delta_r\in\{0,1\}}(-c)^{\delta_1+\cdots+\delta_r}\zeta_t\left(-z_1,\dots,-z_t;c^{\delta_1},\dots,c^{\delta_t}\right)\\
&\qquad\qquad\qquad\cdot\zeta_{r-t}(-z_{t+1},\dots,-z_r;c^{\delta_{t+1}},\dots,c^{\delta_r})\Bigg|\\
&\le\prod_{k=1}^r\frac{1}{|\Gamma(-z_k)|}\int_{(0,\infty)^r}\left\{\prod_{j=1}^t A\exp\left(-\frac12\sum_{k=j}^t u_k\right)\right\}\\
&\qquad\qquad\cdot\left\{\prod_{j=t+1}^r A\exp\left(-\frac12\sum_{k=j}^r u_k\right)\right\}\prod_{l=1}^r u_l^{-\Re(z_l)-1}du_l\\
&=\prod_{k=1}^r\frac{A}{|\Gamma(-z_k)|}\int_{(0,\infty)^r}\exp\left(-\frac12\sum_{j=1}^t\sum_{k=j}^t u_k\right)\exp\left(-\frac12\sum_{j=t+1}^r\sum_{k=j}^r u_k\right)\prod_{l=1}^r u_l^{-\Re(z_l)-1}du_l\\
&=\prod_{k=1}^r\frac{A}{|\Gamma(-z_k)|}\int_{(0,\infty)^r}\exp\left(-\frac12\sum_{k=1}^t ku_k-\frac12\sum_{k=t+1}^r(k-t)u_k\right)\prod_{l=1}^r u_l^{-\Re(z_l)-1}du_l\\
&=\left\{\prod_{k=1}^r\frac{A}{|\Gamma(-z_k)|}\right\}\prod_{k=1}^t\left\{\int_0^\infty\exp\left(-\frac{k}{2}u_k\right)u_k^{-\Re(z_k)-1}du_k\right\}\\
&\qquad\qquad\cdot\prod_{k=t+1}^r\left\{\int_0^\infty\exp\left(-\frac{k-t}{2}u_k\right)u_k^{-\Re(z_k)-1}du_k\right\}.
\end{aligned}$$

Because we have

$$n^{-s}\Gamma(s)=\int_0^\infty\exp(-nu)u^{s-1}du$$

for $\Re(s) > 0$ and $n \in \mathbb{R}_{>0}$ and we get $\Re(z_k) > 0$ for $1 \le k \le r$, we obtain the following inequality on the formula (3.3.5):

$$\begin{aligned}
&\Bigg|\int_{(a_{t+1})\times\cdots\times(a_r)}\left\{\prod_{k=t+1}^r\frac{\Gamma(s_k+z_k)\Gamma(-z_k)}{\Gamma(s_k)}\right\}\frac{1}{(1-c)^r}\sum_{\delta_1,\dots,\delta_r\in\{0,1\}}\\
&\quad\cdot(-c)^{\delta_1+\cdots+\delta_r}\zeta_t\left(-z_1,\dots,-z_t;c^{\delta_1},\dots,c^{\delta_t}\right)\cdot\zeta_{r-t}(-z_{t+1},\dots,-z_r;c^{\delta_{t+1}},\dots,c^{\delta_r})\prod_{l=t+1}^r dz_l\Bigg|\\
&\le\int_{(a_{t+1})\times\cdots\times(a_r)}\left\{\prod_{k=t+1}^r\frac{|\Gamma(s_k+z_k)\Gamma(-z_k)|}{|\Gamma(s_k)|}\right\}\left\{\prod_{k=1}^r\frac{A}{|\Gamma(-z_k)|}\right\}\\
&\qquad\cdot\prod_{k=1}^t\left\{\left(\frac{k}{2}\right)^{\Re(z_k)}\Gamma\big(\Re(z_k)\big)\right\}\prod_{k=t+1}^r\left\{\left(\frac{k-t}{2}\right)^{\Re(z_k)}\Gamma\big(\Re(z_k)\big)\right\}\prod_{l=t+1}^r|dz_l|.
\end{aligned}$$

On the above integral paths, we have $\Re(z_k) = a_k$ $(t+1 \le k \le r)$ and $-z_k = s_k$ $(1 \le k \le t-1)$ and $-z_t = s_t + \sum_{j=t+1}^{r}(s_j+z_j)$. So we put

$$C := A^r \left\{ \prod_{k=t+1}^{r} \frac{1}{|\Gamma(s_k)|} \right\} \left\{ \prod_{k=1}^{t-1} \frac{1}{|\Gamma(-z_k)|} \right\} \cdot \prod_{k=1}^{t} \left\{ \left(\frac{k}{2}\right)^{\Re(z_k)} \Gamma\big(\Re(z_k)\big) \right\} \prod_{k=t+1}^{r} \left\{ \left(\frac{k-t}{2}\right)^{\Re(z_k)} \Gamma\big(\Re(z_k)\big) \right\}.$$

Then this symbol C is independent on $z_{t+1}, \ldots, z_r$. Therefore, we get

$$\Bigg| \int_{(a_{t+1})\times\cdots\times(a_r)} \left\{ \prod_{k=t+1}^{r} \frac{\Gamma(s_k+z_k)\Gamma(-z_k)}{\Gamma(s_k)} \right\} \frac{1}{(1-c)^r} \sum_{\delta_1,\ldots,\delta_r \in \{0,1\}} \cdot (-c)^{\delta_1+\cdots+\delta_r} \zeta_t\left(-z_1,\ldots,-z_t; c^{\delta_1},\ldots,c^{\delta_t}\right) \cdot \zeta_{r-t}(-z_{t+1},\ldots,-z_r; c^{\delta_{t+1}},\ldots,c^{\delta_r}) \prod_{l=t+1}^{r} dz_l \Bigg|$$
$$\le C \int_{(a_{t+1})\times\cdots\times(a_r)} \prod_{k=t+1}^{r} |\Gamma(s_k+z_k)| \frac{1}{\left|\Gamma\left(s_t + \sum_{j=t+1}^{r}(s_j+z_j)\right)\right|} \prod_{l=t+1}^{r} |dz_l|.$$

We have

$$|\Gamma(\sigma+i\tau)| = \sqrt{2\pi}|\tau|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|\tau|}\big(1+O(|\tau|^{-1})\big) \qquad (|\tau|\to\infty),$$

for $|\tau| \ge 1$, where O is the Landau symbol. So by using this equation, we get

$$\int_{(a_{t+1})\times\cdots\times(a_r)} \prod_{k=t+1}^{r} |\Gamma(s_k+z_k)| \frac{1}{\left|\Gamma\left(s_t + \sum_{j=t+1}^{r}(s_j+z_j)\right)\right|} \prod_{l=t+1}^{r} |dz_l| < \infty.$$

We obtain the claim. □

The equation (3.3.3) holds not only for $\zeta_r((s_j);(\gamma_j))$ but also for $\zeta_r^{\mathrm{des}}(s_1,\ldots,s_r)$.

