A new approximate formula for L-functions and its applications to the value distribution of L-functions

(L 関数に対する新しい近似公式とその近似公式による L 関数の値分布論への応用)

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Abstract

In this thesis, we discuss the value distribution of L-functions from several viewpoints. The value distribution of the Riemann zeta-function $\zeta(s)$ is related to the distribution of prime numbers and therefore important in number theory. Recently, this theme in probabilistic aspects based on limit theorems due to Bohr-Jessen and Selberg has developed rapidly by many mathematicians. In this thesis, we show some results related to this theme.

Chapter 1 is the introduction of this thesis, and we survey some of the previous works on the value distribution of zeta and L-functions, and describe some of results in this thesis. In Chapter 2, we prove an approximate formula for the Riemann zeta-function $\zeta(s)$ and its iterated integrals. As applications of the formula, we also prove some results on the value distribution of $\zeta(s)$ and one the relation between the distribution of nontrivial zeros of $\zeta(s)$ and a Dirichlet polynomial. In particular, a result for the value distribution of $\zeta(s)$ contributes to Radziwiłł's conjecture. In Chapter 3, we discuss the large deviations for the distribution function of iterated integrals of the logarithm of the Riemann zeta-function. In Chapter 4, we prove results on denseness of the Riemann zeta-function. In particular, we also give an equivalence between the denseness and the Riemann Hypothesis. This theme is related to Ramachandra's denseness problem, which is the problem to ask whether the values $\zeta(\frac{1}{2} + it)$, $t \in \mathbb{R}$ is dense in $\mathbb{C}$. In Chapter 5, we prove some results for the discrepancy bounds and the large deviations for the distribution function of $\zeta(\sigma + it)$ in the strip $\frac{1}{2} < \sigma < 1$. The result for the large deviations is an improvement on a recent work. In Chapter 6, we discuss the independence of certain L-functions on the critical line. We in this chapter show some results for large deviations in multidimensional central limit theorem due to Bombieri and Hejhal. As application of the results, we also prove results for moments of L-functions. In particular, the results for moments include some new results for the Riemann zeta-function. Finally, in Chapter 7, we discuss the dependence of $\log \zeta(\sigma + it)$ and $\log L(\sigma + it, \chi)$ in the strip $\frac{1}{2} < \sigma < 1$. We show that these functions have a certain dependence property as random variables.

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Contents

Chapter 1 Introduction

In this thesis, we discuss the value distribution of zeta and L-functions such as the Riemann zeta-function. The theme is interesting because that is related to the distribution of zeros and some arithmetic objects involving prime numbers. In fact, there are many studies for this theme such as mean value estimates, limit theorems, order estimates, and omega-estimates. In this chapter, we survey this theme and present some of our results.

1.1 Relations among distribution of values, zeros, and primes

The distribution of prime numbers has interested many people since a long time ago. Riemann first related the distribution of prime numbers to zeros of the function, which is now called the Riemann zeta-function, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1} \quad \text{for} \quad \operatorname{Re} s > 1.$$

Throughout this paper, $s = \sigma + it$ is a complex number with $\sigma, t \in \mathbb{R}$, and the product $\prod_p$ runs over all prime numbers. Riemann in [101] studied the number of prime numbers less than x a given number. In that paper, he also conjectured a rule of the distribution of zeros of the Riemann zeta-function. The conjecture is called the Riemann Hypothesis today.

Conjecture (Riemann Hypothesis (RH)). *All real parts of nontrivial zeros of the Riemann zeta-function are one-half.*

This conjecture is one of the most important and famous open problems in mathematics. The Riemann Hypothesis has a consequence to the distribution of prime numbers. Actually, the Riemann Hypothesis is equivalent to that

$$\pi(x) = \int_2^x \frac{du}{\log u} + O\left(x^{1/2} \log x\right). \tag{1.1}$$

Here, $\pi(x)$ is the number of prime numbers less than x. In particular, this formula implies $p_{n+1} - p_n \ll p_n^{1/2} \log p_n$ with p_n the n-th prime number. Here, we explain some notations. For a complex-valued function f and a positive-valued function $g(x)$, we write $f(x) = O(g(x))$ if there is a constant

$C > 0$ such that $|f(x)| \leq Cg(x)$ for all x in the appropriate domain. The constant C is called the implicit constant. If C depends on a parameter α, we write $f(x) = O_\alpha(g(x))$. Additionally, we can also write $f(x) \ll g(x)$, $f(x) \ll_\alpha g(x)$ in the same meaning as $f(x) = O(g(x))$, $f(x) = O_\alpha(g(x))$ respectively. We write $f(x) \asymp g(x)$ if both $f(x) \ll g(x)$ and $f(x) \gg g(x)$ hold. Moreover, if $\lim_{x\to a} f(x)/g(x) = 0$ with $a \in \mathbb{R} \cup \{\pm\infty\}$, then we write $f(x) = o(g(x))$ (as $x \to a$). Furthermore, $f(x) = \Omega_+(g(x))$ (as $x \to a$) means that $\limsup_{x\to a} f(x)/g(x) > 0$, and also $f(x) = \Omega_-(g(x))$ (as $x \to a$) means that $\liminf_{x\to a} f(x)/g(x) < 0$. If both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold, we write $f(x) = \Omega_\pm(g(x))$. If either $f(x) = \Omega_+(g(x))$ or $f(x) = \Omega_-(g(x))$ holds, we also write $f(x) = \Omega(g(x))$.

We can find that the Riemann Hypothesis implies equation (1.1) by using the formula

$$\pi(x) = \sum_{n \leq \frac{\log x}{\log 2}} \frac{\mu(n)}{n} \left\{ \mathrm{li}(x^{1/n}) - \sum_{|\gamma| \leq T} \mathrm{li}(x^{\rho/n}) + O\left(\frac{x^{1/n}(\log x^{1/n}T)^2}{T \log x^{1/n}} + 1 \right) \right\}. \quad (1.2)$$

Here, $\mu(n)$ is the Möbius function, and the function $\mathrm{li}(e^{x+iy})$ is defined by if $y = 0$,

$$\mathrm{li}(e^x) = \lim_{\varepsilon \downarrow 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{+\varepsilon}^{x} \right) \frac{e^u}{u} du = \lim_{\varepsilon \downarrow 0} \left(\int_{0}^{1-\varepsilon} + \int_{1+\varepsilon}^{e^x} \right) \frac{du}{\log u},$$

and if $y \neq 0$,

$$\mathrm{li}(e^{x+iy}) = \int_{-\infty+iy}^{x+iy} \frac{e^w}{w} dw.$$

Then, it holds that $\mathrm{li}(x^\rho) = \mathrm{li}(x^{\beta+i\gamma}) \ll \frac{x^\beta}{(|\gamma|+1)\log x}$. By using these estimates, we can prove (1.1) under the Riemann Hypothesis. Also, we can easily obtain the inverse implication by using the formulas $\pi(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(u)}{u(\log u)^2} du + O(x^{1/2})$ and $-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \frac{\psi(x)}{x^{s+1}} dx$, where $\psi(x) := \sum_{n \leq x} \Lambda(n)$ with $\Lambda(n)$ the von Mangoldt function.

We also know another conjecture having a consequence for the distribution of prime numbers. In [70], Lindelöf studied the order of magnitude of the Riemann zeta-function and its convexity. In that paper, he also conjectured the following hypothesis.

Conjecture (Lindelöf Hypothesis (LH)). *For any $t \geq 1$, $\varepsilon > 0$,*

$$|\zeta(\tfrac{1}{2} + it)| \ll_\varepsilon t^\varepsilon.$$

This hypothesis is also one of the most famous and challenging open problems in analytic number theory. The statement of this hypothesis is in terms of the value distribution of the Riemann zeta-function, particularly for the order of magnitude of the Riemann zeta-function. On the other hand, it is known that this conjecture is rewritten to a statement of the distribution of zeros of the Riemann zeta-function. In fact, Backlund [2] showed that the following statement (BS) is equivalent to the Lindelöf Hypothesis.

BS: for every $\varepsilon > 0$, the estimate $N(\frac{1}{2}+\varepsilon, T+1) - N(\frac{1}{2}+\varepsilon, T) = o(\log T)$ holds as $T \to +\infty$.

Here, $N(\sigma, T)$ is the number of nontrivial zeros $\rho = \beta + i\gamma$ with $\beta \geq \sigma$, $0 < \gamma < T$ counted with multiplicities. From this equivalence, we see that the Riemann Hypothesis implies the Lindelöf Hypothesis. Additionally, assuming the Riemann Hypothesis, Littlewood [71] showed that $|\zeta(1/2 + it)| \leq \exp(C\frac{\log t}{\log\log t})$ for some constant $C > 0$ that also leads the implication.

Ingham [49] showed that the Lindelöf Hypothesis implies $p_{n+1} - p_n \ll_\varepsilon p_n^{1/2+\varepsilon}$, which is close to the consequence of the Riemann Hypothesis. He showed that the Lindelöf Hypothesis implies

$$N(\sigma, T) \ll_\varepsilon T^{2(1-\sigma)+\varepsilon}, \tag{1.3}$$

and this estimate implies $p_{n+1} - p_n \ll_\varepsilon p_n^{1/2+\varepsilon}$. Estimate (1.3) has not yet proved at present and called the Density Hypothesis (DH) today. The best unconditional result of gaps of primes is $p_{n+1} - p_n \ll p_n^{21/40} = p_n^{\frac{1}{2}+\frac{1}{40}}$ proved by Baker, Harman, and Pintz [3].

On the other hand, there are many difficult open problems on the distribution of prime numbers even under the Riemann Hypothesis. The following two conjectures are typical examples.

Conjecture (Cramér's conjecture)**.**

$$p_{n+1} - p_n \ll (\log p_n)^2.$$

Conjecture (Twin prime conjecture)**.**

$$\liminf_{n\to+\infty}(p_{n+1} - p_n) = 2.$$

Recently, the studies on these conjectures have developed by interesting works [25], [26], [81], [82], and [124]. On the other hand, the best upper bound of the gap of prime numbers is $p_{n+1} - p_n \ll p_n^{1/2}\log p_n$ even under the Riemann Hypothesis. If we would like to develop this direction of research by theory of the zeta-function or using formula (1.2), it requires to understand the distribution of zeros more deeply beyond the Riemann Hypothesis. In other words, we need to understand the distribution of imaginary parts of zeros precisely.

For the distribution of imaginary parts of zeros, the formula

$$N(T) = \frac{1}{\pi}\arg\Gamma\left(\frac{1}{4} + i\frac{T}{2}\right) - \frac{T}{2\pi}\log\pi + S(T) + 2 \tag{1.4}$$

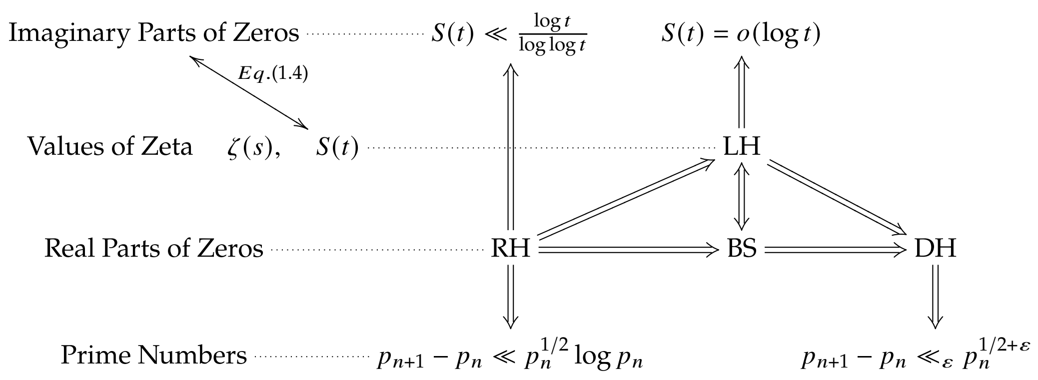
is useful, where $N(T)$ is the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T$ counted with multiplicities. This formula is usually called the Riemann-von

Mangoldt formula. Here, $S(T)$ is defined as $\frac{1}{\pi}\operatorname{Im}\log\zeta(\frac{1}{2}+it) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+it)$, whose branch will be described in the last of this section. By using the Stirling formula, we can calculate terms on the right hand side of (1.4) satisfactorily except for $S(T)$. Hence, it is desirable to understand the behavior of $S(T)$ exactly. From this viewpoint, the function $S(T)$ is interesting, and there are many works. For example, the estimate $S(T) \ll \log T$ was proved by von Mangoldt[1)] in 1905, and Cramér [20] showed that $S(T) = o(\log T)$ as $T \to +\infty$ under the Lindelöf Hypothesis. Moreover, Littlewood [71] established that the Riemann Hypothesis implies $S(T) \ll \frac{\log T}{\log\log T}$. In particular, it holds that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} + O(\log T)$$

unconditionally, and this error term comes from the estimate $S(T) \ll \log T$.

The following is a summary of the above.



Here, we describe the branch of the logarithm of zeta and L-functions F satisfying certain suitable conditions. First, t is equal to neither imaginary parts of zeros nor poles of F, then we choose the branch by the continuation with the initial condition $\lim_{\sigma\to+\infty}\log F(\sigma+it) = 0$. If $t \neq 0$ is equal to an imaginary part of a zero or a pole of F, we take $\log F(\sigma+it) = \lim_{\varepsilon\downarrow 0}\log F(\sigma+i(t-\operatorname{sgn}(t)\varepsilon))$, where sgn is the signum function. If there exists a pole or a zero such that the imaginary part is zero, then we take $\log F(\sigma) = \lim_{\varepsilon\downarrow 0}\log F(\sigma - i\varepsilon)$.

1.2 The distribution function of the Riemann zeta-function

From the observation in the previous section, we are interested in the value distribution of zeta and L-functions. For this theme, the following interesting theorems are known. Throughout this thesis, $\operatorname{meas}(\cdot)$ stands for the Lebesgue measure on $\mathbb{R}$.

1) The author was not able to find the original paper of this result. The source of this information is the textbook by Davenport [21, Section 8]

Theorem (Bohr-Jessen in [8]). *Let $\sigma > \frac{1}{2}$ be fixed. There exists a probability measure P_σ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that for any fixed $V \in \mathbb{R}$*

$$\frac{1}{T} \operatorname{meas}\left\{t \in [T, 2T] \ : \ \log \zeta(\sigma + it) > V\right\} \sim P_\sigma((V, +\infty)) \tag{1.5}$$

as $T \to +\infty$. Moreover, the probability measure P_σ has a probability density function D_σ.

Theorem (Central limit theorem). *For any fixed $V \in \mathbb{R}$, we have*

$$\frac{1}{T} \operatorname{meas}\left\{t \in [T, 2T] \ : \ \frac{\log |\zeta(\frac{1}{2} + it)|}{\sqrt{\frac{1}{2}\log\log T}} > V\right\} \sim \int_V^\infty e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \tag{1.6}$$

as $T \to +\infty$.

Note that the former theorem is a special case in the original their theorem in [8]. From these theorems and more developed results, we can guess the behavior of the Riemann zeta-function. Joyner [55, Theorem 4.3 in Chapter 5] showed that there exist positive constants c_1, c_2 such that

$$\begin{aligned} \exp\left(-(c_1 + o(1))\left(V(\log V)^\sigma\right)^{\frac{1}{1-\sigma}}\right) &\le P_\sigma((V, +\infty)) \\ &\le \exp\left(-(c_2 + o(1))\left(V(\log V)^\sigma\right)^{\frac{1}{1-\sigma}}\right) \end{aligned}$$

for $\frac{1}{2} < \sigma < 1$ as $V \to +\infty$. Moreover, Hattori and Matsumoto [40] showed that $c_1 = c_2 = A(\sigma)$, that is,

$$P_\sigma((V, +\infty)) = \exp\left(-(A(\sigma) + o(1))\left(V(\log V)^\sigma\right)^{\frac{1}{1-\sigma}}\right) \tag{1.7}$$

for $\frac{1}{2} < \sigma < 1$ as $V \to +\infty$. Here, $A(\sigma)$ is expressed by

$$A(\sigma) = \left(\frac{\sigma^{2\sigma}}{(1-\sigma)^{2\sigma-1} G(\sigma)^\sigma}\right)^{\frac{1}{1-\sigma}}, \tag{1.8}$$

where $G(\sigma) = \int_0^\infty \log I_0(u) u^{-1-\frac{1}{\sigma}} du$, and I_0 is the modified 0-th order Bessel function. By these estimates and the classical bound $\zeta'(\sigma + it) \ll (|t| + 2)^c$ with c a positive constant, it seems to be guessed that, for $\frac{1}{2} < \sigma < 1$,

$$\log|\zeta(\sigma + it)| \le C(\sigma) \frac{(\log t)^{1-\sigma}}{(\log\log t)^\sigma} \tag{1.9}$$

for any $t \ge 3$, and

$$\log|\zeta(\sigma + it)| = \Omega\left(\frac{(\log t)^{1-\sigma}}{(\log\log t)^\sigma}\right) \tag{1.10}$$

as $t \to +\infty$. Actually, these estimates coming from this rough observation are believed to hold. In particular, the Ω-estimate has been proved by Montgomery [86].

Similarly, when $\sigma = \frac{1}{2}$, it seems to be guessed that, from central limit theorem (1.6), the classical bound $\zeta'(\frac{1}{2}+it) \ll (|t|+2)^c$, and the estimate $\int_V^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}} \asymp \frac{1}{1+V}e^{-V^2/2}$ for $V \geq 0$,

$$\log|\zeta(\tfrac{1}{2}+it)| \leq C\sqrt{\log t \log\log t} \tag{1.11}$$

for $t \geq 3$, and

$$\log|\zeta(\tfrac{1}{2}+it)| = \Omega\left(\sqrt{\log t \log\log t}\right) \tag{1.12}$$

as $t \to +\infty$. Remark that the upper bound is stronger than the bound of the original Lindelöf Hypothesis. These estimates are also believed to hold, and further there is an interesting work for the constant term by Farmer, Gonek, and Hughes [24]. Moreover, we should also mention that Bondarenko and Seip [10] made a breakthrough for the Ω-estimate of $|\zeta(\frac{1}{2}+it)|$. The above expectations are supported from the viewpoint of large deviations in limit theorems (1.5) and (1.6).

1.3 Moments of the Riemann zeta-function

The study of the moments plays an important role in the study of the value distribution of zeta and L-functions. We define the $2k$-th moment of the Riemann zeta-function by

$$I_k(T) = \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt.$$

It is well known that the Lindelöf Hypothesis is equivalent to, for any $T \geq 1$, $k \in \mathbb{Z}_{\geq 1}$, $\varepsilon > 0$,

$$I_k(T) \ll_{k,\varepsilon} T^{1+\varepsilon}.$$

From this equivalence, we may find the importance of the work for the moments of the Riemann zeta-function. It is natural to ask the precise order of magnitude of moments. For this problem, Keating and Snaith suggested the following interesting conjecture.

Conjecture (Keating and Snaith in [63]). *For any* $k > -\frac{1}{2}$,

$$I_k(T) \sim a(k)f(k)T(\log T)^{k^2}, \tag{1.13}$$

where

$$a(k) = \prod_p \left\{(1-1/p)^{k^2}\left(\sum_{m=0}^\infty \left(\frac{\Gamma(k+m)}{m!\Gamma(k)}\right)^2 p^{-m}\right)\right\},$$

and

$$f(k) = \frac{(G(1+k))^2}{G(1+2k)}.$$

Here, G is the Barnes G-function.

Asymptotic formula (1.13) has been proved only in the cases $k = 0, 1, 2$. The case $k = 0$ is trivial. The cases $k = 1, 2$ were proved by Hardy and Littlewood [37], and by Ingham [48] respectively. However, the other cases have not been proved yet at present. We also note that $I_k(T) = +\infty$ for $k \le -1/2$ which is proved by the existence the zeros in $\frac{1}{2} + it$, $t \in [T, 2T]$ (see [105, Theorem A]). Therefore, it is interesting to ask whether the weaker estimates

$$I_k(T) \gg_k T(\log T)^{k^2}, \tag{1.14}$$

$$I_k(T) \ll_k T(\log T)^{k^2} \tag{1.15}$$

hold for $k > -\frac{1}{2}$. Also for this, there are many works, and we already know the following works.

Theorem. *Estimate* (1.14) *holds in the following situations.*

- $k \in \mathbb{Z}_{\ge 0}$ *by Ramachandra [99],*
- $k \in \mathbb{Q}_{\ge 0}$ *by Heath-Brown [44],*
- $k \ge 0$ *under RH by Ramachandra [100], Heath-Brown [44], independently,*
- $k \le 0$ *under RH by Gonek [32],*
- $k \ge 1$ *by Radziwiłł and Soundararajan [97],*
- $k \ge 0$ *by Heap and Soundararajan [43].*

Theorem. *Estimate* (1.15) *holds in the following situations.*

- $k = \frac{1}{2}$ *by Ramachandra [100],*
- $k = \frac{1}{n}$ *with* $n \in \mathbb{Z}_{\ge 1}$ *by Heath-Brown [44],*
- $0 \le k \le 2$ *under RH by Ramachandra [100], Heath-Brown [44], independently,*
- $0 \le k \le 2 + \frac{2}{11}$ *under RH by Radziwiłł [96],*
- $k \ge 0$ *under RH by Harper [39],*
- $k = 1 + \frac{1}{n}$ *with* $n \in \mathbb{Z}_{\ge 1}$ *by Bettin, Chandee, and Radziwiłł [5],*
- $0 \le k \le 2$ *by Heap, Radziwiłł, and Soundararajan [42].*

By the above results, we see that the lower bound (1.14) has been proved for every $k \geq 0$ unconditionally, and for every $k \in \mathbb{R}$ under RH. On the other hand, there is a gap for the dependence of k between the implicit constants of the above results and the constant $a(k)f(k)$ due to Keating-Snaith. In fact, we see that $a(k)f(k) = e^{-k^2 \log(k+3) - k^2 \log\log(k+3) + O(k^2)}$ for any $k \geq 0$, and particularly $a(k)f(k) \asymp 1$ if $0 \leq k \leq 2$. By contrast, the implicit constant of Radziwiłł and Soundararajan [97] is $\gg e^{-30k^4}$, and the implicit constant of Heap and Soundararajan [43] is $\gg k$ which tends to zero as $k \to 0$. Hence, it would be at present desirable to improve these. If k is a positive integer, the implicit constant has been improved to $\gg e^{-2k^2 \log k + O(k^2)}$ by Conrey and Ghosh [18], and Soundararajan [109]. Moreover, Conrey and Ghosh [17] showed that for any $k \geq 0$, the implicit constant is $\gg e^{-2k^2(\log(k+3)) + O(k^2)}$ under RH. It would be also interesting to improve the bound of their implicit constants at present. Furthermore, the negative moment of the Riemann zeta-function has been established by Gonek [32]. Assuming the Riemann Hypothesis, he showed that $I_k(T) \gg T(\log T)^{k^2}$ for $k \leq 0$, and the implicit constant is absolute. On the other hand, we see that $a(k)f(k) \asymp (1+2k)^{-1}$ for $-\frac{1}{2} < k \leq 0$. Hence, it seems desirable to show that $I_k(T) \gg (1+2k)^{-1} T(\log T)^{k^2}$ uniformly for $-\frac{1}{2} < k \leq 0$. Also, the magnitude of negative moments is unknown unconditionally. For this problem, we give the following unconditional result for the lower bound of negative moments in this thesis.

Theorem 1.1 (Special case of Theorem 6.3)**.** *There exist absolute constants $a > 0$, $B > 0$ such that for any $0 \leq k \leq a$ we have*

$$I_{-k}(T) \gg T + T(\log T)^{k^2 - Bk^3}.$$

This implicit constant is absolute.

This lower bound is weaker than Gonek's and the conjectural lower bound due to the factor of $(\log T)^{-Bk^3}$, but unconditionally.

For the upper bound, Heap, Radziwiłł, and Soundararajan showed that (1.15) for $0 \leq k \leq 2$, and the implicit constant is absolute. Hence, we have already obtained the conjectural upper bound, if not the asymptotic formula due to Keating-Snaith. However, the conjectural upper bound for $k > 2$ has not been proved yet unconditionally. On the other hand, assuming the RH, Harper showed (1.15) for $T \geq \exp_3(Ck)$, where $\exp_\ell$ denote the ℓ-fold iterated exponential throughout this thesis. The implicit constant of his result is $\exp_2(O(k))$, which is so bigger than the conjectural one. Before Harper's work, Soundararajan [111] showed that[2)]

$$I_k(T) \ll kT(\log T)^{k^2 + \varepsilon(T)k^3} \log\log T \tag{1.16}$$

for $k \geq 2$ and $T \geq \exp_3(Ck)$, where $\varepsilon(T) = O\left((\log_3 T)^{-1}\right)$, and the implicit constants are absolute. Toward the improvement for Harper's implicit constant, we in this thesis give another proof of Soundararajan's result, and in

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2)This estimate is little different from Soundararajan's estimate, but one can obtain it just by using his main theorem.

Chapter 6 prove the estimate

$$I_k(T) \ll T(\log T)^{k^2+\varepsilon(T)k^3}.$$

Note that we succeed in removing the factor $\log\log T$, but our implicit constant in $\varepsilon(T)$ may be worse than Soundararajan's. However, we cannot remove the factor $\log\log T$ just by using his main theorem for large deviations. Additionally, one of the important points of our method is that we do not use Soundararajan's main proposition [111, Proposition]. Thanks to that, it is possible to apply our method to the negative moments and to prove Theorem 1.1 unconditionally. Moreover, our method can be also applied to the moments of the imaginary part of the Riemann zeta-function. Precisely, we can prove the following theorem by using our method.

Theorem 1.2 (Special case of Theorem 6.5)**.** *Assume the Riemann Hypothesis. For any $k \in \mathbb{R}$, $\varepsilon > 0$, we have*

$$T(\log T)^{k^2-\varepsilon} \ll_{\varepsilon,k} \int_T^{2T} \exp\left(2k \arg\zeta(\tfrac{1}{2}+it)\right) dt \ll_{\varepsilon,k} T(\log T)^{k^2+\varepsilon}.$$

As we described in Section 1.1, the function $\arg\zeta(\frac{1}{2}+it)(=\pi S(t))$ is related to the distribution of the imaginary parts of nontrivial zeros, and so this estimate is interesting from this viewpoint. Very recently, Najnudel [91] showed that

$$\int_T^{2T} \exp\left(2k \arg\zeta(\tfrac{1}{2}+it)\right) dt \ll_{\varepsilon,k} T(\log T)^{k^2+\varepsilon}$$

for any $k \in \mathbb{R}$ and $\varepsilon > 0$ under the the Riemann Hypothesis. We give another proof of this estimate, and further the method allows us to prove the lower bound too.

1.4 Large deviations in limit theorems for the Riemann zeta-function

In this section, we consider large deviations in limit theorems (1.5) and (1.6). The parameter V in the theorems does not depend on T, but the case when V depends on T has an important application. For example, we would prove estimates (1.9), (1.10), (1.11), and (1.12) if limit theorems (1.5), (1.6) could be true for any V depending on T. However, this sufficient condition would be not correct. Therefore, we would like to know the range of V where the limit theorems hold, and the behavior of the distribution functions in the case when the limit theorems do not hold.

For this problem, Lamzouri [66] showed an effective result for asymptotic formula (1.7) due to Hattori and Matsumoto. Precisely, Lamzouri showed that

$$\frac{1}{T}\operatorname{meas}\left\{t\in[T,2T]\ :\ \log|\zeta(\sigma+it)|>V\right\} \tag{1.17}$$
$$=\exp\left(-A(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{\sigma}{1-\sigma}}\left(1+O\left(\frac{1}{\sqrt{\log V}}+\left(\frac{V\log\log T}{(\log T)^{1-\sigma}}\right)^{(\sigma-\frac{1}{2})/(1-\sigma)}\right)\right)\right)$$

for any fixed $\frac{1}{2}<\sigma<1$ and any $C(\sigma)\le V\le c(\sigma)\frac{(\log T)^{\sigma}}{\log\log T}$ with $C(\sigma)$, $c(\sigma)$ suitable positive constants. Moreover, Lamzouri, Lester, and Radziwiłł [67] showed a result for large deviations in limit formula (1.5). Actually, they proved that asymptotic formula (1.5) holds for $V=o\left(\frac{(\log T)^{1-\sigma}}{(\log\log T)^{\frac{1}{\sigma}}}\right)$. This range is a little narrower than Lamzouri's. In this thesis, we give a result which extends their range.

Theorem 1.3 (Special case of Theorem 5.2)**.** *Asymptotic formula* (1.5) *holds for* $V=o\left(\frac{(\log T)^{1-\sigma}}{\log\log T}\right)$.

For central limit theorem (1.6), Selberg-Tsang [116, Eq. (6.11)] showed that

$$\frac{1}{T}\operatorname{meas}\left\{t\in[T,2T]\ :\ \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}}>V\right\}$$
$$=\int_V^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}+O\left(\frac{(\log_3 T)^2}{\sqrt{\log\log T}}\right).$$

From this formula and the estimate $\int_V^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}\asymp\frac{1}{1+V}\exp\left(-\frac{V^2}{2}\right)$ for $V\ge 0$, asymptotic formula (1.6) holds for $V\le(1-\varepsilon)\sqrt{\log_3 T}$ with ε any fixed constant. Radziwiłł [95] improved this range into $V=o\left((\log\log T)^{1/10}\right)$[3]. He discussed the large deviations of the distribution function of the Dirichlet polynomial $\sum_{p\le X}p^{-1/2-it}$. Actually, he showed that for $V=o(\sqrt{\log\log T})$, $X=T^{1/(\log\log T)^2}$,

$$\frac{1}{T}\operatorname{meas}\left\{t\in[T,2T]\ :\ \frac{\sum_{p\le X}p^{-(1/2+it)}}{\sqrt{\frac{1}{2}\sum_{p\le X}p^{-1}}}>V\right\}\sim\int_V^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}$$

as $T\to+\infty$. By using this asymptotic formula and the mean value estimate of the gap of $\log\zeta(\frac{1}{2}+it)$ and $\sum_{p\le X}p^{-1/2-it}$ by Selberg-Tsang, he proved the result of large deviations. From the above result for the Dirichlet polynomial, Radziwiłł suggested the following conjecture.

[3] In Radziwiłł's paper [95], the range is $V=O\left((\log\log T)^{1/10-\varepsilon}\right)$ with ε any fixed constant, but the range can be easily improved into $V=o\left((\log\log T)^{1/10}\right)$ just by following his argument.

Conjecture (Radziwiłł [95]). *If $V = o\left(\sqrt{\log\log T}\right)$, asymptotic formula* (1.6) *holds. Moreover, when $V \sim k\sqrt{\log\log T}$ with $k > 0$ any fixed constant, there exists a constant $C(k)$ such that*

$$\frac{1}{T}\,\mathrm{meas}\left\{t \in [T, 2T] \;:\; \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} > V\right\} \sim C(k)\int_V^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.$$

We give a result contributing to this conjecture.

Theorem 1.4 (Inequality (2.35) in Theorem 2.5). *For $V = o\left((\log\log T)^{1/6}\right)$, we have*

$$\frac{1}{T}\,\mathrm{meas}\left\{t \in [T, 2T] \;:\; \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} > V\right\} \le (1+o(1))\int_V^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.$$

Some results weaker than such limit theorems have been already proved. We note some of those here. We first mention the trivial upper bound of the distribution of $\zeta(\frac{1}{2}+it)$ coming from the fourth moment due to Ingham. Actually, from Ingham's estimate $I_2(T) \sim \frac{1}{2\pi^2}T(\log T)^4$, we can immediately obtain

$$\begin{aligned}&\frac{1}{T}\,\mathrm{meas}\left\{t \in [T, 2T] \;:\; \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} > V\right\}\\ &\le \left(\frac{1}{2\pi^2}+\varepsilon\right)\exp\left(-4V\sqrt{\tfrac{1}{2}\log\log T} + 4\log\log T\right)\end{aligned}$$

for $V \ge 0$. Also, Jutila [56] showed that, for any $0 \le V \le \log\log T$,

$$\frac{1}{T}\,\mathrm{meas}\left\{t \in [T, 2T] \;:\; \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} > V\right\} \ll \exp\left(-\frac{V^2}{2} + O\left(\frac{V^3}{\sqrt{\log\log T}}\right)\right).$$

This upper bound is bigger than the gaussian integral $\int_V^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}}$, but his range is wider than Radziwiłł's. Recently, Heap and Soundararajan [43] showed that

$$\frac{1}{T}\,\mathrm{meas}\left\{t \in [T, 2T] \;:\; \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} > V\right\} = \exp\left(-\frac{V^2}{2} + O\left(V\log_3 T\right)\right)$$

for $\sqrt{\log\log T}\log_3 T \le V \le 2\log\log T - 2\sqrt{\log\log T}\log_3 T$. This formula is also weaker than Radziwiłł's, but the range is wider than his. Moreover, for the lower bound, Soundararajan [110] showed that, for any $3 \leq V \leq \frac{1}{5}\sqrt{\log T/\log\log T}$,

$$\frac{1}{T}\operatorname{meas}\left\{t \in [T, 2T] \ : \ \log|\zeta(\tfrac{1}{2}+it)| > V\right\} \tag{1.18}$$
$$\gg \frac{1}{(\log T)^4}\exp\left(-10\frac{V^2}{\log\frac{\log T}{8V^2\log V}}\right).$$

Such estimates also have applications such as to the moments $I_k(T)$. Actually, Soundararajan in [111] showed an upper bound of the distribution function of $|\zeta(\frac{1}{2}+it)|$, and proved (1.16) by using the bound. From this background, we give some upper and lower bounds for distribution functions of L-functions in Chapter 6, and show some results for moments of L-functions.

1.5 Iterated integrals of the logarithm of the Riemann zeta-function

In this section, we discuss the iterated integrals of the logarithm of the Riemann zeta-function. Define the functions $\eta_m(\sigma+it)$ and $\tilde{\eta}_m(\sigma+it)$ by the recurrence equations

$$\eta_m(\sigma+it) = \int_0^t \eta_{m-1}(\sigma+iu)du + c_m(\sigma),$$
$$\tilde{\eta}_m(\sigma+it) = \int_\sigma^\infty \tilde{\eta}_m(\alpha+it)d\alpha,$$

where $\eta_0(s) = \tilde{\eta}_0(s) = \log\zeta(s)$, and $c_m(\sigma) = \frac{i^m}{(m-1)!}\int_\sigma^\infty(\alpha-\sigma)\log\zeta(\alpha)d\alpha$. Under this definition, the well known function $S_m(t)$ is defined by $\frac{1}{\pi}\operatorname{Im}\eta_m(\frac{1}{2}+it)$.

Fujii [29] showed that the formula

$$\operatorname{Im}\eta_m(\tfrac{1}{2}+it) \tag{1.19}$$
$$= \operatorname{Im} i^m\tilde{\eta}_m(\tfrac{1}{2}+it) + 2\pi\operatorname{Im}\sum_{k=0}^{m-1}\frac{i^{m-1-k}}{(m-k)!k!}\sum_{\substack{0<\gamma<t \\ \beta>\frac{1}{2}}}(\beta-\tfrac{1}{2})^{m-k}(t-\gamma)^k,$$

which relates the η_m to $\tilde{\eta}_m$. Moreover, he showed that $\operatorname{Im}\tilde{\eta}_m(\frac{1}{2}+it) \ll_m \log t$ for $t \geq 2$ and consequently established that the Riemann Hypothesis is equivalent to the estimate $\operatorname{Im}\eta_m(\frac{1}{2}+iT) = o(T^{m-2})$ for every $m \in \mathbb{Z}_{\geq 3}$. Also, we can show the following proposition.

Proposition 1.1. *Let $m \in \mathbb{Z}_{\geq 1}$. The Lindelöf Hypothesis is equivalent to the estimate* $\operatorname{Re}\tilde{\eta}_m(\frac{1}{2}+it) = o(\log t)$ *as* $t \to +\infty$.

This proposition is a generalization of an unpublished work of Ghosh and Goldston (see pp.334–335 in [114]). They showed that the Lindelöf Hypothesis is equivalent to that the estimate $S_1(t) = o(\log t)$ as $t \to +\infty$. We see that

$S_1(t) = \operatorname{Re} \tilde{\eta}_1(\frac{1}{2} + it)$ from equation (1.19), and so we can regard that Proposition 1.1 is a generalization of the equivalence of Ghosh and Goldston. From these observations, the functions $\eta_m(s)$ and $\tilde{\eta}_m(s)$ are interesting as well as $\zeta(s)$ and $S(t)$, and we discuss the value distribution of these functions as one of the topics in this thesis.

Recently, the study Ω-estimates of $S_m(t)$ have been developed by some articles such as [11], [14], [15] under the Riemann Hypothesis. Those results were shown by the resonance method due to Bondarenko and Seip [10], [11]. On the other hand, as Bondarenko and Seip mentioned in [11], it is desirable that those could be shown unconditionally by proving a stronger result on the measure of extreme values like Soundararajan's result (1.18). Toward this problem, we discuss the large deviations of the distribution function of $\tilde{\eta}_m(s)$ in the critical strip. For example, we give the following theorem.

Theorem 1.5 (Theorem 3.1)**.** *Let $m \in \mathbb{Z}_{\geq 1}$, $\theta \in \mathbb{R}$ be fixed. There exists a positive constant $a_1 = a_1(m)$ such that, for any large numbers T, V with $V \leq a_1 \left(\frac{\log T}{(\log\log T)^{2m+2}}\right)^{\frac{m}{2m+1}}$, we have*

$$\frac{1}{T} \operatorname{meas}\left\{t \in [T, 2T] \ : \ \operatorname{Re} e^{-i\theta} \tilde{\eta}_m(\tfrac{1}{2} + it) > V\right\} \tag{1.20}$$
$$= \exp\left(-2m4^m V^2 (\log V)^{2m} \left(1 + R\right)\right),$$

where the error term R satisfies

$$R \ll_m \frac{V^{2m+1}(\log V)^{2m(m+1)}}{(\log T)^m} + \sqrt{\frac{\log\log V}{\log V}}.$$

This result recovers Tsang's Ω-estimate [117]

$$S_1(T) = \Omega_-\left(\frac{(\log T)^{1/3}}{(\log\log T)^{4/3}}\right)$$

as $T \to +\infty$. This is at present the best unconditional bound. On the other hand, Tsang [118] also showed $S_1(T) = \Omega_+\left(\frac{\sqrt{\log T}}{(\log\log T)^{3/2}}\right)$ unconditionally, and our result cannot recover this estimate. From this problem, it is desirable to prove (1.20) for some larger range of V.

1.6 Ramachandra's denseness problem

As forerunners of the limit theorem of Bohr-Jessen (1.5), Bohr and Courant [7], and Bohr [6] showed the following interesting theorems.

Theorem (Bohr and Courant in 1914 [7])**.** *Let $\frac{1}{2} < \sigma \leq 1$. Then the set $\{\zeta(\sigma + it) \ : \ t \in \mathbb{R}\}$ is dense in the complex plane.*

Theorem (Bohr in 1916 [6]). *Let* $\frac{1}{2} < \sigma \leq 1$. *Then the set* $\{\log \zeta(\sigma + it) \ : \ t \in \mathbb{R}\}$ *is dense in the complex plane.*

Note that the latter theorem is an improvement of former one since the former one is an immediate consequence from the latter theorem. These results are interesting, and there are many developments inspired by these results, such as the Bohr-Jessen limit theorem [8] and Voronin's universality theorem [119]. On the other hand, the value distribution of $\zeta(s)$ on the critical line $\sigma = \frac{1}{2}$ is more difficult, and the following problem is well known.

Problem 1.1. *Is the set* $\{\zeta(\frac{1}{2} + it) \ : \ t \in \mathbb{R}\}$ *dense in the complex plane?*

This problem was first mentioned by Ramachandra (for the history and the present state-of-art of this problem, see [65]). This problem is at present open, and it is difficult to solve this even under the Riemann Hypothesis. For Problem 1.1, there is an interesting study by Kowalski and Nikeghbali [65]. They studied the Fourier transform of the probability measure which represents the probability of $\log \zeta(1/2 + it) \in A$ with A a Borel set. In particular, they gave a sufficient condition that the values $\zeta(1/2 + it)$ for $t \in \mathbb{R}$ are dense in the complex plane (see [65, Corollary 9]). Hence, from their study, we might guess that the answer for Problem 1.1 could be yes. However, as they mentioned in their paper [65], their sufficient condition is rather strong. Therefore, it is also not strange that the answer for Problem 1.1 could be no. Moreover, Garunkštis and Steuding [30] showed that the set of $(\zeta(1/2 + it), \zeta'(1/2 + it))$ for $t \in \mathbb{R}$ is not dense in $\mathbb{C}^2$. As we can see from these works, it seems difficult to decide clearly the answer of Problem 1.1 at present. Hence, it is desirable to obtain some new information for this problem.

In this thesis, we consider the following problem.

Problem 1.2. *Is the set* $\{\log \zeta(1/2 + it) \ : \ t \in \mathbb{R}\}$ *dense in the complex plane?*

This problem is stronger than Problem 1.1 in the sense that if the set $\{\log \zeta(\frac{1}{2} + it) \ : \ t \in \mathbb{R}\}$ is dense in $\mathbb{C}$, then $\{\zeta(\frac{1}{2} + it) \ : \ t \in \mathbb{R}\}$ is also dense in $\mathbb{C}$. Since the function $\eta_m(s)$ is the m-times iterated integral of $\log \zeta(s)$ on the vertical line, we can expect that the function contains information related to the value distribution of $\log \zeta(s)$. In particular, since $\eta_m(1/2 + it)$ is the iterated integral on the critical line, the study of the value distribution of this function might give new information on Problem 1.2. From this background, we study the denseness of the function $\eta_m(s)$ and prove the following theorem in Chapter 4.

Theorem 1.6 (Theorem 4.1). *Let* $1/2 \leq \sigma < 1$. *If the number of zeros* $\rho = \beta + i\gamma$ *with* $\beta > \sigma$ *is finite, then the set*

$$\left\{ \int_0^t \log \zeta(\sigma + it')dt' \ : \ t \in [0, \infty) \right\}$$

is dense in the complex plane. Moreover, for each integer $m \geq 2$, *the following statements are equivalent.*

(I). The Riemann zeta-function does not have zeros whose real part are greater than σ.

(II). The set $\{\eta_m(\sigma + it) \ : \ t \in [0, \infty)\}$ is dense in the complex plane.

From this theorem, we see that the Riemann Hypothesis implies that the set

$$\left\{ \int_0^t \log \zeta(1/2 + it')dt' \ : \ t \in [0, \infty) \right\}$$

is dense in the complex plane. This implication seems to suggest that the answer of Problem 1.2 is yes. Moreover, the equivalence above would be a new type of statement which gives the relation between the denseness of values of the Riemann zeta-function and the Riemann Hypothesis.

Furthermore, we also give a result for the denseness of Dirichlet polynomial. Roughly speaking, the proofs of Bohr and Courant are mainly divided into the following two parts.

Step 1. (Denseness lemma) *The corresponding Dirichlet polynomial to $\zeta(\sigma + it)$ and $\log \zeta(\sigma + it)$ can approximate to any complex numbers.*

Step 2. *For "almost all" t, $\zeta(\sigma + it)$ and $\log \zeta(\sigma + it)$ can be approximated by the corresponding Dirichlet polynomial.*

When $\frac{1}{2} < \sigma \leq 1$, these assertions were shown by Bohr and Courant. Additionally, Step 1 in the case $\sigma = \frac{1}{2}$ was been proved by Kowalski and Nikeghbali [65, Theorem 2] by showing a lower bound of the distribution function of the Dirichlet polynomial $\prod_{p \leq X}(1 - p^{-1/2-it})^{-1}$, which corresponds to $\zeta(\frac{1}{2} + it)$. On the other hand, we give the following theorem.

Theorem 1.7 (Special case of Corollary 6.3)**.** *Put $\sigma(X) = \sqrt{\frac{1}{2}\sum_{p \leq X} p^{-1}}$, and define $R(z, r) := \{u + iv \in \mathbb{C} \ : \ \max\{|\operatorname{Re} z - u|, |\operatorname{Im} z - v|\} < r\}$. For any $0 < \varepsilon \leq 1$, $z \in \mathbb{C}$, and any numbers T, X with $X^{(\log\log X)^{12}} \leq T$, we have*

$$\frac{1}{T} \operatorname{meas}\left\{ t \in [T, 2T] \ : \ \sum_{p \leq X} p^{-1/2-it} \in R(z, \varepsilon) \right\} \sim \iint_{R(z/\sigma(X), \varepsilon/\sigma(X))} e^{-\frac{u^2+v^2}{2}} \frac{dudv}{2\pi}$$

as $X \to +\infty$.

This theorem gives the result on the denseness of the Dirichlet polynomial $P_1(t) = \sum_{p \leq X} p^{-1/2-it}$, and so this theorem advances Ramachandra's problem. On the other hand, there is a gap between this theorem and the denseness lemma because the Dirichlet polynomial corresponding to $\log \zeta(\frac{1}{2} + it)$ is $P_2(t) = -\sum_p \log(1 - p^{-1/2-it})$ or $P_3(t) = \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{1/2+it}\log n}$. Hence, we should consider these Dirichlet polynomials from the viewpoint of Ramachandra's problem. On the other hand, the contribution from the gaps between the Dirichlet polynomials $P_2(t)$, $P_3(t)$ and $P_1(t)$ is not big. From this fact, Theorem 1.7 suggests that the distribution functions of $P_2(t)$, $P_3(t)$ have also similar asymptotic formulas. Actually, the author is considering whether the asymptotic formulas can be proved by the methods in Chapters 5, 6. In particular, such asymptotic formulas for the Distribution functions of $P_2(t)$, $P_3(t)$ would give an improvement of the result of Kowalski and Nikeghbali. From these observations, the author believes it makes sense to state Theorem 1.7 here as one of the progress towards Ramachandra's problem.

1.7 Definition of some classes of L-functions

So far, we surveyed the value distribution of the Riemann zeta-function. Some of the results above can be extended to a certain class of L-functions. For example, Selberg introduced a class of L-functions and gave a theorem [108, Theorem 2] for central limit theorems of L-functions in the class. Today this class is called the Selberg class. To define the class, we introduce some properties for Dirichlet series $F(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$.

(S1) The series $\sum_{n=1}^{\infty} a_F(n)n^{-s}$ is convergent absolutely for $\sigma > 1$.

(S2) (Analytic continuation) There exists $m_F \in \mathbb{Z}_{\geq 0}$ such that $(s-1)^{m_F} F(s)$ is entire of finite order.

(S3) (Functional equation) $F(s)$ satisfies the following functional equation

$$\Phi_F(s) = \omega_F \overline{\Phi_F}(1-s),$$

where $\Phi_F(s) = \gamma(s)F(s)$ and $\gamma(s) = Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j)$, with $\lambda_j > 0$, $Q > 0$, $\mathrm{Re}(\mu_j) \geq 0$, and $|\omega_F| = 1$. Here we use the notation $\overline{\Phi_F}(s) = \overline{\Phi_F(\overline{s})}$.

(S3′) $F(s)$ has the same functional equation as in (S3), where the condition $\mathrm{Re}(\mu_j) \geq 0$ changes to $\mathrm{Re}(\mu_j) \geq -\frac{1}{2}$.

(S4) (Euler product) $F(s)$ can be written as

$$F(s) = \prod_p F_p(s), \quad F_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b_F\,(p^k)}{p^{ks}}\right),$$

where $b_F(n) = 0$ unless $n = p^{\ell}$ with $\ell \in \mathbb{Z}_{\geq 1}$, and $b_F(n) \ll n^{\vartheta_F}$ for some $\vartheta_F < \frac{1}{2}$.

(S5) For any $k \geq 2$,

$$\sum_p \frac{|b_F(p^k)(\log p^k)|^2}{p^k} < +\infty. \tag{1.21}$$

(S6) (Ramanujan conjecture) For every $\varepsilon > 0$, the inequality $a_F(n) \ll_F n^{\varepsilon}$ holds.

The set of L-functions satisfying (S1)–(S4) and (S6) is called the Selberg class denoted by $\mathcal{S}$, and also the set of L-functions not equaling to the identically zero and satisfying (S1), (S2), and (S3) is called the extended Selberg class denoted by $\mathcal{S}^\sharp$. In this thesis, we study the set $\mathcal{S}^\dagger$ consisting of L-functions satisfy (S1), (S2), (S3'), (S4), and (S5). We call $\mathcal{S}^\dagger$ the modified Selberg class. The Ramanujan Conjecture is a strong condition, that implies (S5) together with (S4). Actually, if F is an L-function satisfying (S4) and (S6), then it holds that (cf. [89, Exercise 8.2.9])

$$|b_F(p^\ell)| \ll_{\varepsilon,F} \left(2^\ell - 1\right) p^{\varepsilon\ell}/\ell.$$

Hence, it holds that $\mathcal{S} \subset \mathcal{S}^\dagger$. Axiom (S5) is sometimes called the Hypothesis H, which was introduced by Rudnick and Sarnak [102, equation (1.7)]. The zeros of Φ_F is called the nontrivial zeros of F. It is known that many interesting L-functions belong to these classes. For example, the Riemann zeta-function, Dirichlet L-functions, Dedekind zeta-functions, Hecke L-functions associated with primitive Hecke characters, and L-functions associated with holomorphic newforms of a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ normalized suitably belong to the Selberg class. In general, it is difficult to decide the all of the L-functions in the Selberg class. For this direction, there are interesting works such as [4], [18], [57], [59], [60], and [61]. Additionally, the following conjecture is known (see [92]).

Conjecture (Main conjecture for the Selberg class)**.** *The Selberg class $\mathcal{S}$ coincides with the class of the* $\mathrm{GL}(n)$ *over* $\mathbb{Q}$ *automorphic L-functions.*

From these observations, the Selberg class is an interesting mathematical object.

In general, it is difficult to prove the Ramanujan conjecture for automorphic L-functions. Hence, we study $\mathcal{S}^\dagger$ in this thesis. As one of advantages of this relaxing, it has been proved that automorphic L-functions attached to an irreducible unitary cusp representation of $\mathrm{GL}(n)$ over $\mathbb{Q}$ for $n \le 4$ belong to $\mathcal{S}^\dagger$. This fact was proved by Rudnick and Sarnak in [102] for $n \le 3$ and Kim and Sarnak [64, Appendix 2] for $n = 4$.

1.8 The distribution functions of L-functions

In [108], Selberg suggested some interesting conjectures, and many mathematicians have worked for the conjectures. In particular, he conjectured that the Riemann Hypothesis is generalized to the Selberg class.

Conjecture (Grand Riemann Hypothesis (GRH))**.** *For $F \in \mathcal{S} \setminus \{1\}$, all nontrivial zeros of F lie on the critical line $\sigma = \frac{1}{2}$.*

Similarly to the case of the Riemann zeta-function, this conjecture implies the Grand Lindelöf Hypothesis, which states that $F(\frac{1}{2} + it) \ll_{\varepsilon,F} |t|^\varepsilon$ for $|t| \ge 2$. Moreover, the Grand Lindelöf Hypothesis for F has some consequences to the distribution of zeros of F. For example, we can generalize Backlund's equivalence to the Selberg class and the modified Selberg class. Also, the Riemann-von Mangoldt formula (1.4) is generalized to the modified Selberg class. Let $N_F(T)$ be the number of nontrivial zeros $\rho_F = \beta_F + i\gamma_F$ with $0 \leq \gamma_F < T$ counted with multiplicity. Then formula (1.4) is generalized to

$$N_F(T) = \tag{1.22}$$
$$\frac{1}{\pi}\sum_{j=1}^{k}\left(\arg\Gamma(\tfrac{\lambda_j}{2}+\mu_j+i\lambda_j T)-\arg\Gamma(\tfrac{\lambda_j}{2}+\mu_j)\right)+\frac{\log Q}{\pi}T+S_F(T)-S_F(0)+m_F$$

for $F \in \mathcal{S}^\dagger \setminus \{1\}$, where λ_j, μ_j, and m_F are the numbers in (S2), (S3), and (S3′), and $S_F(t)$ is the function defined by $\frac{1}{\pi}\operatorname{Im}\log F(\frac{1}{2}+it) = \frac{1}{\pi}\arg F(\frac{1}{2}+it)$.

Using standard methods, we can show that $F(\frac{1}{2}+it) \ll_{F,\varepsilon} (|t|+1)^{\frac{d_F}{4}+\varepsilon}$ and $S_F(t) \ll_F \log(|t|+3)$ for $t \in \mathbb{R}$. Here, d_F is the degree of F defined by $2\sum_{j=1}^{k}\lambda_j$ which is an invariant for F, that is, the degree d_F does not depend on the form of the gamma factor $\gamma(s)$ in (S3). In particular, substituting the latter estimate to (1.22) and using the Stirling formula, we have

$$N_F(T) = \frac{T}{2\pi}\log\left(q_F\left(\frac{T}{2\pi}\right)^{d_F}\right) + O_F(\log T) \tag{1.23}$$

for $T \geq 3$. Here, q_F is the conductor of F defined by $(2\pi)^{d_F}Q^2\prod_{j=1}^{k}\lambda_j^{2\lambda_j}$. This number is also an invariant for F. Additionally, the estimate of $S_F(t)$ can be improved into $S_F(t) = o(\log(|t|+3))$ as $|t| \to +\infty$ under the Lindelöf Hypothesis for F. By this improvement, we can improve the error term of (1.23) into $o(\log T)$ under the Lindelöf Hypothesis for F. From these observation, the value distribution of L-functions is important as well as the Riemann zeta-function.

As we mentioned above, Selberg generalized (1.6) to L-functions in the Selberg class. Precisely, he showed that, for all $F \in \mathcal{S}\setminus\{1\}$ satisfying a certain condition,

$$\lim_{T\to+\infty}\frac{1}{T}\operatorname{meas}\left\{t \in [T,2T] \ : \ \frac{\log|F(\frac{1}{2}+it)|}{\sqrt{\frac{n_F}{2}\log\log T}} > V\right\} = \int_V^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}},$$
$$\lim_{T\to+\infty}\frac{1}{T}\operatorname{meas}\left\{t \in [T,2T] \ : \ \frac{\arg F(\frac{1}{2}+it)}{\sqrt{\frac{n_F}{2}\log\log T}} > V\right\} = \int_V^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}},$$

where n_F is a positive integer. Moreover, these formulas are also generalized to the modified Selberg class. We will see this fact in Chapter 6.

Also, the value distribution of L-function in the domain $\sigma > \frac{1}{2}$ have been studied by many mathematicians. Matsumoto [77] generalized limit theorem (1.5) to a class of zeta or L-functions. Combing his result with Potter's result [94, Theorem 1], we obtain the following: if $F \in \mathcal{S}\setminus\{1\}$ having the "polynomial Euler product" satisfies the estimate

$$\int_T^{2T} |F(\sigma_0 + it)|^2 dt \ll T^{1+\varepsilon}$$

for any $\varepsilon > 0$ and for some $\sigma_0 \geq \frac{1}{2}$, then there exists a probability measure $P_{\sigma,F}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\lim_{T\to+\infty} \frac{1}{T} \operatorname{meas}\left\{t \in [T, 2T] \ : \ \log F(\sigma + it) > V\right\} = P_{\sigma,F}((V, +\infty))$$

for $\sigma > \sigma_0$. Thus, we have already succeeded in generalizing the limit theorem (1.5) to a certain extent. On the other hand, there is an obstacle to generalize the existence of the density function of $P_{\sigma,F}$. The details of the obstacle is written in [78]. Some mathematicians [79], [80], [84] have worked this generalization, and succeeded it for some certain L-functions. However, a more generalization, for example to all L-functions in Selberg class, would require further works.

1.9 Independence of L-functions

We observed some properties of L-functions so far. From the results mentioned above, we see that many properties of the Riemann zeta-function can be generalized to L-functions in $\mathcal{S}$ and $\mathcal{S}^\dagger$. Therefore, we find that L-functions in these classes have many common properties. On the other hand, in order to understand each L-function deeply, it seems to be important to study so that the differences of distinct primitive L-functions can be clarified. Here, we say $F \in \mathcal{S} \setminus \{1\}$ primitive if $F = F_1 F_2$ with $F_1, F_2 \in \mathcal{S}$ implies $F_1 = 1$ or $F_2 = 1$. In the classes $\mathcal{S}^\sharp$, $\mathcal{S}^\dagger$, define it in the same way. From this perspective, we also discuss the independence of L-functions in this thesis.

Selberg in [108] mentioned that the distinct primitive L-functions in the Selberg class are "statistically independent" under a strong zero density estimate and an orthonormality conjecture without any precise description for the independence. The strong zero density estimate means that there exists a positive constant κ_F such that, uniformly for any $T \geq 3$ and $\frac{1}{2} \leq \sigma \leq 1$,

$$N_F(\sigma, T) \ll_F T^{1-(2\sigma-1)\kappa_F} \log T. \tag{1.24}$$

Here, $N_F(\sigma, T)$ is the number of nontrivial zeros $\rho_F = \beta_F + i\gamma_F$ of F with $\beta_F \geq \sigma$, $0 \leq \gamma_F \leq T$ counted with multiplicities. The statement of the orthonormality conjecture is the following.

Conjecture (Selberg Orthonormality Conjecture (SOC))**.** *For any primitive L-function $F \in \mathcal{S}$,*

$$\sum_{p \leq X} \frac{|a_F(p)|^2}{p} = \log\log X + O_F(1)$$

for some positive integer n_F and any $X \geq 3$. For any primitive L-functions $F \neq G \in \mathcal{S}$,

$$\sum_{p \leq X} \frac{a_F(p)\overline{a_G(p)}}{p} = O_{F,G}(1)$$

for any $X \geq 3$.

Assuming (1.24), SOC, and other certain conditions, Bombieri and Hejhal [9][4)] established the statistically independence of L-functions by showing that, for any fixed $V_1, \dots, V_r, W_1, \dots, W_r \in \mathbb{R}$,

$$\frac{1}{T} \operatorname{meas} \bigcap_{j=1}^{N} \left\{ t \in [T, 2T] \ : \ \frac{\log |F_j(\frac{1}{2} + it)|}{\sqrt{\frac{n_{F_j}}{2} \log\log T}} > V_j \text{ and } \frac{\arg F_j(\frac{1}{2} + it)}{\sqrt{\frac{n_{F_j}}{2} \log\log T}} > W_j \right\}$$
$$\sim \prod_{j=1}^{N} \int_{V_j}^{\infty} \int_{W_j}^{\infty} e^{-(u^2+v^2)/2} \frac{du dv}{2\pi} \tag{1.25}$$

as $T \to +\infty$, where F_j are L-functions satisfying certain properties, and n_{F_j} are certain positive integers. In particular, this formula leads that the L-functions normalized by the variance $\frac{n_F}{2} \log\log T$ are independent as random variables on the critical line. To understand this independence more deeply, we improve the limit theorem of Bombieri and Hejhal for the direction of large deviations. Here, we omit the precise condition of the theorem which is written in Chapter 6.

Theorem 1.8 (Special case of Theorem 6.1)**.** *Let $r \in \mathbb{Z}_{\geq 1}$. Let $\boldsymbol{F} = (F_1, \dots, F_r) \in (\mathcal{S}^\dagger)^r$, $(\theta_1, \dots, \theta_r) \in \mathbb{R}^r$ satisfying suitable conditions. Then, for any $(V_1, \dots, V_r) \in \mathbb{R}^r$ with $\max_{1 \leq j \leq r} |V_j| = o\left((\log\log T)^{1/10}\right)$, we have*

$$\frac{1}{T} \operatorname{meas} \bigcap_{j=1}^{r} \left\{ t \in [T, 2T] \ : \ \frac{\operatorname{Re} e^{-i\theta_j} \log F_j(\frac{1}{2} + it)}{\sqrt{\frac{n_{F_j}}{2} \log\log T}} > V_j \right\} \sim \prod_{j=1}^{r} \int_{V_j}^{\infty} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}$$

as $T \to +\infty$.

Also, we can obtain an upper bound and a lower bound of the distribution function of $\boldsymbol{F} = (F_1, \dots, F_r)$ in a wider range of $\boldsymbol{V}$ (see Theorems 6.2, 6.4). As one of the application of the bounds, we can obtain the following mean value theorem.

[4)] Bombieri and Hejhal credited Selberg in their paper. The author does not know the meaning, but their result may be an unpublished work of Selberg.

Theorem 1.9 (Special case of Theorems 6.3, 6.5). *Let $\chi_1, \ldots, \chi_r$ be distinct primitive Dirichlet characters. Then there exists some positive constant B depending on χ_j's such that for any small enough positive real number k,*

$$\int_T^{2T} \left(\min_{1 \le j \le r} \left| L(\tfrac{1}{2} + it, \chi_j) \right| \right)^{2k} dt \ll T \frac{(\log T)^{k^2/r + Bk^3}}{(\log\log T)^{(r-1)/2}}, \tag{1.26}$$
$$\int_T^{2T} \left(\max_{1 \le j \le r} \left| L(\tfrac{1}{2} + it, \chi_j) \right| \right)^{-2k} dt \gg T \frac{(\log T)^{k^2/r - Bk^3}}{(\log\log T)^{(r-1)/2}}.$$

The above implicit constants depend on χ_j and k. If we assume the Riemann Hypothesis for $L(s, \chi_j)$, then we have, for any $k \ge 0$ and $\varepsilon > 0$,

$$\int_T^{2T} \left(\min_{1 \le j \le r} \left| L(\tfrac{1}{2} + it, \chi_j) \right| \right)^{2k} dt \ll T (\log T)^{k^2/r + \varepsilon},$$
$$\int_T^{2T} \left(\max_{1 \le j \le r} \left| L(\tfrac{1}{2} + it, \chi_j) \right| \right)^{-2k} dt \gg T (\log T)^{k^2/r - \varepsilon}.$$

The implicit constants depend on χ_j, k, and ε.

From this theorem, we find that the mean value estimate of $\min\{|\zeta(\frac{1}{2} + it)|, |L(\frac{1}{2} + it)|\}$ is strictly smaller than just the mean value estimate $|\zeta(\frac{1}{2} + it)|$. As we mentioned in Section 1.3, it is known unconditionally that, for $0 \le k \le 2$,

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} dt \asymp_k T (\log T)^{k^2}.$$

Moreover, it is expected that $\int_T^{2T} |L(\frac{1}{2} + it, \chi)|^{2k} dt \asymp T(\log T)^{k^2}$. Therefore, the unconditional result (1.26) is new and interesting when $k < B^{-1}/2$. We could regard this fact as one of the new evidence of independence of L-functions.

So far in this section, we observed independence of L-functions on the critical line. It is a natural question to ask the independence in the other domain, particularly in the strip $\frac{1}{2} < \sigma < 1$. One may speculate that L-functions are independence as random variables even in this strip, but this does not hold. This fact was informed to the author by Mine [85]. Roughly speaking Mine's method, we consider the characteristic functions

$$\begin{aligned} &\varphi_{\chi_1, \chi_2}(\xi_1, \xi_2) \\ &:= \lim_{T \to +\infty} \frac{1}{T} \int_T^{2T} \exp\left(i\xi_1 \log|L(\sigma + it, \chi_1)| + i\xi_2 \log|L(\sigma + it, \chi_2)| \right) dt, \end{aligned}$$

and

$$\varphi_{\chi_j}(\xi_j) := \lim_{T \to +\infty} \frac{1}{T} \int_T^{2T} \exp\left(i\xi_j \log|L(\sigma + it, \chi_j)| \right) dt$$

with χ_1, χ_2 distinct primitive Dirichlet characters modulo q. Then we can show that $\varphi_{\chi_1,\chi_2} \neq \varphi_{\chi_1} \cdot \varphi_{\chi_2}$. Hence, $L(\sigma+it,\chi_1)$ and $L(\sigma+it,\chi_2)$ are not independent as random variables, and particularly the equation

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{T} \operatorname{meas} \bigcap_{j=1}^{2} \left\{ t \in [T,2T] \ : \ \log|L(\sigma+it,\chi_j)| > V_j \right\} \\
&= \lim_{T\to+\infty} \left\{ \left(\frac{1}{T} \operatorname{meas} \left\{ t \in [T,2T] \ : \ \log|L(\sigma+it,\chi_1)| > V_1 \right\} \right) \right. \\
&\qquad\qquad \left. \times \left(\frac{1}{T} \operatorname{meas} \left\{ t \in [T,2T] \ : \ \log|L(\sigma+it,\chi_2)| > V_2 \right\} \right) \right\}
\end{aligned}$$

does not hold for some $V_1, V_2 \in \mathbb{R}$. From this fact, it is a natural question to ask how dependent are L-functions in the strip $\frac{1}{2} < \sigma < 1$. We give an answer to this question by extending the results of Hattori-Matsumoto and Lamzouri to joint value distribution of $\zeta(\sigma+it)$ and $L(\sigma+it,\chi)$ with χ a quadratic character.

Theorem 1.10 (Special case of Theorem 7.1)**.** *Let $\frac{1}{2} < \sigma < 1$ be fixed, and let χ be a quadratic character. There exists a positive constant $a = a(\sigma,\chi)$ such that for any large T, V with $V \leq a\frac{(\log T)^{\sigma}}{\log\log T}$, we have*

$$\begin{aligned}
&\frac{1}{T} \operatorname{meas} \left\{ t \in [T,2T] \ : \ \log|\zeta(\sigma+it)| > V \text{ and } \log|L(\sigma+it,\chi)| > V \right\} \\
&= \exp\left(-2^{\frac{\sigma}{1-\sigma}} A(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\sigma}{1-\sigma}} (1+o(1)) \right) \qquad (1.27)
\end{aligned}$$

as $V \to +\infty$, where $A(\sigma)$ is the number defined by (1.8)*. The above implicit constant may depend on σ and χ.*

For any primitive Dirichlet character χ, define the distribution functions

$$\begin{aligned}
&\Phi_T(x,y;\chi) := \\
&\frac{1}{T} \operatorname{meas} \left\{ t \in [T,2T] \ : \ \log|\zeta(\sigma+it)| > x \text{ and } \log|L(\sigma+it,\chi)| > y \right\},
\end{aligned}$$

$$\begin{aligned}
\Psi_T(x) &:= \frac{1}{T} \operatorname{meas} \left\{ t \in [T,2T] \ : \ \log|\zeta(\sigma+it)| > x \right\}, \\
\Psi_T(x;\chi) &:= \frac{1}{T} \operatorname{meas} \left\{ t \in [T,2T] \ : \ \log|L(\sigma+it,\chi)| > x \right\}
\end{aligned}$$

Then, using the method in Chapter 3, one can show that, for any $V \leq c\frac{(\log T)^{1-\sigma}}{\log\log T}$ with $c = c(\chi)$ a small positive constant,

$$\Psi_T(V;\chi) = \exp\left(-A(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\sigma}{1-\sigma}} (1+o(1)) \right)$$

as $V \to +\infty$. From this limit formula and (1.17), if the functions $\log|\zeta(\sigma+it)|$ and $\log|L(\sigma+it,\chi)|$ are independent as random variables, the right hand side of (1.27) must become

$$\exp\left(-2A(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{\sigma}{1-\sigma}}(1+o(1))\right),$$

but it does not hold for any large V when $\frac{1}{2} < \sigma < 1$. Moreover, we can clearly understand the difference between $\Phi_T(V,V;\chi)$ and $\Psi_T(V) \times \Psi_T(V;\chi)$ in the sense

$$\frac{\Psi_T(V) \times \Psi_T(V;\chi)}{\Phi_T(V,V;\chi)} = \exp\left(2\left(2^{\frac{2\sigma-1}{1-\sigma}}-1\right)A(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{\sigma}{1-\sigma}}(1+o(1))\right)$$

when χ is a quadratic character. Hence, when $\frac{1}{2} < \sigma < 1$, the functions $\log|\zeta(\sigma+it)|$ and $\log|L(\sigma+it,\chi)|$ are dependent as random variables. It would be interesting to make an arithmetic meaning for this fact. Also, this result seems unexpected, in view of the previous work of the joint universality theorem due to Lee, Nakamura, and Pańkowski [69]. This fact would suggest that the universality cannot give us information of independence for random variables. From this viewpoint too, the dependence would be interesting.

Chapter 2 Approximate formula for $\log\zeta(s)$, $\eta_m(s)$, $\tilde{\eta}_m(s)$ and its applications

In this chapter, we prove an approximate formula for the Riemann zeta-function. The formula plays an important role throughout this thesis. The contents in this chapter are based on the paper [50].

2.1 Approximate formula for $\log\zeta(s)$, $\eta_m(s)$, and $\tilde{\eta}_m(s)$

Throughout this chapter, we use the following notations.

Notations. Let $s = \sigma + it$ be a complex number with σ, t real numbers, and $\rho = \beta + i\gamma$ be a nontrivial zero of $\zeta(s)$ with β, γ also real numbers. Let $\Lambda(n)$ be the von Mangoldt function.

Let $H \geq 1$ be a real parameter. The function $f : \mathbb{R} \to [0, +\infty)$ is mass one and supported on $[0, 1]$, and further f is a $C^1([0,1])$-function, or for some $d \geq 2$ f belongs to $C^{d-2}(\mathbb{R})$ and is a $C^d([0,1])$-function. For such f's, we define the number $D(f)$, and functions $u_{f,H}$, $v_{f,H}$ by

$$D(f) = \max\{d \in \mathbb{Z}_{\geq 1} \cup \{+\infty\} \mid f \text{ is a } C^d([0,1])\text{-function}\},$$

$u_{f,H}(x) = Hf(H\log(x/e))/x$, and

$$v_{f,H}(y) = \int_y^\infty u_{f,H}(x)dx,$$

respectively. Further, for each integer $m \geq 0$, the function U_m is defined by

$$U_m(z) = \frac{1}{m!}\int_0^\infty \frac{u_{f,H}(x)}{(\log x)^m} E^*_{m+1}(z\log x)dx$$

for $\mathrm{Im}(z) \neq 0$. Here, $E^*_{m+1}(z) = E^*_{m+1}(x+iy)$ is the function of a little modified m-th exponential integral defined by

$$E^*_{m+1}(z) := \int_{x+iy}^{+\infty+iy} (w - (x+iy))^m \frac{e^{-w}}{w}dw = \int_z^\infty (w-z)^m \frac{e^{-w}}{w}dw.$$

When $\mathrm{Im}(z) = 0$, then $U_m(x) = \lim_{\varepsilon\uparrow 0} U_m(x + i\varepsilon)$.

Let $X \geq 3$ be a real parameter. The function $Y_m(s, X)$ is defined by

$$Y_m(s,X) = \begin{cases} \displaystyle\sum_{|s-\rho| \leq 1/\log X} \log((s-\rho)\log X) & m = 0, \\ \displaystyle 2\pi \sum_{k=0}^{m-1} \frac{i^{m-1-k}}{(m-k)!k!} \sum_{\substack{0<\gamma<t \\ \beta>\sigma}} (\beta-\sigma)^{m-k}(t-\gamma)^k & m \geq 1. \end{cases} \tag{2.1}$$

In this paper, we take the branch of $\log z$ by $-\pi \leq \arg(z) < \pi$. Here, we may represent $Y_m(s, X)$ by $Y_m(s)$ in the case $m \geq 1$ since $Y_m(s, X)$ does not depend on X in this case.

Remark 1. From the above definitions, the function $u_{f,H}$ is mass one and supported on $[e, e^{1+1/H}]$, and further $u_{f,H}$ is a $C^1([e, e^{1+1/H}])$-function, or $u_{f,H}$ belongs to $C^{d-2}(\mathbb{R}_{>0})$ and is a $C^d([e, e^{1+1/H}])$-function for some integer $d \geq 2$. We also note that $v_{f,H}$ is a nonnegative continuous function on $\mathbb{R}_{>0}$ and satisfies $v_{f,H}(y) = 0$ for $y \geq e^{1+1/H}$ and $v_{f,H}(y) = 1$ for $0 < y \leq e$.

Remark 2. Note that some remarks for $Y_m(s, X)$. When $m = 0$, the real part of it is always non-positive. When $m = 1$, the function $Y_1(s)$ has the following simple formula

$$Y_1(s) = 2\pi \sum_{\substack{0<\gamma<t \\ \beta>\sigma}} (\beta - \sigma),$$

and its value is always nonnegative and always zero for $\sigma \geq 1/2$ under the Riemann Hypothesis. Next, we suppose $m \geq 2$. Then if the Riemann Hypothesis is true, $Y_m(s)$ is always zero for $\sigma \geq 1/2$. On the other hand, if the Riemann Hypothesis is false, the value of $Y_m(s)$ becomes big in σ close to $1/2$. Actually, there exists a nontrivial zero $\rho_0 = \beta_0 + i\gamma_0$ with $\beta_0 > 1/2$, then we have

$$\begin{aligned} \mathrm{Re}(Y_m(s)) &\geq (\beta_0 - \sigma)t^{m-1} + O\left(t^{m-3}\log t\right), \\ \mathrm{Im}(Y_m(s)) &\geq (\beta_0 - \sigma)t^{m-2} + O\left(t^{m-4}\log t\right) \end{aligned} \tag{2.2}$$

for a fixed σ with $1/2 \leq \sigma < \beta_0$.

Now, we state the main theorem in this chapter.

Theorem 2.1. *Let m, d be nonnegative integers with $d \leq D(f)$, and H, X real parameters with $H \geq 1$, $X \geq 3$. Then, for any $\sigma \in \mathbb{R}$, $t \geq 1$, we have*

$$\eta_m(s) = i^m \sum_{2 \leq n \leq X^{1+1/H}} \frac{\Lambda(n) v_{f,H}\left(e^{\log n/\log X}\right)}{n^s (\log n)^{m+1}} + Y_m(s, X) + R_m(s, X, H).$$

Here the error term $R_m(s, X, H)$ satisfies the estimate

$$R_m(s, X, H) \ll_{f,d} \tag{2.3}$$

$$\frac{X^{2(1-\sigma)} + X^{1-\sigma}}{t(\log X)^{m+1}} \min_{0\le l\le d}\left\{\left(\frac{H}{t\log X}\right)^l\right\} + \frac{1}{(\log X)^m} \sum_{|t-\gamma|\le \frac{1}{\log X}} (X^{2(\beta-\sigma)} + X^{\beta-\sigma})$$

$$+ \frac{1}{(\log X)^{m+1}} \sum_{|t-\gamma|> \frac{1}{\log X}} \frac{X^{2(\beta-\sigma)} + X^{\beta-\sigma}}{|t-\gamma|} \min_{0\le l\le d}\left\{\left(\frac{H}{|t-\gamma|\log X}\right)^l\right\}.$$

Moreover, if the Riemann Hypothesis is true, for $1 \le H \le t/2$, $3 \le X \le t$, we have

$$R_m(s, X, H) \ll_f X^{1/2-\sigma} \frac{\log t}{(\log X)^m}\left(\frac{1}{\log\log t} + \frac{\log(H+2)}{\log X}\right). \tag{2.4}$$

The important point of this theorem is that, by $Y_m(s, X)$, we can express explicitly the contribution of certain zeros which have big influence to $\eta_m(s)$. Actually, from this theorem, we can take out the information of singularities coming from such zeros. Thanks to it, we can prove some results for the Riemann zeta-function. For example, the results are related to the following:

1. An equivalence between the order of magnitude of $\eta_m(s)$ and the zero-free region of $\zeta(s)$,
2. A relation between the prime numbers and the distribution of zeros of $\zeta(s)$ under the Riemann Hypothesis,
3. The value distribution of $\log|\zeta(1/2+it)|$,
4. A mean value theorem involving $\eta_m(s)$,
5. The value distribution of $\eta_m(1/2+it)$.

We will state the details and proofs of these results in the following five sections.

Note some remarks on this theorem. First, when $m = 0$, and H is large, for example $H = X$, this formula becomes an assertion close to the hybrid formula of Gonek, Hughes, and Keating [33, Theorem 1]. In fact, this theorem is proved by calculating the contribution of nontrivial zeros which is based on the following proposition.

Proposition 2.1. *Let m be a nonnegative integer. Then, for $\sigma \in \mathbb{R}$, $t > 0$ we have*

$$\eta_m(s) = i^m \sum_{2\le n\le X^{1+1/H}} \frac{\Lambda(n) v_{f,H}(e^{\log n/\log X})}{n^s (\log n)^{m+1}} + \frac{i^m}{(\log X)^m} U_m((s-1)\log X)$$

$$- \frac{i^m}{(\log X)^m}\sum_{\rho} U_m((s-\rho)\log X) - \frac{i^m}{(\log X)^m}\sum_{n=1}^{\infty} U_m((s+2n)\log X)$$

$$+ 2\pi \sum_{k=0}^{m-1} \frac{i^{m-1-k}}{(m-k)!k!} \sum_{\substack{0<\gamma<t \\ \beta>\sigma}} (\beta-\sigma)^{m-k}(t-\gamma)^k.$$

Here if $m = 0$, then we regard that the third term on the right hand side is zero.

This proposition in the case of $H = X$, $m = 0$ becomes the almost same as their hybrid formula. On the other hand, as we can see from Theorem 2.1, it becomes difficult to obtain a good estimate for the contribution of nontrivial zeros and mean value estimates when H is large. From this reason, we introduce the new parameter H which can control the length of "smoothing functions." Although most of discussions and results in the following are obtained by this theorem in the case H is small, the theorem in the case H is large is also useful when we discuss a Dirichlet polynomial without smoothing functions like $\sum_{p\le X} p^{-1/2-it}$. Actually, we will mention an estimate of this Dirichlet polynomial under the Riemann Hypothesis in inequality (2.25) below.

2.1.1 Preliminary lemmas

Lemma 2.1. *Let m be a positive integer, and let $t > 0$. Then, for any $\sigma \ge 1/2$, we have*

$$\eta_m(\sigma + it) = i^m \tilde{\eta}_m(\sigma + it) + 2\pi \sum_{k=0}^{m-1} \frac{i^{m-1-k}}{(m-k)!k!} \sum_{\substack{0<\gamma<t \\ \beta>\sigma}} (\beta - \sigma)^{m-k} (t - \gamma)^k.$$

Proof. In view of our choice of the branch of $\log\zeta(s)$, it suffices to show this lemma in the case t is not the ordinate of zeros of $\zeta(s)$. We show this lemma by induction on m. When $m = 0$, by using Littlewood's lemma (cf. (9.9.1) in [114]), it holds that

$$\begin{aligned} i\int_0^t \log\zeta(\sigma + it')dt' - \int_\sigma^\infty \log\zeta(\alpha)d\alpha \\ = -\int_\sigma^\infty \log\zeta(\alpha + it)d\alpha + 2\pi i \int_\sigma^\infty N(\alpha, t)d\alpha. \end{aligned} \quad (2.5)$$

Here $N(\sigma, t)$ indicates the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function with $\beta \ge \sigma$, $0 < \gamma < t$ counted with multiplicity. We see that

$$\int_\sigma^\infty N(\alpha, t)d\alpha = \int_\sigma^\infty \sum_{\substack{0<\gamma<t \\ \beta>\alpha}} 1 d\alpha = \sum_{\substack{0<\gamma<t \\ \beta>\sigma}} \int_\sigma^\beta d\alpha = \sum_{\substack{0<\gamma<t \\ \beta>\sigma}} (\beta - \sigma).$$

Therefore, by this formula and the definition of $\eta_m(s)$, we have

$$\eta_1(\sigma + it) = i\int_\sigma^\infty \log\zeta(\alpha + it)d\alpha + 2\pi \sum_{\substack{0<\gamma<t \\ \beta>\sigma}} (\beta - \sigma),$$

which is the assertion of this lemma in the case $m = 1$.

Next we show this lemma in the case $m \geq 2$. Assume that the assertion of this lemma is true at $m-1$. Then, we find that

$$\begin{aligned}
&\int_0^t \eta_{m-1}(\sigma+it')dt' \\
&= \int_0^t \frac{i^{m-1}}{(m-2)!}\int_\sigma^\infty (\alpha-\sigma)^{m-2}\log\zeta(\alpha+it')d\alpha dt' \\
&\qquad + 2\pi\sum_{k=0}^{m-2}\frac{i^{m-2-k}}{(m-1-k)!k!}\int_0^t \sum_{\substack{0<\gamma<t' \\ \beta>\sigma}}(\beta-\sigma)^{m-1-k}(t'-\gamma)^k dt' \\
&= \frac{i^{m-1}}{(m-2)!}\int_\sigma^\infty (\alpha-\sigma)^{m-2}\int_0^t \log\zeta(\alpha+it')dt'd\alpha \\
&\qquad + 2\pi\sum_{k=1}^{m-1}\frac{i^{m-1-k}}{(m-k)!k!}\sum_{\substack{0<\gamma<t \\ \beta>\sigma}}(\beta-\sigma)^{m-k}(t-\gamma)^k. \qquad (2.6)
\end{aligned}$$

Note that the exchange of integration of the first term in the second equation is guaranteed by the absolute convergence of the integral. Applying formula (2.5), we find that

$$\begin{aligned}
&\frac{i^{m-1}}{(m-2)!}\int_\sigma^\infty (\alpha-\sigma)^{m-2}\int_0^t \log\zeta(\alpha+it')dt'd\alpha \\
&= \frac{i^m}{(m-1)!}\int_\sigma^\infty (\alpha-\sigma)^{m-1}\log\zeta(\alpha+it)d\alpha - c_m(\sigma) \\
&\qquad + 2\pi\frac{i^{m-1}}{(m-1)!}\int_\sigma^\infty (\alpha-\sigma)^{m-1}N(\alpha,t)d\alpha,
\end{aligned}$$

and that

$$\int_\sigma^\infty (\alpha-\sigma)^{m-1}N(\alpha,t)d\alpha = \sum_{\substack{0<\gamma<t \\ \beta>\sigma}}\int_\sigma^\beta (\alpha-\sigma)^{m-1}d\alpha = \frac{1}{m}\sum_{\substack{0<\gamma<t \\ \beta>\sigma}}(\beta-\sigma)^m.$$

Hence, by these formulas, (2.6), and the definition of $\eta_m(s)$, we obtain

$$\begin{aligned}
\eta_m(\sigma+it) &= \frac{i^m}{(m-1)!}\int_\sigma^\infty (\alpha-\sigma)^{m-1}\log\zeta(\alpha+it)d\alpha \\
&\qquad + 2\pi\sum_{k=0}^{m-1}\frac{i^{m-1-k}}{(m-k)!k!}\sum_{\substack{0<\gamma<t \\ \beta>\sigma}}(\beta-\sigma)^{m-k}(t-\gamma)^k,
\end{aligned}$$

which completes the proof of this lemma. □

Lemma 2.2. *Let m, d be a nonnegative integers with $d \leq D = D(f)$. Let $z = a+ib$ be a complex number with $b \neq 0$. Set $H \geq 1$ be a real parameter. Then we have*

$$U_m(z) \ll_{f,d} \frac{e^{-(1+1/H)a}+e^{-a}}{|b|}\min_{0\leq l\leq d}\left\{\left(\frac{H}{|b|}\right)^l\right\}.$$

Proof. By the definition of $U_m(z)$, we have

$$U_m(z) = \frac{1}{m!}\int_a^\infty \frac{(\alpha-a)^m}{\alpha+ib}\left(\int_0^\infty u_{f,H}(x)e^{-(\alpha+ib)\log x}dx\right)d\alpha. \tag{2.7}$$

Since $u_{f,H}$ belongs to $C^{D-2}([0,\infty))$ and is a $C^D([e,e^{1+1/H}])$-function and supported on $[e,e^{1+1/H}]$, for $0\le d\le D-1$, we see that

$$\int_0^\infty u_{f,H}(x)e^{-(\alpha+ib)\log x}dx = \int_e^{e^{1+1/H}} \frac{u_{f,H}^{(d)}(x)x^{d-(\alpha+ib)}}{\prod_{l=1}^d\{(\alpha+ib)-l\}}dx. \tag{2.8}$$

Here the estimate $u_{f,H}^{(d)}(x)\ll_{f,d} H^{d+1}$ holds on $x\in[e,e^{1+1/H}]$ for $0\le d\le D$. By this estimate and (2.8), we have

$$\int_0^\infty u_{f,H}(x)e^{-(\alpha+ib)\log x}dx \ll_{f,d} (e^{-(1+\frac{1}{H})\alpha}+e^{-\alpha})\min_{0\le l\le d}\left\{\left(\frac{H}{|b|}\right)^l\right\}$$

for $0\le d\le D-1$. Moreover, by (2.8), we find that

$$\begin{aligned}
&\int_0^\infty u_{f,H}(x)e^{-(\alpha+ib)\log x}dx\\
&=\left[\frac{u_{f,H}^{(D-1)}(x)x^{D-(\alpha+ib)}}{\prod_{l=1}^D\{(\alpha+ib)-l\}}\right]_{x=e}^{x=e^{1+1/H}} + \int_e^{e^{1+1/H}}\frac{u_{f,H}^{(D)}(x)x^{D-(\alpha+ib)}}{\prod_{l=1}^D\{(\alpha+ib)-l\}}dx\\
&\ll_{f,D} (e^{-(1+\frac{1}{H})\alpha}+e^{-\alpha})\left(\frac{H}{|b|}\right)^D.
\end{aligned}$$

By these estimates and (2.7), for $0\le d\le D$, we have

$$\begin{aligned}
U_m(z) &\ll_{f,d} \frac{1}{|b|m!}\min_{0\le l\le d}\left\{\left(\frac{H}{|b|}\right)^l\right\}\int_a^\infty(\alpha-a)^m(e^{-\alpha(1+1/H)}+e^{-\alpha})d\alpha\\
&\ll \frac{e^{-(1+1/H)a}+e^{-a}}{|b|}\min_{0\le l\le d}\left\{\left(\frac{H}{|b|}\right)^l\right\},
\end{aligned}$$

which completes the proof of this lemma. □

Lemma 2.3. *Let m be a nonnegative integer, and let $H\ge 1$. Then, for any complex number $z=a+ib$ with $a\in\mathbb{R}$ and $|b|\le 1$, we have*

$$U_m(z) = \begin{cases} -\frac{1}{m!}(-z)^m\log z + O(1) & \text{if } |z|\le 1,\\ O\left(e^{-(1+1/H)a}+e^{-a}\right) & \text{if } |z|>1.\end{cases} \tag{2.9}$$

In particular, for any positive integer m, we have

$$U_m(z) \ll e^{-(1+1/H)a}+e^{-a} \tag{2.10}$$

for any complex number $z=a+ib$ with $|b|\le 1$. Here, the above implicit constants are absolute.