Proposition 3.3.4. *Let $s_1,\ldots,s_r \in \mathbb{C}$ with $\Re(s_j) > 1$ $(1 \le j \le r)$. Then, for $-\Re(s_k) < a_k < -1$ $(t+1 \le k \le r)$ and $1 \le t \le r-1$, we have*

$$\zeta_r^{\mathrm{des}}(s_1,\ldots,s_r) = \left(\frac{1}{2\pi i}\right)^{r-t} \int_{(a_{t+1})\times\cdots\times(a_r)} \prod_{k=t+1}^{r} \frac{\Gamma(s_k+z_k)\Gamma(-z_k)}{\Gamma(s_k)} \cdot \zeta_t^{\mathrm{des}}\left(s_1,\ldots,s_{t-1}, s_t + \sum_{j=t+1}^{r}(s_j+z_j)\right) \zeta_{r-t}^{\mathrm{des}}(-z_{t+1},\ldots,-z_r) \prod_{l=t+1}^{r} dz_l.$$

Proof. We set $(-z_1,\ldots,-z_t) := \left(s_1,\ldots,s_{t-1}, s_t + \sum_{j=t+1}^{r}(s_j+z_j)\right)$ and $\sigma_k := -\Re(z_k)$ for $1 \le k \le r$. By using Lemma 3.3.1 and the above equation (3.3.3), we get

$$\begin{aligned} &\zeta_r^{\mathrm{des}}(s_1,\ldots,s_r) \\ &= \lim_{\substack{c\to 1 \\ c\in\mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^r} \sum_{\delta_1,\ldots,\delta_r\in\{0,1\}} (-c)^{\delta_1+\cdots+\delta_r} \zeta_r((s_j);(c^{\delta_j})) \\ &= \lim_{\substack{c\to 1 \\ c\in\mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^r} \sum_{\delta_1,\ldots,\delta_r\in\{0,1\}} (-c)^{\delta_1+\cdots+\delta_r} \left(\frac{1}{2\pi i}\right)^{r-t} \int_{(a_{t+1})\times\cdots\times(a_r)} \prod_{k=t+1}^{r} \frac{\Gamma(s_k+z_k)\Gamma(-z_k)}{\Gamma(s_k)} \\ &\quad \cdot \zeta_t\left(-z_1,\ldots,-z_t; c^{\delta_1},\ldots,c^{\delta_t}\right) \cdot \zeta_{r-t}(-z_{t+1},\ldots,-z_r; c^{\delta_{t+1}},\ldots,c^{\delta_r}) \prod_{l=t+1}^{r} dz_l. \end{aligned}$$

$$= \lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}} \left(\frac{1}{2\pi i}\right)^{r-t} \int_{(a_{t+1})\times\cdots\times(a_r)} \prod_{k=t+1}^{r} \frac{\Gamma(s_k+z_k)\Gamma(-z_k)}{\Gamma(s_k)}$$

$$\cdot \frac{1}{(1-c)^r} \sum_{\delta_1,\ldots,\delta_r\in\{0,1\}} (-c)^{\delta_1+\cdots+\delta_r} \zeta_t\left(-z_1,\ldots,-z_t;c^{\delta_1},\ldots,c^{\delta_t}\right) \cdot \zeta_{r-t}(-z_{t+1},\ldots,-z_r;c^{\delta_{t+1}},\ldots,c^{\delta_r}) \prod_{l=t+1}^{r} dz_l.$$

By Lemma 3.3.3 and Lebesgue's dominated convergence theorem, we can commute the limit $\lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}}$ with the integral $\int_{(a_{t+1})\times\cdots\times(a_r)}$. Therefore we have

$$\begin{aligned}
&\zeta_r^{\mathrm{des}}(s_1,\ldots,s_r)\\
&=\left(\frac{1}{2\pi i}\right)^{r-t} \int_{(a_{t+1})\times\cdots\times(a_r)} \prod_{j=t+1}^{r} \frac{\Gamma(s_j+z_j)\Gamma(-z_j)}{\Gamma(s_j)}\\
&\quad\cdot\left\{\lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^t} \sum_{\delta_1,\ldots,\delta_t\in\{0,1\}} (-c)^{\delta_1+\cdots+\delta_t}\zeta_t\left(-z_1,\ldots,-z_t;c^{\delta_1},\ldots,c^{\delta_t}\right)\right\}\\
&\quad\cdot\left\{\lim_{\substack{c\to 1\\ c\in\mathbb{R}\setminus\{1\}}} \frac{1}{(1-c)^{r-t}} \sum_{\delta_{t+1},\ldots,\delta_r\in\{0,1\}} (-c)^{\delta_{t+1}+\cdots+\delta_r}\zeta_{r-t}(-z_{t+1},\ldots,-z_r;c^{\delta_{t+1}},\ldots,c^{\delta_r})\right\} \prod_{l=t+1}^{r} dz_l\\
&=\left(\frac{1}{2\pi i}\right)^{r-t} \int_{(a_{t+1})\times\cdots\times(a_r)} \prod_{k=t+1}^{r} \frac{\Gamma(s_k+z_k)\Gamma(-z_k)}{\Gamma(s_k)} \zeta_t^{\mathrm{des}}(-z_1,\ldots,-z_t)\zeta_{r-t}^{\mathrm{des}}(-z_{t+1},\ldots,-z_r) \prod_{l=t+1}^{r} dz_l.
\end{aligned}$$

So we obtain the claim. □

Proposition 3.3.5. *Let $1\le t\le r$. For $s_1,\ldots,s_t\in\mathbb{C}$ and $k_{t+1},\ldots,k_r\in\mathbb{N}_0$, we have*

$$\begin{aligned}
&\zeta_r^{\mathrm{des}}(s_1,\ldots,s_t,-k_{t+1},\ldots,-k_r)\\
&\quad= \sum_{\substack{i_b+j_b=k_b\\ i_b,j_b\ge 0\\ t+1\le b\le r}} \prod_{a=t+1}^{r} \binom{k_a}{i_a} \zeta_t^{\mathrm{des}}(s_1,\ldots,s_{t-1},s_t-i_{t+1}-\cdots-i_r)\zeta_{r-t}^{\mathrm{des}}(-j_{t+1},\ldots,-j_r).
\end{aligned}$$

Proof. Let $s_1,\ldots,s_r\in\mathbb{C}$ with $\Re(s_j)>1$ $(1\le j\le r)$, $1\le t\le r-1$ and $a_{t+1},\ldots,a_r\in\mathbb{R}$ with $-\Re(s_k)<a_k<-1$ $(t+1\le k\le r)$. We assume $1\le t\le r-1$. To save space, we put

$$f(s_1,\ldots,s_r;z_{t+1},\ldots,z_r) := \zeta_t^{\mathrm{des}}\left(s_1,\ldots,s_{t-1},s_t+\sum_{j=t+1}^{r}(s_j+z_j)\right)\zeta_{r-t}^{\mathrm{des}}(-z_{t+1},\ldots,-z_r).$$

By using Proposition 3.3.4, we have

$$\begin{aligned}
\zeta_r^{\mathrm{des}}(s_1,\ldots,s_r) &= \left(\frac{1}{2\pi i}\right)^{r-t-1} \int_{(a_{t+1})\times\cdots\times(a_{r-1})} \prod_{j=t+1}^{r-1} \frac{\Gamma(s_j+z_j)\Gamma(-z_j)}{\Gamma(s_j)}\\
&\qquad\cdot\left\{\frac{1}{2\pi i}\int_{(a_r)} \frac{\Gamma(s_r+z_r)\Gamma(-z_r)}{\Gamma(s_r)} f(s_1,\ldots,s_r;z_{t+1},\ldots,z_r)dz_r\right\} \prod_{l=t+1}^{r-1} dz_l.
\end{aligned}$$

For $M_r\in\mathbb{N}$ and sufficiently small $\varepsilon_r>0$, we set $\mathcal{D}_r := \{z_r\in\mathbb{C} \mid a_r<\Re(z_r)<M_r-\varepsilon_r\}$. For $z_r\in\mathcal{D}_r$, we have $\Re(s_r+z_r)>0$ by $-\Re(s_r)<a_r<0$. So singularities of the above integrand, which lie on $\mathcal{D}_r$, are only $z_r = 0, 1, \ldots, M_r - 1$. By using the residue theorem, we get

$$\begin{aligned}
&\zeta_r^{\rm des}(s_1, \ldots, s_r) \\
&= \left(\frac{1}{2\pi i}\right)^{r-t-1} \int_{(a_{t+1})\times\cdots\times(a_{r-1})} \prod_{j=t+1}^{r-1} \frac{\Gamma(s_j+z_j)\Gamma(-z_j)}{\Gamma(s_j)} \\
&\quad \cdot \left\{ -\sum_{j_r=0}^{M_r-1} \mathrm{Res}\left[\frac{\Gamma(s_r+z_r)\Gamma(-z_r)}{\Gamma(s_r)} f(s_1, \ldots, s_r; z_{t+1}, \ldots, z_r), z_r = j_r\right] \right. \\
&\quad \left. + \frac{1}{2\pi i} \int_{(M_r-\varepsilon_r)} \frac{\Gamma(s_r+z_r)\Gamma(-z_r)}{\Gamma(s_r)} f(s_1, \ldots, s_r; z_{t+1}, \ldots, z_r) dz_r \right\} \prod_{l=t+1}^{r-1} dz_l.
\end{aligned}$$