Proof. In view of our definition of $U_m(z)$ and $\log z$, it suffices to show this lemma in the case that b is not equal to zero. First, we consider the case $a > 1$. Then we see that

$$\begin{aligned} U_m(z) &= \frac{1}{m!}\int_0^{\infty} u_{f,H}(x)\int_a^{\infty}(\alpha-a)^m\frac{e^{-(\alpha+ib)\log x}}{\alpha+ib}d\alpha dx \\ &\ll \frac{1}{m!}\int_e^{e^{1+1/H}} u_{f,H}(x)\int_a^{\infty}(\alpha-a)^{m-1}e^{-\alpha}dx \ll e^{-a}. \end{aligned}$$

Next, we consider the case $|a| \le 1$. Then we can write

$$\begin{aligned} U_m(z) = \frac{1}{m!}\int_e^{e^{1+1/H}} u_{f,H}(x)\int_a^{1}(\alpha-a)^m\frac{e^{-(\alpha+ib)\log x}}{\alpha+ib}d\alpha dx \\ + \frac{1}{m!}\int_e^{e^{1+1/H}} u_{f,H}(x)\int_1^{\infty}(\alpha-a)^m\frac{e^{-(\alpha+ib)\log x}}{\alpha+ib}d\alpha dx. \end{aligned}$$

We see that the absolute value of the latter term on the right hand side is

$$\le \frac{1}{m!}\int_e^{e^{1+1/H}} u_{f,H}(x)\int_a^{\infty}(\alpha-a)^m e^{-\alpha\log x}d\alpha dx \ll 1.$$

Next, we consider the former term on the right hand side. By the Taylor expansion, it holds that

$$\begin{aligned} &\int_a^1(\alpha-a)^m\frac{e^{-(\alpha+it)\log x}}{\alpha+ib}d\alpha \\ &= \int_a^1\frac{(\alpha-a)^m}{\alpha+ib}d\alpha + \sum_{n=1}^{\infty}\frac{(-\log x)^n}{n!}\int_a^1(\alpha-a)^m(\alpha+ib)^{n-1}d\alpha. \end{aligned}$$

When $n \ge 1$, we find that

$$\left|\int_a^1(\alpha-a)^m(\alpha+ib)^{n-1}d\alpha\right| \le 2^{m+n},$$

and so

$$\sum_{n=1}^{\infty}\frac{(-\log x)^n}{n!}\int_a^1(\alpha-a)^m(\alpha+ib)^{n-1}d\alpha \ll 2^m.$$

Using the binomial expansion, we also find that

$$\begin{aligned} &\int_a^1\frac{(\alpha-a)^m}{\alpha+ib}d\alpha = \sum_{k=0}^{m}\binom{m}{k}(-a-ib)^{m-k}\int_a^1(\alpha+ib)^{k-1}d\alpha \\ &= (-z)^m\left(\log(1+ib)-\log z\right) + \sum_{k=1}^{m}\binom{m}{k}(-z)^{m-k}\left((1+ib)^{k-1}-z^{k-1}\right) \\ &= -(-z)^m\log(z) + O\left(4^m\right). \end{aligned}$$

Therefore, by the above calculations, when $|a| \leq 1$, we obtain

$$U_m(z) = -\frac{1}{m!}(-z)^m \log z + O\left(1\right).$$

Finally, we consider the case $a < -1$. We can write

$$U_0(z) = \int_e^{e^{1+1/H}} u_{f,H}(x) \int_a^{-1} \frac{e^{-(\alpha+ib)\log x}}{\alpha+ib} d\alpha + U_0(-1+ib).$$

Using the result of the previous case, we have $U_0(-1+ib) = -(-1+ib)\log(-1+ib) + O(1) = O(1)$. Also, we can easily see that the first term is $\ll e^{-(1+1/H)a} + e^{-a}$. Hence, we have

$$U_0(z) \ll e^{-(1+1/H)a} + e^{-a}$$

for $a < -1$. When $m \in \mathbb{Z}_{\geq 1}$, it holds that

$$\begin{aligned} U_m(z) &= \frac{1}{m!}\int_e^{e^{1+1/H}} u_{f,H}(x) \int_a^{\infty} (\alpha - a)^m \frac{e^{-(\alpha+ib)\log x}}{\alpha+ib} d\alpha \\ &= \frac{1}{(m-1)!}\int_a^{\infty} (\alpha-a)^{m-1} U_0(\alpha+ib) d\alpha \end{aligned}$$

by integration by parts and Fubini's theorem. Applying the estimate of U_0, we find that

$$\begin{aligned} \frac{1}{(m-1)!}\int_1^{\infty} (\alpha-a)^{m-1} U_0(\alpha+ib) d\alpha &\ll 1, \\ \frac{1}{(m-1)!}\int_a^{-1} (\alpha-a)^{m-1} U_0(\alpha+ib) d\alpha &\ll e^{-(1+1/H)a} + e^{-a}, \end{aligned}$$

and that

$$\begin{aligned} &\frac{1}{(m-1)!}\int_{-1}^{1} (\alpha-a)^{m-1} U_0(\alpha+ib) d\alpha \\ &= -\frac{1}{(m-1)!}\int_{-1}^{1} (\alpha-a)^{m-1} \left(\log(\alpha+ib) + O(1)\right) d\alpha \ll \frac{(|a|+1)^{m-1}}{(m-1)!} \leq e^{-a+1}. \end{aligned}$$

Therefore, we have

$$U_m(z) \ll e^{-(1+1/H)a} + e^{-a},$$

and this implicit constant is absolute.

From the above calculations, we obtain

$$U_m(z) = \begin{cases} -(-z)^m \log z + O\left(1\right) & \text{if} \quad |a| \leq 1, \\ O\left(e^{-(1+1/H)a} + e^{-a}\right) & \text{if} \quad |a| > 1. \end{cases}$$

Now, from the condition $|b| \leq 1$, the formula where $|a|$ is replaced by $|z|$ also holds. Hence, we complete the proof of the estimate (2.9).

Moreover, we can obtain the estimate (2.10) from (2.9) since, for $m \in \mathbb{Z}_{\geq 1}$, the inequality $\frac{1}{m!}(-z)^m \log z \ll 1$ holds for $|z| \leq 1$. Thus, we obtain this lemma. □

Proof of Proposition 2.1. In view of our definition of $U_m(z)$ and $\log\zeta(s)$, it suffices to show this lemma in the case that t is not equal to the ordinate of zeros of $\zeta(s)$. First, we prove this proposition in the case $m = 0$. The proof is the almost same as the proof of Theorem 1 in [33] (see also the proof of Lemma 1 in [9], if necessary). Hence, we only write the rough proof in this case. Let $\tilde{u}(s)$ be the Mellin transform of $u_{f,H}$, that is, $\tilde{u}(s) := \int_0^\infty u_{f,H}(x)x^{s-1}dx$. Since the functions $v_{f,H}(x)$ and $\tilde{u}(s+1)/s$ are Mellin transforms, we find that, for any complex number z,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z}v_{f,H}\left(e^{\log n/\log X}\right)\\
&=\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z}\int_{c-i\infty}^{c+i\infty}\frac{\tilde{u}(w+1)}{w}n^{-w/\log X}dw\\
&=-\frac{1}{2\pi i}\int_{(2-\mathrm{Re}(z))\log X-i\infty}^{(2-\mathrm{Re}(z))\log X+i\infty}\frac{\zeta'}{\zeta}\left(z+\frac{w}{\log X}\right)\frac{\tilde{u}(w+1)}{w}dw.
\end{aligned}$$

By this formula, for $\mathrm{Im}(z) > 0$, we have

$$\begin{aligned}
&\sum_{n\le X^{1+1/H}}\frac{\Lambda(n)}{n^z}v_{f,H}\left(e^{\log n/\log X}\right)\\
&=-\frac{\zeta'}{\zeta}(z)+\frac{1}{1-z}\tilde{u}(1+(1-z)\log X)-\sum_{\rho}\frac{1}{\rho-z}\tilde{u}(1+(\rho-z)\log X)-\\
&\qquad\qquad -\sum_{n=1}^{\infty}\frac{1}{2n-z}\tilde{u}(1+(2n-z)\log X).
\end{aligned}$$

Integrating both sides with respect to z from $\infty+it$ to $\sigma+it\ (=s)$, we obtain

$$\begin{aligned}
&\log\zeta(s) \qquad (2.11)\\
&=\sum_{2\le n\le X^{1+1/H}}\frac{\Lambda(n)}{n^s\log n}v_{f,H}\left(e^{\log n/\log X}\right)\\
&\quad+U_0((s-1)\log X)-\sum_{\rho}U_0((s-\rho)\log X)-\sum_{n=1}^{\infty}U_0((s-2n)\log X).
\end{aligned}$$

Therefore, this theorem holds in the case $m = 0$.

Next we show this proposition for $m \ge 1$. By Lemma 2.1, it suffices to show that

$$\begin{aligned}
&\frac{i^m}{(m-1)!}\int_{\sigma}^{\infty}(\alpha-\sigma)^{m-1}\log\zeta(\alpha+it)d\alpha= \qquad (2.12)\\
&i^m\sum_{2\le n\le X^{1+1/H}}\frac{\Lambda(n)v_{f,H}(e^{\log n/\log X})}{n^s(\log n)^{m+1}}+\frac{i^m}{(\log X)^m}U_m((s-1)\log X)\\
&-\frac{i^m}{(\log X)^m}\sum_{\rho}U_m((s-\rho)\log X)-\frac{i^m}{(\log X)^m}\sum_{n=1}^{\infty}U_m((s-2n)\log X).
\end{aligned}$$

Here, by using formula (2.11), the left hand side on the above equation is

$$
\begin{aligned}
=&i^m \sum_{2\le n\le X^{1+1/H}} \frac{\Lambda(n)v_{f,H}(e^{\log n/\log X})}{n^s(\log n)^{m+1}} \\
&+ \frac{i^m}{(m-1)!}\int_\sigma^\infty (\alpha-\sigma)^{m-1}U_0((\alpha+it-1)\log X)d\alpha \\
&- \frac{i^m}{(m-1)!}\int_\sigma^\infty \sum_\rho (\alpha-\sigma)^{m-1}U_0((\alpha+it-\rho)\log X)d\alpha \\
&- \frac{i^m}{(m-1)!}\int_\sigma^\infty \sum_{n=1}^\infty (\alpha-\sigma)^{m-1}U_0((\alpha+it-2n)\log X)d\alpha.
\end{aligned}
\tag{2.13}
$$

In the following, we will change the above sum and integral, and it is guaranteed by

$$
\sum_\rho \int_\sigma^\infty |(\alpha-\sigma)^{m-1}U_0((\alpha+it-\rho)\log X)|d\alpha < +\infty.
$$

This convergence can be obtained by Lemma 2.2. Further, simple calculations show that, for any $w \in \mathbb{C}$,

$$
\begin{aligned}
&\frac{i^m}{(m-1)!}\int_\sigma^\infty (\alpha-\sigma)^{m-1}U_0((\alpha+it-w)\log X)d\alpha \\
&= \frac{i^m}{(\log X)^m}U_m((s-w)\log X).
\end{aligned}
\tag{2.14}
$$

Hence, by (2.13), (2.14), we obtain formula (2.12), and this completes the proof of this proposition. □

2.1.2 Proof of the approximate formula

Proof of Theorem 2.1. We can immediately obtain estimate (2.3) by Proposition 2.1, Lemma 2.2, and Lemma 2.3. Now we prove estimate (2.4) under the Riemann Hypothesis. It suffices to show

$$
\sum_{\frac{1}{\log X}<|t-\gamma|\le\frac{H}{\log X}} \frac{1}{|t-\gamma|} \ll \log t\left(\frac{\log X}{\log\log t} + \log H\right), \tag{2.15}
$$

and

$$
\sum_{|t-\gamma|>\frac{H}{\log X}} \frac{H}{(t-\gamma)^2\log X} \ll \log t \times \left(\frac{\log X}{H\log\log t} + 1\right) \tag{2.16}
$$

under the Riemann Hypothesis. Assuming the Riemann Hypothesis, the following estimate (cf. Lemma 13.19 in [87])

$$
\tilde{N}\left(t, \frac{1}{\log\log t}\right) \ll \frac{\log t}{\log\log t} \tag{2.17}
$$

holds for $t \geq 5$. By this estimate, for any $1 \leq H \leq \frac{t}{2}$, we find that

$$\sum_{\frac{1}{\log X} < |t-\gamma| \leq \frac{H}{\log X}} \frac{1}{|t-\gamma|} \leq \sum_{k=0}^{[(H-1)\frac{\log\log t}{\log X}]} \sum_{\frac{1}{\log X} + \frac{k}{\log\log t} < |t-\gamma| \leq \frac{1}{\log X} + \frac{k+1}{\log\log t}} \frac{1}{|t-\gamma|}$$
$$\ll \log t \sum_{k=0}^{[(H-1)\frac{\log\log t}{\log X}]} \frac{1}{\frac{\log\log t}{\log X} + k} \leq \log t \left(\frac{\log X}{\log\log t} + \int_0^{(H-1)\frac{\log\log t}{\log X}} \frac{du}{\frac{\log\log t}{\log X} + u} \right)$$
$$= \log t \left(\frac{\log X}{\log\log t} + \log H \right),$$

and that

$$\sum_{|t-\gamma| > \frac{H}{\log X}} \frac{H}{(t-\gamma)^2 \log X} = \sum_{\frac{H}{\log X} < |t-\gamma| \leq \frac{t}{2}} \frac{H}{(t-\gamma)^2 \log X} + O\left(\frac{H}{t \log X} \right)$$
$$\leq \sum_{k=0}^{[\frac{t \log\log t}{2}]} \sum_{\frac{H}{\log X} + \frac{k}{\log\log t} < |t-\gamma| \leq \frac{H}{\log X} + \frac{k+1}{\log\log t}} \frac{H}{(t-\gamma)^2 \log X} + O\left(\frac{H}{t \log X} \right)$$
$$\ll H \log\log t \frac{\log t}{\log X} \sum_{k=0}^{[\frac{t \log\log t}{2}]} \frac{1}{\left(k + \frac{H \log\log t}{\log X}\right)^2} + \frac{H}{t \log X}$$
$$\leq H \log\log t \frac{\log t}{\log X} \left(\left(\frac{\log X}{H \log\log t} \right)^2 + \int_0^\infty \frac{du}{\left(u + \frac{H \log\log t}{\log X}\right)^2} \right) + \frac{H}{t \log X}$$
$$\ll \log t \left(\frac{\log X}{H \log\log t} + 1 \right).$$

Hence, we obtain estimates (2.15), (2.16). □

By Theorem 2.1 and Lemma 2.1, we also obtain an approximate formula for $\tilde{\eta}_m(s)$.

Theorem 2.2. *Let $m \in \mathbb{Z}_{\geq 1}$, and let d be a nonnegative integer with $d \leq D(f)$. Let H, X real parameters with $H \geq 1$, $X \geq 3$. Then, for any $\sigma \in \mathbb{R}$, $t \geq 1$, we have*

$$\tilde{\eta}_m(s) = \sum_{2 \leq n \leq X^{1+1/H}} \frac{\Lambda(n) v_{f,H}\left(e^{\log n/\log X}\right)}{n^s (\log n)^{m+1}} + E_m(s, X, H).$$

Here the error term $E_m(s, X, H)$ satisfies estimates (2.3) *and* (2.4) *under the same conditions as in Theorem 2.1.*

2.2 An equivalence with the zero-free region of $\zeta(s)$

As the first application of the approximate formula, we state a consequence which gives an equivalent condition to the zero-free region of $\zeta(s)$.

Corollary 2.1. *Let $\sigma \geq 1/2$. Then the following three statements* (A), (B), (C) *are equivalent.*

(A). The Riemann zeta-function does not have zeros whose real part are greater than σ.

(B). For a fixed integer $m \geq 2$, the estimate

$$\operatorname{Re} \eta_m(\sigma + iT) = o\left(T^{m-1}\right)$$

holds as $T \to +\infty$.

(C). For a fixed integer $m \geq 3$, the estimate

$$\operatorname{Im} \eta_m(\sigma + iT) = o\left(T^{m-2}\right)$$

holds as $T \to +\infty$.

In particular, for a fixed integer $m \geq 2$, the Riemann Hypothesis is equivalent to that the estimate

$$\eta_m(1/2 + iT) = o\left(T^{m-1}\right)$$

holds as $T \to +\infty$.

This corollary is easily obtained from Theorem 2.1. Actually, we can show it by the following little discussion.

Applying Theorem 2.1 as $X = 3$, $H = 1$, for any positive integer m, we can obtain the formula

$$\eta_m(s) = Y_m(s) + O_m\left(\sum_{\rho} \frac{1}{1 + (t - \gamma)^2}\right).$$

Now, by the well known estimate (cf. p.98 [21])

$$\sum_{\rho} \frac{1}{1 + (t - \gamma)^2} \ll \log t, \tag{2.18}$$

the above O-term is $\ll_m \log t$. Hence, we obtain

$$\eta_m(s) = Y_m(s) + O_m(\log t). \tag{2.19}$$

Thus, from estimates (2.2) and (2.19), we obtain Corollary 2.1.

Fujii [29] established an equivalence for the Riemann Hypothesis and an estimate for $S_m(t)$. He discussed only the behavior of the Riemann zeta-function on the critical line, and this corollary means that his equivalence can be generalized to the critical strip naturally. Moreover, Fujii's result is an equivalence for $S_m(t)$ in the case $m \geq 3$. On the other hand, thanks to the consideration on the real part of iterated integrals of the logarithm of the Riemann zeta-function, we also have the same type of equivalence for $m = 2$.

2.3 A Dirichlet polynomial involving prime numbers and zeros of $\zeta(s)$

In this section, we state some consequences of Theorem 2.1 for a relationship between prime numbers and the distribution of nontrivial zeros of $\zeta(s)$ in short intervals. These consequences are obtained from a principle of taking out the information of singularities coming from certain zeros by using Theorem 2.1.

We define the weighted Dirichlet polynomial $P_f(s, X)$ by

$$P_f(s,X) = \sum_{p \le X^2} \frac{v_{f,1}(e^{\log p/\log X})}{p^s}$$

for $X \ge 3$. Here, the sum runs over prime numbers. Moreover, the function $\tilde{N}(t,h)$ means the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $|t-\gamma| \le h$ counted with multiplicity. Then we can obtain the following theorem.

Theorem 2.3. *Assume the Riemann Hypothesis. Let f be a nonnegative mass one $C^1([0,1])$-function supported in $[0,1]$. Then, for $t \ge 14$, $\log t \le X \le t$, we have*

$$P_f(1/2+it,X) = \log\left(\frac{\log\log t}{\log X}\right) \times \tilde{N}\left(t, \frac{1}{\log X}\right) + \\ + \sum_{\frac{1}{\log X} < |t-\gamma| \le \frac{1}{\log\log t}} \log\left(|t-\gamma|\log\log t\right) + O_f\left(\frac{\log t}{\log\log t}\right). \quad (2.20)$$

In particular, we have

$$\max_{3 \le X \le t} \mathrm{Re}\left(P_f(1/2+it,X)\right) \ll_f \frac{\log t}{\log\log t}, \quad (2.21)$$

$$\max_{3 \le X \le t} \mathrm{Re}\left(-P_f(1/2+it,X)\right) \ll_f \log t, \quad (2.22)$$

and

$$\max_{3 \le X \le t} \left|\mathrm{Im}\left(P_f(1/2+it,X)\right)\right| \ll_f \frac{\log t}{\log\log t}. \quad (2.23)$$

Here we focus on estimates (2.21), (2.23). From these estimates, we would expect that it is possible to improve estimate (2.22) at $\log t/\log\log t$. This expectation is coming from the following discussion. By the randomness of the prime numbers, it is probably true that the numbers $\{t\log p_1\}, \dots, \{t \log p_n\}$ are randomly distributed on $[0,1)$ for $t \ge 1$. Here, $\{x\}$ means the fractional part of x. Hence, the author believes that there is not a big difference among the bounds of the real and imaginary parts of a weighted Dirichlet polynomial like $P_f(s,X)$ and their positive and negative parts. From this observation, the author suggests the following conjecture.

Conjecture 2.1. *Let σ be a real number, and f be a nonnegative mass one $C^1([0,1])$-function supported in $[0,1]$. For sufficiently large $T > 0$,*

$$\max_{14\le t\le T}\max_{3\le X\le t}\mathrm{Re}(P_f(\sigma+it,X)) \asymp \max_{14\le t\le T}\max_{3\le X\le t}\mathrm{Re}(-P_f(\sigma+it,X)),$$
$$\max_{14\le t\le T}\max_{3\le X\le t}\mathrm{Re}(P_f(\sigma+it,X)) \asymp \max_{14\le t\le T}\max_{3\le X\le t}\mathrm{Im}(P_f(\sigma+it,X)),$$

and

$$\max_{14\le t\le T}\max_{3\le X\le t}\mathrm{Im}(P_f(\sigma+it,X)) \asymp \max_{14\le t\le T}\max_{3\le X\le t}\mathrm{Im}(-P_f(\sigma+it,X)).$$

If this conjecture and the Riemann Hypothesis are true, for every certain f, we obtain

$$\max_{3\le X\le t}|P_f(1/2+it,X)| \ll \frac{\log t}{\log\log t} \tag{2.24}$$

from estimates (2.21), (2.23).

Estimate (2.24) can be applied to the distribution of the ordinate of zeros of $\zeta(s)$. If estimate (2.24) and the Riemann Hypothesis are true, by using formula (2.20) as $X = (\log t)^D$, we can obtain the following interesting estimate

$$\tilde{N}\left(t, \frac{1}{D\log\log t}\right) \ll \frac{\log t}{\log D\log\log t}$$

for any $2 \le D \le \log t/\log\log t$. In particular, on the same condition, we can improve the estimate of the multiplicity of zeros of the Riemann zeta-function like the following

$$m(\rho) \ll \frac{\log|\gamma|}{(\log\log|\gamma|)^2},$$

where $m(\rho)$ means the multiplicity of a zero $\rho = \frac{1}{2} + i\gamma$. This upper bound is sharp because the following inequality (see Corollary 1 in [31])

$$m(\rho) \le \left(\frac{1}{2} + o(1)\right)\frac{\log|\gamma|}{\log\log|\gamma|}$$

is the best known upper bound under the Riemann Hypothesis at present. From this observation, the author suggests Conjecture 2.1 as an important open problem.

Furthermore, we will find a deeper fact from the same method as the above discussion. We consider the following estimate

$$\max_{3\le X\le Y(t)}\left|\sum_{p\le X}\frac{1}{p^{1/2+it}}\right| \le M(t), \tag{2.25}$$

where $Y(t)$, $M(t)$ are some monotonically increasing functions with $3 \le Y(t) \le t$, $M(t) \ll \sqrt{Y(t)}/\log Y(t)$. Note that an estimate of Dirichlet polynomial without a mollifier is useful because by partial summation and assuming estimate (2.25), for any certain f, we have $P_f(1/2+it) \ll M(t)$ for $3 \le X \le \sqrt{Y(t)}$. This fact plays an important role in the following discussion in this section.

From the discussion in [24, Section 2.2], we may expect that estimate (2.25) is true with $Y(t) = t$, $M(t) \asymp \sqrt{\log t \log\log t}$. Here, we can obtain some bounds of $Y(t)$ and $M(t)$ under the Riemann Hypothesis. Assuming the Riemann Hypothesis, by using estimate (2.4) as $H = X$, we can show that estimate (2.25) is true when $Y(t) = t$, $M(t) = \log t$. Moreover, we can also show the inequality $M(t) \gg \sqrt{\log t \log\log\log t/\log\log t}$ when the inequality $Y(t) \ge \exp\left(L\sqrt{\log t \log\log t/\log\log\log t}\right)$ holds with L sufficiently large constant. This fact can be shown, for example, by the work of Bondarenko and Seip [11, Theorem 2] and Selberg's formula [107, Theorem 1].

Now, if estimate (2.25) and the Riemann Hypothesis are true, then we can obtain the following theorem.

Theorem 2.4. *Assume the Riemann Hypothesis and estimate* (2.25)*. Let $\psi(t)$ be a function with $3 \le \psi(t) \le \sqrt{Y(t)}$. Let f be a nonnegative mass one $C^1([0,1])$-function supported on $[0,1]$. Then, for $t \ge 14$, $\psi(t) \le X \le t$, we have*

$$P_f(1/2+it, X) = \log\left(\frac{\log\psi(t)}{\log X}\right) \times \tilde{N}\left(t, \frac{1}{\log X}\right) + \\ + \sum_{\frac{1}{\log X} < |t-\gamma| \le \frac{1}{\log\psi(t)}} \log\left(|t-\gamma|\log\psi(t)\right) + O_f\left(M(t) + \frac{\log t}{\log\psi(t)} + \log\log X\right).$$

In particular, if the Riemann Hypothesis and estimate (2.25) *with $Y(t) = t$, $M(t) \asymp \sqrt{\log t \log\log t}$ are true, then by taking $\psi(t) = \exp\left(\sqrt{\frac{\log t}{\log\log t}}\right)$, $X = \exp\left(D\sqrt{\frac{\log t}{\log\log t}}\right)$, we have*

$$\tilde{N}\left(t, \frac{\sqrt{\log\log t}}{D\sqrt{\log t}}\right) \ll \frac{\sqrt{\log t \log\log t}}{\log D} \tag{2.26}$$

for $3 \le D \le \frac{1}{2}\sqrt{\log t \log\log t}$.

By estimate (2.26), assuming the Riemann Hypothesis and estimate (2.25) with $Y(t) = t$, $M(t) \asymp \sqrt{\log t \log\log t}$, we have

$$m(\rho) \ll \sqrt{\frac{\log|\gamma|}{\log\log|\gamma|}}. \tag{2.27}$$

Here, we should mention that, under the same condition, the estimate $m(\rho) \ll \sqrt{\log|\gamma| \log\log|\gamma|}$ immediately follows from Selberg's formula [106, Theorem 1] and the Riemann-von Mangldt formula (1.4), and inequality (2.27) is an improvement of this estimate. Hence, from this observation, we may expect that there is an interesting relationship between the behavior of $\sum_{p \le X} p^{-1/2-it}$ and the distribution of zeros of the Riemann zeta-function.

2.3.1 Preliminary lemmas and the proofs of theorems

We prove Theorems 2.3, 2.4. First, we prepare a standard lemma to prove Theorem 2.3.

Lemma 2.4. *Assume the Riemann Hypothesis. Then, for* $t \ge 14$, $\frac{1}{2} \le \sigma \le \frac{1}{2} + \frac{1}{\log\log t}$,

$$\frac{\zeta'}{\zeta}(s) = \sum_{|t-\gamma| \le 1/\log\log t} \frac{1}{s-\rho} + O\left(\log t\right). \tag{2.28}$$

Proof. This lemma is Lemma 13.20 in [87]. □

Proof of Theorem 2.3. Let $t \ge 14$ and X be a real parameter with $\log t \le X \le t$. By using Theorem 2.1, we have

$$P_f(1/2+it, X) = \log\zeta(1/2+it) - \sum_{|t-\gamma| \le \frac{1}{\log X}} \log\left(|t-\gamma|\log X\right) + O_f\left(\frac{\log t}{\log\log t}\right).$$

By integrating the both sides of (2.28), we obtain

$$\log\zeta\left(\frac{1}{2}+it\right) - \log\zeta\left(\frac{1}{2} + \frac{1}{\log\log t} + it\right)$$
$$= \sum_{|t-\gamma| \le \frac{1}{\log\log t}} \log\left(|t-\gamma|\log\log t\right) + O\left(\frac{\log t}{\log\log t}\right),$$

and by using estimate (13.44) in [87], we obtain

$$\log\zeta\left(\frac{1}{2} + \frac{1}{\log\log t} + it\right) \ll \frac{\log t}{\log\log t}.$$

Hence, we obtain

$$P_f(1/2+it, X) =$$
$$\sum_{|t-\gamma| \le \frac{1}{\log\log t}} \log\left(|t-\gamma|\log\log t\right) - \sum_{|t-\gamma| \le \frac{1}{\log X}} \log\left(|t-\gamma|\log X\right) + O_f\left(\frac{\log t}{\log\log t}\right)$$
$$= \log\left(\frac{\log\log t}{\log X}\right) \times \sum_{|t-\gamma| \le \frac{1}{\log X}} 1 + \sum_{\frac{1}{\log X} < |t-\gamma| \le \frac{1}{\log\log t}} \log\left(|t-\gamma|\log\log t\right)$$
$$+ O_f\left(\frac{\log t}{\log\log t}\right).$$

Thus, we obtain formula (2.20). In particular, estimates (2.21), (2.22), (2.23) are easily obtained by formula (2.20) and estimate (2.17). □

Next, we prepare three lemmas to prove Theorem 2.4. The method of the proofs of these lemmas are probably standard, and so those proofs are briefly.

Lemma 2.5. *Assume the Riemann Hypothesis and* (2.25). *Let* $\psi(t)$ *be a monotonic function with* $3 \le \psi(t) \le \sqrt{Y(t)}$. *Then we have*

$$\tilde{N}\left(t, \frac{1}{\log\psi(t)}\right) \ll M(t) + \frac{\log t}{\log\psi(t)}$$

Proof. For $\sigma \ge \sigma_X := \frac{1}{2} + \frac{1}{\log X}$, by using the following formula (cf. (2.3) in [106])

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n \le X^2} \frac{\Lambda'_X(n)}{n^s} + O\left(X^{1/2-\sigma}\left(\left|\sum_{n \le X^2} \frac{\Lambda'_X(n)}{n^{\sigma_X+it}}\right| + \log t\right)\right), \tag{2.29}$$

we have

$$\frac{\zeta'}{\zeta}(\sigma_X + it) \ll \left|\sum_{n \le X^2} \frac{\Lambda'_X(n)}{n^{\sigma_X+it}}\right| + \log t. \tag{2.30}$$

Here, the function $\Lambda'_X(n)$ is defined by

$$\Lambda'_X(n) = \begin{cases} \Lambda(n) & \text{if } 1 \le n \le X, \\ \Lambda(n)\log(X^2/n)/\log X & \text{if } X \le n \le X^2, \\ 0 & \text{otherwise.} \end{cases}$$

By assuming estimate (2.25) and using partial summation, the right hand side of (2.30) is

$$\ll M(t)\log X + \log t$$

for $X^2 \le Y(t)$. On the other hand, by the following formula

$$\mathrm{Re}\left(\frac{\zeta'}{\zeta}(\sigma + it)\right) = \sum_{|t-\gamma| \le 1} \frac{\sigma - 1/2}{(\sigma - 1/2)^2 + (t-\gamma)^2} + O(\log t),$$

we have

$$\sum_{|t-\gamma| \le 1} \frac{1/\log X}{(1/\log X)^2 + (t-\gamma)^2} \ll M(t)\log X + \log t.$$

Therefore, we have

$$\sum_{|t-\gamma| \le 1/\log X} 1 \ll M(t) + \frac{\log t}{\log X}$$

for $X \le \sqrt{Y(t)}$. Hence by putting $X = \psi(t)$, we obtain this lemma. □

Lemma 2.6. *Assume the Riemann Hypothesis and estimate* (2.25)*. Let* $\psi(t)$ *be a monotonic function with* $3 \leq \psi(t) \leq \sqrt{Y(t)}$*. Then we have*

$$\log \zeta\left(\frac{1}{2} + \frac{1}{\log \psi(t)} + it\right) \ll M(t) + \frac{\log t}{\log \psi(t)}.$$

Proof. By the formula (2.29), we see that

$$\log \zeta\,(\sigma_X + it) = \sum_{2 \leq n \leq X^2} \frac{\Lambda'_X(n)}{n^{\sigma_X + it} \log n} + O\left(\frac{1}{\log X}\left(\left|\sum_{n \leq X^2} \frac{\Lambda'_X(n)}{n^{\sigma_X + it}}\right| + \log t\right)\right).$$

By using the partial summation, the above right hand side is

$$\ll M(t) + \frac{\log t}{\log X}$$

for $X \leq \sqrt{Y(t)}$. Hence by putting $X = \psi(t)$, we obtain this lemma. □

Lemma 2.7. *Assume the Riemann Hypothesis and estimate* (2.25)*. Let* $\psi(t)$ *be a monotonic function with* $3 \leq \psi(t) \leq \sqrt{Y(t)}$*. Then, for* $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{1}{\log \psi(t)}$*,* $t \geq 14$*, we have*

$$\frac{\zeta'}{\zeta}(s) = \sum_{|t-\gamma| \leq \frac{1}{\log \psi(t)}} \frac{1}{s - \rho} + O(M(t) \log \psi(t) + \log t). \tag{2.31}$$

Proof. We can obtain this lemma by using Lemma 2.5 and the same method as in the proof of Lemma 13.20 in [87]. □

Proof of Theorem 2.4. Let $\psi(t) \leq X \leq t$. Using (2.18), Lemma 2.5, and Lemma 2.7, we can find that

$$\sum_{|t-\gamma| > \frac{1}{\log \psi(t)}} \frac{1}{(t-\gamma)^2} \ll \log \psi(t)(M(t) \log \psi(t) + \log t).$$

Therefore, by using this estimate and Theorem 2.1, we have

$$\begin{aligned} &\sum_{2 \leq n \leq X^2} \frac{\Lambda(n) v_{f,1}(e^{\log n / \log X})}{n^{1/2+it} \log n} \\ &= \log \zeta\left(\frac{1}{2} + it\right) - \sum_{|t-\gamma| \leq \frac{1}{\log X}} \log(|t-\gamma| \log X) + O\left(M(t) + \frac{\log t}{\log \psi(t)}\right). \end{aligned} \tag{2.32}$$

On the other hand, by integrating the both sides of (2.31), we find that

$$\begin{aligned} &\log \zeta\left(\frac{1}{2} + it\right) - \log \zeta\left(\frac{1}{2} + \frac{1}{\log \psi(t)} + it\right) \\ &\qquad = \sum_{|t-\gamma| \leq \frac{2}{\log Y(t)}} \log\left(\frac{i(t-\gamma)}{\frac{1}{\log \psi(t)} + i(t-\gamma)}\right) + O\left(M(t) + \frac{\log t}{\log \psi(t)}\right). \end{aligned}$$

Hence, using Lemma 2.5 and Lemma 2.6, we have

$$\log \zeta\left(\frac{1}{2}+it\right) = \sum_{|t-\gamma| \le \frac{1}{\log\psi(t)}} \log\left(|t-\gamma|\log\psi(t)\right) + O\left(M(t) + \frac{\log t}{\log\psi(t)}\right).$$

By this formula, the right hand side of (2.32) is equal to

$$\log\left(\frac{\log\psi(t)}{\log X}\right) \times \tilde{N}\left(t, \frac{1}{\log X}\right) + \sum_{\frac{1}{\log X} < |t-\gamma| \le \frac{1}{\log\psi(t)}} \log\left(|t-\gamma|\log\psi(t)\right)$$
$$+ O\left(M(t) + \frac{\log t}{\log\psi(t)}\right).$$

On the other hand, we see that the left hand side of (2.32) is $= P_f(1/2+it) + O(\log\log X)$, which completes the proof of Theorem 2.4. □

2.4 On the value distribution of $\log|\zeta(1/2+it)|$

In this section, we consider the value distribution of the Riemann zeta-function. Now, we define the set $\mathcal{S}(T,V)$ by

$$\mathcal{S}(T,V) = \left\{t \in [T,2T] \mid \log|\zeta(1/2+it)| > V\right\}.$$

Here, we give a result on the value distribution of $\log|\zeta(1/2+it)|$. There are interesting studies on this theme by Soundararajan [110], [111]. He showed a lower bound and an upper bound of the Lebesgue measure of $\mathcal{S}(T,V)$, and his result for the upper bound is under the Riemann Hypothesis. In [111], he mentioned the question that, in how large range of V, the following estimate

$$\frac{1}{T}\,\mathrm{meas}(\mathcal{S}\,(T,V)) \ll \frac{\sqrt{\log\log T}}{V}\exp\left(-\frac{V^2}{\log\log T}\right) \tag{2.33}$$

holds. Here, the symbol meas$(\cdot)$ stands for the Lebesgue measure. This problem is important because there are some interesting consequences such as the mean value estimate and the Lindelöf Hypothesis. Actually, if estimate (2.33) holds for any large range of V, we can obtain the conjectural estimates

$$\max_{t\in[T,2T]} \log|\zeta(1/2+it)| \ll \sqrt{\log T \log\log T},$$
$$\int_T^{2T} |\zeta(1/2+it)|^{2k} dt \ll T(\log T)^{k^2}.$$

Here, we should mention Jutila's work [56]. He showed unconditionally that the estimate

$$\frac{1}{T}\,\mathrm{meas}(\mathcal{S}(T,V)) \ll \exp\left(-\frac{V^2}{\log\log T}\left(1+O\left(\frac{V}{\log\log T}\right)\right)\right)$$

holds for $0 \le V \le \log\log T$. In particular, as an immediate consequence of this estimate, we have

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T,V)) \ll \exp\left(-\frac{V^2}{\log\log T}\right) \tag{2.34}$$

for $0 \le V \ll (\log\log T)^{2/3}$. This estimate does not slightly reach to estimate (2.33). On the other hand, this estimate was improved by Radziwiłł [95] in the shorter range $V = o\left((\log\log T)^{3/5-\varepsilon}\right)$. In fact, he showed that the following conjecture is true for $V = o\left((\log\log T)^{1/10-\varepsilon}\right)$.

Conjecture (Radziwiłł, [95])**.** *For $V = o\left(\sqrt{\log\log T}\right)$, as $T \to +\infty$*

$$\frac{1}{T}\operatorname{meas}\left(\mathcal{S}\left(T, V\sqrt{\frac{1}{2}\log\log T}\right)\right) \sim \int_V^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.$$

Hence, by his study, estimate (2.33) have been proved for $\sqrt{\log\log T} \ll V = o\left((\log\log T)^{3/5-\varepsilon}\right)$. In this paper, we will extend unconditionally this range for V to $\sqrt{\log\log T} \ll V \ll (\log\log T)^{2/3}$. Moreover, we will also show that the upper bound of Radziwiłł's conjecture is true for $V = o\left((\log\log T)^{1/6}\right)$.

Theorem 2.5. *For $1 \ll V \ll (\log\log T)^{1/6}$, we have*

$$\begin{aligned}\frac{1}{T}&\operatorname{meas}\left(\mathcal{S}\left(T, V\sqrt{\frac{1}{2}\log\log T}\right)\right)\\ &\le (1+o(1))\int_V^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}} + O\left(\frac{V}{(\log\log T)^{1/3}}\exp\left(-\frac{V^2}{2}\right)\right)\end{aligned}$$

as $T \to +\infty$. In particular, for $1 \ll V = o\left((\log\log T)^{1/6}\right)$, we have

$$\frac{1}{T}\operatorname{meas}\left(\mathcal{S}\left(T, V\sqrt{\frac{1}{2}\log\log T}\right)\right) \le (1+o(1))\int_V^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}} \tag{2.35}$$

as $T \to +\infty$, and for any large T, we have

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T,V)) \ll \frac{\sqrt{\log\log T}}{V}\exp\left(-\frac{V^2}{\log\log T}\right) \tag{2.36}$$

for $\sqrt{\log\log T} \ll V \ll (\log\log T)^{2/3}$.

Estimate (2.36) is an improvement of estimate (2.34), and it is expected from Radziwiłł's conjecture that the estimate is best possible.

This theorem will be shown by using a method of Selberg-Tsang [116] and Radziwiłł's method [95]. On the other hand, it would be difficult to prove Theorem 2.5 by using their method only. Actually, the author could not derive this theorem by a method using Lemma 5.4 in [116] which plays an important role in their method. The reason why the author could not derive this theorem by such a method is that the contribution of zeros close to s cannot be well managed. On the other hand, we can ignore the contribution of such zeros by using Theorem 2.1 while considering the upper bound of meas $\mathcal{S}(T, V)$. In fact, the important point in the proof of Theorem 2.5 is that the real part of $Y_0(s, X)$ is always non-positive.

2.4.1 Preliminary lemmas

In this section, we prove Theorem 2.5. We will use the method of Selberg-Tsang [116] in a part of the proof, where the following proposition plays an important role there. Moreover, the proposition also plays an important role in the proof of Theorem 2.6.

Before stating the proposition, we define $\sigma_{X,t}$ and $\Lambda_X(n) = \Lambda(n) w_X(n)$ by

$$\sigma_{X,t} = \frac{1}{2} + 2 \max_{|t-\gamma| \le \frac{X^{3(\beta-1/2)}}{\log X}} \left\{ \beta - \frac{1}{2}, \frac{2}{\log X} \right\}, \tag{2.37}$$

$$w_X(y) = \begin{cases} 1 & \text{if } 1 \le y \le X, \\ \frac{(\log(X^3/y))^2 - 2(\log(X^2/y))^2}{2(\log X)^2} & \text{if } X \le y \le X^2, \\ \frac{(\log(X^3/y))^2}{2(\log X)^2} & \text{if } X^2 \le y \le X^3. \end{cases} \tag{2.38}$$

Then, we can obtain the following proposition.

Proposition 2.2. *Assume $D(f) \ge 2$. Let m be a nonnegative integer, and let X, H be real parameters with $X \ge 3$, $H \ge 1$. Then, for $t \ge 14$, $\sigma \ge 1/2$, the right hand side of* (2.3) *is estimated by*

$$\ll_f \frac{X^{2(1-\sigma)} + X^{1-\sigma}}{t(\log X)^{m+1}} + \\ + H^3 \frac{\sigma_{X,t} - 1/2}{(\log X)^m} (X^{2(\sigma_{X,t}-\sigma)} + X^{\sigma_{X,t}-\sigma}) \left(\left| \sum_{n \le X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}} \right| + \log t \right).$$

Thanks to Proposition 2.2, we can combine the method of Selberg-Tsang with Theorem 2.1.

Proof. By estimate (2.3) and the line symmetry of nontrivial zeros of $\zeta(s)$

with respect to $\sigma = 1/2$, it suffices to show that

$$\sum_{\substack{|t-\gamma| \le \frac{1}{\log X} \\ \beta \ge 1/2}} (X^{2(\beta-\sigma)} + X^{\beta-\sigma}) + \frac{1}{(\log X)^3} \sum_{\substack{|t-\gamma| > \frac{1}{\log X} \\ \beta \ge 1/2}} \frac{X^{2(\beta-\sigma)} + X^{\beta-\sigma}}{|t-\gamma|^3}$$
$$\ll (\sigma_{X,t} - 1/2)(X^{2(\sigma_{X,t}-\sigma)} + X^{\sigma_{X,t}-\sigma}) \left(\left| \sum_{n \le X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}} \right| + \log t \right).$$

If $\beta > \frac{\sigma_{X,t}+1/2}{2}$, then by the definition of $\sigma_{X,t}$ (2.37), we have

$$|t-\gamma| > \frac{X^{3(\beta-1/2)}}{\log X} > 3(\beta - 1/2) > 3|\sigma_{X,t} - \beta|.$$

By these inequalities, we find that

$$\frac{X^{2(\beta-\sigma)} + X^{\beta-\sigma}}{|t-\gamma|^3} \ll \frac{\log X}{X^{3(\beta-1/2)}} \frac{X^{2(\beta-\sigma)} + X^{\beta-\sigma}}{(\sigma_{X,t}-\beta)^2 + (t-\gamma)^2}$$
$$\ll X^{1/2-\sigma} (\log X)^2 \frac{\sigma_{X,t} - 1/2}{(\sigma_{X,t}-\beta)^2 + (t-\gamma)^2}.$$

Next, we suppose $1/2 \le \beta \le \frac{\sigma_{X,t}+1/2}{2}$. Then if $|t-\gamma| > \sigma_{X,t} - 1/2$, we find that

$$\frac{X^{2(\beta-\sigma)} + X^{\beta-\sigma}}{|t-\gamma|^3} \ll (X^{2(\sigma_{X,t}-\sigma)} + X^{\sigma_{X,t}-\sigma})(\log X)^2 \frac{\sigma_{X,t} - 1/2}{(\sigma_{X,t}-\beta)^2 + (t-\gamma)^2},$$

and if $1/\log X < |t-\gamma| \le \sigma_{X,t} - 1/2$, we find that

$$\frac{X^{2(\beta-\sigma)} + X^{\beta-\sigma}}{|t-\gamma|^3} \ll (X^{2(\sigma_{X,t}-\sigma)} + X^{\sigma_{X,t}-\sigma})(\log X)^3 \frac{(\sigma_{X,t} - 1/2)^2}{(\sigma_{X,t}-\beta)^2 + (t-\gamma)^2}.$$

From the above estimates, we have

$$\frac{1}{(\log X)^3} \sum_{\substack{|t-\gamma| > \frac{1}{\log X} \\ \beta \ge 1/2}} \frac{X^{2(\beta-\sigma)} + X^{\beta-\sigma}}{|t-\gamma|^3}$$
$$\ll (\sigma_{X,t} - 1/2)(X^{2(\sigma_{X,t}-\sigma)} + X^{\sigma_{X,t}-\sigma}) \sum_{|t-\gamma| > \frac{1}{\log X}} \frac{\sigma_{X,t} - 1/2}{(\sigma_{X,t}-\beta)^2 + (t-\gamma)^2}. \quad (2.39)$$

Moreover, it holds that

$$\sum_{\substack{|t-\gamma| \le \frac{1}{\log X} \\ \beta \ge 1/2}} (X^{2(\beta-\sigma)} + X^{\beta-\sigma})$$
$$\ll (\sigma_{X,t} - 1/2)(X^{2(\sigma_{X,t}-\sigma)} + X^{\sigma_{X,t}-\sigma}) \sum_{|t-\gamma| \le \frac{1}{\log X}} \frac{\sigma_{X,t} - 1/2}{(\sigma_{X,t}-\beta)^2 + (t-\gamma)^2}.$$

By this estimate and (2.39), we obtain

$$\sum_{\substack{|t-\gamma|\le\frac{1}{\log X}\\ \beta\ge1/2}} (X^{2(\beta-\sigma)} + X^{\beta-\sigma}) + \frac{1}{(\log X)^3}\sum_{\substack{|t-\gamma|>\frac{1}{\log X}\\ \beta\ge1/2}} \frac{X^{2(\beta-\sigma)} + X^{\beta-\sigma}}{|t-\gamma|^3}$$
$$\ll (\sigma_{X,t}-1/2)(X^{2(\sigma_{X,t}-\sigma)} + X^{\sigma_{X,t}-\sigma})\sum_{\rho}\frac{\sigma_{X,t}-1/2}{(\sigma_{X,t}-\beta)^2+(t-\gamma)^2}.$$

Here, we have the following estimates (cf. (4.4) and (4.9) in [107])

$$\sum_{\rho}\frac{\sigma_{X,t}-1/2}{(\sigma_{X,t}-\beta)^2+(t-\gamma)^2} \ll \left|\sum_{n\le X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\right| + \log t.$$

Thus, we obtain this proposition. □

Moreover, we prepare some lemmas.

Lemma 2.8. *Let $T\ge5$, and let $3\le X\le T$. Let k be a positive integer such that $X^k\le T/\log T$. Then, for any complex numbers $a(p)$, we have*

$$\int_0^T\left|\sum_{p\le X}\frac{a(p)}{p^{1/2+it}}\right|^{2k}dt \ll Tk!\left(\sum_{p\le X}\frac{|a(p)|^2}{p}\right)^k.$$

Here, the above sums run over prime numbers.

Proof. This lemma is a little modified assertion of Lemma 3 in [111], and the proof of this lemma is the same as its proof. □

Lemma 2.9. *Let $T\ge5$, and let k be a positive integer, $X\ge3$, $\xi\ge1$ be some parameters with $X^{15}\xi^{10}\le T$. Then, we have*

$$\int_0^T\left(\sigma_{X,t}-\frac{1}{2}\right)^k\xi^{\sigma_{X,t}-1/2}dt \ll T\left(\frac{4^k\xi^{\frac{4}{\log X}}}{(\log X)^k}+\frac{8^k k!}{\log X(\log T)^{k-1}}\right).$$

We omit the proof of this lemma because this lemma is a little modified assertion of Lemma 12 in [107], and the proof of this lemma is the same as its proof. On the other hand, we will give the proof of a general situation (see Lemma 6.4 and its proof).

Lemma 2.10. *Let T be large, $Z\ge3$, and k a positive integer with $k\le\frac{1}{55}\frac{\log T}{\log Z}$. Then we have*

$$\int_0^T\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)}{p^{\sigma_{Z,t}+it}}\right|^{2k}dt \ll Tk^k(C\log Z)^{2k},$$

where C is an absolute positive constant.

Proof. Now, we can write

$$\sum_{p\le Z^3}\frac{\Lambda_Z(p)}{p^{\sigma_{Z,t}+it}}=\sum_{p\le Z^3}\frac{\Lambda_Z(p)}{p^{1/2+it}}-\sum_{p\le Z^3}\frac{\Lambda_Z(p)}{p^{1/2+it}}(1-p^{1/2-\sigma_{Z,t}})$$
$$=\sum_{p\le Z^3}\frac{\Lambda_Z(p)}{p^{1/2+it}}-\int_{1/2}^{\sigma_{Z,t}}\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log p}{p^{\alpha'+it}}d\alpha',$$

and, for $1/2\le\alpha'\le\sigma_{Z,t}$,

$$\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log p}{p^{\alpha'+it}}\right|=Z^{\alpha'-1/2}\left|\int_{\alpha'}^{\infty}Z^{1/2-\alpha}\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}d\alpha\right|$$
$$\le Z^{\sigma_{Z,t}-1/2}\int_{1/2}^{\infty}Z^{1/2-\alpha}\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|d\alpha.$$

Therefore, we have

$$\left|\sum_{p\le X^3}\frac{\Lambda_Y(p)}{p^{\sigma_{Y,t}+it}}\right|\le\left|\sum_{p\le Y^3}\frac{\Lambda_Y(p)}{p^{1/2+it}}\right|+$$
$$+(\sigma_{Y,t}-1/2)Y^{\sigma_{Y,t}-1/2}\int_{1/2}^{\infty}Y^{1/2-\alpha}\left|\sum_{p\le Y^3}\frac{\Lambda_Y(p)\log(Yp)\log p}{p^{\alpha+it}}\right|d\alpha. \quad (2.40)$$

By Lemma 2.8, we have

$$\int_0^T\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)}{p^{1/2+it}}\right|^{2k}dt\ll Tk!(C\log Z)^{2k}. \quad (2.41)$$

On the other hand, by the Cauchy-Schwarz inequality and Lemma 2.9, we find that

$$\int_0^T(\sigma_{Z,t}-1/2)^{2k}Z^{2k(\sigma_{Z,t}-1/2)}\left(\int_{1/2}^{\infty}Z^{1/2-\alpha}\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|d\alpha\right)^{2k}dt$$
$$\le\left(\int_0^T(\sigma_{Z,t}-1/2)^{4k}Z^{4k(\sigma_{Z,t}-1/2)}dt\right)^{1/2}\times$$
$$\times\left(\int_0^T\left(\int_{1/2}^{\infty}Z^{1/2-\alpha}\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|d\alpha\right)^{4k}dt\right)^{1/2}$$
$$\ll\frac{T^{1/2}C^k}{(\log Z)^{2k}}\left(\int_0^T\left(\int_{1/2}^{\infty}Z^{1/2-\alpha}\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|d\alpha\right)^{4k}dt\right)^{1/2}.$$

Moreover, by Hölder's inequality, we have

$$\left(\int_{1/2}^{\infty} Z^{1/2-\alpha}\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|d\alpha\right)^{4k}$$
$$\le\left(\int_{1/2}^{\infty} Z^{1/2-\alpha}d\alpha\right)^{4k-1}\times\left(\int_{1/2}^{\infty} Z^{1/2-\alpha}\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|^{4k}d\alpha\right)$$
$$=\frac{1}{(\log Z)^{4k-1}}\int_{1/2}^{\infty} Z^{1/2-\alpha}\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|^{4k}d\alpha.$$

Therefore, by using Lemma 2.8, we find that

$$\int_0^T\left(\int_{1/2}^{\infty} Z^{1/2-\alpha}\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|d\alpha\right)^{4k}dt$$
$$\le\frac{1}{(\log Z)^{4k-1}}\int_{1/2}^{\infty} Z^{1/2-\alpha}\left(\int_0^T\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|^{4k}dt\right)d\alpha$$
$$\ll\frac{T(2k)!}{(\log Z)^{4k-1}}\int_{1/2}^{\infty} Z^{1/2-\alpha}\left(\sum_{p\le Z^3}\frac{(\log(Zp))^2(\log p)^4}{p^{2\alpha}}\right)^{2k}d\alpha$$
$$\ll Tk^{2k}C^k(\log Z)^{8k+1}\int_{1/2}^{\infty} Z^{1/2-\alpha}d\alpha\le Tk^{2k}C^k(\log Z)^{8k}.$$

Hence, by estimate (2.42), we have

$$\int_0^T(\sigma_{Z,t}-1/2)^{2k}Z^{2k(\sigma_{Z,t}-1/2)}\left(\int_{1/2}^{\infty} Z^{1/2-\alpha}\left|\sum_{p\le Z^3}\frac{\Lambda_Z(p)\log(Zp)\log p}{p^{\alpha+it}}\right|d\alpha\right)^{2k}dt$$
$$\ll Tk^k(C\log Z)^{2k}.$$

Thus, from this estimate and (2.40), (2.41), we obtain this lemma. □

Lemma 2.11. *Let T be large, $X=T^{1/(\log\log T)^2}$. For $1\ll V=o(\sqrt{\log\log T})$, we have*

$$\frac{1}{T}\,\mathrm{meas}\left\{t\in[T,2T]\ :\ \mathrm{Re}\sum_{p\le X}\frac{1}{p^{\frac{1}{2}+it}}>V\sqrt{\frac{1}{2}\sum_{p\le X}\frac{1}{p}}\right\}$$
$$=(1+o(1))\int_V^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}$$

as $T\to+\infty$.

Proof. This lemma is Proposition 1 in [95]. □

2.4.2 Proof of the theorem

Proof of Theorem 2.5. Let T be large, and V a parameter with $\sqrt{\log\log T} \ll V \ll (\log\log T)^{2/3}$. Here, we may assume the inequality $V \le A(\log\log T)^{2/3}$ with A any fixed positive constant. Then, it suffices to show that, as $T \to +\infty$

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T,V)) \le (1+o(1))\int_{\frac{V}{\sqrt{1/2\log\log T}}}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} + O\left(\frac{V}{(\log\log T)^{5/6}}\exp\left(-\frac{V^2}{\log\log T}\right)\right).$$

Let X, Y be parameters with $X = T^{1/(\log\log T)^2} \le Y \le T^{1/100}$. Let f be a fixed function satisfying the condition of this paper and $D(f) \ge 2$. By Theorem 2.1 and Proposition 2.2, for $T \le t \le 2T$, we have

$$\log|\zeta(1/2+it)| \le \operatorname{Re}\sum_{2\le n\le Y^2}\frac{\Lambda(n)v_{f,1}(e^{\log n/\log Y})}{n^{1/2+it}\log n} + C_1(\sigma_{Y,t}-1/2)Y^{2\sigma_{Y,t}-1}\left(\left|\sum_{n\le Y^3}\frac{\Lambda_Y(n)}{n^{\sigma_{Y,t}+it}}\right| + \log T\right), \quad (2.42)$$

where C_1 is an absolute positive constant. Now, we see that

$$\operatorname{Re}\sum_{2\le n\le Y^2}\frac{\Lambda(n)v_{f,1}(e^{\log n/\log Y})}{n^{1/2+it}\log n} = \operatorname{Re}\sum_{p\le X}\frac{1}{p^{1/2+it}} + \operatorname{Re}\sum_{X<p\le Y^2}\frac{v_{f,1}(e^{\log p/\log Y})}{p^{1/2+it}} + \operatorname{Re}\sum_{p\le Y}\frac{v_{f,1}(e^{\log p^2/\log Y})}{p^{1+2it}\log p^2} + \operatorname{Re}\sum_{\substack{p^k\le Y^2\\ k\ge 3}}\frac{\Lambda(p^k)v_{f,1}(e^{\log p^k/\log Y})}{p^{k(1/2+it)}\log p^k},$$

$$\left|\sum_{\substack{p^k\le Y^2\\ k\ge 3}}\frac{\Lambda(p^k)v_{f,1}(e^{\log p^k/\log Y})}{p^{k(1/2+it)}\log p^k}\right| \le \sum_{\substack{p^k\le Y^2\\ k\ge 3}}\frac{\Lambda(p^k)}{p^{k/2}\log p^k} \ll 1,$$

and that

$$\left|\sum_{\substack{p^k\le Y^3\\ k\ge 2}}\frac{\Lambda_Y(p^k)}{p^{k(\sigma_{Y,t}+it)}}\right| \le \sum_{\substack{p^k\le Y^3\\ k\ge 2}}\frac{\log p}{p^{k\sigma_{Y,t}}} \le \log Y + O(1) \le \log T.$$

Hence, we have

$$\operatorname{meas}(\mathcal{S}(T,V)) \le \operatorname{meas}(S_1) + \operatorname{meas}(S_2) + \operatorname{meas}(S_3) + \operatorname{meas}(S_4), \quad (2.43)$$

where the sets S_1, S_2, S_3, S_4 are defined by

$$S_1 := \left\{ t \in [T, 2T] \;\middle|\; \mathrm{Re} \sum_{p \le X} \frac{1}{p^{1/2+it}} > V_1 \right\},$$

$$S_2 := \left\{ t \in [T, 2T] \;:\; \mathrm{Re} \sum_{X < p \le Y^2} \frac{\nu_{f,1}(e^{\log p/\log Y})}{p^{1/2+it}} > V_2 \right\},$$

$$S_3 := \left\{ t \in [T, 2T] \;\middle|\; \mathrm{Re} \sum_{p \le Y} \frac{\nu_{f,1}(e^{\log p^2/\log Y})}{p^{1+2it}} > V_2 \right\},$$

$$S_4 := \left\{ t \in [T, 2T] \;\middle|\; C_1(\sigma_{Y,t} - 1/2) Y^{2\sigma_{Y,t}-1} \left(\left| \sum_{p \le Y^3} \frac{\Lambda_Y(p)}{p^{\sigma_{Y,t}+it}} \right| + 2\log T \right) > V_2 \right\},$$

where $V_1 = V - 3V_2$, and V_2 is a positive parameter with $V_2 \le V/4$. Let k be a positive integer with $k \le \frac{1}{100}\frac{\log T}{\log Y}$. By Lemma 2.11, we find that

$$\int_T^{2T} \left| \sum_{X < p \le Y^2} \frac{\nu_{f,1}(e^{\log p/\log Y})}{p^{1/2+it}} \right|^{2k} dt \ll T \left(C_2 k \log\log\log T \right)^k, \tag{2.44}$$

and that

$$\int_T^{2T} \left| \sum_{p \le X} \frac{\nu_{f,1}(e^{\log p^2/\log X})}{p^{1+2it}} \right|^{2k} dt \ll T k! C_3^k. \tag{2.45}$$

By Lemma 2.9, we have

$$\int_T^{2T} (2C_1)^k (\sigma_{Y,t} - 1/2)^k Y^{2k(\sigma_{Y,t}-1/2)} (\log T)^k dt \ll T \left(\frac{C_4 \log T}{\log Y} \right)^k.$$

Moreover, by Cauchy-Schwarz and Lemmas 2.9, 2.10, we have

$$\begin{aligned}
&\frac{1}{T} \int_T^{2T} C_1^k (\sigma_{Y,t} - 1/2)^k Y^{(2\sigma_{Y,t}-1)k} \left| \sum_{p \le Y^3} \frac{\Lambda_Y(p)}{p^{\sigma_{Y,t}+it}} \right|^k dt \\
&\le \frac{C_1^k}{T} \left(\int_T^{2T} (\sigma_{Y,t} - 1/2)^{2k} Y^{4k(\sigma_{Y,t}-1/2)} dt \right)^{1/2} \times \left(\int_T^{2T} \left| \sum_{p \le Y^3} \frac{\Lambda_Y(p)}{p^{\sigma_{Y,t}+it}} \right|^{2k} dt \right)^{1/2} \\
&\ll \left(\frac{C_5 k^{1/2}}{\log Y} \right)^k.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{1}{T} \int_T^{2T} C_1^k (\sigma_{Y,t} - 1/2)^k Y^{(2\sigma_{Y,t}-1)k} \left(\left| \sum_{p \le Y^3} \frac{\Lambda_Y(p)}{p^{\sigma_{Y,t}+it}} \right| + 2\log T \right)^k dt \\
&\ll \left(\frac{C_6 \log T}{\log Y} \right)^k.
\end{aligned} \tag{2.46}$$

Thus, by estimates (2.44), (2.45), (2.46), the following estimates

$$\frac{1}{T}\,\mathrm{meas}(S_2) \ll \left(\frac{kC_2\log\log\log T}{V_2^2}\right)^k,$$

$$\frac{1}{T}\,\mathrm{meas}(S_3) \ll \left(\frac{kC_3}{V_2^2}\right)^k, \quad \frac{1}{T}\,\mathrm{meas}(S_4) \ll \left(\frac{C_6\log T}{V_2\log Y}\right)^k$$

hold for $X \le Y \le T^{1/100}$, $k \le \frac{1}{100}\frac{\log T}{\log Y}$.

We put $Y = T^{\log\log T/(200C_7V^2)}$ and $k = 2\left[\frac{V^2}{\log\log T}+1\right]$, where C_7 is a constant chosen as satisfying $C_7 \ge 2$ and $C_7V^2/\log\log T \ge 2$. Further, we decide V_2 as $200C_4C_5e^2AV/(\log\log T)^{1/3}$ Then we obtain

$$\frac{\mathrm{meas}(S_2)+\mathrm{meas}(S_3)+\mathrm{meas}(S_4)}{T} \ll \exp\left(-\frac{2V^2}{\log\log T}\log\left(\frac{eA(\log\log T)^{2/3}}{V}\right)\right)$$

for $\sqrt{\log\log T} \ll V \le A(\log\log T)^{2/3}$. Hence, by Lemma 2.11 and inequality (2.43), we have

$$\frac{1}{T}\,\mathrm{meas}(\mathcal{S}(T,V)) \le (1+o(1))\int_{\frac{V_1}{W(T)}}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} + o\left(\int_{\frac{V}{\sqrt{1/2\log\log T}}}^{\infty} e^{-u^2/2}du\right)$$

for $\sqrt{\log\log T} \ll V \le A(\log\log T)^{2/3}$. Here, $W(T)$ indicates

$$W(T) = \sqrt{\frac{1}{2}\sum_{p\le X}p^{-1}} = \sqrt{\frac{1}{2}\log\log T + O\left(\frac{\log\log\log T}{\sqrt{\log\log T}}\right)}.$$

Here, we find that

$$\int_{\frac{V_1}{W(T)}}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} = \int_{\frac{V}{\sqrt{1/2\log\log T}}}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} + \int_{\frac{V_1}{W(T)}}^{\frac{V}{\sqrt{1/2\log\log T}}} e^{-u^2/2}\frac{du}{\sqrt{2\pi}},$$

and that

$$\begin{aligned}\int_{\frac{V_1}{W(T)}}^{\frac{V}{\sqrt{1/2\log\log T}}} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} &\ll \left(\frac{V}{\sqrt{1/2\log\log T}} - \frac{V_1}{W(T)}\right)e^{-\frac{V_1^2}{2W(T)^2}} \\ &\ll \frac{V}{(\log\log T)^{5/6}}e^{-\frac{V^2}{\log\log T}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\frac{1}{T}\,\mathrm{meas}(\mathcal{S}(T,V)) \\ &\qquad \le (1+o(1))\int_{\frac{V}{\sqrt{1/2\log\log T}}}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} + O\left(e^{-\frac{V^2}{\log\log T}}\frac{V}{(\log\log T)^{5/6}}\right)\end{aligned}$$

for $\sqrt{\log\log T} \ll V \le A(\log\log T)^{2/3}$. This completes the proof of Theorem 2.5. □

2.5 A mean value theorem involving $\tilde{\eta}_m(s)$

In this section, we state a certain mean value theorem. There are some interesting applications of the theorem to the value distribution of $\tilde{\eta}_m(s)$.

Theorem 2.6. *Let m be a positive integer. Let k be a positive integer. Let T be large, and $X \geq 3$ with $X \leq T^{\frac{1}{175k}}$. Then, for $\sigma \geq 1/2$, we have*

$$\frac{1}{T}\int_{14}^{T}\left|\tilde{\eta}_m(\sigma+it) - \sum_{2\leq n\leq X}\frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}}\right|^{2k} dt$$
$$\ll 2^k k!\left(\frac{2m+1}{2m} + \frac{C}{\log X}\right)^k \frac{X^{k(1-2\sigma)}}{(\log X)^{2km}} + C^k k^{2k(m+1)}\frac{T^{\frac{1-2\sigma}{175}}}{(\log T)^{2km}}.$$

Here, the above C is an absolute positive constant.

This theorem will give an answer for the question of how much of the function $\eta_m(s)$ can be approximated by the corresponding Dirichlet polynomial. Such a study is often useful. For example, Radziwiłł [95] proved a large deviation theorem for Selberg's limit theorem, and he used Corollary in [116, p.60] to prove his result. The corollary is related with the approximation of $\log\zeta(s)$ by a certain Dirichlet polynomial, and we can regard that Theorem 2.6 corresponds to the corollary. Hence, it is expected to be able to show a limit theorem for $\eta_m(s)$, which is similar to Selberg's limit theorem or the Bohr-Jessen limit theorem, and also its large deviation.

2.5.1 Proof of the theorem

Proof of Theorem 2.6. Let m be a positive integer and f be a fixed function satisfying the condition of this paper and $D(f) \geq 2$. Then, by Theorem 2.2, for $t \geq 14$, $X \leq T^{\frac{1}{175k}} =: Y$, we obtain

$$\left|\tilde{\eta}_m(\sigma+it) - i^m\sum_{2\leq n\leq X}\frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}}\right|^{2k}$$
$$\leq 2^{2k}\left|\sum_{X<n\leq Y^2}\frac{\Lambda(n)v_{f,1}(e^{\log n/\log Y})}{n^{\sigma+it}(\log n)^{m+1}}\right|^{2k} + 2^{2k}|R_m(\sigma+it,Y,1)|^{2k}. \tag{2.47}$$

By using partial summation, Lemma 2.8, and the prime number theorem, we find that

$$\int_0^T\left|\sum_{X<p\leq Y^2}\frac{v_{f,1}(e^{\log p/\log Y})}{p^{\sigma+it}(\log p)^m}\right|^{2k} dt \ll Tk!\left(\sum_{p>X}\frac{1}{p^{2\sigma}(\log p)^{2m}}\right)^k$$
$$\leq Tk!\left(\frac{2m+1}{2m} + \frac{C}{\log X}\right)^k \frac{X^{k(1-2\sigma)}}{(\log X)^{2km}},$$

and that

$$\int_0^T \left| \sum_{X<p^2\le Y^2} \frac{v_{f,1}(e^{\log p^2/\log Y})}{p^{2\sigma+2it}(\log p^2)^m} \right|^{2k} dt \ll Tk! \left(\sum_{p>\sqrt{X}} \frac{1}{p^{4\sigma}(\log p^2)^{2m}} \right)^k$$
$$\le Tk! C^k \frac{X^{k(1-4\sigma)/2}}{(\log X)^{2km}}.$$

Set

$$\psi_3(z,y) := \sum_{\substack{y<p^l\le z \\ l\ge 3}} \log p.$$

Then we can easily obtain the inequality $\psi_3(z,y) \ll z^{1/3}$. By using this inequality and partial summation, we find that

$$\left| \sum_{\substack{X<p^l\le Y^2 \\ l\ge 3}} \frac{v_{f,1}(e^{\log p^l/\log Y})}{l p^{l(\sigma+it)}(\log p^l)^m} \right| \le \int_X^\infty \frac{\sigma \log \xi + m}{\xi^{1+\sigma}(\log \xi)^{m+1}} \psi_3(\xi, X) d\xi \ll \frac{X^{1/3-\sigma}}{(\log X)^m}.$$

Therefore, we have

$$\int_0^T \left| \sum_{\substack{X<p^l\le Y^2 \\ l\ge 3}} \frac{v_{f,1}(e^{\log p^l/\log Y})}{l p^{l(\sigma+it)}(\log p^l)^m} \right|^{2k} dt \ll TC^k \frac{X^{k(2/3-2\sigma)}}{(\log X)^{2km}}.$$

Hence it holds that

$$\int_0^T \left| \sum_{X<n\le Y^2} \frac{\Lambda(n) v_{f,1}(e^{\log n/\log Y})}{n^{1/2+it}(\log n)^{m+1}} \right|^{2k} dt$$
$$\ll Tk! \left(\frac{2m+1}{2m} + \frac{C}{\log X} \right)^k \frac{X^{k(1-2\sigma)}}{(\log X)^{2km}}. \tag{2.48}$$

Next, we consider the integral of $R_m(s,Y,1)$. By Proposition 2.2, we have

$$\int_{14}^T |R_m(\sigma+it, Y, 1)|^{2k} dt \ll \left(Ck^{2(m+1)} \right)^k \times \frac{T^{1-\sigma} + T^{(1-\sigma)/2}}{(\log T)^{2k(m+1)}} +$$
$$+ \frac{(Ck^{2m})^k Y^{(1-2\sigma)k}}{(\log T)^{2km}} \int_{14}^T \left\{ \left(\sigma_{Y,t} - \frac{1}{2} \right) Y^{2\sigma_{Y,t}-1} \left(\left| \sum_{n\le Y^3} \frac{\Lambda_Y(n)}{n^{\sigma_{Y,t}+it}} \right| + \log t \right) \right\}^{2k} dt,$$

where $\Lambda_Y(n) = \Lambda(n) w_Y(n)$, and $w_Y(n)$ is given by (2.38). By Lemma 2.9, we find that

$$\int_{14}^T \left(\sigma_{Y,t} - \frac{1}{2} \right)^{2k} Y^{8k(\sigma_{Y,t}-1/2)} (\log t)^{2k} dt \ll T(Ck^2)^k,$$

and that, by using the Cauchy-Schwarz inequality and applying Lemmas 2.9, 2.10,

$$\begin{aligned}
&\int_{14}^{T} \left(\sigma_{Y,t} - \frac{1}{2}\right)^{2k} Y^{8k(\sigma_{Y,t}-1/2)} \left| \sum_{n \le Y^3} \frac{\Lambda_Y(n)}{n^{\sigma_{Y,t}+it}} \right|^{2k} dt \\
&\le \left(\int_{14}^{T} \left(\sigma_{Y,t} - \frac{1}{2}\right)^{4k} Y^{16k(\sigma_{Y,t}-1/2)} dt \right)^{1/2} \times \left(\int_{14}^{T} \left| \sum_{n \le Y^3} \frac{\Lambda_Y(n)}{n^{\sigma_{Y,t}+it}} \right|^{4k} dt \right)^{1/2} \\
&\ll T(Ck)^k.
\end{aligned}$$

Therefore, we obtain

$$\int_{0}^{T} \left\{ \left(\sigma_{Y,t} - \frac{1}{2}\right) Y^{2\sigma_{Y,t}-1} \left(\left| \sum_{n \le Y^3} \frac{\Lambda_Y(n)}{n^{\sigma_{Y,t}+it}} \right| + \log(t+2) \right) \right\}^{2k} dt \ll T(Ck^2)^k.$$

Hence, we have

$$\int_{14}^{T} |R_m(\sigma + it, Y, 1)|^{2k} dt \ll T^{1+\frac{1-2\sigma}{175}} \frac{C^k k^{2k(m+1)}}{(\log T)^{2km}}.$$

Thus, from this estimate, (2.47), and (2.48), we obtain Theorem 2.6. □

2.6 An upper bound of the distribution function of $\tilde{\eta}_m(\frac{1}{2}+it)$

In this section, we consider the value distribution of $\tilde{\eta}_m(1/2+it)$. There are many studies on the value distribution of the Riemann zeta-function and other L-functions.

We discuss a measure for the difference between $\tilde{\eta}_m(1/2+it)$ and the corresponding Dirichlet polynomial. We are interested in the exact value distribution of $\tilde{\eta}_m(1/2+it)$ and $S_m(t)$. Here our aim is to establish a theorem for $\eta_m(1/2+it)$ and $S_m(t)$ similar to the results of Jutila [56], Radziwiłł [95], and Soundararajan [111] on the large deviation of the Riemann zeta-function. The motivation of this study in the present paper is to search for the exact bound of $\tilde{\eta}_m(1/2+it)$.

We define the set $\mathscr{T}_m(T, X, V)$ by

$$\left\{ t \in [T, 2T] \ : \ \left| \tilde{\eta}_m(1/2+it) - \sum_{2 \le n \le X} \frac{\Lambda(n)}{n^{\frac{1}{2}+it}(\log n)^{m+1}} \right| > V \right\}.$$

We obtain the following result which evaluates the difference between $\eta_m(1/2+it)$ and the corresponding Dirichlet polynomial.

Theorem 2.7. *Let m be a positive integer, and let T, X be large with $X^{135} \le T$. If V satisfies the inequality $2(\log X)^{-m} \le V \le c_0(\log T)^{\frac{m}{2m+1}}(\log X)^{-\frac{2m^2+2m}{2m+1}}$, then we have*

$$\frac{1}{T}\,\mathrm{meas}(\mathscr{T}_m(T, X, V)) \ll \exp\left(-\frac{m}{4(m+1)} V^2 (\log X)^{2m} \left(1 - \frac{C}{\log X}\right) \right).$$

If V satisfies $c_0(\log T)^{\frac{m}{2m+1}}(\log X)^{-\frac{2m^2+2m}{2m+1}} \leq V \leq \log T/(\log X)^{m+1}$*, then we have*

$$\frac{1}{T}\operatorname{meas}(\mathscr{T}_m(T,X,V)) \ll \exp\left(-c_1 V^{\frac{1}{m+1}}(\log T)^{\frac{m}{m+1}}\right).$$

Moreover, if the Riemann Hypothesis is true, then we have

$$\frac{1}{T}\operatorname{meas}(\mathscr{T}_m(T,X,V)) \ll \exp\left(-c_2 V^{\frac{1}{m+1}}(\log T)^{\frac{m}{m+1}}\log\left(e\frac{V^{\frac{2m+1}{2m+2}}(\log X)^m}{(\log T)^{\frac{m}{2m+2}}}\right)\right) \tag{2.49}$$

for $(\log T)^{\frac{m}{2m+1}}(\log X)^{-\frac{2m^2+2m}{2m+1}} \leq V \leq \log T/(\log X)^{m+1}$*. Here the numbers c_0, c_1, c_2, C are some absolute positive constants.*

This theorem can be applied to the value distribution of $\eta_m(s)$ on the critical line. For example, we can obtain the following results from this theorem.

Corollary 2.2. *Let T, V be large numbers. If* $V \leq (\log T)^{\frac{m}{2m+1}}(\log\log T)^{-\frac{2m^2+2m}{2m+1}}$*, then we have*

$$\frac{1}{T}\operatorname{meas}\left\{t\in[T,2T]\ :\ |\tilde{\eta}_m(\tfrac{1}{2}+it)|>V\right\} \ll \exp\left(-c_5 V^2(\log V)^{2m}\right). \tag{2.50}$$

If $V \geq (\log T)^{\frac{m}{2m+1}}(\log\log T)^{-\frac{2m^2+2m}{2m+1}}$*, then we have*

$$\frac{1}{T}\operatorname{meas}\left\{t\in[T,2T]\ :\ |\tilde{\eta}_m(\tfrac{1}{2}+it)|>V\right\} \ll \exp\left(-c_6 V^{\frac{1}{m+1}}(\log T)^{\frac{m}{m+1}}\right). \tag{2.51}$$

Here c_5, c_6 are some absolute positive constants.

Corollary 2.3. *Assume the Riemann Hypothesis. Let m be a positive integer, and let T, V be numbers with $T, V \geq T_0(m)$, where $T_0(m)$ is a sufficiently large number depending only on m. If* $V \geq (\log T)^{\frac{m}{2m+1}}(\log\log T)^{-\frac{2m^2+2m}{2m+1}}$*, then we have*

$$\frac{1}{T}\operatorname{meas}\left\{t\in[T,2T]\mid |\tilde{\eta}_m(\tfrac{1}{2}+it)|>V\right\} \ll \exp\left(-c_7 V^{\frac{1}{m+1}}(\log T)^{\frac{m}{m+1}}\log\left(e\frac{V^{\frac{2m+1}{2m+2}}(\log V)^m}{(\log T)^{\frac{m}{2m+2}}}\right)\right).$$

Here c_7 is an absolute positive constant.

These assertions can be obtained by the following argument. Now, we see that $\sum_{2\leq n\leq V}\frac{\Lambda(n)}{n^{1/2+it}(\log n)^{m+1}} \ll_m \frac{V^{1/2}}{(\log V)^{m+1}}$. Hence, for sufficiently large V, we find that

$$\operatorname{meas}\left\{t\in[T,2T]\mid |\tilde{\eta}_m(\tfrac{1}{2}+it)|>V\right\} \leq \operatorname{meas}(\mathscr{T}_1(T,V,V/2))$$

unconditionally, and that

$$\mathrm{meas}\left\{t \in [T, 2T] \mid |\tilde{\eta}_m(\tfrac{1}{2} + it)| > V\right\} \le \mathrm{meas}(\mathscr{T}_m(T, V, V/2))$$

under the Riemann Hypothesis. Further, the estimate $\tilde{\eta}_m(\frac{1}{2} + it) \ll_m \log t$ holds unconditionally, and the estimate $\tilde{\eta}_m(1/2 + it) \ll_m \log t/(\log\log t)^{m+1}$ holds under the Riemann Hypothesis. By these inequalities and Theorem 2.7, we can obtain Corollary 2.2 and Corollary 2.3.

It could be expected that the function $\sqrt{V \log T}$ in the exponential on the right hand side of (2.51) is sharp as an unconditional result by the following discussion. Actually, if there is a function $\omega(T, V)$ with $\lim_{T \to +\infty} \omega(T, V) = +\infty$ or $\lim_{V \to +\infty} \omega(T, V) = +\infty$ such that the left hand side of (2.51) is $\ll \exp(-\omega(T, V)\sqrt{V \log T})$, then the Lindelöf Hypothesis holds. Moreover, estimate (2.51) matches the well known inequality $S_1(t) \ll \log t$.

We are also interested in that estimate (2.50) holds in how large range of V. If the estimates hold for any large V, then we have $\eta_m(1/2 + it) \ll_m \sqrt{\log t}/(\log\log t)^m$. Although the necessary condition of this implication is rather strong, the author guesses that it could be true. Hence the author expects the inequality for $\eta_m(1/2 + it)$ could be also true.

2.6.1 Proof of the theorem

Proof of Theorem 2.7. Let m be a positive integer. Let X, T be sufficiently large numbers with $X \le T^{\frac{1}{175k}}$. Set V be any positive number. By Theorem 2.6, there exists a positive number $C_1 > 3$ such that

$$\mathrm{meas}(\mathscr{T}_m(T, X, V)) \ll \sqrt{k}\left(\frac{4k(1 + \frac{1}{m} + \frac{C_1}{\log X})}{eV^2(\log X)^{2m}}\right)^k + \left(\frac{C_1 k^{2(m+1)}}{V^2(\log T)^{2m}}\right)^k. \quad (2.52)$$

Here, if V satisfies $2(\log X)^{-m} \le V \le c_0(\log T)^{\frac{m}{2m+1}}(\log X)^{-\frac{2m^2+2m}{2m+1}}$, then we choose $k = [V^2(\log X)^{2m}/4(1+1/m)]$, where c_0 is an absolute positive constant satisfying $c_0 \le e^{-1}C_1^{1/(4m+2)}$. Then, by (2.52), we have

$$\mathrm{meas}(\mathscr{T}_m(T, X, V)) \ll \exp\left(-\frac{m}{4(m+1)}V^2(\log X)^{2m}\left(1 - \frac{C'}{\log X}\right)\right). \quad (2.53)$$

If V satisfies $c_0(\log T)^{\frac{m}{2m+1}}(\log X)^{-\frac{2m^2+2m}{2m+1}} \le V \le \frac{\log T}{(\log X)^{m+1}}$, then we choose $k = [(eC_1)^{-\frac{1}{m+1}} V^{\frac{1}{m+1}} (\log T)^{\frac{m}{m+1}}]$. Then, by (2.52), we have

$$\mathrm{meas}(\mathscr{T}_m(T, X, V)) \ll \exp\left(-c_1 V^{\frac{1}{m+1}} (\log T)^{\frac{m}{m+1}}\right). \quad (2.54)$$

Thus, from estimates (2.53) and (2.54), we obtain this theorem.

Next, we show (2.49) under the Riemann Hypothesis. Let f be a fixed function satisfying the condition of this paper and $D(f) \ge 2$. By Theorem

2.1 as $H = 1$, for $X \le Z \le T$, we have

$$\eta_m(\sigma + it) - i^m \sum_{2 \le n \le X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \\ = i^m \sum_{X < n \le Z^2} \frac{\Lambda(n) v_{f,1}(e^{\log n / \log Z})}{n^{\sigma+it}(\log n)^{m+1}} + R_m(\sigma + it, Z, 1). \quad (2.55)$$

Since we assume the Riemann Hypothesis, by using Proposition 2.2, it holds that there exists some constant $C_3 > 1$ such that for any $3 \le Z \le T$, $t \in [T, 2T]$,

$$|R_m(1/2 + it, Z, 1)| \le \frac{C_3}{2} \left(\frac{1}{(\log Z)^{m+1}} \left| \sum_{p \le Z^3} \frac{w_Z(p) \log p}{p^{\frac{1}{2} + \frac{4}{\log Z} + it}} \right| + \frac{\log T}{(\log Z)^{m+1}} \right),$$

where w_Z is defined by (2.38). Therefore, by letting $Z = \exp\left(\left(C_3 \frac{\log T}{V} \right)^{\frac{1}{m+1}} \right)$, we have

$$|R_m(1/2 + it, Z; u)| \le \frac{V}{2 \log T} \left| \sum_{p \le Z^2} \frac{w_Z(p) \log p}{p^{\frac{1}{2} + \frac{4}{\log Z} + it}} \right| + \frac{V}{2}$$

for $t \in [T, 2T]$. Note that the inequality $V \le \frac{\log T}{(\log X)^{m+1}}$ implies $X \le Z$. Hence, by formula (2.55), when $V \le \frac{\log T}{(\log X)^{m+1}}$, we have

$$\mathrm{meas}(\mathcal{T}_m(T, X, V)) \le \mathrm{meas}(S_1) + \mathrm{meas}(S_2). \quad (2.56)$$

Here, the sets S_1 and S_2 are defined by

$$S_1 := \left\{ t \in [T, 2T] \;\middle|\; \left| \sum_{X < n \le Z^2} \frac{\Lambda(n) v_{f,1}(e^{\log n / \log Z})}{n^{1/2+it}(\log n)^{m+1}} \right| > \frac{V}{4} \right\},$$
$$S_2 := \left\{ t \in [T, 2T] \;\middle|\; \frac{V}{2 \log T} \left| \sum_{p \le Z^3} \frac{w_Z(p) \log p}{p^{\frac{1}{2} + \frac{4}{\log Z} + it}} \right| > \frac{V}{4} \right\}.$$

By the same calculation as (2.48), we obtain

$$\frac{1}{T} \int_T^{2T} \left| \sum_{X < n \le Z^2} \frac{\Lambda(n) v_{f,1}(e^{\log n / \log Z})}{n^{1/2+it}(\log n)^{m+1}} \right|^{2k} dt \ll \frac{C^k k!}{(\log X)^{2mk}}. \quad (2.57)$$

On the other hand, by Lemma 2.8 and the prime number theorem, we find that

$$\frac{1}{T} \int_T^{2T} \left(\frac{V}{2 \log T} \left| \sum_{p \le Z^3} \frac{w_Z(p) \log p}{p^{\frac{1}{2} + \frac{4}{\log Z} + it}} \right| \right)^{2k} dt \ll C^k k! \left(\frac{V}{\log T} \right)^{\frac{2m}{m+1} k}$$

for $k \leq c_0 V^{\frac{1}{m+1}} (\log T)^{\frac{m}{m+1}}$. Here c_0 is a small positive constant. Therefore, by this estimate and (2.57), we obtain the following estimates

$$\frac{\mathrm{meas}(S_1) + \mathrm{meas}(S_2)}{T} \ll \left(\frac{C_4 k^{1/2}}{V (\log X)^m} \right)^{2k} + \left(\frac{C_4 k^{1/2}}{V} \left(\frac{V}{\log T} \right)^{m/(m+1)} \right)^{2k},$$

where C_4 is a sufficiently large positive constant. Hence, by these esitmates and (2.56), when $V \leq \frac{\log T}{(\log X)^{m+1}}$, we have

$$\mathrm{meas}(\mathscr{T}_m(T, X, V)) \ll \left(\frac{C_4 k^{1/2}}{V (\log X)^m} \right)^{2k}.$$

Since V satisfies $(\log T)^{\frac{m}{2m+1}} (\log X)^{-\frac{2m^2+2m}{2m+1}} \leq V \leq \frac{C_0 \log T}{(\log X)^{m+1}}$, we have, by choosing $k = [(eC_4)^{-2} V^{\frac{1}{m+1}} (\log T)^{\frac{m}{m+1}}]$,

$$\mathrm{meas}(\mathscr{T}_m(T, X, V)) \ll \exp\left(-c_4 V^{\frac{1}{m+1}} (\log T)^{\frac{m}{m+1}} \log\left(e \frac{V^{\frac{2m+1}{2m+2}} (\log X)^m}{(\log T)^{\frac{m}{2m+2}}} \right) \right).$$

Thus, we obtain estimate (2.49) under the Riemann Hypothesis. $\square$

Chapter 3 On the value distribution of $\tilde{\eta}_m(s)$ in the critical strip

In this chapter, we discuss the value distribution of $\tilde{\eta}_m(s)$ in the critical strip. The contents in this chapter are based on the paper [51].

3.1 Results of large deviations of the distribution function of $\tilde{\eta}_m(s)$

Now, we define the set $\mathcal{S}_{m,\theta}(T, V; \sigma)$ by

$$\mathcal{S}_{m,\theta}(T, V; \sigma) := \left\{ t \in [T, 2T] \ : \ \mathrm{Re}\, e^{-i\theta} \tilde{\eta}_m(\sigma + it) > V \right\}.$$

Then we show the following theorem.

Theorem 3.1. *Let $m \in \mathbb{Z}_{\geq 1}$, $\theta \in \mathbb{R}$ be fixed. There exists a positive constant $a_1 = a_1(m)$ such that, for any large numbers T, V with $V \leq a_1 \left(\frac{\log T}{(\log\log T)^{2m+2}} \right)^{\frac{m}{2m+1}}$, we have*

$$\frac{1}{T} \mathrm{meas}(\mathcal{S}_{m,\theta}(T, V; 1/2)) = \exp\left(-2m 4^m V^2 (\log V)^{2m} (1 + R) \right),$$

where the error term R satisfies

$$R \ll_m \frac{V^{2m+1} (\log V)^{2m(m+1)}}{(\log T)^m} + \sqrt{\frac{\log\log V}{\log V}}.$$

Theorem 3.1 contains the unconditional best result $S_1(t) = \Omega_{-}\left(\frac{(\log t)^{1/3}}{(\log\log t)^{4/3}} \right)$ due to Tsang [117]. Actually, we can immediately obtain the following corollary.

Corollary 3.1. *Let $m \in \mathbb{Z}_{\geq 1}$, $\theta \in \mathbb{R}$ be fixed. Then we have*

$$\mathrm{Re}\, e^{-i\theta} \tilde{\eta}_m(1/2 + it) = \Omega_{\pm}\left(\frac{(\log t)^{\frac{m}{2m+1}}}{(\log\log t)^{\frac{2m^2+2m}{2m+1}}} \right).$$

To prove Theorem 3.1, we show the result for the value distribution of the Dirichlet polynomial.

Proposition 3.1. *Let $m \in \mathbb{Z}_{\geq 1}$, $\theta \in \mathbb{R}$ be fixed. There exist positive constants $a_2 = a_2(m)$, $a_3 = a_3(m)$ such that for large numbers T, V, X with $V \leq a_2 \frac{\sqrt{\log T}}{(\log\log T)^{m+1/2}}$, and $V^4 \leq X \leq T^{a_3/V^2(\log V)^{2m}}$, we have*

$$\frac{1}{T}\operatorname{meas}\left\{t \in [T, 2T] \ : \ \operatorname{Re} e^{-i\theta} \sum_{p \leq X} \frac{1}{p^{1/2+it}(\log p)^m} > V\right\}$$
$$= \exp\left(-\frac{2m4^m V^2(\log V)^{2m}}{1 - \left(\frac{\log V^2}{\log X}\right)^m}\left(1 + O_m\left(\sqrt{\frac{\log\log V}{\log V}}\right)\right)\right).$$

Moreover, our method of the proof of the above assertions can be also applied to the case $\frac{1}{2} < \sigma < 1$. Actually, we can obtain the theorem which is an analogue of the works due to Lamzouri [66]. We define $A_m(\sigma)$ by

$$A_m(\sigma) = \left(\frac{\sigma^{2\sigma}}{(1-\sigma)^{2\sigma-1+m}G(\sigma)^{\sigma}}\right)^{\frac{1}{1-\sigma}}. \tag{3.1}$$

Here, $G(\sigma) = \int_0^\infty \log I_0(u) u^{-1-\frac{1}{\sigma}} du$, and I_0 is the modified 0-th Bessel function defined by $I_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(z\cos\theta) d\theta = \sum_{n=0}^{\infty}(z/2)^{2n}/(n!)^2$.

Theorem 3.2. *Let $m \in \mathbb{Z}_{\geq 0}$, $\frac{1}{2} < \sigma < 1$, and $\theta \in \mathbb{R}$ be fixed. There exists a positive constant $a_4 = a_4(\sigma, m)$ such that, for any large numbers T, V with $V \leq a_4 \frac{(\log T)^{1-\sigma}}{(\log\log T)^{m+1}}$, we have*

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}_{m,\theta}(T, V; \sigma)) = \exp\left(-A_m(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}} (1 + R)\right),$$

where the error term R satisfies the estimate

$$R \ll_{\sigma,m} \sqrt{\frac{1 + m\log\log V}{\log V}}. \tag{3.2}$$

When $m = 0$, the asymptotic formula of this type was firstly proved by Hattori and Matsumoto[1)] [40]. They showed that, for $\frac{1}{2} < \sigma < 1$,

$$\lim_{T\to+\infty}\frac{1}{T}\operatorname{meas}\left(\bigcup_{j=0}^{3} \mathcal{S}_{0,\frac{\pi}{2}j}(T, V; \sigma)\right) \tag{3.3}$$
$$= \exp\left(-A_0(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\sigma}{1-\sigma}} (1 + o(1))\right)$$

as $V \to +\infty$. Note that the parameter V in their asymptotic formula is not effective with respect to T. Theorem 3.2 can recover this asymptotic formula

[1)] There is a difference of the range of t between ours and theirs, but it seems not essential. Precisely, our range of t is $t \in [T, 2T]$, and theirs is $t \in [-T, T]$.

effectively. Actually, we see that

$$\frac{1}{T}\operatorname{meas}\left(\mathcal{S}_{0,0}(T,V;\sigma)\right) \leq \frac{1}{T}\operatorname{meas}\left(\bigcup_{j=0}^{3}\mathcal{S}_{0,\frac{\pi}{2}j}(T,V;\sigma)\right)$$
$$\leq \frac{1}{T}\sum_{j=0}^{3}\operatorname{meas}\left(\mathcal{S}_{0,\frac{\pi}{2}j}(T,V;\sigma)\right),$$

and both sides are equal to $\exp\left(-A_0(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{\sigma}{1-\sigma}}(1+R)\right)$ from Theorem 3.2. Here, the error term R satisfies (3.2). Hence, we can improve (3.3) to the effective form. On the other hand, it seems this improvement has been essentially obtained by Lamzouri's work [66]. After the study of Hattori-Matsumoto, Lamzouri [66] showed an effective asymptotic formula in the case $\theta = 0$ only. Though he did not mention, we can also prove his theorem for any $\theta \in \mathbb{R}$ by just using his method. Therefore, we may say that the above improvement has been already given by Lamzouri.

Now, we state the proposition corresponding to Proposition 3.1, which plays an important role in Theorem 3.2.

Proposition 3.2. *Let $m \in \mathbb{Z}_{\geq 0}$, $\frac{1}{2} < \sigma < 1$, and $\theta \in \mathbb{R}$ be fixed. There exist positive constants $a_5 = a_5(\sigma, m)$, $a_6 = a_6(\sigma, m)$ such that for large numbers T, X, V with $V \leq a_5\frac{(\log T)^{1-\sigma}}{(\log\log T)^{m+1}}$ and $V^{\frac{4\sigma}{1-\sigma}} \leq X \leq T^{a_6/V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}}$, we have*

$$\frac{1}{T}\operatorname{meas}\left\{t \in [T,2T] \ :\ \operatorname{Re} e^{-i\theta}\sum_{p\leq X}\frac{1}{p^{\sigma+it}(\log p)^m} > V\right\}$$
$$= \exp\left(-A_m(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\left(1 + O_{\sigma,m}\left(\sqrt{\frac{1+m\log\log V}{\log V}}\right)\right)\right).$$

Here, we describe the method of the proofs of Theorem 3.1 and Theorem 3.2 roughly. These theorems are analogues of Lamzouri's result, but we cannot adopt directly his method. He used the Euler product of the Riemann zeta-function and the generalized divisor function to estimate a Dirichlet polynomial. However, $\tilde{\eta}_m(s)$ does not have the representation of Euler product when $m \geq 1$, and so we cannot apply directly his method. To avoid this obstacle the author uses Radziwiłł's method [95] to estimate the Dirichlet polynomial.

3.2 Preliminaries

In this section, we prepare some lemmas.

Lemma 3.1. *Let $\theta \in \mathbb{R}$ be fixed. For any $n \in \mathbb{Z}_{\geq 2}$, we write $n = q_1^{\alpha_1}\dots q_r^{\alpha_r}$, where q_j are distinct prime numbers. Then we have*

$$\frac{1}{T}\int_{T}^{2T}\prod_{j=1}^{r}\left(\cos(t\log q_j + \theta)\right)^{\alpha_j}dt = f(n) + O\left(\frac{n}{T}\right)$$

for any $T > 0$. Here, f is the multiplicative function defined by $f(p^\alpha) = 2^{-\alpha}\binom{\alpha}{\alpha/2}$ for a prime power p^α, and we regard that $\binom{\alpha}{\alpha/2} = 0$ if α is odd.

Proof. This lemma is a special case of Lemma 6.7, and so we omit this proof. □

Lemma 3.2. *Let $m \in \mathbb{Z}_{\geq 0}$, $\frac{1}{2} \leq \sigma < 1$ be fixed. Let $X \geq 3$, and T be large. Then, for any positive integer k, we have*

$$\frac{1}{T}\int_T^{2T}\left(\mathrm{Re}\left(e^{-i\theta}\sum_{p\leq X}\frac{1}{p^{\sigma+it}(\log p)^m}\right)\right)^k dt$$
$$= \frac{k!}{2\pi i}\oint_{|w|=R}\frac{1}{w^{k+1}}\prod_{p\leq X} I_0\left(\frac{w}{p^\sigma(\log p)^m}\right)dw + O\left(\frac{X^{2k}}{T}\right).$$

Here, R is any positive number, and I_0 is the modified 0-th order Bessel function.

Proof. Define the multiplicative function $g_X(n)$ as, for every prime number p and $\alpha \in \mathbb{Z}_{\geq 1}$, $g_X(p^\alpha) = 1/\alpha!(\log p)^{\alpha m}$ if $p \leq X$, and $g_X(p^\alpha) = 0$ otherwise. By Lemma 3.1, we find that

$$\frac{1}{T}\int_T^{2T}\left(\mathrm{Re}\left(e^{-i\theta}\sum_{p\leq X}\frac{1}{p^{\sigma+it}(\log p)^m}\right)\right)^k dt$$
$$= \frac{1}{T}\sum_{p_1,\dots,p_k\leq X}\frac{\int_T^{2T}\cos(t\log p_1+\theta)\cdots\cos(t\log p_k+\theta)dt}{(p_1\cdots p_k)^\sigma(\log p_1\cdots\log p_k)^m}$$
$$= \sum_{p_1,\dots,p_k\leq X}\frac{f(p_1\cdots p_k)}{(p_1\cdots p_k)^\sigma(\log p_1\cdots\log p_k)^m} + O\left(\frac{X^{2k}}{T}\right).$$

From this equation and the definition of g_X, we have

$$\frac{1}{T}\int_T^{2T}\left(\mathrm{Re}\sum_{p\leq X}\frac{1}{p^{\sigma+it}(\log p)^m}\right)^k dt = k!\sum_{\Omega(n)=k}\frac{f(n)}{n^\sigma}g_X(n) + O\left(\frac{X^{2k}}{T}\right).$$

By Cauchy's integral formula, the above is equal to

$$\frac{k!}{2\pi i}\oint_{|w|=R}\sum_{n=1}^\infty\frac{f(n)}{n^\sigma}g_X(n)w^{\Omega(n)}\frac{dw}{w^{k+1}} + O\left(\frac{X^{2k}}{T}\right).$$

Since the functions f, g_X, and $w^{\Omega(n)}$ are multiplicative, this main term is

$$= \frac{k!}{2\pi i}\oint_{|w|=R}\frac{1}{w^{k+1}}\prod_{p\leq X}\left(\sum_{l=0}^\infty\left(\frac{(w/2p^\sigma(\log p)^m)^{2l}}{(l!)^2}\right)\right)dw$$
$$= \frac{k!}{2\pi i}\oint_{|w|=R}\frac{1}{w^{k+1}}\prod_{p\leq X} I_0\left(\frac{w}{p^\sigma(\log p)^m}\right)dw,$$

which completes the proof of this lemma. □

Lemma 3.3. *Let m be a fixed positive interger. For $x \geq 3$, $X \geq x^3$, we have*

$$\prod_{p\leq X} I_0\left(\frac{x}{\sqrt{p}(\log p)^m}\right)$$
$$= \exp\left(\frac{x^2}{8m(2\log x)^{2m}}\left(1-\left(\frac{\log x^2}{\log X}\right)^{2m} + O\left(\frac{\log\log x}{\log x}\right)\right)\right).$$

Proof. By the Taylor expansion of I_0 and the prime number theorem, we find that

$$\prod_{\frac{x^2}{(\log x)^{2m}}<p\leq X} I_0\left(\frac{x}{\sqrt{p}(\log p)^m}\right)$$
$$= \exp\left(\sum_{\frac{x^2}{(\log x)^{2m}}<p\leq X}\left(\frac{x^2}{4p(\log p)^{2m}} + O_m\left(\frac{x^4}{p^2(\log p)^{4m}}\right)\right)\right)$$
$$= \exp\left(\frac{x^2}{8m(2\log x)^{2m}}\left(1-\left(\frac{\log x^2}{\log X}\right)^{2m} + O_m\left(\frac{\log\log x}{\log x}\right)\right)\right). \tag{3.4}$$

On the other hand, by using the inequality $I_0(x) \leq \exp(x)$ and the prime number theorem, it holds that

$$\prod_{p\leq \frac{x^2}{(\log x)^{2m}}} I_0\left(\frac{x}{\sqrt{p}(\log p)^m}\right) \leq \exp\left(x\sum_{p\leq \frac{x^2}{(\log x)^{2m}}}\frac{1}{\sqrt{p}(\log p)^m}\right)$$
$$\leq \exp\left(O_m\left(\frac{x^2}{(\log x)^{2m+1}}\right)\right).$$

From this estimate and equation (3.4), we obtain this lemma. □

Lemma 3.4. *Let $\frac{1}{2} < \sigma < 1$, $m \in \mathbb{Z}_{\geq 0}$ be fixed. Then, for large x, $X \geq x^3$, we have*

$$\prod_{p\leq X} I_0\left(\frac{x}{p^\sigma(\log p)^m}\right) = \exp\left(\frac{\sigma^{\frac{m}{\sigma}}G(\sigma)x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}}\left(1+O\left(\frac{1+m\log\log x}{\log x}\right)\right)\right).$$

Proof. We take the numbers y_0, y_1 as satisfying the equations $y_0^\sigma(\log y_0)^m = x^{1/2}$, $y_1^\sigma(\log y_1)^m = x^{3/2}$, respectively. Then, it holds that $y_0 \asymp_m x^{\frac{1}{2\sigma}}(\log x)^{-\frac{m}{\sigma}}$, $y_1 \asymp_m x^{\frac{3}{2\sigma}}(\log x)^{-\frac{m}{\sigma}}$, and the estimate $X \gg y_1$ also holds. By the Taylor expansion of I_0 and the prime number theorem, we find that

$$\sum_{p\leq X}\log I_0\left(\frac{x}{p^\sigma(\log p)^m}\right) = \sum_{p\leq y_1}\log I_0\left(\frac{x}{p^\sigma(\log p)^m}\right) + O_{m,\sigma}\left(\frac{x^{\frac{3-2\sigma}{2\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}}\right).$$

By the inequality $I_0(x) \le \exp(x)$, it holds that

$$\sum_{p \le y_0} \log I_0\left(\frac{x}{p^\sigma (\log p)^m}\right) \le \sum_{p \le y_0} \frac{x}{p^\sigma (\log p)^m} \ll_{m,\sigma} \frac{x^{\frac{1-\sigma}{2\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}}.$$

From these estimates, one has

$$\begin{aligned} &\sum_{p \le X} \log I_0\left(\frac{x}{p^\sigma (\log p)^m}\right) \qquad (3.5)\\ &= \sum_{y_0 < p \le y_1} \log I_0\left(\frac{x}{p^\sigma (\log p)^m}\right) + O_{m,\sigma}\left(\frac{1}{(\log x)^{\frac{m}{\sigma}+1}}\left(x^{\frac{3-2\sigma}{2\sigma}} + x^{\frac{1+\sigma}{2\sigma}}\right)\right). \end{aligned}$$

By using partial summation and estimates of I_0, we obtain

$$\begin{aligned} &\sum_{y_0 < p \le y_1} \log I_0\left(\frac{x}{p^\sigma (\log p)^m}\right) \qquad (3.6)\\ &= -\int_{y_0+}^{y_1+} \pi(\xi)\left(\frac{d}{d\xi} \log I_0\left(\frac{x}{\xi^\sigma (\log \xi)^m}\right)\right) d\xi + O_m\left(\frac{x^{\frac{1+\sigma}{2\sigma}} + x^{\frac{3-2\sigma}{2\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}}\right). \end{aligned}$$

Applying the basic formula $\pi(\xi) = \int_2^\xi \frac{du}{\log u} + O(\xi e^{-c\sqrt{\log \xi}})$, we find that the first term on the right hand side is equal to

$$\int_{y_0}^{y_1} \frac{\log I_0\left(\frac{x}{\xi^\sigma (\log \xi)^m}\right)}{\log \xi} d\xi + O\left(\int_{y_0}^{y_1} e^{-c\sqrt{\log \xi}} \log I_0\left(\frac{x}{\xi^\sigma (\log \xi)^m}\right) d\xi\right). \qquad (3.7)$$

Note that we used the monotonicity of I_0 in the above deforming. By the estimate $I_0(x) \le \exp(x)$ and the Taylor expansion of $I_0(z)$, we find that

$$\begin{aligned} &\int_{y_0}^{y_1} e^{-c\sqrt{\log \xi}} \log I_0\left(\frac{x}{\xi^\sigma (\log \xi)^m}\right) d\xi\\ &\ll_m x \int_{y_0}^{\frac{x^{1/\sigma}}{(\log x)^{m/\sigma}}} \frac{d\xi}{\xi^\sigma (\log \xi)^{2m+3}} + x^2 \int_{\frac{x^{1/\sigma}}{(\log x)^{m/\sigma}}}^\infty \frac{d\xi}{\xi^{2\sigma} (\log \xi)^{2m+3}}\\ &\ll \frac{x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m}{\sigma}+2}}. \end{aligned}$$

Finally, we consider the first term of (3.7). By making the change of variables $u = \frac{x}{\xi^\sigma (\log \xi)^m}$, hard but not difficult calculations can lead that the first term of (3.7) is equal to

$$\begin{aligned} &\sigma^{m/\sigma} x^{1/\sigma} \int_{x^{-1/2}}^{x^{1/2}} \frac{(1 + O_m(\frac{m \log\log x}{\log x})) \log I_0(u)}{u^{1+\frac{1}{\sigma}} (\log (x/u))^{\frac{m}{\sigma}+1}} du\\ &= \sigma^{m/\sigma} x^{1/\sigma} \int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}} (\log (x/u))^{\frac{m}{\sigma}+1}} du + O_{m,\sigma}\left(\frac{m x^{1/\sigma} \log\log x}{(\log x)^{\frac{m}{\sigma}+2}}\right). \end{aligned}$$

Since $\frac{1}{(\log(x/u))^{m/\sigma+1}} = \frac{1+O_m(|\log u|/\log x)}{(\log x)^{m/\sigma+1}}$ for $x^{-1/2} \le u \le x^{1/2}$, we find that

$$\int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}(\log(x/u))^{\frac{m}{\sigma}+1}} du$$
$$= \frac{1}{(\log x)^{\frac{m}{\sigma}+1}} \int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}} du + O_m\left(\frac{1}{(\log x)^{\frac{m}{\sigma}+2}} \int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)|\log u|}{u^{1+\frac{1}{\sigma}}} du\right).$$

Moreover, by $I_0(x) \le \exp(x)$ and the Taylor expansion of I_0, it holds that

$$\int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}} du = \int_0^\infty \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}} du + O_\sigma\left(x^{\frac{1-2\sigma}{2\sigma}} + x^{\frac{\sigma-1}{2\sigma}}\right),$$

and that

$$\int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)|\log u|}{u^{1+\frac{1}{\sigma}}} du \ll_\sigma 1$$

for $\frac{1}{2} < \sigma < 1$. From the above calculations, equation (3.6) is

$$= \frac{\sigma^{\frac{m}{\sigma}} G(\sigma) x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}} \left(1 + O\left(\frac{1 + m \log\log x}{\log x}\right)\right).$$

Hence, by estimates (3.5), (3.6), (3.7), we obtain this lemma. □

Lemma 3.5. *Let $m \in \mathbb{Z}_{\ge 0}$, $\frac{1}{2} \le \sigma < 1$ be fixed with $(m, \sigma) \ne (0, 1/2)$. Let T, W be large numbers. Put $\kappa(\sigma) = 0$ if $\sigma = 1/2$, $\kappa(\sigma) = \sigma$ otherwise. Define the set $\mathcal{A} = \mathcal{A}(T, X, W; \sigma, m)$ by*

$$\mathcal{A} = \left\{ t \in [T, 2T] \ : \ \left| \sum_{p \le X} \frac{1}{p^{\sigma+it}(\log p)^m} \right| \le W \right\}. \tag{3.8}$$

Then, there exists a small positive constant $b_1 = b_1(\sigma, m) \le 1$ such that for any $3 \le X \le T^{1/W^{\frac{1}{1-\sigma}}(\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}}}$,

$$\frac{1}{T} \operatorname{meas}([T, 2T] \setminus \mathcal{A}) \ll \exp\left(-b_1 W^{\frac{1}{1-\sigma}} (\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}}\right).$$

Proof. Using the prime number theorem, we can obtain

$$\sum_{p \le k(\log k)^{2-\kappa(\sigma)}} \frac{1}{p^{\sigma+it}(\log p)^m} \ll_m \frac{k^{1-\sigma}}{(\log k)^{m+\kappa(\sigma)}}.$$

By Lemma 2.8, we have

$$\frac{1}{T} \int_T^{2T} \left| \sum_{k(\log k)^{2-\kappa(\sigma)} < p \le X} \frac{1}{p^{\sigma+it}(\log p)^m} \right|^{2k} dt$$
$$\ll k! \left(\sum_{p > k(\log k)^{2-\kappa(\sigma)}} \frac{1}{p^{2\sigma}(\log p)^{2m}} \right)^k \le \left(C_1 \frac{k^{1-\sigma}}{(\log k)^{m+\kappa(\sigma)}} \right)^{2k}$$

for $X^k \le T^{1/2}$, where $C_1 = C_1(\sigma, m)$ is a positive constant. Therefore, when $X^k \le T^{1/2}$ it holds that

$$\frac{1}{T}\int_T^{2T} \left| \sum_{p \le X} \frac{1}{p^{\sigma+it}(\log p)^m} \right|^{2k} dt \le \left(C_2 \frac{k^{1-\sigma}}{(\log k)^{m+\kappa(\sigma)}} \right)^{2k} \tag{3.9}$$

for some constant $C_2 = C_2(\sigma, m) > 0$. Hence, we have

$$\frac{1}{T}\,\mathrm{meas}([T, 2T] \setminus \mathcal{A}) \le \left(C_2 \frac{k^{1-\sigma}}{W(\log k)^{m+\kappa(\sigma)}} \right)^{2k}.$$

Choosing $k = [cW^{\frac{1}{1-\sigma}}(\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}}]$ with $c = c(\sigma, m)$ a suitably small constant, we obtain this lemma. □

Lemma 3.6. *Assume the same situation as in Lemma 3.5. There exists a small positive constant $b_2 = b_2(\sigma, m)$ such that for $3 \le x \le b_2 W^{\frac{\sigma}{1-\sigma}}(\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}}$, $x^3 \le X \le T^{1/W^{\frac{1}{1-\sigma}}(\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}}}$, we have*

$$\begin{aligned}
&\frac{1}{T}\int_{\mathcal{A}} \exp\left(x \,\mathrm{Re}\, e^{-i\theta} \sum_{p \le X} \frac{1}{p^{\sigma+it}(\log p)^m} \right) dt \\
&= \prod_{p \le X} I_0\left(\frac{x}{p^{\sigma}(\log p)^m} \right) + O\left(\exp\left(-xW\right)\right).
\end{aligned}$$

Proof. By the definition of $\mathcal{A}$ and the Stirling formula, we have

$$\begin{aligned}
&\int_A \exp\left(x \,\mathrm{Re}\, e^{-i\theta} \sum_{p \le X} \frac{1}{p^{\sigma+it}(\log p)^m} \right) dt \\
&= \sum_{k \le Y} \frac{x^k}{k!} \int_A \left(\mathrm{Re} \sum_{p \le X} \frac{e^{-i\theta}}{p^{\sigma+it}(\log p)^m} \right)^k dt + O\left(T \sum_{k > Y} \frac{1}{\sqrt{k}} \left(\frac{exW}{k} \right)^k \right),
\end{aligned} \tag{3.10}$$

where $Y = e^2 xW$. Here, an easy calculation for geometric sequence shows that the above O-term is $\ll T\exp\left(-e^2 xW\right)$. By using the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}
\int_A \left(\mathrm{Re} \sum_{p \le X} \frac{e^{-i\theta}}{p^{\sigma+it}(\log p)^m} \right)^k dt &= \int_T^{2T} \left(\mathrm{Re} \sum_{p \le X} \frac{e^{-i\theta}}{p^{\sigma+it}(\log p)^m} \right)^k dt + \\
&+ O\left((\mathrm{meas}([T, 2T] \setminus \mathcal{A}))^{1/2} \left(\int_T^{2T} \left| \sum_{p \le X} \frac{1}{p^{\sigma+it}(\log p)^m} \right|^{2k} dt \right)^{1/2} \right).
\end{aligned}$$

When $b_2 \le e^{-2}$, from estimate (3.9) and Lemma 3.5, this O-term is

$$\ll T \exp\left(-\frac{b_1}{2} W^{\frac{1}{1-\sigma}} (\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}} \right) \left(C_2 \frac{k^{1-\sigma}}{(\log k)^{m+\kappa(\sigma)}} \right)^k$$

for $k \leq Y$, where $C_2 = C_2(\sigma, m)$ is a positive constant. Also, it holds that

$$\sum_{0 \leq k \leq Y} \frac{x^k}{k!} \left(C_2 \frac{k^{1-\sigma}}{(\log k)^{m+\kappa(\sigma)}} \right)^k \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left(C_2 \frac{xY^{1-\sigma}}{(\log Y)^{m+\kappa(\sigma)}} \right)^k$$
$$\leq \exp\left(2b_2^{2-\sigma} C_2 W^{\frac{1}{1-\sigma}} (\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}} \right)$$

for any sufficiently large W. Therefore, choosing b_2 suitably small, we find that the right hand side is $\leq \exp\left(\frac{b_1}{6} W^{\frac{1}{1-\sigma}} (\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}}\right)$. Hence, we obtain

$$\sum_{k \leq Y} \frac{x^k}{k!} \int_A \left(\operatorname{Re} \sum_{p \leq X} \frac{e^{-i\theta}}{p^{\sigma+it} (\log p)^m} \right)^k dt$$
$$= \sum_{k \leq Y} \frac{x^k}{k!} \int_T^{2T} \left(\operatorname{Re} \sum_{p \leq X} \frac{e^{-i\theta}}{p^{\sigma+it} (\log p)^m} \right)^k dt +$$
$$+ O\left(T \exp\left(-\frac{b_1}{3} W^{\frac{1}{1-\sigma}} (\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}} \right) \right).$$

From these estimates, the left hand side of (3.10) is equal to

$$\sum_{k \leq Y} \frac{x^k}{k!} \int_T^{2T} \left(\operatorname{Re} \sum_{p \leq X} \frac{e^{-i\theta}}{p^{\sigma+it} (\log p)^m} \right)^k dt + O\left(T \exp\left(-e^2 xW \right) \right) \tag{3.11}$$

for any sufficiently large W when b_2 is suitably small. By Lemma 3.2, this main term is equal to

$$\frac{T}{2\pi i} \oint_{|w|=ex} \sum_{k \leq Y} \frac{x^k}{w^{k+1}} \prod_{p \leq X} I_0\left(\frac{w}{p^{\sigma} (\log p)^m} \right) dw. \tag{3.12}$$

By Lemmas 3.3 and 3.4, there exists a constant $C_4 = C_4(\sigma, m) > 0$ such that

$$\left| \prod_{p \leq X} I_0(w/p^{\sigma} (\log p)^m) \right| \leq I_0(R/p^{\sigma} (\log p)^m) \leq \exp\left(C_4 \frac{x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m+\kappa(\sigma)}{\sigma}}} \right).$$

Choosing b_2 as a suitably small constant, the right hand side is $\ll \exp(xW)$. Moreover, since we see that

$$\left| \sum_{k > Y} \frac{x^k}{w^{k+1}} \right| \ll \exp\left(-e^2 xW \right),$$

it holds that

$$\left| \sum_{k > Y} \frac{x^k}{w^{k+1}} \prod_{p \leq X} I_0\left(\frac{w}{\sqrt{p} (\log p)^m} \right) \right| \leq \exp\left(-xW \right)$$

for $|w| = ex$. Hence, (3.12) is equal to

$$\frac{T}{2\pi i}\oint_{|w|=ex}\sum_{k\leq Y}\frac{1}{w-x}\prod_{p\leq X} I_0\left(\frac{w}{p^{\sigma}(\log p)^m}\right) dw + O\left(T\exp\left(-xW\right)\right).$$

Thus, by this formula and equation (3.11) and using Cauchy's integral formula, we obtain

$$\begin{aligned}&\frac{1}{T}\int_A \exp\left(x \operatorname{Re} e^{-i\theta}\sum_{p\leq X}\frac{1}{p^{\sigma+it}(\log p)^m}\right) dt\\ &= \prod_{p\leq X} I_0\left(\frac{x}{p^{\sigma}(\log p)^m}\right) + O\left(\exp\left(-xW\right)\right),\end{aligned}$$

which completes the proof of this lemma. □

Lemma 3.7. *Let $m \in \mathbb{Z}_{\geq 1}$, $\frac{1}{2} \leq \sigma < 1$ be fixed. Let T be large, $X \geq 3$, and $\Delta > 0$. Define the set $\mathcal{B} = \mathcal{B}(T, X, \Delta; \sigma)$ by*

$$\mathcal{B} = \left\{ t \in [T, 2T] \ :\ \left|\tilde{\eta}_m(\sigma+it) - \sum_{2\leq n\leq X}\frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}}\right| \leq \Delta X^{1/2-\sigma}\right\}.$$

Then, for $0 < \Delta \leq \left(\frac{\log T}{(\log X)^{2(m+1)}}\right)^{\frac{m}{2m+1}}$, we have

$$\frac{1}{T}\operatorname{meas}([T, 2T]\setminus\mathcal{B}) \leq \exp\left(-b_3\Delta^2(\log X)^{2m}\right),$$

and for $\left(\frac{\log T}{(\log X)^{2(m+1)}}\right)^{\frac{m}{2m+1}} \leq \Delta \leq \frac{\log T}{(\log X)^{m+1}}$, we have

$$\frac{1}{T}\operatorname{meas}([T, 2T]\setminus\mathcal{B}) \leq \exp\left(-b_4(\Delta(\log T)^m)^{1/(m+1)}\right).$$

Here, b_3, b_4 are absolute positive constants.

Proof. By Lemma 2.1 and Theorem 2.6, we have

$$\begin{aligned}&\frac{1}{T}\int_T^{2T}\left|\tilde{\eta}_m(\sigma+it) - \sum_{2\leq n\leq X}\frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}}\right|^{2k} dt\\ &\ll C^k k!\frac{X^{k(1-2\sigma)}}{(\log X)^{2km}} + C^k k^{2k(m+1)}\frac{T^{\frac{1-2\sigma}{135}}}{(\log T)^{2km}}\end{aligned}$$

for $3 \leq X \leq T^{\frac{1}{135k}}$, where C is an absolute positive constant. Therefore, we obtain

$$\frac{1}{T}\operatorname{meas}([T, 2T]\setminus\mathcal{B}) \ll \left(\frac{Ck^{1/2}}{\Delta(\log X)^m}\right)^{2k} + \left(\frac{Ck^{m+1}}{\Delta(\log T)^m}\right)^{2k}.$$

When $\Delta \leq \left(\frac{\log T}{(\log X)^{2(m+1)}}\right)^{\frac{m}{2m+1}}$, putting $k = [c\Delta^2(\log X)^{2m}] + 1$ with c a suitably small constant, we have

$$\frac{1}{T}\,\mathrm{meas}([T, 2T] \setminus \mathcal{B}) \leq \exp\left(-b_3\Delta^2(\log X)^{2m}\right)$$

for some absolute constant $b_3 > 0$. When the inequality $\left(\frac{\log T}{(\log X)^{2(m+1)}}\right)^{\frac{m}{2m+1}} \leq \Delta \leq \frac{\log T}{(\log X)^{m+1}}$ holds, by choosing $k = \left[c(\Delta(\log T)^m)^{\frac{1}{m+1}}\right] + 1$ with c a suitably small constant, we have

$$\frac{1}{T}\,\mathrm{meas}([T, 2T] \setminus \mathcal{B}) \leq \exp\left(-b_4(\Delta(\log T)^m)^{1/(m+1)}\right)$$

for some absolute constant $b_4 > 0$. Thus, we obtain this lemma. □

3.3 Proofs of Proposition 3.1 and Theorem 3.1

In this section, we prove Proposition 3.1 and Theorem 3.1.

Proof of Proposition 3.1. Let $m \in \mathbb{Z}_{\geq 1}$, $\theta \in \mathbb{R}$ be fixed. Let T, V be large numbers with $V \leq a_2\frac{\sqrt{\log T}}{(\log\log T)^{m+\frac{1}{2}}}$, and let X be a real parameter with $V^4 \leq X \leq T^{a_3/V^2(\log V)^{2m}}$. Here, $a_2 = a_2(m)$, $a_3 = a_3(m)$ are positive constants to be chosen later. Moreover, let $W > 0$, $3 \leq x \leq b_2W(\log W)^{2m}$ be numbers to be chosen later, where $b_2 = b_2(1/2, m)$ is the same constant as in Lemma 3.6. Put

$$\mathcal{S}^*(T, V) := \left\{t \in \mathcal{A} \ : \ \mathrm{Re}\, e^{-i\theta}\sum_{p \leq X}\frac{1}{p^{1/2+it}(\log p)^m} > V\right\}.$$

Here, the set $\mathcal{A} = \mathcal{A}(T, X, W; 1/2, m)$ is defined by (3.8). Then, for $x > 0$, we have

$$\int_A \exp\left(x\,\mathrm{Re}\, e^{-i\theta}\sum_{p \leq X}\frac{1}{p^{1/2+it}(\log p)^m}\right)dt = x\int_{-\infty}^{\infty} e^{xv}\,\mathrm{meas}(\mathcal{S}^*(T, v))dv.$$

By this equation and Lemma 3.6, it holds that

$$\frac{1}{T}\int_{-\infty}^{\infty} e^{xv}\,\mathrm{meas}(\mathcal{S}^*(T, v))dv = \frac{1}{x}\prod_{p \leq X} I_0\left(\frac{x}{p^{\sigma}(\log p)^m}\right) + O\left(\frac{1}{x}\exp\left(-xW\right)\right)$$

when $x^3 \leq X \leq T^{1/W^2(\log W)^{2m}}$. Therefore, by Lemma 3.3, we obtain

$$\begin{aligned}&\frac{1}{T}\int_{-\infty}^{\infty} e^{xv}\,\mathrm{meas}(\mathcal{S}^*(T, v))dv \\ &= \exp\left(\frac{x^2}{8m(2\log x)^{2m}}\left(1 - \left(\frac{\log x^2}{\log X}\right)^{2m} + O_m\left(\frac{\log\log x}{\log x}\right)\right)\right)\end{aligned} \tag{3.13}$$

for $x^3 \le X \le T^{1/W^2(\log W)^{2m}}$. Now, we decide the parameters x, W as satisfying the equations

$$V = \frac{2x}{8m(2\log x)^{2m}}\left(1 - \left(\frac{\log x^2}{\log X}\right)^{2m}\right),$$

and $W = 8m4^m K_1 V$, respectively. The constant $K_1 = K_1(m)$ is defined as $K_1 = \max\{b_1^{-1}, b_2^{-1}\}$, and b_1 is the same constant as in Lemma 3.5. Then, this x satisfies

$$x = \frac{4m4^m}{1 - (\log V^2/\log X)^{2m}} V(\log V)^{2m}\left(1 + O_m(\log\log V/\log V)\right),$$

and hence we can take out x from the range $3 \le x \le b_2 W(\log W)^{2m}$ for any large V. Also, when a_2, a_3 are suitably small, the inequalities $x^3 \le T^{1/W^2(\log W)^{2m}}$ and $x^3 \le X \le T^{1/W^2(\log W)^{2m}}$ hold for any large V. Moreover, by using Lemma 3.5, $\mathrm{meas}([T, 2T] \setminus \mathcal{A}) \le T\exp\left(-8m4^m V^2(\log V)^{2m}\right)$ holds. Therefore, we obtain

$$\begin{aligned}
&\frac{1}{T}\,\mathrm{meas}\left\{t \in [T, 2T] \ :\ \mathrm{Re}\, e^{-i\theta}\sum_{p \le X}\frac{1}{p^{1/2+it}(\log p)^m} > V\right\}\\
&= \frac{1}{T}\,\mathrm{meas}(\mathcal{S}^*(T, V)) + O\left(\frac{1}{T}\,\mathrm{meas}([T, 2T] \setminus \mathcal{A})\right)\\
&= \frac{1}{T}\,\mathrm{meas}(\mathcal{S}^*(T, V)) + O\left(\exp\left(-8m4^m V^2(\log V)^{2m}\right)\right). \qquad (3.14)
\end{aligned}$$

Put $\varepsilon = K_2\sqrt{\log\log x/\log x}$ with $K_2 = K_2(m)$ a sufficiently large constant. Then, by using equation (3.13), we find that

$$\begin{aligned}
&\int_{-\infty}^{V(1-\varepsilon)} e^{xv}\,\mathrm{meas}(\mathcal{S}^*(T, v))dv \le e^{\varepsilon x V(1-\varepsilon)}\int_{-\infty}^{\infty} e^{x(1-\varepsilon)v}\,\mathrm{meas}(\mathcal{S}^*(T, v))dv\\
&= T\exp\left(\frac{x^2}{8m(2\log x)^{2m}}\left(1 - \left(\frac{\log x^2}{\log X}\right)^{2m} - \frac{\varepsilon^2}{3} + O_m\left(\frac{\log\log x}{\log x}\right)\right)\right)\\
&\le \frac{1}{3}\int_{-\infty}^{\infty} e^{xv}\,\mathrm{meas}(\mathcal{S}^*(T, v))dv.
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
&\int_{V(1+\varepsilon)}^{\infty} e^{xv}\,\mathrm{meas}(\mathcal{S}^*(T, v))dv \le e^{-\varepsilon x V(1+\varepsilon)}\int_{-\infty}^{\infty} e^{x(1+\varepsilon)v}\,\mathrm{meas}(\mathcal{S}^*(T, v))dv\\
&= T\exp\left(\frac{x^2}{8m(2\log x)^{2m}}\left(1 - \left(\frac{\log x^2}{\log X}\right)^{2m} - \frac{\varepsilon^2}{3} + O_m\left(\frac{\log\log x}{\log x}\right)\right)\right)\\
&\le \frac{1}{3}\int_{-\infty}^{\infty} e^{xv}\,\mathrm{meas}(\mathcal{S}^*(T, v))dv.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{1}{T}\int_{V(1-\varepsilon)}^{V(1+\varepsilon)} e^{xv}\operatorname{meas}(\mathcal{S}^*(T,v))dv\\
&= \exp\left(\frac{x^2}{8m(2\log x)^{2m}}\left(1-\left(\frac{\log x^2}{\log X}\right)^{2m} + O_m\left(\frac{\log\log x}{\log x}\right)\right)\right).
\end{aligned}$$

Moreover, since $\operatorname{meas}(\mathcal{S}^*(T,v))$ is a nonincreasing function with respect to v and $\int_{V(1-\varepsilon)}^{V(1+\varepsilon)} e^{xv}dv = \exp(xV(1+O(\varepsilon)))$, it holds that

$$\begin{aligned}
&\frac{1}{T}\operatorname{meas}(\mathcal{S}^*(T,V(1+\varepsilon)))\\
&\leq \exp\left(-\frac{x^2}{8m(2\log x)^{2m}}\left(1-\left(\frac{\log x^2}{\log X}\right)^{2m} + O_m\left(\sqrt{\frac{\log\log x}{\log x}}\right)\right)\right)\\
&\leq \frac{1}{T}\operatorname{meas}(\mathcal{S}^*(T,V(1-\varepsilon))).
\end{aligned}$$

In particular, since x satisfies

$$x = 4mV(2\log V)^{2m}\left\{\left(1+(\log x^2/\log X)^{2m}\right)^{-1} + O_m(\log\log V/\log V)\right\},$$

the second term of the above inequalities is equal to

$$\exp\left(-\frac{2m4^m}{1-\left(\frac{\log V^2}{\log X}\right)^m}V^2(\log V)^{2m}\left(1+O_m\left(\sqrt{\frac{\log\log V}{\log V}}\right)\right)\right).$$

Additionally, if we change the above V to $V(1+O(\varepsilon))$, the above form does not change. Hence, we obtain

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}^*(T,V)) = \exp\left(-\frac{2m4^mV^2(\log V)^{2m}}{1-\left(\frac{\log V^2}{\log X}\right)^m}\left(1+O_m\left(\sqrt{\frac{\log\log V}{\log V}}\right)\right)\right).$$

By this equation and (3.14), we complete the proof of Proposition 3.1. □

Proof of Theorem 3.1. Let T, V be sufficiently large parameters satisfying $V \leq a_1\left(\frac{\log T}{(\log\log T)^{2m+2}}\right)^{\frac{m}{2m+1}}$, where $a_1 = a_1(m)$ is a suitably small constant to be chosen later. Let a_3, b_4 be the same constants as in Proposition 3.1 and Lemma 3.7. Put $X = T^{b_5/V^2(\log V)^{2m}}$ with $b_5 = \min\{a_3, b_4(4m4^m)^{-1}\}$. Note that this X satisfies the inequality $X \geq \exp\left((\log T)^{\frac{1}{2m+1}-\varepsilon}\right) \geq V^4$ when T is large. Then, applying Lemma 3.7 as $\Delta = \frac{\log T}{(\log X)^{m+1}} = \frac{V^{2m+2}(\log V)^{2m(m+1)}}{b_5^{m+1}(\log T)^m}$, we find that there exists a set $\mathcal{B} \subset [T, 2T]$ such that $\operatorname{meas}([T, 2T] \setminus \mathcal{B}) \leq T \exp\left(-4m4^m V^2 (\log V)^{2m}\right)$, and for all $t \in \mathcal{B}$

$$\left|\tilde{\eta}_m(1/2 + it) - \sum_{p \leq X} \frac{1}{p^{1/2+it}(\log p)^m}\right| \leq \left(\frac{V^{2m+1}(\log V)^{2m(m+1)}}{b_6^{m+1}(\log T)^m} + \frac{c}{V}\right) V$$
$$=: \delta_m V,$$

say. Here the constant c indicates the value $\sum_{p^k, k \geq 2} \frac{1}{p^{k/2}(\log p^k)^m}$. Now, we decide the number a_1 such that $\delta_m \leq 1/2$. Then, it holds that

$$\begin{aligned}
&\operatorname{meas}\left\{t \in \mathcal{B} \ : \ \operatorname{Re} e^{-i\theta} \sum_{p \leq X} \frac{1}{p^{1/2+it}(\log p)^m} > V(1 + \delta_m)\right\} \\
&\leq \operatorname{meas}\left\{t \in \mathcal{B} \ : \ \operatorname{Re} e^{-i\theta} \tilde{\eta}_m(1/2 + it) > V\right\} \\
&\leq \operatorname{meas}\left\{t \in \mathcal{B} \ : \ \operatorname{Re} e^{-i\theta} \sum_{p \leq X} \frac{1}{p^{1/2+it}(\log p)^m} > V(1 - \delta_m)\right\}.
\end{aligned}$$

Hence, by these inequalities and Proposition 3.1, we have

$$\begin{aligned}
&\frac{1}{T} \operatorname{meas}\left\{t \in \mathcal{B} \ : \ \operatorname{Re} e^{-i\theta} \tilde{\eta}_m(1/2 + it) > V\right\} = \\
&\exp\left(-2m4^m V^2 (\log V)^{2m} \left(1 + O_m\left(\frac{V^{2m+1}(\log V)^{2m(m+1)}}{(\log T)^m} + \sqrt{\frac{\log\log V}{\log V}}\right)\right)\right).
\end{aligned}$$

Thus, by this equation and $\operatorname{meas}([T, 2T] \setminus \mathcal{B}) \leq T \exp\left(-4m4^m V^2 (\log V)^{2m}\right)$, we complete the proof of Theorem 3.1. □

3.4 Proofs of Proposition 3.2 and Theorem 3.2

Some parts in the proof of Proposition 3.2 are written briefly because many points are similar to the proof of Proposition 3.1.

Proof of Proposition 3.2. Let $m \in \mathbb{Z}_{\geq 0}$, $\frac{1}{2} < \sigma < 1$ be fixed. Let T, V be large numbers with $V \leq a_5 \frac{(\log T)^{1-\sigma}}{(\log\log T)^{m+1}}$, and let X be a real parameter with $V^{\frac{4}{1-\sigma}} \leq X \leq T^{a_6 / V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}}}$. Here $a_5 = a_5(\sigma, m)$, $a_6 = a_6(\sigma, m)$ are positive constants to be chosen later. Moreover, let $W > 0$, $3 \leq x \leq b_2 W^{\frac{\sigma}{1-\sigma}} (\log W)^{\frac{m+\sigma}{1-\sigma}}$ be numbers to be chosen later. Here, $b_2 = b_2(\sigma, m)$ is the same constant as in Lemma 3.6. Put

$$\mathcal{S}_\sigma^*(T, V) := \left\{t \in \mathcal{A} \ : \ \operatorname{Re} e^{-i\theta} \sum_{p \leq X} \frac{1}{p^{\sigma+it}(\log p)^m} > V\right\},$$

where $\mathcal{A} = \mathcal{A}(T, X, V; \sigma, m)$ is the set defined by (3.8). Using Lemmas 3.4, 3.6, and the equation

$$\int_{\mathcal{A}} \exp\left(x \operatorname{Re} e^{-i\theta} \sum_{p \leq X} \frac{1}{p^\sigma (\log p)^m}\right) = x \int_{-\infty}^{\infty} e^{xv} \operatorname{meas}(\mathcal{S}_\sigma^*(T, v)) dv,$$

we obtain

$$\frac{1}{T}\int_{-\infty}^{\infty} e^{xv}\operatorname{meas}(\mathcal{S}_\sigma^*(T,v))dv \tag{3.15}$$
$$= \exp\left(\frac{\sigma^{\frac{m}{\sigma}}G(\sigma)x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}}\left(1+O\left(\frac{1+m\log\log x}{\log x}\right)\right)\right)$$

for $x^3 \leq X \leq T^{1/W^{\frac{1}{1-\sigma}}(\log W)^{\frac{m+\sigma}{1-\sigma}}}$. Here, we decide the parameters x, W as the numbers satisfying the equations

$$V = \frac{\sigma^{\frac{m}{\sigma}}G(\sigma)x^{\frac{1}{\sigma}-1}}{\sigma(\log x)^{\frac{m}{\sigma}+1}},$$

and $W = \left(2\frac{A_m(\sigma)}{1-\sigma}K_3\right)^{\frac{1-\sigma}{\sigma}}V$, respectively. The constant $K_3 = K_3(\sigma,m)$ is defined as $K_3 = \max\{b_1^{-1}, b_2^{-1}\}$, where b_1 is the same constant as in Lemma 3.5. Then, this x satisfies $x = \frac{A_m(\sigma)}{1-\sigma}V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}(1+O(\log\log V/\log V))$, and so we can pick up this x from the range $3 \leq x \leq b_2 W^{\frac{\sigma}{1-\sigma}}(\log W)^{\frac{m+\sigma}{1-\sigma}}$ for any large V. Also, choosing a_5, a_6 as suitably small constants, we find that the inequalities $x^3 \leq T^{1/W^{\frac{1}{1-\sigma}}(\log W)^{\frac{m+\sigma}{1-\sigma}}}$ and $x^3 \leq X \leq T^{1/W^{\frac{1}{1-\sigma}}(\log W)^{\frac{m+\sigma}{1-\sigma}}}$ hold for any large V. Moreover, by Lemma 3.5, the inequality $\operatorname{meas}([T,2T]\setminus\mathcal{A}) \leq T\exp\left(-2A_m(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right)$ holds.

Putting $\varepsilon = K_4\sqrt{\frac{1+m\log\log x}{\log x}}$ with $K_4 = K_4(\sigma,m)$ a suitably large constant and using equation (3.15), we have

$$\int_{-\infty}^{V(1-\varepsilon)} e^{xv}\operatorname{meas}(\mathcal{S}_\sigma^*(T,v;X))dv \leq \frac{1}{3}\int_{-\infty}^{\infty} e^{xv}\operatorname{meas}(\mathcal{S}_\sigma^*(T,v;X))dv,$$

and

$$\int_{V(1+\varepsilon)}^{\infty} e^{xv}\operatorname{meas}(\mathcal{S}_\sigma^*(T,v;X))dv \leq \frac{1}{3}\int_{-\infty}^{\infty} e^{xv}\operatorname{meas}(\mathcal{S}_\sigma^*(T,v;X))dv.$$

Therefore, we obtain

$$\frac{1}{T}\int_{(1-\varepsilon)V}^{(1+\varepsilon)V} e^{xv}\operatorname{meas}(\mathcal{S}_\sigma^*(T,v))dv$$
$$= \exp\left(\frac{\sigma^{\frac{m}{\sigma}}G(\sigma)x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}}\left(1+O\left(\frac{1+m\log\log x}{\log x}\right)\right)\right).$$

Moreover, since $\operatorname{meas}(\mathcal{S}^*(T,v))$ is a nonincreasing function, and the equation $\int_{V(1-\varepsilon)}^{V(1+\varepsilon)} e^{xv}dv = \exp(xV(1+O(\varepsilon)))$ holds, we have

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}^*(T,V(1+\varepsilon);X))$$
$$\leq \exp\left(-\frac{1-\sigma}{\sigma}\frac{\sigma^{\frac{m}{\sigma}}G(\sigma)x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}}(1+O(\varepsilon))\right)$$
$$\leq \frac{1}{T}\operatorname{meas}(\mathcal{S}^*(T,V(1-\varepsilon);X)).$$

In particular, as x is the solution of equation (3.15), the above second term is equal to

$$\exp\left(-A_m(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}(1+R)\right),$$

where

$$R \ll \sqrt{\frac{1+m\log\log x}{\log x}} \ll \sqrt{\frac{1+m\log\log V}{\log V}}.$$

Additionally, if we change the above V to $V(1+O(\varepsilon))$, the above form does not change. Thus, we obtain

$$\begin{aligned}&\frac{1}{T}\operatorname{meas}(\mathcal{S}^*(T,V;X))\\&=\exp\left(-A_m(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\left(1+O\left(\sqrt{\frac{1+m\log\log V}{\log V}}\right)\right)\right).\end{aligned}$$

By this equation and $\operatorname{meas}([T,2T]\setminus\mathcal{A}) \leq T\exp\left(-2A_m(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right)$, we obtain Proposition 3.2. □

Proof of Theorem 3.2. We show only the case $m \geq 1$ because the case $m = 0$ can be shown similarly by use of Lemma 2.2 in [36] instead of Lemma 3.7.

Let $m \in \mathbb{Z}_{\geq 1}$, $1/2 < \sigma < 1$. Let a_5, a_6, and b_4 be the same constants as in Proposition 3.2 and Lemma 3.7. Let T, V be sufficiently large positive numbers satisfying the inequality $V \leq a_4\frac{(\log T)^{1-\sigma}}{(\log\log T)^{m+1}}$, where $a_4 = a_4(\sigma,m)$ is a suitably small constant less than a_5 to be chosen later. Put $X = T^{b_6/V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}}$ with $b_6 = \min\{a_6, b_4(2A_m(\sigma))^{-1}\}$. Then we decide the number a_4 as satisfying $X^{\sigma-1/2} \geq (\log T)^6$. Applying Lemma 3.7 as $\Delta = \frac{\log T}{(\log X)^{m+1}} = \frac{\left(V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right)^{m+1}}{b_6^{m+1}(\log T)^m}$, we find that there exists a set $\mathcal{B} \subset [T,2T]$ such that $\operatorname{meas}([T,2T]\setminus\mathcal{B}) \leq T\exp\left(-2A_m(\sigma)V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right)$, and for all $t \in \mathcal{B}$

$$\left|\tilde{\eta}_m(\sigma+it) - \sum_{p\leq X}\frac{1}{p^{\sigma+it}(\log p)^m}\right| \leq \frac{\left(V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right)^{m+1}}{X^{\sigma-1/2}b_6^{m+1}(\log T)^m} + c.$$

Here, $c = \sum_{p^k, k\geq 2}\frac{\Lambda(p^k)}{p^{k\sigma}(\log p^k)^{m+1}}$. Therefore, the right hand side is $\leq K_4$ with $K_4 = K_4(m,\sigma)$ a positive constant. Then, it holds that

$$\begin{aligned}&\operatorname{meas}\left\{t\in\mathcal{B} \;:\; \operatorname{Re} e^{-i\theta}\sum_{p\leq X}\frac{1}{p^{\sigma+it}(\log p)^m} > V(1+K_4V^{-1})\right\}\\&\leq \operatorname{meas}\left\{t\in\mathcal{B} \;:\; \operatorname{Re} e^{-i\theta}\tilde{\eta}_m(\sigma+it) > V\right\}\\&\leq \operatorname{meas}\left\{t\in\mathcal{B} \;:\; \operatorname{Re} e^{-i\theta}\sum_{p\leq X}\frac{1}{p^{\sigma+it}(\log p)^m} > V(1-K_4V^{-1})\right\}.\end{aligned}$$

Hence, by these inequalities and Proposition 3.1, we have

$$\begin{aligned}
&\frac{1}{T} \operatorname{meas}\left\{t \in \mathcal{B} \ : \ \operatorname{Re} e^{-i\theta} \tilde{\eta}_m(\sigma + it) > V\right\} \\
&= \exp\left(-A_m(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}} \left(1 + O\left(\sqrt{\frac{1 + m \log\log V}{\log V}}\right)\right)\right).
\end{aligned}$$

By this equation and $\operatorname{meas}([T, 2T] \setminus \mathcal{B}) \leq T \exp\left(-2A_m(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}}\right)$, we complete the proof of Theorem 3.2. □

Chapter 4 Denseness of $\eta_m(s)$ and $\tilde{\eta}_m(s)$

In this chapter, we prove some results for the denseness of $\eta_m(s)$ and $\tilde{\eta}_m(s)$. The contents in this chapter are based on the paper [22].

4.1 Results

The main results in this chapter is the following.

Theorem 4.1. *Let $1/2 \le \sigma < 1$. If the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\beta > \sigma$ is finite, then the set*

$$\left\{ \int_0^t \log \zeta(\sigma + it') dt' \ : \ t \in [0, \infty) \right\}$$

is dense in the complex plane. Moreover, for each integer $m \ge 2$, the following statements are equivalent.

(I) The Riemann zeta-function does not have any zero whose real part is greater than σ.

(II) The set $\{\eta_m(\sigma + it) \ : \ t \in [0, \infty)\}$ is dense in the complex plane.

From this theorem, we see that the Riemann Hypothesis implies that the set

$$\left\{ \int_0^t \log \zeta(1/2 + it') dt' \ : \ t \in [0, \infty) \right\}$$

is dense in the complex plane. This implication seems to suggest that the values $\zeta(\frac{1}{2} + it)$ in $t \in \mathbb{R}$ are dense in $\mathbb{C}$. Moreover, the equivalence above would be a new type of statement which gives the relation between the denseness of values of the Riemann zeta-function and the Riemann Hypothesis.

Here, we mention the plan of the proof of Theorem 4.1 briefly. Recall that the function $\tilde{\eta}_m(\sigma + it)$ is defined by the recurrence equation

$$\tilde{\eta}_m(\sigma + it) = \int_\sigma^\infty \tilde{\eta}_{m-1}(\alpha + it) d\alpha,$$

where $\tilde{\eta}_0(\sigma + it) = \log \zeta(\sigma + it)$. This function is the m-times iterated integral of $\log \zeta(\sigma + it)$ on the horizontal line. Our main focus in this paper is better understanding of Ramachanra's denseness problem, and the value-distribution of $\eta_m(1/2+it)$. However, the function $\tilde{\eta}_m(s)$ is regular in the same region as in the case of $\log\zeta(s)$, and also some properties of this function are similar to $\log\zeta(s)$. Additionally, by Proposition 1.1, the behavior $\tilde{\eta}_m(s)$ on the critical line is directly related to the Lindelöf Hypothesis. From this observation, this function would be an interesting object itself, and we obtain the following theorem unconditionally.

Theorem 4.2. *Let $1/2 \leq \sigma < 1$, and m be a positive integer. Let T_0 be any positive number. Then the set*

$$\left\{\tilde{\eta}_m(\sigma + it) \ : \ t \in [T_0, \infty)\right\}$$

is dense in the complex plane.

Theorem 4.1 can be obtained from Theorem 4.2 and Lemma 2.1. Hence, our first purpose is to show Theorem 4.2. In the proof of Theorem 4.2, the following two propositions play an important role.

The function $\mathrm{Li}_m(z)$ indicates the polylogarithmic function defined as $\sum_{n=1}^{\infty} \frac{z^n}{n^m}$ for $|z| < 1$.

Proposition 4.1. *Let m be a positive integer. Then for any $\sigma \geq 1/2$, $T \geq X^{135}$, $\varepsilon > 0$, we have*

$$\lim_{X \to +\infty} \frac{1}{T} \operatorname{meas}\left\{t \in [0, T] \ : \ \left|\tilde{\eta}_m(\sigma + it) - \sum_{p \leq X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma - it})}{(\log p)^m}\right| < \varepsilon\right\} = 1.$$

The important point of this proposition is that $\tilde{\eta}_m(s)$ can be approximated by the Dirichlet polynomial even on the critical line. To prove this proposition, we must control exactly the contribution of nontrivial zeros of $\zeta(s)$, and we therefore need a strong zero density estimate of the Riemann zeta-function like Selberg's result [107, Theorem 1]. More precisely, we require that there exist numbers $c > 0$, $A < 2m + 1$ such that

$$N(\sigma, T) \ll T^{1 - c(\sigma - 1/2)} (\log T)^A$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$. Here, $N(\sigma, T)$ is the number of zeros of $\zeta(s)$ with multiplicity satisfying $\beta > \sigma$ and $0 < \gamma \leq T$. Therefore, to prove Proposition 4.1, we need a strong zero density estimate comparable to the assumption by Bombieri and Hejhal [9]. On the other hand, when we discuss the denseness of $\tilde{\eta}_m(s)$ for fixed $\frac{1}{2} < \sigma < 1$, it suffices to use the weaker estimate

$$N(\sigma, T) \ll T^{1 - c(\sigma - 1/2) + \varepsilon}$$

for every $\varepsilon > 0$. Hence, there is an essential difference of the depth between the discussion in the case $\frac{1}{2} < \sigma < 1$ and that in the case $\sigma = \frac{1}{2}$ in Proposition 4.1. We will explain this discussion more closely later.

In contrast, we can prove the following proposition by almost the same method as in [6], [7].

Proposition 4.2. *Let m be a positive integer, $1/2 \le \sigma < 1$. Let a be any complex number, and ε be any positive number. If we take a sufficiently large number $N_0 = N_0(m, \sigma, a, \varepsilon)$, then, for any integer $N \ge N_0$, there exists some Jordan measurable set $\Theta_0 = \Theta_0(m, \sigma, a, \varepsilon, N) \subset [0, 1)^{\pi(N)}$ with* $\mathrm{meas}(\Theta_0) > 0$ *such that*

$$\left| \sum_{p \le N} \frac{\mathrm{Li}_{m+1}(p^{-\sigma} \exp(-2\pi i \theta_p))}{(\log p)^m} - a \right| < \varepsilon.$$

for any $\underline{\theta} = \left(\theta_{p_n}\right)_{n=1}^{\pi(N)} \in \Theta_0$.

Roughly speaking, Proposition 4.1 means that $\tilde{\eta}_m(\sigma + it)$ "almost" equals the finite sum of polylogarithmic functions when the number of the terms of the sum is sufficiently large, and Proposition 4.2 that any complex number can be approximated by the finite sum of polylogarithmic functions when the number of the terms of the sum is sufficiently large.

Bohr developed his denseness results with Jessen from the viewpoint of probability theory in [8]. Following their method, the author will continue our study in the next chapter. They will give deeper results such as an analog of Lamzouri's study [66] and the study of Lamzouri, Lester, and Radziwiłł [67].

4.2 Approximation of $\eta_m(s)$ and $\tilde{\eta}_m(s)$ by Dirichlet polynomials

In this section, we prove Proposition 4.1. In order to prove it, we prepare two lemmas.

Lemma 4.1. *Let m be a positive integer, and $\sigma \ge 1/2$. Let T be large. Then, for $3 \le X \le T^{\frac{1}{135}}$, we have*

$$\frac{1}{T} \int_{14}^{T} \left| \tilde{\eta}_m(\sigma + it) - \sum_{2 \le n \le X} \frac{\Lambda(n)}{n^{\sigma + it} (\log n)^{m+1}} \right|^2 dt \ll \frac{X^{1-2\sigma}}{(\log X)^{2m}}.$$

This lemma is a special case the following lemma.

Lemma 4.2. *Let m, k be positive integers. Let T be large, and $X \ge 3$ with $X \le T^{\frac{1}{135k}}$. Then, for $\sigma \ge 1/2$, we have*

$$\int_{14}^{T} \left| \tilde{\eta}_m(\sigma + it) - \sum_{2 \le n \le X} \frac{\Lambda(n)}{n^{\sigma + it} (\log n)^{m+1}} \right|^{2k} dt$$
$$\ll 2^k k! \left(\frac{2m+1}{2m} + \frac{C}{\log X} \right)^k \frac{X^{k(1-2\sigma)}}{(\log X)^{2km}} + C^k k^{2k(m+1)} \frac{T^{\frac{1-2\sigma}{135}}}{(\log T)^{2km}}. \quad (4.1)$$

This lemma is Theorem 2.6. As we mentioned in the previous section, the proof of this lemma requires a strong zero density estimate like Selberg's result. In fact, if we only knew the estimate

$$N(\sigma, T) \ll T^{1 - c(\sigma - 1/2)} (\log T)^A$$

for some $c > 0$, $A \geq 1$, then the right hand side of (4.1) in the case $k = 1$ becomes

$$O\left(\frac{X^{1-2\sigma}}{(\log X)^{2m}} + \frac{T^{\frac{1-2\sigma}{135}}}{(\log T)^{2m+1-A}}\right).$$

Hence, the power of the logarithmic factor of the zero density estimate plays an important role in the case $\sigma = 1/2$.

Lemma 4.3. *Let m be an integer, $\sigma \geq 1/2$. Let T be large. Then for $3 \leq X \leq T^{1/4}$, we have*

$$\frac{1}{T}\int_0^T \left|\sum_{p \leq X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}}\right|^2 dt \ll \frac{X^{1-2\sigma}}{(\log X)^{2m+1}},$$

where the function $\Lambda(n)$ is the von Mangoldt function.

Proof. By definitions of the polylogarithmic function and the von Mangoldt function, we find that

$$\sum_{p \leq X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} = \sum_{p \leq X} \sum_{k > \frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m}$$

$$= \sum_{p \leq X} \sum_{\frac{\log X}{\log p} < k \leq 3\frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m} + O\left(\frac{X^{1-3\sigma}}{(\log X)^m}\right).$$

Here, we can write

$$\left|\sum_{p \leq X} \sum_{\frac{\log X}{\log p} < k \leq 3\frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m}\right|^2$$

$$= \sum_{p \leq X} \sum_{\frac{\log X}{\log p} < k \leq 3\frac{\log X}{\log p}} \frac{p^{-2k\sigma}}{k^{2(m+1)}(\log p)^{2m}} +$$

$$+ \sum_{p_1 \leq X} \sum_{p_2 \leq X} \sum_{\substack{\frac{\log X}{\log p_1} < k_1 \leq 3\frac{\log X}{\log p_1} \\ (p_1,k_1) \neq (p_2,k_2)}} \sum_{\frac{\log X}{\log p_2} < k_2 \leq 3\frac{\log X}{\log p_2}} \frac{(p_1^{k_1} p_2^{k_2})^{-\sigma}(p_1^{k_1}/p_2^{k_2})^{-it}}{(k_1 k_2)^{m+1}(\log p_1 \log p_2)^m}.$$

Therefore, it holds that

$$
\begin{aligned}
&\int_0^T \left| \sum_{p \le X} \sum_{\frac{\log X}{\log p} < k \le 3\frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m} \right|^2 dt \\
&= T \sum_{p \le X} \sum_{\frac{\log X}{\log p} < k \le 3\frac{\log X}{\log p}} \frac{p^{-2k\sigma}}{k^{2(m+1)}(\log p)^{2m}} + \\
&\qquad\qquad + O\left(X^3 \left(\sum_{p \le X} \sum_{\frac{\log X}{\log p} < k \le 3\frac{\log X}{\log p}} \frac{1}{p^{k\sigma} k^{m+1}(\log p)^m} \right)^2 \right) \\
&\ll T \frac{X^{1-2\sigma}}{(\log X)^{2m+1}} + \frac{X^{5-2\sigma}}{(\log X)^{2(m+1)}} \ll T \frac{X^{1-2\sigma}}{(\log X)^{2m+1}}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\int_0^T \left| \sum_{p \le X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2 \le n \le X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 dt \\
&\ll \int_0^T \left| \sum_{p \le X} \sum_{\frac{\log X}{\log p} < k \le 3\frac{\log X}{\log p}} \frac{p^{-k(\sigma+it)}}{k^{m+1}(\log p)^m} \right|^2 dt + T \frac{X^{2-6\sigma}}{(\log X)^{2m}} \ll T \frac{X^{1-2\sigma}}{(\log X)^{2m+1}},
\end{aligned}
$$

which completes the proof of this lemma. □

Proof of Proposition 4.1. By Lemma 4.1 and Lemma 4.3, for $X \le T^{1/135}$, we find that

$$
\begin{aligned}
&\frac{1}{T} \int_{14}^T \left| \tilde{\eta}_m(\sigma + it) - \sum_{p \le X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} \right|^2 dt \\
&\ll \frac{1}{T} \int_{14}^T \left| \tilde{\eta}_m(\sigma + it) - \sum_{2 \le n \le X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 dt \\
&\qquad\qquad + \frac{1}{T} \int_{14}^T \left| \sum_{p \le X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} - \sum_{2 \le n \le X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 dt \\
&\ll \frac{X^{1-2\sigma}}{(\log X)^{2m}}.
\end{aligned}
$$

By using this estimate, for any fixed $\varepsilon > 0$, we have

$$
\begin{aligned}
&\frac{1}{T} \mathrm{meas} \left\{ t \in [0, T] \ : \ \left| \tilde{\eta}_m(\sigma + it) - \sum_{p \le X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m} \right| \ge \varepsilon \right\} \\
&\qquad\qquad \ll \frac{X^{1-2\sigma}}{\varepsilon^2 (\log X)^{2m}} + \frac{1}{T}.
\end{aligned}
$$

Hence, for any $T \geq X^{135}$, it holds that

$$\frac{1}{T}\,\mathrm{meas}\left\{t \in [0,T] \ : \ \left|\tilde{\eta}_m(\sigma+it) - \sum_{p \leq X} \frac{\mathrm{Li}_{m+1}(p^{-\sigma-it})}{(\log p)^m}\right| \geq \varepsilon\right\} \to 0$$

as $X \to +\infty$. Thus, we obtain Proposition 4.1. □

4.3 Proof of the denseness lemma for corresponding to $\tilde{\eta}_m(s)$

In this section, we prove Proposition 4.2 by the method of [62, VIII.3], [120]. First of all, we will show the following elementary geometric lemma.

Lemma 4.4. *Let N be a positive integer larger than two. Suppose that the positive numbers $r_1, r_2, \ldots, r_N$ satisfy the condition*

$$r_{n_0} \leq \sum_{\substack{n=1 \\ n \neq n_0}}^{N} r_n, \tag{4.2}$$

where $r_{n_0} = \max\{r_n \mid n = 1, 2, \ldots, N\}$. Then we have

$$\left\{\sum_{n=1}^{N} r_n \exp(-2\pi i \theta_n) \in \mathbb{C} \ : \ \theta_n \in [0,1)\right\} = \left\{z \in \mathbb{C} \ : \ |z| \leq \sum_{n=1}^{N} r_n\right\}.$$

Proof. By Proposition 3.3 in [13], it immediately follows that

$$\left\{\sum_{n=1}^{N} r_n \exp(-2\pi i \theta_n) \in \mathbb{C} \ : \ \theta_n \in [0,1)\right\}$$

is the closed circle with center origin and radius $\sum_{n=1}^{N} r_n$. Note that their T_n becomes zero under assumption (4.2). □

Next, we introduce the following definitions.

Definition 1. Let m be a positive integer and $\mathcal{M}$ a finite subset of the set of prime numbers. For $\sigma \geq 1/2$ and $\underline{\theta} = (\theta_p)_{p \in \mathcal{M}} \in [0,1)^{\mathcal{M}}$, we define the functions

$$\phi_{m,\mathcal{M}}(\sigma, \underline{\theta}) := \sum_{p \in \mathcal{M}} \frac{\exp(-2\pi i \theta_p)}{p^{\sigma}(\log p)^m},$$

$$\tilde{\eta}_{m,\mathcal{M}}(\sigma, \underline{\theta}) := \sum_{p \in \mathcal{M}} \frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i \theta_p))}{(\log p)^m} = \sum_{p \in \mathcal{M}} \sum_{k=1}^{\infty} \frac{\exp(-2\pi i k \theta_p)}{k^{m+1} p^{k\sigma}(\log p)^m},$$

respectively.

Definition 2. Let p_n be the n-th prime number. Put

$$\underline{\theta}^{(0)} = \left(\theta^{(0)}_{p_n}\right)_{n\in\mathbb{N}} = (0, 1/2, 0, 1/2, \ldots) \in [0,1)^{\mathbb{N}},$$

and

$$\gamma_{m,\sigma} = \sum_p \sum_{k=1}^{\infty} \frac{\exp(-2\pi i k\theta^{(0)}_p)}{k^{m+1} p^{k\sigma} (\log p)^m}.$$

Note that the series for $\gamma_{m,\sigma}$ is convergent for $\sigma \geq 1/2$.

Proof of Proposition 4.2. Fix a complex number a and $1/2 \leq \sigma < 1$. Let U be a positive real parameter. We take a sufficiently large number $N = N(U, m, \sigma, a)$ for which the estimates

$$|a - \gamma_{m,\sigma}| \leq \sum_{p\in\mathcal{M}} \frac{1}{p^{\sigma}(\log p)^m},$$
$$\frac{1}{p^{\sigma}_{\min}(\log p_{\min})^m} \leq \sum_{p\in\mathcal{M}\setminus\{p_{\min}\}} \frac{1}{p^{\sigma}(\log p)^m}$$

are satisfied, where $\mathcal{M} = \mathcal{M}(U, N)$ is defined as $\{p \ : \ p \text{ is prime}, U < p \leq N\}$, and $p_{\min}$ is the minimum of $\mathcal{M}$. Note that the existence of such N is guaranteed by $\sum_p \frac{1}{p^{\sigma}(\log p)^m} = \infty$. Then, by Lemma 4.4, the function

$$\varphi_{m,\mathcal{M}}(\sigma, \cdot) \ : \ [0,1)^{\mathcal{M}} \ni \underline{\theta} \longmapsto \varphi_{m,\mathcal{M}}(\sigma, \underline{\theta}) \in \left\{ z \in \mathbb{C} \ : \ |z| \leq \sum_{p\in\mathcal{M}} \frac{1}{p^{\sigma}(\log p)^m} \right\}$$

is surjective. Hence, there exists some $\underline{\theta}^{(1)} = \underline{\theta}(m, \sigma, U, N)^{(1)} = (\theta^{(1)}_p)_{p\in\mathcal{M}} \in [0,1)^{\mathcal{M}}$ such that

$$\phi_{m,\mathcal{M}}(\sigma, \underline{\theta}^{(1)}) = a - \gamma_{m,\sigma}.$$

By using the prime number theorem, we find that

$$\begin{aligned}\tilde{\eta}_{m,\mathcal{M}}(\sigma, \underline{\theta}^{(1)}) &= \phi_{m,\mathcal{M}}(\sigma, \underline{\theta}^{(1)}) + \sum_{p\in\mathcal{M}} \sum_{k=2}^{\infty} \frac{\exp(-2\pi i k\theta^{(1)}_p)}{k^{m+1} p^{k\sigma} (\log p)^m} \\ &= a - \gamma_{m,\sigma} + O\left(\frac{1}{(\log U)^m}\right).\end{aligned}$$

For any prime number p, we put

$$\theta^{(2)}_p = \begin{cases} \theta^{(0)}_p & \text{if} \quad p \notin \mathcal{M}, \\ \theta^{(1)}_p & \text{if} \quad p \in \mathcal{M}. \end{cases}$$

Then it holds that

$$\begin{aligned}
&\sum_{p\le N}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(2)}))}{(\log p)^m}\\
=&\sum_{p\in\mathcal{M}}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(1)}))}{(\log p)^m}+\sum_{p\le U}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(0)}))}{(\log p)^m}\\
=&\tilde{\eta}_{m,\mathcal{M}}(\sigma,\underline{\theta}^{(1)})+\gamma_{m,\sigma}+\sum_{p>U}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(0)}))}{(\log p)^m},
\end{aligned}$$

and additionally, by using the prime number theorem and simple calculations of alternating series,

$$\begin{aligned}
\sum_{p>U}\frac{\mathrm{Li}_{m+1}(p^{-\sigma}\exp(-2\pi i\theta_p^{(0)}))}{(\log p)^m}&=\sum_{p>U}\frac{\exp(-2\pi i\theta_p^{(0)})}{p^{\sigma}(\log p)^m}+O\left(\sum_{p>U}\frac{1}{p^{2\sigma}(\log p)^m}\right)\\
&\ll\frac{1}{(\log U)^m}.
\end{aligned}$$

Hence, by taking a sufficiently large $U=U(\varepsilon)$ and noting the continuity of the function $\sum_{p\le N}\frac{\mathrm{Li}_{m+1}(p^{\sigma}\exp(-2\pi i\theta_p))}{(\log p)^m}$ with respect to $(\theta_p)_{p\le N}\in[0,1)^{\pi(N)}$, we obtain this proposition. □

4.4 Proof of the denseness of $\tilde{\eta}_m(s)$

In this section, we prove Theorem 4.2. Here, we use the following lemma related with Kronecker's approximation theorem.

Lemma 4.5. *Let A be a Jordan measurable subregion of $[0,1)^N$, and $a_1,\dots,a_N$ be real numbers linearly independent over $\mathbb{Q}$. Set, for any $T>0$,*

$$I(T,A)=\{t\in[0,T]\ :\ (\{a_1t\},\dots,\{a_Nt\})\in A\}.$$

Then we have

$$\lim_{T\to+\infty}\frac{\mathrm{meas}(I(T,A))}{T}=\mathrm{meas}(A).$$

Proof. This lemma is Theorem 1 of Appendix 8 in [62] □

Let us start the proof of Theorem 4.2.

Proof of Theorem 4.2. Let $\varepsilon>0$ be any small number, a any fixed complex number, $\frac{1}{2}\le\sigma<1$, and let T_0 be any positive number. Define $S_M(\theta_1,\dots,\theta_M;\sigma,m)$ and $S_{M,N}(\theta_{M+1},\dots,\theta_N;\sigma,m)$ by

$$S_M(\theta_1,\dots,\theta_M;\sigma,m)=\sum_{n\le M}\frac{\mathrm{Li}_{m+1}(p_n^{-\sigma}e^{-2\pi i\theta_n})}{(\log p_n)^m},$$

$$S_{M,N}(\theta_{M+1},\dots,\theta_N;\sigma,m)=\sum_{M<n\le N}\frac{\mathrm{Li}_{m+1}(p_n^{-\sigma}e^{-2\pi i\theta_n})}{(\log p_n)^m}.$$

Then, by Proposition 4.2, we can take a sufficiently large $M_0 = M_0(m, \sigma, a, \varepsilon)$ so that for any $M \geq M_0$, there exists some Jordan measurable subset $\Theta_1^{(M)} = \Theta_1^{(M)}(m, \sigma, a, \varepsilon, M)$ of $[0,1)^M$ such that $\delta_M := \mathrm{meas}(\Theta_1^{(M)}) > 0$ and

$$|S_M(\theta_1, \ldots, \theta_M; \sigma, m) - a| < \varepsilon$$

for any $(\theta_1, \ldots, \theta_M) \in \Theta_1^{(M)}$. We also find that

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 |S_{M,N}(\theta_{M+1}, \ldots, \theta_N; \sigma, m)|^2 d\theta_{M+1} \cdots d\theta_N \\
&= \int_0^1 \cdots \int_0^1 \left| \sum_{M<n\leq N} \sum_{k=1}^{\infty} \frac{p_n^{-\sigma k} e^{-2\pi i k \theta_n}}{k^{m+1}(\log p_n)^m} \right|^2 d\theta_{M+1} \cdots d\theta_N \\
&= \sum_{M<n_1\leq N} \sum_{M<n_2\leq N} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left\{ \frac{(p_{n_1} p_{n_2})^{-\sigma k}}{(k_1 k_2)^{m+1} (\log p_{n_1} \log p_{n_2})^m} \times \right. \\
&\qquad\qquad \left. \times \int_0^1 \cdots \int_0^1 e^{-2\pi i (k_1 \theta_{n_1} - k_2 \theta_{n_2})} d\theta_{M+1} \cdots d\theta_N \right\} \\
&= \sum_{M<n\leq N} \sum_{k=1}^{\infty} \frac{1}{k^{2(m+1)} p_n^{2\sigma k} (\log p_n)^{2m}} \ll \sum_{M<n\leq N} \frac{1}{p_n (\log p_n)^{2m}}.
\end{aligned}$$

Note that the last sum tends to zero as $M \to +\infty$. Therefore, there exists some large number $M_1 = M_1(m, \varepsilon)$ such that, for any $N > M \geq M_1$, it holds that

$$\mathrm{meas}\left(\left\{ (\theta_{M+1}, \ldots, \theta_N) \in [0,1)^{N-M} \ : \ |S_{M,N}(\theta_{M+1}, \ldots, \theta_N; \sigma, m)| < \varepsilon \right\} \right) > \frac{1}{2}.$$

Here we denote the set of the content of meas$(\cdot)$ in the above inequality by $\Theta_2^{(M,N)} = \Theta_2^{(M,N)}(M, N, \varepsilon)$.

We put $M_2 = \max\{M_0, M_1\}$ and $\Theta_3 = \Theta_1^{(M_2)} \times \Theta_2^{(M_2,N)}$ for any $N > M_2$. Then Θ_3 is a subset of $[0,1)^N$ satisfying $\mathrm{meas}(\Theta_3) > \delta_{M_2}/2$. Hence, putting

$$\mathcal{I}(T) = \left\{ t \in [T_0, T] \ : \ \left(\left\{ \frac{t}{2\pi} \log p_1 \right\}, \ldots, \left\{ \frac{t}{2\pi} \log p_N \right\} \right) \in \Theta_3 \right\}$$

and applying Lemma 4.5, for any positive integer $N > M_2$, there exists some large number $T_N > T_0$ such that $\mathrm{meas}(\mathcal{I}(T)) > \delta_{M_2} T/2$ holds for any $T \geq T_N$. On the other hand, by Proposition 4.1, there exists a large number $N_0 = N_0(\varepsilon, \delta_{M_2})$ such that

$$\mathrm{meas}\left\{ t \in [T_0, T] \ : \ \left| \tilde{\eta}_m(\sigma + it) - \sum_{n\leq N} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma - it})}{(\log p_n)^m} \right| < \varepsilon \right\} > (1 - \delta_{M_2}/4)T$$

for any $N \geq N_0$, $T \geq p_N^{135}$.

Therefore, for any $N \geq \max\{N_0, M_2+1\}$, $T \geq \max\{T_N, p_N^{135}\}$, there exists some $t_0 \in [T_0, T]$ such that

$$\left(\left\{ \frac{t_0}{2\pi} \log p_1 \right\}, \dots, \left\{ \frac{t_0}{2\pi} \log p_N \right\} \right) \in \Theta_3,$$

and

$$\left| \tilde{\eta}_m(\sigma + it_0) - \sum_{n \leq N} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma - it_0})}{(\log p_n)^m} \right| < \varepsilon.$$

Then we have

$$\begin{aligned}
&|\tilde{\eta}_m(\sigma + it_0) - a| \\
&\leq \left| \tilde{\eta}_m(\sigma + it_0) - \sum_{n \leq N} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma} e^{-it_0 \log p_n})}{(\log p_n)^m} \right| + \left| \sum_{n \leq M_2} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma} e^{-it_0 \log p_n})}{(\log p_n)^m} - a \right| \\
&\quad + \left| \sum_{M_2 < n \leq N} \frac{\mathrm{Li}_{m+1}(p_n^{-\sigma} e^{-it_0 \log p_n})}{(\log p_n)^m} \right| < 3\varepsilon.
\end{aligned}$$

This completes the proof of Theorem 4.2. □

4.5 Proof of the denseness of $\eta_m(s)$

In this section, we prove Theorem 4.1. Here, we prepare the following lemma.

Lemma 4.6. *Let $\sigma \geq 1/2$ and m be a positive integer. Then we have*

$$\eta_m(s) = Y_m(s) + O_m(\log t),$$

where Y_m is defined by (2.1).

Proof. This lemma is equation (2.2). □

Proof of Theorem 4.1. First, we show Theorem 4.1 in the case $m = 1$. If the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\beta > \sigma$ is finite, then there exists a sufficiently large T_0 such that $Y_1(\sigma + it) \equiv b$ for $t \geq T_0$, where b is a positive real number. Therefore, by Lemma 2.1, we have

$$\int_0^t \log \zeta(\sigma + it') dt' = i\tilde{\eta}_1(\sigma + it) + b$$

for any $t \geq T_0$. By this formula, we obtain

$$\begin{aligned}
\left\{ \int_0^t \log \zeta(\sigma + it') dt' \ : \ t \in [0, \infty) \right\} &\supset \left\{ \int_0^t \log \zeta(\sigma + it') dt' \ : \ t \in [T_0, \infty) \right\} \\
&= \left\{ i\tilde{\eta}_1(\sigma + it) + b \ : \ t \in [T_0, \infty) \right\}.
\end{aligned}$$

If a set $A \subset \mathbb{C}$ is dense in $\mathbb{C}$, then for any $c_1 \in \mathbb{C}\setminus\{0\}$ and $c_2 \in \mathbb{C}$, the set $\{c_1 a + c_2 \mid a \in A\}$ is also dense in $\mathbb{C}$. By this fact and Theorem 4.2, the set $\{i\tilde{\eta}_1(\sigma+it)+b \mid t \in [T_0,\infty)\}$ is dense in $\mathbb{C}$. Thus, the set $\left\{\int_0^t \log\zeta(\sigma+it')dt' \mid t \in [0,\infty)\right\}$ is dense in $\mathbb{C}$ under this assumption.

Next, for $m \in \mathbb{Z}_{\geq 2}$, we show the equivalence of (I) and (II). The implication (I) $\Rightarrow$ (II) is clear since the equation $\eta_m(\sigma+it) = i^m\tilde{\eta}_m(\sigma+it)$ holds by assuming (I).