(By the same arguments as in [Mat], the second term above converges). By using the fact that the residue of gamma function $\Gamma(s)$ at $s = -j$ is $\frac{(-1)^j}{j!}$, we have

$$\mathrm{Res}\left[\frac{\Gamma(s_r+z_r)\Gamma(-z_r)}{\Gamma(s_r)}, z_r = j_r\right] = (s_r + j_r - 1)\cdots s_r \cdot \frac{(-1)^{j_r}}{j_r!} = \binom{-s_r}{j_r}.$$

So we obtain

$$\begin{aligned}
&\zeta_r^{\rm des}(s_1, \ldots, s_r) \\
&= \left(\frac{1}{2\pi i}\right)^{r-t-1} \int_{(a_{t+1})\times\cdots\times(a_{r-1})} \prod_{j=t+1}^{r-1} \frac{\Gamma(s_j+z_j)\Gamma(-z_j)}{\Gamma(s_j)} \\
&\quad \cdot \left\{ \sum_{j_r=0}^{M_r-1} \binom{-s_r}{j_r} f(s_1, \ldots, s_r; z_{t+1}, \ldots, z_{r-1}, j_r) \right. \\
&\quad \left. + \frac{1}{2\pi i \Gamma(s_r)} \int_{(M_r-\varepsilon_r)} \Gamma(s_r+z_r)\Gamma(-z_r) f(s_1, \ldots, s_r; z_{t+1}, \ldots, z_r) dz_r \right\} \prod_{l=t+1}^{r-1} dz_l.
\end{aligned}$$

By setting $s_r = -k_r$ and $M_r = k_r + 1$ for $k_r \in \mathbb{N}_0$, because of $\frac{1}{\Gamma(-k_r)} = 0$, we get

$$\begin{aligned}
\zeta_r^{\rm des}(s_1, \ldots, s_{r-1}, -k_r) &= \left(\frac{1}{2\pi i}\right)^{r-t-1} \int_{(a_{t+1})\times\cdots\times(a_{r-1})} \prod_{j=t+1}^{r-1} \frac{\Gamma(s_j+z_j)\Gamma(-z_j)}{\Gamma(s_j)} \\
&\quad \cdot \left\{ \sum_{j_r=0}^{k_r} \binom{k_r}{j_r} f(s_1, \ldots, s_{r-1}, -k_r; z_{t+1}, \ldots, z_{r-1}, j_r) \right\} \prod_{l=t+1}^{r-1} dz_l.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&\zeta_r^{\rm des}(s_1, \ldots, s_{r-2}, -k_{r-1}, -k_r) \\
&= \left(\frac{1}{2\pi i}\right)^{r-t-2} \int_{(a_{t+1})\times\cdots\times(a_{r-2})} \prod_{j=t+1}^{r-2} \frac{\Gamma(s_j+z_j)\Gamma(-z_j)}{\Gamma(s_j)} \\
&\cdot \left\{ \sum_{j_r=0}^{k_r} \sum_{j_{r-1}=0}^{k_{r-1}} \binom{k_r}{j_r}\binom{k_{r-1}}{j_{r-1}} f(s_1, \ldots, s_{r-2}, -k_{r-1}, -k_r; z_{t+1}, \ldots, z_{r-2}, j_{r-1}, j_r) \right\} \prod_{l=t+1}^{r-2} dz_l.
\end{aligned}$$

By repeating the above computation, we get

$$\begin{aligned}
&\zeta_r^{\mathrm{des}}(s_1,\ldots,s_{t+1},-k_{t+2},\ldots,-k_r)\\
&=\frac{1}{2\pi i}\int_{(a_{t+1})}\frac{\Gamma(s_{t+1}+z_{t+1})\Gamma(-z_{t+1})}{\Gamma(s_{t+1})}\\
&\cdot\left\{\sum_{j_r=0}^{k_r}\cdots\sum_{j_{t+2}=0}^{k_{t+2}}\binom{k_r}{j_r}\cdots\binom{k_{t+2}}{j_{t+2}}f(s_1,\ldots,s_{t+1},-k_{t+2},\ldots,-k_r;z_{t+1},j_{t+2},\ldots,j_r)\right\}dz_{t+1}.
\end{aligned}$$

By carrying out the above computation again, lastly we obtain

$$\begin{aligned}
&\zeta_r^{\mathrm{des}}(s_1,\ldots,s_t,-k_{t+1},\ldots,-k_r)\\
&=\sum_{j_r=0}^{k_r}\cdots\sum_{j_{t+1}=0}^{k_{t+1}}\binom{k_r}{j_r}\cdots\binom{k_{t+1}}{j_{t+1}}f(s_1,\ldots,s_t,-k_{t+1},\ldots,-k_r;j_{t+1},\ldots,j_r).
\end{aligned}$$

Therefore, we get the proposition for $(s_1,\ldots,s_r)\in\mathbb{C}^r$ with $\Re(s_j)>1$. Because the function $\zeta_r^{\mathrm{des}}(s_1,\ldots,s_r)$ is analytic on $\mathbb{C}^r$ we get the claim for $(s_1,\ldots,s_r)\in\mathbb{C}^r$. $\square$

Lemma 3.3.6. *Let $f,g:\mathbb{C}\times\mathbb{Z}^q\to\mathbb{C}$ be maps ($q\in\mathbb{N}$). We assume that*

$$g(s;-l_1,\ldots,-l_q)=\sum_{\substack{i_b+j_b=l_b\\ i_b,j_b\geq 0\\ 1\leq b\leq q}}\left\{\prod_{a=1}^{q}\binom{l_a}{i_a}\right\}\cdot f(s-i_1-\cdots-i_q;-j_1,\ldots,-j_q)\tag{3.3.6}$$

for $s\in\mathbb{C}$ and $l_1,\ldots,l_q\in\mathbb{N}_0$. Then we have

$$f(s;-l_1,\ldots,-l_q)=\sum_{\substack{i_b+j_b=l_b\\ i_b,j_b\geq 0\\ 1\leq b\leq q}}\left\{\prod_{a=1}^{q}(-1)^{i_a}\binom{l_a}{i_a}\right\}\cdot g(s-i_1-\cdots-i_q;-j_1,\ldots,-j_q)\tag{3.3.7}$$

for $s\in\mathbb{C}$ and $l_1,\ldots,l_q\in\mathbb{N}_0$.