In the following, we show the inverse implication (II) $\Rightarrow$ (I). By Lemma 4.6, if (I) is false, then the estimate $|\eta_m(\sigma+it)| \gg_m t^{m-1}$ holds. Therefore, for some $T_2 > 0$, we have

$$\overline{\{\eta_m(\sigma+it) \ : \ t \in [T_2,\infty)\}} \subset \mathbb{C} \setminus \{z \in \mathbb{C} \ : \ |z| \leq 1\}\,.$$

Here, $\bar{A}$ means the closure of the set A. Since $\{\eta_m(\sigma+it) \mid t \in [0,T_2]\}$ is a piecewise smooth curve of finite length, $\mu\left(\overline{\{\eta_m(\sigma+it) \ : \ t \in [0,T_2]\}}\right) = 0$. Here μ is the Lebesgue measure in $\mathbb{C}$. Therefore, we obtain

$$\{z \in \mathbb{C} \ : \ |z| \leq 1\} \not\subset \overline{\{\eta_m(\sigma+it) \ : \ t \in [0,T_2]\}}.$$

Hence, if (I) is false, then the set $\{\eta_m(\sigma+it) \ : \ t \in [0,\infty)\}$ is not dense in $\mathbb{C}$. Thus, we obtain the implication (II) $\Rightarrow$ (I). □

Chapter 5 Discrepancy estimate and Large deviations for the Riemann zeta-function

The purpose of this chapter is to study the distribution of values $\tilde{\eta}_m(\sigma+it)$ as $t \in \mathbb{R}$ varies. The contents in this chapter are based on the paper [23].

5.1 Results

For a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{C}$, define

$$\mathbb{P}_T(f(t) \in A) := \frac{1}{T} \operatorname{meas}\{t \in [T, 2T] \ : \ f(t) \in A\},$$

where $T > 0$ and $A \in \mathcal{B}(\mathbb{C})$. We consider the probability measure $\mathbb{P}_T(\tilde{\eta}_m(\sigma + it) \in A)$. Let $\mathcal{P}$ be the set of prime numbers. Let $\{X(p)\}_{p\in\mathcal{P}}$ be a sequence of independent random variables on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ uniformly distributed on the unit circle in the complex plane. Define

$$\tilde{\eta}_m(\sigma, X) = \sum_p \tilde{\eta}_{m,p}(\sigma, X(p)), \tag{5.1}$$

where

$$\tilde{\eta}_{m,p}(\sigma, w) = \frac{\mathrm{Li}_{m+1}(p^{-\sigma}w)}{(\log p)^m} = \sum_{k=1}^{\infty} \frac{w^k}{kp^{k\sigma}(\log p^k)^m} \tag{5.2}$$

for $w \in \mathbb{C}$ with $|w| = 1$. One can show that (5.1) converges almost surely if $\sigma > 1/2$ and $m \in \mathbb{Z}_{\geq 0}$; see Lemma 5.1. The first main result of this chapter presents a discrepancy bound for the value distribution of $\tilde{\eta}_m(\sigma+it)$, that is, an upper bound for the quantity

$$D_{\sigma,m}(T) = \sup_{\mathcal{R}} \left|\mathbb{P}_T(\tilde{\eta}_m(\sigma+it) \in \mathcal{R}) - \mathbb{P}(\tilde{\eta}_m(\sigma, X) \in \mathcal{R})\right|,$$

where $\mathcal{R}$ runs through all rectangle in $\mathbb{C}$ with edges parallel to the axes.

Theorem 5.1. *Let $1/2 < \sigma < 1$ and $m \in \mathbb{Z}_{\geq 0}$. Then we have*

$$D_{\sigma,m}(T) \ll_{\sigma,m} \frac{1}{(\log T)^{\sigma}(\log\log T)^m}.$$

Next, we consider the large deviation for the values $\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma+it)$ with any angle $\alpha \in \mathbb{R}$.

Theorem 5.2. *Let $1/2 < \sigma < 1$ and $m \in \mathbb{Z}_{\geq 0}$. There exists a positive constant $a = a(\sigma, m)$ such that for large T, τ with $\tau \leq a(\log T)^{1-\sigma}(\log\log T)^{-m-1}$ we have*

$$\begin{aligned}
&\mathbb{P}_T\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma+it) > \tau\right)\\
&= \mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right)\left(1 + O\left(\frac{\tau^{\frac{\sigma}{1-\sigma}}(\log\tau)^{\frac{\sigma+m}{1-\sigma}}}{(\log T)^{\sigma}(\log\log T)^m}\right)\right)
\end{aligned}$$

for any $\alpha \in \mathbb{R}$. Here, the implicit constant depends on σ and m.

5.2 Mean value results

Denote by $\mathcal{A}$ the set of pairs (σ, m) such that

$$\mathcal{A} = \{(\sigma, m) \ : \ \sigma \geq 1/2 \text{ and } m \in \mathbb{Z}_{\geq 0}\} \setminus \{(\tfrac{1}{2}, 0)\}$$

For $1/2 \leq \sigma < 1$, we put

$$\tau(\sigma) = \begin{cases} \sigma & \text{if} \quad 1/2 < \sigma < 1, \\ 0 & \text{if} \quad \sigma = 1/2. \end{cases}$$

We define

$$\begin{aligned}
P_{m,Y}(\sigma+it) &= \sum_{p^k \leq Y} \frac{p^{-itk}}{kp^{k\sigma}(\log p^k)^m},\\
P_{m,Y}(\sigma, X) &= \sum_{p^k \leq Y} \frac{X(p)^k}{kp^{k\sigma}(\log p^k)^m}
\end{aligned}$$

for $(\sigma, m) \in \mathcal{A}$ and $Y \geq 3$. The following mean value result for $P_{m,Y}(\sigma+it)$ and $P_{m,Y}(\sigma, X)$ is useful to study the value distribution of $\tilde{\eta}_m(\sigma+it)$.

Proposition 5.1. *Let $(\sigma, m) \in \mathcal{A}$ with $\sigma < 1$. Let $T, V > 0$ be large. Denote by $A_T = A_T(V, Y; \sigma, m)$ the set*

$$A_T = \left\{t \in [0, T] \ : \ |P_{m,Y}(\sigma+it)| \leq V\right\} \tag{5.3}$$

for $Y \geq 3$. If we further suppose that

$$3 \leq Y \leq \exp\left(\frac{\log T}{V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}}\right) \tag{5.4}$$

holds, then there exist positive constants $b_1 = b_1(\sigma, m)$ and $b_2 = b_2(\sigma, m)$ such that for any complex numbers z_1, z_2 with $|z_1|, |z_2| \leq b_1 V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}$ we have

$$\begin{aligned}
&\frac{1}{T}\int_{A_T} \exp\left(z_1 P_{m,Y}(\sigma+it) + z_2\overline{P_{m,Y}(\sigma+it)}\right) dt\\
&= \mathbb{E}\left[\exp\left(z_1 P_{m,Y}(\sigma, X) + z_2\overline{P_{m,Y}(\sigma, X)}\right)\right] + E,
\end{aligned}$$

where E is estimated as

$$E \ll \frac{1}{T}\left(V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}Y\right)^{V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}} + \exp\left(-b_2 V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right).$$

After showing some preliminary lemmas in Section 5.2.1, we derive another mean value result which plays a key role in the proof of Theorem 5.1.

Proposition 5.2. *Let $1/2 < \sigma < 1$ and $m \in \mathbb{Z}_{\geq 0}$. For $A \geq 1$, there exists a positive constant $b = b(\sigma, m, A)$ such that for any complex number w with $|w| \leq b(\log T)^{\sigma}(\log\log T)^m$ we have*

$$\frac{1}{T}\int_0^T \exp(i\langle \tilde{\eta}_m(\sigma+it), w\rangle)dt = \mathbb{E}\left[\exp(i\langle \tilde{\eta}_m(\sigma, X), w\rangle)\right] + O\left(\frac{1}{(\log T)^A}\right),$$

where the inner product $\langle \cdot, \cdot \rangle$ is defined by $\langle z, w\rangle = \operatorname{Re} z \operatorname{Re} w + \operatorname{Im} z \operatorname{Im} w$ for $z, w \in \mathbb{C}$.

5.2.1 Preliminaries

Lemma 5.1. *Let $(\sigma, m) \in \mathcal{A}$. Then the series of* (5.1) *converges almost surely.*

Proof. For any prime number p, we have

$$\mathbb{E}\left[\tilde{\eta}_{m,p}(\sigma, X(p))\right] = \sum_{k=1}^{\infty} \frac{\mathbb{E}\left[X(p)^k\right]}{k p^{k\sigma}(\log p^k)^m} = 0,$$

where the change of the sum and expectation is justified by Fubini's theorem. By the prime number theorem, we further obtain

$$\sum_p \mathbb{E}\left[\left|\tilde{\eta}_{m,p}(\sigma, X(p))\right|^2\right] \ll \sum_p \frac{1}{p^{2\sigma}(\log p)^{2m}} < \infty$$

since $(\sigma, m) \in \mathcal{A}$. Thus the assertion follows from [75, Theorem 17.3.I]. □

Lemma 5.2. *Let $(\sigma, m) \in \mathcal{A}$ with $\sigma < 1$. Let $T \geq 5$ and $Y \geq 3$. For any $k, \ell \in \mathbb{Z}_{\geq 1}$, we have*

$$\frac{1}{T}\int_0^T (P_{m,Y}(\sigma+it))^k \left(\overline{P_{m,Y}(\sigma+it)}\right)^{\ell} dt = \mathbb{E}\left[\left(P_{m,Y}(\sigma, X)\right)^k \left(\overline{P_{m,Y}(\sigma, X)}\right)^{\ell}\right] + O\left(\frac{Y^{2(k+\ell)}}{T}\right).$$

Proof. We see that

$$\begin{aligned}
&\int_0^T (P_{m,Y}(\sigma+it))^k \left(\overline{P_{m,Y}(\sigma+it)}\right)^\ell dt \\
&= \sum_{\substack{p_1^{a_1},\dots,p_k^{a_k}\le Y \\ q_1^{b_1},\dots,q_\ell^{b_\ell}\le Y}} \frac{1}{a_1 p_1^{a_1\sigma}(\log p_1^{a_k})^m \cdots a_k p_k^{a_k\sigma}(\log p_k^{a_k})^m} \\
&\qquad\qquad \times \frac{1}{b_1 q_1^{b_1\sigma}(\log q_1^{b_1})^m \cdots b_\ell q_\ell^{b_\ell\sigma}(\log q_\ell^{b_\ell})^m} \int_0^T \left(\frac{q_1^{b_1}\cdots q_\ell^{b_\ell}}{p_1^{a_1}\cdots p_k^{a_k}}\right)^{it} dt \\
&= S_1 + S_2,
\end{aligned}$$

where S_1 is the sum over the terms with $p_1^{a_1}\cdots p_k^{a_k} = q_1^{b_1}\cdots q_\ell^{b_\ell}$, and S_2 is the sum over the other terms. Here, for $p_1^{a_1}\cdots p_k^{a_k} \neq q_1^{b_1}\cdots q_\ell^{b_\ell}$, it holds that

$$\int_0^T \left(\frac{q_1^{b_1}\cdots q_\ell^{b_\ell}}{p_1^{a_1}\cdots p_k^{a_k}}\right)^{it} dt \ll Y^{k+\ell},$$

and hence we have

$$S_2 \ll Y^{k+\ell}\left(\sum_{p^a\le Y}\frac{1}{ap^{a\sigma}(\log p^a)^m}\right)^{k+\ell} \ll_m Y^{2(k+\ell)}.$$

We can also write

$$\begin{aligned}
\frac{1}{T}S_1 = &\sum_{\substack{p_1^{a_1},\dots,p_k^{a_k}\le Y \\ q_1^{b_1},\dots,q_\ell^{b_\ell}\le Y \\ p_1^{a_1}\cdots p_k^{a_k}=q_1^{b_1}\cdots q_\ell^{b_\ell}}} \frac{1}{a_1 p_1^{a_1\sigma}(\log p_1^{a_k})^m \cdots a_k p_k^{a_k\sigma}(\log p_k^{a_k})^m} \\
&\qquad\qquad \times \frac{1}{b_1 q_1^{b_1\sigma}(\log q_1^{b_1})^m \cdots b_\ell q_\ell^{b_\ell\sigma}(\log q_\ell^{b_\ell})^m}.
\end{aligned} \tag{5.5}$$

On the other hand, it holds that

$$\begin{aligned}
&\mathbb{E}\left[\left(P_{m,Y}(\sigma,X)\right)^k \left(\overline{P_{m,Y}(\sigma,X)}\right)^\ell\right] \\
&= \sum_{\substack{p_1^{a_1},\dots,p_k^{a_k}\le Y \\ q_1^{b_1},\dots,q_\ell^{b_\ell}\le Y}} \frac{1}{a_1 p_1^{a_1\sigma}(\log p_1^{a_k})^m \cdots a_k p_k^{a_k\sigma}(\log p_k^{a_k})^m} \\
&\qquad \times \frac{1}{b_1 q_1^{b_1\sigma}(\log q_1^{b_1})^m \cdots b_\ell q_\ell^{b_\ell\sigma}(\log q_\ell^{b_\ell})^m}\mathbb{E}\left[\frac{X(p_1)^{a_1}\cdots X(p_k)^{a_k}}{X(q_1)^{b_1}\cdots X(q_\ell)^{b_\ell}}\right].
\end{aligned}$$

Since $X(p)$'s are independent and uniformly distribution on the unit circle in $\mathbb{C}$, it holds that

$$\mathbb{E}\left[\frac{X(p_1)^{a_1}\cdots X(p_k)^{a_k}}{X(q_1)^{b_1}\cdots X(q_\ell)^{b_\ell}}\right] = \begin{cases} 1 & \text{if } p_1^{a_1}\cdots p_k^{a_k} = q_1^{b_1}\cdots q_\ell^{b_\ell}, \\ 0 & \text{if } p_1^{a_1}\cdots p_k^{a_k} \neq q_1^{b_1}\cdots q_\ell^{b_\ell}. \end{cases} \tag{5.6}$$

Therefore, we deduce from (5.5) the equation

$$\mathbb{E}\left[\left(P_{m,Y}(\sigma, X)\right)^k \left(\overline{P_{m,Y}(\sigma, X)}\right)^\ell\right] = TS_1,$$

which completes the proof of the lemma. □

Lemma 5.3. *Let $\{a(p)\}_{p\in\mathcal{P}}$ be any complex sequence. Let $T \geq 5$ and $Y \geq 3$. For $k \in \mathbb{Z}_{\geq 1}$ with $Y^k \leq T(\log T)^{-1}$, we have*

$$\frac{1}{T}\int_0^T \left|\sum_{p\leq Y} a(p)p^{-it}\right|^{2k} dt \ll k!\left(\sum_{p\leq Y}|a(p)|^2\right)^k.$$

Additionally, for any $k \in \mathbb{Z}_{\geq 1}$, we have

$$\mathbb{E}\left[\left|\sum_{p\leq Y} a(p)X(p)\right|^{2k}\right] \leq k!\left(\sum_{p\leq Y}|a(p)|^2\right)^k.$$

Proof. The former assertion is Lemma 2.8. We prove the latter assertion. By equation (5.6), we see that

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{p\leq Y} a(p)X(p)\right|^{2k}\right] \\
&= \sum_{\substack{p_1,\ldots,p_k\leq Y\\ q_1,\ldots,q_k\leq Y}} a(p_1)\cdots a(p_k)\overline{a(q_1)\cdots a(q_k)}\,\mathbb{E}\left[\frac{X(p_1)\cdots X(p_k)}{X(q_1)\cdots X(q_k)}\right] \\
&\leq k! \sum_{p_1,\ldots,p_k\leq Y} |a(p_1)|^2\cdots|a(p_k)|^2 \leq k!\left(\sum_{p\leq Y}|a(p)|^2\right)^k,
\end{aligned}$$

which completes the proof of the lemma. □

Lemma 5.4. *Let $(\sigma, m) \in \mathcal{A}$ with $\sigma < 1$. Let $T > 0$ be large and $Y \geq 3$. There exists a positive constant $C = C(\sigma, m)$ such that*

$$\frac{1}{T}\int_0^T |P_{m,Y}(\sigma+it)|^{2k} dt \ll \left(\frac{Ck^{1-\sigma}}{(\log 2k)^{m+\tau(\sigma)}}\right)^{2k} \tag{5.7}$$

for $k \in \mathbb{Z}_{\geq 1}$ with $Y^k \leq T(\log T)^{-1}$. Additionally, we have

$$\mathbb{E}\left[|P_{m,Y}(\sigma, X)|^{2k}\right] \ll \left(\frac{Ck^{1-\sigma}}{(\log 2k)^{m+\tau(\sigma)}}\right)^{2k} \tag{5.8}$$

for any $k \in \mathbb{Z}_{\geq 1}$.

Proof. Suppose that the inequality $k \log 2k < Y$ holds. Then we see that

$$\begin{aligned}
&\int_0^T |P_{m,Y}(\sigma + it)|^{2k} dt \\
&\leq 9^k \Bigg(\int_0^T \Bigg| \sum_{p \leq k \log 2k} \frac{1}{p^{\sigma + it} (\log p)^m} \Bigg|^{2k} dt \\
&\qquad + \int_0^T \Bigg| \sum_{k \log 2k < p \leq Y} \frac{1}{p^{\sigma + it} (\log p)^m} \Bigg|^{2k} dt + C^{2k} T \Bigg),
\end{aligned}$$

where C is an absolute positive constant. By Lemma 5.3 and the prime number theorem, it holds that

$$\begin{aligned}
\frac{1}{T} \int_0^T \Bigg| \sum_{k \log 2k < p \leq Y} \frac{1}{p^{\sigma + it} (\log p)^m} \Bigg|^{2k} dt &\ll k! \Bigg(\sum_{k \log 2k < p \leq Y} \frac{1}{p^{2\sigma} (\log p)^{2m}} \Bigg)^k \\
&\ll \left(\frac{C_1 k^{1-\sigma}}{(\log 2k)^{m + \tau(\sigma)}} \right)^{2k},
\end{aligned}$$

where C_1 is a positive constant which may depend on σ and m. Furthermore, by the prime number theorem it follows that

$$\begin{aligned}
\frac{1}{T} \int_0^T \Bigg| \sum_{p \leq k \log 2k} \frac{1}{p^{\sigma + it} (\log p)^m} \Bigg|^{2k} dt &\ll \Bigg(\sum_{p \leq k \log 2k} \frac{1}{p^{\sigma} (\log p)^{m}} \Bigg)^{2k} \\
&\ll \left(\frac{C_2 k^{1-\sigma}}{(\log 2k)^{m + \sigma}} \right)^{2k},
\end{aligned}$$

where C_2 is also a positive constant which may depend on σ and m. From the above estimates, we obtain estimate (5.7). If the inequality $Y \leq k \log 2k$ holds, then we have

$$P_{m,Y}(\sigma + it) \ll \sum_{p \leq Y} \frac{1}{p^{\sigma} (\log p)^m} \ll \frac{C_2 k^{1-\sigma}}{(\log 2k)^{m+\sigma}}$$

by the prime number theorem. Hence, estimate (5.7) follows in this case. Similarly, we can prove estimate (5.8). □

Lemma 5.5. *Let $(\sigma, m) \in \mathcal{A}$ with $\sigma < 1$. Let $T, V > 0$ be large. There exists a small positive constant $c = c(\sigma, m)$ such that*

$$\mathbb{P}_T \left(|P_{m,Y}(\sigma + it)| > V \right) \leq \exp\left(-c V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m + \tau(\sigma)}{1-\sigma}} \right) \tag{5.9}$$

if $Y \geq 3$ satisfies (5.4)*. Additionally, we have*

$$\mathbb{P} \left(|P_{m,Y}(\sigma, X)| > V \right) \leq \exp\left(-c V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m + \tau(\sigma)}{1-\sigma}} \right) \tag{5.10}$$

for any $Y \geq 3$.

Proof. Note that (5.4) implies the condition $Y^k \le T(\log T)^{-1}$. Then we derive from (5.7) the estimate

$$\begin{aligned}\mathbb{P}_T\left(|P_{m,Y}(\sigma+it)|>V\right) &\le \frac{1}{V^{2k}}\frac{1}{T}\int_0^T |P_{m,Y}(\sigma+it)|^{2k}\,dt\\ &\ll \frac{1}{V^{2k}}\left(\frac{C(\sigma,m)k^{1-\sigma}}{(\log 2k)^{m+\tau(\sigma)}}\right)^{2k}.\end{aligned}$$

Hence, choosing $k=\left[c_1 V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right]$ with c_1 a suitably small constant depending on σ and m, we obtain inequality (5.9). Similarly, by using (5.8), we see that

$$\begin{aligned}\mathbb{P}\left(|P_{m,Y}(\sigma,X)|>V\right) &\le \frac{1}{V^{2k}}\mathbb{E}\left[|P_{m,Y}(\sigma,X)|^{2k}\right]\\ &\ll \frac{1}{V^{2k}}\left(\frac{C(\sigma,m)k^{1-\sigma}}{(\log 2k)^{m+\tau(\sigma)}}\right)^{2k}\end{aligned}$$

holds for any $Y\ge 3$. Thus again choosing $k=\left[c_1 V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right]$, we obtain inequality (5.10). □

Lemma 5.6. *Let $(\sigma,m)\in\mathcal{A}$, and let $T>0$ be large. For $Y\ge 3$ and $W>0$, we denote by $B_T=B_T(Y,W;\sigma,m)$ the set*

$$B_T=\left\{t\in[14,T]\ :\ |\tilde{\eta}_m(\sigma+it)-P_{m,Y}(\sigma+it)|\le WY^{\frac{1}{2}-\sigma}\right\}. \tag{5.11}$$

There exists a small positive constant c such that

$$\frac{1}{T}\operatorname{meas}([14,T]\setminus B_T)\ll \exp\left(-cW^2(\log Y)^{2m}\right)$$

for $0<W\le\left((\log T)(\log Y)^{-2(m+1)}\right)^{\frac{m}{2m+1}}$. Moreover, we have

$$\frac{1}{T}\operatorname{meas}([14,T]\setminus B_T)\ll \exp\left(-c(W(\log T)^m)^{\frac{1}{m+1}}\right) \tag{5.12}$$

for $\left((\log T)(\log Y)^{-2(m+1)}\right)^{\frac{m}{2m+1}}\le W\le(\log T)(\log Y)^{-(m+1)}$.

Proof. When $m\in\mathbb{Z}_{\ge1}$, this lemma is a little modification of Lemma 5.6, and the proof is the same as that for Lemma 5.6. When $m=0$, this can be also proved similarly by using [116, Corollary in page 60] or Proposition 6.6 in Chapter 6. □

Lemma 5.7. *Let $(\sigma,m)\in\mathcal{A}$. Let $Y\ge 3$ and $W>0$. There exists a small positive constant c such that*

$$\mathbb{P}\left(|\tilde{\eta}_m(\sigma,X)-P_{m,Y}(\sigma,X)|>WY^{\frac{1}{2}-\sigma}\right)\le \exp\left(-cW^2(\log Y)^{2m}\right).$$

Proof. We see that

$$\mathbb{E}\left[|\tilde{\eta}_m(\sigma,X)-P_{m,Y}(\sigma,X)|^{2k}\right]=\mathbb{E}\left[\left|\sum_{p^\ell>Y}\frac{X(p)^\ell}{\ell p^{\ell\sigma}(\log p^\ell)^m}\right|^{2k}\right].$$

Additionally, we find that

$$\sum_{\substack{p^\ell>Y\\ \ell\geq 2}}\frac{X(p)^\ell}{\ell p^{\ell\sigma}(\log p^\ell)^m}\ll\frac{1}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}.$$

Therefore, it holds that

$$\begin{aligned}&\mathbb{E}\left[|\tilde{\eta}_m(\sigma,X)-P_{m,Y}(\sigma,X)|^{2k}\right]\\&=\mathbb{E}\left[\left|\sum_{p>Y}\frac{X(p)}{p^{\sigma}(\log p)^m}\right|^{2k}\right]+O\left(\left(\frac{C^k}{Y^{2\sigma-1}(\log Y)^{2m}}\right)^k\right)\end{aligned}$$

for some constant $C>0$. Similarly to the proof of Lemma 5.3, we obtain

$$\mathbb{E}\left[\left|\sum_{p>Y}\frac{X(p)}{p^{\sigma}(\log p)^m}\right|^{2k}\right]\leq k!\left(\sum_{p>Y}\frac{1}{p^{2\sigma}(\log p)^{2m}}\right)^k\leq\left(\frac{kC}{Y^{2\sigma-1}(\log Y)^{2m}}\right)^k.$$

Hence, it follows that

$$\mathbb{E}\left[|\tilde{\eta}_m(\sigma,X)-P_{m,Y}(\sigma,X)|^{2k}\right]\leq\left(\frac{kC}{Y^{2\sigma-1}(\log Y)^{2m}}\right)^k.$$

This inequality leads that

$$\begin{aligned}&\mathbb{P}\left(|\tilde{\eta}_m(\sigma,X)-P_{m,Y}(\sigma,X)|>WY^{\frac{1}{2}-\sigma}\right)\\&\leq\frac{1}{(WY^{\frac{1}{2}-\sigma})^{2k}}\mathbb{E}\left[|\tilde{\eta}_m(\sigma,X)-P_{m,Y}(\sigma,X)|^{2k}\right]\leq\left(\frac{kC}{W^2(\log Y)^{2m}}\right)^k.\end{aligned}$$

Choosing $k=[e^{-1}C^{-1}W^2(\log Y)^{2m}]$, we obtain this lemma. □

Lemma 5.8. *Let* $(\sigma,m)\in\mathcal{A}$. *For* $Y\geq 3$ *and* $w\in\mathbb{C}$, *we have*

$$\mathbb{E}\left[\exp(i\langle\tilde{\eta}_m(\sigma,X),w\rangle)\right]=\mathbb{E}\left[\exp\left(i\langle P_{m,Y}(\sigma,X),w\rangle\right)\right]+O\left(\frac{|w|}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}\right).$$

Proof. By the definition of $\tilde{\eta}_m(\sigma,X)$, we see that

$$\begin{aligned}&\mathbb{E}\left[\exp(i\langle\tilde{\eta}_m(\sigma,X),w\rangle)\right]\\&=\mathbb{E}\left[\exp\left(i\langle P_{m,Y}(\sigma,X),w\rangle+i\langle\sum_{p^k>Y}\frac{X(p)^k}{kp^{k\sigma}(\log p^k)^m},w\rangle\right)\right].\end{aligned}\tag{5.13}$$

Since the estimate

$$\sum_{\substack{p^k>Y\\k\geq 2}} \frac{X(p)^k}{kp^{k\sigma}(\log p^k)^m} \ll \frac{1}{Y^{2\sigma-1}(\log Y)^m}$$

holds, we have

$$\langle \sum_{p^k>Y} \frac{X(p)^k}{kp^{k\sigma}(\log p^k)^m}, w\rangle = \langle \sum_{p>Y} \frac{X(p)}{p^{\sigma}(\log p)^m}, w\rangle + O\left(\frac{|w|}{Y^{2\sigma-1}(\log Y)^m}\right)$$

by applying the Cauchy-Schwarz inequality $|\langle z, w\rangle| \leq |z||w|$. Furthermore, by the inequality $|e^{ib} - e^{ia}| \leq |b-a|$ for $a, b \in \mathbb{R}$, the left hand side of (5.13) is equal to

$$\mathbb{E}\left[\exp\left(i\langle P_{m,Y}(\sigma, X), w\rangle + i\langle \sum_{p>Y} \frac{X(p)}{p^{\sigma}(\log p)^m}, w\rangle\right)\right] + O\left(\frac{|w|}{Y^{2\sigma-1}(\log Y)^m}\right).$$

From the independence of $X(p)$'s, we see that the above expectation is equal to

$$\mathbb{E}\left[\exp\left(i\langle P_{m,Y}(\sigma, X), w\rangle\right)\right] \times \mathbb{E}\left[\exp\left(i\langle \sum_{p>Y} \frac{X(p)}{p^{\sigma}(\log p)^m}, w\rangle\right)\right]. \tag{5.14}$$

Moreover, by the Cauchy-Schwarz inequality and the inequality $|e^{ix}-1| \leq |x|$ for $x \in \mathbb{R}$, we find that

$$\begin{aligned}&\left|\mathbb{E}\left[\exp\left(i\langle \sum_{Y<p\leq Z} \frac{X(p)}{p^{\sigma}(\log p)^m}, w\rangle\right)\right] - 1\right|\\ &\leq |w|\mathbb{E}\left[\left|\sum_{Y<p\leq Z} \frac{X(p)}{p^{\sigma}(\log p)^m}\right|\right] \leq |w|\left(\mathbb{E}\left[\left|\sum_{Y<p\leq Z} \frac{X(p)}{p^{\sigma}(\log p)^m}\right|^2\right]\right)^{1/2}\end{aligned}$$

for any $Z > Y$. Applying Lemma 5.3, the last is

$$\leq |w|\left(\sum_{Y<p\leq Z} \frac{1}{p^{2\sigma}(\log p)^{2m}}\right)^{1/2} \ll \frac{|w|}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}.$$

Therefore, by Lebesgue's dominated convergence theorem, it holds that

$$\begin{aligned}&\mathbb{E}\left[\exp\left(i\langle \sum_{p>Y} \frac{X(p)}{p^{\sigma}(\log p)^m}, w\rangle\right)\right]\\ &= \lim_{Z\to\infty} \mathbb{E}\left[\exp\left(i\langle \sum_{Y<p\leq Z} \frac{X(p)}{p^{\sigma}(\log p)^m}, w\rangle\right)\right] = 1 + O\left(\frac{|w|}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}\right),\end{aligned}$$

and hence, (5.14) is equal to

$$\mathbb{E}\left[\exp\left(i\langle P_{m,Y}(\sigma, X), w\rangle\right)\right] + O\left(\frac{|w|}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}\right).$$

Thus, the left hand side of (5.13) is also equal to the above. □

5.2.2 Proofs of mean value results

Proof of Proposition 5.1. Let $(\sigma, m) \in \mathscr{A}$ be fixed. Suppose that Y satisfies inequality (5.4). By the definition of the set $A_T = A_T(V, Y; \sigma, m)$, we find that

$$\begin{aligned}
&\int_{A_T} \exp\left(z_1 P_{m,Y}(\sigma+it) + z_2 \overline{P_{m,Y}(\sigma+it)}\right) dt \qquad (5.15)\\
&= \sum_{\substack{k+\ell \le Z \\ k,\ell \in \mathbb{Z}_{\ge 0}}} \frac{z_1^k z_2^\ell}{k!\ell!} \int_{A_T} P_{m,Y}(\sigma+it)^k \overline{P_{m,Y}(\sigma+it)}^\ell dt + O\left(T \sum_{\substack{k+\ell > Z \\ k,\ell \in \mathbb{Z}_{\ge 0}}} \frac{|z_1|^k |z_2|^\ell}{k!\ell!} V^{k+\ell} \right),
\end{aligned}$$

where $Z = c_3 V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}$, and c_3 is a small constant decided later. For $|z_1|, |z_2| \le 2^{-1} e^{-2} c_3 V^{\frac{\sigma}{1-\sigma}} (\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}} = 2^{-1} e^{-2} V^{-1} Z$, it holds that

$$\begin{aligned}
\sum_{\substack{k+\ell > Z \\ k,\ell \in \mathbb{Z}_{\ge 0}}} \frac{|z_1|^k |z_2|^\ell}{k! l!} V^{k+\ell} &\le \sum_{n > Z} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(2^{-1} e^{-2} Z\right)^n = \sum_{n > Z} \frac{1}{n!} \left(e^{-2} Z\right)^n \\
&\ll \sum_{n > Z} e^{-n} \ll \exp\left(-c_3 V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right) \qquad (5.16)
\end{aligned}$$

by the Stirling formula. On the other hand, we can write

$$\begin{aligned}
&\int_{A_T} P_{m,Y}(\sigma+it)^k \overline{P_{m,Y}(\sigma+it)}^\ell dt \\
&= \int_0^T P_{m,Y}(\sigma+it)^k \overline{P_{m,Y}(\sigma+it)}^\ell dt - \int_{[0,T]\setminus A_T} P_{m,Y}(\sigma+it)^k \overline{P_{m,Y}(\sigma+it)}^\ell dt.
\end{aligned}$$

Recall that $Y^{k+\ell} \le T (\log T)^{-1}$ is satisfied for $k+\ell \le Z$ if c_3 is sufficiently small. By using the Cauchy-Schwarz inequality and estimates (5.7), (5.9), we have

$$\begin{aligned}
&\frac{1}{T} \int_{[0,T]\setminus A_T} P_{m,Y}(\sigma+it)^k \overline{P_{m,Y}(\sigma+it)}^\ell dt \\
&\le \left(\frac{1}{T} \mathrm{meas}([0,T] \setminus A_T)\right)^{1/2} \left(\frac{1}{T} \int_0^T \left|P_{m,Y}(\sigma+it)\right|^{2(k+\ell)} dt\right)^{1/2} \\
&\ll \exp\left(-\frac{c_1}{2} V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right) \left(\frac{C(m,\sigma)(k+\ell)^{1-\sigma}}{(\log 2(k+\ell))^{m+\tau(\sigma)}}\right)^{k+\ell} \\
&\ll \exp\left(-\frac{c_1}{2} V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right) \left(\frac{C(m,\sigma) Z^{1-\sigma}}{(\log 2Z)^{m+\tau(\sigma)}}\right)^{k+\ell}
\end{aligned}$$

for $1 \leq k+\ell \leq Z$. We note that the same is true for $k=\ell=0$ by estimate (5.9). Therefore, we have

$$\begin{aligned}
&\frac{1}{T}\sum_{\substack{k+\ell\leq Z\\ k,\ell\in\mathbb{Z}_{\geq 0}}}\frac{z_1^k z_2^\ell}{k!\ell!}\int_{[0,T]\setminus A_T} P_{m,Y}(\sigma+it)^k\overline{P_{m,Y}(\sigma+it)}^\ell\,dt\\
&\ll \exp\left(-\frac{c_1}{2}V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right)\sum_{\substack{0\leq k+\ell\leq Z\\ k,\ell\in\mathbb{Z}_{\geq 0}}}\frac{1}{k!\ell!}\left(2^{-1}C'c_3V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right)^{k+\ell}\\
&\ll \exp\left(-\frac{c_1}{2}V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right)\exp\left(C'c_3V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right),
\end{aligned}$$

where $C' > C(m,\sigma)+1$ is a positive constant not depending on V and c_3. Hence, choosing $c_3 = c_1/4C'$, we obtain

$$\begin{aligned}
&\frac{1}{T}\sum_{\substack{k+\ell\leq Z\\ k,\ell\in\mathbb{Z}_{\geq 0}}}\frac{z_1^k z_2^\ell}{k!\ell!}\int_{[0,T]\setminus A_T} P_{m,Y}(\sigma+it)^k\overline{P_{m,Y}(\sigma+it)}^\ell\,dt\\
&\ll \exp\left(-\frac{c_1}{4}V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right) \ll \exp\left(-c_3V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right). \quad (5.17)
\end{aligned}$$

Thus, by (5.15), (5.16), and (5.17), we have

$$\begin{aligned}
&\frac{1}{T}\int_{A_T}\exp\left(z_1P_{m,Y}(\sigma+it)+z_2\overline{P_{m,Y}(\sigma+it)}\right)dt \quad (5.18)\\
&=\frac{1}{T}\sum_{\substack{k+\ell\leq Z\\ k,\ell\in\mathbb{Z}_{\geq 0}}}\frac{z_1^k z_2^\ell}{k!\ell!}\int_0^T P_{m,Y}(\sigma+it)^k\overline{P_{m,Y}(\sigma+it)}^\ell\,dt\\
&\quad + O\left(\exp\left(-c_3V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right)\right).
\end{aligned}$$

Applying Lemma 5.2 to the integral on the right hand side, we see that its first term is equal to

$$\begin{aligned}
&\sum_{\substack{k+\ell\leq Z\\ k,\ell\in\mathbb{Z}_{\geq 0}}}\mathbb{E}\left[\frac{z_1^k z_2^\ell}{k!\ell!}P_{m,Y}(\sigma,X)^k\overline{P_{m,Y}(\sigma,X)}^\ell\right] + O\left(\frac{1}{T}\left(V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}Y\right)^{2Z}\right)\\
&=\mathbb{E}\left[\exp\left(z_1P_{m,Y}(\sigma,X)+z_2\overline{P_{m,Y}(\sigma,X)}\right)\right]\\
&\quad -\sum_{\substack{k+\ell> Z\\ k,\ell\in\mathbb{Z}_{\geq 0}}}\frac{z_1^k z_2^\ell}{k!\ell!}\mathbb{E}\left[P_{m,Y}(\sigma,X)^k\overline{P_{m,Y}(\sigma,X)}^\ell\right]\\
&\quad + O\left(\frac{1}{T}\left(V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}Y\right)^{2Z}\right).
\end{aligned}$$

By using the Cauchy-Schwarz inequality and estimate (5.8), we obtain

$$\mathbb{E}\left[P_{m,Y}(\sigma,X)^k\overline{P_{m,Y}(\sigma,X)}^\ell\right] \ll \left(\frac{C(\sigma,m)(k+\ell)^{1-\sigma}}{(\log 2(k+\ell))^{m+\tau(\sigma)}}\right)^{k+\ell}.$$

By this estimate, a calculation similar to (5.16) shows that

$$\sum_{\substack{k+\ell>Z\\ k,\ell\in\mathbb{Z}_{\geq 0}}} \frac{z_1^k z_2^\ell}{k!\ell!}\mathbb{E}\left[P_{m,Y}(\sigma,X)^k\overline{P_{m,Y}(\sigma,X)}^\ell\right] \ll \exp\left(-c_3 V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right).$$

Hence, the left hand side of (5.18) is equal to

$$\mathbb{E}\left[\exp\left(z_1 P_{m,Y}(\sigma,X)+z_2\overline{P_{m,Y}(\sigma,X)}\right)\right] \\ +O\left(\frac{1}{T}\left(V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}Y\right)^{2Z}+\exp\left(-c_3 V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right)\right),$$

which completes the proof of Proposition 5.1. □

Proof of Proposition 5.2. Suppose that Y satisfies (5.4), and let $w=u+iv$ be a complex number with $u,v\in\mathbb{R}$. Let $0<W\leq(|w|+1)^{-1}$ be a parameter chosen later. Then we see that

$$\frac{1}{T}\int_0^T \exp\left(i\langle\tilde{\eta}_m(\sigma+it),w\rangle\right)dt \\ =\frac{1}{T}\int_{B_T}\exp\left(i\langle\tilde{\eta}_m(\sigma+it),w\rangle\right)dt+O\left(\frac{1}{T}(\mathrm{meas}([0,T]\setminus B_T)\right). \tag{5.19}$$

From the definition of the set $B_T=B_T(Y,W;\sigma,m)$, we can write

$$\exp\left(i\langle\tilde{\eta}_m(\sigma+it),w\rangle\right)=\exp\left(i\langle P_{m,Y}(\sigma+it),w\rangle\right)+O\left(|w|WY^{\frac{1}{2}-\sigma}\right)$$

for all $t\in B_T$ by using the inequality $|e^{ib}-e^{ia}|\leq|b-a|$. By this formula, the integral on the right hand side of (5.19) is equal to

$$\frac{1}{T}\int_{B_T}\exp\left(i\langle P_{m,Y}(\sigma+it),w\rangle\right)dt+O\left(|w|WY^{\frac{1}{2}-\sigma}\right) \\ =\frac{1}{T}\int_{A_T}\exp\left(i\langle P_{m,Y}(\sigma+it),w\rangle\right)dt \\ +O\left(|w|WY^{\frac{1}{2}-\sigma}+\frac{1}{T}\left\{\mathrm{meas}([0,T]\setminus A_T)+\mathrm{meas}([0,T]\setminus B_T)\right\}\right).$$

Therefore, by this formula and Lemma 5.5, the left hand side of (5.19) is

$$=\frac{1}{T}\int_{A_T}\exp\left(i\langle P_{m,Y}(\sigma+it),w\rangle\right)dt \\ +O\left(|w|WY^{\frac{1}{2}-\sigma}+\exp\left(-cV^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right)+\frac{1}{T}\mathrm{meas}([0,T]\setminus B_T)\right).$$

Here, applying Proposition 5.1 to the integral on the right hand side with $z_1=\frac{i}{2}(u-iv)$, $z_2=\frac{i}{2}(u+iv)$, the above integral is equal to

$$\mathbb{E}\left[\exp(i\langle P_{m,Y}(\sigma,X),w\rangle)\right] \\ +O\left(\frac{1}{T}\left(V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}Y\right)^{V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}}+\exp\left(-cV^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right)\right)$$

for $|w| \le cV^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}$. Moreover, applying Lemma 5.8, we find that

$$\mathbb{E}\left[\exp(i\langle P_{m,Y}(\sigma,X),w\rangle)\right] = \mathbb{E}\left[\exp(i\langle \tilde{\eta}_m(\sigma,X),w\rangle)\right] + O\left(\frac{|w|}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}\right).$$

Hence, for $|w| \le cV^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}$, we obtain

$$\begin{aligned}
&\frac{1}{T}\int_0^T \exp\left(i\langle \tilde{\eta}_m(\sigma+it),w\rangle\right)dt - \mathbb{E}\left[\exp(i\langle \tilde{\eta}_m(\sigma,X),w\rangle)\right] \\
&\ll |w|WY^{\frac{1}{2}-\sigma} + \exp\left(-cV^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}\right) \\
&\quad + \frac{1}{T}\operatorname{meas}([0,T]\setminus B_Y(T,W)) \\
&\quad + \frac{1}{T}\left(V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}Y\right)^{V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\tau(\sigma)}{1-\sigma}}} + \frac{|w|}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}.
\end{aligned}$$

Choosing $W = (\log T)^{1/2}$, $Y = (\log T)^{\frac{A+2}{\sigma-1/2}}$, and $V = c_3\frac{(\log T)^{\frac{1}{1-\sigma}}}{(\log\log T)^{m+1}}$ with c_3 a small constant depending on σ, m, and A, we have

$$\frac{1}{T}\int_0^T \exp\left(i\langle \tilde{\eta}_m(\sigma+it),w\rangle\right)dt = \mathbb{E}\left[\exp(i\langle \tilde{\eta}_m(\sigma,X),w\rangle)\right] + O\left(\frac{1}{(\log T)^A}\right)$$

for $|w| \le c_4(\log T)^\sigma(\log\log T)^m$ from estimate (5.12). Here, c_4 is a small constant determined from c_3. Thus, we complete the proof of Proposition 5.2. □

5.3 Probability density function for $\tilde{\eta}_m(\sigma, X)$

The goal of this section is the following proposition.

Proposition 5.3. *Let $(\sigma,m)\in\mathcal{A}$. There exists a continuous function $D_{\sigma,m}:\mathbb{C}\to\mathbb{R}_{\ge 0}$ such that*

$$\mathbb{P}(\tilde{\eta}_m(\sigma,X)\in A) = \int_A D_{\sigma,m}(z)\,|dz|,$$

for all $A\in\mathcal{B}(\mathbb{C})$, where $|dz| = (2\pi)^{-1}dxdy$ for $z = x+iy$. Furthermore, the following properties hold.

(i) *Let $m\ge 1$. If $1/2\le\sigma<1$, then $D_{\sigma,m}(z)>0$ for all $z\in\mathbb{C}$. If $\sigma\ge 1$, then $D_{\sigma,m}$ is compactly supported.*

(ii) *Let $m=0$. If $1/2<\sigma\le 1$, then $D_{\sigma,m}(z)>0$ for all $z\in\mathbb{C}$. If $\sigma>1$, then $D_{\sigma,m}$ is compactly supported.*

(iii) *Let $(\sigma,m)\in\mathcal{A}$. For any $a>0$, we have*

$$\int_{\mathbb{C}} e^{a|z|}D_{\sigma,m}(z)\,|dz| < \infty.$$

The distribution of $\tilde{\eta}_m(\sigma, X)$ is the probabilistic measure defined as

$$\mu_{\sigma,m}(A) = \mathbb{P}(\tilde{\eta}_m(\sigma, X) \in A) \tag{5.20}$$

for $A \in \mathcal{B}(\mathbb{C})$. Let p be a prime number. We also define

$$\mu_{\sigma,m,p}(A) = \mathbb{P}(\tilde{\eta}_{m,p}(\sigma, X(p)) \in A)$$

for $A \in \mathcal{B}(\mathbb{C})$, where $\tilde{\eta}_{m,p}(\sigma, X(p))$ is defined from (5.2).

Lemma 5.9. *Let $(\sigma, m) \in \mathcal{A}$. The convolution measure*

$$\nu_{\sigma,m,N} = \mu_{\sigma,m,p_1} * \cdots * \mu_{\sigma,m,p_N} \tag{5.21}$$

converges weakly to $\mu_{\sigma,m}$ as $N \to \infty$, where p_n indicates the n-th prime number. Furthermore, the convergence is absolute in the sense that it converges to $\mu_{\sigma,m}$ in any order of terms of the convolution.

Proof. Recall that $\tilde{\eta}_{m,p}(\sigma, X(p))$ and $\tilde{\eta}_{m,q}(\sigma, X(q))$ are independent if p and q are distinct prime numbers. Hence, the distribution of $\tilde{\eta}_{m,p}(\sigma, X(p)) + \tilde{\eta}_{m,q}(\sigma, X(q))$ equals to $\mu_{\sigma,m,p} * \mu_{\sigma,m,q}$. More generally, we see that

$$\nu_{\sigma,m,N}(A) = \mathbb{P}\left(\sum_{n \le N} \tilde{\eta}_{m,p_n}(\sigma, X(p_n)) \in A\right)$$

for all $A \in \mathcal{B}(\mathbb{C})$. By Lemma 5.1, $\sum_{n \le N} \tilde{\eta}_{m,p_n}(\sigma, X(p_n)) \to \tilde{\eta}_m(\sigma, X)$ in law as $N \to \infty$, i.e. $\nu_{\sigma,m,N} \to \mu_{\sigma,m}$ weakly. The absoluteness of the convergence follows from Jessen-Wintner [54, Theorem 6]. □

In general, the support of a probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ is defined as

$$\mathrm{supp}(P) = \left\{ z \in \mathbb{C} \ : \ P(A) > 0 \text{ for any } A \in \mathcal{B}(\mathbb{C}) \text{ with } z \in A^i \right\},$$

where A^i is the interior of A. We know that $\mathrm{supp}(P)$ is a non-empty closed subset of $\mathbb{C}$. Applying Lemma 5.9, we can study the support of $\mu_{\sigma,m}$.

Lemma 5.10. *Let $(\sigma, m) \in \mathcal{A}$ and $\mu_{\sigma,m}$ be the probability measure defined as* (5.20).

(i) *Let $m \ge 1$. If $1/2 \le \sigma < 1$, then $\mathrm{supp}(\mu_{\sigma,m}) = \mathbb{C}$. If $\sigma \ge 1$, then $\mathrm{supp}(\mu_{\sigma,m})$ is a compact subset of $\mathbb{C}$.*

(ii) *Let $m = 0$. If $1/2 < \sigma \le 1$, then $\mathrm{supp}(\mu_{\sigma,m}) = \mathbb{C}$. If $\sigma > 1$, then $\mathrm{supp}(\mu_{\sigma,m})$ is a compact subset of $\mathbb{C}$.*

Proof. Let $\{A_N\}_{N \in \mathbb{N}}$ be a sequence of subsets of $\mathbb{C}$. We denote by $\lim_{N \to \infty} A_N$ the set of all points in $\mathbb{C}$ that may be represented in at least one way as the limit of a sequence of points $a_n \in A_n$. For $A, B \subset \mathbb{C}$, define $A + B$ by

$\{a+b\ :\ a\in A, b\in B\}$. Then Jessen and Wintner [54, Theorem 3] proved that

$$\operatorname{supp}(P) = \lim_{N\to\infty} \left(\operatorname{supp}(P_1) + \cdots + \operatorname{supp}(P_N)\right)$$

if the convolution measure $P_1 * \cdots * P_N$ converges weakly to P as $N \to \infty$. Applying further [54, Theorem 14] with $P_n = \mu_{\sigma,m,p_n}$ for $m = 0$, we obtain assertion (ii). Now, we consider the case $m \geq 1$. By Lemma 5.9, we have

$$\operatorname{supp}(\mu_{\sigma,m}) = \lim_{N\to\infty} \left(\operatorname{supp}(\mu_{\sigma,m,p_1}) + \cdots + \operatorname{supp}(\mu_{\sigma,m,p_N})\right).$$

Note that the support of every $\mu_{\sigma,m,p}$ is determined as

$$\operatorname{supp}(\mu_{\sigma,m,p}) = \left\{\tilde{\eta}_{m,p}(\sigma, e^{i\theta})\ :\ \theta \in [0, 2\pi)\right\}$$

by the definition. First, we let $1/2 \leq \sigma < 1$. In this case we apply [112, Theorem 5.4] to deduce that for any $z \in \mathbb{C}$, $N_0 \geq 1$, and $\varepsilon > 0$, we have

$$\left| \left(z - \sum_{n<N_0} \frac{\operatorname{Li}_{m+1}(p_n^{-\sigma})}{(\log p_n)^m} \right) - \sum_{N_0<n\leq N} \frac{p_n^{-\sigma} e^{i\theta_n}}{(\log p_n)^m} \right| < \varepsilon$$

with some $N = N(z, N_0, \varepsilon) > N_0$ and $\{\theta_n\}_{N_0<n\leq N} \in [0, 2\pi)^{N-N_0}$. We also derive

$$\left| \sum_{N_0<n\leq N} \frac{\operatorname{Li}_{m+1}(p_n^{-\sigma} e^{i\theta_n})}{(\log p_n)^m} - \sum_{N_0<n\leq N} \frac{p_n^{-\sigma} e^{i\theta_n}}{(\log p_n)^m} \right| < \varepsilon$$

if N_0 is sufficiently large. These imply $\operatorname{supp}(\mu_{\sigma,m}) = \mathbb{C}$ for $1/2 \leq \sigma < 1$. Next, we let $\sigma \geq 1$. Then we have

$$\sum_{n=1}^{\infty} \tilde{\eta}_{m,p_n}(\sigma, e^{i\theta_n}) \ll \sum_{n=1}^{\infty} \frac{1}{p_n \log p_n} < \infty$$

for any $\{\theta_n\}_{n\in\mathbb{N}} \in [0, 2\pi)^{\mathbb{N}}$ in this case. Hence, $\operatorname{supp}(\mu_{\sigma,m})$ is included in a bounded disk, which completes the proof. □

Lemma 5.11. *Let* $(\sigma, m) \in \mathcal{A}$. *Then the expected value* $\mathbb{E}\left[\exp\left(a|\tilde{\eta}_m(\sigma, X)|\right)\right]$ *is finite for all* $a > 0$.

Proof. Since $\sum_{p<y} \tilde{\eta}_{m,p}(\sigma, X(p)) \to \tilde{\eta}_m(\sigma, X)$ in law as $y \to \infty$, we have, by Fatou's lemma,

$$\mathbb{E}\left[\Phi(\tilde{\eta}_m(\sigma, X))\right] \leq \liminf_{y\to\infty} \mathbb{E}\left[\Phi\left(\sum_{p<y} \tilde{\eta}_{m,p}(\sigma, X(p))\right)\right] \tag{5.22}$$

for any continuous function $\Phi : \mathbb{C} \to \mathbb{R}_{\geq 0}$. In particular, we can take the function $\Phi(z) = \exp(\pm a \operatorname{Re} z)$. In this case, we have $\Phi(z+w) = \Phi(z)\Phi(w)$, and therefore the equation

$$\mathbb{E}\left[\Phi\left(\sum_{p<y} \tilde{\eta}_{m,p}(\sigma, X(p))\right)\right] = \prod_{p<y} \mathbb{E}\left[\Phi\left(\tilde{\eta}_{m,p}(\sigma, X(p))\right)\right] \tag{5.23}$$

holds since $\{X(p)\}_{p\in\mathcal{P}}$ is a set of independent variables. If we suppose $p \geq a^{1/\sigma}$, then the estimate

$$\Phi\left(\tilde{\eta}_{m,p}(\sigma, X(p))\right) = 1 \pm a \operatorname{Re} \tilde{\eta}_{m,p}(\sigma, X(p)) + O\left(\frac{a^2}{p^{2\sigma}(\log p)^{2m}}\right)$$

follows by the Taylor expansion. This implies

$$\mathbb{E}\left[\Phi\left(\tilde{\eta}_{m,p}(\sigma, X(p))\right)\right] = 1 + O\left(\frac{a^2}{p^{2\sigma}(\log p)^{2m}}\right)$$

since $\mathbb{E}\left[\operatorname{Re} \tilde{\eta}_{m,p}(\sigma, X(p))\right]$ vanishes. Recall that $\sum_p p^{-2\sigma}(\log p)^{-2m}$ is finite if $(\sigma, m) \in \mathcal{A}$. Hence, we conclude that the infinite product

$$\prod_p \mathbb{E}\left[\Phi\left(\tilde{\eta}_{m,p}(\sigma, X(p))\right)\right]$$

converges, and that $\mathbb{E}\left[\Phi(\tilde{\eta}_m(\sigma, X))\right]$ is finite by (5.22) and (5.23). From the above, we deduce

$$\begin{aligned}&\mathbb{E}\left[\exp(a|\operatorname{Re} \tilde{\eta}_m(\sigma, X)|)\right]\\ &\leq \mathbb{E}\left[\exp(a \operatorname{Re} \tilde{\eta}_m(\sigma, X))\right] + \mathbb{E}\left[\exp(-a \operatorname{Re} \tilde{\eta}_m(\sigma, X))\right] < \infty.\end{aligned}$$

One can prove that $\mathbb{E}\left[\exp(a|\operatorname{Im} \tilde{\eta}_m(\sigma, X)|)\right]$ is finite by replacing the function Φ by $\Phi(z) = \exp(\pm a \operatorname{Im} z)$. By the Cauchy-Schwarz inequality, we conclude that

$$\begin{aligned}&\mathbb{E}\left[\exp\left(a|\tilde{\eta}_m(\sigma, X)|\right)\right]\\ &\leq \sqrt{\mathbb{E}\left[\exp\left(2a|\operatorname{Re} \tilde{\eta}_m(\sigma, X)|\right)\right]}\sqrt{\mathbb{E}\left[\exp\left(2a|\operatorname{Im} \tilde{\eta}_m(\sigma, X)|\right)\right]} < \infty\end{aligned}$$

as desired. □

The characteristic function of $\mu_{\sigma,m}$ of (5.20) is represented as

$$\begin{aligned}\Lambda(w; \mu_{\sigma,m}) &:= \mathbb{E}\left[\exp(iu \operatorname{Re} \tilde{\eta}_m(\sigma, X) + iv \operatorname{Im} \tilde{\eta}_m(\sigma, X))\right]\\ &= \prod_p \mathbb{E}\left[\exp(iu \operatorname{Re} \tilde{\eta}_{m,p}(\sigma, X(p)) + iv \operatorname{Im} \tilde{\eta}_{m,p}(\sigma, X(p)))\right]\\ &= \prod_p \Lambda(w; \mu_{\sigma,m,p})\end{aligned}$$

for $w = u + iv$ since $\mu_{\sigma,m,p_1} * \cdots * \mu_{\sigma,m,p_N} \to \mu_{\sigma,m}$ weakly as $N \to \infty$. Applying this infinite product expression, we prove the following result.

Lemma 5.12. *Let $(\sigma, m) \in \mathcal{A}$. Suppose that $|w| \geq c(\sigma, m)$ with a large constant $c(\sigma, m) > 0$. Then we have*

$$|\Lambda(w; \mu_{\sigma,m})| \leq \exp\left(-|w|^{1/(2\sigma)}\right).$$

Proof. Since $|\Lambda(w;\mu_{\sigma,m,p})| \leq 1$ for every p, the inequality

$$\left|\Lambda(w;\mu_{\sigma,m})\right| \leq \prod_{p\in\mathscr{P}} \left|\Lambda(w;\mu_{\sigma,m,p})\right| \tag{5.24}$$

holds for any subset $\mathscr{P} \subset \mathcal{P}$. Put $P(M) = M|w|^{1/\sigma}$ for $M \geq 1$. By the Taylor expansion of $\exp(z)$, we obtain

$$\begin{aligned}
&\exp(iu \operatorname{Re}\tilde{\eta}_{m,p}(\sigma, X(p)) + iv \operatorname{Im}\tilde{\eta}_{m,p}(\sigma, X(p)))\\
&= 1 + iu \operatorname{Re}\tilde{\eta}_{m,p}(\sigma, X(p)) + iv \operatorname{Im}\tilde{\eta}_{m,p}(\sigma, X(p))\\
&\qquad + \frac{1}{2}\{iu \operatorname{Re}\tilde{\eta}_{m,p}(\sigma, X(p)) + iv \operatorname{Im}\tilde{\eta}_{m,p}(\sigma, X(p))\}^2 + O\left(\frac{(|u|+|v|)^3}{p^{3\sigma}(\log p)^{3m}}\right)
\end{aligned}$$

for $p > P(M_1)$ with some $M_1 \geq 1$. We have

$$\mathbb{E}\left[\operatorname{Re}\tilde{\eta}_{m,p}(\sigma, X(p))\right] = \mathbb{E}\left[\operatorname{Im}\tilde{\eta}_{m,p}(\sigma, X(p))\right] = 0,$$

$$\mathbb{E}\left[\operatorname{Re}\tilde{\eta}_{m,p}(\sigma, X(p)) \operatorname{Im}\tilde{\eta}_{m,p}(\sigma, X(p))\right] = 0,$$

and

$$\mathbb{E}\left[\left(\operatorname{Re}\tilde{\eta}_{m,p}(\sigma, X(p))\right)^2\right] = \mathbb{E}\left[\left(\operatorname{Im}\tilde{\eta}_{m,p}(\sigma, X(p))\right)^2\right] = \frac{1}{2}\frac{\operatorname{Li}_{2m+2}(p^{-2\sigma})}{(\log p)^{2m}}.$$

Therefore, the characteristic function $\Lambda(w;\mu_{\sigma,m,p})$ is evaluated as

$$\Lambda(w;\mu_{\sigma,m,p}) = 1 - \frac{|w|^2}{4}\frac{\operatorname{Li}_{2m+2}(p^{-2\sigma})}{(\log p)^{2m}} + O\left(\frac{|w|^3}{p^{3\sigma}(\log p)^{3m}}\right).$$

Hence, we deduce the asymptotic formula

$$\log\left|\Lambda(w;\mu_{\sigma,m,p})\right| = -\frac{|w|^2}{4}\frac{\operatorname{Li}_{2m+2}(p^{-2\sigma})}{(\log p)^{2m}} + O\left(\frac{|w|^3}{p^{3\sigma}(\log p)^{3m}}\right)$$

if $p > P(M_2)$ with some $M_2 \geq M_1$. We notice that the inequalities

$$\begin{aligned}
\frac{|w|^2}{4}\frac{\operatorname{Li}_{2m+2}(p^{-2\sigma})}{(\log p)^{2m}} &\geq \frac{1}{4}\frac{|w|^2}{p^{2\sigma}(\log p)^{2m}},\\
\frac{|w|^3}{p^{3\sigma}(\log p)^{3m}} &\leq \frac{1}{M}\frac{|w|^2}{p^{2\sigma}(\log p)^{2m}}
\end{aligned}$$

are satisfied for $p > P(M)$ with any $M \geq 1$. Hence, there exists an absolute constant $M_3 \geq M_2$ such that the inequality

$$\log\left|\Lambda(w;\mu_{\sigma,m,p})\right| \leq -\frac{1}{8}\frac{|w|^2}{p^{2\sigma}(\log p)^{2m}}$$

holds for $p > P(M_3)$. Therefore, taking $\mathscr{P} = \mathcal{P}_{>P(M_3)}$ in (5.24), we deduce

$$\left|\Lambda(w;\mu_{\sigma,m})\right| \leq \exp\left(-\frac{|w|^2}{8}\sum_{p>P(M_3)}\frac{1}{p^{2\sigma}(\log p)^{2m}}\right) \leq \exp\left(-|w|^{1/(2\sigma)}\right)$$

if $|w| > c(\sigma, m)$ with some large constant $c(\sigma, m) > 0$. □

Proof of Proposition 5.3. By Lemma 5.12, we see that

$$\int_{\mathbb{C}} |\Lambda(w; \mu_{\sigma,m})| \, |dw| < \infty.$$

Therefore the probability measure $\mu_{\sigma,m}$ is absolutely continuous in the sense that it is represented as

$$\mu_{\sigma,m}(A) = \int_A D_{\sigma,m}(z) \, |dz|$$

for all $A \in \mathcal{B}(\mathbb{C})$ with some non-negative Lebesgue measurable function $D_{\sigma,m}$. By Levy's inversion formula, we can determine one of such functions as

$$D_{\sigma,m}(z) = \int_{\mathbb{C}} \Lambda(w; \mu_{\sigma,m}) \exp(-i\langle z, w \rangle) \, |dw|, \tag{5.25}$$

which is a continuous function. We prove properties (i)–(iii). Remark that the support of the function $D_{\sigma,m}$ is equal to $\mathrm{supp}(\mu_{\sigma,m})$ studied in Lemma 5.10. Let $m \geq 1$. Then the fact that $D_{\sigma,m}$ is compactly supported for $\sigma \geq 1$ is a direct consequence of the lemma. Let $1/2 \leq \sigma < 1$. To prove the positivity of $D_{\sigma,m}(z)$, we define two probabilistic measures

$$\nu^{\flat}_{\sigma,m,N} = \mu_{\sigma,m,2} * \mu_{\sigma,m,p^{\flat}_1} * \cdots * \mu_{\sigma,m,p^{\flat}_N} \quad \text{and} \quad \nu^{\#}_{\sigma,m,N} = \mu_{\sigma,m,p^{\#}_1} * \cdots * \mu_{\sigma,m,p^{\#}_N}$$

as analogues of (5.21), where $p^{\flat}_n$ is the n-th prime number congruent to 1 (mod 4), and $p^{\#}_n$ is the n-th prime number congruent to $-1 \pmod 4$. Then it can be proved that $\nu^{\flat}_{\sigma,m,N}$ and $\nu^{\#}_{\sigma,m,N}$ converge weakly to some probability measures $\mu^{\flat}_{\sigma,m}$ and $\mu^{\#}_{\sigma,m}$ as $N \to \infty$, respectively. One can check that the limit measures $\mu^{\flat}_{\sigma,m}$ and $\mu^{\#}_{\sigma,m}$ satisfy many of the same properties as $\mu_{\sigma,m}$ described above. In particular, we have

$$\mathrm{supp}(\mu^{\flat}_{\sigma,m}) = \mathrm{supp}(\mu^{\#}_{\sigma,m}) = \mathbb{C}$$

for $1/2 \leq \sigma < 1$ along the same line of Lemma 5.10. Furthermore, we obtain

$$|\Lambda(w; \mu^{\flat}_{\sigma,m})|, \ |\Lambda(w; \mu^{\#}_{\sigma,m})| \leq \exp\left(-|w|^{1/(2\sigma)}\right)$$

as analogues of Lemma 5.12. Therefore, there exist non-negative continuous functions $D^{\flat}_{\sigma,m}$ and $D^{\#}_{\sigma,m}$ such that

$$\mu^{\flat}_{\sigma,m}(A) = \int_A D^{\flat}_{\sigma,m}(z) \, |dz| \quad \text{and} \quad \mu^{\#}_{\sigma,m}(A) = \int_A D^{\#}_{\sigma,m}(z) \, |dz|$$

for all $A \in \mathcal{B}(\mathbb{C})$, whose supports are equal to $\mathbb{C}$ if $1/2 \leq \sigma < 1$. Recall that $\nu^{\flat}_{\sigma,m,N} * \nu^{\#}_{\sigma,m,N}$ converges weakly to $\mu_{\sigma,m}$ as $N \to \infty$ by Lemma 5.9. Hence, we deduce the equation

$$D_{\sigma,m}(z) = \int_{\mathbb{C}} D^{\flat}_{\sigma,m}(z - w) D^{\#}_{\sigma,m}(w) |dw|$$

for any $z \in \mathbb{C}$. Since the functions $D^{\flat}_{\sigma,m}$ and $D^{\#}_{\sigma,m}$ are continuous and are non-zeros on every disk on $\mathbb{C}$, we see that $D_{\sigma,m}(z) > 0$ for any $z \in \mathbb{C}$. Hence, the proof of assertion (i) is completed. We note that assertion (ii) is just a consequence of [54, Theorem 14]. Finally, we have

$$\int_{\mathbb{C}} e^{a|z|} D_{\sigma,m}(z)\,|dz| = \mathbb{E}\left[\exp\left(a|\tilde{\eta}_m(\sigma,X)|\right)\right] < \infty$$

for any $a > 0$ by Lemma 5.11. Thus, we complete the proof of assertion (iii). □

Let $(\sigma,m) \in \mathcal{A}$. By Lemma 5.11, we see that the moment-generating function

$$\mathbb{E}\left[\exp(s\,\mathrm{Re}(e^{-i\alpha}, \tilde{\eta}_m(\sigma,X)))\right]$$

exists for any $s \in \mathbb{C}$. The following lemma is used in the proof of Theorem 5.2.

Lemma 5.13. *Let $(\sigma,m) \in \mathcal{A}$. For $Y \geq 3$ and $s = \kappa + it$ with $|s| \leq Y^{\sigma-\frac{1}{2}}(\log Y)^m$, we have*

$$\begin{aligned}\mathbb{E}\left[\exp(s\,\mathrm{Re}\,e^{-i\alpha}\tilde{\eta}_m(\sigma,X))\right] &= \mathbb{E}\left[\exp\left(s\,\mathrm{Re}\,e^{-i\alpha}P_{m,Y}(\sigma,X)\right)\right] \\ &\quad + O\left(\mathbb{E}\left[\exp(\kappa\,\mathrm{Re}\,e^{-i\alpha}\tilde{\eta}_m(\sigma,X))\right]\frac{|s|}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}\right).\end{aligned}$$

Proof. Since the estimate

$$\sum_{\substack{p^k > Y \\ k \geq 2}} \frac{X(p)^k}{k p^{k\sigma}(\log p^k)^m} \ll \frac{1}{Y^{2\sigma-1}(\log Y)^m}$$

holds, we have

$$\begin{aligned}&\mathbb{E}\left[\exp(s\,\mathrm{Re}\,e^{-i\alpha}\tilde{\eta}_m(\sigma,X))\right] \\ &= \mathbb{E}\left[\exp\left(s\,\mathrm{Re}\,e^{-i\alpha}P_{m,Y}(\sigma,X) + s\sum_{p>Y}\frac{\mathrm{Re}\,e^{-i\alpha}X(p)}{p^{\sigma}(\log p)^m}\right.\right. \\ &\qquad\qquad \left.\left. + O\left(\frac{|s|}{Y^{2\sigma-1}(\log Y)^m}\right)\right)\right] \qquad (5.26) \\ &= \mathbb{E}\left[\exp\left(s\,\mathrm{Re}\,e^{-i\alpha}P_{m,Y}(\sigma,X) + s\sum_{p>Y}\frac{\mathrm{Re}\,e^{-i\alpha}X(p)}{p^{\sigma}(\log p)^m}\right)\right] \\ &\quad + O\left(\mathbb{E}\left[\kappa\,\mathrm{Re}\,e^{-i\alpha}P_{m,Y}(\sigma,X) + \kappa\sum_{p>Y}\frac{\mathrm{Re}\,e^{-i\alpha}X(p)}{p^{\sigma}(\log p)^m}\right]\frac{|s|}{Y^{2\sigma-1}(\log Y)^m}\right).\end{aligned}$$

Note that the independence of $X(p)$'s yields

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(s\operatorname{Re} e^{-i\alpha}P_{m,Y}(\sigma,X)+s\sum_{p>Y}\frac{\operatorname{Re} e^{-i\alpha}X(p)}{p^{\sigma}(\log p)^m}\right)\right]\\
&=\mathbb{E}\left[\exp\left(s\operatorname{Re} e^{-i\alpha}P_{m,Y}(\sigma,X)\right)\right]\times\prod_{p>Y}\mathbb{E}\left[\exp\left(s\frac{\operatorname{Re} e^{-i\alpha}X(p)}{p^{\sigma}(\log p)^m}\right)\right]. \qquad (5.27)
\end{aligned}$$

Furthermore, if $p > Y$, we find that the inequality

$$\left|s\frac{\operatorname{Re} e^{-i\alpha}X(p)}{p^{\sigma}(\log p)^m}\right|\leq\frac{|s|}{p^{\sigma}(\log p)^m}<1$$

holds for $|s| \leq Y^{\sigma-\frac{1}{2}}(\log Y)^m$. From the Taylor expansion of $\exp(z)$ we deduce

$$\begin{aligned}
\mathbb{E}\left[\exp\left(s\frac{\operatorname{Re} e^{-i\alpha}X(p)}{p^{\sigma}(\log p)^m}\right)\right]&=\mathbb{E}\left[1+s\frac{\operatorname{Re} e^{-i\alpha}X(p)}{p^{\sigma}(\log p)^m}+O\left(\frac{|s|^2}{p^{2\sigma}(\log p)^{2m}}\right)\right]\\
&=1+O\left(\frac{|s|^2}{p^{2\sigma}(\log p)^{2m}}\right)
\end{aligned}$$

since the expected value $\mathbb{E}\left[\operatorname{Re}(e^{-i\alpha}X(p))\right]$ vanishes. Therefore, by equation (5.27), the formula

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(s\operatorname{Re} e^{-i\alpha}P_{m,Y}(\sigma,X)+s\sum_{p>Y}\frac{\operatorname{Re} e^{-i\alpha}X(p)}{p^{\sigma}(\log p)^m}\right)\right]\\
&=\mathbb{E}\left[\exp\left(s\operatorname{Re} e^{-i\alpha}P_{m,Y}(\sigma,X)\right)\right]\\
&\quad+O\left(\mathbb{E}\left[\exp\left(\kappa\operatorname{Re} e^{-i\alpha}P_{m,Y}(\sigma,X)\right)\right]\frac{|s|}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}\right)
\end{aligned}$$

holds for $|s| \leq Y^{\sigma-\frac{1}{2}}(\log Y)^m$. Inserting this to (5.26), we finally obtain

$$\begin{aligned}
&\mathbb{E}\left[\exp(s\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma,X))\right]\\
&=\mathbb{E}\left[\exp\left(s\operatorname{Re} e^{-i\alpha}P_{m,Y}(\sigma,X)\right)\right]\left(1+O\left(\frac{|s|}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}\right)\right),
\end{aligned}$$

which yields the result. □

5.4 Discrepancy bounds: Proof of Theorem 5.1

In this paper, we derive discrepancy bounds by applying Esseen's inequality.

Lemma 5.14 (Sadikova [103]). *Let P, Q be probabilistic measures on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ with the distribution functions*

$$F(s,t)=P((-\infty,s]\times(-\infty,t]) \quad \text{and} \quad G(s,t)=Q((-\infty,s]\times(-\infty,t]).$$

Denote by $f(u,v)$ and $g(u,v)$ the characteristic functions of P and Q, that is,

$$f(u,v) = \int_{\mathbb{R}^2} e^{i(ux+vy)}\, dP(x,y) \quad \textit{and} \quad g(u,v) = \int_{\mathbb{R}^2} e^{i(ux+vy)}\, dQ(x,y).$$

Furthermore, we put

$$\hat{f}(u,v) = f(u,v) - f(u,0)f(0,v) \quad \textit{and} \quad \hat{g}(u,v) = g(u,v) - g(u,0)g(0,v).$$

Suppose that $G(s,t)$ is partially differentiable and that $G_s(s,t)$ and $G_t(s,t)$ are bounded on $\mathbb{R}^2$. Then we have

$$\begin{aligned}\sup_{(s,t)\in\mathbb{R}^2} |F(s,t) - G(s,t)| &\leq \frac{2}{(2\pi)^2} \iint_{[-R,R]^2} \left| \frac{\hat{f}(u,v) - \hat{g}(u,v)}{uv} \right| dudv \\ &+ \frac{2}{\pi}\int_{-R}^{R} \left| \frac{f(u,0) - g(u,0)}{u} \right| du + \frac{2}{\pi}\int_{-R}^{R} \left| \frac{f(0,v) - g(0,v)}{v} \right| dv \\ &+ \left(6\sqrt{2} + 8\sqrt{3} + \frac{48}{\pi} \right) (A_1 + A_2) R^{-1} \end{aligned} \tag{5.28}$$

for all $R > 0$, where $A_1 = \sup_{(s,t)\in\mathbb{R}^2} |G_s(s,t)|$ and $A_2 = \sup_{(s,t)\in\mathbb{R}^2} |G_t(s,t)|$.

To begin with, we prepare the following result.

Lemma 5.15. *Let $(\sigma, m) \in \mathcal{A}$, and let $T > 0$ be large. We have*

$$\frac{1}{T}\int_0^T |\tilde{\eta}_m(\sigma + it)|^2 dt \ll_{\sigma,m} 1.$$

Proof. Let $X = T^{1/135}$, and put

$$R_m(\sigma + it; X) = \tilde{\eta}_m(\sigma + it) - \sum_{2\leq n\leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}}.$$

Then we see that

$$|\tilde{\eta}_m(\sigma + it)|^2 \leq 4\left| \sum_{2\leq n\leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 + 4|R_m(\sigma + it; X)|^2.$$

By Lemma 2.1 and Theorem 2.6, we have

$$\frac{1}{T}\int_0^T |R_m(\sigma + it; X)|^2 dt \ll \frac{T^{\frac{1-2\sigma}{135}}}{(\log T)^{2m}}.$$

Also, we can write

$$\begin{aligned}&\left| \sum_{2\leq n\leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 \\ &= \sum_{p^k\leq X} \frac{1}{k^{2(m+1)}p^{2k\sigma}(\log p)^{2m}} \\ &\quad + \sum_{\substack{p^k, q^\ell \leq X \\ (p,k)\neq(q,\ell)}} \frac{1}{k^{m+1}p^{k\sigma}(\log p)^m} \frac{1}{\ell^{m+1}q^{\ell\sigma}(\log q)^m} \left(p^k q^{-\ell}\right)^{it}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_0^T \left| \sum_{2\le n\le X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right|^2 dt \\
&= T \sum_{p^k\le X} \frac{1}{k^{2(m+1)} p^{2k\sigma} (\log p)^{2m}} \\
&\quad + \sum_{\substack{p^k, q^l \le X \\ (p,k)\neq(q,l)}} \frac{1}{(k\ell)^{m+1} (p^k q^\ell)^\sigma (\log p \log q)^m} O\left(\left| \log(p^k q^{-\ell}) \right|^{-1} \right).
\end{aligned}$$

The first sum on the right hand side is $= O_{\sigma,m}(1)$. Next, it holds that $\left|\log(p^k q^{-\ell})\right|^{-1} \ll X$ when $p^k, q^l \le X$ and $p^k \neq q^l$. Hence, the second sum is

$$\ll X \left(\sum_{p^k\le X} \frac{1}{k^{2(m+1)} p^{2k\sigma} (\log p)^{2m}} \right)^2 \ll X^2.$$

From the above estimates, we obtain this lemma. □

Proof of Theorem 5.1. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we apply Lemma 5.14 with

$$P(A) = \mathbb{P}_T(\tilde{\eta}_m(\sigma+it) \in A) \quad \text{and} \quad Q(A) = \mathbb{P}(\tilde{\eta}_m(\sigma, X) \in A).$$

In this case, the distribution function of Q is given by

$$G(s,t) = \int_{-\infty}^{s} \int_{-\infty}^{t} D_{\sigma,m}(x+iy)\,|dz|$$

by Proposition 5.3. Hence, it is partially differentiable, and we have

$$\begin{aligned}
\sup_{(s,t)\in\mathbb{R}^2} |G_s(s,t)| &\le \sup_{s\in\mathbb{R}} \int_{-\infty}^{\infty} D_{\sigma,m}(s+iy)\,dy < \infty, \\
\sup_{(s,t)\in\mathbb{R}^2} |G_t(s,t)| &\le \sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} D_{\sigma,m}(x+it)\,dx < \infty.
\end{aligned}$$

Furthermore, the characteristic functions of P and Q are given by

$$\begin{aligned}
f(u,v) &= \frac{1}{T} \int_0^T \exp\left(i\langle \tilde{\eta}_m(\sigma+it), u+iv \rangle \right) dt, \\
g(u,v) &= \mathbb{E}\left[\exp(i\langle \tilde{\eta}_m(\sigma, X), u+iv \rangle) \right].
\end{aligned}$$

We begin by considering the estimate of the first integral on the right hand side of (5.28). Let $r = (\log T)^{-2}$ and define

$$U = \left\{ (u,v) \in [-R,R]^2 \;:\; |u| > r \text{ and } |v| > r \right\}.$$

Then we have

$$\begin{aligned}&\iint_U \left|\frac{\hat{f}(u,v)-\hat{g}(u,v)}{uv}\right| dudv \\ &\ll \left(\log\frac{R}{r}\right)^2 \sup_{(u,v)\in[-R,R]^2} |\hat{f}(u,v)-\hat{g}(u,v)|. \end{aligned} \tag{5.29}$$

We estimate the difference $|\hat{f}(u,v)-\hat{g}(u,v)|$ as follows. First, we have

$$|\hat{f}(u,v)-\hat{g}(u,v)| \le |f(u,v)-g(u,v)| + |f(u,0)-g(u,0)| + |f(0,v)-g(0,v)|$$

by the definition. Then Proposition 5.2 yields

$$|f(u,v)-g(u,v)| \ll (\log T)^{-A}$$

for $(u,v)\in U$ if we take $R=\frac{1}{\sqrt{2}}b(\log T)^{\sigma}(\log\log T)^m$. One can prove the same estimate for $|f(u,0)-g(u,0)|$ and $|f(0,v)-g(0,v)|$. Inserting these estimates to (5.29), we obtain

$$\iint_U \left|\frac{\hat{f}(u,v)-\hat{g}(u,v)}{uv}\right| dudv \ll (\log T)^{-A}(\log\log T)^2. \tag{5.30}$$

Next, we consider the case $(u,v)\notin U$. We have

$$\begin{aligned}\hat{f}(u,v) &= \{f(u,v)-f(u,0)-f(0,v)+1\} - (f(u,0)-1)(f(0,v)-1) \\ &= \int_{\mathbb{R}^2} (e^{ixu}-1)(e^{iyv}-1)\, dP(x,y) \\ &\quad - \int_{\mathbb{R}^2} (e^{ixu}-1)\, dP(x,y)\cdot \int_{\mathbb{R}^2} (e^{iyv}-1)\, dP(x,y).\end{aligned}$$

Recall that $e^{i\theta}-1 \ll |\theta|$ holds for any $\theta\in\mathbb{R}$. Then we deduce

$$\hat{f}(u,v) \ll |uv| \int_{\mathbb{R}^2} (x^2+y^2)\, dP(x,y) \ll_{\sigma,m} |uv| \tag{5.31}$$

by Lemma 5.15. Furthermore, we see that

$$\hat{g}(u,v) \ll |uv| \int_{\mathbb{R}^2} (x^2+y^2)\, dQ(x,y) = |uv| \int_{\mathbb{C}} |z|^2 D_{\sigma,m}(z)|dz| \tag{5.32}$$

holds similarly. Recall that the integral

$$I = \int_{\mathbb{C}} |z|^2 D_{\sigma,m}(z)|dz|$$

is finite by Proposition 5.3 (iii). As a result, the estimate

$$\begin{aligned}\iint_{[-R,R]^2\setminus U} \left|\frac{\hat{f}(u,v)-\hat{g}(u,v)}{uv}\right| dudv &\ll_{\sigma,m} \iint_{[-R,R]^2\setminus U} 1\, dudv \\ &\ll (\log T)^{\sigma-2}(\log\log T)^m \end{aligned} \tag{5.33}$$

holds. We proceed to the second integral on the right hand side of (5.28). We divide the integral as

$$\int_{-R}^{R}\left|\frac{f(u,0)-g(u,0)}{u}\right|du=\left(\int_{-R}^{-r}+\int_{-r}^{r}+\int_{r}^{R}\right)\left|\frac{f(u,0)-g(u,0)}{u}\right|du.$$

By an argument similar to (5.30), we obtain

$$\left(\int_{-R}^{-r}+\int_{r}^{R}\right)\left|\frac{f(u,0)-g(u,0)}{u}\right|du\ll(\log T)^{-A}(\log\log T).$$

For the integral over $[-r,r]$, we use the estimate

$$f(u,0)-g(u,0)=\int_{\mathbb{R}^2}(e^{ixu}-1)\,dP(x,y)-\int_{\mathbb{R}^2}(e^{ixu}-1)\,dQ(x,y)\ll_{\sigma,m}|u|$$

similarly to (5.31) and (5.32). It yields

$$\int_{-r}^{-r}\left|\frac{f(u,0)-g(u,0)}{u}\right|du\ll_{\sigma,m}(\log T)^{-2}.$$

Note that the same estimates are valid for $f(0,v)-g(0,v)$. Therefore, we obtain

$$\begin{aligned}\int_{-R}^{R}\left|\frac{f(u,0)-g(u,0)}{u}\right|du+\int_{-R}^{R}\left|\frac{f(0,v)-g(0,v)}{v}\right|dv \\ \ll(\log T)^{-A}(\log\log T)+(\log T)^{-2}.\end{aligned}\tag{5.34}$$

Combining (5.30), (5.33), and (5.34), we conclude

$$\begin{aligned}\sup_{(s,t)\in\mathbb{R}^2}|F(s,t)-G(s,t)|&\ll_{\sigma,m}(\log T)^{-A}(\log\log T)^2\\&\qquad+(\log T)^{\sigma-2}(\log\log T)^{m}+(\log T)^{-\sigma}(\log\log T)^{-m}\\&\ll(\log T)^{-\sigma}(\log\log T)^{-m}\end{aligned}$$

by Lemma 5.14. Finally, using the inequality

$$\sup_{\mathcal{R}}\left|\mathbb{P}_T(\tilde{\eta}_m(\sigma+it)\in\mathcal{R})-\mathbb{P}(\tilde{\eta}_m(\sigma,X)\in\mathcal{R})\right|\le4\sup_{(s,t)\in\mathbb{R}^2}|F(s,t)-G(s,t)|,$$

we obtain the desired upper bound for $D_{\sigma,m}(T)$.

□

5.5 Preliminaries for the results on large deviations

5.5.1 Results on polylogarithms

Let $m\in\mathbb{Z}$ and $\alpha\in\mathbb{R}$. We define

$$\lambda_r(\theta;m,\alpha)=\operatorname{Re}e^{-i\alpha}\operatorname{Li}_m(re^{i\theta})=\sum_{k=1}^{\infty}\frac{r^k}{k^m}\cos(k\theta-\alpha)$$

for $\theta \in \mathbb{R}$, where $0 < r \leq 1/\sqrt{2}$ is a real number. By the definition, the function $\lambda_r(\theta; m, \alpha)$ satisfies the differential relation

$$\lambda_r'(\theta; m, \alpha) = \lambda_r(\theta; m-1, \alpha - \pi/2). \tag{5.35}$$

We begin with the following lemma on zeros of $\lambda_r(\theta; m, \alpha)$.

Lemma 5.16. *Let $m \geq 0$ and $\alpha \in \mathbb{R}$. For any fixed real number $0 < r \leq 1/\sqrt{2}$, the function $\lambda_r(\theta; m, \alpha)$ has exactly two zeros in the interval $[0, 2\pi)$.*

Proof. We prove this lemma by induction on m. Note that $\mathrm{Li}_0(z) = z/(1-z)$. Therefore, we see that $\lambda_r(\theta; 0, \alpha) = 0$ if and only if $\cos(\theta - \alpha) = r\cos\alpha$. There exist only two such θ's. Let $m \in \mathbb{Z}_{\geq 1}$. We assume that $\lambda_r(\theta; m, \alpha)$ has exactly two zeros in the interval $[0, 2\pi)$ for any $\alpha \in \mathbb{R}$, $0 < r \leq 1/\sqrt{2}$. We have $\lambda_r(\theta; m, \alpha) = \lambda_r'(\theta; m+1, \alpha + \pi/2)$ by relation (5.35). Note that the function $\lambda_r(\theta; m+1, \alpha+\pi/2)$ is smooth and periodic. Thus $\lambda_r'(\theta; m+1, \alpha+\pi/2)$ vanishes at least twice in the period. Hence, there exist at least two zeros of $\lambda_r(\theta; m, \alpha)$ in $[0, 2\pi)$. If there were three zeros of $\lambda_r(\theta; m, \alpha)$ in $[0, 2\pi)$, then we saw that $\lambda_r'(\theta; m, \alpha)$ has also three zeros in $[0, 2\pi)$ by Rolle's theorem. However, it implies that the function $\lambda_r(\theta; m-1, \alpha - \pi/2)$ has three zeros in $[0, 2\pi)$ by (5.35), which contradicts the assumption of induction. □

Let $m \in \mathbb{Z}_{\geq 1}$ and $\alpha \in \mathbb{R}$. Denote by θ_1 and θ_2 the zeros of $\lambda_r'(\theta; m, \alpha)$ with $0 \leq \theta_1 < \theta_2 < 2\pi$. Then we have $\lambda_r(\theta_1; m, \alpha) \neq \lambda_r(\theta_2; m, \alpha)$; otherwise we have the third zero of $\lambda_r'(\theta; m, \alpha)$ between θ_1 and θ_2. Furthermore, we obtain the following result as a consequence of Lemma 5.16.