Proof. Firstly, we prove this claim in the case of $q=1$ by induction on l_1. The case of $l_1=0$ is obvious. We assume the equation (3.3.7) for $q=1$ and $l_1\leq l-1$ $(l\in\mathbb{N})$. When $l_1=l$, from the equation (3.3.6), we have

$$f(s;-l)=g(s;-l)-\sum_{j=0}^{l-1}\binom{l}{j}f(s-l+j;-j).$$

By using the equation (3.3.7), we get

$$\begin{aligned}
&=g(s;-l)-\sum_{j=0}^{l-1}\binom{l_0}{j}\left\{\sum_{k=0}^{j}(-1)^k\binom{j}{k}g(s-l+j-k;-j+k)\right\}\\
&=g(s;-l)-\sum_{j=0}^{l-1}\sum_{k=0}^{j}(-1)^k\binom{l}{j}\binom{j}{k}g(s-l+j-k;-j+k).
\end{aligned}$$

By putting $i = j - k \quad (0 \leq i \leq l-1)$, we have

$$
\begin{aligned}
&= g(s;-l) - \sum_{i=0}^{l-1} \sum_{j=i}^{l-1} (-1)^{j-i} \binom{l}{j} \binom{j}{j-i} g(s-l+i;-i) \\
&= g(s;-l) - \sum_{i=0}^{l-1} \left\{ \sum_{j=i}^{l} (-1)^{j-i} \binom{l}{j} \binom{j}{i} - (-1)^{l-i} \binom{l}{i} \right\} g(s-l+i;-i) \\
&= g(s;-l) + \sum_{i=0}^{l-1} (-1)^{l-i} \binom{l}{i} g(s-l+i;-i) \\
&= \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} g(s-l+i;-i).
\end{aligned}
$$

Secondly, we prove the claim for $q \geq 2$. From the equation (3.3.6), we have

$$
\begin{aligned}
&g(s;-l_1,\ldots,-l_q) \\
&= \sum_{i_1+j_1=l_1} \binom{l_1}{i_1} \left[\sum_{i_2+j_2=l_2} \binom{l_2}{i_2} \left[\cdots \left[\sum_{i_q+j_q=l_q} \binom{l_q}{i_q} f(s-i_1-\cdots-i_q;-j_1,\ldots,-j_q) \right] \cdots \right]\right].
\end{aligned}
$$

By using Lemma 3.3.6 as $q = 1$, we get

$$
\begin{aligned}
&\sum_{i_1+j_1=l_1} (-1)^{i_1} \binom{l_1}{i_1} g(s-i_1;-j_1,-l_2\ldots,-l_q) \\
&= \sum_{i_2+j_2=l_2} \binom{l_2}{i_2} \left[\cdots \left[\sum_{i_q+j_q=l_q} \binom{l_q}{i_q} f(s-i_2-\cdots-i_q;-l_1,-j_2,\ldots,-j_q) \right] \cdots \right].
\end{aligned}
$$

By using Lemma 3.3.6 as $q = 1$ again, we have

$$
\begin{aligned}
&\sum_{i_2+j_2=l_2} (-1)^{i_2} \binom{l_2}{i_2} \left[\sum_{i_1+j_1=l_1} (-1)^{i_1} \binom{l_1}{i_1} g(s-i_1-i_2;-j_1,-j_2,-l_3\ldots,-l_q) \right] \\
&= \sum_{i_3+j_3=l_3} \binom{l_3}{i_3} \left[\cdots \left[\sum_{i_q+j_q=l_q} \binom{l_q}{i_q} f(s-i_3-\cdots-i_q;-l_1,-l_2-j_3,\ldots,-j_q) \right] \cdots \right].
\end{aligned}
$$

Therefore, by using Lemma 3.3.6 repeatedly, we obtain the claim. $\square$

By Proposition 3.3.5 and Lemma 3.3.6, we obtain the following theorem.

Theorem 3.3.7. *For $s_1,\ldots,s_p \in \mathbb{C}$ and $l_1,\ldots,l_q \in \mathbb{N}_0$, we have*

$$
\begin{aligned}
&\zeta_p^{\mathrm{des}}(s_1,\ldots,s_p)\zeta_q^{\mathrm{des}}(-l_1,\ldots,-l_q) \\
&= \sum_{\substack{i_b+j_b=l_b \\ i_b,j_b\geq 0 \\ 1\leq b\leq q}} \prod_{a=1}^{q} (-1)^{i_a} \binom{l_a}{i_a} \zeta_{p+q}^{\mathrm{des}}(s_1,\ldots,s_{p-1},s_p-i_1-\cdots-i_q,-j_1,\ldots,-j_q).
\end{aligned}
$$

Proof. By putting $r = p + q$, $t = p$ and $(k_{t+1}, \dots, k_r) := (l_1, \dots, l_q)$ in Proposition 3.3.5, we have

$$\begin{aligned}
&\zeta_{p+q}^{\mathrm{des}}(s_1, \dots, s_p, -l_1, \dots, -l_q) \\
&= \sum_{\substack{i_b + j_b = l_b \\ i_b, j_b \geq 0 \\ 1 \leq b \leq q}} \prod_{a=1}^{q} \binom{l_a}{i_a} \zeta_p^{\mathrm{des}}(s_1, \dots, s_{p-1}, s_p - i_1 - \cdots - i_q) \zeta_q^{\mathrm{des}}(-j_1, \dots, -j_q).
\end{aligned}$$

By applying Lemma 3.3.6 to the above equation with

$$\begin{aligned}
g(s; -l_1, \dots, -l_q) &= \zeta_{p+q}^{\mathrm{des}}(s_1, \dots, s_{p-1}, s, -l_1, \dots, -l_q), \\
f(s; -l_1, \dots, -l_q) &= \zeta_p^{\mathrm{des}}(s_1, \dots, s_{p-1}, s) \zeta_q^{\mathrm{des}}(-l_1, \dots, -l_q),
\end{aligned}$$

we get the theorem. $\square$

Chapter 4

Comparison problem

In §4.1, we treat the problem posed by Singer ([S]) which is on a comparison problem between the renormalized values of shuffle type and harmonic type. In §4.2, we settle the problem by giving a universal presentation of the renormalized values of [EMS17] as finite linear combinations of any renormalized values of harmonic type (Theorem 4.2.7).

4.1 Renormalized values of harmonic type

In this section, we reformulate a certain problem between renormalized values posed in the final line of [S] as Problem 4.1.6. We start with the following problem.

Problem 4.1.1 (Renormalization problem of MZVs (cf. [S, Problem 1]))**.** Extend MZVs to all integer points such that

(A). the values coincide with the special values of analytic continuation of MZFs,

(B). the harmonic relations are preserved.

Based on [EMSZ], we recall the solutions of this problem. Let $\mathcal{H} := \mathbb{Q}\langle z_k \mid k \in \mathbb{Z}\rangle$ be the non-commutative polynomial algebra with the empty word $\mathbf{1}$ generated by the letters z_k. Then $(\mathcal{H}, *, \Delta)$ is a Hopf algebra. Here, the product $*$ is the harmonic product, which is given by $w * \mathbf{1} := \mathbf{1} * w := w$ and

$$z_k w * z_l w' := z_k(w * z_l w') + z_l(z_k w * w') + z_{k+l}(w * w'), \tag{4.1.1}$$

for $k, l \in \mathbb{Z}$ and words w, w' in $\mathcal{H}$, and the coproduct Δ is the deconcatenation coproduct.

Definition 4.1.2 ([EMSZ, Definition 4.2])**.** We call a word $w = z_{k_1} \cdots z_{k_r}$ in $\mathcal{H}$ *non-singular* if all of the following conditions hold:

$$\begin{aligned}
&k_r \neq 1, \\
&k_{r-1} + k_r \neq 2, 1, 0, -2, -4, \dots, \\
&k_{r-i+1} + \cdots + k_r \neq i - n \quad (3 \leq i \leq r,\ n \in \mathbb{N}_0).
\end{aligned}$$

We denote $N \subset \mathcal{H}$ to be the $\mathbb{C}$-vector space spanned by all non-singular words.

We define the $\mathbb{C}$-linear map $\zeta^* : N \to \mathbb{C}$ by

$$\zeta^*(z_{k_1} \cdots z_{k_r}) := \zeta(k_1, \ldots, k_r),$$

for $z_{k_1} \cdots z_{k_r} \in N$, where the right hand side is the special values of analytic continuation of MZF. We put $G_{\mathbb{C}}$ to be the set of all algebra homomorphisms from $\mathcal{H}$ to $\mathbb{C}$, and put the convolution product $\star : G_{\mathbb{C}} \otimes G_{\mathbb{C}} \to G_{\mathbb{C}}$ by

$$f \star g := m \circ (f \otimes g) \circ \Delta,$$

for any $f, g \in G_{\mathbb{C}}$, where m is the ordinary product of $\mathbb{C}$. Then $(G_{\mathbb{C}}, \star)$ forms a group.