Lemma 5.17. *Let $m \geq 1$ and $\alpha \in \mathbb{R}$. For $0 < r \leq 1/\sqrt{2}$, there exist real numbers $\theta_1 = \theta_1(m, \alpha, r)$ and $\theta_2 = \theta_2(m, \alpha, r)$ with $\theta_1 < \theta_2 < \theta_1 + 2\pi$ such that the function $\lambda_r(\theta; m, \alpha)$ is decreasing for $\theta_1 \leq \theta \leq \theta_2$ and is increasing for $\theta_2 \leq \theta \leq \theta_1 + 2\pi$.*

Proof. Let $0 \leq \tilde{\theta}_1 < \tilde{\theta}_2 < 2\pi$ be the zeros of $\lambda_r'(\theta; m, \alpha)$. If $\lambda_r(\tilde{\theta}_1; m, \alpha) > \lambda_r(\tilde{\theta}_2; m, \alpha)$, then we have

$$\lambda_r'(\theta; m, \alpha) \begin{cases} < 0 & \text{for } \tilde{\theta}_1 < \theta < \tilde{\theta}_2, \\ > 0 & \text{for } \tilde{\theta}_2 < \theta < \tilde{\theta}_1 + 2\pi \end{cases}$$

since there exists no zero except for $\tilde{\theta}_1$ and $\tilde{\theta}_2$ by Lemma 5.16. Then the result follows by taking $\theta_j = \tilde{\theta}_j$. In the case of $\lambda_r(\tilde{\theta}_1; m, \alpha) < \lambda_r(\tilde{\theta}_2; m, \alpha)$, we take $\theta_1 = \tilde{\theta}_2$ and $\theta_2 = \tilde{\theta}_1 + 2\pi$. Then we obtain the desired result similarly. □

Lemma 5.18. *Let $m \geq 0$ and $\alpha \in \mathbb{R}$. We have*

$$\lambda_r^{(n)}(\theta; m, \alpha) \ll n!\, r$$

uniformly for $0 < r \leq 1/\sqrt{2}$, $n \geq 0$, and $\theta \in \mathbb{R}$.

Proof. By (5.35) and the definition of $\lambda_r(\theta; m, \alpha)$, we have

$$\lambda_r^{(n)}(\theta; m, \alpha) = \lambda_r(\theta; m-n, \alpha - n/2) \ll \sum_{k=1}^{\infty} k^n r^k =: S_n(r).$$

We prove the upper bound $S_n(r) \ll n!\,r$ by induction on n. The bound is elementary for $n = 0$. If $n \geq 1$, we have

$$(1-r)S_n(r) = r + \sum_{k=1}^{\infty} \{(k+1)^n - k^n\} r^{k+1} = r\left(1 + \sum_{j=0}^{n-1} \binom{n}{j} S_j(r)\right).$$

Hence, the desired estimate on $S_n(r)$ holds by the assumption of induction. □

Let $\mathcal{B}$ denote

$$\mathcal{B} = \left\{\left(m, (0, 1/\sqrt{2}]\right) \;:\; m \in \mathbb{Z}_{m\geq 4}\right\} \cup \{(m, (0, 0.15]) \;:\; m = 0, 1, 2, 3\}.$$

Lemma 5.19. *Let $m \geq 0$ and $\alpha \in \mathbb{R}$. Denote by $\theta_1 = \theta_1(m, \alpha, r)$ and $\theta_2 = \theta_2(m, \alpha, r)$ the real numbers of Proposition 5.17 for $0 < r \leq 1/\sqrt{2}$.*

(i) We have $|\lambda_r''(\theta_1; m, \alpha)| \gg r$ for $(m, I) \in \mathcal{B}$ and $r \in I$.

(ii) There exists an absolute constant $d > 0$ such that $\theta_2 - \theta_1 > d$ for $(m, I) \in \mathcal{B}$ and $r \in I$.

(iii) For any $0.15 < r \leq 1/\sqrt{2}$ and $m = 0, 1, 2, 3$, there exists a positive integer $n_1 = n_1(m, \alpha, r)$ such that $\lambda_r^{(n)}(\theta_1; m, \alpha) = 0$ for $1 \leq n \leq 2n_1 - 1$ and $\lambda_r^{(2n_1)}(\theta_1; m, \alpha) < 0$.

Proof. The third assertion follows from Lemma 5.17. Since $\lambda_r'(\theta_1; m, \alpha) = 0$, we have

$$|\sin(\theta_1 - \alpha)| = \left| -\sum_{k=2}^{\infty} \frac{r^{k-1}}{k^{m-1}} \sin(k\theta_1 - \alpha) \right| \leq \sum_{k=2}^{\infty} \frac{r^{k-1}}{k^{m-1}} \leq f_m(r),$$

and

$$f_m(r) = \begin{cases} \dfrac{\sqrt{2}}{8} = 0.176\ldots & \text{if } m \in \mathbb{Z}_{\geq 4}, \\ \dfrac{2r - r^2}{(1-r)^2} & \text{if } m = 0, 1, 2, 3 \end{cases}$$

for $r \in I$ due to $0 < \zeta(3) - 1 < 1/4$. Since $f_m(0.15) = 0.384\ldots$ holds for $m = 0, 1, 2, 3$, we have

$$|\cos(\theta_1 - \alpha)| = \sqrt{1 - \sin(\theta_1 - \alpha)^2} \geq \sqrt{1 - f_m(r)^2} \geq \sqrt{1 - (0.39)^2} =: c_1$$

holds for $r \in I$. Furthermore, we obtain

$$\left|\lambda_r''(\theta_1; m, \alpha) + r\cos(\theta_1 - \alpha)\right| \leq \sum_{k=2}^{\infty} \frac{r^k}{k^{m-2}} \leq r g_m(r),$$

$$g_m(r) = \begin{cases} \dfrac{\sqrt{2}}{3} & \text{if} \quad m \in \mathbb{Z}_{\geq 4}, \\ \dfrac{4r - 3r^2 + r^3}{(1-r)^3} & \text{if} \quad m = 0, 1, 2, 3 \end{cases}$$

for $r \in I$ due to $0 < \zeta(2) - 1 < 2/3$. If we suppose $\cos(\theta_1 - \alpha) \leq -\sqrt{1 - f_m(r)^2}$, then we have

$$\lambda_r''(\theta_1; m, \alpha) > \left(\sqrt{1 - f_m(r)^2} - g_m(r)\right) r =: h_m(r) r,$$

$$h_m(r) \geq \begin{cases} 1/2 & \text{if} \quad m \in \mathbb{Z}_{\geq 4}, \\ h_m(0.15) = 0.0507\ldots & \text{if} \quad m = 0, 1, 2, 3. \end{cases}$$

for $r \in I$, which contradicts with the fact that $\lambda_r(\theta; m, \alpha)$ takes the maximum value at $\theta = \theta_1$. Thus we have $\cos(\theta_1 - \alpha) > \sqrt{1 - f_m(r)^2} \geq c_1$, and therefore

$$\lambda_r''(\theta_1; m, \alpha) < -\left(\sqrt{1 - f_m(r)^2} - g_m(r)\right) r = -h_m(r) r \leq -0.0507 \cdots \times r.$$

Since $\left|\lambda_r''(\theta_1; m, \alpha)\right| = -\lambda_r''(\theta_1; m, \alpha)$, we obtain the first assertion.

On the other hand, we have $\cos(\theta_2 - \alpha) < -\sqrt{1 - f_m(r)^2} \leq -c_1$ by similar calculations. Putting

$$d = \inf\left\{|\omega_1 - \omega_2| \ : \ \omega_1 \in \cos^{-1}\left([c_1, 1]\right), \omega_2 \in \cos^{-1}([-1, -c_1])\right\} > 0,$$

we have the second assertion. □

In what follows, we take $r = p^{-\sigma}$ with $p \geq 2$ and $\sigma \geq 1/2$. We study the function

$$F_{\sigma,m,p}(s; \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(\frac{s}{(\log p)^m} \lambda_{p^{-\sigma}}(\theta; m+1, \alpha)\right) d\theta. \tag{5.36}$$

Proposition 5.4. *Let $(\sigma, m) \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. We take $s = \kappa + it \in \mathbb{C}$ with $\kappa > c$ and $|t| \leq \kappa$, where $c > 0$ is sufficiently large. Then we have the followings;*
(i) *Suppose that $(m, I) \in \mathcal{B}$, $p^{-\sigma} \in I$, and $p^{\sigma}(\log p)^m \leq \kappa(\log \kappa)^{-6}$ are satisfied. Then we have*

$$\begin{aligned} &F_{\sigma,m,p}(s; \alpha) \\ &= \exp\left(\frac{s}{(\log p)^m} \lambda_{p^{-\sigma}}(\theta_1; m+1, \alpha)\right) \sqrt{\frac{(\log p)^m}{2\pi s |\lambda_{p^{-\sigma}}''(\theta_1; m+1, \alpha)|}} \\ &\quad \times \left\{1 + \sum_{\ell=1}^{2N-1} \sum_{k=3\ell}^{2(N+l)-1} A_{\ell,k}(\lambda_{m+1,\alpha}, N) \left(\frac{p^{\sigma}(\log p)^m}{s}\right)^{(k/2)-\ell} \right. \\ &\qquad \left. + O_N\left(\left(\frac{p^{\sigma}(\log p)^m}{\kappa}\right)^N (\log \kappa)^{6N}\right)\right\}, \end{aligned}$$

where

$$
\begin{aligned}
&A_{\ell,k}(\lambda_{m+1,\alpha}, N)\\
&= \frac{2g_k}{\ell!\sqrt{\pi}} \left(\frac{2p^{-\sigma}}{|\lambda''_{p^{-\sigma}}(\theta_1; m+1, \alpha)|} \right)^{k/2}\\
&\quad \times \sum_{\substack{3 \le j_1, \dots, j_\ell \le 2N+1;\\ j_1 + \cdots + j_\ell = k}} \frac{\lambda^{(j_1)}_{p^{-\sigma}}(\theta_1; m+1, \alpha) \cdots \lambda^{(j_\ell)}_{p^{-\sigma}}(\theta_1; m+1, \alpha)}{j_1! \cdots j_\ell!} p^{\ell\sigma}
\end{aligned}
$$

and $g_k = \int_0^\infty x^k \exp(-x^2)dx$. *In addition, we have* $A_{\ell,k}(\lambda_{m+1,\alpha}, N) \ll_N 1$.
(ii) *Suppose that* $p^{-\sigma} \in (0.15, 1/\sqrt{2}]$ *and* $m = 0, 1, 2, 3$ *are satisfied. Then we have*

$$
\begin{aligned}
&F_{\sigma,m,p}(s; \alpha)\\
&= \frac{g_{0,n_1}}{\pi} \exp\left(\frac{s}{(\log p)^m} \lambda_{p^{-\sigma}}(\theta_1; m+1, \alpha) \right) \left(\frac{(2n_1!)(\log p)^m}{s|\lambda^{(2n_1)}_{p^{-\sigma}}(\theta_1; m+1, \alpha)|} \right)^{\frac{1}{2n_1}} \times\\
&\quad \times \left\{ 1 + \sum_{\ell=1}^{2n_1 N - 1} \sum_{k=(2n_1+1)\ell}^{2n_1(N+\ell)-1} B_{\ell,k}(\lambda_{m+1,\alpha}, N) s^{\ell - \frac{k}{2n_1}} + \right.\\
&\quad \left. + O_{m,\sigma,N}\left(\kappa^{-N} (\log \kappa)^{2n_1(2n_1+1)N} \right) \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
&B_{\ell,k}(\lambda_{m+1,\alpha}, N)\\
&= \frac{g_{k,n_1}}{\ell! g_{0,n_1} (\log p)^{m\ell}} \left(\frac{(2n_1)!(\log p)^m}{|\lambda^{(2n_1)}_{p^{-\sigma}}(\theta_1; m+1, \alpha)|} \right)^{\frac{k}{2n_1}} \times\\
&\quad \times \sum_{\substack{2n_1+1 \le j_1, \dots, j_\ell \le 2n_1(N+1)-1;\\ j_1 + \cdots + j_\ell = k}} \frac{\lambda^{(j_1)}_{p^{-\sigma}}(\theta_1; m+1, \alpha) \cdots \lambda^{(j_\ell)}_{p^{-\sigma}}(\theta_1; m+1, \alpha)}{j_1! \cdots j_\ell!},
\end{aligned}
$$

and $g_{k,n_1} = \int_0^\infty x^k \exp(-x^{2n_1})dx$. *In addition, we have* $B_{\ell,k}(\lambda_{m+1,\alpha}, N) \ll_{m,\sigma,N} 1$.

Proof. We will use the saddle point method of asymptotic expansions. First, we prove the case $(m, I) \in \mathcal{B}$ and $p^{-\sigma} \in I$. For simplicity, we write

$$\lambda(\theta) = \lambda_{p^{-\sigma}}(\theta; m+1, \alpha).$$

By the periodicity of $\lambda(\theta)$, we have $F_{\sigma,m,p}(s; \alpha) = I_1 + I_2$, where

$$I_1 = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \exp\left(\frac{s}{(\log p)^m} \lambda(\theta) \right) d\theta$$

and

$$I_2 = \frac{1}{2\pi}\int_{\theta_2}^{\theta_1+2\pi} \exp\left(\frac{s}{(\log p)^m}\lambda(\theta)\right) d\theta.$$

We start with the estimate of I_1. Put

$$\varepsilon = (\log \kappa)\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}}$$

and divide the integral I_1 into two parts:

$$I_1 = \frac{1}{2\pi}\left(\int_{\theta_1}^{\theta_1+\varepsilon} + \int_{\theta_1+\varepsilon}^{\theta_2}\right) \exp\left(\frac{s}{(\log p)^m}\lambda(\theta)\right) d\theta$$

Note that we have $\varepsilon < d$ when κ is large, where d is the absolute constant in Lemma 5.19. Since $\lambda'_{p^{-\sigma}}(\theta_1; m+1, \alpha) = 0$, we obtain

$$\begin{aligned}
&\int_{\theta_1}^{\theta_1+\varepsilon} \exp\left(\frac{s}{(\log p)^m}\lambda(\theta)\right) d\theta \\
&= \exp\left(\frac{s}{(\log p)^m}\lambda(\theta_1)\right)\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}} \times \\
&\times \int_0^{\log\kappa} \exp\left(\frac{s}{(\log p)^m}\sum_{k=3}^{\infty}\frac{\lambda^{(k)}(\theta_1)}{k!}\left(\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}}x\right)^k\right)\exp\left(-\frac{s}{\kappa}x^2\right) dx
\end{aligned}$$

by making change of variables. By Lemmas 5.18 and 5.19, we have

$$\begin{aligned}
&\exp\left(\frac{s}{(\log p)^m}\sum_{k=3}^{\infty}\frac{\lambda^{(k)}(\theta_1)}{k!}\left(\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}}x\right)^k\right) \\
&= \exp\left(\frac{s}{(\log p)^m}\sum_{k=3}^{2N+1}\frac{\lambda^{(k)}(\theta_1)}{k!}\left(\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}}x\right)^k\right) \times \\
&\times \left(1 + O_N\left(\left(\frac{p^s\,(\log p)^m}{\kappa}\right)^N (\log\kappa)^{2N+2}\right)\right)
\end{aligned}$$

for $0 \le x \le \log\kappa$ since we have the estimate

$$\begin{aligned}
&\frac{s}{(\log p)^m}\sum_{k=2N+2}^{\infty}\frac{\lambda^{(k)}(\theta_1)}{k!}\left(\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}}x\right)^k \\
&\ll \frac{\kappa}{p^\sigma(\log p)^m}\sum_{k=2N+2}^{\infty}\left(C\frac{p^\sigma(\log p)^m}{\kappa}\right)^{k/2} x^k \ll_N \left(\frac{p^\sigma(\log p)^m}{\kappa}\right)^N (\log\kappa)^{2N+2},
\end{aligned}$$

where C is a some positive absolute constant. By the Taylor series expansion, we have

$$\exp\left(\frac{s}{(\log p)^m}\sum_{k=3}^{2N+1}\frac{\lambda^{(k)}(\theta_1)}{k!}\left(\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}}x\right)^k\right)$$

$$=1+\sum_{l=1}^{2N-1}\frac{1}{l!}\left(\frac{s}{(\log p)^m}\right)^l\left(\sum_{k=3}^{2N+1}\frac{\lambda^{(k)}(\theta_1)}{k!}\left(\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}}x\right)^k\right)^l$$

$$+O_N\left(\left(\frac{p^\sigma(\log p)^m}{\kappa}\right)^N(\log\kappa)^{6N}\right)$$

for $0\le x\le\log\kappa$. The above second sum is equal to

$$\sum_{\ell=1}^{2N-1}\sum_{k=3}^{2(N+\ell)-1}\frac{a_{\ell,k}(p,\lambda_{m+1,\alpha},N)}{\ell!}\left(\frac{s}{(\log p)^m}\right)^\ell\left(\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}\right)^{k/2}x^k$$

$$+O_N\left(\left(\frac{p^\sigma(\log p)^m}{\kappa}\right)^N(\log\kappa)^{6N-2}\right)$$

by Lemma 5.18, where

$$a_{\ell,k}(\lambda_{m+1,\alpha},N)=\sum_{\substack{3\le j_1,\ldots,j_\ell\le 2N+1;\\ j_1+\cdots+j_\ell=k}}\frac{\lambda^{(j_1)}(\theta_1)\cdots\lambda^{(j_\ell)}(\theta_1)}{j_1!\cdots j_\ell!}.$$

Therefore we obtain

$$\int_0^{\log\kappa}\exp\left(\frac{s}{(\log p)^m}\sum_{k=3}^{\infty}\frac{\lambda^{(k)}(\theta_1)}{k!}\left(\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}}x\right)^k\right)\exp\left(-\frac{s}{\kappa}x^2\right)dx$$

$$=g_0(s,\kappa)+$$

$$+\sum_{\ell=1}^{2N-1}\sum_{k=3}^{2(N+\ell)-1}\frac{a_{\ell,k}(p,\lambda_{m+1,\alpha},N)}{\ell!}\left(\frac{s}{(\log p)^m}\right)^\ell\left(\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}\right)^{k/2}g_k(s,\kappa)$$

$$+O_N\left(\left(\frac{p^\sigma(\log p)^m}{\kappa}\right)^N(\log\kappa)^{6N}\right)+E_1,$$

where $g_k(s,\kappa)$ is defined by

$$g_k(s,\kappa)=\int_0^\infty x^k\exp\left(-\frac{s}{\kappa}x^2\right)dx,$$

and the error term E_1 is estimated as $E_1\ll_N\exp\left(-(\log\kappa)^2/2\right)$ by using Lemma 5.18 and the bound

$$\int_{\log\kappa}^\infty x^k\exp\left(-x^2\right)\ll_k\exp\left(-\frac{3}{4}(\log\kappa)^2\right).$$

By using the equation $g_k(s,\kappa) = (\kappa/s)^{(k+1)/2} g_k$ with $g_k = \int_0^\infty x^k \exp(-x^2)dx$, we obtain

$$\int_0^{\log\kappa} \exp\left(\frac{s}{(\log p)^m}\sum_{k=3}^{\infty}\frac{\lambda^{(k)}(\theta_1)}{k!}\left(\sqrt{\frac{2(\log p)^m}{\kappa|\lambda''(\theta_1)|}}x\right)^k\right)\exp\left(-\frac{s}{\kappa}x^2\right)dx$$

$$=\frac{\sqrt{\pi}}{2}\sqrt{\frac{\kappa}{s}}\Bigg(1+\sum_{\ell=1}^{2N-1}\sum_{k=3\ell}^{2(N+\ell)-1}A_{\ell,k}(p,\lambda_{m+1,\alpha},N)\left(\frac{p^\sigma(\log p)^m}{s}\right)^{(k/2)-\ell}+$$

$$+O_N\left(\left(\frac{p^\sigma(\log p)^m}{\kappa}\right)^N(\log\kappa)^{6N}\right)\Bigg),$$

where

$$A_{\ell,k}(p,\lambda_{m+1,\alpha},N)$$
$$=\frac{2g_k}{\ell!\sqrt{\pi}}\left(\frac{2p^{-\sigma}}{|\lambda''(\theta_1)|}\right)^{k/2}\sum_{\substack{3\le j_1,\dots,j_\ell\le 2N+1;\\ j_1+\cdots+j_\ell=k}}\frac{\lambda^{(j_1)}(\theta_1)\cdots\lambda^{(j_\ell)}(\theta_1)}{j_1!\cdots j_\ell!}p^{\ell\sigma}.$$

Therefore we deduce the asymptotic formula

$$\frac{1}{2\pi}\int_{\theta_1}^{\theta_1+\varepsilon}\exp\left(\frac{s}{(\log p)^m}\lambda(\theta)\right)d\theta$$
$$=\frac{1}{2}\exp\left(\frac{s}{(\log p)^m}\lambda(\theta_1)\right)\sqrt{\frac{(\log p)^m}{2\pi s|\lambda''(\theta_1)|}}\times$$
$$\times\Bigg\{1+\sum_{\ell=1}^{2N-1}\sum_{k=3\ell}^{2(N+\ell)-1}A_{\ell,k}(p,\lambda_{m+1,\alpha},N)\left(\frac{p^\sigma(\log p)^m}{s}\right)^{(k/2)-\ell}+$$
$$+O_N\left(\left(\frac{p^\sigma(\log p)^m}{\kappa}\right)^N(\log\kappa)^{6N}\right)\Bigg\}.$$

We estimate the integral I_1 whose integral interval is restricted to $[\theta_1+\varepsilon,\theta_2]$. We can write

$$\int_{\theta_1+\varepsilon}^{\theta_2}\exp\left(\frac{s}{(\log p)^m}\lambda(\theta)\right)d\theta$$
$$=\exp\left(\frac{s}{(\log p)^m}\lambda(\theta_1)\right)\int_{\theta_1+\varepsilon}^{\theta_2}\exp\left(\frac{s}{(\log p)^m}(\lambda(\theta)-\lambda(\theta_1))\right)d\theta.$$

Since $\lambda(\theta)$ is decreasing for $\theta_1+\varepsilon\le\theta\le\theta_2$, we find that

$$\lambda(\theta)-\lambda(\theta_1)\le\lambda(\theta_1+\varepsilon)-\lambda(\theta_1)\le-\frac{|\lambda''(\theta_1)|}{4}\varepsilon^2.$$

Therefore we obtain

$$\int_{\theta_1+\varepsilon}^{\theta_2}\exp\left(\frac{s}{(\log p)^m}(\lambda(\theta)-\lambda(\theta_1))\right)d\theta \ll \exp\left(-\frac{|\lambda''(\theta_1)|}{4}\varepsilon^2\right)$$
$$=\exp\left(-\frac{(\log\kappa)^2}{2}\right).$$

Thus, we have the asymptotic formula for I_1. Applying the same calculations to the integral I_2, we have the asymptotic formula for $F_{\sigma,m,p}(s;\alpha)$ in this case. The estimate $A_{\ell,k}(\lambda_{m+1,\alpha}, N) \ll_N 1$ follows from Lemma 5.18.

Next, we will prove the second assertion. Let $p^{-\sigma} \in (0.15, 1/\sqrt{2}]$ and $m = 0, 1, 2, 3$. Note that $1/\sqrt{p} \le 0.15$ implies $1/0.15^2 = 44.444.. \le p$. Thus it is enough to prove the case $m = 0, 1, 2, 3$ and $2 \le p \le 44$ to complete the proof. Since the patterns we should consider are finite, the implicit constant appearing in the error term depend only on m, σ, N when we carry out similar calculations to the above. Therefore we obtain the second assertion. □

We see that $F_{\sigma,m,p}(s,\alpha)$ is holomorphic and non-zero on the region

$$\Delta_c = \{s = \kappa + it \ : \ \kappa > c, \ |t| \le \kappa\}$$

under the assumption on Proposition 5.4. Therefore, we may define

$$f_{\sigma,m,p}(s,\alpha) = \log F_{\sigma,m,p}(s,\alpha)$$

for $s \in \Delta_c$, where the branch is taken so that $f_{\sigma,m,p}(s,\alpha)$ is real valued on the positive real axis. The function $f_{\sigma,m,p}(s,\alpha)$ is holomorphic on Δ_c, and we obtain the following result.

Corollary 5.1. *Let $(\sigma, m) \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. We take $s = \kappa + it \in \mathbb{C}$ with $\kappa > c$ and $|t| \le \kappa$, where $c > 0$ is sufficiently large. Suppose that $p^{\sigma}(\log p)^m \le \kappa(\log \kappa)^{-6}$ is satisfied. Then we have*

$$f_{\sigma,m,p}(s;\alpha) \ll \frac{\kappa}{p^{\sigma}(\log p)^m}, \qquad f'_{\sigma,m,p}(s;\alpha) \ll \frac{1}{p^{\sigma}(\log p)^m},$$

and for all $n \ge 2$,

$$f^{(n)}_{\sigma,m,p}(s;\alpha) \ll \frac{2^n n!}{\kappa^n}.$$

Proof. We only prove the case $(m, I) \in \mathcal{B}$ and $p^{-\sigma} \in I$. We have

$$\begin{aligned} f_{\sigma,m,p}(s;\alpha) = &\frac{s}{(\log p)^m}\lambda_{p^{-\sigma}}(\theta_1; m+1, \alpha) \\ &- \frac{1}{2}\log\left(\frac{s|\lambda''_{p^{-\sigma}}(\theta_1; m+1,\alpha)|}{(\log p)^m}\right) + C + h_{\sigma,m,p}(s;\alpha), \end{aligned}$$

for $s \in \Delta$, where C is a real number, and

$$h_{\sigma,m,p}(s;\alpha) \ll \frac{1}{(\log \kappa)^{5/2}}.$$

By the Cauchy integral formula, we have

$$h^{(n)}_{\sigma,m,p}(\kappa;\alpha) = \frac{1}{2\pi i}\int_{|z-\kappa|=\kappa/2} \frac{h_{\sigma,m,p}(z;\alpha)}{(z-\kappa)^{n+1}}dz \ll \frac{2^n n!}{\kappa^n(\log \kappa)^{5/2}}$$

for $\kappa > c$. By differentiating $h_{\sigma,m,p}(s;\alpha)$ and substituting $s = \kappa$, we have the conclusion. The other cases can be obtained by a similar argument. □

5.5.2 Basic properties of the Bessel function I_0

We further prepare some lemmas on Bessel functions. Put

$$\Delta = \{z = x + iy \ : \ x \geq 0, |y| \leq x\}. \tag{5.37}$$

Lemma 5.20. *We have $|I_0(z)| \asymp I_0(x)$ for all $z \in \Delta$.*

Proof. The inequality $|I_0(z)| \leq I_0(x)$ is deduced from the definition. We prove that $|I_0(x)/I_0(z)|$ is bounded if $x \geq 0$ and $|y| \leq x$. Recall that the asymptotic formula (see [121, pp. 74, 198])

$$I_0(z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 + O\left(|z|^{-1}\right)\right) \tag{5.38}$$

holds if $\operatorname{Re} z > 0$. Hence, we see that there exists an absolute constant $R > 0$ such that

$$\left|\frac{I_0(x)}{I_0(z)}\right| \leq 2\sqrt{\frac{|z|}{x}} \leq 2\sqrt{2}$$

if $|z| > R$ and $z \in \Delta$. Since $|I_0(x)/I_0(z)|$ is bounded if $|z| \leq R$ and $z \in \Delta$, we complete the proof. □

Recall that the modified Bessel function $I_0(z)$ is non-zero and holomorphic for $\operatorname{Re} z > 0$. Therefore, we may define

$$g(z) = \log I_0(z)$$

as a holomorphic function on $\operatorname{Re} z > 0$, whose values are real on the real axis.

Lemma 5.21. *We have the following statements;*

(i) *We have $g(z) = z^2/4 + O\left(|z|^4\right)$ for $|z| \leq 1$.*

(ii) *Let δ be a positive number. We have $g(z) \ll_\delta |z|$ for $\operatorname{Re}(z) > \delta$ and $z \in \Delta$.*

Proof. By the Taylor expansion of $\exp(z)$,

$$I_0(z) = 1 + E(z)z^2, \quad E(z) = \frac{1}{4} + \sum_{n=2}^{\infty} \frac{1}{n!}\left(\frac{z}{2}\right)^{2(n-1)}$$

holds. Since the estimate $|E(z)| < 1$ holds for $|z| \leq 1$, we have

$$\log I_0(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} E(z)^k = \frac{z^2}{4} + O\left(|z|^4\right)$$

for $|z| \leq 1$. Hence, we have the first assertion. The second assertion immediately follows from the asymptotic formula (5.38). This completes the proof. □

Lemma 5.22. *For any $z \in \mathbb{C}$, we have*

$$g(z) \ll \begin{cases} |z|^2 & \text{if } 0 \le |z| \le 1, \\ |z| & \text{if } |z| \ge 1, \end{cases} \qquad g'(z) \ll \begin{cases} |z| & \text{if } 0 \le |z| \le 1, \\ 1 & \text{if } |z| \ge 1, \end{cases}$$

and for $n \ge 2$,

$$g^{(n)}(z) \ll \begin{cases} n! & \text{if } 0 \le |z| \le 1, \\ 2^n n! |z|^{1-n} & \text{if } |z| \ge 1. \end{cases}$$

Proof. The first and the second estimates follow from [67, Lemma 7.4]. We also know that $g(z) \ll |z|$ holds. Hence, the third estimate follows by Cauchy's integral formula. □

5.6 Cumulant-generating functions

Let $(\sigma, m) \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. We consider the moment-generating function

$$F_{\sigma,m}(s;\alpha) = \mathbb{E}\left[\exp(s \operatorname{Re} e^{-i\alpha} \tilde{\eta}_m(\sigma, X))\right] \tag{5.39}$$

for $s = \kappa + it \in \mathbb{C}$. Note that $F_{\sigma,m}(\kappa;\alpha) > 0$ if $\kappa \in \mathbb{R}$ by the definition. We define the cumulant-generating function

$$f_{\sigma,m}(\kappa;\alpha) = \log F_{\sigma,m}(\kappa;\alpha)$$

for $\kappa \in \mathbb{R}$, which is a real analytic function. In this section, we will show the asymptotic formulas for $f_{\sigma,m}^{(n)}(\kappa;\alpha)$ for $1/2 < \sigma < 1$ and $m \in \mathbb{Z}_{\ge 0}$.

Proposition 5.5. *Let $1/2 < \sigma < 1$ and $m \in \mathbb{Z}_{\ge 0}$. There exists a small constant $c_m > 0$ such that for $n \in \mathbb{Z}_{\ge 0}$ we have*

$$f_{\sigma,m}^{(n)}(\kappa;\alpha) = \sigma^{\frac{m}{\sigma}} g_n(\sigma) \frac{\kappa^{\frac{1}{\sigma}-n}}{(\log \kappa)^{\frac{m}{\sigma}+1}} \left(1 + O\left(2^n (n+1)^3 \frac{\log\log \kappa}{\log \kappa}\right)\right)$$

if $\kappa \ge \kappa_0(\sigma, m)$, where $\kappa_0(\sigma, m) > 0$ is a large constant, and

$$g_n(\sigma) = \int_0^\infty \frac{g^{(n)}(u)}{u^{(1/\sigma)+1-n}} \, du$$

with $g(z) = \log I_0(z)$ as above. The implicit constant depends only on σ and m.

Let $(\sigma, m) \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. Note that the the function $F_{\sigma,m}(s;\alpha)$ of (5.39) satisfies

$$F_{\sigma,m}(s;\alpha) = \prod_p F_{\sigma,m,p}(s;\alpha) \tag{5.40}$$

by the independence of $X(p)$'s, where $F_{\sigma,m,p}(s;\alpha)$ is the function of (5.36). Put

$$\Delta_{m,\sigma} = \left\{ s = \kappa + it \ : \ \kappa > L_{m,\sigma}, \ |t| \le \kappa \right\},$$

where $L_{m,\sigma}$ is a sufficiently large constant depending on m and σ.

Lemma 5.23. *Let $(\sigma, m) \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. Suppose that $\kappa p^{-2\sigma}(\log p)^{-m} \leq \delta$ is satisfied with a positive small absolute constant δ. Then we have*

$$f_{\sigma,m,p}(s;\alpha) = g\left(\frac{s}{p^{\sigma}(\log p)^m}\right) + O\left(\frac{\kappa}{p^{2\sigma}(\log p)^m}\right)$$

for $s = \kappa + it \in \Delta$. Here, the region Δ is given by (5.37)*.*

Proof. We can write

$$\tilde{\eta}_{m,p}(\sigma, X(p)) = \frac{X(p)}{p^{\sigma}(\log p)^m} + E_{\sigma,m}(p), \quad E_{\sigma,m}(p) \ll \frac{1}{p^{2\sigma}(\log p)^m}.$$

Recalling that $|s| \leq 2\kappa$ holds for every $s = \kappa + it \in \Delta$, we have

$$\exp\left(s \operatorname{Re}(e^{-i\alpha} E_{\sigma,m}(p))\right) = 1 + O\left(\frac{\kappa}{p^{2\sigma}(\log p)^m}\right)$$

for $s \in \Delta$ if p satisfies $\kappa p^{-2\sigma}(\log p)^{-m} \leq \delta$. Hence, $F_{\sigma,m,p}(s;\alpha)$ can be calculated as

$$\begin{aligned} F_{\sigma,m,p}(s;\alpha) &= \mathbb{E}\left[\exp\left(s\frac{\operatorname{Re}(e^{-i\alpha}X(p))}{p^{\sigma}(\log p)^m}\right)\right] \\ &\quad + \mathbb{E}\left[\exp\left(s\frac{\operatorname{Re}(e^{-i\alpha}X(p))}{p^{\sigma}(\log p)^m}\right) O\left(\frac{\kappa}{p^{2\sigma}(\log p)^m}\right)\right] \\ &= I_0\left(\frac{s}{p^{\sigma}(\log p)^m}\right) + O\left(I_0\left(\frac{\kappa}{p^{\sigma}(\log p)^m}\right)\frac{\kappa}{p^{2\sigma}(\log p)^m}\right). \end{aligned}$$

By Lemma 5.20, we have

$$F_{\sigma,m,p}(s;\alpha) = I_0\left(\frac{s}{p^{\sigma}(\log p)^m}\right)\left(1 + O\left(\frac{\kappa}{p^{2\sigma}(\log p)^m}\right)\right). \tag{5.41}$$

Therefore, if δ is sufficiently small, $F_{\sigma,m,p}(s;\alpha) \neq 0$ for $s \in \Delta$. Hence, we define $f_{\sigma,m,p}(s;\alpha) = \log F_{\sigma,m,p}(s;\alpha)$ as before. We have by (5.41) the formula

$$f_{\sigma,m,p}(s;\alpha) = g\left(\frac{s}{p^{\sigma}(\log p)^m}\right) + O\left(\frac{\kappa}{p^{2\sigma}(\log p)^m}\right).$$

This completes the proof. □

Proof of Proposition 5.5. First, we will show the asymptotic formula

$$f_{\sigma,m}(s;\alpha) = \sigma^{\frac{m}{\sigma}} g_0(\sigma) \frac{s^{\frac{1}{\sigma}}}{(\log \kappa)^{\frac{m}{\sigma}+1}}\left(1 + O\left(\frac{\log\log\kappa}{\log\kappa}\right)\right) \tag{5.42}$$

for $s = \kappa + it \in \Delta_{m,\sigma}$ and $\alpha \in \mathbb{R}$. Let y_1 and y_2 be the parameters determined by

$$\frac{\kappa}{y_1^{2\sigma}} = \delta \quad \text{and} \quad \frac{\kappa}{y_2^{\sigma}} = \left(\frac{1}{\log\kappa}\right)^{\frac{\sigma}{2\sigma-1}},$$

where $\delta > 0$ is the constant in Lemma 5.23. Using formula (5.40) along with Corollary 5.1, Lemmas 5.22 and 5.23, we have

$$f_{\sigma,m}(s;\alpha) = \sum_{y_1<p\le y_2} g\left(\frac{s}{p^\sigma(\log p)^m}\right) + E_1, \tag{5.43}$$

where

$$\begin{aligned}E_1 &\ll \sum_{p\le y_1} \frac{\kappa}{p^\sigma(\log p)^m} + \sum_{p>y_1} \frac{\kappa}{p^{2\sigma}(\log p)^m} + \sum_{p>y_2} \frac{\kappa^2}{p^{2\sigma}(\log p)^{2m}} \\ &\ll_{m,\sigma} \frac{\kappa {y_1}^{1-\sigma}}{(\log y_1)^{m+1}} + \frac{\kappa^2 {y_2}^{1-2\sigma}}{(\log y_2)^{2m+1}}.\end{aligned}$$

Since $y_1 \asymp_\sigma \kappa^{\frac{1}{2\sigma}}$ and $y_2 = \kappa^{\frac{1}{\sigma}}(\log\kappa)^{\frac{1}{2\sigma-1}}$, we obtain

$$E_1 \ll_{\sigma,m} \frac{\kappa^{\frac{1}{2\sigma}+\frac{1}{2}}}{(\log\kappa)^{m+1}} + \frac{\kappa^{\frac{1}{\sigma}}}{(\log\kappa)^{2m+2}} \ll \frac{\kappa^{\frac{1}{\sigma}}}{(\log\kappa)^{\frac{m}{\sigma}+2}} \tag{5.44}$$

if κ is large enough.

The main term comes from the terms for $y_1 < p \le y_2$. Recall that the asymptotic formula

$$\pi(y) = \int_2^y \frac{dt}{\log t} + O\left(ye^{-8\sqrt{\log y}}\right)$$

holds. Then, by partial summation, we have

$$\sum_{y_1<p\le y_2} g\left(\frac{s}{p^\sigma(\log p)^m}\right) = \int_{y_1}^{y_2} g\left(\frac{s}{y^\sigma(\log y)^m}\right)\frac{dy}{\log y} + E_2,$$

where

$$\begin{aligned}E_2 \ll &\left|g\left(\frac{s}{{y_1}^\sigma(\log y_1)^m}\right)\right| y_1 e^{-8\sqrt{\log y_1}} + \left|g\left(\frac{s}{{y_2}^\sigma(\log y_2)^m}\right)\right| y_2 e^{-8\sqrt{\log y_2}} \\ &+ \kappa\int_{y_1}^{y_2}\left|g'\left(\frac{s}{{y_1}^\sigma(\log y_1)^m}\right)\right| \frac{e^{-8\sqrt{\log y}}}{y^\sigma(\log y)^m}\,dy.\end{aligned}$$

Recall further that we have $|s{y_2}^{-\sigma}(\log y_2)^{-m}| \le 1$, and that $\left|s{y_1}^{-\sigma}(\log y_1)^{-m}\right| \le 1$ is sufficiently large. Hence, estimate (5.38) and Lemma 5.21 give

$$\left|g\left(\frac{s}{{y_1}^\sigma(\log y_1)^m}\right)\right| y_1 e^{-8\sqrt{\log y_1}} \ll \frac{\kappa e^{-8\sqrt{\log y_1}}}{{y_1}^{\sigma-1}(\log y_1)^m} \ll_{\sigma,m} \frac{\kappa^{\frac{1}{2\sigma}+\frac{1}{2}}}{(\log\kappa)^m}e^{-4\sqrt{\log\kappa}}$$

and

$$\left|g\left(\frac{s}{{y_2}^\sigma(\log y_2)^m}\right)\right| y_2 e^{-8\sqrt{\log y_2}} \ll \frac{\kappa^2 e^{-8\sqrt{\log y_2}}}{{y_2}^{2\sigma-1}(\log y_2)^m} \ll_{\sigma,m} \frac{\kappa^{\frac{1}{\sigma}}}{(\log\kappa)^{2m+1}}e^{-8\sqrt{\log\kappa}}.$$

The third term is estimated as

$$\kappa \int_{y_1}^{y_2} \left| g'\left(\frac{s}{{y_1}^{\sigma}(\log y_1)^m}\right)\right| \frac{e^{-8\sqrt{\log y}}}{y^{\sigma}(\log y)^m}\,dy$$
$$\ll \kappa e^{-8\sqrt{\log y_1}} \int_1^{y_2} \frac{dy}{y^{\sigma}(\log y)^m} \ll_{\sigma,m} \frac{\kappa^{\frac{1}{\sigma}}}{(\log \kappa)^{\frac{m}{\sigma}}} e^{-4\sqrt{\log \kappa}}$$

by Lemma 5.22. As a result, the error term E_2 is estimated as

$$E_2 \ll_{\sigma,m} \frac{\kappa^{\frac{1}{\sigma}}}{(\log \kappa)^{\frac{m}{\sigma}+2}}.$$

Next, making change of variables, we obtain

$$\int_{y_1}^{y_2} g\left(\frac{s}{p^{\sigma}(\log p)^m}\right) \frac{dy}{\log y}$$
$$= \sigma^{\frac{m}{\sigma}} \kappa^{\frac{1}{\sigma}} \int_{u_2}^{u_1} \frac{g\left(\frac{su}{\kappa}\right)}{u^{\frac{1}{\sigma}+1}(\log(\frac{\kappa}{u}))^{\frac{m}{\sigma}+1}} du \left(1 + O_m\left(\frac{\log\log \kappa}{\log \kappa}\right)\right),$$

where we put

$$u_1 = \frac{\kappa}{y_1^{\sigma}(\log y_1)^m} \quad \text{and} \quad u_2 = \frac{\kappa}{y_2^{\sigma}(\log y_2)^m}.$$

Since it holds that

$$\left(\log\left(\frac{\kappa}{u}\right)\right)^{\frac{m}{\sigma}+1} = \frac{1}{(\log \kappa)^{\frac{m}{\sigma}+1}} \left(1 + O_{\sigma,m}\left(\frac{|\log u|}{\log \kappa}\right)\right)$$

for $u_1 \le u \le u_2$, and the estimate

$$\int_0^{\infty} \frac{\left|g\left(\frac{su}{\kappa}\right) \log u\right|}{u^{\frac{1}{\sigma}+1}} du \ll_{\sigma} 1$$

also holds, the integral is calculated as

$$\int_{y_1}^{y_2} g\left(\frac{s}{p^{\sigma}(\log p)^m}\right) \frac{dy}{\log y}$$
$$= \sigma^{\frac{m}{\sigma}} \frac{\kappa^{\frac{1}{\sigma}}}{(\log \kappa)^{\frac{m}{\sigma}+1}} \int_{u_2}^{u_1} \frac{g\left(\frac{su}{\kappa}\right)}{u^{\frac{1}{\sigma}+1}} du \left(1 + O_{\sigma,m}\left(\frac{\log\log \kappa}{\log \kappa}\right)\right).$$

Finally, we see that the estimates

$$\int_0^{u_2} \frac{g\left(\frac{su}{\kappa}\right)}{u^{\frac{1}{\sigma}+1}} du \ll_{\sigma} u_2^{2-\frac{1}{\sigma}} \ll_{\sigma,m} \frac{1}{\log \kappa},$$
$$\int_{u_1}^{\infty} \frac{g\left(\frac{su}{\kappa}\right)}{u^{\frac{1}{\sigma}+1}} du \ll_{\sigma} u_1^{1-\frac{1}{\sigma}} \ll_{\sigma,m} \kappa^{\frac{1}{2}-\frac{1}{2\sigma}}$$

hold by Lemma 5.22, and therefore, the asymptotic formula

$$\int_{u_2}^{u_1} \frac{g\left(\frac{su}{\kappa}\right)}{u^{\frac{1}{\sigma}+1}}\, du = \int_0^\infty \frac{g\left(\frac{su}{\kappa}\right)}{u^{\frac{1}{\sigma}+1}}\, du + O_{\sigma,m}\left(\frac{1}{\log \kappa}\right)$$

follows. From the above, and by using the equation

$$\int_0^\infty \frac{g\left(\frac{su}{\kappa}\right)}{u^{\frac{1}{\sigma}+1}}\, du = \frac{s^{\frac{1}{\sigma}}}{\kappa^{\frac{1}{\sigma}}} \int_0^\infty \frac{g(u)}{u^{\frac{1}{\sigma}+1}}\, du,$$

we conclude

$$\sum_{y_1 < p \le y_2} g\left(\frac{s}{p^\sigma (\log p)^m}\right) = \sigma^{\frac{m}{\sigma}} g_0(\sigma) \frac{s^{\frac{1}{\sigma}}}{(\log \kappa)^{\frac{m}{\sigma}+1}} \left(1 + O_{\sigma,m}\left(\frac{\log\log\kappa}{\log\kappa}\right)\right). \tag{5.45}$$

Combining (5.43), (5.44), and (5.45), we obtain the asymptotic formula (5.42).

Let κ be large enough depending on σ and m. Then we have

$$f_{\sigma,m}(z;\alpha) = \sigma^{\frac{m}{\sigma}} g_0(\sigma) \frac{z^{\frac{1}{\sigma}}}{(\log\kappa)^{\frac{m}{\sigma}+1}} + h_{\sigma,m}(z;\alpha),$$

where

$$h_{\sigma,m}(z;\alpha) \ll_{\sigma,m} \frac{\kappa^{\frac{1}{\sigma}}}{(\log\kappa)^{\frac{m}{\sigma}+1}} \frac{\log\log\kappa}{\log\kappa}$$

for $|z-\kappa| \le \kappa/2$ by the asymptotic formula (5.42). By Cauchy's integral formula, we have

$$\begin{aligned} h_{\sigma,m}^{(n)}(\kappa;\alpha) &= f_{\sigma,m}^{(n)}(\kappa;\alpha) - \sigma^{\frac{m}{\sigma}} G_n(\sigma) g_0(\sigma) \frac{\kappa^{\frac{1}{\sigma}-n}}{(\log\kappa)^{\frac{m}{\sigma}+1}} \\ &= \frac{n!}{2\pi i} \int_{|z-\kappa|=\kappa/2} \frac{h_{\sigma,m}(z;\alpha)}{(z-\kappa)^{n+1}}\, dz \\ &\ll_{\sigma,m} \frac{2^n n! \kappa^{\frac{1}{\sigma}-n}}{(\log\kappa)^{\frac{m}{\sigma}+1}} \frac{\log\log\kappa}{\log\kappa}, \end{aligned}$$

where $G_n(\sigma) = \prod_{j=0}^{n-1}(\frac{1}{\sigma} - j)$. Using the equation $g_n(\sigma) = G_n(\sigma) g_0(\sigma)$ and the estimate $|g_n(\sigma)| \gg_\sigma (n-3)!$ for $n \ge 3$, we have the conclusion. □

5.7 Further results on probability density functions

5.7.1 Preliminaries

Lemma 5.24. *Let $(\sigma, m) \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. Suppose that $s = \kappa + it$ satisfies $\kappa > c(\sigma, m)$ with a large constant $c(\sigma, m) > 0$. Then we have*

$$\frac{|F_{\sigma,m}(s;\alpha)|}{F_{\sigma,m}(\kappa;\alpha)} \le \exp\left(-|t|^{1/(2\sigma)}\right)$$

for $|t| \ge \kappa/3$.

Proof. We see that

$$\frac{|F_{\sigma,m}(s;\alpha)|}{F_{\sigma,m}(\kappa;\alpha)} \leq \prod_{M_1|t|^{1/\sigma} \leq p \leq |t|^{2/\sigma}} \frac{\left|\mathbb{E}\left[\exp(s \operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p)))\right]\right|}{\mathbb{E}\left[\exp(\kappa \operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p)))\right]}. \tag{5.46}$$

If we suppose that $s = \kappa + it$ satisfies $\kappa > c(\sigma, m)$ and $|t| \geq \kappa/3$, then $|s| \leq 4|t|$. By the Taylor expansion of $\exp(z)$, we obtain

$$\begin{aligned}
&\exp(s \operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p))) \\
&= 1 + s \operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p)) + \frac{1}{2}\left(s \operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p))\right)^2 \\
&\quad + O\left(\frac{|t|^3}{p^{3\sigma}(\log p)^{3m}}\right)
\end{aligned}$$

for $p > M_1|t|^{1/\sigma}$ with suitably large M_1. It holds that

$$\mathbb{E}\left[\operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p))\right] = 0,$$

and

$$\mathbb{E}\left[\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p))\right)^2\right] = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(kp^{k\sigma}(\log p^k)^m)^2}.$$

By these formulas, it follows that

$$\begin{aligned}
&\mathbb{E}\left[\exp(s \operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p)))\right] \\
&= 1 + \frac{s^2}{4}\sum_{k=1}^{\infty}\frac{1}{(kp^{k\sigma}(\log p^k)^m)^2} + O\left(\frac{|t|^3}{p^{3\sigma}(\log p)^{3m}}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{\left|\mathbb{E}\left[\exp(s \operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p)))\right]\right|}{\mathbb{E}\left[\exp(\kappa \operatorname{Re} e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p)))\right]} \\
&= \left|1 + \frac{2i\kappa t - t^2}{4}\sum_{k=1}^{\infty}\frac{1}{(kp^{k\sigma}(\log p^k)^m)^2} + O\left(\frac{|t|^3}{p^{3\sigma}(\log p)^{3m}}\right)\right|
\end{aligned}$$

for $M_1|t|^{1/\sigma} \leq p \leq |t|^{2/\sigma}$. In particular, when M_1 is sufficiently large, it also holds that

$$\left|\frac{2i\kappa t - t^2}{4}\sum_{k=1}^{\infty}\frac{1}{(kp^{k\sigma}(\log p^k)^m)^2} + O\left(\frac{|t|^3}{p^{3\sigma}(\log p)^{3m}}\right)\right| \leq \frac{1}{2}.$$

From these results and inequality (5.46), we obtain

$$\begin{aligned}
&\frac{|F_{\sigma,m}(s;\alpha)|}{F_{\sigma,m}(\kappa;\alpha)} \\
&\leq \exp\left(\sum_{M_1|t|^{1/\sigma}\leq p\leq |t|^{2/\sigma}} \operatorname{Re}\log\left(1+\frac{2i\kappa t-t^2}{4}\sum_{k=1}^{\infty}\frac{1}{(kp^{k\sigma}(\log p^k)^m)^2}\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.\left. + O\left(\frac{|t|^3}{p^{3\sigma}(\log p)^{3m}}\right)\right)\right) \\
&= \exp\left(\sum_{M_1|t|^{1/\sigma}\leq p\leq |t|^{2/\sigma}} \operatorname{Re}\left(\frac{2i\kappa t-t^2}{4}\sum_{k=1}^{\infty}\frac{1}{(kp^{k\sigma}(\log p^k)^m)^2}\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.\left. + O\left(\frac{|t|^3}{p^{3\sigma}(\log p)^{3m}}\right)\right)\right) \\
&\leq \exp\left(\sum_{M_1|t|^{1/\sigma}\leq p\leq |t|^{2/\sigma}}\left(\frac{-t^2}{2p^{2\sigma}(\log p)^{2m}} + O\left(\frac{|t|^3}{p^{3\sigma}(\log p)^{3m}}\right)\right)\right) \\
&\leq \exp\left(-t^2\sum_{M_1|t|^{1/\sigma}\leq p\leq |t|^{2/\sigma}}\frac{1}{4p^{2\sigma}(\log p)^{2m}}\right) \leq \exp\left(-|t|^{1/2\sigma}\right)
\end{aligned}$$

when M_1 is sufficiently large. Thus, we obtain this lemma. □

As a final preliminary lemma, we prove the following result.

Lemma 5.25. *Let $(\sigma, m) \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. For each $\tau > 0$, there exists a unique real number $\kappa = \kappa(\tau; \sigma, m, \alpha) > 0$ such that*

$$f'_{\sigma,m}(\kappa;\alpha) = \tau. \tag{5.47}$$

Furthermore, we have $\kappa \to \infty$ as $\tau \to \infty$.

Proof. Since $f_{\sigma,m}(\kappa;\alpha) = \log F_{\sigma,m}(\kappa;\alpha)$, we have

$$f'_{\sigma,m}(\kappa;\alpha) = \frac{F'_{\sigma,m}(\kappa;\alpha)}{F_{\sigma,m}(\kappa;\alpha)}.$$

In particular, we obtain

$$f'_{\sigma,m}(0;\alpha) = \frac{F'_{\sigma,m}(0;\alpha)}{F_{\sigma,m}(0;\alpha)} = \mathbb{E}\left[Y\right] = 0,$$

where we define $Y = \operatorname{Re}(e^{-i\alpha}\tilde{\eta}_{m,p}(\sigma, X(p)))$. Therefore it is sufficient to show that $f''_{\sigma,m}(\kappa;\alpha) > 0$ for $\kappa > 0$ for the proof of the result. Note that we have

$$\begin{aligned}
f''_{\sigma,m}(\kappa;\alpha) &= \frac{F''_{\sigma,m}(\kappa;\alpha)F_{\sigma,m}(\kappa;\alpha) - F'_{\sigma,m}(\kappa;\alpha)^2}{F_{\sigma,m}(\kappa;\alpha)^2} \\
&= \frac{1}{F_{\sigma,m}(\kappa;\alpha)}\mathbb{E}\left[\left(Y - f'_{\sigma,m}(\kappa;\alpha)\right)^2\exp(\kappa Y)\right] > 0.
\end{aligned}$$

Hence, the result follows. □

5.7.2 A transformation of the density function

Let $(\sigma, m) \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. We define a non-negative continuous function $D_{\sigma,m}(x;\alpha)$ as

$$D_{\sigma,m}(x;\alpha) = \int_{\mathbb{R}} D_{\sigma,m}(e^{i\alpha}(x+iy)) \frac{dy}{\sqrt{2\pi}},$$

where $D_{\sigma,m}(z)$ is the probability density function determined by (5.25). Then the function $D_{\sigma,m}(x;\alpha)$ satisfies

$$\begin{aligned}\mathbb{E}\left[\Phi\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X)\right)\right] &= \int_{\mathbb{C}} \Phi\left(\operatorname{Re} e^{-i\alpha} z\right) D_{\sigma,m}(z)\,|dz| \\ &= \int_{\mathbb{R}} \Phi(x) D_{\sigma,m}(x;\alpha)\,|dx|\end{aligned}$$

for all Lebesgue measurable functions $\Phi(x)$, where $|dx| = (2\pi)^{-1/2}dx$. Hence, $D_{\sigma,m}(x;\alpha)$ is again a probability density function, whose moment-generating function is given by

$$F_{\sigma,m}(s;\alpha) = \int_{\mathbb{R}} e^{sx} D_{\sigma,m}(x;\alpha)\,|dx| \tag{5.48}$$

which agrees with (5.39). In this section, we study the function

$$D^{\tau}_{\sigma,m}(x;\alpha) = \frac{e^{\kappa\tau}}{F_{\sigma,m}(\kappa;\alpha)} e^{\kappa x} D_{\sigma,m}(x+\tau;\alpha),$$

where $\tau > 0$ and $\kappa = \kappa(\tau;\sigma,m,\alpha)$ is a positive real number satisfying (5.47).

Lemma 5.26. *Let $(\sigma, m) \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. For $\tau > 0$, the function $D^{\tau}_{\sigma,m}(x;\alpha)$ is a probability density function, whose Fourier transform is given by*

$$\widetilde{D}^{\tau}_{\sigma,m}(t;\alpha) := \int_{\mathbb{R}} e^{itx} D^{\tau}_{\sigma,m}(x;\alpha)\,|dx| = e^{-it\tau}\frac{F_{\sigma,m}(\kappa+it;\alpha)}{F_{\sigma,m}(\kappa;\alpha)}.$$

Proof. By the definition, we have

$$\int_{\mathbb{R}} D^{\tau}_{\sigma,m}(x;\alpha)\,|dx| = \frac{1}{F_{\sigma,m}(\kappa;\alpha)} \int_{-\infty}^{\infty} e^{\kappa(x+\tau)} D_{\sigma,m}(x+\tau;\alpha)\,|dx| = 1$$

due to (5.48). Similarly, the Fourier transform of $D^{\tau}_{\sigma,m}(x;\alpha)$ is calculated as

$$\begin{aligned}\widetilde{D}^{\tau}_{\sigma,m}(t;\alpha) &= \frac{1}{F_{\sigma,m}(\kappa;\alpha)} \int_{-\infty}^{\infty} e^{\kappa(x+\tau)+itx} D_{\sigma,m}(x+\tau;\alpha)\,|dx| \\ &= \frac{1}{F_{\sigma,m}(\kappa;\alpha)} \int_{-\infty}^{\infty} e^{\kappa x + it(x-\tau)} D_{\sigma,m}(x;\alpha)\,|dx| \\ &= e^{-it\tau}\frac{F_{\sigma,m}(\kappa+it;\alpha)}{F_{\sigma,m}(\kappa;\alpha)},\end{aligned}$$

which completes the proof. □

By Lemma 5.24, we find that $\widetilde{D}^{\tau}_{\sigma,m}(t;\alpha)$ is absolutely integrable over $\mathbb{R}$. Hence, the function $D^{\tau}_{\sigma,m}(x;\alpha)$ can be recovered by the inversion formula

$$D^{\tau}_{\sigma,m}(x;\alpha) = \int_{\mathbb{R}} \widetilde{D}^{\tau}_{\sigma,m}(t;\alpha)e^{-itx}\,|dt|. \tag{5.49}$$

Next, we apply (5.49) to obtain an asymptotic formula for $D^{\tau}_{\sigma,m}(x;\alpha)$.

Proposition 5.6. *Let* $1/2 < \sigma < 1$, $m \in \mathbb{Z}_{\geq 0}$, *and* $\alpha \in \mathbb{R}$. *For* $\tau > 0$, *we take* $\kappa = \kappa(\tau;\sigma,m,\alpha) > 0$ *satisfying* (5.47). *Then we have*

$$D^{\tau}_{\sigma,m}(x;\alpha) = \frac{1}{\sqrt{2\pi f''_{\sigma,m}(\kappa;\alpha)}}\left\{\exp\left(-\frac{x^2}{2f''_{\sigma,m}(\kappa;\alpha)}\right) + O\left(\kappa^{-\frac{1}{2\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}\right)\right\}$$

for all $x \in \mathbb{R}$ *if* $\tau > 0$ *is large enough. The implicit constant depends only on* σ *and* m.

Proof. First, applying formula (5.49), we deduce

$$D^{\tau}_{\sigma,m}(x;\alpha) = \int_{-\kappa/3}^{\kappa/3} \widetilde{D}^{\tau}_{\sigma,m}(t;\alpha)e^{-itx}\,|dt| + E_1, \tag{5.50}$$

where $E_1 \ll \sqrt{\kappa}\exp(-\sqrt{\kappa/3})$ by Lemmas 5.24 and 5.26. In order to estimate the integral in (5.50), we define a holomorphic function $G(z)$ as

$$\begin{aligned} G(z) &= \exp\left(-\tau z - \frac{f''_{\sigma,m,Y}(\kappa;\alpha)}{2}z^2\right)\frac{F_{\sigma,m,Y}(z+\kappa;\alpha)}{F_{\sigma,m,Y}(\kappa;\alpha)} \\ &= 1 + \sum_{n=3}^{\infty}\frac{a_n}{n!}z^n. \end{aligned} \tag{5.51}$$

Note that the coefficients a_n are calculated as

$$a_n = \sum_{k=1}^{\lfloor n/3 \rfloor}\frac{1}{k!}\sum_{\substack{n_1+\cdots n_k=n \\ \forall j,\ n_j \geq 3}}\binom{n}{n_1,\ldots,n_k} f^{(n_1)}_{\sigma,m}(\kappa;\alpha)\cdots f^{(n_k)}_{\sigma,m}(\kappa;\alpha)$$

since $G(z)$ is also expressed as

$$\begin{aligned} G(z) &= \exp\left(f_{\sigma,m}(z+\kappa;\alpha) - f_{\sigma,m}(\kappa;\alpha) - f'_{\sigma,m}(\kappa;\alpha)z - \frac{f''_{\sigma,m}(\kappa;\alpha)}{2}z^2\right) \\ &= \exp\left(\sum_{n=3}^{\infty}\frac{f^{(n)}_{\sigma,m}(\kappa;\alpha)}{n!}z^n\right) \end{aligned}$$

near the origin. Then, by Lemma 5.26, we have

$$\widetilde{D}^{\tau}_{\sigma,m}(t;\alpha) = \exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)G(it).$$

Hence, we obtain

$$\int_{-\kappa/3}^{\kappa/3} \tilde{D}^{\tau}_{\sigma,m}(t;\alpha)e^{-itx}\,|dt| = \int_{-\kappa/3}^{\kappa/3} \exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right) e^{-itx}\,|dt| + E_2, \quad (5.52)$$

where

$$\begin{aligned} E_2 &= \int_{-\kappa/3}^{\kappa/3} \exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)(G(it)-1)\,e^{-itx}\,|dt| \\ &\ll \int_0^{\kappa/3} \exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)\left(\sum_{n=3}^{\infty}\frac{|a_n|}{n!}t^n\right)dt. \end{aligned}$$

We further evaluate the error term E_2 as follows. Notice that we have

$$\sum_{n=3}^{\infty}\frac{|a_n|}{n!}t^n \leq \exp\left(\sum_{n=3}^{\infty}\frac{\left|f^{(n)}_{\sigma,m}(\kappa;\alpha)\right|}{n!}t^n\right) - 1$$

for $0 \leq t \leq \kappa/3$. Furthermore, it is deduced from Proposition 5.5 that

$$\sum_{n=3}^{\infty}\frac{\left|f^{(n)}_{\sigma,m}(\kappa;\alpha)\right|}{n!}t^n \ll_{\sigma,m} \frac{\kappa^{\frac{1}{\sigma}}}{(\log\kappa)^{\frac{m}{\sigma}+1}}\sum_{n=3}^{\infty}\left(\frac{2t}{\kappa}\right)^n \ll \frac{\kappa^{\frac{1}{\sigma}-3}}{(\log\kappa)^{\frac{m}{\sigma}+1}}t^3.$$

Hence, there exists a constant $C_{\sigma,m} > 0$ such that

$$\sum_{n=3}^{\infty}\frac{|a_n|}{n!}t^n \leq \sum_{n=1}^{\infty}\frac{1}{n!}\left(C_{\sigma,m}\frac{\kappa^{\frac{1}{\sigma}-3}}{(\log\kappa)^{\frac{m}{\sigma}+1}}t^3\right)^n, \quad (5.53)$$

which deduces

$$\begin{aligned} E_2 &\ll \sum_{n=1}^{\infty}\frac{1}{n!}\left(C_{\sigma,m}\frac{\kappa^{\frac{1}{\sigma}-3}}{(\log\kappa)^{\frac{m}{\sigma}+1}}t^3\right)^n \int_0^{\kappa/3}\exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)t^{3n}\,dt \\ &\ll \frac{1}{\sqrt{f''_{\sigma,m}(\kappa;\alpha)}}\sum_{n=1}^{\infty}\frac{\Gamma(\frac{3n+1}{2})}{n!}\left(2\sqrt{2}C_{\sigma,m}\frac{\kappa^{\frac{1}{\sigma}-3}}{\sqrt{f''_{\sigma,m}(\kappa;\alpha)}^3(\log\kappa)^{\frac{m}{\sigma}+1}}\right)^n. \end{aligned}$$

By Proposition 5.5, we see that

$$\frac{\kappa^{\frac{1}{\sigma}-3}}{\sqrt{f''_{\sigma,m}(\kappa;\alpha)}^3(\log\kappa)^{\frac{m}{\sigma}+1}} \ll_{\sigma,m} \kappa^{-\frac{1}{2\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}$$

holds. Therefore we arrive at the estimate

$$E_2 \ll_{\sigma,m} \frac{1}{\sqrt{f''_{\sigma,m}(\kappa;\alpha)}}\kappa^{-\frac{1}{2\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}$$

if $\kappa > 0$ is large enough. Finally, we obtain

$$\begin{aligned}
&\int_{-\kappa/3}^{\kappa/3} \exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right) e^{-itx}\,|dt| \qquad (5.54)\\
&= \int_{-\infty}^{\infty} \exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right) e^{-itx}\,|dt| + E_3\\
&= \frac{1}{\sqrt{2\pi f''_{\sigma,m}(\kappa;\alpha)}} \exp\left(-\frac{x^2}{2f''_{\sigma,m}(\kappa;\alpha)}\right) + E_3,
\end{aligned}$$

where

$$E_3 \ll \int_{\kappa/3}^{\infty} \exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right) dt \ll \exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{18}\kappa^2\right).$$

Since $f''_{\sigma,m}(\kappa;\alpha) \ll_{\sigma,m} \kappa^{\frac{1}{\sigma}-2}(\log\kappa)^{-\frac{m}{\sigma}}$ due to Proposition 5.5, we obtain the result by combining (5.50), (5.52), and (5.54). □

5.8 Large deviations: Proof of Theorem 5.2

5.8.1 Preliminaries

In this section, we prove the following proposition.

Proposition 5.7. *Let $1/2 < \sigma < 1$, $m \in \mathbb{Z}_{\geq 0}$, and $\alpha \in \mathbb{R}$. For $\tau > 0$, we take $\kappa = \kappa(\tau;\sigma,m,\alpha) > 0$ satisfying (5.47). Then we have*

$$\begin{aligned}
&\mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma,X) > \tau\right) \qquad (5.55)\\
&= \frac{F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}}{\kappa\sqrt{2\pi f''_{\sigma,m}(\kappa;\alpha)}}\left\{1 + O\left(\kappa^{-\frac{1}{2\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}\right)\right\}
\end{aligned}$$

if $\tau > 0$ is large enough. The implicit constant depends only on σ and m.

We prepare some lemmas toward the proof of Proposition 5.7.

Lemma 5.27 (Granville–Soundararajan [35])**.** *Let $\lambda > 0$ be a real number . For $y > 0$ and $c > 0$, we have*

$$\begin{aligned}
0 &\leq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s\left(\frac{e^{\lambda s}-1}{\lambda s}\right)\frac{ds}{s} - \chi(y)\\
&\leq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s\left(\frac{e^{\lambda s}-1}{\lambda s}\right)\left(\frac{1-e^{-\lambda s}}{s}\right) ds,
\end{aligned}$$

where $\chi(y) = 1$ if $y > 1$ and $\chi(y) = 0$ otherwise.

By Lemma 5.27, we obtain

$$\begin{aligned}
0 &\leq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{\sigma,m}(s;\alpha)e^{-\tau s}\left(\frac{e^{\lambda s}-1}{\lambda s^2}\right) ds - \mathbb{P}\left(\operatorname{Re}(e^{-i\alpha}\tilde{\eta}_m(\sigma,X)) > \tau\right)\\
&\leq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{\sigma,m}(s;\alpha)e^{-\tau s}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)\left(\frac{1-e^{-\lambda s}}{s}\right) ds.
\end{aligned}$$

Then the main term in (5.55) comes from the following integral.

Lemma 5.28. *With the assumptions of Proposition 5.7, we have*

$$\frac{1}{2\pi i}\int_{\kappa-i\kappa/3}^{\kappa+i\kappa/3} F_{\sigma,m}(s;\alpha)e^{-\tau s}\left(\frac{e^{\lambda s}-1}{\lambda s^2}\right)ds$$
$$=\frac{F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}}{\kappa\sqrt{2\pi f''_{\sigma,m}(\kappa;\alpha)}}\left\{1+O\left(\kappa^{-\frac{1}{2\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}\right)\right\},$$

where we take $\lambda=\kappa^{-3}$, and the implicit constant depends only on σ and m.

Proof. Let $G(z)$ be the function defined as (5.51). Then we see that

$$F_{\sigma,m}(\kappa+it;\alpha)e^{-\tau(\kappa+it)}=F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}\exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)G(it)$$

by definition. Furthermore, we obtain

$$\frac{e^{\lambda(\kappa+it)}-1}{\lambda(\kappa+it)^2}=\frac{1}{\kappa}\left(1-i\frac{t}{\kappa}+O\left(\lambda\kappa+\frac{t^2}{\kappa^2}\right)\right)$$

for $|t|\leq\kappa/3$. Hence, the integral is calculated as

$$\frac{1}{2\pi i}\int_{\kappa-i\kappa/3}^{\kappa+i\kappa/3} F_{\sigma,m}(s;\alpha)e^{-\tau s}\left(\frac{e^{\lambda s}-1}{\lambda s^2}\right)ds$$
$$=\frac{F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}}{\kappa}\frac{1}{2\pi i}\int_{-\kappa/3}^{\kappa/3}\exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)$$
$$\times G(it)\left(1-i\frac{t}{\kappa}+O\left(\lambda\kappa+\frac{t^2}{\kappa^2}\right)\right)dt$$
$$=\frac{F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}}{\kappa}\left\{\frac{1}{2\pi}\int_{-\kappa/3}^{\kappa/3}\exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)dt\right.$$
$$+\frac{1}{2\pi}\int_{-\kappa/3}^{\kappa/3}\exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)\left(-i\frac{t}{\kappa}+O\left(\lambda\kappa+\frac{t^2}{\kappa^2}\right)\right)dt$$
$$+\frac{1}{2\pi}\int_{-\kappa/3}^{\kappa/3}\exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)\left(\sum_{n=3}^{\infty}\frac{a_n}{n!}(it)^n\right)$$
$$\left.\times\left(1-i\frac{t}{\kappa}+O\left(\lambda\kappa+\frac{t^2}{\kappa^2}\right)\right)dt\right\}$$
$$=\frac{F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}}{\kappa}(I_1+I_2+I_3),$$

say. We have

$$I_1=\frac{1}{\sqrt{2\pi f''_{\sigma,m}(\kappa;\alpha)}}\left\{1+O\left(\exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{18}\kappa^2\right)\right)\right\}$$

and

$$I_2\ll\frac{1}{\sqrt{f''_{\sigma,m}(\kappa;\alpha)}}\left(\lambda\kappa+\frac{1}{\kappa^2 f''_{\sigma,m}(\kappa;\alpha)}\right).$$

Since $\lambda = \kappa^{-3}$, the contributions of I_1 and I_2 are evaluated as

$$\frac{F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}}{\kappa}(I_1+I_2) = \frac{F_{\sigma,m}(\kappa;\alpha)e^{-\kappa\tau}}{\kappa\sqrt{2\pi f''_{\sigma,m}(\kappa;\alpha)}}\left\{1+O\left(\kappa^{-\frac{1}{\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}\right)\right\}$$

by using Proposition 5.5. As for I_3, we use inequality (5.53) to derive

$$\begin{aligned} I_3 &\ll \sum_{n=1}^{\infty}\frac{1}{n!}\left(C_{\sigma,m}\frac{\kappa^{\frac{1}{\sigma}-3}}{(\log\kappa)^{\frac{m}{\sigma}+1}}\right)^n\int_0^{\kappa/3}\exp\left(-\frac{f''_{\sigma,m}(\kappa;\alpha)}{2}t^2\right)t^{3n}\,dt \\ &\ll \frac{1}{\sqrt{f''_{\sigma,m}(\kappa;\alpha)}}\sum_{n=1}^{\infty}\frac{\Gamma(\frac{3n+1}{2})}{n!}\left(2\sqrt{2}C_{\sigma,m}\frac{\kappa^{\frac{1}{\sigma}-3}}{\sqrt{f''_{\sigma,m}(\kappa;\alpha)}^{3}(\log\kappa)^{\frac{m}{\sigma}+1}}\right)^n. \end{aligned}$$

By Proposition 5.5, we see that

$$\frac{\kappa^{\frac{1}{\sigma}-3}}{\sqrt{f''_{\sigma,m}(\kappa;\alpha)}^{3}(\log\kappa)^{\frac{m}{\sigma}+1}} \ll_{\sigma,m} \kappa^{-\frac{1}{2\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}$$

holds. Therefore we arrive at the estimate

$$I_3 \ll_{\sigma,m} \frac{1}{\sqrt{f''_{\sigma,m}(\kappa;\alpha)}}\kappa^{-\frac{1}{2\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}$$

if $\kappa > 0$ is large enough, which yields the result. □

Proof of Proposition 5.7. The remaining work is to evaluate the integrals

$$E_1 = \frac{1}{2\pi i}\int_{\kappa\pm i\kappa/3}^{\kappa\pm i\infty}F_{\sigma,m}(s;\alpha)e^{-\tau s}\left(\frac{e^{\lambda s}-1}{\lambda s^2}\right)ds$$

and

$$E_2 = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}F_{\sigma,m}(s;\alpha)e^{-\tau s}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)\left(\frac{1-e^{-\lambda s}}{s}\right)ds. \tag{5.56}$$

Since we take $\lambda = \kappa^{-3}$, we obtain

$$\frac{e^{\lambda s}-1}{\lambda s^2} \ll \frac{\kappa^3}{t^2} \quad\text{and}\quad \left(\frac{e^{\lambda s}-1}{\lambda s}\right)\left(\frac{1-e^{-\lambda s}}{s}\right) \ll \begin{cases}\kappa^{-3} & \text{if } |t|\le\kappa/3,\\ \kappa^3 t^{-2} & \text{if } |t|>\kappa/3.\end{cases}$$

For the integral E_1, we apply Lemma 5.24 to derive

$$\begin{aligned} E_1 &\ll F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}\int_{\kappa/3}^{\infty}\exp(-t^{1/(2\sigma)})\frac{\kappa^3}{t^2}\,dt \\ &\ll F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}\exp\left(-\frac{1}{2}\kappa^{1/(2\sigma)}\right). \end{aligned}$$

Hence, Proposition 5.5 yields

$$E_1 \ll \frac{F_{\sigma,m}(\kappa;\alpha)e^{-\kappa\tau}}{\kappa\sqrt{f''_{\sigma,m}(\kappa;\alpha)}}\kappa^{-\frac{1}{2\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}.$$

Next, we split the integral of the right hand side of (5.56) as

$$E_2 = \frac{1}{2\pi i}\left(\int_{\kappa-i\kappa/3}^{\kappa+i\kappa/3} + \int_{\kappa\pm i\kappa/3}^{\kappa\pm i\infty}\right) = E_{21} + E_{22}.$$

Then we have

$$E_{21} \ll F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}\int_{-\kappa/3}^{\kappa/3}\kappa^{-3}\,dt \ll F_{\sigma,m}(\kappa;\alpha)e^{-\tau\kappa}\kappa^{-2}.$$

The integral E_{22} can be estimated along the same lines as in the case of E_1. Hence, we deduce

$$E_2 \ll \frac{F_{\sigma,m}(\kappa;\alpha)e^{-\kappa\tau}}{\kappa\sqrt{f''_{\sigma,m}(\kappa;\alpha)}}\kappa^{-\frac{1}{2\sigma}}(\log\kappa)^{\frac{1}{2}(\frac{m}{\sigma}+1)}$$

by Proposition 5.5. The conclusion follows from the above estimates on E_1 and E_2 together with Lemma 5.28. □

Corollary 5.2. *Let $1/2 < \sigma < 1$, $m \in \mathbb{Z}_{\geq 0}$, and $\alpha \in \mathbb{R}$. For large $\tau > 0$, we have*

$$\begin{aligned}&\mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right)\\ &= \exp\left(-A_m(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{m+\sigma}{1-\sigma}}\left(1 + O\left(\frac{\log\log\tau}{\log\tau}\right)\right)\right),\end{aligned}$$

where $A_m(\sigma)$ is defined as (3.1).

Proof. By Proposition 5.5, one can estimate $\kappa = \kappa(\tau;\sigma,m,\alpha)$ of Lemma 5.25 as

$$\kappa = C_m(\sigma)\tau^{\frac{\sigma}{1-\sigma}}(\log\tau)^{\frac{m+\sigma}{1-\sigma}}\left(1 + O\left(\frac{\log\log\tau}{\log\tau}\right)\right)$$

if $\tau > 0$ is large enough, where

$$C_m(\sigma) = \left(\frac{\sigma}{(1-\sigma)^{\frac{m}{\sigma}+1}g_1(\sigma)}\right)^{\frac{\sigma}{1-\sigma}}.$$

Inserting it and $g_1(\sigma) = \sigma^{-1}(1-\sigma)^{1-1/\sigma}G(\sigma)$ to (5.55), we obtain the corollary. □

5.8.2 Proof of the result for large deviations

Define

$$G(u) = \frac{2u}{\pi} + \frac{2(1-u)u}{\tan(\pi u)} \quad \text{and} \quad f_{c,d}(u) = \frac{1}{2}(e^{-2\pi iuc} - e^{-2\pi iud})$$

with $c, d \in \mathbb{R}$. For a set A, we denote the indicator function of A by $\mathbf{1}_A$.

Lemma 5.29. *Let $L > 0$. Let $c, d \in \mathbb{R}$ with $c < d$. For any $x \in \mathbb{R}$, we have*

$$\begin{aligned}\mathbf{1}_{(c,d)}(x) = \operatorname{Im} \int_0^L G\left(\frac{u}{L}\right) e^{2\pi iux} f_{c,d}(u) \frac{du}{u} \\ + O\left(\left(\frac{\sin(\pi L(x-c))}{\pi L(x-c)}\right)^2 + \left(\frac{\sin(\pi L(x-d))}{\pi L(x-d)}\right)^2\right).\end{aligned}$$

Proof. This lemma is equation (6.1) in [67], which is proved essentially in [116]. □

Let $\mathcal{X} \subset [0, T]$ be a Lebesgue measurable set. We define

$$\mathbb{P}_T^{\mathcal{X}}(f(t) \in A) = \frac{1}{T} \operatorname{meas}\{t \in \mathcal{X} \ : \ f(t) \in A\}$$

for $A \in \mathcal{B}(\mathbb{R})$, where $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function. Denote by μ and ν the measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\begin{aligned}\mu(A) &= \mathbb{P}_T^{A_T}\left(\operatorname{Re} e^{-i\alpha} P_{m,Y}(\sigma + it) \in A\right), \\ \nu(A) &= \mathbb{P}\left(\operatorname{Re} e^{-i\alpha} \tilde{\eta}_m(\sigma, X) \in A\right),\end{aligned}$$

respectively, where $A_T = A_T(V, Y; \sigma, m)$ is given by (5.3), and Y, V are functions of T determined later. Towards the proof of Theorem 5.2, we further define the measures P and Q as

$$P(A) = \int_A e^{\kappa u}\, d\mu(u), \quad Q(A) = \int_A e^{\kappa u}\, d\nu(u)$$

for $A \in \mathcal{B}(\mathbb{R})$, where κ is a real number chosen later. Then, for any $\tau > 0$, we obtain

$$\mu((\tau, \infty)) = \int_\tau^\infty e^{-\kappa u}\, dP(u) = \int_{\kappa\tau}^\infty e^{-x} P((\tau, x/\kappa))dx, \tag{5.57}$$

and

$$\nu((\tau, \infty)) = \int_\tau^\infty e^{-\kappa u}\, dQ(u) = \int_{\kappa\tau}^\infty e^{-x} Q((\tau, x/\kappa))dx. \tag{5.58}$$

We begin with estimating the difference between P and Q.

Lemma 5.30. *Let* $1/2 < \sigma < 1$, $m \in \mathbb{Z}_{\geq 0}$, *and* $\alpha \in \mathbb{R}$. *Suppose that we have*

$$V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}} \leq \varepsilon Y^{\sigma-\frac{1}{2}}(\log Y)^m$$

with a small constant $\varepsilon > 0$. *There exists a positive constant* $b_2 = b_2(\sigma, m)$ *such that for any* $|\kappa| \leq b_2 V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}$ *we have*

$$P((c,d)) = Q((c,d)) + E,$$

for any $c, d \in \mathbb{R}$ *with* $c < d$, *where*

$$E \ll_{\sigma,m} \left(\frac{1+\kappa}{V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}} + \frac{\kappa^2}{V^{\frac{2\sigma}{1-\sigma}}(\log V)^{\frac{2(m+\sigma)}{1-\sigma}}} + \frac{V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}}{Y^{\sigma-\frac{1}{2}}(\log Y)^m}\right) F_{\sigma,m}(\kappa;\alpha)$$
$$+ \left(V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}} Y\right)^{V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}} + \exp\left(-b_2 V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right). \tag{5.59}$$

Proof. Let $L = b_2 V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}$. By Lemma 5.29 and the definition of P, we can write

$$P((c,d)) = \int_{-\infty}^{\infty} e^{\kappa x} \operatorname{Im} \int_0^L G\left(\frac{u}{L}\right) e^{2\pi i u x} f_{c,d}(u) \frac{du}{u} d\mu(x) + E_1, \tag{5.60}$$

where

$$E_1 \ll \int_{\mathbb{R}} \left\{\left(\frac{\sin(\pi L(u-c))}{\pi L(u-c)}\right)^2 + \left(\frac{\sin(\pi L(u-d))}{\pi L(u-d)}\right)^2\right\} e^{\kappa u} d\mu(u).$$

First, we estimate E_1. For $z \in \mathbb{C}$, we define

$$M(z) = \int_{\mathbb{R}} e^{zu} d\mu(u).$$

Then it holds that

$$\begin{aligned} M(z) &= \frac{1}{T}\int_{A_T} \exp\left(z \operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma+it)\right) dt \\ &= \frac{1}{T}\int_{A_T} \exp\left(\frac{z}{2} e^{-i\alpha}\tilde{\eta}_m(\sigma+it) + \frac{z}{2} e^{i\alpha}\overline{\tilde{\eta}_m(\sigma+it)}\right) dt. \end{aligned}$$

For any $\ell, x \in \mathbb{R}$, we can write

$$\begin{aligned} \left(\frac{\sin(\pi L(u-\ell))}{\pi L(u-\ell)}\right)^2 &= \frac{2}{L^2}\int_0^L (L-\xi)\cos(2\pi(u-\ell)\xi) d\xi \\ &= \frac{2}{L^2} \operatorname{Re} \int_0^L (L-\xi) e^{2\pi i (u-\ell)\xi} d\xi, \end{aligned}$$

and therefore we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\frac{\sin(\pi L(u-\ell))}{\pi L(u-\ell)} \right)^2 e^{\kappa u} d\mu(u) \\
&= \frac{2}{L^2} \int_0^L (L-\xi) \int_{\mathbb{R}} e^{\kappa u + 2\pi i (u-\ell)\xi} d\mu(u) d\xi \\
&= \frac{2}{L^2} \int_0^L (L-\xi) e^{-2\pi i \ell \xi} M(\kappa + 2\pi i \xi) d\xi. \qquad (5.61)
\end{aligned}$$

Here, we decide b_2 as $b_1/4$, where b_1 is the same constant as in Proposition 5.1. We can apply Proposition 5.1, and obtain

$$M(\kappa + 2\pi i \xi) = \mathbb{E}\left[\exp\left((\kappa + 2\pi i \xi) \operatorname{Re} e^{-i\alpha} P_{m,Y}(\sigma, X)\right)\right] + O(E_2)$$

for $|\xi| \le L$, where

$$E_2 = \frac{1}{T} \left(V^{\frac{\sigma}{1-\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}} Y \right)^{V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}}} + \exp\left(-b_2 V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}} \right).$$

Applying further Lemma 5.13, we derive

$$M(\kappa + 2\pi i \xi) = F_{\sigma,m}(\kappa + 2\pi i \xi; \alpha) + O(E_2 + E_3), \qquad (5.62)$$

where

$$E_3 = F_{\sigma,m}(\kappa; \alpha) \frac{|\kappa + 2\pi i \xi|}{Y^{\sigma - \frac{1}{2}} (\log Y)^m} \le F_{\sigma,m}(\kappa; \alpha) \frac{V^{\frac{1}{1-\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}}}{Y^{\sigma - \frac{1}{2}} (\log Y)^m}.$$

By this formula, Lemma 5.24, and the inequality

$$|F_{\sigma,m}(\kappa + 2\pi i \xi; \alpha)| \le F_{\sigma,m}(\kappa; \alpha),$$

we find that

$$\begin{aligned}
&\frac{2}{L^2} \int_0^L (L-\xi) e^{-2\pi i \ell \xi} M(\kappa + 2\pi i \xi) d\xi \\
&\ll \frac{F_{\sigma,m}(\kappa; \alpha)}{L^2} \left(\int_0^\kappa (L-\xi) d\xi + \frac{1}{L^2} \int_\kappa^L (L-\xi) \exp(-\xi^{1/(2\sigma)}) d\xi \right) + E_2 + E_3 \\
&\ll \left(\frac{1+\kappa}{L} + \frac{\kappa^2}{L^2} \right) F_{\sigma,m}(\kappa; \alpha) + E_2 + E_3.
\end{aligned}$$

From this estimate and equation (5.61), we obtain

$$\int_{\mathbb{R}} \left(\frac{\sin(\pi L(u-c))}{\pi L(u-c)} \right)^2 e^{\kappa u} d\mu(u) \ll \left(\frac{1+\kappa}{L} + \frac{\kappa^2}{L^2} \right) F_{\sigma,m}(\kappa; \alpha) + E_2 + E_3,$$

and

$$\int_{\mathbb{R}} \left(\frac{\sin(\pi L(v-d))}{\pi L(v-d)} \right)^2 e^{\kappa u} d\mu(u) \ll \left(\frac{1+\kappa}{L} + \frac{\kappa^2}{L^2} \right) F_{\sigma,m}(\kappa; \alpha) + E_2 + E_3.$$

Thus, we can estimate the error term E_1 on the right hand side of equation (5.60) by

$$E_1 \ll \left(\frac{1+\kappa}{L} + \frac{\kappa^2}{L^2}\right) F_{\sigma,m}(\kappa;\alpha) + E_2 + E_3. \tag{5.63}$$

Next, we calculate the main term of (5.60). Using Fubini's theorem, we find that the main term is equal to

$$\operatorname{Im} \int_0^L G\left(\frac{u}{L}\right) \frac{f_{c,d}(u)}{u} M(\kappa + 2\pi i u) du.$$

By equation (5.62) and the estimates $G(x) \ll 1$ for $0 \le x \le 1$ and $|f_{c,d}(u)/u| \le |d-c|$, this is further equal to

$$\operatorname{Im} \int_0^L G\left(\frac{u}{L}\right) \frac{f_{c,d}(u)}{u} F_{\sigma,m}(\kappa + 2\pi i u;\alpha) du + O(L(d-c)(E_2+E_3)). \tag{5.64}$$

Since we can write

$$F_{\sigma,m}(\kappa + 2\pi i u;\alpha) = \int_{\mathbb{R}} e^{(\kappa+2\pi i u)\xi} d\nu(\xi),$$

we find that, by using Fubini's theorem, (5.64) equals to

$$\operatorname{Im} \int_{\mathbb{R}} \int_0^L G\left(\frac{u}{L}\right) e^{2\pi i u \xi} f_{c,d}(u) \frac{du}{u} e^{\kappa \xi} d\nu(\xi) + O(L(d-c)(E_2+E_3)).$$

Applying Lemma 5.29 again, this is equal to

$$Q(\mathscr{R}) + O\left(L(d-c)(E_2+E_3) + E_4\right),$$

where

$$E_4 \ll \int_{\mathbb{R}} \left\{ \left(\frac{\sin(\pi L(u-c))}{\pi L(u-c)}\right)^2 + \left(\frac{\sin(\pi L(u-d))}{\pi L(u-d)}\right)^2 \right\} e^{\kappa u} d\nu(u).$$

Similarly to the proof of (5.63), we can obtain

$$E_4 \ll \left(\frac{1+\kappa_1}{L} + \frac{\kappa_1^2}{L^2}\right) F_{\sigma,m}(\kappa;\alpha).$$

Thus, we obtain this lemma. □

Proposition 5.8. *Let $1/2 < \sigma < 1$, $m \in \mathbb{Z}_{\ge 0}$, and $\alpha \in \mathbb{R}$. Let $T > 0$ be large enough. We take the functions Y and V as*

$$\begin{aligned} Y &= (\log T)^B, \\ V &= (\log T)^{1-\sigma} (\log\log T)^{-m-1} \end{aligned} \tag{5.65}$$

with $B = \frac{6}{\sigma - 1/2}$. *Then there exists a constant* $b_m(\sigma) > 0$ *such that*

$$\begin{aligned}&\mathbb{P}_T^{A_T}\left(\operatorname{Re} e^{-i\alpha} P_{m,Y}(\sigma+it) > \tau\right)\\ &= \mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right)\left(1 + O\left(\frac{\tau^{\frac{\sigma}{1-\sigma}}(\log\tau)^{\frac{m+\sigma}{1-\sigma}}}{V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}}\right)\right)\end{aligned}$$

in the range $1 \ll \tau \le b_m(\sigma)V$.

Proof. By equation (5.57), we have

$$\mathbb{P}_T^{A_T}\left(\operatorname{Re} e^{-i\alpha} P_{m,Y}(\sigma+it) > \tau\right) = \int_{\kappa\tau}^{\infty} e^{-x} P((\tau, x/\kappa))dx.$$

Hence, by Lemma 5.30, this is equal to

$$\int_{\kappa\tau}^{\infty} e^{-x} Q((\tau, x/\kappa))dx + O\left(e^{-\kappa\tau} E\right),$$

where E satisfies estimate (5.59). By equation (5.58), this main term is equal to $\mathbb{P}\left(\operatorname{Re}(e^{-i\alpha}\tilde{\eta}_m(\sigma, X)) > \tau\right)$. Thus, we complete the proof. □

Proof of Theorem 5.2. Define Y, V by (5.65). Let $B_T = B_T(Y, W; \sigma, m)$ be the set given by (5.11). Then we have

$$\begin{aligned}&\mathbb{P}_T\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma+it) > \tau\right)\\ &= \mathbb{P}_T^{B_T}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma+it) > \tau\right) + O\left(\frac{1}{T}\operatorname{meas}([0,T]\setminus B_T)\right)\\ &\le \mathbb{P}_T^{B_T}\left(\operatorname{Re} e^{-i\alpha} P_{m,Y}(\sigma+it) > \tau - \varepsilon\right) + O\left(\frac{1}{T}\operatorname{meas}([0,T]\setminus B_T)\right)\end{aligned}$$

by the definition of B_T, where we put $\varepsilon = WY^{\frac{1}{2}-\sigma}$. Furthermore, we obtain

$$\begin{aligned}&\mathbb{P}_T^{B_T}\left(\operatorname{Re} e^{-i\alpha} P_{m,Y}(\sigma+it) > \tau - \varepsilon\right)\\ &= \mathbb{P}_T^{A_T}\left(\operatorname{Re} e^{-i\alpha} P_{m,Y}(\sigma+it) > \tau - \varepsilon\right)\\ &\qquad + O\left(\frac{1}{T}\operatorname{meas}([0,T]\setminus A_T) + \frac{1}{T}\operatorname{meas}([0,T]\setminus B_T)\right).\end{aligned}$$

Then, the asymptotic formula

$$\begin{aligned}&\mathbb{P}_T^{A_T}\left(\operatorname{Re} e^{-i\alpha} P_{m,Y}(\sigma+it) > \tau - \varepsilon\right)\\ &= \mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau - \varepsilon\right)\left(1 + O\left(\frac{\tau^{\frac{\sigma}{1-\sigma}}(\log\tau)^{\frac{m+\sigma}{1-\sigma}}}{V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}}\right)\right)\end{aligned}$$

follows from Proposition 5.8. Recall that $\mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau + \gamma\right)$ is represented as

$$\mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau + \gamma\right) = F_{\sigma,m}(\kappa;\alpha)e^{-\kappa\tau}\int_{\gamma}^{\infty} e^{-\kappa x} D_{\sigma,m}^{\tau}(x;\alpha)\,|dx|$$

for all $\gamma \in \mathbb{R}$, which implies

$$\begin{aligned}
&\mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau - \varepsilon\right) - \mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right) \\
&= F_{\sigma,m}(\kappa;\alpha)e^{-\kappa\tau}\int_{-\varepsilon}^{0} e^{-\kappa x}E_{\sigma,m}(x;\alpha)\,|dx| \\
&\ll \mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right)\cdot \kappa\varepsilon e^{\kappa\varepsilon}
\end{aligned}$$

by Propositions 5.6 and 5.7. Therefore we deduce

$$\begin{aligned}
&\mathbb{P}_T\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma + it) > \tau\right) \\
&\le \mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right)\left(1 + O\left(\frac{\tau^{\frac{\sigma}{1-\sigma}}(\log\tau)^{\frac{m+\sigma}{1-\sigma}}}{V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}} + \kappa\varepsilon e^{\kappa\varepsilon}\right)\right) \\
&\quad + O\left(\frac{1}{T}\operatorname{meas}([0,T]\setminus A_T) + \frac{1}{T}\operatorname{meas}([0,T]\setminus B_T)\right).
\end{aligned}$$

Similarly one can obtain

$$\begin{aligned}
&\mathbb{P}_T\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma + it) > \tau\right) \\
&\ge \mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right)\left(1 + O\left(\frac{\tau^{\frac{\sigma}{1-\sigma}}(\log\tau)^{\frac{m+\sigma}{1-\sigma}}}{V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}} + \kappa\varepsilon e^{\kappa\varepsilon}\right)\right) \\
&\quad + O\left(\frac{1}{T}\operatorname{meas}([0,T]\setminus A_T) + \frac{1}{T}\operatorname{meas}([0,T]\setminus B_T)\right),
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\mathbb{P}_T\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma + it) > \tau\right) \\
&= \mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right)\left(1 + O\left(\frac{\tau^{\frac{\sigma}{1-\sigma}}(\log\tau)^{\frac{m+\sigma}{1-\sigma}}}{V^{\frac{\sigma}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}} + \kappa\varepsilon e^{\kappa\varepsilon}\right)\right) \\
&\quad + O\left(\frac{1}{T}\operatorname{meas}([0,T]\setminus A_T) + \frac{1}{T}\operatorname{meas}([0,T]\setminus B_T)\right).
\end{aligned}$$

We choose the function W as

$$W = (\log T)(\log\log T)^{-m-1}.$$

Then we find $\kappa\varepsilon e^{\kappa\varepsilon} \ll (\log T)^{-2}$ for $1 \ll \tau \le b_m(\sigma, A)V$. Let $c = c(\sigma, m)$ be a small positive constant for which both (5.10) and (5.12) are valid. Then, by Corollary 5.2, we have

$$\mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right)^{-1} \ll \exp\left(\frac{c}{2}V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right)$$

in the range $1 \ll \tau \le b_m(\sigma)V$ with $b_m(\sigma) > 0$ small enough. By this estimate and Lemmas 5.5, 5.6, we obtain

$$\frac{1}{T}\operatorname{meas}([0,T]\setminus A_T) \ll \mathbb{P}\left(\operatorname{Re} e^{-i\alpha}\tilde{\eta}_m(\sigma, X) > \tau\right)\exp\left(-\frac{c}{2}V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right)$$

and

$$\frac{1}{T}\,\mathrm{meas}([0,T]\setminus B_T) \ll \mathbb{P}\left(\mathrm{Re}\, e^{-i\alpha}\tilde{\eta}_m(\sigma,X) > \tau\right)\exp\left(-\frac{c}{2}V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right).$$

Since we have

$$\exp\left(-\frac{c}{2}V^{\frac{1}{1-\sigma}}(\log V)^{\frac{m+\sigma}{1-\sigma}}\right) \ll (\log T)^{-A},$$

the desired result follows. □

Chapter 6 Joint value distributions of L-functions on the critical line $\sigma = 1/2$

In this chapter, we discuss the joint value distribution of L-functions. The contents in this chapter are based on the paper [52]. We consider the L-functions belonging to the modified Selberg class $\mathcal{S}^\dagger$. Additionally, to study the joint value distribution of functions in $\mathcal{S}^\dagger$, we need the following assumption $\mathscr{A}$.

Assumption ($\mathscr{A}$). *An r-tuple of L-functions $\boldsymbol{F} = (F_1, \dots, F_r)$ with $F_j \in \mathcal{S}^\dagger$ and an r of the numbers $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in \mathbb{R}^r$ satisfy $\mathscr{A}$ if and only if $\boldsymbol{F}$, $\boldsymbol{\theta}$ satisfy the following properties.*

(A1) (Selberg Orthonormality Conjecture) For any F_j, we have

$$\sum_{p \le x} \frac{|a_{F_j}(p)|^2}{p} = n_{F_j} \log\log x + O_{F_j}(1), \tag{6.1}$$

for some positive constant n_{F_j} and $x \ge 3$. For any pair $F_i \ne F_j$,

$$\sum_{p \le x} \frac{a_{F_i}(p)\overline{a_{F_j}(p)}}{p} = O_{F_i, F_j}(1).$$

(A2) For every component F_i, there is at most one L-function such that $F_i = F_j$ with $i \ne j$, and then $|\theta_i - \theta_j| = \frac{\pi}{2}$.

(A3) (Zero Density Estimate) For every F_j, there exists a positive constant κ_{F_j} such that, uniformly for any $T \ge 3$ and $1/2 \le \sigma \le 1$,

$$N_{F_j}(\sigma, T) \ll_{F_j} T^{1 - \kappa_{F_j}(\sigma - 1/2)} \log T, \tag{6.2}$$

where $N_F(\sigma, T)$ is the number of nontrivial zeros $\rho_F = \beta_F + i\gamma_F$ of $F \in \mathcal{S}^\dagger$ with $\beta_F \ge \sigma$ and $0 \le \gamma_F \le T$.

Remark 3. The Selberg Orthonormality Conjecture has been proved for L-functions associated with cuspidal automorphic representations of $GL(n)$ unconditionally for $n \le 4$ by Liu and Ye in [73], (see [1, 72]) and in full generality in [74].

Remark 4. The zero density estimate like (6.2) for the Riemann zeta function and Dirichlet L-functions was established by Selberg [107] and Fujii [28] respectively. For GL(2) L-functions, (A3) has been established by Luo [76] for holomorphic cusp forms of the full modular group. Also, some weaker estimates have been proved by Ford and Zaharescu [27, Section 7] for other congruence subgroups of GL(2), and further by Sankaranarayanan and Sengupta [104] for Maaß cusp forms. If we assume the Riemann Hypothesis for F, then (A3) holds for any $\kappa_F > 0$.

Remark 5. It is natural to assume (A2). This allows us to consider the joint distribution of $\log|F(s)|$ and $\operatorname{Im}\log F(s)$. It can be seen that $\operatorname{Re} e^{-i\theta_1}\log F(s)$ and $\operatorname{Re} e^{-i\theta_2}\log F(s)$ can not be independent when $\theta_1 - \theta_2 \not\equiv \frac{\pi}{2} \pmod{2\pi}$.

6.1 Results

Before we state our theorems, we need some notation. Let r be a fixed positive integer. For $\boldsymbol{V} = (V_1, \dots, V_r) \in \mathbb{R}^r$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in [0, 2\pi]^r$, and $\boldsymbol{F} = (F_1, \dots, F_r) \in (\mathcal{S}^\dagger)^r$ satisfying assumption $\mathscr{A}$, we define

$$\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta}) := \left\{ t \in [T, 2T] \ : \ \frac{\operatorname{Re} e^{-i\theta_j}\log F_j(\frac{1}{2}+it)}{\sqrt{\frac{n_{F_j}}{2}\log\log T}} \geq V_j \text{ for } j = 1, \dots, r \right\},$$

where the constants n_{F_j} are defined in (6.1). Let $\alpha_{\boldsymbol{F}} := \min\{2r, \frac{1-2\vartheta_{\boldsymbol{F}}}{2\vartheta_{\boldsymbol{F}}}\}$, where $\vartheta_{\boldsymbol{F}} = \max_{1\leq j\leq r}\vartheta_{F_j}$ as defined in (S4). Here $\alpha_{\boldsymbol{F}} = 2r$ if $\vartheta_{\boldsymbol{F}} = 0$. We denote $\|\boldsymbol{z}\| = \max_{1\leq j\leq r}|z_j|$. Throughout this paper, we write $\log_3 x$ for $\log\log\log x$.

The following theorem extends the result of Bombieri and Hejhal [9], where we show (1.25) holds for a larger range of V.

Theorem 6.1. *Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in \mathbb{R}^r$, $\boldsymbol{F} = (F_1, \dots, F_r) \in (\mathcal{S}^\dagger)^r$ satisfying assumption $\mathscr{A}$. Let $A \geq 1$ be a fixed constant. For any large T and any $\boldsymbol{V} = (V_1, \dots, V_r) \in \mathbb{R}^r$ with $\|\boldsymbol{V}\| \leq A(\log\log T)^{1/10}$, we have*

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) = (1 + R_1)\prod_{j=1}^{r}\int_{V_j}^{\infty} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}}, \tag{6.3}$$

where

$$R_1 \ll_{\boldsymbol{F},A} \frac{(\|\boldsymbol{V}\|^4 + (\log_3 T)^2)(\|\boldsymbol{V}\| + 1)}{\sqrt{\log\log T}} + \frac{\prod_{k=1}^{r}(1 + |V_k|)}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}$$

Moreover, if $\theta_1, \dots, \theta_r \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\|\boldsymbol{V}\| \leq A(\log\log T)^{1/6}$ we have

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) \leq (1 + R_2)\prod_{j=1}^{r}\int_{V_j}^{\infty} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}}, \tag{6.4}$$

and

$$R_2 \ll_{\boldsymbol{F},A} \frac{(\|\boldsymbol{V}\|^2 + (\log_3 T)^2)(\|\boldsymbol{V}\|+1)}{\sqrt{\log\log T}} + \frac{\prod_{k=1}^r (1+|V_k|)}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}} \tag{6.5}$$

Furthermore, if $\theta_1, \dots, \theta_r \in [\frac{\pi}{2}, \frac{3\pi}{2}]$*,* $\|\boldsymbol{V}\| \le c(\log\log T)^{1/6}$*, and* $\prod_{j=1}^r (1+|V_j|) \le c(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}$ *with* $c = c(\boldsymbol{F}) > 0$ *small enough, we have*

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) \ge (1 - R_3) \prod_{j=1}^r \int_{V_j}^\infty e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}}. \tag{6.6}$$

Here, the error term R_3 *is* $\ll_{\boldsymbol{F}}$ *the right hand side of* (6.5).

Remark 6. For $r = 1$, $F_1 = \zeta$, and $\theta_1 = 0$, the asymptotic for $\|\boldsymbol{V}\| \ll (\log\log T)^{10}$ in (6.3) was obtained by Radziwiłł [95], and the bound for $\|\boldsymbol{V}\| \ll (\log\log T)^{1/6}$ in (6.4) was obtained in Theorem 2.5.

It is reasonable to conjecture that the asymptotic in (6.3) holds for $\|\boldsymbol{V}\| = o(\sqrt{\log\log T})$ as speculated in [95] for $\zeta(s)$. If we are only concerned with upper and lower bounds, we could extend the range of $\|\boldsymbol{V}\|$ further.

Theorem 6.2. *Let* $\boldsymbol{F} = (F_1, \dots, F_r) \in (\mathcal{S}^\dagger \setminus \{1\})^r$ *and* $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ *satisfying assumption* $\mathscr{A}$*. Let* T *be large. There exist some positive constants* $a_1 = a_1(\boldsymbol{F})$*,* $a_2 = a_2(\boldsymbol{F})$ *such that if* $\boldsymbol{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$*, we have*

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) \ll_{\boldsymbol{F}}$$
$$\left\{\left(\prod_{j=1}^r \frac{1}{1+V_j}\right) + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right\} \exp\left(-\frac{V_1^2 + \cdots + V_r^2}{2} + O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|^3}{\sqrt{\log\log T}}\right)\right)$$

for any $\boldsymbol{V} = (V_1, \dots, V_r) \in (\mathbb{R}_{\ge 0})^r$ *satisfying* $\|\boldsymbol{V}\| \le a_1(1 + V_m^{1/2})(\log\log T)^{1/4}$ *with* $V_m = \min_{1\le j \le r} V_j$*, and if* $\boldsymbol{\theta} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]^r$*, we have*

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta}))$$
$$\gg_{\boldsymbol{F}} \left(\prod_{j=1}^r \frac{1}{1+V_j}\right) \exp\left(-\frac{V_1^2 + \cdots + V_r^2}{2} + O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|^3}{\sqrt{\log\log T}}\right)\right)$$

for $\|\boldsymbol{V}\| \le a_1(1 + V_m^{1/2})(\log\log T)^{1/4}$ *with* $\prod_{j=1}^r (1+V_j) \le a_2(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}$.

Substituting $\boldsymbol{V} = \left(\frac{V}{\sqrt{\frac{n_{F_1}}{2}\log\log T}}, \dots, \frac{V}{\sqrt{\frac{n_{F_r}}{2}\log\log T}}\right)$ to Theorem 6.2, we obtain the following corollary.

Corollary 6.1. *Let $\boldsymbol{F} = (F_1, \ldots, F_r) \in (\mathcal{S}^\dagger \backslash \{1\})^r$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_r) \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ satisfying assumption $\mathscr{A}$. Put $h_{\boldsymbol{F}} = n_{F_1}^{-1} + \cdots + n_{F_r}^{-1}$. There exists a small constant $a_3 = a_3(\boldsymbol{F})$ such that if $\boldsymbol{\theta} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^r$, we have*

$$\frac{1}{T} \operatorname{meas}\left\{t \in [T, 2T] \ : \ \min_{1 \le j \le r} \operatorname{Re} e^{-i\theta_j} \log F_j(\tfrac{1}{2} + it) > V\right\}$$
$$\ll_{\boldsymbol{F}} \left(\frac{1}{(1 + V/\sqrt{\log\log T})^r} + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}}\right)$$
$$\times \exp\left(-h_{\boldsymbol{F}} \frac{V^2}{\log\log T}\left(1 + O_{\boldsymbol{F}}\left(\frac{V}{\log\log T}\right)\right)\right)$$

for any $0 \le V \le a_3 \log\log T$, and if $\boldsymbol{\theta} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]^r$, we have

$$\frac{1}{T} \operatorname{meas}\left\{t \in [T, 2T] \ : \ \min_{1 \le j \le r} \operatorname{Re} e^{-i\theta_j} \log F_j(\tfrac{1}{2} + it) > V\right\} \tag{6.7}$$
$$\gg_{\boldsymbol{F}} \frac{1}{(1 + V/\sqrt{\log\log T})^r} \exp\left(-h_{\boldsymbol{F}} \frac{V^2}{\log\log T}\left(1 + O_{\boldsymbol{F}}\left(\frac{V}{\log\log T}\right)\right)\right)$$

for any $0 \le V \le a_3 \min\{\log\log T, (\log\log T)^{\frac{\alpha_{\boldsymbol{F}}}{r} + \frac{1}{2} + \frac{1}{2r}}\}$.

Remark 7. When $r = 1$, $F_1 = \zeta$, and $\theta_1 = 0$, Jutila [56], using bounds on moments of $\zeta(s)$, has proved

$$\frac{1}{T} \operatorname{meas}\left\{t \in [T, 2T] \ : \ \log|\zeta(\tfrac{1}{2} + it)| > V\right\}$$
$$\ll \exp\left(-\frac{V^2}{\log\log T}\left(1 + O\left(\frac{V}{\log\log T}\right)\right)\right)$$

uniformly for $0 \le V \le \log\log T$. Our Theorem 6.2 slightly improves this bound by a factor of $\frac{\sqrt{\log\log T}}{V}$ when $\sqrt{\log\log T} \le V \le a_3 \log\log T$ for some small constant a_3.

The extended range of V allows us to prove the following mean value theorem.

Theorem 6.3. *Let $\boldsymbol{F} = (F_1, \ldots, F_r) \in (\mathcal{S}^\dagger)^r$ satisfying assumption $\mathscr{A}$. Let $h_{\boldsymbol{F}} = n_{F_1}^{-1} + \cdots + n_{F_r}^{-1}$. Then there exist some positive constants $a_4 = a_4(\boldsymbol{F})$ and $B = B(\boldsymbol{F})$ such that for any $0 < k \le a_4$, we have*

$$\int_T^{2T} \exp\left(2k \min_{1 \le j \le r} \operatorname{Re} e^{-\theta_j} \log F_j(\tfrac{1}{2} + it)\right) dt$$
$$\ll_{\boldsymbol{F}} T(\log T)^{k^2/h_{\boldsymbol{F}} + Bk^3} \left(\frac{k\sqrt{\log\log T}}{1 + (k\sqrt{\log\log T})^r} + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}}\right) \tag{6.8}$$

when $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^r$*, and if* $\vartheta_{\boldsymbol{F}} \le \frac{1}{r+1}$*, we have*

$$\int_T^{2T} \exp\left(2k \min_{1\le j\le r} \operatorname{Re} e^{-i\theta_j} \log F_j(\tfrac{1}{2}+it)\right) dt$$
$$\gg_{\boldsymbol{F}} T(\log T)^{\frac{k^2}{h_{\boldsymbol{F}}}-Bk^3} \frac{k\sqrt{\log\log T}}{1+(k\sqrt{\log\log T})^r} \tag{6.9}$$

when $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]^r$*. Here, the above implicit constants depend only on* $\boldsymbol{F}$*. In particular, if* $\vartheta_{\boldsymbol{F}} \le \frac{1}{r+1}$*, it holds that, for any* $0 < k \le a_4$*,*

$$\int_T^{2T} \left(\min_{1\le j\le r} |F_j(\tfrac{1}{2}+it)|\right)^{2k} dt \ll_{k,\boldsymbol{F}} T\frac{(\log T)^{\frac{k^2}{h_{\boldsymbol{F}}}+Bk^3}}{(\log\log T)^{(r-1)/2}},$$

$$\int_T^{2T} \left(\max_{1\le j\le r} |F_j(\tfrac{1}{2}+it)|\right)^{-2k} dt \gg_{\boldsymbol{F}} T\frac{(\log T)^{\frac{k^2}{h_{\boldsymbol{F}}}-Bk^3}}{(\log\log T)^{(r-1)/2}},$$

$$T\frac{(\log T)^{\frac{k^2}{h_{\boldsymbol{F}}}-Bk^3}}{(\log\log T)^{(r-1)/2}} \ll_{k,\boldsymbol{F}} \int_T^{2T} \exp\left(2k \min_{1\le j\le r} \operatorname{Im} \log F_j(\tfrac{1}{2}+it)\right) dt$$
$$\ll_{k,\boldsymbol{F}} T\frac{(\log T)^{\frac{k^2}{h_{\boldsymbol{F}}}+Bk^3}}{(\log\log T)^{(r-1)/2}},$$

and

$$T\frac{(\log T)^{\frac{k^2}{h_{\boldsymbol{F}}}-Bk^3}}{(\log\log T)^{(r-1)/2}} \ll_{k,\boldsymbol{F}} \int_T^{2T} \exp\left(-2k \max_{1\le j\le r} \operatorname{Im} \log F_j(\tfrac{1}{2}+it)\right) dt$$
$$\ll_{k,\boldsymbol{F}} T\frac{(\log T)^{\frac{k^2}{h_{\boldsymbol{F}}}+Bk^3}}{(\log\log T)^{(r-1)/2}}.$$

Remark 8. It is conjectured that

$$\int_0^T |F(\tfrac{1}{2}+it)|^{2k} dt \sim C(F,k)T(\log T)^{k^2}$$

for some constant $C(F,k)$ as $T \to \infty$, see [16]. It is also expected that the values of distinct primitive L-functions are uncorrelated, which leads to the conjecture

$$\int_0^T |F_1(\tfrac{1}{2}+it)|^{2k_1} \cdots |F_r(\tfrac{1}{2}+it)|^{2k_r} dt \sim C(\boldsymbol{F},\boldsymbol{k})T(\log T)^{k_1^2+\cdots+k_r^2}$$

for some constant $C(\boldsymbol{F},\boldsymbol{k})$ as $T \to \infty$. This has be established for product of two Dirichlet L-functions for $k_1 = k_2 = 1$ (see [41, 88, 115] and for some degree two L-functions when $k = 1$ and $r = 1$ (see [34, 122, 123]). For higher degree L-functions and higher values of k, obtaining the asymptotic formula seems to be beyond the scope of current techniques. But an upper bound of this kind has been established by by Milinovich and Turnage-Butterbaugh [83] for automorphic L-functions of $GL(n)$ under the Riemann Hypothesis for these L-functions. Our result give some further evidence that distinct primitive L-functions are "statistically independent".

The range for k in Theorem 6.3 is small due to the fact that $\|\boldsymbol{V}\|$ in Theorem 6.2 is only allowed to be a small multiple of $\log\log T$. However, if we assume the Riemann Hypothesis for the corresponding L-functions, then we can improve the upper bound for $\|\boldsymbol{V}\|$ in Theorem 6.2 and thus obtain better bound for Theorem 6.3 for all k.

Theorem 6.4. *Let $\boldsymbol{F} = (F_1, \dots, F_r) \in (\mathcal{S}^\dagger)^r$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in \mathbb{R}^r$ satisfying assumption $\mathscr{A}$, and additionally assume that the Riemann Hypothesis is true for $F_1, \dots, F_r$. Let T be large, and $\boldsymbol{V} = (V_1, \dots, V_r) \in (\mathbb{R}_{\geq 3})^r$ satisfying $\|\boldsymbol{V}\| \leq a_5 V_m^{1/2} (\log\log T)^{1/4} (\log_3 T)^{1/2}$ with $V_m = \min_{1\leq j\leq r} V_j$, where $a_5 = a_5(\boldsymbol{F})$ is a small constant. If $\boldsymbol{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]^r$, we have*

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) \tag{6.10}$$
$$\ll_{\boldsymbol{F}} \left(\frac{1}{V_1 \cdots V_r} + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}} \right) \times \exp\left(-\frac{V_1^2 + \cdots + V_r^2}{2} + O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|^3}{\sqrt{\log\log T}\log\|\boldsymbol{V}\|} \right) \right),$$

and if $\boldsymbol{\theta} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]^r$ and $\prod_{j=1}^r V_j \leq a_6 (\log\log T)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}$ with $a_6 = a_6(\boldsymbol{F})$ a suitably small constant, we have

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) \tag{6.11}$$
$$\gg_{\boldsymbol{F}} \frac{1}{V_1 \cdots V_r} \exp\left(-\frac{V_1^2 + \cdots + V_r^2}{2} - O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|^3}{\sqrt{\log\log T}\log\|\boldsymbol{V}\|} \right) \right).$$

Moreover, there exist some positive constants $a_7 = a_7(\boldsymbol{F})$, $a_8 = a_8(\boldsymbol{F})$ such that for any $\boldsymbol{V} \in (\mathbb{R}_{\geq 3})^r$ with $\|\boldsymbol{V}\| \geq \sqrt{\log\log T}$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) \tag{6.12}$$
$$\ll_{\boldsymbol{F}} \exp\left(-a_7 \|\boldsymbol{V}\|^2 \right) + \exp\left(-a_8 \|\boldsymbol{V}\| \sqrt{\log\log T}\log\|\boldsymbol{V}\| \right).$$

With $r = 1$, Theorem 6.4 slightly improves the bound in [83, Proposition 4.1] in the range of the following corollary.

Corollary 6.2. *Let $F \in \mathcal{S}^\dagger$, and assume the Riemann Hypothesis for F. Let $A \geq 1$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, for any real number V with $\sqrt{\log\log T} \leq V \leq A(\log\log T)^{2/3}(\log_3 T)^{1/3}$, we have*

$$\frac{1}{T}\operatorname{meas}\left\{t \in [T, 2T] \,:\, \operatorname{Re} e^{-i\theta}\log F(\tfrac{1}{2}+it)| > V\right\}$$
$$\ll_{A,F} \frac{\sqrt{\log\log T}}{V}\exp\left(-\frac{V^2}{n_F \log\log T}\right).$$

as $T \to \infty$.

An application of Theorem 6.4 yields the following mean value theorem.

Theorem 6.5. *Let $\boldsymbol{F} = (F_1, \dots, F_r) \in (\mathcal{S}^\dagger)^r$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta) \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ satisfying assumption $\mathscr{A}$, and additionally assume that the Riemann Hypothesis is true for $F_1, \dots, F_r$. Then, there exists some positive constant $B = B(\boldsymbol{F})$ such that for any $k \geq 0$, and any $T \geq \exp\exp\exp(Ck)$ with $C = C(\boldsymbol{F})$ a large constant, if $\boldsymbol{\theta} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have*

$$\int_T^{2T} \exp\left(2k \min_{1\leq j\leq r} \operatorname{Re} e^{-\theta_j}\log F_j(\tfrac{1}{2}+it)\right) dt \tag{6.13}$$
$$\ll_{k,\boldsymbol{F}} T + T(\log T)^{k^2/h_{\boldsymbol{F}}+Bk^3\varepsilon(T)}\left(\frac{k\sqrt{\log\log T}}{1+(k\sqrt{\log\log T})^r} + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right),$$

and if $\boldsymbol{\theta} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and $\vartheta_{\boldsymbol{F}} < \frac{1}{r+1}$, we have

$$\int_T^{2T} \exp\left(2k \min_{1\leq j\leq r} \operatorname{Re} e^{-\theta_j}\log F_j(\tfrac{1}{2}+it)\right) dt$$
$$\gg_{k,\boldsymbol{F}} T + T(\log T)^{k^2/h_{\boldsymbol{F}}-Bk^3\varepsilon(T)}\frac{k\sqrt{\log\log T}}{1+(k\sqrt{\log\log T})^r}. \tag{6.14}$$

Here, $\varepsilon(T) = (\log_3 T)^{-1}$. In particular, if $\vartheta_{\boldsymbol{F}} < \frac{1}{r+1}$, it holds that, for any $k \geq 0$, $\varepsilon > 0$,

$$\int_T^{2T}\left(\min_{1\leq j\leq r}|F_j(\tfrac{1}{2}+it)|\right)^{2k} dt \ll_{\varepsilon,k,\boldsymbol{F}} T(\log T)^{k^2/h_{\boldsymbol{F}}+\varepsilon},$$

$$\int_T^{2T}\left(\max_{1\leq j\leq r}|F_j(\tfrac{1}{2}+it)|\right)^{-2k} dt \gg_{\varepsilon,k,\boldsymbol{F}} T(\log T)^{k^2/h_{\boldsymbol{F}}-\varepsilon},$$

$$T(\log T)^{k^2/h_{\boldsymbol{F}}-\varepsilon} \ll_{\varepsilon,k,\boldsymbol{F}} \int_T^{2T}\exp\left(2k\min_{1\leq j\leq r}\operatorname{Im}\log F_j(\tfrac{1}{2}+it)\right)dt$$
$$\ll_{\varepsilon,k,\boldsymbol{F}} T(\log T)^{k^2/h_{\boldsymbol{F}}+\varepsilon},$$

and

$$T(\log T)^{k^2/h_{\boldsymbol{F}}-\varepsilon} \ll_{\varepsilon,k,\boldsymbol{F}} \int_T^{2T}\exp\left(-2k\max_{1\leq j\leq r}\operatorname{Im}\log F_j(\tfrac{1}{2}+it)\right)dt$$
$$\ll_{\varepsilon,k,\boldsymbol{F}} T(\log T)^{k^2/h_{\boldsymbol{F}}+\varepsilon}.$$

To prove the above theorems, we consider the Dirichlet polynomials associated with F. Let $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$, $\boldsymbol{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$, and $\boldsymbol{F} = (F_1, \ldots, F_r) \in (\mathcal{S}^\dagger)^r$ satisfying assumption $\mathscr{A}$. We define

$$P_F(s, X) := \sum_{p \le X} \frac{a_F(p)}{p^s},$$

$$\sigma_F(X) := \sqrt{\frac{1}{2} \sum_{p \le X} \frac{|a_F(p)|^2}{p}}, \tag{6.15}$$

$$\tau_{i,j}(X) = \tau_{i,j}(X; \boldsymbol{F}, \boldsymbol{\theta}) := \frac{1}{2} \operatorname{Re}\left(e^{-i(\theta_i - \theta_j)} \sum_{p \le X} \frac{a_{F_i}(p)\overline{a_{F_j}(p)}}{p} \right), \tag{6.16}$$

$$K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z}) := \sum_{j=1}^{r} z_j a_{F_j}(p) e^{-i\theta_j} \sum_{k=1}^{r} z_k \overline{a_{F_k}(p) e^{-i\theta_k}}, \tag{6.17}$$

$$\begin{aligned} \Xi_X(\boldsymbol{x}) &= \Xi_X(\boldsymbol{x}; \boldsymbol{F}, \boldsymbol{\theta}) \\ &:= \exp\left(\sum_{1 \le l_1 < l_2 \le r} x_{l_1} x_{l_2} \tau_{l_1, l_2}(X) \right) \prod_p \frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{x})/p} \right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{x})/4p \right)}, \end{aligned} \tag{6.18}$$

and

$$\mathcal{S}_X(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta}) := \left\{ t \in [T, 2T] \ : \ \frac{\operatorname{Re} e^{-i\theta_j} P_{F_j}(\frac{1}{2} + it, X)}{\sigma_{F_j}(X)} > V_j \text{ for } j = 1, \ldots, r \right\}.$$

Here, $I_0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(z \cos\theta) d\theta = \sum_{n=0}^{\infty} (z/2)^{2n}/(n!)^2$ is the modified 0-th order Bessel function. The convergence of the infinite product of (6.18) is shown in Lemma 6.10.

We have the following the joint large deviations results for Dirichlet polynomials.

Proposition 6.1. *Assume $\boldsymbol{F} = (F_1, \ldots, F_r)$ be an r-tuple of L-functions and $\boldsymbol{\theta} \in [0, 2\pi]^r$ satisfy (S4), (A1), and (A2). Let T, X be large numbers with $X^{(\log\log X)^{4(r+1)}} \le T$. Then, there exists some positive constant $a_9 = a_9(\boldsymbol{F})$ such that for $\boldsymbol{V} = (V_1, \ldots, V_r) \in \mathbb{R}^r$ with $|V_j| \le a_9 \sigma_{F_j}(X)$,*

$$\begin{aligned} &\frac{1}{T} \operatorname{meas}(\mathcal{S}_X(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) \\ &= \left(1 + O_{\boldsymbol{F}}\left(\frac{\prod_{k=1}^{r}(1 + |V_k|)}{(\log\log X)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}} + \frac{1 + \|\boldsymbol{V}\|^2}{\log\log X} \right) \right) \prod_{j=1}^{r} \int_{V_j}^{\infty} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}. \end{aligned}$$

We could improve the range V_j in Propositions 6.1 with a weaker error term.

Proposition 6.2. *Assume $\boldsymbol{F} = (F_1, \dots, F_r)$ be an r-tuple of L-functions and $\boldsymbol{\theta} \in [0, 2\pi]^r$ satisfying (S4), (A1), and (A2). Let T, X be large numbers satisfying $X^{(\log\log X)^{4(r+1)}} \leq T$. Then for any $\boldsymbol{V} = (V_1, \dots, V_r) \in (\mathbb{R}_{\geq 0})^r$ with $\|\boldsymbol{V}\| \leq (\log\log X)^{2r}$, we have*

$$\begin{aligned} &\frac{1}{T}\operatorname{meas}(\mathcal{S}_X(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) \qquad (6.19)\\ &= (1 + E) \times \Xi_X\left(\tfrac{V_1}{\sigma_{F_1}(X)}, \dots, \tfrac{V_r}{\sigma_{F_r}(X)}\right) \prod_{j=1}^{r} \int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}, \end{aligned}$$

where E satisfies

$$E \ll_{\boldsymbol{F}} \exp\left(C\left(\frac{\|\boldsymbol{V}\|}{\sqrt{\log\log X}}\right)^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\left\{\frac{\prod_{k=1}^{r}(1+V_k)}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}} + \frac{1}{\sqrt{\log\log X}}\right\}.$$

Remark 9. In contrast to Proposition 6.1, we allow V_j to be of size $C\sqrt{\log\log X}$ for arbitrarily large C, which is important in the proof of Theorem 6.5. We can prove an estimate similar to (6.19) for larger V_j, where we need to change the value of X suitably in this case. However, our main purpose is to prove Theorems 6.1, 6.2, and 6.4, and the case of larger V_j is not required in the their proofs. For this reason, we give only the case $\|\boldsymbol{V}\| \leq (\log\log X)^{2r}$ for simplicity.

When $r = 2$ and $F_1 = F_2$, we can improve the error term in Proposition 6.1 slightly, which has a consequence for Ramachandra's denseness problem.

Proposition 6.3. *Let $\boldsymbol{F} = (F, F)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ satisfy (S4), (A1), and (A2). Let T, X be large numbers with $X^{(\log\log X)^{12}} \leq T$. Then, there exists some positive constant $a_{10} = a_{10}(F)$ such that for $\boldsymbol{V} = (V_1, V_2) \in \mathbb{R}^r$ with $|V_j| \leq a_{10}\sigma_F(X)$,*

$$\begin{aligned} &\frac{1}{T}\operatorname{meas}(\mathcal{S}_X(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta}))\\ &= \left(1 + O_F\left(\frac{(1+|V_1|)(1+|V_2|)}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}} + \frac{1+\|\boldsymbol{V}\|^4}{(\log\log X)^2}\right)\right)\int_{V_2}^{\infty}\int_{V_1}^{\infty} e^{-\frac{u^2+v^2}{2}}\frac{du dv}{2\pi}. \end{aligned}$$

Corollary 6.3. *Let $F \in \mathcal{S}^\dagger$ satisfying (A1), (A2), and the estimate $|a_F(p)| \ll_F p^{\vartheta_F}$ for some $\vartheta_F \in [0, \frac{1}{3})$. For any $0 < \varepsilon \leq 1$, $z \in \mathbb{C}$, and any large numbers T, X with $X^{(\log\log X)^{12}} \leq T$, we have*

$$\begin{aligned} &\frac{1}{T}\operatorname{meas}\{t \in [T, 2T] \ : \ P_F(1/2 + it, X) \in R(z, \varepsilon)\}\\ &\sim \iint_{R(z/\sigma_F(X), \varepsilon/\sigma_F(X))} e^{-\frac{u^2+v^2}{2}}\frac{du dv}{2\pi} \end{aligned}$$

as $X \to +\infty$. Recall that $R(z, r) := \{u + iv \in \mathbb{C} \ : \ \max\{|\operatorname{Re} z - u|, |\operatorname{Im} z - v|\} < r\}$.

6.2 Approximate formulas for L-functions

In this section, we give an approximate formula for $\log F(s)$. In the following, we use the same notation as in Chapter 2. We also define $\Lambda_F(n)$ as the von Mangoldt function associated with F defined by $\Lambda_F(n) = b_F(n)\log n$. Additionally, define $\sigma_{X,t}(F)$ for $F \neq 1$ and $w_X(n)$ by

$$\sigma_{X,t}(F) = \frac{1}{2} + 2 \max_{|t-\gamma_F| \leq \frac{X^{3|\beta_F - 1/2|}}{\log X}} \left\{ \beta_F - \frac{1}{2}, \frac{2}{\log X} \right\}. \tag{6.20}$$

Then we have the following theorem, which is a generalization of Theorem 2.1 in the case when F is the Riemann zeta function $\zeta(s)$ and $m = 0$.

Theorem 6.6. *Let $F \in \mathcal{S}^\dagger$. Let d be a nonnegative integer with $d \leq D(f)$, and H, X real parameters with $H \geq 1$, $X \geq 3$. Then, for any $\sigma \geq 1/2$, $t \geq 14$, we have*

$$\begin{aligned} \log F(s) = &\sum_{2 \leq n \leq X^{1+1/H}} \frac{\Lambda_F(n) v_{f,H}\left(e^{\log n/\log X}\right)}{n^s \log n} \\ &+ \sum_{|s-\rho_F| \leq \frac{1}{\log X}} \log((s-\rho_F)\log X) + R_F(s, X, H). \end{aligned} \tag{6.21}$$

Here, the error term $R_F(s, X, H)$ satisfies the estimate

$$\begin{aligned} R_F(s, X, H) \ll_{f,d} &\frac{m_F(X^{2(1-\sigma)} + X^{1-\sigma})}{t \log X} + \sum_{|t-\gamma_F| \leq \frac{1}{\log X}} (X^{2(\beta_F - \sigma)} + X^{\beta_F - \sigma}) \\ &+ \frac{1}{\log X} \sum_{|t-\gamma_F| > \frac{1}{\log X}} \frac{X^{2(\beta_F-\sigma)} + X^{\beta_F - \sigma}}{|t-\gamma_F|} \min_{0 \leq l \leq d} \left\{ \left(\frac{H}{|t-\gamma_F|\log X} \right)^l \right\}, \end{aligned} \tag{6.22}$$

where the number m_F is the nonnegative integer such that the function $(s-1)^{m_F} F(s)$ is entire. Moreover, when $D(f) \geq 2$, we also have

$$\begin{aligned} &R_F(s, X, H) \ll_f m_F \frac{X^{2(1-\sigma)} + X^{1-\sigma}}{t \log X} + \\ &H^3(\sigma_{X,t}(F) - 1/2)(X^{2(\sigma_{X,t}(F)-\sigma)} + X^{\sigma_{X,t}(F)-\sigma}) \left(\left| \sum_{n \leq X^3} \frac{\Lambda_F(n) w_X(n)}{n^{\sigma_{X,t}(F)+it}} \right| + d_F \log t \right) \end{aligned} \tag{6.23}$$

for $|t| \geq t_0(F)$ with $t_0(F)$ a sufficiently large constant depending on F.

Remark 10. Note that we choose the branch of $\log(s-\rho_F)$ as follows. If $t \neq \gamma_F$, then $-\pi < \arg(s-\rho_F) < \pi$, and if $t = \gamma_F$, then $\arg(s-\rho_F) = \lim_{\varepsilon \uparrow 0} \arg(\sigma - \beta_F + i\varepsilon)$.

Remark 11. Theorem 6.6 is a modification and generalization of the hybrid formula by Gonek, Hughes, and Keating [33] to apply the method of Selberg-Tsang [116]. Our formula has the advantage of being able to estimate to contribution from zeros close to s over the original their formula [116, Lemma 5.4]. Actually, we can find the sign of the contribution from zeros to s by the form $\sum_{|s-\rho_F|\le\frac{1}{\log X}}\log((s-\rho_F)\log X)$. This fact plays an important role in the proof of theorems in Section 6.1.

Theorem 6.6 can be obtained by the same method as the proof of Theorem 2.1, where we need the following proposition instead of Proposition 2.1.

Proposition 6.4. *Let $F \in \mathcal{S}^\dagger$. Let $X \ge 3$, $H \ge 1$ be real parameters. Then, for any $s \in \mathbb{C}$, we have*

$$\log F(s) = $$
$$\sum_{2\le n\le X^{1+1/H}} \frac{\Lambda_F(n) v_{f,H}(e^{\log n/\log X})}{n^s \log n} + m_F^*(U_0((s-1)\log X) + U_0(s\log X))$$
$$- \sum_{\substack{\rho_F \\ \rho_F\neq 0,1}} U_0((s-\rho_F)\log X) - \sum_{n=0}^{\infty}\sum_{j=1}^{k} U_0((s+(n+\mu_j)/\lambda_j)\log X),$$

where the number m_F^* is the integer such that the function $(s-1)^{m_F^*}F(s)$ is entire and not equal to zero at $s = 1$.

Using Theorem 6.6, we obtain the following propositions.

Proposition 6.5. *Let $F \in \mathcal{S}^\dagger$ satisfying* (6.1) *and (A3). Put $\delta_F = \min\{\frac{1}{4}, \frac{\kappa_F}{20}\}$ with κ_F the positive constant in* (6.2). *Let $\sigma \ge 1/2$, and T be large. Let $0 < \varepsilon < 1/2$ be given. Then, there exist positive constants $A_1 = A_1(F)$ such that for any $k \in \mathbb{Z}_{\ge 1}$, $3 \le X \le Y := T^{\delta_F/k}$,*

$$\frac{1}{T}\int_T^{2T}\left|\log F(\sigma+it) - \sum_{2\le n\le X}\frac{\Lambda_F(n)}{n^{\sigma+it}\log n} - \sum_{|\sigma+it-\rho_F|\le\frac{1}{\log Y}}\log((\sigma+it-\rho_F)\log Y)\right|^{2k} dt$$
$$\le A_1^k k^{2k} T^{(1-2\sigma)\delta_F} + A_1^k k! X^{k(1-2\sigma)} + A_1^k k!\left(\sum_{X<p\le Y^2}\frac{|a_F(p)|^2}{p^{2\sigma}}\right)^k.$$

Proposition 6.6. *Suppose the same situation as Proposition 6.5. Then, there exists a positive constant $A_2 = A_2(F)$ such that for any $k \in \mathbb{Z}_{\ge 1}$, $3 \le X \le Y := T^{\delta_F/k}$,*

$$\frac{1}{T}\int_T^{2T}\left|\log F(\sigma+it) - \sum_{2\le n\le X}\frac{\Lambda_F(n)}{n^{\sigma+it}\log n}\right|^{2k} dt$$
$$\le A_2^k k^{4k} T^{(1-2\sigma)\delta_F} + A_2^k k! X^{k(1-2\sigma)} + A_2^k k!\left(\sum_{X<p\le Y^2}\frac{|a_F(p)|^2}{p^{2\sigma}}\right)^k.$$

Proposition 6.6 has been essentially proved by Selberg [108, Theorem 1]. However, there are some differences from his, so we give the proof of Proposition 6.6 for completeness.

6.2.1 Proof of the approximate formula for L-functions

We give the proofs of Proposition 6.4 and Theorem 6.6, but the proofs are almost the same as the proofs of Proposition 2.1 and Theorem 2.1. Therefore, we give the sketches only.

Lemma 6.1. *Let $F \in \mathcal{S}^\sharp \setminus \{1\}$. For all $s \in \mathbb{C}$ neither equaling to pole nor a zero of F, we have*

$$\frac{F'}{F}(\sigma+it) = \sum_{\substack{\rho_F \\ \rho_F \neq 0,1}} \left(\frac{1}{s-\rho_F} + \frac{1}{\rho_F}\right) + \gamma_F - m_F\left(\frac{1}{s-1}+\frac{1}{s}\right) - \log Q \quad (6.24)$$
$$- \sum_{n=0}^{\infty}\sum_{j=1}^{k} \lambda_j \frac{\Gamma'}{\Gamma}\left(\lambda_j s + \mu_j\right),$$

where γ_F is a complex number and satisfies $\mathrm{Re}(\gamma_F) = -\mathrm{Re}\sum_{\rho_F}(1/\rho_F)$. In particular, for $|t| \geq t_0(F)$, we have

$$\frac{F'}{F}(\sigma+it) = \sum_{\substack{\rho_F \\ \rho_F \neq 0,1}} \left(\frac{1}{s-\rho_F} + \frac{1}{\rho_F}\right) + O(d_F \log|t|). \quad (6.25)$$

Proof. We obtain equation (6.24) by the same method as the proof of [87, eq. (10.29)]. Moreover, by applying the Stirling formula to equation (6.24), we can also obtain equation (6.25). □

Lemma 6.2. *Let $F \in \mathcal{S}^\dagger$. For $|t| \geq t_0(F)$, $1 \leq H \leq \frac{|t|}{2}$, we have*

$$\sum_{|t-\gamma_F| \leq H} 1 \ll d_F H \log|t|, \quad (6.26)$$

$$\sum_{|t-\gamma_F| \geq H} \frac{1}{(t-\gamma_F)^2} \ll \frac{d_F \log|t|}{H}. \quad (6.27)$$

Proof. Applying the Stirling formula to Lemma 6.1, for $|t| \geq t_0(F)$, we have

$$\mathrm{Re}\left(\frac{F'}{F}(H+it)\right) = \sum_{\rho} \frac{H-\beta_F}{(H-\beta_F)^2 + (t-\gamma_F)^2} + O\left(d_F \log|t|\right).$$

On the other hand, it holds that

$$\sum_{\rho_F} \frac{H-\beta_F}{(H-\beta_F)^2+(t-\gamma_F)^2} \gg \sum_{|t-\gamma_F| \leq H} \frac{1}{H},$$
$$\sum_{\rho_F} \frac{H-\beta_F}{(H-\beta_F)^2+(t-\gamma_F)^2} \gg \sum_{|t-\gamma_F| \geq H} \frac{H}{(t-\gamma_F)^2}.$$

Since $b_F(n)\log n \ll_F n^{1/2}$ from (S4), we find that

$$|(F'/F)(H+it)| = \left|\sum_{n=2}^{\infty} b_F(n)\log n/n^{H+it}\right| \ll_F \zeta(H-1/2) - 1 \ll 2^{-H}.$$

Hence we obtain (6.26) and (6.27) for $H \geq 2$. In addition, we immediately obtain these inequality the case $1 \leq H \leq 2$ from (6.26) and (6.27) in the case $H = 2$. □

Lemma 6.3. *Let $F \in \mathcal{S}^{\dagger}$. For any $T \geq t_0(F)$, there exists some $t \in [T, T+1]$ such that, uniformly for $1/2 \leq \sigma \leq 2$,*

$$\frac{F'}{F}(\sigma + it) \ll_F (\log T)^2.$$

Proof. Using Lemma 6.2, we obtain this lemma by the same argument as the proof of [87, Lemma 12.2]. □

Proof of Proposition 6.4. By using Lemma 6.3, we obtain Proposition 6.4 in the same method as the proof of Proposition 2.1. □

Proof of Theorem 6.6. Equation (6.21) and estimate (6.22) are immediately obtained from Lemmas 2.2, 2.3 in the case $m = 0$ and Proposition 6.4. Hence, it suffices to show estimate (6.23) on the range $|t| \geq t_0(F)$. Following the proof of Proposition 2.1, we see that it suffices to check

$$\sum_{\rho_F} \frac{\sigma_{X,t}(F) - 1/2}{(\sigma_{X,t}(F) - \beta_F)^2 + (t - \gamma_F)^2} \ll \left| \sum_{n \leq X^3} \frac{\Lambda_F(n) w_X(n)}{n^{\sigma_{X,t}(F)+it}} \right| + d_F \log |t|, \quad (6.28)$$

which can be shown by the same proofs as in [107, eq. (4.4); eq. (4.7)] by using equation (6.25) instead of [107, Lemma 11]. □

6.2.2 Proofs of mean value results for L-functions

The next lemma is an analogue and a generalization of [116, Lemma 5.2].

Lemma 6.4. *Let $F \in \mathcal{S}^{\dagger} \setminus \{1\}$ be an L-function satisfying (A3). Let T be large, and κ_F be the positive constant in (6.2). For $k \in \mathbb{Z}_{\geq 1}$, $3 \leq X \leq T^{2/3}$, $\xi \geq 1$ with $X\xi \leq T^{\kappa_F/4}$, we have*

$$\int_T^{2T} \left(\sigma_{X,t}(F) - \frac{1}{2} \right)^k \xi^{\sigma_{X,t}(F)-1/2} dt \ll_F T \left(\frac{4^k \xi^{\frac{4}{\log X}}}{(\log X)^k} + \frac{C^k k!}{\log X (\log T)^{k-1}} \right),$$

where $C = C(F)$ is a positive constant.

Proof. By definition (6.20) of $\sigma_{X,t}(F)$, we obtain

$$\int_T^{2T} \left(\sigma_{X,t}(F) - \frac{1}{2} \right)^k \xi^{\sigma_{X,t}(F)-1/2} dt \tag{6.29}$$
$$\leq T \xi^{\frac{4}{\log X}} \left(\frac{4}{\log X} \right)^k + \frac{2^k}{\log X} \sum_{\substack{T - \frac{X^{3|\beta_F - 1/2|}}{\log X} \leq \gamma_F \leq 2T + \frac{X^{3|\beta_F - \frac{1}{2}|}}{\log X} \\ \beta_F \geq 1/2}} \left(\beta_F - \frac{1}{2} \right)^k (X^3 \xi^2)^{\beta_F - \frac{1}{2}}.$$

From the equation

$$
\begin{aligned}
&\left(\beta_F - \frac{1}{2}\right)^k (X^3\xi^2)^{\beta_F - \frac{1}{2}} \\
&= \int_{\frac{1}{2}}^{\beta_F} \left\{(\sigma - 1/2)^k \log (X^3\xi^2) + k\,(\sigma - 1/2)^{k-1}\right\} (X^3\xi^2)^{\sigma-\frac{1}{2}} d\sigma,
\end{aligned}
$$

we find that

$$
\begin{aligned}
&\sum_{\substack{T - \frac{X^3|\beta_F - 1/2|}{\log X} \le \gamma_F \le 2T + \frac{X^3|\beta_F - \frac{1}{2}|}{\log X} \\ \beta_F \ge 1/2}} \left(\beta_F - \frac{1}{2}\right)^k (X^3\xi^2)^{\beta_F - \frac{1}{2}} \\
&\le \sum_{\substack{0 \le \gamma_F \le 3T \\ \beta_F \ge 1/2}} \int_{\frac{1}{2}}^{\beta_F} \left\{(\sigma - 1/2)^k \log (X^3\xi^2) + k\,(\sigma - 1/2)^{k-1}\right\} (X^3\xi^2)^{\sigma-\frac{1}{2}} d\sigma \\
&\le \int_{\frac{1}{2}}^{1} \left\{(\sigma - 1/2)^k \log (X^3\xi^2) + k\,(\sigma - 1/2)^{k-1}\right\} (X^3\xi^2)^{\sigma-\frac{1}{2}} \sum_{\substack{0 \le \gamma_F \le 3T \\ \beta_F \ge \sigma}} 1 d\sigma \\
&= \int_{\frac{1}{2}}^{1} \left\{(\sigma - 1/2)^k \log (X^3\xi^2) + k\,(\sigma - 1/2)^{k-1}\right\} (X^3\xi^2)^{\sigma-\frac{1}{2}} N_F(\sigma, 3T) d\sigma.
\end{aligned}
$$

By assumption (A3), we can use the estimate $N_F(\sigma, T) \ll_F T^{1-\kappa_F(\sigma-1/2)} \log T$, and so, for $X\xi \le T^{\kappa_F/4}$, the above most right hand side is

$$
\begin{aligned}
&\ll_F T \log T \int_{\frac{1}{2}}^{1} \left\{(\sigma - 1/2)^k \log (X^3\xi^2) + k\,(\sigma - 1/2)^{k-1}\right\} \left(\frac{X^3\xi^2}{T^{\kappa_F}}\right)^{\sigma-\frac{1}{2}} d\sigma \\
&\ll T \frac{C^k k!}{(\log T)^{k-1}}
\end{aligned}
$$

for some $C = C(F) > 0$. Hence, by this estimate and inequality (6.29), we obtain this lemma. □

Lemma 6.5. *Let $F \in \mathcal{S}^\dagger$ be an L-function satisfying* (6.1) *and (A3). Let T be large. Put $\delta_F = \min\{\frac{1}{4}, \frac{\kappa_F}{20}\}$ with κ_F the positive constant in* (6.2). *For any $k \in \mathbb{Z}_{\ge 1}$, $X \ge 3$ with $X \le T^{\delta_F/k}$, we have*

$$
\int_T^{2T} \left| \sum_{n \le X^3} \frac{\Lambda_F(n) w_X(n)}{n^{\sigma_{X,t}(F)+it}} \right|^{2k} dt \ll T C^k k^k (\log X)^{2k},
$$

where w_X is the smoothing function defined by (2.38)*, and $C = C(F)$ is a positive constant.*

Proof. For brevity, we write $\sigma_{X,t}(F)$ as $\sigma_{X,t}$ in this proof. We see that

$$
\sum_{n \le X^3} \frac{\Lambda_F(n) w_X(n)}{n^{\sigma_{X,t}+it}} = \sum_{\ell \ge 1} \sum_{p \le X^{3/\ell}} \frac{\Lambda_F(p^\ell) w_X(p^\ell)}{p^{\ell(\sigma_{X,t}+it)}}
$$

Let $\delta_1 = \delta_1(F) > 0$ be a positive constant for which the estimate

$$\Lambda_F(p^\ell) \ll_F p^{\ell(1/2-\delta_1)} \tag{6.30}$$

holds. The existence of such a constant is guaranteed by condition (S4). Put $K_1 = 2\delta_1^{-1}$. Using the inequality $0 \le w_X(n) \le 1$, we find that for any $\sigma \ge \frac{1}{2}$

$$\begin{aligned}\left|\sum_{\ell > K_1}\sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(\sigma+it)}}\right| &\le \sum_{\ell > K_1}\sum_{p} \frac{|\Lambda_F(p^\ell)|}{p^{\ell/2}} \\ &\ll_F \sum_{p}\sum_{\ell > K_1} \frac{1}{p^{\delta_1 \ell}} \ll_F \sum_{p} \frac{1}{p^2} \ll 1.\end{aligned}$$

Hence, it holds that for any $\sigma \ge \frac{1}{2}$

$$\sum_{n \le X^3} \frac{\Lambda_F(n)w_X(n)}{n^{\sigma+it}} = \sum_{1 \le \ell \le K_1}\sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(\sigma+it)}} + O_F(1). \tag{6.31}$$

For the inner sum, we write

$$\begin{aligned}&\sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(\sigma_{X,t}+it)}} \\ &= \sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(1/2+it)}} - \sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(1/2+it)}}(1 - p^{\ell(1/2-\sigma_{X,t})}) \\ &= \sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(1/2+it)}} - \int_{1/2}^{\sigma_{X,t}} \sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)\log p^\ell}{p^{\ell(\alpha'+it)}} d\alpha',\end{aligned}$$

and, for $1/2 \le \alpha' \le \sigma_{X,t}$,

$$\begin{aligned}&\left|\sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)\log p^\ell}{p^{\ell(\alpha'+it)}}\right| \\ &= \left|X^{\alpha'-1/2}\int_{\alpha'}^{\infty} X^{1/2-\alpha} \sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}} d\alpha\right| \\ &\le X^{\sigma_{X,t}-1/2}\int_{1/2}^{\infty} X^{1/2-\alpha}\left|\sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right| d\alpha.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left|\sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(\sigma_{X,t}+it)}}\right| \\ &\le \left|\sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(1/2+it)}}\right| \\ &\quad + (\sigma_{X,t} - \tfrac{1}{2})X^{\sigma_{X,t}-1/2}\int_{1/2}^{\infty} X^{1/2-\alpha}\left|\sum_{p^\ell \le X^3} \frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right| d\alpha,\end{aligned}$$

which together with equation (6.31) yields

$$\begin{aligned}
&\Big|\sum_{n\le X^3}\frac{\Lambda_F(n)w_X(n)}{n^{\sigma_{X,t}+it}}\Big|^{2k} \qquad (6.32)\\
&\le C_1^k+(2K_1)^{2k}\sum_{1\le\ell\le K_1}\left\{\Big|\sum_{p\le X^{3/\ell}}\frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(1/2+it)}}\Big|^{2k}\right.\\
&\left.+\left((\sigma_{X,t}-\tfrac{1}{2})X^{\sigma_{X,t}-\frac{1}{2}}\int_{\frac{1}{2}}^{\infty}X^{\frac{1}{2}-\alpha}\Big|\sum_{p\le X^{3/\ell}}\frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\Big|d\alpha\right)^{2k}\right\}
\end{aligned}$$

for some constant $C_1=C_1(F)>0$. By Lemma 2.8, it holds that

$$\sum_{1\le\ell\le K_1}\int_T^{2T}\Big|\sum_{p\le X^{3/\ell}}\frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(1/2+it)}}\Big|^{2k}dt\ll Tk!\sum_{1\le\ell\le K_1}\left(\sum_{p\le X^{3/\ell}}\frac{|\Lambda_F(p^\ell)|^2}{p^\ell}\right)^k$$

for $X\le T^{\delta_F/k}\le T^{1/4k}$. By the definition of the von Mangoldt function $\Lambda_F(n)$, we can rewrite (1.21) to

$$\sum_p\frac{|\Lambda_F(p^\ell)|^2}{p^\ell}\ll_F 1 \qquad (6.33)$$

for every $\ell\ge 2$. Hence, we obtain

$$\sum_{2\le\ell\le K_1}\left(\sum_{p\le X^{3/\ell}}\frac{|\Lambda_F(p^\ell)|^2}{p^\ell}\right)^k\le C_2^k$$

for some constant $C_2=C_2(F)>0$. Additionally, by using (6.1), partial summation, and applying the fact $\Lambda_F(p)=a_F(p)\log p$, we find that

$$\begin{aligned}
&\sum_{p\le X^3}\frac{|\Lambda_F(p)|^2}{p}\\
&=(\log X^3)^2\sum_{p\le X^3}\frac{|a_F(p)|^2}{p}-\int_2^{X^3}\frac{2\log\xi}{\xi}\sum_{p\le\xi}\frac{|a_F(p)|^2}{p}d\xi\\
&=n_F(\log X^3)^2\log\log X^3-2n_F\int_2^{X^3}\frac{\log\xi\times\log\log\xi}{\xi}d\xi+O_F\left((\log X)^2\right)\\
&=n_F(\log X^3)^2\log\log X^3-n_F(\log X^3)^2\log\log X^3+O_F\left((\log X)^2\right)\\
&\ll_F(\log X)^2. \qquad (6.34)
\end{aligned}$$

Hence, we have

$$\sum_{1\le\ell\le K_1}\int_T^{2T}\Big|\sum_{p\le X^{3/\ell}}\frac{\Lambda_F(p^\ell)w_X(p^\ell)}{p^{\ell(1/2+it)}}\Big|^{2k}dt\ll TC_3^kk^k(\log X)^{2k} \qquad (6.35)$$

for some constant $C_3 = C_3(F) > 0$.

Next, we estimate the integral of the last term on the right hand side of (6.32). By the Cauchy-Schwarz inequality and Lemma 6.4, when $\delta_F \leq \kappa_F/20$, it holds that

$$\int_T^{2T}\left((\sigma_{X,t}-1/2)X^{\sigma_{X,t}-\frac{1}{2}}\int_{\frac{1}{2}}^{\infty}X^{\frac{1}{2}-\alpha}\left|\sum_{p^\ell\leq X^3}\frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right|d\alpha\right)^{2k}dt$$
$$\leq\left(\int_T^{2T}(\sigma_{X,t}-1/2)^{4k}X^{4k(\sigma_{X,t}-1/2)}dt\right)^{1/2}\times$$
$$\times\left(\int_T^{2T}\left(\int_{1/2}^{\infty}X^{1/2-\alpha}\left|\sum_{p^\ell\leq X^3}\frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right|d\alpha\right)^{4k}dt\right)^{\frac{1}{2}}$$
$$\ll\frac{T^{\frac{1}{2}}C_4^k}{(\log X)^{2k}}\left(\int_T^{2T}\left(\int_{1/2}^{\infty}X^{1/2-\alpha}\left|\sum_{p^\ell\leq X^3}\frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right|d\alpha\right)^{4k}dt\right)^{\frac{1}{2}}, \tag{6.36}$$

for some constant $C_4 = C_4(F) > 0$. Moreover, by Hölder's inequality, we have

$$\left(\int_{1/2}^{\infty}X^{\frac{1}{2}-\alpha}\left|\sum_{p^\ell\leq X^3}\frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right|d\alpha\right)^{4k}$$
$$\leq\left(\int_{1/2}^{\infty}X^{\frac{1}{2}-\alpha}d\alpha\right)^{4k-1}\times\int_{1/2}^{\infty}X^{\frac{1}{2}-\alpha}\left|\sum_{p^\ell\leq X^3}\frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right|^{4k}d\alpha$$
$$=\frac{1}{(\log X)^{4k-1}}\int_{1/2}^{\infty}X^{1/2-\alpha}\left|\sum_{p^\ell\leq X^3}\frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right|^{4k}d\alpha.$$

Therefore, by using Lemma 2.8, we find that

$$\int_T^{2T}\left(\int_{1/2}^{\infty}X^{\frac{1}{2}-\alpha}\left|\sum_{p\leq X^{3/\ell}}\frac{\Lambda_F(p^\ell)w_X(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right|d\alpha\right)^{4k}dt$$
$$\leq\frac{1}{(\log X)^{4k-1}}\int_{1/2}^{\infty}X^{1/2-\alpha}\left(\int_T^{2T}\left|\sum_{p\leq X^{3/\ell}}\frac{\Lambda_F(p^\ell)\log(Xp^\ell)\log p^\ell}{p^{\ell(\alpha+it)}}\right|^{4k}dt\right)d\alpha$$
$$\ll\frac{T(2k)!}{(\log X)^{4k-1}}\int_{1/2}^{\infty}X^{1/2-\alpha}\left(\sum_{p\leq X^{3/\ell}}\frac{|\Lambda_F(p^\ell)|^2(\log(Xp^\ell))^2(\log p^\ell)^2}{p^{2\ell\alpha}}\right)^{2k}d\alpha$$
$$\ll\frac{T(2k!)}{(\log X)^{4k}}\left(\sum_{p\leq X^{3/\ell}}\frac{|\Lambda_F(p^\ell)|^2(\log(Xp^\ell))^2(\log p^\ell)^2}{p^\ell}\right)^{2k}.$$

From (6.34) we see that

$$\sum_{p\le X^3}\frac{|\Lambda_F(p)|^2(\log{(Xp)})^2(\log p)^2}{p}\ll_F(\log X)^6.$$

Also by (6.33), we have

$$\sum_{p\le X^{3/\ell}}\frac{|\Lambda_F(p^\ell)|^2(\log{(Xp^\ell)})^2(\log p^\ell)^2}{p^\ell}\ll_F(\log X)^4$$

for every $\ell\ge 2$. Therefore, we obtain

$$\int_T^{2T}\left(\int_{1/2}^{\infty}X^{\frac{1}{2}-\alpha}\left|\sum_{p\le X^{3/\ell}}\frac{\Lambda_F(p^\ell)w_X(p^\ell)\log{(Xp^\ell)}\log p^\ell}{p^{\ell(\alpha+it)}}\right|d\alpha\right)^{4k}dt$$
$$\ll T(2k)!C_5^k(\log X)^{8k}$$

for some $C_5=C_5(F)>0$. By this estimate and (6.36), we have

$$\sum_{1\le\ell\le K_1}\int_T^{2T}\Bigg((\sigma_{X,t}-1/2)X^{\sigma_{X,t}-\frac{1}{2}}$$
$$\times\int_{\frac{1}{2}}^{\infty}X^{\frac{1}{2}-\alpha}\left|\sum_{p\le X^{3/l}}\frac{\Lambda_F(p^l)w_X(p^l)\log{(Xp^l)}\log p^l}{p^{l(\alpha+it)}}\right|d\alpha\Bigg)^{2k}dt$$

$$\ll TC_6^kk^k(\log X)^{2k}$$

for some constant $C_6=C_6(F)>0$. Combining this with (6.32) and (6.35), we complete the proof of Lemma 6.5. □

Lemma 6.6. *Let $F\in\mathcal{S}^\dagger$ be an L-function satisfying* (6.1) *and (A3). Let $\sigma\ge 1/2$, T be large. Let κ_F, δ_F be the same constants as in Lemma 6.5. There exists a positive constant $C=C(F)$ such that for any $k\in\mathbb{Z}_{\ge1}$, $3\le X\le T^{\delta_F/k}$,*

$$\int_T^{2T}\left(\sum_{|\sigma+it-\rho_F|\le\frac{1}{\log X}}1\right)^{2k}dt\le C^kT^{1-(2\sigma-1)\delta_F+\frac{8\delta_F}{\log X}}\left(\frac{\log T}{\log X}\right)^{2k},\qquad(6.37)$$

and

$$\int_T^{2T}\left|\sum_{|\sigma+it-\rho_F|\le\frac{1}{\log X}}\log((\sigma+it-\rho_F)\log X)\right|^{k}dt$$
$$\le(Ck)^kT^{1-(\sigma-1/2)\delta_F+\frac{4\delta_F}{\log X}}\left(\frac{\log T}{\log X}\right)^{k+\frac{1}{2}}.\qquad(6.38)$$

Proof. From (6.20), there are no zeros of F with $|\sigma + it - \rho_F| \le \frac{1}{\log X}$ when $\sigma \ge \sigma_{X,t}(F)$. Put $\xi := T^{\delta_F/k}$. Note that $\xi \ge 1$ because we suppose that $3 \le X \le T^{\delta_F/k} = \xi$. From these facts, we have

$$\sum_{|\sigma+it-\rho_F|\le\frac{1}{\log X}} 1 \le \xi^{\sigma_{X,t}(F)-\sigma} \sum_{|t-\gamma_F|\le\frac{1}{\log X}} 1$$

for $\sigma \ge 1/2$. By definition (6.20) and the line symmetry of nontrivial zeros of F, we find that

$$\sum_{|t-\gamma_F|\le\frac{1}{\log X}} 1 \le 2 \sum_{\substack{|t-\gamma_F|\le\frac{1}{\log X}\\ \beta_F\ge 1/2}} 1 \ll \sum_{\substack{|t-\gamma_F|\le\frac{1}{\log X}\\ \beta_F\ge 1/2}} \frac{(\sigma_{X,t}(F)-1/2)^2}{(\sigma_{X,t}(F)-\beta_F)^2+(t-\gamma_F)^2}.$$

Applying estimate (6.28) to the above right hand side, we obtain

$$\sum_{|t-\gamma_F|\le\frac{1}{\log X}} 1 \ll (\sigma_{X,t}(F)-1/2)\left(\left|\sum_{n\le X^3}\frac{\Lambda_F(n)w_X(n)}{n^{\sigma_{X,t}(F)+it}}\right| + d_F\log T\right) \quad (6.39)$$

for $t \in [T,2T]$. Noting $X\xi^{2k} \le T^{\kappa_F/4}$ and using Lemmas 6.4, 6.5, we have

$$\int_T^{2T} (\sigma_{X,t}(F)-1/2)^{2k}\xi^{2k(\sigma_{X,t}(F)-\sigma)}\left(\left|\sum_{n\le X^3}\frac{\Lambda_F(n)w_X(n)}{n^{\sigma_{X,t}(F)+it}}\right|^{2k} + (\log T)^{2k}\right)dt$$

$$\le \xi^{2k(1/2-\sigma)}\Bigg\{(\log T)^{2k}\int_T^{2T}(\sigma_{X,t}(F)-1/2)^{2k}\xi^{2k(\sigma_{X,t}(F)-1/2)}dt+$$

$$\left(\int_T^{2T}(\sigma_{X,t}(F)-1/2)^{2k}\xi^{2k(\sigma_{X,t}(F)-1/2)}dt\right)^{1/2}\left(\int_T^{2T}\left|\sum_{n\le X^3}\frac{\Lambda_F(n)w_X(n)}{n^{\sigma_{X,t}(F)+it}}\right|^{2k}dt\right)^{1/2}\Bigg\}$$

$$\le C^k\xi^{2k(1/2-\sigma)}\left(T\xi^{\frac{8k}{\log X}}\left(\frac{\log T}{\log X}\right)^{2k} + T\xi^{\frac{4k}{\log X}}k^{k/2}\right)$$

$$\le C_2^k T^{1-(2\sigma-1)\delta_F+\frac{8\delta_F}{\log X}}\left(\frac{\log T}{\log X}\right)^{2k} \quad (6.40)$$

for some constant $C_2 = C_2(F) > 0$. Hence, we obtain estimate (6.37).

Next, we show estimate (6.38). We find that

$$\left|\sum_{|\sigma+it-\rho_F|\le\frac{1}{\log X}} \log\left((\sigma+it-\rho_F)\log X\right)\right| \le (g_X(s)+\pi)\times\sum_{|\sigma+it-\rho_F|\le\frac{1}{\log X}} 1,$$

where $g_X(s) = g_X(\sigma+it)$ is the function defined by

$$g_X(s) = \begin{cases} \log\left(\frac{1}{|(\sigma+it-\rho_s)\log X|}\right) & \text{if there exists a zero } \rho_F \text{ with } |\sigma+it-\rho_F| \le \frac{1}{\log X}, \\ 0 & \text{otherwise.}\end{cases}$$

Here, ρ_s indicates the zero of F nearest from $s = \sigma + it$. By using the Cauchy-Schwarz inequality and estimate (6.37), we obtain

$$\int_T^{2T} \left| \sum_{|\sigma+it-\rho_F| \le \frac{1}{\log X}} \log((\sigma + it - \rho_F)\log X) \right|^k dt \tag{6.41}$$
$$\le C_3^k \left(\int_T^{2T} g_X(\sigma+it)^{2k} dt + \pi^{2k} T \right)^{1/2} \times T^{\frac{1}{2}-(\sigma-1/2)\delta_F + \frac{4\delta_F}{\log X}} \left(\frac{\log T}{\log X} \right)^k$$

for some constant $C_3 = C_3(F) > 0$. Moreover, we find that

$$\begin{aligned}
\int_T^{2T} g_X(s)^{2k} dt &\le \int_T^{2T} \sum_{|\sigma+it-\rho_F| \le \frac{1}{\log X}} \left(\log\left(\frac{1}{|\sigma + it - \rho_F| \log X} \right) \right)^{2k} dt \\
&\le \sum_{T - \frac{1}{\log X} \le \gamma_F \le 2T + \frac{1}{\log X}} \int_{\gamma_F - \frac{1}{\log X}}^{\gamma_F + \frac{1}{\log X}} \left(\log\left(\frac{1}{|t - \gamma_F| \log X} \right) \right)^{2k} dt \\
&\ll \frac{1}{\log X} \sum_{T-1 \le \gamma_F \le 2T+1} \int_0^1 \left(\log\left(\frac{1}{\ell} \right) \right)^{2k} d\ell \\
&\ll_F T \frac{\log T}{\log X} \int_0^1 \left(\log\left(\frac{1}{\ell} \right) \right)^{2k} d\ell.
\end{aligned}$$

By induction, we can easily confirm that the last integral is equal to $(2k)!$. Hence, we obtain

$$\int_T^{2T} g_X(s)^{2k} dt \ll_F (2k)! T \frac{\log T}{\log X}.$$

By substituting this estimate to inequality (6.41), we obtain this lemma. □

Proof of Proposition 6.5. Let f be a fixed function satisfying the condition of this paper (see Notation) and $D(f) \ge 2$. Let $\sigma \ge 1/2$, T be large, $k \in \mathbb{Z}_{\ge 1}$, and X, Y be parameters with $3 \le X \le Y := T^{\delta_F / k}$, where $\delta_F = \min\{\frac{\kappa_F}{20}, \frac{1}{4}\}$. It holds that $Y^{2(\sigma_{Y,t}(F)-\sigma)} + Y^{\sigma_{Y,t}(F)-\sigma} = Y^{2(1/2-\sigma)} \cdot Y^{2(\sigma_{Y,t}(F)-1/2)} + Y^{1/2-\sigma} \cdot Y^{\sigma_{Y,t}(F)-1/2} \le 2Y^{1/2-\sigma} \cdot Y^{2(\sigma_{Y,t}(F)-1/2)}$ for $\sigma \ge 1/2$. Using this inequality and estimate (6.23) as $H = 1$, we find that there exists a positive constant $C_1 = C_1(F)$ such that

$$\left| \log F(\sigma+it) - \sum_{2 \le n \le X} \frac{\Lambda_F(n)}{n^{\sigma+it} \log n} - \sum_{|\sigma+it-\rho_F| \le \frac{1}{\log Y}} \log((\sigma+it-\rho_F)\log Y) \right|^{2k} \tag{6.42}$$
$$\le C_1^k \left| \sum_{X < n \le Y^2} \frac{\Lambda_F(n) v_{f,1}(e^{\log n / \log Y})}{n^{\sigma+it} \log n} \right|^{2k}$$
$$+ C_1^k Y^{2k(1/2-\sigma)} (\sigma_{Y,t}(F) - 1/2)^{2k} Y^{4k(\sigma_{Y,t}(F)-1/2)} \left(\left| \sum_{n \le Y^3} \frac{\Lambda_F(n) w_Y(n)}{n^{\sigma_{Y,t}(F)+it}} \right| + \log T \right)^{2k}$$

for $t \in [T, 2T]$. Following the proof of estimate (6.40), we obtain

$$Y^{k(1-2\sigma)} \int_T^{2T} (\sigma_{Y,t}(F) - 1/2)^{2k} Y^{4k(\sigma_{Y,t}(F)-1/2)} \left(\left| \sum_{n \le Y^3} \frac{\Lambda_F(n) w_Y(n)}{n^{\sigma_{Y,t}(F)+it}} \right| + \log T \right)^{2k} dt$$

$$\ll C^k T^{1-(2\sigma-1)\delta_F + \frac{16\delta_F}{\log Y}} \left(\frac{\log T}{\log Y} \right)^{2k} \ll T^{1-(2\sigma-1)\delta_F} C_2^k k^{2k} \tag{6.43}$$

for some positive constant $C_2 = C_2(F)$. Similarly to the proof of (6.31), we have

$$\sum_{\substack{X < p^\ell \le Y^2 \\ \ell > K_1}} \frac{\Lambda_F(p^\ell) v_{f,1}(e^{\log p^\ell / \log Y})}{p^{\ell(\sigma+it)} \log p^\ell} \ll_F X^{1/2-\sigma},$$

where K_1 is the same constant as in the proof of Lemma 6.5. Here, we used the inequality $|v_{f,1}(e^{\log p^\ell / \log Y})| \le 1$. Therefore,

$$\left| \sum_{X < n \le Y^2} \frac{\Lambda_F(n) v_{f,1}(e^{\log n / \log Y})}{n^{\sigma+it} \log n} \right|^{2k}$$

$$\le K_1^{2k} \sum_{\ell \le K_1} \left| \sum_{X^{1/\ell} \le p \le Y^{2/\ell}} \frac{\Lambda_F(p^\ell) v_{f,1}(e^{\log p^\ell / \log Y})}{p^{\ell(\sigma+it)} \log p^\ell} \right|^{2k} + C_3^k X^{(1-2\sigma)k}$$

for some positive constant $C_3 = C_3(F)$. Using Lemma 2.8, we have

$$\frac{1}{T} \int_T^{2T} \left| \sum_{X < n \le Y^2} \frac{\Lambda_F(n) v_{f,1}(e^{\log n / \log Y})}{n^{\sigma+it} \log n} \right|^{2k} dt$$

$$\ll K_1^{2k} k! \sum_{1 \le \ell \le K_1} \left(\sum_{X^{1/\ell} < p \le Y^{2/\ell}} \frac{|\Lambda_F(p^\ell)|^2}{p^{2\sigma\ell} (\log p^\ell)^2} \right)^k + C_3^k X^{-(2\sigma-1)k}. \tag{6.44}$$

Moreover, by estimate (6.30), it holds that $\sum_p \frac{|\Lambda_F(p^\ell)|^2}{p^\ell (\log p^\ell)^2} \ll 1$ for $\ell \ge 2$, and thus we obtain

$$\sum_{X^{1/\ell} < p \le Y^{2/\ell}} \frac{|\Lambda_F(p^\ell)|^2}{p^{2\sigma\ell} (\log p^\ell)^2} \le X^{1-2\sigma} \sum_p \frac{|\Lambda_F(p^\ell)|^2}{p^\ell (\log p^\ell)^2} \ll_F X^{1-2\sigma}. \tag{6.45}$$

Combining (6.45), (6.42), and (6.43), we obtain Proposition 6.5. □

Proof of Proposition 6.6. By Proposition 6.5, it suffices to show that there exists a positive constant $A_3 = A_3(F)$ such that

$$\int_T^{2T} \left| \sum_{|\sigma+it-\rho_F| \le \frac{1}{\log Y}} \log((\sigma + it - \rho_F) \log Y) \right|^{2k} \le T^{1-(2\sigma-1)\delta_F} A_3^k k^{4k}$$

with $Y = T^{\delta_F / k}$, and this estimate can be obtained by Lemma 6.6. □

6.3 Distribution functions of Dirichlet polynomials on the critical line

In this section, we assume that $\boldsymbol{F} = (F_1, \dots, F_r)$ is an r-tuple of L-functions, and that $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in \mathbb{R}^r$.

6.3.1 Approximate formulas for moment generating functions I

We first show the following proposition, which gives formulas for moment generating functions.

Proposition 6.7. *Assume that $\boldsymbol{F}$, $\boldsymbol{\theta}$ satisfy (S4), (A1), and (A2). Let T, X be large numbers with $X^{(\log\log X)^{4(r+1)}} \le T$. For any $\boldsymbol{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$ with $\|\boldsymbol{z}\| \le 2(\log\log X)^{2r}$,*

$$\frac{1}{T}\int_{\mathcal{A}} \exp\left(\sum_{j=1}^{r} z_j \operatorname{Re}\left(e^{-i\theta_j} P_{F_j}(\tfrac{1}{2}+it, X)\right)\right) dt$$
$$= \prod_{p \le X} I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})/p}\right) + O\left(\exp\left(-6^{-1}(\log\log X)^{4(r+1)}\right)\right),$$

where $K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})$ is define by (6.17), *and $\mathcal{A}$ is a subset of $[T, 2T]$ defined by* (6.49) *satisfying* $\operatorname{meas}([T,2T] \setminus \mathcal{A}) \ll T\exp\left(-e^{-1}(\log\log X)^{4(r+1)}\right)$.

We prepare the proof this Proposition 6.7 with some lemmas.

Lemma 6.7. *Let $\boldsymbol{w} = \{w_{j,p}\}_{1\le j\le r, p\in\mathcal{P}}$ be a complex sequence, where $\mathcal{P}$ is the set of all prime numbers, and $\boldsymbol{\psi} = \{\psi_{j,p}\}_{1\le j \le r, p\in\mathcal{P}}$ a real sequence. For all $n \in \mathbb{Z}_{\ge 2}$ written as $n = q_1^{\alpha_1}\cdots q_s^{\alpha_s}$ with q_j distinct prime numbers, we have*

$$\frac{1}{T}\int_{T}^{2T} \prod_{m=1}^{s}\left(\sum_{j=1}^{r} w_{j,q_m}\cos(t\log q_m + \psi_{j,q_m})\right)^{\alpha_m} dt$$
$$= f_{\boldsymbol{w},\boldsymbol{\psi}}(n) + O\left(\frac{r^{\Omega(n)} n}{T}\prod_{m=1}^{s}(|w_{1,q_m}| + \cdots + |w_{r,q_m}|)^{\alpha_m}\right).$$

Here, $\Omega(n)$ is the number of the prime factors of n, and $f_{\boldsymbol{w},\boldsymbol{\psi}}$ is the multiplicative function defined by

$$f_{\boldsymbol{w},\boldsymbol{\psi}}(p^\alpha) = \frac{1}{2^\alpha}\binom{\alpha}{\alpha/2}\left(\sum_{j=1}^{r} w_{j,p}e^{i\psi_{j,p}}\right)^{\frac{\alpha}{2}}\left(\sum_{j=1}^{r} w_{j,p}\overline{e^{i\psi_{j,p}}}\right)^{\frac{\alpha}{2}}.$$

The number $\binom{\alpha}{\alpha/2}$ is the binomial coefficient, and we define $\binom{\alpha}{\alpha/2} = 0$ if α is odd.