Definition 4.1.3 ([EMSZ, Definition 4.5])**.** We define the set $X_{\mathbb{C},\zeta^*}$ of all solutions of Problem 4.1.1 by

$$X_{\mathbb{C},\zeta^*} := \{\phi \in G_{\mathbb{C}} \mid \phi|_N = \zeta^*\},$$

and we define the set $T_{\mathbb{C}}$ called the *renormalization group* by

$$T_{\mathbb{C}} := \{\phi \in G_{\mathbb{C}} \mid \phi|_N = 0\}.$$

Theorem 4.1.4 ([EMSZ] (cf. [S, Theorem 16]))**.** *We have:*

(a). *The set $T_{\mathbb{C}}$ forms a subgroup of $(G_{\mathbb{C}}, \star)$.*

(b). *The left group action*

$$\begin{aligned} T_{\mathbb{C}} \times X_{\mathbb{C},\zeta^*} &\longrightarrow X_{\mathbb{C},\zeta^*}, \\ (\alpha, \phi) &\longmapsto \alpha \star \phi \end{aligned}$$

is free and transitive.

(c). *The cardinality of the set $X_{\mathbb{C},\zeta^*}$ is infinite.*

We put $\mathcal{H}_{\leq 0}$ to be the subalgebra of $\mathcal{H}$ generated by $\{z_k \mid k \in \mathbb{Z}_{\leq 0}\}$. Then it is immediate that $\mathcal{H}_{\leq 0}$ forms a Hopf subalgebra of $(\mathcal{H}, *, \Delta)$. We define $G_{\leq 0}$ to be the set of all restrictions of elements in $G_{\mathbb{C}}$ to $\mathcal{H}_{\leq 0}$.

Definition 4.1.5. We define the set $X_{\leq 0}$ of *renormalized values (at non-positive integer points) of harmonic type* by

$$X_{\leq 0} := \{\phi \in G_{\leq 0} \mid \phi|_{N \cap \mathcal{H}_{\leq 0}} = \zeta^*\}.$$

By using this, we reformulate the problem which is mentioned in [S] as follows.

Problem 4.1.6 (The final line of [S])**.** Which renormalized value of harmonic type have an explicit relationship with the renormalized values $\zeta_{\text{EMS}}(-k_1, \ldots, -k_r)$ (defined in Definition 2.2.4)?

Remark 4.1.7. We recall that the renormalized values (denoted by $\zeta_{\text{GZ}}(k_1, \ldots, k_r)$) in [GZ] are defined for $k_1, \ldots, k_r \in \mathbb{Z}_{\leq 0}$, and the ones (denoted by $\zeta_{\text{MP}}(k_1, \ldots, k_r)$) in [MP] are defined on $k_1, \ldots, k_r \in \mathbb{Z}$. Hence, $\zeta_{\text{MP}}(k_1, \ldots, k_r)$ can be regarded as an element of $X_{\mathbb{C},\zeta^*}$ but it is not clear whether there is an element ϕ of $X_{\mathbb{C},\zeta^*}$ such that

$$\phi(z_{k_1} \cdots z_{k_r}) = \zeta_{\text{GZ}}(k_1, \ldots, k_r),$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\leq 0}$. In any case, we have elements $\mathfrak{z}_{\text{GZ}}$ and $\mathfrak{z}_{\text{MP}}$ of $X_{\leq 0}$ which satisfy

$$\mathfrak{z}_{\text{GZ}}(z_{k_1} \cdots z_{k_r}) = \zeta_{\text{GZ}}(k_1, \ldots, k_r), \qquad \mathfrak{z}_{\text{MP}}(z_{k_1} \cdots z_{k_r}) = \zeta_{\text{MP}}(k_1, \ldots, k_r),$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\leq 0}$.

4.2 Explicit relationship

In this section, we settle Problem 4.1.6 in Theorem 4.2.7. From now on, we assume that $\mathfrak{z}$ is an element of $X_{\leq 0}$, that is, $\mathfrak{z}$ is an algebra homomorphism from $\mathcal{H}_{\leq 0}$ to $\mathbb{C}$ and $\mathfrak{z}$ satisfies

$$\mathfrak{z}|_{N\cap\mathcal{H}_{\leq 0}} = \zeta^*. \tag{4.2.1}$$

By extension of scalars $\mathbb{C}[[t_1,\dots,t_r]]\otimes_{\mathbb{C}}\mathcal{H} = \mathcal{H}_{\leq 0}[[t_1,\dots,t_r]]$, we sometimes regard $\mathfrak{z}$ as a map from $\mathcal{H}_{\leq 0}[[t_1,\dots,t_r]]$ to $\mathbb{C}[[t_1,\dots,t_r]]$.

Remark 4.2.1. Because z_{-k} ($k\geq 0$) is an element of the vector space N (introduced in Definition 4.1.2), we have

$$\mathfrak{z}(z_{-k}) = \zeta(-k),$$

for $k\geq 0$.

Let $T := \{t_i\}_{i\in\mathbb{N}}$. We put $T_{\mathbb{Z}}$ to be the free $\mathbb{Z}$-module generated by all elements of T, that is, $T_{\mathbb{Z}}$ is defined by

$$T_{\mathbb{Z}} := \left\{ \sum_{i=1}^{n} a_i t_i \;\middle|\; n\in\mathbb{N}, a_i\in\mathbb{Z} \right\}.$$

We define $T_{\mathbb{Z}}^{\bullet}$ to be the non-commutative free monoid generated by all elements of $T_{\mathbb{Z}}$ with the empty word $\emptyset$. We denote each element $\omega = u_1\cdots u_r\in T_{\mathbb{Z}}^{\bullet}$ with $u_1,\dots,u_r\in T_{\mathbb{Z}}$ by $\omega = [u_1,\dots,u_r]$ as a sequence and we denote the concatenation uv with $u,v\in T_{\mathbb{Z}}^{\bullet}$ by $[u,v]$. The length of $\omega = [u_1,\dots,u_r]$ is defined to be $l(\omega) = r$. We set $\mathcal{A}_T := \mathbb{C}\langle T_{\mathbb{Z}}\rangle$ to be the non-commutative polynomial ring generated by $T_{\mathbb{Z}}$. We define the harmonic product $* : \mathcal{A}_T^{\otimes 2}\to\mathcal{A}_T$ by $\emptyset * w := w * \emptyset := w$ and

$$[u_1,w_1]*[u_2,w_2] := [u_1, w_1*[u_2,w_2]] + [u_2,[u_1,w_1]*w_2] + [u_1+u_2, w_1*w_2], \tag{4.2.2}$$

for $w,w_1,w_2\in T_{\mathbb{Z}}^{\bullet}$ and $u_1,u_2\in T_{\mathbb{Z}}$. Then the pair $(\mathcal{A}_T,*)$ is a commutative, associative, unital $\mathbb{C}$-algebra. We define [1] the family $\{\mathrm{QSh}\binom{\omega;\eta}{\alpha}\}_{\omega,\eta,\alpha\in T_{\mathbb{Z}}^{\bullet}}$ in $\mathbb{Z}$ by

$$\omega*\eta = \sum_{\alpha\in T_{\mathbb{Z}}^{\bullet}} \mathrm{QSh}\binom{\omega;\eta}{\alpha}\alpha.$$

Example 4.2.2. For $r\geq 1$, we have

$$\begin{aligned}[t_{r+1}]*[t_1,\dots,t_r] = &\sum_{j=1}^{r+1}[t_1,\dots,t_{j-1},t_{r+1},t_j,\dots,t_r]\\ &+\sum_{j=1}^{r}[t_1,\dots,t_{j-1},t_{r+1}+t_j,t_{j+1},\dots,t_r].\end{aligned} \tag{4.2.3}$$

Definition 4.2.3. For $r\geq 1$, we define the generating functions $Z_*(t_1,\dots,t_r)$ of the family $\{\mathfrak{z}(z_{k_1}\cdots z_{k_r})\in\mathbb{C}\mid k_1,\dots,k_r\in\mathbb{Z}_{\leq 0}\}$ by

$$Z_*(t_1,\dots,t_r) := \sum_{k_1,\dots,k_r\geq 0}\frac{(-t_1)^{k_1}\cdots(-t_r)^{k_r}}{k_1!\cdots k_r!}\mathfrak{z}(z_{-k_1}\cdots z_{-k_r})\in\mathbb{C}[[t_1,\dots,t_r]]. \tag{4.2.4}$$

[1] The harmonic product is sometimes called the quasi-shuffle product. The symbol QSh comes from this name.

We put $g : \mathcal{A}_T \to \cup_{r\geq 1}\mathbb{C}[[t_1,\dots,t_r]]$ to be the $\mathbb{C}$-linear map defined by $g(\emptyset) := 1$ and

$$g\big([u_1,\dots,u_r]\big) := Z_*(u_1,\dots,u_r),$$

for $r \geq 1$ and $u_1,\dots,u_r \in T_{\mathbb{Z}}$. Then the following lemma holds.

Lemma 4.2.4. *The map g is an algebra homomorphism, that is, we have*

$$g\big(\omega * \eta\big) = g\big(\omega\big)g\big(\eta\big) \tag{4.2.5}$$

for any $\omega, \eta \in T_{\mathbb{Z}}^{\bullet}$.

Proof. For $r \geq 1$, we put

$$\widetilde{Z_*}(t_1,\dots,t_r) := \sum_{k_1,\dots,k_r\geq 0} \frac{(-t_1)^{k_1}\cdots(-t_r)^{k_r}}{k_1!\cdots k_r!} z_{-k_1}\cdots z_{-k_r} \in \mathcal{H}_{\leq 0}[[t_1,\dots,t_r]].$$

Because we have $\mathfrak{z}\left(\widetilde{Z_*}(t_1,\dots,t_r)\right) = Z_*(t_1,\dots,t_r)$ and $\mathfrak{z}$ is an algebra homomorphism, we have

$$\begin{aligned}\mathfrak{z}\left(\widetilde{Z_*}(t_1,\dots,t_r) * \widetilde{Z_*}(t_{r+1},\dots,t_{r+s})\right) &= Z_*(t_1,\dots,t_r)Z_*(t_{r+1},\dots,t_{r+s})\\ &= g\big([t_1,\cdots,t_r]\big)g\big([t_{r+1},\cdots,t_{r+s}]\big),\end{aligned}$$

for $r, s \geq 1$. Hence, it is sufficient to prove

$$\widetilde{Z_*}(t_1,\dots,t_r) * \widetilde{Z_*}(t_{r+1},\dots,t_{r+s}) = \sum_{\alpha\in T_{\mathbb{Z}}^{\bullet}} \mathrm{QSh}\binom{[t_1,\cdots,t_r];[t_{r+1},\cdots,t_{r+s}]}{\alpha}\widetilde{Z_*}(\alpha), \tag{4.2.6}$$

for $r, s \geq 1$. We have

$$\begin{aligned}&\widetilde{Z_*}(t_1,\dots,t_r) * \widetilde{Z_*}(t_{r+1},\dots,t_{r+s})\\ =&\sum_{k_1,\dots,k_{r+s}\geq 0} \frac{(-t_1)^{k_1}\cdots(-t_{r+s})^{k_{r+s}}}{k_1!\cdots k_{r+s}!}(z_{-k_1}\cdots z_{-k_r} * z_{-k_{r+1}}\cdots z_{-k_{r+s}}).\end{aligned}$$

Here, by definition (4.1.1), we calculate

$$\begin{aligned}=&\sum_{k_1,\dots,k_{r+s}\geq 0}\big\{z_{-k_1}(z_{-k_2}\cdots z_{-k_r} * z_{-k_{r+1}}\cdots z_{-k_{r+s}}) + z_{-k_{r+1}}(z_{-k_1}\cdots z_{-k_r} * z_{-k_{r+2}}\cdots z_{-k_{r+s}})\\ &\qquad + z_{-k_1-k_{r+1}}(z_{-k_2}\cdots z_{-k_r} * z_{-k_{r+2}}\cdots z_{-k_{r+s}})\big\}\frac{(-t_1)^{k_1}\cdots(-t_{r+s})^{k_{r+s}}}{k_1!\cdots k_{r+s}!}\\ =&\widetilde{Z_*}(t_1)\left\{\widetilde{Z_*}(t_2,\cdots,t_r) * \widetilde{Z_*}(t_{r+1},\cdots,t_{r+s})\right\} + \widetilde{Z_*}(t_{r+1})\left\{\widetilde{Z_*}(t_1,\cdots,t_r) * \widetilde{Z_*}(t_{r+2},\cdots,t_{r+s})\right\}\\ &+\left(\sum_{k_1,k_{r+1}\geq 0}\frac{(-t_1)^{k_1}(-t_{r+1})^{k_{r+1}}}{k_1!k_{r+1}!}z_{-k_1-k_{r+1}}\right)\left\{\widetilde{Z_*}(t_2,\cdots,t_r) * \widetilde{Z_*}(t_{r+2},\cdots,t_{r+s})\right\}\\ =&\widetilde{Z_*}(t_1)\left\{\widetilde{Z_*}(t_2,\cdots,t_r) * \widetilde{Z_*}(t_{r+1},\cdots,t_{r+s})\right\} + \widetilde{Z_*}(t_{r+1})\left\{\widetilde{Z_*}(t_1,\cdots,t_r) * \widetilde{Z_*}(t_{r+2},\cdots,t_{r+s})\right\}\\ &+\widetilde{Z_*}(t_1+t_{r+1})\left\{\widetilde{Z_*}(t_2,\cdots,t_r) * \widetilde{Z_*}(t_{r+2},\cdots,t_{r+s})\right\}.\end{aligned}$$

By induction hypothesis, we get

$$
\begin{aligned}
=&\widetilde{Z_*}(t_1)\sum_{\alpha\in T_{\mathbb{Z}}^{\bullet}}\mathrm{QSh}\binom{[t_2,\cdots,t_r];[t_{r+1},\cdots,t_{r+s}]}{\alpha}\widetilde{Z_*}(\alpha)\\
&+\widetilde{Z_*}(t_{r+1})\sum_{\alpha\in T_{\mathbb{Z}}^{\bullet}}\mathrm{QSh}\binom{[t_1,\cdots,t_r];[t_{r+2},\cdots,t_{r+s}]}{\alpha}\widetilde{Z_*}(\alpha)\\
&\quad+\widetilde{Z_*}(t_1+t_{r+1})\sum_{\alpha\in T_{\mathbb{Z}}^{\bullet}}\mathrm{QSh}\binom{[t_2,\cdots,t_r];[t_{r+2},\cdots,t_{r+s}]}{\alpha}\widetilde{Z_*}(\alpha).
\end{aligned}
$$

Here, by the definition of $\widetilde{Z_*}$, we see that $\widetilde{Z_*}(t)\widetilde{Z_*}(\alpha)=\widetilde{Z_*}(t,\alpha)$ holds for $t\in T_{\mathbb{Z}}$ and $\alpha\in T_{\mathbb{Z}}^{\bullet}$. Therefore, we have

$$
\begin{aligned}
=&\sum_{\alpha\in T_{\mathbb{Z}}^{\bullet}}\mathrm{QSh}\binom{[t_2,\cdots,t_r];[t_{r+1},\cdots,t_{r+s}]}{\alpha}\widetilde{Z_*}([t_1,\alpha])\\
&+\sum_{\alpha\in T_{\mathbb{Z}}^{\bullet}}\mathrm{QSh}\binom{[t_1,\cdots,t_r];[t_{r+2},\cdots,t_{r+s}]}{\alpha}\widetilde{Z_*}([t_{r+1},\alpha])\\
&\quad+\sum_{\alpha\in T_{\mathbb{Z}}^{\bullet}}\mathrm{QSh}\binom{[t_2,\cdots,t_r];[t_{r+2},\cdots,t_{r+s}]}{\alpha}\widetilde{Z_*}([t_1+t_{r+1},\alpha]).
\end{aligned}
$$