Proof. Let p be a prime number, and α a positive integer. Then we find that

$$\left(\sum_{j=1}^{r} w_{j,p}\cos(t\log p+\psi_{j,p})\right)^{\alpha} = \sum_{1\le j_1,\dots,j_\alpha\le r} w_{j_1,p}\cdots w_{j_\alpha,p}\prod_{k=1}^{\alpha}\cos(t\log p+\psi_{j_k,p}),$$

and that

$$\begin{aligned}\prod_{k=1}^{\alpha}\cos(t\log p+\psi_{j_k,p}) &= \frac{1}{2^\alpha}\prod_{k=1}^{\alpha}\left(e^{i(t\log p+\psi_{j_k,p})}+e^{-i(t\log p+\psi_{j_k,p})}\right)\\ &= \frac{1}{2^\alpha}\sum_{\varepsilon_1,\dots,\varepsilon_\alpha\in\{-1,1\}} e^{i\varepsilon_1(t\log p+\psi_{j_1,p})+\cdots+i\varepsilon_\alpha(t\log p+\psi_{j_\alpha,p})}\\ &= \frac{1}{2^\alpha}\sum_{\substack{\varepsilon_1,\dots,\varepsilon_\alpha\in\{-1,1\}\\ \varepsilon_1+\cdots+\varepsilon_\alpha=0}} e^{i(\varepsilon_1\psi_{j_1,p}+\cdots+\varepsilon_\alpha\psi_{j_\alpha,p})}+E_1,\end{aligned}$$

where E_1 is the sum whose the number of terms is less than 2^α, and the form of each term is $\delta e^{it\beta\log p}$ with $|\delta|\le 2^{-\alpha}$ and $1\le|\beta|\le\alpha$. We define that the first sum on the right hand side is zero if α is odd. Therefore, we can write

$$\left(\sum_{j=1}^{r} w_{j,p}\cos(t\log p+\psi_{j,p})\right)^{\alpha} = \frac{1}{2^\alpha}\sum_{1\le j_1,\dots,j_\alpha\le r} w_{j_1,p}\cdots w_{j_\alpha,p}\sum_{\substack{\varepsilon_1,\dots,\varepsilon_\alpha\in\{-1,1\}\\ \varepsilon_1+\cdots+\varepsilon_\alpha=0}} e^{i(\varepsilon_1\psi_{j_1,p}+\cdots+\varepsilon_\alpha\psi_{j_\alpha,p})}+E_2.$$

Here, E_2 is the sum of which the number of terms is less than $(2r)^\alpha$, and the form of each term is $\delta' e^{it\beta\log p}$ with $|\delta'|\le 2^{-\alpha}(\sum_{j=1}^{r}|w_{j,p}|)^\alpha$ and $1\le|\beta|\le\alpha$. Moreover, the first term on the right hand side is rewritten as

$$\begin{aligned}&\frac{1}{2^\alpha}\sum_{\substack{\varepsilon_1,\dots,\varepsilon_\alpha\in\{-1,1\}\\ \varepsilon_1+\cdots+\varepsilon_\alpha=0}}\ \sum_{1\le j_1,\dots,j_\alpha\le r}\left(w_{j_1,p}e^{i\varepsilon_1\psi_{j_1,p}}\right)\cdots\left(w_{j_\alpha,p}e^{i\varepsilon_\alpha\psi_{j_\alpha,p}}\right)\\ &= \frac{1}{2^\alpha}\sum_{\substack{\varepsilon_1,\dots,\varepsilon_\alpha\in\{-1,1\}\\ \varepsilon_1+\cdots+\varepsilon_\alpha=0}}\left(\sum_{j=1}^{r} w_{j,p}e^{i\psi_{j,p}}\right)^{\frac{\alpha}{2}}\left(\sum_{j=1}^{r} w_{j,p}\overline{e^{i\psi_{j,p}}}\right)^{\frac{\alpha}{2}}\\ &= \frac{1}{2^\alpha}\binom{\alpha}{\alpha/2}\left(\sum_{j=1}^{r} w_{j,p}e^{i\psi_{j,p}}\right)^{\frac{\alpha}{2}}\left(\sum_{j=1}^{r} w_{j,p}\overline{e^{i\psi_{j,p}}}\right)^{\frac{\alpha}{2}} = f_{\boldsymbol{w},\boldsymbol{\psi}}(p^\alpha).\end{aligned}$$

Thus, we obtain

$$\prod_{m=1}^{s}\left(\sum_{j=1}^{r} w_{j,q_m}\cos(t\log q_m+\psi_{j,q_m})\right)^{\alpha_m} = f_{\boldsymbol{w},\boldsymbol{\psi}}(n)+E_3,$$

where E_3 is the sum whose the number of terms is less than $\prod_{m=1}^{s}(2r)^{\alpha_m}$, and the form of each term is $\delta'' e^{it(\beta_1 \log q_1 + \cdots + \beta_s \log q_s)}$ with $0 \le |\beta_j| \le \alpha_j$ and $\beta_u \neq 0$ for some $1 \le u \le s$. Here, δ'' is a complex number independent of t, and satisfies $|\delta''| \le W := \prod_{m=1}^{s} 2^{-\alpha_m}(\sum_{j=1}^{r}|w_{j,q_m}|)^{\alpha_m}$. Since $|\beta_1 \log q_1 + \cdots + \beta_s \log q_s| \gg n^{-1}$, the integral of each term of E_3 is bounded by Wn. Hence, by this bound of E_3 and the bound for the number of terms of E_3, we have

$$\int_T^{2T} E_3 dt \ll Wn \prod_{m=1}^{s}(2r)^{\alpha_m} = r^{\Omega(n)} n \prod_{m=1}^{s}(\sum_{j=1}^{r}|w_{j,q_m}|)^{\alpha_m},$$

which completes the proof of the lemma. □

Lemma 6.8. *Let $\boldsymbol{a}(p) = (a_1(p), \dots, a_r(p))$ be an r-tuple of sequences with $\{a_j(p)\}$ a complex sequence over prime numbers. Let $X \ge 3$, and T be large. Let $\boldsymbol{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$. Put*

$$K_{\boldsymbol{a}}(p, \boldsymbol{z}) = \sum_{j=1}^{r} z_j a_j(p) \sum_{k=1}^{r} z_k \overline{a_k(p)}. \tag{6.46}$$

Then, for $k \in \mathbb{Z}_{\ge 1}$, $R > 0$, we have

$$\frac{1}{T}\int_T^{2T}\left(\sum_{j=1}^{r} z_j \operatorname{Re}\sum_{p \le X}\frac{a_j(p)}{p^{1/2+it}}\right)^k dt =$$

$$\frac{k!}{2\pi i}\oint_{|w|=R}\frac{1}{w^{k+1}}\prod_{p \le X} I_0\left(w\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p}\right)dw + O\left(\frac{1}{T}\left(r\|\boldsymbol{z}\|\sum_{p \le X}\|\boldsymbol{a}(p)\|_1\sqrt{p}\right)^k\right).$$

Here, the symbol $\|\cdot\|_1$ means the L_1-norm, that is, $\|\boldsymbol{a}(p)\|_1 = |a_1(p)| + \cdots + |a_r(p)|$.

Note that we do not need to consider the branch of $I_0(\sqrt{z})$ since the function is an entire function on the complex plane.

Proof. We write

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T}\left(\sum_{j=1}^{r} z_j \operatorname{Re}\sum_{p \le X}\frac{a_j(p)}{p^{1/2+it}}\right)^k dt \qquad (6.47)\\
&= \frac{1}{T}\int_T^{2T}\left(\sum_{p \le X}\frac{1}{\sqrt{p}}\sum_{j=1}^{r} z_j |a_j(p)|\cos(t\log p - \arg a_j(p))\right)^k dt\\
&= \frac{1}{T}\sum_{p_1,\dots,p_k \le X}\frac{1}{\sqrt{p_1\cdots p_k}}\int_T^{2T}\prod_{l=1}^{k}\sum_{j=1}^{r} z_j |a_j(p_l)|\cos(t\log p_l - \arg a_j(p_l))dt.
\end{aligned}$$

In order to use Lemma 6.7, we put $\boldsymbol{w} = \{w_{j,p}\}_{1 \le j \le r, p \in \mathcal{P}}$ where $w_{j,p} = z_j|a_j(p)|$, and $\boldsymbol{\psi} = \{\psi_{j,p}\}_{1 \le j \le r, p \in \mathcal{P}}$ with $\psi_{j,p} = -\arg a_j(p)$. For $n = p_1 \cdots p_k = q_1^{\alpha_1}\cdots q_s^{\alpha_s}$

with q_m's being distinct prime numbers, we find that

$$\begin{aligned} r^{\Omega(n)}\prod_{m=1}^{s}(|w_{1,q_m}|+\cdots+|w_{r,q_m}|)^{\alpha_m} &= r^k\prod_{m=1}^{s}(|z_1a_1(q_m)|+\cdots+|z_ra_r(q_m)|)^{\alpha_m} \\ &\leq r^k\prod_{m=1}^{s}(\|\boldsymbol{z}\|\cdot\|\boldsymbol{a}(q_m)\|_1)^{\alpha_m} \\ &= (r\|\boldsymbol{z}\|)^k\prod_{l=1}^{k}\|\boldsymbol{a}(p_l)\|_1. \end{aligned}$$

Therefore, by Lemma 6.7, integral (6.47) becomes

$$\begin{aligned} &= \sum_{p_1,\ldots,p_k\leq X}\frac{f_{\boldsymbol{w},\psi}(p_1\cdots p_k)}{\sqrt{p_1\cdots p_k}} + O\left(\frac{(r\|\boldsymbol{z}\|)^k}{T}\prod_{l=1}^{k}\sum_{p_l\leq X}\|\boldsymbol{a}(p_l)\|_1\sqrt{p_l}\right) \\ &= \sum_{p_1,\ldots,p_k\leq X}\frac{f_{\boldsymbol{w},\psi}(p_1\cdots p_k)}{\sqrt{p_1\cdots p_k}} + O\left(\frac{1}{T}\left(r\|\boldsymbol{z}\|\sum_{p\leq X}\|\boldsymbol{a}(p)\|_1\sqrt{p}\right)^k\right). \end{aligned}$$

This gives

$$\begin{aligned} &\frac{1}{T}\int_T^{2T}\left(\sum_{j=1}^{r}z_j\operatorname{Re}\sum_{p\leq X}\frac{a_j(p)}{p^{1/2+it}}\right)^k dt \qquad (6.48) \\ &= k!\sum_{\Omega(n)=k}\frac{f_{\boldsymbol{w},\psi}(n)}{\sqrt{n}}g_X(n) + O\left(\frac{1}{T}\left(r\|\boldsymbol{z}\|\sum_{p\leq X}\|\boldsymbol{a}(p)\|_1\sqrt{p}\right)^k\right), \end{aligned}$$

where $g_X(n)$ is the multiplicative function defined as $g_X(p^\alpha) = 1/\alpha!$ for $p \leq X$ and 0 otherwise. By Cauchy's integral formula, the right hand side of (6.48) becomes

$$\frac{k!}{2\pi i}\oint_{|w|=R}\sum_{n=1}^{\infty}\frac{f_{\boldsymbol{w},\psi}(n)}{\sqrt{n}}g_X(n)w^{\Omega(n)}\frac{dw}{w^{k+1}} + O\left(\frac{1}{T}\left(r\|\boldsymbol{z}\|\sum_{p\leq X}\|\boldsymbol{a}(p)\|_1\sqrt{p}\right)^k\right).$$

Note that we exchanged the order of the integral and the series in the above deformation, but it is guaranteed by their absolute convergence. Since the functions $f_{\boldsymbol{w},\psi}$, g_X, and $w^{\Omega(n)}$ are multiplicative, we find that

$$\sum_{n=1}^{\infty}\frac{f_{\boldsymbol{w},\psi}(n)}{\sqrt{n}}g_X(n)w^{\Omega(n)} = \prod_{p\leq X}\sum_{\alpha=0}^{\infty}\frac{f_{\boldsymbol{w},\psi}(p^\alpha)}{p^{\alpha/2}\alpha!}w^\alpha = \prod_{p\leq X}\sum_{\alpha=0}^{\infty}\frac{f_{\boldsymbol{w},\psi}(p^{2\alpha})}{(2\alpha)!p^\alpha}w^{2\alpha}.$$

Using the definition of $f_{\boldsymbol{w},\psi}(n)$, we can write

$$\begin{aligned} \sum_{\alpha=0}^{\infty}\frac{f_{\boldsymbol{w},\psi}(p^{2\alpha})}{(2\alpha)!p^\alpha}w^{2\alpha} &= \sum_{\alpha=0}^{\infty}\frac{1}{(\alpha!)^2}\left(\frac{w^2}{4p}\sum_{j=1}^{r}z_ja_{F_j}(p)e^{-i\theta_j}\sum_{k=1}^{r}z_k\overline{a_{F_k}(p)e^{-i\theta_k}}\right)^\alpha \\ &= I_0\left(w\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p}\right), \end{aligned}$$

which completes the lemma. □

Lemma 6.9. *Let T, X be large with $X^{(\log\log X)^{4(r+1)}} \leq T$. Define the set $\mathcal{A} = \mathcal{A}(T, X, \boldsymbol{F})$ by*

$$\mathcal{A} = \bigcap_{j=1}^{r} \left\{ t \in [T, 2T] \ : \ \frac{\left| P_{F_j}(\tfrac{1}{2} + it, X) \right|}{\sigma_{F_j}(X)} \leq (\log\log X)^{2(r+1)} \right\}. \tag{6.49}$$

Then we have

$$\frac{1}{T} \operatorname{meas}([T, 2T] \setminus \mathcal{A}) \ll_{\boldsymbol{F}} \exp\left(-e^{-1} (\log\log X)^{4(r+1)} \right).$$

Proof. By Lemma 2.8, we have

$$\frac{1}{T} \int_{T}^{2T} \left| P_{F_j}(\tfrac{1}{2} + it, X) \right|^{2k} dt \ll k! \left(\sum_{p \leq X} \frac{|a_{F_j}(p)|^2}{p} \right)^{k} = (k \sigma_{F_j}(X)^2)^k \tag{6.50}$$

for $3 \leq X \leq T^{1/2k}$. Therefore, it holds that

$$\frac{1}{T} \operatorname{meas} \left\{ t \in [T, 2T] \ : \ \frac{\left| P_{F_j}(\tfrac{1}{2} + it, X) \right|}{\sigma_{F_j}(X)} > (\log\log X)^{2(r+1)} \right\}$$
$$\ll \left(\frac{k}{(\log\log X)^{4(r+1)}} \right)^{k}.$$

Hence, we obtain

$$\frac{1}{T} \operatorname{meas}([T, 2T] \setminus \mathcal{A}) \ll r \times \left(\frac{k}{(\log\log X)^{4(r+1)}} \right)^{k}.$$

Choosing $k = \lfloor e^{-1} (\log\log X)^{4(r+1)} \rfloor$, we obtain this lemma. □

Proof of Proposition 6.7. Let T, X be large numbers such that $X^{(\log\log X)^{4(r+1)}} \leq T$. Let $\boldsymbol{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$ with $\|\boldsymbol{z}\| \leq 2(\log\log X)^{2r}$. From (6.49), we have

$$\begin{aligned}
&\frac{1}{T} \int_{\mathcal{A}} \exp\left(\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2} + it, X) \right) dt \\
&= \frac{1}{T} \sum_{0 \leq k \leq Y} \frac{1}{k!} \int_{\mathcal{A}} \left(\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2} + it, X) \right)^{k} dt \\
&\qquad + O\left(\sum_{k > Y} \frac{1}{k!} \left(C (\log\log X)^{2r+5/2} \|\boldsymbol{z}\| \right)^{k} \right)
\end{aligned}$$

with $Y = \frac{1}{4}(\log\log X)^{4(r+1)}$. Here, $C = C(\boldsymbol{F})$ is some positive constant. We see that this O-term is $\ll \exp\left(-(\log\log X)^{4(r+1)}\right)$. By the Cauchy-Schwarz inequality, we find that

$$\frac{1}{T}\int_{\mathcal{A}}\left(\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2}+it, X)\right)^k dt$$
$$= \frac{1}{T}\int_{T}^{2T}\left(\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2}+it, X)\right)^k dt$$
$$+ O\left(\frac{1}{T}(\operatorname{meas}([T,2T]\setminus\mathcal{A}))^{1/2}\left(\int_{T}^{2T}\left|\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2}+it, X)\right|^{2k} dt\right)^{1/2}\right).$$

By Lemma 6.9, estimate (6.50), and bounds for $\|\boldsymbol{z}\|$, this O-term is

$$\ll \exp\left(-(2e)^{-1}(\log\log X)^{4(r+1)}\right)\left(C_1 k^{1/2}(\log\log X)^{2r+1/2}\right)^k$$

for $0 \le k \le Y$, where $C_1 = C_1(\boldsymbol{F}) > 0$ is a positive constant. Therefore, it holds that

$$\frac{1}{T}\int_{\mathcal{A}}\exp\left(\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2}+it, X)\right) dt \tag{6.51}$$
$$= \frac{1}{T}\sum_{0\le k\le Y}\frac{1}{k!}\int_{T}^{2T}\left(\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2}+it, X)\right)^k dt$$
$$+ O\left(\exp\left(-(2e)^{-1}(\log\log X)^{4(r+1)}\right)\sum_{0\le k\le Y}\frac{(C_1 e(\log\log X)^{2r+1/2})^k}{k^{k/2}}\right).$$

When X is sufficiently large, it follows that

$$\sum_{0\le k\le Y}\frac{(C_1 e(\log\log X)^{2r+1/2})^k}{k^{k/2}} = \sum_{0\le k\le (\log\log X)^{4r+2}}\frac{(C_1 e(\log\log X)^{2r+1/2})^k}{k^{k/2}} + O(1)$$
$$\ll \exp\left((\log\log X)^{4r+3}\right).$$

Hence, the O-term of (6.51) is $\ll \exp\left(-\frac{1}{6}(\log\log X)^{4(r+1)}\right)$. Moreover, applying Lemma 6.8 as $a_j(p) = a_{F_j}(p)e^{-i\theta_j}$, we find that

$$\frac{1}{T}\sum_{0\le k\le Y}\frac{1}{k!}\int_{T}^{2T}\left(\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2}+it, X)\right)^k dt =$$
$$\frac{1}{2\pi i}\int_{|w|=e}\sum_{0\le k\le Y}\frac{1}{w^{k+1}}\prod_{p\le X} I_0\left(w\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) dw + O_{\boldsymbol{F}}\left(\frac{1}{T}\sum_{0\le k\le Y}\frac{1}{k!}\left(C_2 X^2\right)^k\right)$$

for some $C_2 = C_2(\boldsymbol{F}) > 0$. Note that we in this deformation of the O-term used the estimate $|a_{F_j}(p)| \ll_{F_j} p^{1/2}$ which is deduced from the equation $a_{F_j}(p) = b_{F_j}(p)$ and axiom (S4). By noting the range of X, the O-term is $\ll_{\boldsymbol{F}} T^{-1/2}$. Hence, by substituting these estimations to equation (6.51), we obtain

$$\frac{1}{T}\int_{\mathcal{A}} \exp\left(\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2}+it, X)\right) dt \tag{6.52}$$
$$= \frac{1}{2\pi i}\int_{|w|=e} \sum_{0\le k\le Y} \frac{1}{w^{k+1}} \prod_{p\le X} I_0\left(w\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) dw$$
$$+ O\left(\exp\left(-6^{-1}(\log\log X)^{4(r+1)}\right)\right).$$

By inequality (6.67) and noting the range of $\|\boldsymbol{z}\|$, we find that

$$\left|\prod_{p\le X} I_0\left(w\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)\right| \le \exp\left((\log\log X)^{4r+2}\right)$$

for $|w| = e$. Additionally, for $|w| = e$, we have

$$\left|\sum_{k>Y} \frac{1}{w^{k+1}}\right| \ll \exp\left(-4^{-1}(\log\log X)^{4(r+1)}\right).$$

Therefore, we obtain

$$\sum_{k>Y} \frac{1}{w^{k+1}} \prod_{p\le X} I_0\left(w\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) dw \ll \exp\left(-6^{-1}(\log\log X)^{4(r+1)}\right).$$

By this inequality, the right hand side of equation (6.52) is equal to

$$\frac{1}{2\pi i}\oint_{|w|=e} \prod_{p\le X} I_0\left(w\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) \frac{dw}{w-1} + O\left(\exp\left(-6^{-1}(\log\log X)^{2(r+1)}\right)\right).$$

In particular, since the function $\prod_{p\le X} I_0\left(w\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)$ is entire with respect to w, this is equal to

$$\prod_{p\le X} I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) + O\left(\exp\left(-6^{-1}(\log\log X)^{2(r+1)}\right)\right),$$

completes the proof of Proposition 6.7. □

Next, we give some lemmas to help estimate the main term in Proposition 6.7.

Lemma 6.10. *Assume that $\boldsymbol{F}$ satisfies (S4), (A1), and (A2). Put*

$$\Psi(\boldsymbol{z}) = \Psi(\boldsymbol{z};\boldsymbol{F},\boldsymbol{\theta}) := \prod_{p} \frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}.$$

Then, Ψ is analytic on $\mathbb{C}^r$, and satisfies

$$|\Psi(\boldsymbol{z})| \leq \left| \prod_{j=1}^{r} \exp\left(-\frac{z_j^2}{2}\sigma_{F_j}\left(|z_j|\right)^2 + O_{\boldsymbol{F}}\left(|z_j|^2 + |z_j|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) \right) \right| \tag{6.53}$$

for any $\boldsymbol{z} = (z_1, \ldots, z_r) \in \mathbb{C}$, and

$$\Psi(\boldsymbol{x}) = \prod_{j=1}^{r} \exp\left(-\frac{x_j^2}{2}\sigma_{F_j}\left(x_j\right)^2 + O_{\boldsymbol{F}}\left(x_j^2 + x_j^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) \right) \tag{6.54}$$

for $\boldsymbol{x} \in (\mathbb{R}_{\geq 0})^r$. Moreover, for any $\boldsymbol{z} = (x_1+iu_1, \ldots, x_r+iu_r) \in \mathbb{C}^r$ satisfying $x_j \geq 0$ and $u_j \in \mathbb{R}$ with $\|\boldsymbol{u}\| \leq 1$, we have

$$\Psi(\boldsymbol{z}) = \Psi(x_1, \ldots, x_r) \prod_{j=1}^{r} \left(1 + O_{\boldsymbol{F}}\left(|u_j| \exp\left(D_1 \|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) \right) \right). \tag{6.55}$$

Here, $D_1 = D_1(\boldsymbol{F})$ is a positive constant.

Proof. First, we show that Ψ is analytic on $\mathbb{C}^r$. It suffices to show that, for every compact set $D \subset \mathbb{C}^r$, the infinite product is convergent uniformly for $\boldsymbol{z} \in D$. By the definitions of $K_{\boldsymbol{F},\boldsymbol{\theta}}$ and $\vartheta_{\boldsymbol{F}}$, it holds that, for any $\boldsymbol{z} \in D$,

$$|K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})| \leq C^2 \|\boldsymbol{z}\|^2 \|a_{\boldsymbol{F}}(p)\|_1^2 \ll_{D,\boldsymbol{F}} p^{2\vartheta_{\boldsymbol{F}}}, \tag{6.56}$$

for some positive constant $C = C(\boldsymbol{F}) > 0$. Therefore, for any prime $p > p_0(D, \boldsymbol{F})$ with $p_0(D, \boldsymbol{F})$ sufficiently large depending on D, we can write

$$\begin{aligned}
\frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})/p}\right)}{\exp(K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})/4p)} &= \sum_{m=0}^{\infty} \frac{K_{\boldsymbol{F},\boldsymbol{\theta}}(p, z)^m}{4^m p^m (m!)^2} \times \left(\sum_{n=0}^{\infty} \frac{K_{\boldsymbol{F},\boldsymbol{\theta}}(p, z)^n}{4^n p^n n!} \right)^{-1} \\
&= 1 + O_{D,\boldsymbol{F}}\left(\frac{\|a_{\boldsymbol{F}}(p)\|_1^2}{p^{2-2\vartheta_{\boldsymbol{F}}}} \right)
\end{aligned} \tag{6.57}$$

uniformly for $\boldsymbol{z} \in D$. Since we assume (A1) for $\boldsymbol{F}$, it holds that $\sum_p \frac{\|a_{\boldsymbol{F}}(p)\|_1^2}{p^{2-2\vartheta_{\boldsymbol{F}}}} < +\infty$. Hence, the infinity product is convergent uniformly for $\boldsymbol{z} \in D$.

Next, we prove (6.53). Put $M = (C\|\boldsymbol{z}\|)^{\frac{2}{1-2\vartheta_{\boldsymbol{F}}}}$. Here, C is the same constant as in (6.56). Then we divide the range of the product as

$$\prod_{p} \frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})/4p\right)} = \left(\prod_{p \leq M} \times \prod_{p > M} \right) \frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})/4p\right)}.$$

Using the Taylor expansion of I_0, we see that

$$\left| I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})/p}\right) \right| \leq I_0\left(\sqrt{|K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})|/p}\right).$$

From this inequality, (6.56), and the inequality $|I_0(z)| \leq \exp(|z|)$, it holds that

$$\left|\prod_{p\leq M} I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)\right| \leq \exp\left(C\|\boldsymbol{z}\| \sum_{j=1}^{r}\sum_{p\leq M}\frac{|a_{F_j}(p)|}{\sqrt{p}}\right).$$

Using assumption (A1) and the Cauchy-Schwarz inequality, we find that

$$\sum_{p\leq M}\frac{|a_{F_j}(p)|}{\sqrt{p}} \leq \left(\sum_{p\leq M} 1\right)^{1/2}\left(\sum_{p\leq M}\frac{|a_{F_j}(p)|^2}{p}\right)^{1/2} \ll_{\boldsymbol{F}} \|\boldsymbol{z}\|^{\frac{1}{1-2\vartheta_{\boldsymbol{F}}}}, \tag{6.58}$$

and thus

$$\begin{aligned}\left|\prod_{p\leq M} I_0\left(w\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)\right| &\leq \exp\left(O_{\boldsymbol{F}}\left(\|\boldsymbol{z}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right)\\ &\leq \prod_{j=1}^{r}\exp\left(O_{\boldsymbol{F}}\left(|z_j|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right).\end{aligned}$$

Additionally, since

$$\begin{aligned}&K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})\\ &= \sum_{j=1}^{r} z_j a_{F_j}(p)e^{-i\theta_j}\sum_{k=1}^{r} z_k\overline{a_{F_k}(p)e^{-i\theta_k}}\\ &= \sum_{j=1}^{r} z_j^2|a_{F_j}(p)|^2 + 2\sum_{1\leq l_1<l_2\leq r} z_{l_1}z_{l_2}\operatorname{Re}\left(e^{-i(\theta_{l_1}-\theta_{l_2})}a_{F_{l_1}}(p)\overline{a_{F_{l_2}}(p)}\right),\end{aligned} \tag{6.59}$$

we see that, using (A1) and (A2),

$$\prod_{p\leq M}\exp\left(-K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right) = \exp\left(-\sum_{j=1}^{r}\frac{z_j^2}{2}\sigma_{F_j}(M)^2 + O_{\boldsymbol{F}}(\|\boldsymbol{z}\|^2)\right). \tag{6.60}$$

If $|z_j| \leq \|\boldsymbol{z}\|^{1/2}$, then $z_j^2\sigma_{F_j}(M)^2 \ll_{\boldsymbol{F}} \|\boldsymbol{z}\|^2$. If $|z_j| > \|\boldsymbol{z}\|^{1/2}$, then we use (A1) to obtain

$$\frac{z_j^2}{2}\sigma_{F_j}(M)^2 = \frac{z_j^2}{2}\sigma_{F_j}(|z_j|)^2 + \frac{z_j^2}{4}\sum_{|z_j|<p\leq M}\frac{|a_{F_j}|^2}{p} = \frac{z_j^2}{2}\sigma_{F_j}(|z_j|)^2 + O_{\boldsymbol{F}}\left(\|\boldsymbol{z}\|^2\right).$$

From this observation and (6.60), we find that

$$\prod_{p\leq M}\exp\left(-K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right) = \prod_{j=1}^{r}\exp\left(-\frac{z_j^2}{2}\sigma_{F_j}(|z_j|)^2 + O_{\boldsymbol{F}}(|z_j|^2)\right). \tag{6.61}$$

Hence, we obtain

$$\left|\prod_{p\le M}\frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}\right| \tag{6.62}$$
$$\le\left|\prod_{j=1}^{r}\exp\left(-\frac{z_j^2}{2}\sigma_{F_j}\left(|z_j|\right)^2+O_{\boldsymbol{F}}\left(|z_j|^2+|z_j|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right)\right|.$$

When $p>M$ and C is sufficiently large, it holds from the Taylor expansion that

$$\left|\frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}-1\right|\le\frac{1}{2},$$

and that

$$\frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}=1+O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{z}\|^4}{p^2}\sum_{j=1}^{r}|a_{F_j}(p)|^4\right).$$

Therefore, we have

$$\sum_{p>M}\log\left|\frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}\right|$$
$$=\operatorname{Re}\sum_{p>M}\log\frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}\ll_{\boldsymbol{F}}\sum_{p>M}\frac{\|\boldsymbol{z}\|^4}{p^2}\sum_{j=1}^{r}|a_{F_j}(p)|^4.$$

By (6.1), we find that

$$\begin{aligned}\sum_{p>M}\frac{|a_{F_j}(p)|^4}{p^2}&\ll_{\boldsymbol{F}}\sum_{p>M}\frac{|a_{F_j}(p)|^2}{p}\cdot p^{2\vartheta_{\boldsymbol{F}}-1}\\&=\sum_{M<p\le M^2}\frac{|a_{F_j}(p)|^2}{p}\cdot p^{2\vartheta_{\boldsymbol{F}}-1}+O\left(\sum_{p>M^2}\frac{|a_{F_j}(p)|^2}{p}\cdot p^{2\vartheta_{\boldsymbol{F}}-1}\right)\\&\le M^{2\vartheta_{\boldsymbol{F}}-1}\sum_{M<p\le M^2}\frac{|a_{F_j}(p)|^2}{p}+O_{\boldsymbol{F}}\left(M^{2(2\vartheta_{\boldsymbol{F}}-1)}\log\log M\right)\\&\ll_{\boldsymbol{F}}M^{2\vartheta_{\boldsymbol{F}}-1}.\end{aligned}\tag{6.63}$$

Hence, we obtain

$$\prod_{p>M}\frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}=\exp\left(O_{\boldsymbol{F}}\left(\|\boldsymbol{z}\|^2\right)\right)=\prod_{j=1}^{r}\exp\left(O_{\boldsymbol{F}}\left(|z_j|^2\right)\right).\tag{6.64}$$

Combing this estimate and (6.62), we obtain estimate (6.53).

Next, we show (6.54). We see that $\exp(x/2) \ll I_0(x) \leq \exp(x)$ for $x \geq 0$ since $\exp(x/2) \ll \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4}\exp(x\cos\theta)d\theta \leq I_0(x) \leq \exp(x)$. It follows from this inequality and (6.56) that, for $M = (C\|\boldsymbol{x}\|)^{\frac{2}{1-2\vartheta_{\boldsymbol{F}}}}$,

$$\prod_{p\leq M} I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{x})/p}\right) = \exp\left(O_{\boldsymbol{F}}\left(\|\boldsymbol{x}\|\sum_{j=1}^{r}\sum_{p\leq M}\frac{|a_{F_j}(p)|}{\sqrt{p}}\right)\right).$$

Similarly to (6.58) and by this equation, we have

$$\prod_{p\leq M} I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{x})/p}\right) = \prod_{j=1}^{r}\exp\left(O_{\boldsymbol{F}}\left(|x_j|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right).$$

We can calculate the other parts similarly to the proof of (6.53), and obtain (6.54).

Finally, we prove equation (6.55). Since Ψ is analytic on $\mathbb{C}^r$, we can write

$$\begin{aligned}
&\Psi(x_1+iu_1,\ldots,x_r+iu_r)\\
&=\sum_{n=0}^{\infty}\sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\ldots,k_r\geq 0}}\frac{1}{k_1!\cdots k_r!}\frac{\partial^n\Psi(x_1,\ldots,x_r)}{\partial z_1^{k_1}\cdots\partial z_r^{k_r}}(iu_1)^{k_1}\cdots(iu_r)^{k_r}.
\end{aligned}$$

It follows from estimates (6.53) and (6.54) that

$$|\Psi(z_1,\ldots,z_r)| \ll_{\boldsymbol{F}} \Psi(x_1,\ldots,x_r)\exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)$$

for some $C = C(\boldsymbol{F}) > 0$ when $|z_1-x_1| = \cdots |z_r-x_r| = 2$. Using this estimate and Cauchy's integral formula, we find that

$$\begin{aligned}
&\frac{\partial^n\Psi(x_1,\ldots,x_r)}{\partial z_1^{k_1}\cdots\partial z_r^{k_r}}\\
&=\frac{k_1!\cdots k_r!}{(2\pi i)^r}\int_{|z_r-x_r|=2}\cdots\int_{|z_1-x_1|=2}\frac{\Psi(z_1,\ldots,z_r)}{(z_1-x_1)^{k_1}\cdots(z_r-x_r)^{k_r}}dz_1\cdots dz_r\\
&\ll_{\boldsymbol{F}} 2^{-(k_1+\cdots+k_r)}k_1!\cdots k_r!\Psi(x_1,\ldots,x_r)\exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right),
\end{aligned}$$

where C is a positive constant depending only on $\boldsymbol{F}$. Hence, when $\|\boldsymbol{u}\| \leq 1$, we have

$$\begin{aligned}
\Psi(x_1+iu_1,\ldots,x_r+iu_r) &= \Psi(x_1,\ldots,x_r)\left(1+O_{\boldsymbol{F}}\left(\|\boldsymbol{u}\|\exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\theta_{\boldsymbol{F}}}}\right)\right)\right)\\
&= \Psi(x_1,\ldots,x_r)\prod_{j=1}^{r}\left(1+O_{\boldsymbol{F}}\left(|u_j|\exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right)\right),
\end{aligned}$$

which completes the proof of (6.55). □

Lemma 6.11. *For $\boldsymbol{x} = (x_1, \ldots, x_r) \in (\mathbb{R}_{\geq 0})^r$, we have*

$$\Xi_X(\boldsymbol{x}) = \prod_{j=1}^{r} \exp\left(-\frac{x_j^2}{2}\sigma_{F_j}(x_j)^2 + O_{\boldsymbol{F}}\left(x_j^2 + x_j^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right), \tag{6.65}$$

and

$$\Xi_X(\boldsymbol{x}) \geq \prod_{j=1}^{r} \exp\left(-\frac{x_j^2}{2}\sigma_{F_j}(x_j)^2 - O_{\boldsymbol{F}}\left(x_j^2\right)\right). \tag{6.66}$$

Proof. By formula (6.54) and the boundedness of $\tau_{i,j}(X)$, we find that

$$\begin{aligned}\Xi_X(\boldsymbol{x}) &= \prod_{j=1}^{r} \exp\left(-\frac{x_j^2}{2}\sigma_{F_j}(|x_j|)^2 + O_{\boldsymbol{F}}\left(\|\boldsymbol{x}\|^2 + x_j^2 + x_j^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right)\\ &= \prod_{j=1}^{r} \exp\left(-\frac{x_j^2}{2}\sigma_{F_j}(|x_j|)^2 + O_{\boldsymbol{F}}\left(x_j^2 + x_j^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right),\end{aligned}$$

which completes the proof of (6.65).

Next, we consider estimate (6.66). By (6.64), we obtain

$$\prod_{p>M} \frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{x})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{x})/4p\right)} = \prod_{j=1}^{r} \exp\left(O_{\boldsymbol{F}}\left(|x_j|^2\right)\right),$$

where $M = (C\|\boldsymbol{x}\|)^{\frac{2}{1-2\vartheta_{\boldsymbol{F}}}}$. Using this equation and the boundedness of $\tau_{i,j}(X)$, we have

$$\Xi_X(\boldsymbol{x}) = \prod_{p\leq M} \frac{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{x})/p}\right)}{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{x})/4p\right)} \times \prod_{j=1}^{r} \exp\left(O_{\boldsymbol{F}}\left(|x_j|^2\right)\right).$$

Additionally, it follows from estimate (6.61) that

$$\prod_{p\leq M} \exp\left(-K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{x})/4p\right) = \prod_{j=1}^{r} \exp\left(-\frac{x_j^2}{2}\sigma_{F_j}(|x_j|)^2 + O_{\boldsymbol{F}}\left(|x_j|^2\right)\right).$$

Hence, using the above equations and the inequality $I_0(x) \geq 1$ for $x \in \mathbb{R}$, we complete the proof of (6.66). □

Lemma 6.12. *Assume that $\boldsymbol{F}$ satisfies (S4), (A1), and (A2). For $\boldsymbol{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$, $X \geq C(\|\boldsymbol{z}\| + 3)^{\frac{2}{1-2\vartheta_{\boldsymbol{F}}}}$ with $C = C(\boldsymbol{F})$ a sufficiently large positive constant, we have*

$$\begin{aligned}&\left|\prod_{p\leq X} I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)\right| \\ &\leq \left|\prod_{j=1}^{r} \exp\left(\frac{z_j^2}{2}\left(\sigma_{F_j}(X)^2 - \sigma_{F_j}(|z_j|\right)^2 + O_{\boldsymbol{F}}\left(|z_j|^2 + |z_j|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right)\right|,\end{aligned} \tag{6.67}$$

where $\sigma_{F_j}(X)$ is defined by (6.15)*. Moreover, there exists a positive constant $b_2 = b_2(\boldsymbol{F})$ such that, for any $X \geq 3$ and any $\boldsymbol{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$ with $\|\boldsymbol{z}\| \leq b_2$, we have*

$$\prod_{p \leq X} I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) = \prod_{j=1}^{r}\left(1 + O_{\boldsymbol{F}}\left(|z_j|^2\right)\right)\exp\left(\frac{z_j^2}{2}\sigma_{F_j}(X)^2\right). \quad (6.68)$$

Furthermore, for any $\boldsymbol{z} = (x_1 + iu_1, \ldots, x_r + iu_r) \in \mathbb{C}^r$ with $x_j, u_j \in \mathbb{R}$ and $\|\boldsymbol{u}\| \leq 1$, and any $X \geq C(\|\boldsymbol{x}\| + 3)^{\frac{2}{1-2\vartheta_{\boldsymbol{F}}}}$ with $C = C(\boldsymbol{F})$ a sufficiently large positive constant, we have

$$\prod_{p \leq X} I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) \quad (6.69)$$
$$= \Xi_X(\boldsymbol{x})\prod_{j=1}^{r}\left(1 + O_{\boldsymbol{F}}\left(|u_j|\exp\left(D_1\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right) + \frac{|z_j|^4}{\log X}\right)\right)\exp\left(\frac{z_j^2}{2}\sigma_{F_j}(X)^2\right),$$

where Ξ_X is the function defined by (6.18)*, and D_1 is the same constant as in Lemma 6.10.*

Proof. First, we prove (6.67). It holds that

$$\prod_{p \leq X} I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)$$
$$= \Psi(\boldsymbol{z})\prod_{p \leq X}\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right) \times \prod_{p > X}\frac{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}.$$

By (6.64), we have

$$\prod_{p > X}\frac{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)} = \prod_{j=1}^{r}\exp\left(O_{\boldsymbol{F}}\left(|z_j|^2\right)\right)$$

when $X \geq C\left(\|\boldsymbol{z}\| + 3\right)^{\frac{2}{1-2\vartheta_{\boldsymbol{F}}}}$ with C a suitably large constant. Also, as in the proof of (6.60), we find that

$$\prod_{p \leq X}\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right) = \prod_{j=1}^{r}\exp\left(\frac{z_j^2}{2}\sigma_{F_j}(X)^2 + O_{\boldsymbol{F}}\left(|z_j|^2\right)\right).$$

Combing the above two estimates and Lemma 6.10, we have estimate (6.67).

Next, we prove (6.68). From the estimate $a_{F_j}(p) \ll p^{\vartheta_{\boldsymbol{F}}}$ for some $\vartheta_{\boldsymbol{F}} \in [0, 1/2)$, there exists a positive constant $b_1 = b_1(\boldsymbol{F})$ such that for any $z_1, \ldots, z_r \in \mathbb{C}$ with $|z_1|, \ldots, |z_r| \leq b_1$, the inequality $|\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}| \leq 1$ holds for all primes p. Then, we find that

$$\left|I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) - 1\right| \leq \frac{1}{2},$$

and that, from the Taylor expansion of I_0,

$$I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) = 1 + \frac{1}{4p}K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z}) + O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{z}\|_1^4}{p^2}\sum_{j=1}^{r}|a_{F_j}(p)|^4\right).$$

Similarly to the proof of estimate (6.63), we see that $\sum_{p\le X}\frac{|a_{F_j}(p)|^4}{p^2} \ll_{\boldsymbol{F}} 1$. Therefore, we obtain

$$\begin{aligned}\sum_{p\le X}\log I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) &= \sum_{p\le X}\left(\frac{1}{4p}K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z}) + O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{z}\|_1^4}{p^2}\sum_{j=1}^{r}|a_{F_j}(p)|^4\right)\right)\\ &= \sum_{p\le X}\frac{1}{4p}K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z}) + O_{\boldsymbol{F}}\left(\|\boldsymbol{z}\|_1^4\right).\end{aligned}$$

Using equation (6.59), we also have

$$\sum_{p\le X}\frac{1}{4p}K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z}) = \sum_{j=1}^{r}\frac{z_j^2}{2}\sigma_{F_j}(X)^2 + \sum_{1\le l_1<l_2\le r} z_{l_1}z_{l_2}\tau_{l_1,l_2}(X), \tag{6.70}$$

where $\tau_{i,j}(X)$ is defined by (6.16). By Assumptions (A1) and (A2), it holds that $\tau_{l_1,l_2}(X) \ll_{\boldsymbol{F}} 1$ for all $1\le l_1<l_2\le r$, and so we obtain

$$\sum_{1\le l_1<l_2\le r} z_{l_1}z_{l_2}\tau_{l_1,l_2}(X) \ll_{\boldsymbol{F}} \|\boldsymbol{z}\|_1^2.$$

Hence, we have

$$\begin{aligned}\sum_{p\le X}\log I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right) &= \sum_{j=1}^{r}\frac{z_j^2}{2}\sigma_{F_j}(X)^2 + O_{\boldsymbol{F}}\left(\|\boldsymbol{z}\|_1^2\right)\\ &= \sum_{j=1}^{r}\frac{z_j^2}{2}\left(\sigma_{F_j}(X)^2 + O_{\boldsymbol{F}}(1)\right),\end{aligned}$$

which completes the proof of (6.68).

Finally, we prove (6.69). The left hand side of (6.69) can be written as

$$\Psi(\boldsymbol{z})\exp\left(\sum_{p\le X}\frac{1}{4p}K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})\right)\prod_{p>X}\frac{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}.$$

Similarly to (6.57), we find that

$$\begin{aligned}\prod_{p>X}\frac{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)} &= \exp\left(\sum_{p>X}\log\left(\frac{\exp\left(K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/4p\right)}{I_0\left(\sqrt{K_{\boldsymbol{F},\boldsymbol{\theta}}(p,\boldsymbol{z})/p}\right)}\right)\right)\\ &= \exp\left(O\left(\sum_{p>X}\frac{\|\boldsymbol{z}\|_1^4\|a_{\boldsymbol{F}}(p)\|_1^2}{p^{2-2\vartheta_{\boldsymbol{F}}}}\right)\right).\end{aligned}$$

for $X \geq C(\|\boldsymbol{x}\|+3)^{\frac{2}{1-2\vartheta_{\boldsymbol{F}}}}$ with $C = C(\boldsymbol{F})$ sufficiently large. By assumption (A1) and partial summation, the last is $= 1 + O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{z}\|_1^4 \log\log X}{X^{1-2\vartheta_{\boldsymbol{F}}}}\right) = 1 + O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{z}\|^4}{\log X}\right) = \prod_{j=1}^r \left(1 + O_{\boldsymbol{F}}\left(\frac{|z_j|^4}{\log X}\right)\right)$. Moreover, by equation (6.70), we see that

$$\exp\left(\sum_{p\leq X} \frac{1}{4p} K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z})\right) = \exp\left(\sum_{j=1}^r \frac{z_j^2}{2}\sigma_{F_j}(X) + \sum_{1\leq l_1<l_2\leq r} z_{l_1}z_{l_2}\tau_{l_1,l_2}(X)\right).$$

In particular, by assumptions (A1) and (A2), the estimate $\tau_{l_1,l_2}(X) \ll_{\boldsymbol{F}} 1$ holds for all $1 \leq l_1 < l_2 \leq r$, and so the above is equal to

$$\exp\left(\sum_{1\leq l_1<l_2\leq r} x_{l_1}x_{l_2}\tau_{l_1,l_2}(X)\right) \prod_{j=1}^r (1 + O_{\boldsymbol{F}}(|u_j| \cdot \|\boldsymbol{x}\| + u_j^2)) \exp\left(\frac{z_j^2}{2}\sigma_{F_j}(X)\right).$$

Additionally, we have

$$\Psi(\boldsymbol{z}) = \Psi(\boldsymbol{x}) \prod_{j=1}^r \left(1 + O_{\boldsymbol{F}}\left(|u_j| \exp\left(D_1\|\boldsymbol{x}\|^{(2-2\vartheta_{\boldsymbol{F}})/(1-2\vartheta_{\boldsymbol{F}})}\right)\right)\right)$$

by Lemma 6.10. From the above estimates and the definition of Ξ_X (6.18), we also obtain formula (6.69). Thus, we complete the proof of this lemma. □

6.3.2 Completion of the proofs of Propositions 6.1, 6.2

Before starting the proofs of Propositions 6.1, 6.2 we introduce some notation. Define the $\mathbb{R}^r$-valued function $\boldsymbol{F}_{\boldsymbol{\theta},X}(t)$ by

$$\boldsymbol{F}_{\boldsymbol{\theta},X}(t) = (\operatorname{Re} e^{-i\theta_1} P_{F_1}(1/2 + it, X), \ldots, \operatorname{Re} e^{-i\theta_r} P_{F_r}(1/2 + it, X)),$$

and $\mu_{T,\boldsymbol{F}}$ the measure on $\mathbb{R}^r$ by $\mu_{T,\boldsymbol{F}}(B) := \frac{1}{T}\operatorname{meas}(\boldsymbol{F}_{\boldsymbol{\theta},X}^{-1}(B) \cap \mathcal{A})$ for $B \in \mathcal{B}(\mathbb{R}^r)$. Put $y_j = V_j\sigma_{F_j}(X)$. Then we find that

$$\begin{aligned}&\frac{1}{T}\operatorname{meas}(\mathcal{S}_X(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})) \qquad (6.71)\\ &= \mu_{T,\boldsymbol{F}}((y_1, \infty) \times \cdots \times (y_r, \infty)) + O_{\boldsymbol{F}}\left(\exp\left(-e^{-1}(\log\log X)^{4(r+1)}\right)\right)\end{aligned}$$

by the estimate $\operatorname{meas}([T, 2T] \setminus \mathcal{A}) \ll_{\boldsymbol{F}} T\exp\left(-e^{-1}(\log\log X)^{4(r+1)}\right)$. For $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$, put

$$\nu_{T,\boldsymbol{F},\boldsymbol{x}}(B) := \int_B e^{x_1\xi_1+\cdots+x_r\xi_r} d\mu_{T,\boldsymbol{F}}(\boldsymbol{\xi})$$

for $B \in \mathcal{B}(\mathbb{R}^r)$. Note that $\nu_{T,\boldsymbol{F},\boldsymbol{x}}$ is a measure on $\mathbb{R}^r$, and has a finite value for every $B \in \mathcal{B}(\mathbb{R}^r)$, $\boldsymbol{x} \in \mathbb{R}^r$, $X \geq 3$ in the sense

$$\nu_{T,\boldsymbol{F},\boldsymbol{x}}(B) \leq \nu_{T,\boldsymbol{F},\boldsymbol{x}}(\mathbb{R}^r) = \frac{1}{T}\int_{\mathcal{A}} \exp\left(\sum_{j=1}^r x_j \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2} + it, X)\right) dt < +\infty.$$

Under the above notation, we state and prove three lemmas.

Lemma 6.13. *For $x_1, \ldots, x_r > 0$, we have*

$$\begin{aligned}&\mu_{T,\boldsymbol{F}}((y_1,\infty)\times\cdots\times(y_r,\infty))\\&=\int_{x_r y_r}^{\infty}\cdots\int_{x_1 y_1}^{\infty} e^{-(\tau_1+\cdots+\tau_r)}\nu_{T,\boldsymbol{F},\boldsymbol{x}}((y_1,\tau_1/x_1)\times\cdots\times(y_r,\tau_r/x_r))d\tau_1\cdots d\tau_r.\end{aligned}$$

Proof. For every $B \in \mathcal{B}(\mathbb{R}^r)$, it holds that

$$\mu_{T,\boldsymbol{F}}(B) = \int_B e^{-(x_1 v_1+\cdots+x_r v_r)} d\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\boldsymbol{v}).$$

By Fubini's theorem, we find that

$$\begin{aligned}&\int_{(y_1,\infty)\times\cdots\times(y_r,\infty)} e^{-(x_1 v_1+\cdots+x_r v_r)} d\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\boldsymbol{v})\\&=\int_{(y_1,\infty)\times\cdots\times(y_r,\infty)}\left(\int_{x_r v_r}^{\infty}\cdots\int_{x_1 v_1}^{\infty} e^{-(\tau_1+\cdots+\tau_r)}d\tau_1\cdots d\tau_r\right)d\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\boldsymbol{v})\\&=\int_{x_r y_r}^{\infty}\cdots\int_{x_1 y_1}^{\infty} e^{-(\tau_1+\cdots+\tau_r)}\left(\int_{(y_1,\tau_1/x_1)\times\cdots\times(y_r,\tau_r/x_r)} 1 d\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\boldsymbol{v})\right)d\tau_1\cdots d\tau_r\\&=\int_{x_r y_r}^{\infty}\cdots\int_{x_1 y_1}^{\infty} e^{-(\tau_1+\cdots+\tau_r)}\nu_{T,\boldsymbol{F},\boldsymbol{x}}((y_1,\tau_1/x_1)\times\cdots\times(y_r,\tau_r/x_r))d\tau_1\cdots d\tau_r.\end{aligned}$$

□

The next lemma is a generalization of [67, Lemma 6.2] in multidimensions. Define

$$G(u) = \frac{2u}{\pi} + \frac{2(1-u)u}{\tan \pi u}, \quad f_{c,d}(u) = \frac{e^{-2\pi i c u} - e^{-2\pi i d u}}{2}.$$

For a set A, we denote the indicator function of A by $\mathbf{1}_A$.

Lemma 6.14. *Let L be a positive number. Let $c_1, \ldots, c_r, d_1, \ldots, d_r$ be real numbers with $c_j < d_j$. Put $\mathscr{R} = (c_1, d_1)\times\cdots\times(c_r, d_r) \subset \mathbb{R}^r$. For any $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_r) \in \mathbb{R}^r$, we have*

$$\mathbf{1}_{\mathscr{R}}(\boldsymbol{\xi}) = W_{L,\mathscr{R}}(\boldsymbol{\xi}) + O_r\left(\sum_{j=1}^{r}\left\{\left(\frac{\sin(\pi L(\xi_j - c_j))}{\pi L(\xi_j - c_j)}\right)^2 + \left(\frac{\sin(\pi L(\xi_j - d_j))}{\pi L(\xi_j - d_j)}\right)^2\right\}\right),$$

where $W_{L,\mathscr{R}}(\boldsymbol{\xi})$ is defined as if r is even,

$$\frac{i^r}{2^{r-1}}\sum_{j=1}^{r}(-1)^{j-1}\operatorname{Re}\prod_{h=1}^{r}\int_0^L G\left(\frac{u}{L}\right)e^{2\pi i\varepsilon_j(h)u\xi_h} f_{c_h,d_h}(\varepsilon_j(h)u)\frac{du}{u},$$

if r is odd,

$$\frac{i^{r+1}}{2^{r-1}}\sum_{j=1}^{r}(-1)^{j-1}\operatorname{Im}\prod_{h=1}^{r}\int_0^L G\left(\frac{u}{L}\right)e^{2\pi i\varepsilon_j(h)u\xi_h} f_{c_h,d_h}(\varepsilon_j(h)u)\frac{du}{u}.$$

Here, $\varepsilon_j(h) = 1$ if $1 \le h \le j-1$, and $\varepsilon_j(h) = -1$ otherwise.

Proof. We use the following formula (cf. [67, equation (6.1)])

$$\mathbf{1}_{(c_h,d_h)}(\xi_h) = \operatorname{Im}\int_0^L G\left(\frac{u}{L}\right) e^{2\pi i u\xi_h} f_{c_h,d_h}(u)\frac{du}{u} + O\left(\left(\frac{\sin(\pi L(\xi_h-c_h))}{\pi L(\xi_h-c_h)}\right)^2 + \left(\frac{\sin(\pi L(\xi_h-d_h))}{\pi L(\xi_h-d_h)}\right)^2\right),$$

which leads to the estimate $\operatorname{Im}\int_0^L G\left(\frac{u}{L}\right) e^{2\pi i u\xi_j} f_{c_j,d_j}(u)\frac{du}{u} \ll 1$. Therefore, we obtain

$$\mathbf{1}_{\mathscr{R}}(\boldsymbol{\xi}) = \prod_{h=1}^r \operatorname{Im}\int_0^L G\left(\frac{u}{L}\right) e^{2\pi i u\xi_h} f_{c_h,d_h}(u)\frac{du}{u} + O_r\left(\sum_{j=1}^r\left\{\left(\frac{\sin(\pi L(\xi_j-c_j))}{\pi L(\xi_j-c_j)}\right)^2 + \left(\frac{\sin(\pi L(\xi_j-d_j))}{\pi L(\xi_j-d_j)}\right)^2\right\}\right). \tag{6.72}$$

For any complex numbers $w_1,\dots,w_r$, we observe that

$$\operatorname{Im}(w_1)\cdots\operatorname{Im}(w_r) = \frac{i^r}{2^r}\sum_{j=1}^r(-1)^{j-1}\left(w_1\cdots w_{j-1}\overline{w_j\cdots w_r} + (-1)^r\overline{w_1\cdots\overline{w_j\cdots w_r}}\right).$$

In particular, if r is even, then

$$\operatorname{Im}(w_1)\cdots\operatorname{Im}(w_r) = \frac{i^r}{2^{r-1}}\operatorname{Re}\sum_{j=1}^r(-1)^{j-1}w_1\cdots w_{j-1}\overline{w_j\cdots w_r},$$

and if r is odd, then

$$\operatorname{Im}(w_1)\cdots\operatorname{Im}(w_r) = \frac{i^{r+1}}{2^{r-1}}\operatorname{Im}\sum_{j=1}^r(-1)^{j-1}w_1\cdots w_{j-1}\overline{w_j\cdots w_r}.$$

Substituting these to (6.72), we obtain Lemma 6.14. □

Lemma 6.15. *Suppose that $\boldsymbol{F}$, $\boldsymbol{\theta}$ satisfy (S4), (A1), and (A2). Let $c_1,\dots,c_r$, $d_1,\dots,d_r$ be real numbers with $c_j < d_j$. Put $\mathscr{R} = (c_1,d_1)\times\cdots\times(c_r,d_r)$. Let T, X be large numbers depending on $\boldsymbol{F}$ and satisfying $X^{(\log\log X)^{4(r+1)}} \le T$. Then for any $\boldsymbol{x} = (x_1,\dots,x_r)\in\mathbb{R}^r$ satisfying $\|\boldsymbol{x}\| \le (\log\log X)^{2r}$, we have*

$$\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\mathscr{R}) = \Xi_X(\boldsymbol{x})\left(\prod_{h=1}^r e^{\frac{x_h^2}{2}\sigma_{F_h}(X)^2}\right)\times\left\{\prod_{j=1}^r\int_{x_j\sigma_{F_j}(X)-\frac{d_j}{\sigma_{F_j}(X)}}^{x_j\sigma_{F_j}(X)-\frac{c_j}{\sigma_{F_j}(X)}} e^{-v^2/2}\frac{dv}{\sqrt{2\pi}} + E_1\right\}, \tag{6.73}$$

where the error term E_1 satisfies

$$E_1 \ll_{\boldsymbol{F}} \frac{\exp\left(D_2\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac12}} + \frac{\exp\left(D_2\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)}{\sqrt{\log\log X}}\prod_{h=1}^r\frac{d_h-c_h}{\sigma_{F_h}(X)}$$

for some constant $D_2 = D_2(\boldsymbol{F}) > 0$. Moreover, if $\|\boldsymbol{x}\| \leq b_4$ with $b_4 = b_4(\boldsymbol{F}) > 0$ sufficiently small, we have

$$\begin{aligned}
&\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\mathscr{R}) \qquad (6.74)\\
&= \left(\prod_{h=1}^{r} e^{\frac{x_h^2}{2}\sigma_{F_h}(X)^2}\right) \times \left\{\prod_{j=1}^{r} \int_{x_j\sigma_{F_j}(X)-\frac{d_j}{\sigma_{F_j}(X)}}^{x_j\sigma_{F_j}(X)-\frac{c_j}{\sigma_{F_j}(X)}} e^{-v^2/2}\frac{dv}{\sqrt{2\pi}} + E_2\right\},
\end{aligned}$$

where the error term E_2 satisfies

$$E_2 \ll_{\boldsymbol{F}} \frac{1}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}} + \sum_{k=1}^{r}\left(\frac{x_k^2(d_k-c_k)}{\sigma_{F_k}(X)} + \frac{1}{\sigma_{F_k}(X)^2}\right)\prod_{\substack{h=1\\h\neq k}}^{r}\frac{d_h-c_h}{\sigma_{F_h}(X)}.$$

Proof. We show formula (6.73). Put $L = b_5(\log\log X)^{\alpha_{\boldsymbol{F}}}$ with $b_5 = b_5(\boldsymbol{F})$ a small positive constant to be chosen later. It follows from Lemma 6.14 that

$$\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\mathscr{R}) = \int_{\mathbb{R}^r} W_{L,\mathscr{R}}(\boldsymbol{\xi})e^{x_1\xi_1+\cdots+x_r\xi_r}\,d\mu_{T,\boldsymbol{F}}(\boldsymbol{\xi}) + E, \qquad (6.75)$$

where the error term E satisfies the estimate

$$\begin{aligned}
&E \ll_r\\
&\sum_{j=1}^{r}\int_{\mathbb{R}^r}\left\{\left(\frac{\sin(\pi L(\xi_j-c_j))}{\pi L(\xi_j-c_j)}\right)^2 + \left(\frac{\sin(\pi L(\xi_j-d_j))}{\pi L(\xi_j-d_j)}\right)^2\right\} e^{x_1\xi_1+\cdots+x_r\xi_r}\,d\mu_{T,\boldsymbol{F}}(\boldsymbol{\xi}).
\end{aligned}$$

First, we estimate E. For $\boldsymbol{z} = (z_1,\ldots,z_r) \in \mathbb{C}^r$, define

$$M_T(\boldsymbol{z}) = \int_{\mathbb{R}^r} e^{z_1\xi_1+\cdots+z_r\xi_r}\,d\mu_{T,\boldsymbol{F}}(\boldsymbol{\xi}).$$

Then, it holds that

$$M_T(\boldsymbol{z}) = \frac{1}{T}\int_{\mathcal{A}} \exp\left(\sum_{j=1}^{r} z_j \operatorname{Re} e^{-i\theta_j}P_{F_j}(\tfrac{1}{2}+it,X)\right)dt.$$

Put $\boldsymbol{w} = (w_1,\ldots,w_r) = (x_1+iu_1,\ldots,x_r+iu_r)$ with $u_j \in \mathbb{R}$. When $\|(u_1,\ldots,u_r)\| \leq L$ holds, we have

$$\begin{aligned}
&|M_T(\boldsymbol{w})|\\
&\leq \left|\prod_{j=1}^{r}\exp\left(\frac{(x_j+iu_j)^2}{2}\left(\sigma_{F_j}(X)^2 - \sigma_{F_j}(|w_j|)^2\right) + O_{\boldsymbol{F}}\left(|x_j+iu_j|^2 + |x_j+iu_j|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right)\right|\\
&\qquad + O_{\boldsymbol{F}}\left(\exp\left(-6^{-1}(\log\log X)^{4(r+1)}\right)\right)\\
&\leq \exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\prod_{j=1}^{r}\exp\left(\frac{x_j^2}{2}\left(\sigma_{F_j}(X)^2 - \sigma_{F_j}(|w_j|)^2\right) - \frac{u_j^2}{3}\sigma_{F_j}(X)^2 + O_{\boldsymbol{F}}\left(u_j^2 L^{\frac{2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\right)
\end{aligned}$$

by Proposition 6.7 and (6.67), where $C = C(\boldsymbol{F})$ is some positive constant. Additionally, by (6.65), we find that

$$\prod_{j=1}^{r} \exp\left(-\frac{x_j^2}{2}\sigma_{F_j}(|w_j|)^2\right) \le \prod_{j=1}^{r} \exp\left(-\frac{x_j^2}{2}\sigma_{F_j}(|x_j|)^2\right)$$
$$\ll_{\boldsymbol{F}} \Xi_X(\boldsymbol{x}) \exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)$$

for some $C = C(\boldsymbol{F}) > 0$. Recall that $\alpha_{\boldsymbol{F}} = \min\{2r, \frac{1-2\vartheta_{\boldsymbol{F}}}{2\vartheta_{\boldsymbol{F}}}\}$. Hence, the inequality $L^{\frac{2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \le (2b_5)^{\frac{2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \log\log X$ holds. Therefore, when b_5 is sufficiently small, we have

$$|M_T(x_1 + iu_1, \dots, x_r + iu_r)| \tag{6.76}$$
$$\ll_{\boldsymbol{F}} \Xi_X(\boldsymbol{x}) \exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right) \prod_{j=1}^{r} \exp\left(\left(\frac{x_j^2}{2} - \frac{u_j^2}{4}\right)\sigma_{F_j}(X)^2\right)$$

for $\|(u_1, \dots, u_r)\| \le L$. For any $\ell, \xi \in \mathbb{R}$, we can write

$$\left(\frac{\sin(\pi L(\xi - \ell))}{\pi L(\xi - \ell)}\right)^2 = \frac{2}{L^2}\int_0^L (L-u)\cos(2\pi(\xi - \ell)u)du$$
$$= \frac{2}{L^2}\operatorname{Re}\int_0^L (L-u)e^{2\pi i(\xi-\ell)u}du. \tag{6.77}$$

Thus

$$\int_{\mathbb{R}^r}\left(\frac{\sin(\pi L(\xi_j - \ell))}{\pi L(\xi_j - \ell)}\right)^2 e^{x_1\xi_1+\cdots+x_r\xi_r}d\mu_{T,\boldsymbol{F}}(\boldsymbol{\xi})$$
$$= \frac{2}{L^2}\operatorname{Re}\int_0^L (L-u)\int_{\mathbb{R}^r} e^{2\pi i(\xi_j-\ell)u}e^{x_1\xi_1+\cdots+x_r\xi_r}d\mu_{T,\boldsymbol{F}}(\boldsymbol{\xi})du$$
$$= \frac{2}{L^2}\operatorname{Re}\int_0^L e^{-2\pi i\ell u}(L-u)M_T(x_1, \dots, x_{j-1}, x_j + 2\pi iu, x_{j+1}, \dots, x_r)du,$$

which, by (6.76), is

$$\ll_{\boldsymbol{F}} \Xi_X(\boldsymbol{x})\frac{\exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)}{L^2}\left(\prod_{k=1}^{r}\exp\left(\frac{x_k^2}{2}\sigma_{F_k}(X)\right)\right)$$
$$\times\int_0^L (L-u)\exp\left(-(\pi\sigma_{F_j}(X)u)^2\right)du$$
$$\ll \Xi_X(\boldsymbol{x})\frac{\exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)}{L\sigma_{F_j}(X)}\prod_{k=1}^{r}\exp\left(\frac{x_k^2}{2}\sigma_{F_k}(X)^2\right).$$

It then follows that equation (6.75) satisfies

$$\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\mathscr{R}) = \int_{\mathbb{R}^r} W_{L,\mathscr{R}}(\boldsymbol{\xi}) e^{x_1\xi_1+\cdots+x_r\xi_r}\, d\mu_{T,\boldsymbol{F}}(\boldsymbol{\xi}) + E \tag{6.78}$$

with

$$E \ll_{\boldsymbol{F}} \Xi_X(\boldsymbol{x}) \frac{\exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)}{L\sqrt{\log\log X}} \prod_{k=1}^{r} \exp\left(\frac{x_k^2}{2}\sigma_{F_k}(X)^2\right).$$

For the main term in (6.78), it is enough to calculate

$$\int_{\mathbb{R}^r} \left(\prod_{h=1}^{r} \int_0^L G\left(\frac{u}{L}\right) e^{2\pi i \varepsilon_j(h) u \xi_h} f_{c_h,d_h}(\varepsilon_j(h)u) \frac{du}{u}\right) e^{x_1\xi_1+\cdots+x_r\xi_r}\, d\mu_{T,\boldsymbol{F}}(\boldsymbol{\xi}) \tag{6.79}$$

for every fixed $1 \le j \le r$. Using Fubini's theorem, we find that (6.79) is equal to

$$\int_0^L \cdots \int_0^L \left(\prod_{h=1}^{r} G\left(\frac{u_h}{L}\right) \frac{f_{c_h,d_h}(\varepsilon_j(h)u_h)}{u_h}\right) \\ \times M_T\left(x_1 + 2\pi i \varepsilon_j(1)u_1, \dots, x_r + 2\pi i \varepsilon_j(r)u_r\right) du_1 \cdots du_r.$$

Next we divide the range of this integral as

$$\int_0^L \cdots \int_0^L = \int_0^1 \cdots \int_0^1 + \sum_{k=0}^{r-1} \int \cdots \int_{D_k},$$

where

$$\int \cdots \int_{D_k} = \int_0^1 \cdots \int_0^1 \int_1^L \overbrace{\int_0^L \cdots \int_0^L}^{k}.$$

By estimate (6.76) and the estimates $\frac{f_{c,d}(\pm u)}{u} \ll d-c, G(u/L) \ll 1$ for $0 \le u \le L$, the integral over D_{r-k} for $1 \le k \le r$ is

$$\ll_{\boldsymbol{F}} \Xi_X(\boldsymbol{x}) \exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right) \left(\prod_{h=1}^{k-1} (d_h - c_h) \int_0^1 \exp\left(\left(\frac{x_h^2}{2} - (\pi u)^2\right) \sigma_{F_h}(X)^2\right) du\right) \\ \times (d_k - c_k) \int_1^L \exp\left(\left(\frac{x_k^2}{2} - (\pi u)^2\right) \sigma_{F_k}(X)^2\right) du \\ \times \left(\prod_{h=k+1}^{r} (d_h - c_h) \int_0^L \exp\left(\left(\frac{x_h^2}{2} - (\pi u)^2\right) \sigma_{F_h}(X)^2\right) du\right) \\ \ll_{\boldsymbol{F}} \Xi_X(\boldsymbol{x}) \exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right) e^{-\sigma_{F_k}(X)^2} \prod_{h=1}^{r} \frac{d_h - c_h}{\sigma_{F_h}(X)} \exp\left(\frac{x_h^2}{2}\sigma_{F_h}(X)^2\right).$$

Hence, integral (6.79) is equal to

$$\int_0^1 \cdots \int_0^1 \left(\prod_{h=1}^{r} G\left(\frac{u_h}{L}\right) \frac{f_{c_h,d_h}(\varepsilon_j(h)u_h)}{u_h} \right) \tag{6.80}$$
$$\times M_T\left(x_1 + 2\pi i \varepsilon_j(1)u_1, \ldots, x_r + 2\pi i \varepsilon_j(r)u_r\right) du_1 \cdots du_r$$
$$+ O_{\boldsymbol{F}}\left(\sum_{k=1}^{r} \Xi_X(\boldsymbol{x}) \exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) e^{-\sigma_{F_k}(X)^2} \prod_{h=1}^{r} \frac{d_h - c_h}{\sigma_{F_h}(X)} \exp\left(\frac{x_h^2}{2} \sigma_{F_h}(X)^2 \right) \right).$$

When $\|(u_1, \ldots, u_r)\| \leq 1$, it follows from Proposition 6.7 and equation (6.69) that

$$M_T\left(x_1 + 2\pi i \varepsilon_j(1)u_1, \ldots, x_r + 2\pi i \varepsilon_j(r)u_r\right)$$
$$= \Xi_X(\boldsymbol{x}) \prod_{h=1}^{r} \left(1 + O_{\boldsymbol{F}}\left(|u_h| \exp\left(D_1 \|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) + \frac{x_j^4 + u_j^4}{\log X} \right) \right)$$
$$\times \exp\left(\frac{x_h^2 + 4\pi i \varepsilon_j(h) x_h u_h - 4\pi^2 u_h^2}{2} \sigma_{F_h}(X)^2 \right)$$
$$+ O_{\boldsymbol{F}}\left(\exp\left(-6^{-1} (\log\log X)^{4(r+1)} \right) \right)$$

Note the last O term could be bounded above by

$$\left| \frac{1}{\log X} \Xi_X(\boldsymbol{x}) \prod_{h=1}^{r} \exp\left(\frac{x_h^2 + 4\pi i \varepsilon_j(h) x_h u_h - 4\pi^2 u_h^2}{2} \sigma_{F_h}(X)^2 \right) \right|$$
$$\geq \frac{1}{\log X} \prod_{h=1}^{r} \exp\left(-\frac{x_h^2}{2} \left(\sigma_{F_h}(X)^2 - \sigma_{F_h}(x_h)^2 + O_{\boldsymbol{F}}(1) \right) - 2\pi^2 \sigma_{F_h}(X)^2 \right)$$
$$\geq \exp\left(-6^{-1} (\log\log X)^{4(r+1)} \right),$$

using the lower bound for $\Xi_X(\boldsymbol{x})$ in (6.66) and the range of $\boldsymbol{x}$ and u_i. Therefore, the integral of (6.80) is equal to

$$\Xi_X(\boldsymbol{x}) \prod_{h=1}^{r} \int_0^1 \left\{ 1 + O_{\boldsymbol{F}}\left(u \exp\left(D_1 \|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) + \frac{x_h^4 + u_h^4 + 1}{\log X} \right) \right\} \times$$
$$\times G\left(\frac{u}{L}\right) f_{c_h,d_h}(\varepsilon_j(h)u) \exp\left(\frac{x_h^2 + 4\pi i \varepsilon_j(h) x_h u - 4\pi^2 u^2}{2} \sigma_{F_h}(X)^2 \right) \frac{du}{u}. \tag{6.81}$$

Since $G(u/L) \ll 1$ and $\frac{f_{c_h,d_h}(\pm u)}{u} \ll d_h - c_h$, we find that

$$\int_0^1 \left(u \exp\left(D_1 \|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) + \frac{x_h^4 + u^4 + 1}{\log X} \right) G\left(\frac{u}{L}\right) f_{c_h,d_h}(\varepsilon_j(h)u) \times \exp\left(\frac{x_h^2 + 4\pi i \varepsilon_j(h) x_h u - 4\pi^2 u^2}{2} \sigma_{F_h}(X)^2 \right) \frac{du}{u}$$

$$\ll \exp\left(\frac{x_h^2}{2} \sigma_{F_h}(X)^2 \right) (d_h - c_h) \times \int_0^1 \left(u \exp\left(D_1 \|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) + \frac{x_h^4 + u^4 + 1}{\log X} \right) \exp\left(-2\pi^2 u^2 \sigma_{F_h}(X)^2 \right) du$$

$$\ll_{\boldsymbol{F}} \exp\left(D_1 \|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) \times \exp\left(\frac{x_h^2}{2} \sigma_{F_h}(X)^2 \right) \frac{d_h - c_h}{\sigma_{F_h}(X)^2},$$

and that

$$\int_0^1 G\left(\frac{u}{L}\right) f_{c_h,d_h}(\varepsilon_j(h)u) \exp\left(\frac{x_h^2 + 4\pi i \varepsilon_j(h) x_h u - 4\pi^2 u^2}{2} \sigma_{F_h}(X)^2 \right) \frac{du}{u}$$

$$\ll (d_h - c_h) \exp\left(\frac{x_h^2}{2} \sigma_{F_h}(X)^2 \right) \int_0^1 \exp\left(-2\pi^2 u^2 \sigma_{F_h}(X)^2 \right) du$$

$$\ll \exp\left(\frac{x_h^2}{2} \sigma_{F_h}(X)^2 \right) \frac{d_h - c_h}{\sigma_{F_h}(X)}.$$

Moreover, we find that

$$\int_1^L G\left(\frac{u}{L}\right) f_{c_h,d_h}(\varepsilon_j(h)u) \exp\left(\frac{x_h^2 + 4\pi i \varepsilon_j(h) x_h u - 4\pi^2 u^2}{2} \sigma_{F_h}(X)^2 \right) \frac{du}{u}$$

$$\ll_{\boldsymbol{F}} \exp\left(\frac{x_h^2}{2} \sigma_{F_h}(X)^2 \right) \frac{d_h - c_h}{\sigma_{F_h}(X)} e^{-\sigma_{F_h}(X)^2}.$$

From these estimates and (6.81), integral (6.80) is equal to

$$\Xi_X(\boldsymbol{x}) \prod_{h=1}^r \int_0^L G\left(\frac{u}{L}\right) f_{c_h,d_h}(\varepsilon_j(h)u) \exp\left(\frac{x_h^2 + 4\pi i \varepsilon_j(h) x_h u - 4\pi^2 u^2}{2} \sigma_{F_h}(X)^2 \right) \frac{du}{u}$$
$$+ O_{\boldsymbol{F}}\left(\Xi_X(\boldsymbol{x}) \exp\left(D_1 \|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}} \right) \prod_{j=1}^r \exp\left(\frac{x_j^2}{2} \sigma_{F_j}(X)^2 \right) \frac{1}{\sqrt{\log\log X}} \prod_{h=1}^r \frac{d_h - c_h}{\sigma_{F_h}(X)} \right).$$

Using the well known formula

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iv\xi} e^{-\eta v^2} dv = \frac{1}{\sqrt{2\eta}} \exp\left(-\frac{\xi^2}{4\eta} \right), \tag{6.82}$$

we can rewrite the above main term as

$$\Xi_X(\boldsymbol{x})\left(\prod_{h=1}^{r}\frac{\exp\left(\frac{x_h^2}{2}\sigma_{F_h}(X)^2\right)}{\sqrt{2\pi}}\right)\int_{\mathbb{R}^r}e^{-(v_1^2+\cdots+v_r^2)/2}$$

$$\times\left\{\prod_{h=1}^{r}\int_0^L G\left(\frac{u}{L}\right)e^{2\pi i\varepsilon_j(h)u(x_h\sigma_{F_h}(X)^2-v_h\sigma_{F_h}(X))}f_{c_h,d_h}(\varepsilon_j(h)u)\frac{du}{u}\right\}d\boldsymbol{v}.$$

Combining this with (6.80), we see that integral (6.79) is equal to

$$\Xi_X(\boldsymbol{x})\left(\prod_{h=1}^{r}\frac{\exp\left(\frac{x_h^2}{2}\sigma_{F_h}(X)^2\right)}{\sqrt{2\pi}}\right)\int_{\mathbb{R}^r}e^{-(v_1^2+\cdots+v_r^2)/2}$$

$$\times\left\{\prod_{h=1}^{r}\int_0^L G\left(\frac{u}{L}\right)e^{2\pi i\varepsilon_j(h)u(x_h\sigma_{F_h}(X)^2-v_h\sigma_{F_h}(X))}f_{c_h,d_h}(\varepsilon_j(h)u)\frac{du}{u}\right\}d\boldsymbol{v}+$$

$$O_{\boldsymbol{F}}\left(\Xi_X(\boldsymbol{x})\exp\left(D_1\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\prod_{j=1}^{r}\exp\left(\frac{x_j^2}{2}\sigma_{F_j}(X)^2\right)\frac{1}{\sqrt{\log\log X}}\prod_{h=1}^{r}\frac{d_h-c_h}{\sigma_{F_h}(X)}\right).$$

Substituting this equation to the definition of $W_{L,\mathscr{R}}$ and using Lemma 6.14 and equation (6.78), we obtain

$$\begin{aligned}&\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\mathscr{R})\\&=\Xi_X(\boldsymbol{x})\left(\prod_{h=1}^{r}\frac{\exp\left(\frac{x_h^2}{2}\sigma_{F_h}(X)^2\right)}{\sqrt{2\pi}}\right)\\&\quad\times\left\{\int_{\mathbb{R}^r}e^{-\frac{v_1^2+\cdots+v_r^2}{2}}\mathbf{1}_{\mathscr{R}}\left(x_1\sigma_{F_1}(X)^2-v_1\sigma_{F_1}(X),\ldots,x_r\sigma_{F_r}(X)^2-v_r\sigma_{F_r}(X)\right)d\boldsymbol{v}\right.\\&\qquad\qquad\left.+E_3+E_4\right\},\end{aligned}$$

where E_3 and E_4 satisfy

$$\begin{aligned}E_3\ll_{\boldsymbol{F}}\sum_{j=1}^{r}\int_{\mathbb{R}^r}&\left\{\left(\frac{\sin(\pi L(x_j\sigma_{F_j}(X)^2-v_j\sigma_{F_j}(X)-c_j))}{\pi L(x_j\sigma_{F_j}(X)^2-v_j\sigma_{F_j}(X)-c_j)}\right)^2\right.\\&\left.+\left(\frac{\sin(\pi L(x_j\sigma_{F_j}(X)^2-v_j\sigma_{F_j}(X)-d_j))}{\pi L(x_j\sigma_{F_j}(X)^2-v_j\sigma_{F_j}(X)-d_j)}\right)^2\right\}\times e^{-(v_1^2+\cdots+v_r^2)/2}d\boldsymbol{v},\end{aligned}$$

and

$$E_4\ll_{\boldsymbol{F}}\frac{\exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}+\frac{\exp\left(C\|\boldsymbol{x}\|^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)}{\sqrt{\log\log X}}\prod_{h=1}^{r}\frac{d_h-c_h}{\sigma_{F_h}(X)}$$

for some constant $C = C(\boldsymbol{F}) > 0$. By equation (6.77), it holds that, for any $\ell \in \mathbb{R}$,

$$\begin{aligned}&\int_{\mathbb{R}^r} \left(\frac{\sin(\pi L(x_j\sigma_{F_j}(X)^2 - v_j\sigma_{F_j}(X) - \ell))}{\pi L(x_j\sigma_{F_j}(X)^2 - v_j\sigma_{F_j}(X) - \ell)}\right)^2 \times e^{-(v_1^2+\cdots+v_r^2)/2} d\boldsymbol{v}\\ &= \frac{2}{L^2}\operatorname{Re}\int_0^L (L-\alpha)\left(\int_{\mathbb{R}^r} e^{2\pi i(x_j\sigma_{F_j}(X)^2 - v_j\sigma_{F_j}(X)-\ell)\alpha} e^{-(v_1^2+\cdots+v_r^2)/2} d\boldsymbol{v}\right) d\alpha\\ &= \frac{2(2\pi)^{(r-1)/2}}{L^2}\operatorname{Re}\int_0^L (L-\alpha) e^{2\pi i(x_j\sigma_{F_j}(X)^2-\ell)\alpha}\left(\int_{\mathbb{R}} e^{-2\pi i v\sigma_{F_j}(X)\alpha} e^{-v^2/2} dv\right) d\alpha,\end{aligned}$$

which, by (6.82), becomes

$$\begin{aligned}&= \frac{2(2\pi)^{r/2}}{L^2}\operatorname{Re}\int_0^L (L-\alpha) e^{2\pi i(x_j\sigma_{F_j}(X)^2-\ell)\alpha}\exp\left(-2\pi^2\alpha^2\sigma_{F_j}(X)^2\right) d\alpha\\ &\ll_{\boldsymbol{F}} \frac{1}{L\sigma_{F_j}(X)} \ll_{\boldsymbol{F}} \frac{1}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}.\end{aligned}$$

Hence, we have $E_3 \ll_{\boldsymbol{F}} \frac{1}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}$. Finally, by simple calculations, we can write

$$\begin{aligned}&\int_{\mathbb{R}^r} e^{-\frac{v_1^2+\cdots+v_r^2}{2}} \mathbf{1}_{\mathscr{R}}\left(x_1\sigma_{F_1}(X)^2 - v_1\sigma_{F_1}(X), \ldots, x_r\sigma_{F_r}(X)^2 - v_r\sigma_{F_r}(X)\right) d\boldsymbol{v}\\ &= \prod_{j=1}^r \int_{x_j\sigma_{F_j}(X) - \frac{d_j}{\sigma_{F_j}(X)}}^{x_j\sigma_{F_j}(X) - \frac{c_j}{\sigma_{F_j}(X)}} e^{-v^2/2}\frac{dv}{\sqrt{2\pi}}\end{aligned}$$

and this completes the proof of (6.73).

Next, we consider (6.74). Using Proposition 6.7 and equation (6.68), we have

$$\begin{aligned}&M_T\left(x_1 + 2\pi i\varepsilon_j(1)u_1, \ldots, x_r + 2\pi i\varepsilon_j(r)u_r\right)\\ &= \prod_{h=1}^r \left(1 + O_{\boldsymbol{F}}\left(|x_h + iu_h|^2\right)\right)\exp\left(\frac{x_h^2 + 4\pi i\varepsilon_j(h)x_h u_h - 4\pi^2 u_h^2}{2}\sigma_{F_h}(X)^2\right)\\ &\qquad + O_{\boldsymbol{F}}\left(\exp\left(-6^{-1}(\log\log X)^{4(r+1)}\right)\right)\end{aligned}$$

when $\|\boldsymbol{x}\|$, $\|\boldsymbol{u}\|$ are sufficiently small. By using this equation, we can prove (6.74) similarly to the proof of (6.73). □

Proof of Proposition 6.1. We firstly prove Proposition 6.1 in the case V_j's are nonnegative. Let $\boldsymbol{x} = (x_1, \ldots, x_r) \in (\mathbb{R}_{>0})^r$ satisfying $\|\boldsymbol{x}\| \leq b_4$ with b_4 the same number as in Lemma 6.15. By Lemma 6.13 and equation (6.74), we have

$$\mu_{T,\boldsymbol{F}}((y_1,\infty)\times\cdots\times(y_r,\infty))$$
$$=\prod_{j=1}^{r} e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2}\int_{x_jy_j}^{\infty} e^{-\tau}\int_{x_j\sigma_{F_j}(X)-\frac{\tau/x_j}{\sigma_{F_j}(X)}}^{x_j\sigma_{F_j}(X)-\frac{y_j}{\sigma_{F_j}(X)}} e^{-v^2/2}\frac{dv}{\sqrt{2\pi}}d\tau$$
$$+O_{\boldsymbol{F}}\left(\frac{1}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\prod_{j=1}^{r} e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2-x_jy_j}+E\times\prod_{j=1}^{r} e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2}\right),$$

where

$$E=$$
$$\sum_{k=1}^{r}\int_{x_ry_r}^{\infty}\cdots\int_{x_1y_1}^{\infty} e^{-(\tau_1+\cdots+\tau_r)}\left(\frac{x_k^2(\frac{\tau_k}{x_k}-y_k)}{\sigma_{F_k}(X)}+\frac{1}{\sigma_{F_k}(X)^2}\right)\prod_{\substack{h=1\\h\neq k}}^{r}\frac{\frac{\tau_h}{x_h}-y_h}{\sigma_{F_h}(X)}d\tau_1\cdots d\tau_r.$$

Now, simple calculations lead that

$$\int_{x_jy_j}^{\infty} e^{-\tau}\int_{x_j\sigma_{F_j}(X)-\frac{\tau/x_j}{\sigma_{F_j}(X)}}^{x_j\sigma_{F_j}(X)-\frac{y_j}{\sigma_{F_j}(X)}} e^{-v^2/2}\frac{dv}{\sqrt{2\pi}}d\tau=\exp\left(-\frac{x_j^2}{2}\sigma_{F_j}(X)^2\right)\int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}$$

since $y_j=V_j\sigma_{F_j}(X)$. Therefore, we obtain

$$\mu_{T,\boldsymbol{F}}((y_1,\infty)\times\cdots\times(y_r,\infty))=$$
$$\prod_{j=1}^{r}\int_{V_j}^{\infty} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}}+O_{\boldsymbol{F}}\left(\frac{1}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\prod_{j=1}^{r} e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2-x_jy_j}+E\prod_{j=1}^{r} e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2}\right).$$

Here, we decide x_j's as $x_j=\max\{1,V_j\}/\sigma_{F_j}(X)$, where V_j's must satisfy the inequality $V_j\leq R\sigma_{F_j}(X)$. Then, we see that

$$e^{-x_jy_j}=e^{-\frac{x_j^2}{2}\sigma_{F_j}(X)^2}e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2-x_jy_j}\ll e^{-\frac{x_j^2}{2}\sigma_{F_j}(X)^2}e^{-V_j^2/2}. \tag{6.83}$$

This estimate leads that

$$\frac{1}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\prod_{j=1}^{r} e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2-x_jy_j}$$
$$\ll_r \frac{e^{-(V_1^2+\cdots+V_r^2)/2}}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\ll_r \frac{1}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\prod_{j=1}^{r}(1+V_j)\int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.$$

Moreover, since it holds that

$$\int_{x_jy_j}^{\infty}\left(\frac{\tau}{x_j}-y_j\right)e^{-\tau}d\tau=\frac{e^{-x_jy_j}}{x_j}, \tag{6.84}$$

we have

$$\int_{x_r y_r}^{\infty}\cdots\int_{x_1 y_1}^{\infty} e^{-(\tau_1+\cdots+\tau_r)}\left(\frac{x_k^2(d_k-c_k)}{\sigma_{F_k}(X)}+\frac{1}{\sigma_{F_k}(X)^2}\right)\prod_{\substack{h=1\\h\neq k}}^{r}\frac{\tau_h/x_h-y_h}{\sigma_{F_h}(X)}d\tau_1\cdots d\tau_r$$

$$= x_k^2\prod_{j=1}^{r}\frac{1}{\sigma_{F_j}(X)}\int_{x_j y_j}^{\infty}\left(\frac{\tau}{x_j}-y_j\right)e^{-\tau}d\tau$$

$$+\frac{e^{-x_k y_k}}{\sigma_{F_k}(X)^2}\prod_{\substack{j=1\\j\neq k}}^{r}\frac{1}{\sigma_{F_j}(X)}\int_{x_j y_j}^{\infty}\left(\frac{\tau}{x_j}-y_j\right)e^{-\tau}d\tau$$

$$= x_k^2\prod_{j=1}^{r}\frac{e^{-x_j y_j}}{x_j\sigma_{F_j}(X)}+\frac{x_k}{\sigma_{F_k}(X)}\prod_{j=1}^{r}\frac{e^{-x_j y_j}}{x_j\sigma_{F_j}(X)}. \tag{6.85}$$

for every $1\leq k\leq r$. By estimate (6.83) and $x_j\sigma_{F_j}(X)\asymp 1+V_j$, we can write

$$\frac{e^{-x_j y_j}}{x_j\sigma_{F_j}(X)}\ll e^{-\frac{x_j^2}{2}\sigma_{F_j}(X)^2}\frac{e^{-V_j^2/2}}{1+V_j}\ll e^{-\frac{x_j^2}{2}\sigma_{F_j}(X)^2}\int_{V_j}^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.$$

By this observation, (6.85) is

$$\ll_r \left(x_k^2+\frac{x_k}{\sigma_{F_k}(X)}\right)\prod_{j=1}^{r}e^{-\frac{x_j^2}{2}\sigma_{F_j}(X)^2}\int_{V_j}^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}$$

$$\ll_{\boldsymbol{F}} \frac{1+V_k^2}{\log\log X}\prod_{j=1}^{r}e^{-\frac{x_j^2}{2}\sigma_{F_j}(X)^2}\int_{V_j}^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.$$

Hence, we have

$$E\prod_{j=1}^{r}e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2}\ll_{\boldsymbol{F}}\frac{1+\|\boldsymbol{V}\|^2}{\log\log X}\prod_{j=1}^{r}\int_{V_j}^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.$$

From the above estimations, we obtain

$$\mu_{T,\boldsymbol{F}}((y_1,\infty)\times\cdots\times(y_r,\infty))=$$

$$\prod_{j=1}^{r}\int_{V_j}^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}+O_{\boldsymbol{F}}\left(\left\{\frac{\prod_{k=1}^{r}(1+V_k)}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac12}}+\frac{1+\|\boldsymbol{V}\|^2}{\log\log X}\right\}\prod_{j=1}^{r}\int_{V_j}^{\infty}e^{-u^2/2}\frac{du}{\sqrt{2\pi}}\right)$$

for $0\leq V_j\leq b_2\sigma_{F_j}(X)$. Thus, by this formula and (6.71), we complete the proof of Proposition 6.1 in the case V_j's are nonnegative.