By using the definition (4.2.2), we get

$$
=\sum_{\alpha\in T_{\mathbb{Z}}^{\bullet}}\mathrm{QSh}\binom{[t_1,\cdots,t_r];[t_{r+1},\cdots,t_{r+s}]}{\alpha}\widetilde{Z_*}(\alpha).
$$

Hence, we obtain (4.2.6). □

In order to prove Proposition 4.2.6, we prepare Lemma 4.2.5. For $r,i\in\mathbb{N}$ with $i\le r$, we define $\mathcal{P}(r,i)$ to be the set of all surjective maps from $\{1,\dots,r\}$ to $\{1,\dots,i\}$. For any element $\sigma\in\mathcal{P}(r,i)$ and $1\le k\le i$, we put

$$
t_{\sigma^{-1}(k)}:=\sum_{n\in\sigma^{-1}(k)}t_n.
$$

We note that $\mathcal{P}(r,r)$ is equal to the symmetric group of degree r, and we note that $\#\mathcal{P}(r,1)=1$, that is, the only element $\sigma\in\mathcal{P}(r,1)$ is given by $\sigma(k):=1$ for $1\le k\le r$. [2]

Lemma 4.2.5. *Let $r\ge 1$. Then, for $1\le i\le r+1$, the summation*

$$
\sum_{\sigma\in\mathcal{P}(r+1,i)}[t_{\sigma^{-1}(1)},\dots,t_{\sigma^{-1}(i)}]
$$

[2] In example 4.2.8, we explicitly compute $\mathcal{P}(r,i)$ for $r=2,3$.

is equal to

$$\sum_{j=1}^{i}\left\{\sum_{\tau\in\mathcal{P}(r,i-1)}[t_{\tau^{-1}(1)},\dots,t_{\tau^{-1}(j-1)},t_{r+1},t_{\tau^{-1}(j)},\dots,t_{\tau^{-1}(i-1)}]\right.$$
$$\left.+\sum_{\tau\in\mathcal{P}(r,i)}[t_{\tau^{-1}(1)},\dots,t_{\tau^{-1}(j-1)},t_{r+1}+t_{\tau^{-1}(j)},t_{\tau^{-1}(j+1)},\dots,t_{\tau^{-1}(i)}]\right\}.$$

Here, for $i=0$ and $r+1$, we put $\mathcal{P}(r,i)$ to be the empty set.

Proof. When $i=1$, we have

$$\sum_{\sigma\in\mathcal{P}(r+1,1)}[t_{\sigma^{-1}(1)}]=[t_1+\cdots+t_{r+1}]=\sum_{\tau\in\mathcal{P}(r,1)}[t_{r+1}+t_{\tau^{-1}(1)}].$$

Hence, we get the claim for $i=1$. When $2\le i\le r$, take an element $\sigma\in\mathcal{P}(r+1,i)$. Then there uniquely exists $j\in\{1,\dots,i\}$ such that $\sigma(r+1)=j$. If $\#\sigma^{-1}(j)=1$, there uniquely exists $\tau\in\mathcal{P}(r,i-1)$ which satisfies

$$\sigma^{-1}(k)=\begin{cases}\tau^{-1}(k) & (1\le k\le j-1),\\ \tau^{-1}(k-1) & (j\le k\le i).\end{cases}$$

On the other hand, if $\#\sigma^{-1}(j)\ge 2$, there uniquely exists $\tau\in\mathcal{P}(r,i)$ which satisfies

$$\sigma^{-1}(k)=\begin{cases}\tau^{-1}(k)\cup\{r+1\} & (k=j),\\ \tau^{-1}(k) & (k\ne j).\end{cases}$$

Hence, we get the claim for $2\le i\le r$. When $i=r+1$, take an element $\sigma\in\mathcal{P}(r+1,r+1)=\mathfrak{S}_{r+1}$. Then there uniquely exists $j\in\{1,\dots,r+1\}$ such that $\sigma(r+1)=j$, and for any $1\le k\le r+1$, we have $\#\sigma^{-1}(k)=1$, that is, we get

$$\left\{\sigma^{-1}(1),\dots,\sigma^{-1}(j-1),\sigma^{-1}(j+1),\dots,\sigma^{-1}(r+1)\right\}=\{1,\dots,r\}.$$

So there uniquely exists $\tau\in\mathfrak{S}_r=\mathcal{P}(r,r)$ such that

$$\left(\sigma^{-1}(1),\dots,\sigma^{-1}(j-1),\sigma^{-1}(j+1),\dots,\sigma^{-1}(r+1)\right)=\left(\tau^{-1}(1),\dots,\tau^{-1}(r)\right).$$

Therefore, we get the claim for $i=r+1$. Hence, we finish the proof. □

Proposition 4.2.6 ([H00, Lemma 5.2; $q=1$])**.** *For $r\ge 1$, we have*

$$Z_*(t_1)\cdots Z_*(t_r)=\sum_{i=1}^{r}\sum_{\sigma\in\mathcal{P}(r,i)}Z_*\left(t_{\sigma^{-1}(1)},\dots,t_{\sigma^{-1}(i)}\right). \tag{4.2.7}$$

Proof. We prove this claim by induction on r. When $r=1$, the element $\sigma\in\mathcal{P}(1,1)$ is only the identity map, i.e., $\sigma^{-1}(1)=\{1\}$. Hence, the right hand side of (4.2.7) is equal to $Z_*(t_1)$. Assume that the equation (4.2.7) holds for $r=r_0\ge 1$. When $r=r_0+1$, by multiplying $Z_*(t_{r_0+1})$ to the both sides of (4.2.7) for $r=r_0$, we have

$$\begin{aligned}&Z_*(t_{r_0+1})\sum_{i=1}^{r_0}\sum_{\sigma\in\mathcal{P}(r_0,i)}Z_*\left(t_{\sigma^{-1}(1)},\dots,t_{\sigma^{-1}(i)}\right)\\ &=\sum_{i=1}^{r_0}\sum_{\sigma\in\mathcal{P}(r_0,i)}g\left([t_{r_0+1}]*[t_{\sigma^{-1}(1)},\dots,t_{\sigma^{-1}(i)}]\right).\end{aligned}$$

By Example 4.2.2, we calculate

$$= \sum_{i=1}^{r_0} \sum_{\sigma \in \mathcal{P}(r_0,i)} g \left(\sum_{j=1}^{i+1} [t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(j-1)}, t_{r_0+1}, t_{\sigma^{-1}(j)}, \ldots, t_{\sigma^{-1}(i)}] \right.$$
$$\left. + \sum_{j=1}^{i} [t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(j-1)}, t_{r_0+1} + t_{\sigma^{-1}(j)}, t_{\sigma^{-1}(j+1)}, \ldots, t_{\sigma^{-1}(i)}] \right).$$

By decomposing each summations, we have

$$= \sum_{\sigma \in \mathcal{P}(r_0,r_0)} g \left(\sum_{j=1}^{r_0+1} [t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(j-1)}, t_{r_0+1}, t_{\sigma^{-1}(j)}, \ldots, t_{\sigma^{-1}(r_0)}] \right)$$
$$+ \sum_{i=1}^{r_0-1} \sum_{\sigma \in \mathcal{P}(r_0,i)} g \left(\sum_{j=1}^{i+1} [t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(j-1)}, t_{r_0+1}, t_{\sigma^{-1}(j)}, \ldots, t_{\sigma^{-1}(i)}] \right)$$
$$+ \sum_{i=2}^{r_0} \sum_{\sigma \in \mathcal{P}(r_0,i)} g \left(\sum_{j=1}^{i} [t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(j-1)}, t_{r_0+1} + t_{\sigma^{-1}(j)}, t_{\sigma^{-1}(j+1)}, \ldots, t_{\sigma^{-1}(i)}] \right)$$
$$+ \sum_{\sigma \in \mathcal{P}(r_0,1)} g \left([t_{r_0+1} + t_{\sigma^{-1}(1)}] \right).$$

By applying Lemma 4.2.5 for $r = r_0$ and $i = r_0 + 1$ (resp. $i = 1$) to the first term (resp. the fourth term), we get

$$= g \left(\sum_{\sigma \in \mathcal{P}(r_0+1,r_0+1)} [t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(r_0+1)}] \right)$$
$$+ \sum_{i=2}^{r_0} g \left(\sum_{j=1}^{i} \left\{ \sum_{\sigma \in \mathcal{P}(r_0,i-1)} [t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(j-1)}, t_{r_0+1}, t_{\sigma^{-1}(j)}, \ldots, t_{\sigma^{-1}(i-1)}] \right. \right.$$
$$\left. \left. + \sum_{\sigma \in \mathcal{P}(r_0,i)} [t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(j-1)}, t_{r_0+1} + t_{\sigma^{-1}(j)}, t_{\sigma^{-1}(j+1)}, \ldots, t_{\sigma^{-1}(i)}] \right\} \right)$$
$$+ g \left(\sum_{\sigma \in \mathcal{P}(r+1,1)} [t_{\sigma^{-1}(1)}] \right).$$

By applying Lemma 4.2.5 for $r = r_0$ and $2 \leq i \leq r_0$ to the second term, we get

$$Z_*(t_{r_0+1}) \sum_{i=1}^{r_0} \sum_{\sigma \in \mathcal{P}(r_0,i)} Z_* \left(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(i)} \right) = \sum_{i=1}^{r_0+1} g \left(\sum_{\sigma \in \mathcal{P}(r_0+1,i)} [t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(i)}] \right)$$
$$= \sum_{i=1}^{r_0+1} \sum_{\sigma \in \mathcal{P}(r_0+1,i)} Z_* \left(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(i)} \right).$$

Hence, we obtain the claim. □

By using above proposition, we get a universal presentation of $Z_{\text{EMS}}(t_1,\dots,t_r)$ (defined by (0.2.1)) by any generating functions of renormalized values of harmonic type.

Theorem 4.2.7. *Let $\mathfrak{z}$ be a renormalized values of harmonic type (cf. Definition 4.1.5), and let Z_* be the generating function of $\mathfrak{z}$ given by (4.2.4). Then for $r \geq 1$, we have*

$$Z_{\text{EMS}}(t_1,\dots,t_r) = \sum_{i=1}^{r} \sum_{\sigma\in\mathcal{P}(r,i)} Z_*\left(u_{\sigma^{-1}(1)},\dots,u_{\sigma^{-1}(i)}\right). \tag{4.2.8}$$

Here, $u_{\sigma^{-1}(k)}$ is defined by

$$u_{\sigma^{-1}(k)} := \sum_{n\in\sigma^{-1}(k)} u_n,$$

for $u_i := t_i + \cdots + t_r$ $(1 \leq i \leq r)$.

Proof. By Remark 4.2.1, we have

$$Z_{\text{EMS}}(t_1) = Z_*(t_1).$$

Therefore, by the equation (2.5.2), we have

$$Z_{\text{EMS}}(t_1,\dots,t_r) = \prod_{i=1}^{r} Z_{\text{EMS}}(t_i+\cdots+t_r) = \prod_{i=1}^{r} Z_*(t_i+\cdots+t_r).$$

Therefore, by putting $u_i := t_i + \cdots + t_r$ $(1 \leq i \leq r)$ and by using the equation (4.2.7), we obtain

$$Z_{\text{EMS}}(t_1,\dots,t_r) = \sum_{i=1}^{r} \sum_{\sigma\in\mathcal{P}(r,i)} Z_*\left(u_{\sigma^{-1}(1)},\dots,u_{\sigma^{-1}(i)}\right).$$

Hence, we finish the proof. □

In the following example, we denote $\sigma \in \mathcal{P}(r,i)$ by

$$\begin{pmatrix} 1 & \cdots & r \\ \sigma(1) & \cdots & \sigma(r) \end{pmatrix}.$$

Example 4.2.8. When $r = 2$, we have

$$\mathcal{P}(2,2) = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}, \quad \mathcal{P}(2,1) = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \right\},$$

so we get

$$Z_{\text{EMS}}(t_1,t_2) = Z_*(t_1+t_2,t_2) + Z_*(t_2,t_1+t_2) + Z_*(t_1+2t_2).$$

When $r = 3$, we have $\mathcal{P}(3,1) = \left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right) \right\}$, and $\mathcal{P}(3,2)$ is given by

$$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \right\},$$

and $\mathcal{P}(3,3)$ is given by

$$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}.$$

Hence, we get

$$\begin{aligned}Z_{\text{EMS}}(t_1,t_2,t_3) &= Z_*(t_1+t_2+t_3,t_2+t_3,t_3)+Z_*(t_1+t_2+t_3,t_3,t_2+t_3)\\&+Z_*(t_2+t_3,t_1+t_2+t_3,t_3)+Z_*(t_2+t_3,t_3,t_1+t_2+t_3)\\&+Z_*(t_3,t_1+t_2+t_3,t_2+t_3)+Z_*(t_3,t_2+t_3,t_1+t_2+t_3)\\&+Z_*(t_1+t_2+2t_3,t_2+t_3)+Z_*(t_2+t_3,t_1+t_2+2t_3)\\&+Z_*(t_2+2t_3,t_1+t_2+t_3)+Z_*(t_1+t_2+t_3,t_2+2t_3)\\&+Z_*(t_1+2t_2+2t_3,t_3)+Z_*(t_3,t_1+2t_2+2t_3)\\&+Z_*(t_1+2t_2+3t_3).\end{aligned}$$

Corollary 4.2.9. *The equation* (4.2.8) *holds for $Z_*=Z_{\text{GZ}}$ and Z_{MP} defined by*

$$\begin{aligned}Z_{\text{GZ}}(t_1,\dots,t_r) &:= \sum_{k_1,\dots,k_r=0}^{\infty}\frac{(-t_1)^{k_1}\cdots(-t_r)^{k_r}}{k_1!\cdots k_r!}\zeta_{\text{GZ}}(-k_1,\dots,-k_r),\\Z_{\text{MP}}(t_1,\dots,t_r) &:= \sum_{k_1,\dots,k_r=0}^{\infty}\frac{(-t_1)^{k_1}\cdots(-t_r)^{k_r}}{k_1!\cdots k_r!}\zeta_{\text{MP}}(-k_1,\dots,-k_r).\end{aligned}$$

Hence, the renormalized values $\zeta_{\text{EMS}}(-k_1,\dots,-k_r)$ can be represented by a finite linear combination of either $\zeta_{\text{GZ}}(-k_1,\dots,-k_r)$ or $\zeta_{\text{GZ}}(-k_1,\dots,-k_r)$.

Proof. We recall elements $\mathfrak{z}_{\text{GZ}}$ and $\mathfrak{z}_{\text{MP}}$ of $X_{\leq 0}$ in Remark 4.1.7. These elements satisfy

$$\mathfrak{z}_{\text{GZ}}(z_{k_1}\cdots z_{k_r})=\zeta_{\text{GZ}}(k_1,\dots,k_r),\qquad \mathfrak{z}_{\text{MP}}(z_{k_1}\cdots z_{k_r})=\zeta_{\text{MP}}(k_1,\dots,k_r),$$

for $k_1,\dots,k_r\in\mathbb{Z}_{\leq 0}$. Hence, we get the claim. $\square$

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