In order to finish the proof of Proposition 6.1, we consider the negative cases. It suffices to show that, for the case $-b\sigma_{F_1}(X)\leq V_1\leq 0$ and $0\leq V_j\leq b\sigma_{F_j}(X)$,

$$\frac{1}{T}\operatorname{meas}(\mathcal{S}_X(T,(-V_1,V_2,\ldots,V_r);\boldsymbol{F},\boldsymbol{\theta}))$$

$$=\left(1+O_{\boldsymbol{F}}\left(\frac{\prod_{k=1}^{r}(1+|V_k|)}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac12}}+\frac{1+\|\boldsymbol{V}\|^2}{\log\log X}\right)\right)\prod_{j=1}^{r}\int_{V_j}^{\infty}e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}}$$

since other cases can be shown by similarly by induction. By the definition of the set $\mathcal{S}_X(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta})$, it holds that

$$\begin{aligned}\mathcal{S}_X(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta}) = \mathcal{S}_X(T, (V_2, \dots, V_r); (F_2, \dots, F_r), (\theta_2, \dots, \theta_r))\\ \setminus \mathcal{S}_X(T, (-V_1 - 0, V_2, \dots, V_r); \boldsymbol{F}, (\pi - \theta_1, \theta_2, \dots, \theta_r)),\end{aligned}$$

where we regard that if $r = 1$, the first set on the right hand side is $[T, 2T]$. Therefore, from the nonnegative cases, we have

$$\begin{aligned}&\frac{1}{T}\operatorname{meas}(\mathcal{S}_X(T, (-V_1, V_2, \dots, V_r); \boldsymbol{F}, \boldsymbol{\theta})) \qquad (6.86)\\ &= \frac{1}{T}\operatorname{meas}(\mathcal{S}_X(T, (V_2, \dots, V_r); (F_2, \dots, F_r), (\theta_2, \dots, \theta_r)))\\ &\qquad - \frac{1}{T}\operatorname{meas}(\mathcal{S}_X(T, (-V_1 - 0, V_2, \dots, V_r); \boldsymbol{F}, (\pi - \theta_1, \theta_2, \dots, \theta_r)))\\ &= (1 + E_1)\prod_{j=2}^{r}\int_{V_j}^{\infty} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}} - (1 + E_2)\prod_{j=1}^{r}\int_{|V_j|}^{\infty} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}}.\end{aligned}$$

Here, E_1 and E_2 satisfy

$$\begin{aligned}E_1 &\ll_{\boldsymbol{F}} \frac{\prod_{k=2}^{r}(1 + V_k)}{(\log\log X)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}} + \frac{1 + \|(V_2, \dots, V_r)\|^2}{\log\log X},\\ E_2 &\ll_{\boldsymbol{F}} \frac{\prod_{k=1}^{r}(1 + V_k)}{(\log\log X)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}} + \frac{1 + \|(V_1, \dots, V_r)\|^2}{\log\log X}.\end{aligned}$$

Hence, we find that (6.86) is equal to

$$\begin{aligned}&\left(1 - \int_{-V_1}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}\right)\prod_{j=2}^{r}\int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}\\ &\qquad + E_1\prod_{j=2}^{r}\int_{V_j}^{\infty} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}} + E_2\prod_{j=1}^{r}\int_{V_j}^{\infty} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}}\\ &= \left(1 + O_{\boldsymbol{F}}\left(\frac{\prod_{k=1}^{r}(1 + |V_k|)}{(\log\log X)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}} + \frac{1 + \|\boldsymbol{V}\|^2}{\log\log X}\right)\right)\prod_{j=1}^{r}\int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.\end{aligned}$$

Thus, we also obtain the negative cases of Proposition 6.1. □

Proof of Proposition 6.2. Let $\boldsymbol{V} = (V_1, \dots, V_r) \in (\mathbb{R}_{\geq 0})^r$ satisfying the inequality $\|\boldsymbol{V}\| \leq (\log\log X)^{2r}$, and put $x_j = \max\{1, V_j\}/\sigma_{F_j}(X)$. Similarly to the proof of Proposition 6.1 by using (6.73) instead of (6.74), we obtain

$$\begin{aligned}&\mu_{T,\boldsymbol{F}}((y_1, \infty) \times \cdots \times (y_r, \infty))\\ &= \Xi_X(\boldsymbol{x})\Bigg\{\prod_{j=1}^{r}\int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}\\ &+ O_{\boldsymbol{F}}\left(\exp\left(C\left(\frac{\|\boldsymbol{V}\|}{\sqrt{\log\log X}}\right)^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\frac{\prod_{k=1}^{r}(1 + V_k)}{(\log\log X)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}}\prod_{j=1}^{r}\int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} + E\right)\Bigg\}\end{aligned}$$

for $0 \le V_j \le (\log\log X)^{2r}$, where

$$E = \exp\left(C\left(\frac{\|\boldsymbol{V}\|}{\sqrt{\log\log X}}\right)^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\frac{1}{\sqrt{\log\log X}} \times \prod_{j=1}^{r} \frac{e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2}}{\sigma_{F_j}(X)} \int_{x_j}^{y_j}\left(\frac{\tau}{x_j} - y_j\right) e^{-\tau} d\tau.$$

Here, $C = C(\boldsymbol{F})$ is a positive constant. Moreover, using (6.84) we have

$$E = \exp\left(C\left(\frac{\|\boldsymbol{V}\|}{\sqrt{\log\log X}}\right)^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\frac{1}{\sqrt{\log\log X}} \prod_{j=1}^{r} \frac{e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2 - x_j y_j}}{x_j\sigma_{F_j}(X)}$$

By estimate (6.83) and $x_j\sigma_{F_j}(X) \asymp 1 + V_j$, we can write

$$\frac{e^{\frac{x_j^2}{2}\sigma_{F_j}(X)^2 - x_j y_j}}{x_j\sigma_{F_j}(X)} \ll \frac{e^{-V_j^2/2}}{1+V_j} \ll \int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.$$

By this observation, we have

$$E \ll_r \exp\left(C\left(\frac{\|\boldsymbol{V}\|}{\sqrt{\log\log X}}\right)^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\frac{1}{\sqrt{\log\log X}} \prod_{j=1}^{r} \int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.$$

From the above estimations, we obtain

$$\begin{aligned}
&\mu_{T,\boldsymbol{F}}((y_1,\infty)\times\cdots\times(y_r,\infty)) \\
&= \Xi_X(\boldsymbol{x}) \prod_{j=1}^{r}\int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} \\
&\times \left\{1 + O_{\boldsymbol{F}}\left(\exp\left(C\left(\frac{\|\boldsymbol{V}\|}{\sqrt{\log\log X}}\right)^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\left\{\frac{\prod_{k=1}^{r}(1+V_k)}{(\log\log X)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}} + \frac{1}{\sqrt{\log\log X}}\right\}\right)\right\}.
\end{aligned}$$

In particular, by the definition of Ξ_X (6.18), assumptions (A1), (A2), and Lemma 6.10, it holds that

$$\Xi_X(\boldsymbol{x}) = \left(1 + O_{\boldsymbol{F}}\left(\frac{1}{\sqrt{\log\log X}}\right)\right)\Xi_X\left(\tfrac{V_1}{\sigma_{F_1}(X)}, \dots, \tfrac{V_r}{\sigma_{F_r}(X)}\right).$$

Thus, by these formulas and (6.71), we obtain Proposition 6.2 when $\|\boldsymbol{V}\| \le (\log\log X)^{2r}$. □

6.3.3 Proofs of a sharp error term of distribution functions

In this section, we prove Proposition 6.3. The proof and the proofs of some lemmas are written roughly because those many points are similar to the proofs of Proposition 6.1. When $\boldsymbol{F} = (F, F) \in (\mathcal{S}^\dagger \setminus \{1\})^2$, $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ satisfy (A2) (i.e. $|\theta_1 - \theta_2| = \pi/2$ in this case), we can write

$$K_{\boldsymbol{F},\boldsymbol{\theta}}(p, \boldsymbol{z}) = \sum_{j=1}^{2} z_j a_F(p) e^{-i\theta_j} \sum_{k=1}^{2} z_k \overline{a_F(p) e^{-i\theta_k}} = (z_1^2 + z_2^2)|a_F(p)|^2. \quad (6.87)$$

Thanks to this equation, we can improve formula (6.68) and Lemma 6.15 to the following lemmas. We omit the proofs of those because the lemmas can be shown similarly the proofs of formula (6.68) and Lemma 6.15 just by using equation (6.87).

Lemma 6.16. *Let $\boldsymbol{F} = (F, F) \in (\mathcal{S}^\dagger \setminus \{1\})^2$ and $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ satisfying (S4), (A1), and (A2). There exists a positive $b = b(F)$ such that for any $\boldsymbol{z} = (z_1, z_2) \in \mathbb{C}^2$ with $\|\boldsymbol{z}\| \le b$ we have*

$$\begin{aligned}&\prod_{p \le X} I_0\left(\sqrt{K_{F,\theta}(p, \boldsymbol{z})/p}\right)\\ &= \exp\left(\frac{z_1^2}{2}\sigma_F(X)^2 + O_F(|z_1|^4)\right) \exp\left(\frac{z_2^2}{2}\sigma_F(X)^2 + O_F(|z_2|^4)\right).\end{aligned}$$

Lemma 6.17. *Suppose that $\boldsymbol{F} = (F, F) \in (\mathcal{S}^\dagger \setminus \{1\})^2$ and $\boldsymbol{\theta} \in \mathbb{R}^2$ satisfy (S4), (A1), and (A2). Let c_1, c_2, d_1, d_2 be real numbers with $c_j < d_j$. Put $\mathscr{R} = (c_1, d_1) \times (c_2, d_2) \subset \mathbb{R}^2$. Let T, X be large numbers with $X^{(\log\log X)^{4(r+1)}} \le T$. Then, there exists a positive constant $b_2 = b_2(\boldsymbol{F})$ such that for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ with $\|\boldsymbol{x}\| \le b_2$, we have*

$$\nu_{T,\boldsymbol{F},\boldsymbol{x}}(\mathscr{R}) = e^{\frac{x_1^2+x_2^2}{2}\sigma_F(X)^2} \times \left\{\prod_{j=1}^{2} \int_{x_j\sigma_F(X) - \frac{d_j}{\sigma_F(X)}}^{x_j\sigma_F(X) - \frac{c_j}{\sigma_F(X)}} e^{-v^2/2} \frac{dv}{\sqrt{2\pi}} + E\right\},$$

where the error term E satisfies

$$E \ll_F \frac{1}{(\log\log X)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}} + \sum_{k=1}^{2} \left(\frac{x_k^4(d_k - c_k)}{\sigma_F(X)} + \frac{1}{\sigma_F(X)^4}\right) \prod_{\substack{h=1 \\ h \ne k}}^{2} \frac{d_h - c_h}{\sigma_F(X)}.$$

Proof of Proposition 6.3. Using Lemma 6.17, we can prove Proposition 6.3 in the same way as Proposition 6.1. □

Proof of Corollary 6.3. Let $0 < \varepsilon \le 1$ and $z = x + iy \in \mathbb{C}$. We may assume $|z| + 2 \le a_7\sigma_F(X)$, where a_7 is the same constant as in Theorem 6.3. and Proposition 6.3 in the case $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, we obtain

$$\begin{aligned}&\frac{1}{T}\operatorname{meas}\left\{t \in [T, 2T] \ : \ \|P_F(\tfrac{1}{2} + it, X) - z\| < \varepsilon\right\}\\ &= \int_{\frac{x-\varepsilon}{\sigma_F(X)}}^{\frac{x+\varepsilon}{\sigma_F(X)}} \int_{\frac{y-\varepsilon}{\sigma_F(X)}}^{\frac{y+\varepsilon}{\sigma_F(X)}} e^{-\frac{u^2+v^2}{2}} \frac{du dv}{2\pi} + O_F\left(\frac{1}{(\log\log X)^{\alpha_{\boldsymbol{F}}+1/2}} + \frac{1}{(\log\log X)^2}\right).\end{aligned}$$

Since we assume that $|a_F(p)| \ll_F p^{\vartheta_F}$ for some $\vartheta_F \in [0, 1/3)$, the inequality $\alpha_{\boldsymbol{F}} \geq 1/2 + c_0$ holds for a constant $0 < c_0 < 1$. Therefore, the above O-term is $\leq \frac{C}{(\log\log X)^{1+c_0}}$ for some $C = C(F) > 0$. Thus, we complete the proof of Corollary 6.3. □

6.4 Proofs of the unconditional results for moments of L-functions

Lemma 6.18. *Suppose the same situation as Proposition 6.5. Let $r \in \mathbb{Z}_{\geq 1}$ be given. There exists a positive constant $A_4 = A_4(F, r)$ such that for $X = T^{1/(\log\log T)^{4(r+1)}}$, $Y = T^{\delta_F/k}$, $k \in \mathbb{Z}_{\geq 1}$ with $k \leq \delta_F(\log\log T)^{4(r+1)}$,*

$$\frac{1}{T}\int_T^{2T} \left|\log F(\tfrac{1}{2}+it) - P_F(\tfrac{1}{2}+it, X) - \sum_{|\frac{1}{2}+it-\rho_F| \leq \frac{1}{\log Y}} \log\left((\tfrac{1}{2}+it-\rho_F)\log Y\right)\right|^{2k} dt$$

$$\leq A_4^k k^{2k} + A_4^k k!(\log_3 T)^k,$$

and

$$\frac{1}{T}\int_T^{2T} \left|\log F(\tfrac{1}{2}+it) - P_F(\tfrac{1}{2}+it, X)\right|^{2k} dt \leq A_4^k k^{4k} + A_4^k k!(\log_3 T)^k.$$

Proof. By Proposition 6.5, it suffices to show that

$$\sum_{X < p \leq Y^2} \frac{|a_F(p)|^2}{p} \ll_{F,r} \log_3 T, \tag{6.88}$$

and that

$$\frac{1}{T}\int_T^{2T} \left|\sum_{\substack{p^\ell \leq X \\ \ell \geq 2}} \frac{\Lambda_F(p^\ell)}{p^{\ell(1/2+it)}\log p^\ell}\right|^{2k} dt \leq C^k k! \tag{6.89}$$

for some constant $C = C(F) > 0$. where $Y = T^{\delta_F/k}$. Using formula (6.1), we find that

$$\sum_{X < p < Y^2} \frac{|a_F(p)|^2}{p} = n_F\left(\log\log Y^2 - \log\log X\right) + O_F(1) \ll_F \log_3 T.$$

Thus, we obtain estimate (6.88).

Next, we show estimate (6.89). Similarly to the proof of (6.31), we obtain

$$\sum_{\substack{p^\ell \leq X \\ \ell \geq 2}} \frac{\Lambda_F(p^\ell)}{p^{\ell(1/2+it)}\log p^\ell} = \sum_{2 \leq \ell \leq K_1} \sum_{p^\ell \leq X} \frac{\Lambda_F(p^\ell)}{p^{\ell(1/2+it)}\log p^\ell} + O_F(1),$$

where K_1 is the same constant as in Lemma 6.5. Therefore, we have

$$\frac{1}{T}\int_T^{2T}\Bigg|\sum_{\substack{p^\ell\le X\\ \ell\ge 2}}\frac{\Lambda_F(p^\ell)}{p^{\ell(1/2+it)}\log p^\ell}\Bigg|^{2k}dt$$
$$\le C_1^k\sum_{2\le\ell\le K_1}\frac{1}{T}\int_T^{2T}\Bigg|\sum_{p\le X^{1/\ell}}\frac{\Lambda_F(p^\ell)}{p^{\ell(1/2+it)}\log p^\ell}\Bigg|^{2k}dt+C_1^k$$

for some $C_1 = C_1(F) > 0$. Moreover, by Lemma 2.8, it holds that

$$\frac{1}{T}\int_T^{2T}\Bigg|\sum_{p\le X^{1/\ell}}\frac{\Lambda_F(p^\ell)}{p^{\ell(1/2+it)}\log p^\ell}\Bigg|^{2k}dt\ll \ell k!\left(\sum_{p\le X^{1/\ell}}\frac{|\Lambda_F(p^\ell)|^2}{p^\ell(\log p^\ell)^2}\right)^k.$$

Since $\sum_p\frac{|\Lambda_F(p^\ell)|^2}{p^\ell(\log p^\ell)^2}\ll 1$ holds by (6.30), we obtain

$$\frac{1}{T}\int_T^{2T}\Bigg|\sum_{\substack{p^\ell\le X\\ \ell\ge 2}}\frac{\Lambda_F(p^\ell)}{p^{\ell(1/2+it)}\log p^\ell}\Bigg|^{2k}dt\le C_2^k k!,$$

which completes the proof of (6.89). □

Proof of Theorem 6.1. We consider (6.3) and (6.4)–(6.6) separately.

Proof of (6.3). Let T be large. Put $X = T^{1/(\log\log T)^{4(r+1)}}$. Let $A\ge 1$ be a fixed arbitrary constant. Let the set $\mathcal{E}_j$ be

$$\mathcal{E}_j := \left\{t\in[T,2T]\ :\ \left|\log F_j(\tfrac12+it)-\sum_{p\le X}\frac{a_{F_j}(p)}{p^{1/2+it}}\right|\ge\mathcal{L}\right\}.$$

From Lemma 6.18, we have

$$\operatorname{meas}(\mathcal{E}_j)\ll T\mathcal{L}_j^{-2k}A_5^k(k^{4k}+k^k(\log_3 T)^k)$$

for all j with $A_5 := A_5(\boldsymbol{F}) = \max_{1\le j\le r} A_4(F_j,r)+1$, where $A_4(F_j,r)$ has the same meaning as in Lemma 6.18. Here the parameter $\mathcal{L}$ satisfying $\mathcal{L}\ge(2A_5\log_3 T)^{2/3}$ will be chosen later. Set $k=\lfloor\mathcal{L}^{1/2}/eA_5^{1/4}\rfloor$ so that $\operatorname{meas}(\mathcal{E}_j)\ll T\exp(-c_1\mathcal{L}^{1/2})$ for some $c_1>0$. Therefore except on the set $\mathcal{E} := \bigcup_{j=1}^r\mathcal{E}_j$ with measure $O_r(T\exp(-c_1\mathcal{L}^{1/2}))$, we have

$$\operatorname{Re} e^{-i\theta_j}\log F_j(\tfrac12+it)=\operatorname{Re} e^{-i\theta_j}P_{F_j}(\tfrac12+it,X)+\beta_j(t)\mathcal{L}\tag{6.90}$$

with $|\beta_j(t)|\le 1$ for all $j=1,\dots,r$. By (6.90) and Proposition 6.1, the measure of $t\in[T,2T]\backslash\mathcal{E}$ such that for all $j=1,\dots,r$

$$\operatorname{Re} e^{-i\theta_j}\log F_j(\tfrac12+it)\ge V_j\sqrt{\frac{n_{F_j}}{2}\log\log T}\tag{6.91}$$

is at least (since $\beta_j(t) \le 1$

$$T\left(1+O_{\boldsymbol{F}}\left(\frac{\prod_{j=1}^r(1+|V_k|+\frac{\mathcal{L}}{\sqrt{\log\log T}})}{(\log\log T)^{\alpha_F+\frac{1}{2}}}+\frac{1+\|\boldsymbol{V}\|^2+\frac{\mathcal{L}^2}{\log\log T}}{\log\log T}\right)\right) \times\prod_{j=1}^r\int_{\sigma_{F_j}(X)^{-1}(V_j\sqrt{(n_{F_j}/2)\log\log T}+\mathcal{L})}^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}} \tag{6.92}$$

for $\frac{\mathcal{L}}{\sqrt{\log\log T}}, \|\boldsymbol{V}\| \le c\sqrt{\log\log T}$ with c sufficiently small. Similarly, the measure of $t\in[T,2T]\backslash\mathcal{E}$ such that (6.91) holds is at most

$$T\left(1+O_{\boldsymbol{F}}\left(\frac{\prod_{k=1}^r(1+|V_k|+\frac{\mathcal{L}}{\sqrt{\log\log T}})}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}+\frac{1+\|\boldsymbol{V}\|^2+\frac{\mathcal{L}^2}{\log\log T}}{\log\log T}\right)\right) \times\prod_{j=1}^r\int_{\sigma_{F_j}(X)^{-1}(V_j\sqrt{(n_{F_j}/2)\log\log T}-\mathcal{L})}^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}} \tag{6.93}$$

for $\frac{\mathcal{L}}{\sqrt{\log\log T}}, \|\boldsymbol{V}\| \le c\sqrt{\log\log T}$. By using equation (6.1), we find that

$$\sigma_{F_j}(X)=\sqrt{\frac{n_{F_j}}{2}\log\log T}+O_{r,F_j}\left(\frac{\log_3 T}{\sqrt{\log\log T}}\right),$$

and so we also have

$$\sigma_{F_j}(X)^{-1}=\frac{1}{\sqrt{\frac{n_{F_j}}{2}\log\log T}}\left(1+O_{r,F_j}\left(\frac{\log_3 T}{\log\log T}\right)\right). \tag{6.94}$$

Therefore, when $|V_j|\le\sqrt{\frac{\log\log T}{\log_3 T}}$, $(|V_j|+1)\mathcal{L}\le B_1\sqrt{\log\log T}$ with $B_1>0$ a constant to be chosen later, we find that

$$\begin{aligned}
&\int_{\sigma_{F_j}(X)^{-1}(V_j\sqrt{(n_{F_j}/2)\log\log T}\pm\mathcal{L})}^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}}\\
&=\int_{V_j}^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}}+\int_{V_j+O_{r,F_j}(|V_j|\frac{\log_3 T}{\log\log T}+\frac{\mathcal{L}}{\sqrt{\log\log T}})}^{V_j} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}\\
&=\int_{V_j}^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}}+O_{r,F_j,B_1}\left(\left(\frac{|V_j|\log_3 T}{\log\log T}+\frac{\mathcal{L}}{\sqrt{\log\log T}}\right)e^{-V_j^2/2}\right)\\
&=\left(1+O_{r,F_j,B_1}\left(\frac{|V_j|(|V_j|+1)\log_3 T+\mathcal{L}(|V_j|+1)\sqrt{\log\log T}}{\log\log T}\right)\right)\int_{V_j}^\infty e^{-u^2/2}\frac{du}{\sqrt{2\pi}}.
\end{aligned}$$

Hence, choosing $\mathcal{L} = 2c_1^{-2}r^2(\|\boldsymbol{V}\|^4 + (\log_3 T)^2)$, we find that (6.92) and (6.93) become

$$T\left(1 + O_{\boldsymbol{F},B_1}\left(\frac{(\|\boldsymbol{V}\|^4 + (\log_3 T)^2)(\|\boldsymbol{V}\| + 1)}{\sqrt{\log\log T}} + \frac{\prod_{k=1}^r(1 + |V_k|)}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right)\right) \times \prod_{j=1}^r \int_{V_j}^{\infty} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}},$$

and

$$T\exp(-c_1\mathcal{L}^{1/2}) \ll T\exp(-r(\|\boldsymbol{V}\|^2 + \log_3 T)) \ll T\frac{1}{\log\log T}\prod_{j=1}^r \int_{V_j}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}$$

when $\|\boldsymbol{V}\| \le c_1^2 r^{-2} B_1(\log\log T)^{1/10}$. Choosing $B_1 = Ac_1^{-2}r^2$, we have

$$\frac{1}{T}\operatorname{meas}\left(\bigcap_{j=1}^r\left\{t \in [T, 2T] \ : \ \frac{\operatorname{Re} e^{-i\theta_j}\log F_j(1/2 + it)}{\sqrt{\frac{n_{F_j}}{2}\log\log T}} \ge V_j\right\}\right)$$
$$= \left(1 + O_{\boldsymbol{F},A}\left(\frac{(\|\boldsymbol{V}\|^4 + (\log_3 T)^2)(\|\boldsymbol{V}\| + 1)}{\sqrt{\log\log T}} + \frac{\prod_{k=1}^r(1 + |V_k|)}{(\log\log T)^{\alpha_F + 1/2}}\right)\right) \times \prod_{j=1}^r \int_{V_j}^{\infty} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}}$$

for $\|\boldsymbol{V}\| \le A(\log\log T)^{1/10}$. Thus, we complete the proof of (6.3). □

Proof of (6.4) *and* (6.6). Let $X = T^{1/(\log\log T)^{4(r+1)}}$ and let $\mathcal{B}_j$ be the set of $t \in [T, 2T]$ such that

$$\left|\log F_j(\tfrac{1}{2} + it) - P_{F_j}(\tfrac{1}{2} + it, X) - \sum_{|1/2+it-\rho_{F_j}| \le \frac{1}{\log Y_j}} \log((\tfrac{1}{2} + it - \rho_{F_j})\log Y)\right| \ge \mathcal{L}.$$

By Lemma 6.18, we know

$$\operatorname{meas}(\mathcal{B}_j) \le T\mathcal{L}^{-2k}A_5^k(k^{2k} + k^k(\log_3 T)^k),$$

where $A_5 = \max_{1\le j\le r} A_4(F_j, r) + 2$ and $A_4(F_j, r)$ has the same meaning as in Lemma 6.18. By taking $k = \lfloor \mathcal{L}/\sqrt{A_5 e}\rfloor$, we have that $\operatorname{meas}(\mathcal{B}_j) \le T\exp(-c_2\mathcal{L})$ for some $c_2 > 0$ as long as $\mathcal{L} \ge 2A_5\log_3 T$. Therefore, it follows that for $t \in [T, 2T] \setminus \bigcup_{j=1}^r \mathcal{B}_j$

$$\operatorname{Re} e^{-i\theta_j}\log F_j(1/2 + it)$$
$$= \operatorname{Re} e^{-i\theta_j}P_{F_j}(\tfrac{1}{2} + it, X) + \sum_{|1/2+it-\rho_{F_j}| \le \frac{1}{\log Y_j}} \operatorname{Re} e^{-i\theta_j}\log((\tfrac{1}{2} + it - \rho_{F_j})\log Y_j) + \beta_j(t)\mathcal{L}$$

holds for all $1 \le j \le r$ with some $|\beta_j(t)| \le 1$. Let C_j be the set of $t \in [T, 2T]$ such that

$$\sum_{|1/2-it-\rho_{F_j}|\le\frac{1}{\log Y_j}} \operatorname{Re} e^{-\theta_j} \log((\tfrac{1}{2}+it-\rho_{F_j})\log Y_j) \ge \mathcal{L}.$$

When $\theta_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $|1/2+it-\rho_{F_j}| \le \frac{1}{\log Y_j}$, we find that

$$\begin{aligned}&\operatorname{Re} e^{-i\theta_j} \log((\tfrac{1}{2}+it-\rho_{F_j})\log Y_j)\\ &= \cos\theta_j \log|(\tfrac{1}{2}+it-\rho_{F_j})\log Y_j| + \sin\theta_j \arg((\tfrac{1}{2}+it-\rho_{F_j})\log Y_j) \le \pi.\end{aligned}$$

Hence, we have

$$\sum_{|1/2+it-\rho_{F_j}|\le\frac{1}{\log Y_j}} \operatorname{Re} e^{-i\theta_j} \log((\tfrac{1}{2}+it-\rho_{F_j})\log Y_j) \le \pi \sum_{|1/2+it-\rho_{F_j}|\le\frac{1}{\log Y_j}} 1,$$

and thus by Lemma 6.6

$$\operatorname{meas}(C_j) \le C^k k^{2k} T \mathcal{L}^{-2k}$$

for some constant $C = C(F_j) > 0$. By choosing $k = \lfloor \mathcal{L}/\sqrt{Ce} \rfloor$, we have $\operatorname{meas}(C_j) \le T\exp(-c_3\mathcal{L})$ for some $c_3 > 0$. Now we have that the measure of $t \in [T, 2T] \setminus \bigcup_{j=1}^r (\mathcal{B}_j \cup C_j)$ such that

$$\operatorname{Re} e^{-\theta_j} \log F_j(\tfrac{1}{2}+it) \ge V_j \sqrt{\frac{n_{F_j}}{2} \log\log T}$$

is bounded by the measure of the set $t \in [T, 2T]$ such that

$$\operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2}+it, X) \ge V_j \sqrt{\frac{n_{F_j}}{2} \log\log T} - 2\mathcal{L}. \tag{6.95}$$

From Proposition 6.1, we know (6.95) holds with measure

$$\begin{aligned}T\left(1 + O_{\boldsymbol{F}}\left(\frac{\prod_{k=1}^r (1+|V_k| + \frac{\mathcal{L}}{\sqrt{\log\log T}})}{(\log\log T)^{\alpha_F+\frac{1}{2}}} + \frac{1+\|\boldsymbol{V}\|^2 + \frac{\mathcal{L}^2}{\log\log T}}{\log\log T}\right)\right)&\\ \times \prod_{j=1}^r \int_{\sigma_{F_j}(X)^{-1}(V_j\sqrt{(n_{F_j}/2)\log\log T}-2\mathcal{L})}^{\infty} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}&\end{aligned} \tag{6.96}$$

for $\frac{\mathcal{L}}{\sqrt{\log\log T}}, \|\boldsymbol{V}\| \le c\sqrt{\log\log T}$ with c sufficiently small. Choosing $\mathcal{L} = c_4^{-1} r(\|\boldsymbol{V}\|^2 + 2A_5 \log_3 T)$, we see that (6.96) becomes

$$\begin{aligned}T\left(1 + O_{\boldsymbol{F},A}\left(\frac{(\|\boldsymbol{V}\|^2+\log_3 T)(\|\boldsymbol{V}\|+1)}{\sqrt{\log\log T}} + \frac{\prod_{k=1}^r(1+|V_k|)}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right)\right)&\\ \times \prod_{j=1}^r \int_{V_j}^{\infty} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}}&\end{aligned}$$

for $\|\boldsymbol{V}\| \le A(\log\log T)^{1/6}$. This completes the proof of (6.4).

The proof of (6.6) is similar by noting that when $\theta_j \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and $|\frac{1}{2} + it - \rho_{F_j}| \le \frac{1}{\log Y_j}$,

$$\begin{aligned}&\operatorname{Re} e^{-i\theta_j} \log((\tfrac{1}{2} + it - \rho_{F_j}) \log Y_j) \\ &= \cos\theta_j \log|(\tfrac{1}{2} + it - \rho_{F_j}) \log Y_j| + \sin\theta_j \arg((\tfrac{1}{2} + it - \rho_{F_j}) \log Y_j) \ge -\pi,\end{aligned}$$

and thus the set of $t \in [T, 2T]$ such that

$$\sum_{|1/2+it-\rho_{F_j}|\le \frac{1}{\log Y_j}} \operatorname{Re} e^{-\theta_j} \log((\tfrac{1}{2} + it - \rho_{F_j}) \log Y_j) \le -\mathcal{L}$$

has measure bounded by $C^k k^{2k} T \mathcal{L}^{-2k}$ for some constant $C = C(F_j)$. □

Proof of Theorem 6.2. Let $X = T^{1/(\log\log T)^{4(r+1)}}$. Let $a_1 = a_1(\boldsymbol{F}) > 0$ be a sufficiently small constant to be chosen later, Let $\boldsymbol{V} = (V_1, \dots, V_r) \in (\mathbb{R}_{\ge 0})^r$ such that $\|\boldsymbol{V}\| \le a_1(1 + V_m^{1/2})(\log\log T)^{1/4}$ with $V_m := \min_{1\le j\le r} V_j$.

We consider the case when $\boldsymbol{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]^r$ first. Similarly to the proof of (6.4) (see (6.95)), we find that the measure of the set of $t \in [T, 2T]$ except for a set of measure $T\exp(-c_4\mathcal{L})$ ($\mathcal{L} \gg \log_3 T$) such that $\operatorname{Re} e^{-i\theta_j} \log F_j\left(\frac{1}{2} + it\right) \ge V_j\sqrt{\frac{n_{F_j}}{2}\log\log T}$ is at most the measure of the set $t \in [T, 2T]$ such that

$$\operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2} + it, X) \ge V_j\sqrt{\frac{n_{F_j}}{2}\log\log T} - 2\mathcal{L}.$$

From Proposition 6.1, the measure of $t \in [T, 2T]$ satisfying this inequality for all $j = 1, \dots, r$ is equal to

$$\begin{aligned}T&\left(1 + O_{\boldsymbol{F}}\left(\frac{\prod_{k=1}^r (1 + V_k + \frac{\mathcal{L}}{\sqrt{\log\log T}})}{(\log\log T)^{\alpha_F + \frac{1}{2}}} + \frac{1 + \|\boldsymbol{V}\|^2 + \frac{\mathcal{L}^2}{\log\log T}}{\log\log T}\right)\right) \\ &\qquad\qquad \times \prod_{j=1}^r \int_{\sigma_{F_j}(X)^{-1}(V_j\sqrt{(n_{F_j}/2)\log\log T} - 2\mathcal{L})}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} \qquad (6.97)\end{aligned}$$

for $\frac{\mathcal{L}}{\sqrt{\log\log T}}, \|\boldsymbol{V}\| \le c\sqrt{\log\log T}$ with c sufficiently small.

Now, we choose $\mathcal{L} = 2rc_4^{-1}\|\boldsymbol{V}\|^2 + \log_3 T$ and a_1 small enough so that we have the inequalities $\|\boldsymbol{V}\| \le a_9\sqrt{\log\log T}$ and $4\mathcal{L} \le (1 + V_j)\sqrt{\frac{n_{F_j}}{2}\log\log T}$ for all $j = 1, \dots, r$, where a_9 is the same constant as in Proposition 6.1. Then, by equation (6.94) and the estimate $\int_V^\infty e^{-u^2/2} du \ll \frac{1}{1+V} e^{-V^2/2}$ for $V \ge 0$, we obtain

$$\int_{\sigma_{F_j}(X)^{-1}(V_j\sqrt{(n_{F_j}/2)\log\log T}-2\mathcal{L})}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}$$
$$\ll_{r,F_j} \frac{1}{1+V_j}\exp\left(-\frac{1}{2}\left(V_j + O_{r,F_j}\left(\frac{\mathcal{L}}{\sqrt{\log\log T}} + V_j\frac{\log_3 T}{\log\log T}\right)\right)^2\right)$$
$$\ll_{\boldsymbol{F}} \frac{1}{1+V_j}\exp\left(-\frac{V_j^2}{2} + O_{\boldsymbol{F}}\left(\frac{V_j\|\boldsymbol{V}\|^2}{\sqrt{\log\log T}} + \frac{\|\boldsymbol{V}\|^4}{\log\log T}\right)\right).$$

Hence, when $0 \le V_1, \dots, V_r \le a\sqrt{\log\log T}$ with a sufficiently small, (6.97) is

$$\ll_{\boldsymbol{F}} T\left\{\left(\prod_{j=1}^{r}\frac{1}{1+V_j}\right) + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right\}$$
$$\times\exp\left(-\frac{V_1^2+\dots+V_r^2}{2} + O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|^3}{\sqrt{\log\log T}}\right)\right).$$

Moreover, we have

$$T\exp(-c_4\mathcal{L}) \le T\exp(-2r\|\boldsymbol{V}\|^2) \le T\prod_{j=1}^{r}\exp\left(-2(V_1^2+\dots+V_r^2)\right)$$
$$\ll T\prod_{j=1}^{r}\frac{1}{1+V_j}\exp\left(-\frac{V_j^2}{2}\right).$$

Similarly when $\boldsymbol{\theta} \in [\frac{\pi}{2}, \frac{3\pi}{2}]^r$, except for a set of measure $T\exp(-c_4\mathcal{L})$ ($\mathcal{L} \gg \log_3 T$), the measure of $t \in [T, 2T]$ such that

$$\operatorname{Re} e^{-i\theta_j}\log F_j(\tfrac{1}{2}+it) \ge V_j\sqrt{\tfrac{n_{F_j}}{2}\log\log T}$$

is at least the measure of $t \in [T, 2T]$ such that

$$\operatorname{Re} e^{-i\theta_j}P_{F_j}(\tfrac{1}{2}+it) \ge V_j\sqrt{\tfrac{n_{F_j}}{2}\log\log T} + 2\mathcal{L}$$

When $4\mathcal{L} \le V_j\sqrt{\frac{n_{F_j}}{2}\log\log T}$, we have the measure of t satisfying the above inequality for all $j = 1, \dots, r$ is (by Proposition 6.1)

$$T\left(1 + O_{\boldsymbol{F}}\left(\frac{\prod_{k=1}^{r}(1+V_k+\frac{\mathcal{L}}{\sqrt{\log\log T}})}{(\log\log T)^{\alpha_F+\frac{1}{2}}} + \frac{1+\|\boldsymbol{V}\|^2+\frac{\mathcal{L}^2}{\log\log T}}{\log\log T}\right)\right)$$
$$\times\prod_{j=1}^{r}\int_{\sigma_{F_j}(X)^{-1}(V_j\sqrt{(n_{F_j}/2)\log\log T}+2\mathcal{L})}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}, \quad (6.98)$$

which can be bounded by

$$\gg_{\boldsymbol{F}} T\left(\prod_{j=1}^{r}\frac{1}{1+V_j}\right)\exp\left(-\frac{V_1^2+\cdots+V_r^2}{2}+O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|\mathcal{L}}{\sqrt{\log\log T}}\right)\right)$$

when $\prod_{j=1}^{r}(1+V_j) \le c(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}$ with $c = c(\boldsymbol{F}) > 0$ a suitably small constant. Choose $\mathcal{L} = 2r\|\boldsymbol{V}\|^2 + \log_3 T$ and a_1 small enough so that $\|\boldsymbol{V}\| \le a_6\sqrt{\log\log T}$ and $4\mathcal{L} \le V_j\sqrt{\frac{n_{F_j}}{2}\log\log T}$ hold for all $j = 1, \dots, r$, where a_6 is the same constant as in Proposition 6.1. Then (6.98) is

$$\gg_{\boldsymbol{F}} T\left(\prod_{j=1}^{r}\frac{1}{1+V_j}\right)\exp\left(-\frac{V_1^2+\cdots+V_r^2}{2}+O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|^3}{\sqrt{\log\log T}}\right)\right),$$

which completes the proof of Theorem 6.2. □

We prepare a lemma to prove Theorem 6.3.

Lemma 6.19. *Let $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $F \in \mathcal{S}^\dagger$ satisfying* (6.1) *and (A3). There exist positive constants $a_{11} = a_{11}(F)$, $a_{12} = a_{12}(F)$ such that for any large V,*

$$\begin{aligned}&\frac{1}{T}\operatorname{meas}\left\{t \in [T, 2T] \ : \ \operatorname{Re} e^{-i\theta}\log F(\tfrac{1}{2}+it) > V\right\}\\ &\le \exp\left(-a_{11}\frac{V^2}{\log\log T}\right) + \exp\left(-a_{12}V\right).\end{aligned}$$

Proof. We can show that, for $t \in [T, 2T]$, the inequality $\operatorname{Re} e^{-i\theta}\log F(1/2+it) \le C_1\log T$ with $C_1 = C_1(F) > 0$ a suitably large constant by using Theorem 6.6 in the case $X = 3$, $H = 1$ and estimate (6.26). Hence, this lemma holds when $V \ge C_1 \log T$ with $C_1 = C_1(F) > 0$. In the following, we consider the case $V \le C_1\log T$. Similarly to the proof of Lemma 6.18, we obtain

$$\int_T^{2T}\left|\log F(\tfrac{1}{2}+it) - P_F(\tfrac{1}{2}+it, X) - \sum_{|\frac{1}{2}+it-\rho_F|\le\frac{1}{\log X}}\log\left((\tfrac{1}{2}+it-\rho_F)\log X\right)\right|^{2k}dt$$
$$\le TA_4^k k^{2k} + TA_4^k k!(\log\log T)^k$$

for $X = T^{\delta_F/k}$. Additionally, by using Lemma 2.8 and Lemma 6.6, we obtain

$$\int_T^{2T}|P_F(\tfrac{1}{2}+it, X)|^{2k}dt \le T\left(Ck\log\log T\right)^k,$$

and

$$\int_T^{2T}\left(\sum_{|\frac{1}{2}+it-\rho_F|\le\frac{1}{\log Y}}1\right)^{2k}dt \le TC^k k^{2k}.$$

When $V \le \log\log T$, we choose $k = \lfloor cV^2/\log\log T \rfloor$, and when $V \ge \log\log T$, we choose $k = \lfloor cV \rfloor$. Here, c is a suitably small constant depending only on F. Then, by the above inequalities and $\operatorname{Re} e^{-i\theta} \log\left((\frac{1}{2} + it - \rho_F) \log X\right) \le \pi$, we obtain

$$\frac{1}{T} \operatorname{meas}\left\{t \in [T, 2T] \ : \ \operatorname{Re} e^{-i\theta} \log F(\tfrac{1}{2} + it) > V\right\}$$
$$\le \exp\left(-c_5 \frac{V^2}{\log\log T}\right) + \exp\left(-c_6 V\right),$$

which completes the proof of Lemma 6.19. □

Proof of Theorem 6.3. Let $0 \le k \le a_3$ with $a_3 > 0$ suitably small to be chosen later. Put $\phi_{\boldsymbol{F}}(t) = \min_{1 \le j \le r} \operatorname{Re} e^{-i\theta_j} \log F_j(1/2 + it)$ and

$$\Phi_{\boldsymbol{F}}(T, V) := \operatorname{meas}\left\{t \in [T, 2T] \ : \ \phi_{\boldsymbol{F}}(t) > V\right\}.$$

Then we have

$$\int_T^{2T} \exp\left(2k\phi_{\boldsymbol{F}}(t)\right) dt = \int_{-\infty}^{\infty} 2k e^{2kV} \Phi_{\boldsymbol{F}}(T, V) dV. \tag{6.99}$$

We consider the case when $\boldsymbol{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]^r$ first. From Theorem 6.2, it follows that, for any $0 \le V \le a_{13} \log\log T$ with $a_{13} = a_{13}(\boldsymbol{F})$ a suitably small constant,

$$\begin{aligned} &\Phi_{\boldsymbol{F}}(T, V) \qquad\qquad (6.100)\\ &\ll_{\boldsymbol{F}} T \left(\frac{1}{1 + (V/\sqrt{\log\log T})^r} + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}} + \frac{1}{2}}} \right) \\ &\qquad\qquad \times \exp\left(-h_{\boldsymbol{F}} \frac{V^2}{\log\log T} + C_1 \frac{V^3}{(\log\log T)^2}\right) \end{aligned}$$

for some constant $C_1 = C_1(\boldsymbol{F}) > 0$. Moreover, by Lemma 6.19, it holds that

$$\Phi_{\boldsymbol{F}}(T, V) \le T \exp\left(-a_{11} \frac{V^2}{\log\log T}\right) + T \exp\left(-a_{12} V\right) \tag{6.101}$$

for any large V. Now we choose $a_3 = \min\{a_{11} a_{13}/4, a_{12}/4\}$. Put $D_1 = 4a_{11}^{-1}$. We divide the integral on the right hand side of (6.99) to

$$\left(\int_{-\infty}^{0} + \int_{0}^{D_1 k \log\log T} + \int_{D_1 k \log\log T}^{\infty} \right) 2k e^{2kV} \Phi_{\boldsymbol{F}}(T, V) dV =: I_1 + I_2 + I_3,$$

say. We use the trivial bound $\Phi_{\boldsymbol{F}}(T, V) \le T$ to obtain $I_1 \le T$. Also, by inequality (6.101), it follows that

$$\begin{aligned} I_3 &\le T \int_{D_1 k \log\log T}^{\infty} 2k \left\{ \exp\left(\left(-a_7 \frac{V}{\log\log T} + 2k \right) V \right) + e^{(-a_8 + 2k)V} \right\} dV \\ &\le T \int_0^{\infty} 4k e^{-2kV} dV \le 2T. \end{aligned}$$

Moreover, using inequality (6.100), we find that

$$
\begin{aligned}
I_2 &\ll_{\boldsymbol{F}} T\int_0^{D_1 k\log\log T}(E_1+E_2)\exp\left(2kV-h_{\boldsymbol{F}}\frac{V^2}{\log\log T}+C_1\frac{V^3}{(\log\log T)^2}\right)dV\\
&\ll T(\log T)^{k^2/h_{\boldsymbol{F}}+C_1D_1^3k^3}\\
&\qquad\times\int_0^{D_1 k\log\log T}(E_1+E_2)\exp\left(-\frac{h_{\boldsymbol{F}}}{\log\log T}\left(V-\frac{k}{h_{\boldsymbol{F}}}\log\log T\right)^2\right)dV,
\end{aligned}
$$

where $E_1=\frac{k}{1+(V/\sqrt{\log\log T})^r}$ and $E_2=\frac{k}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}$. We see that

$$
\begin{aligned}
&\int_0^{D_1 k\log\log T}E_2\exp\left(-\frac{h_{\boldsymbol{F}}}{\log\log T}\left(V-\frac{k}{h_{\boldsymbol{F}}}\log\log T\right)^2\right)dV\\
&\le\frac{k}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{h_{\boldsymbol{F}}}{\log\log T}V^2\right)dV\ll_{\boldsymbol{F}}\frac{k}{(\log\log T)^{\alpha_{\boldsymbol{F}}}}.
\end{aligned}
$$

Also, we write

$$
\begin{aligned}
&\int_0^{D_1 k\log\log T}E_1\exp\left(-\frac{h_{\boldsymbol{F}}}{\log\log T}\left(V-\frac{k}{h_{\boldsymbol{F}}}\log\log T\right)^2\right)dV\\
&=\left(\int_0^{\frac{k}{2h_{\boldsymbol{F}}}\log\log T}+\int_{\frac{k}{2h_{\boldsymbol{F}}}\log\log T}^{D_1 k\log\log T}\right)k\frac{\exp\left(-\frac{h_{\boldsymbol{F}}}{\log\log T}\left(V-\frac{k}{h_{\boldsymbol{F}}}\log\log T\right)^2\right)}{1+(V/\sqrt{\log\log T})^r}dV\\
&=: I_{2,1}+I_{2,2},
\end{aligned}
$$

say. We find that

$$
\begin{aligned}
I_{2,2}&\ll_{\boldsymbol{F}}\frac{k}{1+(k\sqrt{\log\log T})^r}\int_{-\infty}^{\infty}\exp\left(-\frac{h_{\boldsymbol{F}}}{\log\log T}V^2\right)dV\\
&\ll_{\boldsymbol{F}}\frac{k\sqrt{\log\log T}}{1+(k\sqrt{\log\log T})^r},
\end{aligned}
$$

and that

$$
\begin{aligned}
I_{2,1}&\le\int_{\frac{k}{2h_{\boldsymbol{F}}}\log\log T}^{\frac{k}{h_{\boldsymbol{F}}}\log\log T}k\exp\left(-\frac{h_{\boldsymbol{F}}}{\log\log T}V^2\right)dV\\
&\le\sqrt{\frac{\log\log T}{h_{\boldsymbol{F}}}}\int_{\frac{k}{2\sqrt{h_{\boldsymbol{F}}}}\sqrt{\log\log T}}^{\infty}ke^{-u^2}du.
\end{aligned}
$$

If $k\le(\log\log T)^{-1/2}$, the last is clearly $\ll_{\boldsymbol{F}}1$. If $k\ge(\log\log T)^{-1/2}$, we use the estimate $\int_x^\infty e^{-u^2}du\ll x^{-1}e^{-x^2}$ to obtain

$$
I_{2,1}\le\sqrt{\frac{\log\log T}{h_{\boldsymbol{F}}}}\int_{\frac{k}{2\sqrt{h_{\boldsymbol{F}}}}\sqrt{\log\log T}}^{\infty}ke^{-u^2}du\ll 1.
$$

Hence, we obtain

$$I_2 \tag{6.102}$$
$$\ll_{\boldsymbol{F}} T + kT(\log T)^{k^2/h_{\boldsymbol{F}}+C_1D_1^3k^3}\left(\frac{\sqrt{\log\log T}}{1+(k\sqrt{\log\log T})^r} + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right).$$

Combing this estimate and the estimates for I_1, I_3, we complete the proof of (6.8).

Next, we consider the case $\boldsymbol{\theta} \in [\frac{\pi}{2}, \frac{3\pi}{2}]^r$. By equation (6.99), estimate (6.7), and positivity of $\Phi_{\boldsymbol{F}}$, we have

$$\begin{aligned}&\int_T^{2T} \exp\left(2k\phi_{\boldsymbol{F}}(t)\right) dt\\ &\geq \int_0^1 2ke^{2kV}\Phi_{\boldsymbol{F}}(T,V)dV + \int_{\frac{k}{h_{\boldsymbol{F}}}\log\log T}^{\frac{k}{h_{\boldsymbol{F}}}\log\log T+\sqrt{\log\log T}} 2ke^{2kV}\Phi_{\boldsymbol{F}}(T,V)dV.\end{aligned}$$

By estimate (6.7), the first integral on the right hand side is $\gg_{\boldsymbol{F}} T$, and the second integral on the right hand side is

$$\begin{aligned}&\gg_{\boldsymbol{F}} \frac{kT}{1+(k\sqrt{\log\log T})^r}\\ &\quad\times\int_{\frac{k}{h_{\boldsymbol{F}}}\log\log T}^{\frac{k}{h_{\boldsymbol{F}}}\log\log T+\sqrt{\log\log T}} \exp\left(2kV - h_{\boldsymbol{F}}\frac{V^2}{\log\log T} - C_1\frac{V^3}{(\log\log T)^2}\right)dV\\ &\geq \frac{kT(\log T)^{\frac{k^2}{h_{\boldsymbol{F}}}-C_2k^3}}{1+(k\sqrt{\log\log T})^r}\\ &\qquad\times\int_{\frac{k}{h_{\boldsymbol{F}}}\log\log T}^{\frac{k}{h_{\boldsymbol{F}}}\log\log T+\sqrt{\log\log T}} \exp\left(-\frac{h_{\boldsymbol{F}}}{\log\log T}\left(V-\frac{k}{h_{\boldsymbol{F}}}\log\log T\right)^2\right)\\ &\gg_{\boldsymbol{F}} kT(\log T)^{\frac{k^2}{h_{\boldsymbol{F}}}-C_2k^3}\frac{\sqrt{\log\log T}}{1+(k\sqrt{\log\log T})^r},\end{aligned}$$

where $C_2 \geq 0$ is some constant depending on $\boldsymbol{F}$. Hence, we also obtain Theorem 6.3 in the case $\boldsymbol{\theta} \in [\frac{\pi}{2}, \frac{3\pi}{2}]^r$. □

6.5 Proofs of the conditional results for moments of L-functions

Proof of Theorem 6.4. Let $\boldsymbol{F} \in (\mathcal{S}^\dagger)^r$ and $\boldsymbol{\theta} \in [-\frac{\pi}{2}, \frac{3\pi}{2}]^r$ satisfying $\mathscr{A}$. Let T be a sufficiently large constant depending on $\boldsymbol{F}$. Set $Y = T^{K_1/\mathcal{L}}$ where $K_1 = K_1(\boldsymbol{F}) > 0$ is a suitably large constant and $\mathcal{L} \geq (\log_3 T)^2$ is a large parameter to be chosen later. Let f be a fixed function satisfying the condition of this paper (see Notation) and $D(f) \geq 2$. Assuming the Riemann Hypothesis for $F_1, \ldots, F_r$, we apply Theorem 6.6 with $X = Y$, $H = 1$ to obtain

$$\log F_j(\tfrac{1}{2} + it) = \sum_{2 \leq n \leq Y^2} \frac{\Lambda_{F_j}(n) v_{f,1}(e^{\log n/\log Y})}{n^{1/2+it} \log n} + \sum_{|1/2+it-\rho_{F_j}| \leq \frac{1}{\log Y}} \log((\tfrac{1}{2} + it - \rho_{F_j}) \log Y) + R_{F_j}(\tfrac{1}{2} + it, Y, 1),$$

where

$$\left|R_{F_j}(\tfrac{1}{2} + it, Y, 1)\right| \leq C_0 \left(\frac{1}{\log Y} \left| \sum_{n \leq Y^3} \frac{\Lambda_{F_j}(n) w_Y(n)}{n^{\frac{1}{2} + \frac{4}{\log Y} + it}} \right| + \frac{d_{F_j} \log T}{\log Y} \right)$$

for any $t \in [T, 2T]$. Here C_0 is a positive constant depending only on f. Moreover, when $\theta_j \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it holds that

$$\operatorname{Re} e^{-i\theta_j} \sum_{|1/2+it-\rho_{F_j}| \leq \frac{1}{\log Y}} \log((\tfrac{1}{2} + it - \rho_{F_j}) \log Y) \leq \pi \sum_{|1/2+it-\rho_{F_j}| \leq \frac{1}{\log Y}} 1,$$

and when $\theta_j \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, it holds that

$$\operatorname{Re} e^{-i\theta_j} \sum_{|1/2+it-\rho_{F_j}| \leq \frac{1}{\log Y}} \log((\tfrac{1}{2} + it - \rho_{F_j}) \log Y) \geq -\pi \sum_{|1/2+it-\rho_{F_j}| \leq \frac{1}{\log Y}} 1.$$

Hence, there exists some positive constant $C_1 > 0$ such that we have (by (6.39)),

$$\begin{aligned} &\operatorname{Re} e^{-i\theta_j} \sum_{|1/2+it-\rho_{F_j}| \leq \frac{1}{\log Y}} \log((\tfrac{1}{2} + it - \rho_{F_j}) \log Y) \\ &\leq C_1 \left(\frac{1}{\log Y} \left| \sum_{n \leq Y^3} \frac{\Lambda_{F_j}(n) w_Y(n)}{n^{\frac{1}{2} + \frac{4}{\log Y} + it}} \right| + \frac{d_{F_j} \log T}{\log Y} \right) \end{aligned}$$

when $\theta_j \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and

$$\begin{aligned} &\operatorname{Re} e^{-i\theta_j} \sum_{|1/2+it-\rho_{F_j}| \leq \frac{1}{\log Y}} \log((\tfrac{1}{2} + it - \rho_{F_j}) \log Y) \\ &\geq -C_1 \left(\frac{1}{\log Y} \left| \sum_{n \leq Y^3} \frac{\Lambda_{F_j}(n) w_Y(n)}{n^{\frac{1}{2} + \frac{4}{\log Y} + it}} \right| + \frac{d_{F_j} \log T}{\log Y} \right) \end{aligned}$$

when $\theta_j \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Taking $K_1 = 2(C_0 + C_1) \max_{1 \leq j \leq r} d_{F_j}$, we find that there exists some positive constant C_2 depending on f such that for any $t \in [T, 2T]$

and all $j = 1, \dots, r$,

$$\operatorname{Re} e^{-i\theta_j} \log F_j(\tfrac{1}{2} + it) \le \operatorname{Re} e^{-i\theta_j} \sum_{p \le Y^2} \frac{a_{F_j}(p) v_{f,1}(e^{\log p/\log Y})}{p^{1/2+it}} \tag{6.103}$$
$$+ \Bigg| \sum_{\substack{p^\ell \le Y^2 \\ \ell \ge 2}} \frac{\Lambda_{F_j}(p^\ell) v_{f,1}(e^{\log p^\ell/\log Y})}{p^{\ell(1/2+it)} \log p^\ell} \Bigg|$$
$$+ \frac{C_2}{\log Y} \Bigg| \sum_{n \le Y^3} \frac{\Lambda_{F_j}(n) w_Y(n)}{n^{\frac{1}{2} + \frac{4}{\log Y} + it}} \Bigg| + \frac{\mathcal{L}}{2}$$

when $\theta_j \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and

$$\operatorname{Re} e^{-i\theta_j} \log F_j(\tfrac{1}{2} + it) \ge \operatorname{Re} e^{-i\theta_j} \sum_{p \le Y^2} \frac{a_{F_j}(p) v_{f,1}(e^{\log p/\log Y})}{p^{1/2+it}}$$
$$- \Bigg| \sum_{\substack{p^\ell \le Y^2 \\ \ell \ge 2}} \frac{\Lambda_{F_j}(p^\ell) v_{f,1}(e^{\log p^\ell/\log Y})}{p^{\ell(1/2+it)} \log p^\ell} \Bigg|$$
$$- \frac{C_2}{\log Y} \Bigg| \sum_{n \le Y^3} \frac{\Lambda_{F_j}(n) w_Y(n)}{n^{\frac{1}{2} + \frac{4}{\log Y} + it}} \Bigg| - \frac{\mathcal{L}}{2}$$

when $\theta_j \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Put $X = Y^{1/(\log\log T)^{4(r+1)}}$. By Lemma 2.8 and assumption (A1), we obtain

$$\int_T^{2T} \Bigg| \sum_{X < p \le Y^2} \frac{a_{F_j}(p) v_{f,1}(e^{\log p/\log Y})}{p^{1/2+it}} \Bigg|^{2k} dt \ll T k! \Bigg(\sum_{X < p \le Y^2} \frac{|a_{F_j}(p)|^2}{p} \Bigg)^k$$
$$\le T C_3^k k^k \left(\log_3 T\right)^k$$

for some constant $C_3 = C_3(F_j, r) > 0$. Similarly to the proofs of estimates (6.44) and (6.45), we can show that for any integer k with $1 \le k \le \mathcal{L}/4K_1$

$$\int_T^{2T} \Bigg| \sum_{\substack{p^\ell \le Y^2 \\ \ell \ge 2}} \frac{\Lambda_{F_j}(p^\ell) v_{f,1}(e^{\log p^\ell/\log Y})}{p^{\ell(1/2+it)} \log p^\ell} \Bigg|^{2k} dt \le T C_4^k k^k$$

for some constant $C_4 = C_4(F_j) > 0$. Moreover, by Lemma 6.5, we have

$$\int_T^{2T} \left(\frac{C_1}{\log Y} \Bigg| \sum_{n \le Y^3} \frac{\Lambda_{F_j}(n) w_Y(n)}{n^{\frac{1}{2} + \frac{4}{\log Y} + it}} \Bigg| \right)^{2k} dt \le T C_5^k k^k$$

for any integer k with $1 \le k \le \mathcal{L}/4K_1$ and for some constant $C_5 = C_5(F_j) > 0$. Here the assumptions in Lemma 6.5 is satisfied as we can take κ_F arbitrarily large. Therefore, the set of $t \in [T, 2T]$ such that for all $j = 1, \dots, r$,

$$\frac{\mathcal{L}}{2} \le \Bigg| \sum_{X < p \le Y^2} \frac{a_{F_j}(p) v_{f,1}(e^{\log p/\log Y})}{p^{1/2+it}} \Bigg| + \Bigg| \sum_{\substack{p^\ell \le Y^2 \\ \ell \ge 2}} \frac{\Lambda_{F_j}(p^\ell) v_{f,1}(e^{\log p^\ell/\log Y})}{p^{\ell(1/2+it)} \log p^\ell} \Bigg| + \frac{C_2}{\log Y} \Bigg| \sum_{n \le Y^3} \frac{\Lambda_{F_j}(n) w_Y(n)}{n^{\frac{1}{2} + \frac{4}{\log Y} + it}} \Bigg|$$

has a measure bounded by $T\mathcal{L}^{-2k} C_6^k k^k \left(\log_3 T\right)^k$ with $C_6 = C_6(\boldsymbol{F}) > 0$ a suitably large constant. Choosing $k = \lfloor c_1 \mathcal{L} \rfloor$ with c_1 suitably small depending only on $\boldsymbol{F}$, we find that there exists a set $\mathcal{X} \subset [T, 2T]$ with

$$\operatorname{meas}(\mathcal{X}) \le T \exp\left(-c_1 \mathcal{L} \log\left(\frac{\mathcal{L}}{\log_3 T}\right)\right) \tag{6.104}$$

such that for any $t \in [T, 2T] \setminus \mathcal{X}$ and any $j = 1, \dots, r$,

$$\operatorname{Re} e^{-i\theta_j} \log F_j(\tfrac{1}{2} + it) \le \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2} + it, X) + \mathcal{L} \tag{6.105}$$

when $\theta_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and

$$\operatorname{Re} e^{-i\theta_j} \log F_j(\tfrac{1}{2} + it) \ge \operatorname{Re} e^{-i\theta_j} P_{F_j}(\tfrac{1}{2} + it, X) - \mathcal{L}$$

when $\theta_j \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

First, we show estimates (6.10) and (6.11). Suppose that $\boldsymbol{V}$ satisfies $\|\boldsymbol{V}\| \le a_5 V_m^{1/2} (\log\log T)^{1/4} (\log_3 T)^{1/2}$, where a_5 is a sufficiently small positive constant. Set $\mathcal{L} = 4rc_1^{-1}\left(\frac{\|\boldsymbol{V}\|^2}{\log\|\boldsymbol{V}\|} + (\log_3 T)^4\right)$. Then we can verify from (6.104) that $\operatorname{meas}(\mathcal{X}) \ll_{\boldsymbol{F}} T \exp\left(-2r\|\boldsymbol{V}\|^2\right)$. Moreover, when $\theta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the measure of $t \in [T, 2T] \setminus \mathcal{X}$ such that

$$\operatorname{Re} e^{-i\theta_j} \log F_j(\tfrac{1}{2} + it) \ge V_j \sqrt{\frac{n_{F_j}}{2} \log\log T}$$

is bounded above by the measure of $t \in [T, 2T]$ such that

$$\frac{\operatorname{Re} e^{-i\theta_j} P_{F_j}(\frac{1}{2} + it, X)}{\sigma_{F_j}(X)} \ge V_j - C_{\boldsymbol{F}}\left(\frac{\mathcal{L}}{\sqrt{\log\log T}} + \frac{V_j \log_3 T}{\log\log T}\right).$$

where $C_{\boldsymbol{F}}$ is some positive constant and we used (6.94) for $\sigma_{F_j}(X)^{-1}$. Similarly when $\theta_i \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, the measure of $t \in [T, 2T] \setminus \mathcal{X}$ such that

$$\operatorname{Re} e^{-i\theta_j} \log F_j(\tfrac{1}{2} + it) \ge V_j \sqrt{\frac{n_{F_j}}{2} \log\log T}$$

is bounded below by the measure of $t \in [T, 2T]$ such that

$$\frac{\operatorname{Re} e^{-i\theta_j} P_{F_j}(\frac{1}{2} + it, X)}{\sigma_{F_j}(X)} \ge V_j + C_{\boldsymbol{F}}\left(\frac{\mathcal{L}}{\sqrt{\log\log T}} + \frac{V_j \log_3 T}{\log\log T}\right).$$

Here, we choose a_3 so that $C_{\boldsymbol{F}}\left(\frac{\mathcal{L}}{\sqrt{\log\log T}}+\frac{V_j\log_3 T}{\log\log T}\right)\le\frac{V_m}{2}$. From these observations, the estimate $\int_V^\infty e^{-u^2/2}du \asymp \frac{1}{1+V}e^{-V^2/2}$ for $V\ge 0$, estimate (6.65), and Proposition 6.2, we find that if $\theta_j\in[-\frac{\pi}{2},\frac{\pi}{2}]$,

$$
\begin{aligned}
&\frac{1}{T}\operatorname{meas}(\mathcal{S}(T,\boldsymbol{V};\boldsymbol{F},\boldsymbol{\theta}))\\
&\ll_{\boldsymbol{F}} \frac{1}{T}\operatorname{meas}(\mathcal{X})\\
&+\left(\frac{1}{V_1\cdots V_r}+\frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right)\\
&\quad\times\prod_{j=1}^{r}\exp\Bigg\{-\frac{V_j^2}{2}-\frac{V_j^2}{2\sigma_{F_j}(X)^2}\sigma_{F_j}\left(\frac{V_j^2}{\sigma_{F_j}(X)^2}\right)^2\\
&\qquad\qquad+O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|\mathcal{L}}{\sqrt{\log\log T}}+\frac{\mathcal{L}^2}{\log\log T}+\left(\frac{\|\boldsymbol{V}\|}{\sqrt{\log\log T}}\right)^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\Bigg\}\\
&\ll_{\boldsymbol{F}}\left(\frac{1}{V_1\cdots V_r}+\frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right)\prod_{j=1}^{r}\exp\left(-\frac{V_j^2}{2}+O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|^3}{\sqrt{\log\log T}\log\|\boldsymbol{V}\|}\right)\right)
\end{aligned}
$$

for $\|\boldsymbol{V}\|\le a_5V_m^{1/2}(\log\log T)^{1/4}(\log_3 T)^{1/2}$. Hence, we obtain estimate (6.10). Similarly, we can also find that if $\theta_j\in[\frac{\pi}{2},\frac{3\pi}{2}]$,

$$
\begin{aligned}
&\frac{1}{T}\operatorname{meas}(\mathcal{S}(T,\boldsymbol{V};\boldsymbol{F},\boldsymbol{\theta}))\\
&\gg_{\boldsymbol{F}}\left(\frac{1}{V_1\cdots V_r}+\frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right)\\
&\quad\times\prod_{j=1}^{r}\exp\Bigg\{-\frac{V_j^2}{2}-\frac{V_j^2}{2\sigma_{F_j}(X)^2}\sigma_{F_j}\left(\frac{V_j^2}{\sigma_{F_j}(X)^2}\right)^2\\
&\qquad\qquad-O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|\mathcal{L}}{\sqrt{\log\log T}}+\frac{\mathcal{L}^2}{\log\log T}+\left(\frac{\|\boldsymbol{V}\|}{\sqrt{\log\log T}}\right)^{\frac{2-2\vartheta_{\boldsymbol{F}}}{1-2\vartheta_{\boldsymbol{F}}}}\right)\Bigg\}\\
&-\frac{1}{T}\operatorname{meas}(\mathcal{X})\\
&\gg_{\boldsymbol{F}}\frac{1}{V_1\cdots V_r}\exp\left(-\frac{V_1^2+\cdots+V_r^2}{2}-O_{\boldsymbol{F}}\left(\frac{\|\boldsymbol{V}\|^3}{\sqrt{\log\log T}\log\|\boldsymbol{V}\|}\right)\right)
\end{aligned}
$$

for $\|\boldsymbol{V}\|\le a_5V_m^{1/2}(\log\log T)^{1/4}(\log_3 T)^{1/2}$ satisfying the inequality $\prod_{j=1}^{r}V_j\le a_6(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}$ with $a_6=a_6(\boldsymbol{F})>0$ a suitably small constant. Hence, we also obtain (6.11).

Now we consider (6.12), where $\theta_j\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Putting $\mathcal{L}=\frac{4K_1\log T}{\log\log T}$, we see that $Y=(\log T)^{1/4}$, and hence there exists a positive constant $A=A(\boldsymbol{F})$ such that the right hand side of (6.103) is $\le A\frac{\log T}{\sqrt{\log\log T}}\sqrt{\frac{n_{F_j}}{2}\log\log T}$ uniformly for any $t \in [T, 2T]$ and all $j = 1, \dots, r$. Thus, we may assume $\|\boldsymbol{V}\| \le A\frac{\log T}{\sqrt{\log\log T}}$. We first consider the case when $\sqrt{\log\log T} \le \|\boldsymbol{V}\| \le A\frac{\log T}{\sqrt{\log\log T}}$. Set $\mathcal{L} = b_1\frac{\|\boldsymbol{V}\|}{2}\sqrt{\log\log T}$, where b_1 is some small positive constant such that the inequality $Y \ge 3$ holds. Then we see (6.104) becomes

$$\operatorname{meas}(\mathcal{X}) \ll_{\boldsymbol{F}} T\exp\left(-c_2\|\boldsymbol{V}\|\sqrt{\log\log T}\log\|\boldsymbol{V}\|\right)$$

for some constant $c_2 = c_2(\boldsymbol{F}) > 0$. Using Lemma 2.8, we have, uniformly for any $j = 1, \dots, r$,

$$\int_T^{2T} |P_{F_j}(\tfrac{1}{2}+it, X)|^{2k} dt \ll_{\boldsymbol{F}} T(C_6 k\log\log T)^k \tag{6.106}$$

for any integer k with $1 \le k \le \mathcal{L}\log\log T$ and some $C_6 = C_6(\boldsymbol{F})$. Combing (6.106) and (6.105), we obtain

$$\begin{aligned}
&\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta}))\\
&\ll \min_{1\le j\le r}\frac{1}{T}\operatorname{meas}\left\{t \in [T, 2T] \ : \ \operatorname{Re} e^{-i\theta_j}\log F_j(\tfrac{1}{2}+it) > V_j\right\}\\
&\ll_{\boldsymbol{F}} \|\boldsymbol{V}\|^{-2k}C_7^k k^k + \exp\left(-c_2\|\boldsymbol{V}\|\sqrt{\log\log T}\log\|\boldsymbol{V}\|\right).
\end{aligned}$$

When $\|\boldsymbol{V}\| \le \log\log T$, we choose $k = \lfloor c_3\|\boldsymbol{V}\|^2\rfloor$, and when $\|\boldsymbol{V}\| > \log\log T$, we choose $k = \lfloor c_3\|\boldsymbol{V}\|\sqrt{\log\log T}\rfloor$, where c_3 is a suitably small positive constant depending only on $\boldsymbol{F}$. Then, it follows that

$$\begin{aligned}
&\frac{1}{T}\operatorname{meas}(\mathcal{S}(T, \boldsymbol{V}; \boldsymbol{F}, \boldsymbol{\theta}))\\
&\ll_{\boldsymbol{F}} \exp\left(-c_4\|\boldsymbol{V}\|^2\right) + \exp\left(-c_5\|\boldsymbol{V}\|\sqrt{\log\log T}\log\|\boldsymbol{V}\|\right),
\end{aligned}$$

which completes the proof of (6.12). □

Proof of Theorem 6.5. Let T be large, and put $\varepsilon(T) = (\log_3 T)^{-1}$. Let $k \ge 0$. We recall equation (6.99), which is

$$\int_T^{2T}\exp\left(2k\phi_{\boldsymbol{F}}(t)\right)dt = \int_{-\infty}^{\infty} 2ke^{2kV}\Phi_{\boldsymbol{F}}(T, V)dV.$$

We divide the integral on the right hand side to

$$\left(\int_{-\infty}^{0} + \int_{(0}^{D_2 k\log\log T} + \int_{D_2 k\log\log T}^{\infty}\right) 2ke^{2kV}\Phi_{\boldsymbol{F}}(T, V)dV =: I_4 + I_5 + I_6,$$

say. Here, $D_2 = D_2(\boldsymbol{F})$ is a suitably large positive constant. Now we consider the case when $\theta_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We use the trivial bound $\Phi_{\boldsymbol{F}}(T,V) \le T$ to obtain $I_4 \le T$. Applying estimate (6.12), we find that the estimate

$$\Phi_{\boldsymbol{F}}(T,V) \ll_{\boldsymbol{F}} T \exp(-4kV)$$

holds for $V \ge D_2 k \log\log T$ when $T \ge \exp\exp\exp(Ck)$ and C, D_2 are suitably large depending only on $\boldsymbol{F}$. Therefore, we have

$$I_6 \ll_{\boldsymbol{F}} T \int_{D_2 \log\log T}^{\infty} 2ke^{-2kV} dV \ll T.$$

By estimate (6.10), we find that

$$\begin{aligned}
&\Phi_{\boldsymbol{F}}(T,V)\\
&\ll_{k,\boldsymbol{F}} T\left(\frac{1}{1+(V/\sqrt{\log\log T})^r} + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right)\\
&\qquad\qquad\qquad \times \exp\left(-\frac{h_{\boldsymbol{F}} V^2}{\log\log T} + \frac{C_1 V^3}{(\log\log T)^2 \log_3 T}\right)\\
&\ll T\left(\frac{1}{1+(V/\sqrt{\log\log T})^r} + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right)(\log T)^{C_1 D_2^3 k^3 \varepsilon(T)}\\
&\qquad\qquad\qquad \times \exp\left(-h_{\boldsymbol{F}} \frac{V^2}{\log\log T}\right)
\end{aligned}$$

for $(\log\log T)^{2/3} \le V \le D_2 k \log\log T$. Here, $C_1 = C_1(\boldsymbol{F})$ is some positive constant. Similarly to the proof of (6.102) by using this estimate, we obtain

$$I_5 \ll_{\boldsymbol{F}} T + T(\log T)^{k^2/h_{\boldsymbol{F}} + Bk^3\varepsilon(T)} \left(\frac{k\sqrt{\log\log T}}{1+(k\sqrt{\log\log T})^r} + \frac{1}{(\log\log T)^{\alpha_{\boldsymbol{F}}+\frac{1}{2}}}\right).$$

Hence, we obtain (6.13).

For estimate (6.14), it holds from the positivity of $\Phi_{\boldsymbol{F}}(T,V)$ and equation (6.99) that

$$\int_T^{2T} \exp(2k\phi_{\boldsymbol{F}}(t))\, dt \gg_k \int_{\frac{k}{h_{\boldsymbol{F}}}\log\log T}^{\frac{k}{h_{\boldsymbol{F}}}\log\log T + \sqrt{\log\log T}} e^{2kV} \Phi_{\boldsymbol{F}}(T,V) dV.$$

When $\theta_i \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, assuming $\vartheta_{\boldsymbol{F}} < \frac{1}{r+1}$, we use (6.11) to obtain

$$\begin{aligned}
\Phi_{\boldsymbol{F}}(T,V) &\gg_{k,\boldsymbol{F}} \frac{T}{1+(V/\sqrt{\log\log T})^r} \exp\left(-h_{\boldsymbol{F}}\frac{V^2}{\log\log T} - \frac{CV^3}{(\log\log T)^2 \log_3 T}\right)\\
&\gg_{\boldsymbol{F}} \frac{T(\log T)^{-C_2 k^3 \varepsilon(T)}}{1+(V/\sqrt{\log\log T})^r} \exp\left(-h_{\boldsymbol{F}}\frac{V^2}{\log\log T}\right)
\end{aligned}$$

for $\frac{k}{h_{\boldsymbol{F}}}\log\log T \le V \le \frac{k}{h_{\boldsymbol{F}}}\log\log T + \sqrt{\log\log T}$. Here, $C_2 = C_2(\boldsymbol{F})$ is a positive constant. Similarly to the proof of (6.9) by using this estimate and the bound $\Phi_{\boldsymbol{F}}(T,V) \gg_{\boldsymbol{F}} T$ for $0 \le V \le 1$, we can also obtain (6.14). □

6.6 Conclusion remarks

From the result for large deviations, it seems to expected that

$$\int_T^{2T} \left(\min_{1\le j\le r} |F_j(\tfrac12 + it)| \right)^{2k} dt \asymp_k T + T\frac{(\log T)^{k^2/h_{\boldsymbol{F}}}}{(\log\log T)^{(r-1)/2}}.$$

We may be interested in whether, using our method and Harper's [39], we can improve our mean value theorem into

$$\int_T^{2T} \left(\min_{1\le j\le r} |F_j(\tfrac12 + it)| \right)^{2k} dt \ll_k T + T\frac{(\log T)^{k^2/h_{\boldsymbol{F}}}}{(\log\log T)^{(r-1)/2}} \qquad \text{for } k \ge 0.$$

Our method requires a strong zero density estimate for L-functions. Unfortunately, the estimate has not been proved yet for many L-functions. Therefore, we may be interested in whether we can prove our large deviations results to avoid the estimate by using the method of Laurinčikas [68] or Radziwiłł-Soundararajan [98]. On the other hand, Hsu and Wong [46] proved a joint central limit theorem (for fixed V_j) for Dirichlet L-functions by using the method of Radziwiłł-Soundararajan. However, their method requires essentially that Dirichlet coefficients satisfy $|a(n)| \le 1$ (in this case $|\chi(n)| \le 1$), hence also requires Ramanujan conjecture when we consider generalization to automorphic L-functions.

In this chapter, we showed that, for certain k,

$$\int_T^{2T} \left(\min_{1\le j\le r} |F_j(\tfrac12 + it)| \right)^{2k} dt \ll_{k,\boldsymbol{F}} T(\log T)^{k^2/h_{\boldsymbol{F}} + Bk^3},$$

and

$$\int_T^{2T} \left(\min_{1\le j\le r} |F_j(\tfrac12 + it)| \right)^{2k} dt \ll_{k,\boldsymbol{F}} T(\log T)^{k^2/h_{\boldsymbol{F}} + \varepsilon}$$

under GRH. Moreover, we can also show that, using Theorems 6.2, 6.4,

$$\int_T^{2T} \left(\max_{1\le j\le r} |F_j(\tfrac12 + it)| \right)^{2k} dt \ll_{k,\boldsymbol{F}} T(\log T)^{n_{\boldsymbol{F}} k^2 + Bk^3},$$

$$\int_T^{2T} \prod_{1\le j\le r} |F_j(\tfrac12 + it)|^{2k_j} dt \ll_{k,\boldsymbol{F}} T(\log T)^{n_{F_1} k_1^2 + \cdots + n_{F_r} k_r^2 + B\boldsymbol{k}^3}$$

for any small $k, k_1, \dots, k_r \ge 0$ with $\boldsymbol{k} = \max_{1\le j\le r} k$, and

$$\int_T^{2T} \left(\max_{1\le j\le r} |F_j(\tfrac12 + it)| \right)^{2k} dt \ll_{k,\boldsymbol{F},\varepsilon} T(\log T)^{n_{\boldsymbol{F}} k^2 + \varepsilon},$$

$$\int_T^{2T} \prod_{1\le j\le r} |F_j(\tfrac12 + it)|^{2k_j} dt \ll_{k,\boldsymbol{F},\varepsilon} T(\log T)^{n_{F_1} k_1^2 + \cdots + n_{F_r} k_r^2 + \varepsilon}$$

for any $k, k_1, \dots, k_r \ge 0$ under GRH, where $n_{\boldsymbol{F}} = \max_{1\le j\le r} n_{F_j}$.

Finally, we should mention that our method also recovers the work of Heuberger-Kropf [45] for higher dimensional quasi-power theorem, and it is probably possible to improve their work in the direction of large deviations by our method.

Chapter 7 Dependence of $\zeta(\sigma+it)$ and $L(\sigma+it,\chi)$ in the strip $1/2<\sigma<1$

In this chapter, we discuss the joint value distribution of L-functions in the Selberg class in the strip $1/2 < \sigma < 1$. The contents in this chapter are based on the paper [53].

7.1 Results

In this section, we state our result for the dependence of the Riemann zeta-function and Dirichlet L-functions associated with a quadratic character. We consider the measure of the set

$$\begin{aligned}&\mathcal{S}(T,\boldsymbol{V};\chi,\sigma,\theta) := \\ &\left\{t\in[T,2T] \ : \ \operatorname{Re} e^{-i\theta}\log\zeta(\sigma+it) > V_1 \text{ and } \operatorname{Re} e^{-i\theta}\log L(\sigma+it,\chi) > V_2\right\}.\end{aligned}$$

When $\sigma=\frac{1}{2}$, the measure is discussed in the previous chapter, and so we in this chapter focus on the case $\frac{1}{2}<\sigma<1$. The main theorem in this chapter is the following.

Theorem 7.1. *Let $\frac{1}{2}<\sigma<1$, and let χ be a quadratic Dirichlet character. Then, there exists a positive constant $a_1=a_1(\sigma,\chi)$ such that, for any large numbers T, V_1 satisfying $V_1\le a_1\frac{(\log T)^{1-\sigma}}{\log\log T}$, we have*

$$\begin{aligned}&\frac{1}{T}\operatorname{meas}(\mathcal{S}(T,\boldsymbol{V};\chi,\sigma,\theta)) \\ &= \exp\left(-2^{\frac{\sigma}{1-\sigma}}A(\sigma)V_1^{\frac{1}{1-\sigma}}(\log V_1)^{\frac{\sigma}{1-\sigma}}(1+o(1))\right)\end{aligned}$$

with $V_2=V_1(1+o(1))$ as $V_1\to+\infty$.

From this theorem, we find that $\log|\zeta(\sigma+it)|$ and $\log|L(\sigma+it,\chi)|$ are dependent as random variables for every $\frac{1}{2}<\sigma<1$. Moreover, we can also obtain the following corollary.

Corollary 7.1. *Let $\frac{1}{2}<\sigma<1$, and let χ be a quadratic character. Then we have*

$$\min\left\{\log|\zeta(\sigma+it)|, \log|L(\sigma+it,\chi)|\right\} = \Omega_{\pm}\left(\frac{(\log t)^{1-\sigma}}{\log\log t}\right)$$

as $t\to+\infty$.

Similarly to the previous chapter, to prove Theorem 7.1, we firstly calculate certain Dirichlet polynomials. We define

$$\begin{aligned}&\mathcal{S}_X(T, V_1, V_2; \chi, \sigma, \theta)\\&:= \left\{ t \in [T, 2T] \ : \ \mathrm{Re} \sum_{p \le X} \frac{e^{-i\theta}}{p^{\sigma + it}} > V_1, \text{ and } \mathrm{Re} \sum_{p \le X} \frac{e^{-i\theta}\chi(p)}{p^{\sigma+it}} > V_2 \right\}.\end{aligned}$$

Then, we can show the following proposition.

Proposition 7.1. *Let $L \ge 2$, and χ be a quadratic Dirichlet character. There exists positive constant $a_2 = a_2(\sigma, \chi, L)$ such that, for any large numbers T, V_1, $X = (\log T)^L$ with $V_1 \le a_2 \frac{(\log T)^{1-\sigma}}{\log\log T}$, we have*

$$\begin{aligned}&\frac{1}{T}\mathrm{meas}(\mathcal{S}_X(T, V_1, V_2; \chi, \sigma, \theta))\\&= \exp\left(-2^{\frac{\sigma}{1-\sigma}} A(\sigma) V_1^{\frac{1}{1-\sigma}} (\log V_1)^{\frac{\sigma}{1-\sigma}} (1 + o(1)) \right)\end{aligned}$$

with $V_2 = V_1(1 + o(1))$ as $V_1 \to +\infty$.

7.2 Approximate formulas for moment generating functions II

In this section, we give an approximate formula for characteristic functions of an r-tuple of Dirichlet polynomials in general cases. In this section, $\boldsymbol{a}(p) = (a_1(p), \dots, a_r(p))$ is a fixed r-tuple of bounded sequences on the prime numbers. For every $w, z_1, \dots, z_r \in \mathbb{C}$, $\sigma \in \mathbb{R}$, and prime number p, we define $K_{\boldsymbol{a}}(p, \boldsymbol{z})$ by (6.46) and

$$P_j(\sigma + it, X) = \sum_{p \le X} \frac{a_j(p)}{p^{\sigma+it}}.$$

It is the goal of this section to prove the following proposition.

Proposition 7.2. *Let $\frac{1}{2} < \sigma < 1$, $L \ge 1$ be fixed. There exist positive constants $b_1 = b_1(\boldsymbol{a}, \sigma, L)$, $b_2 = b_2(\boldsymbol{a}, \sigma, L)$, $b_3 = b_3(\boldsymbol{a}, \sigma, L)$ such that, for large T, $X = (\log T)^L$, and $\boldsymbol{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$ with $\|\boldsymbol{z}\| \le b_1 (\log T)^{\sigma}$, we have*

$$\begin{aligned}&\frac{1}{T}\int_{\mathcal{A}} \exp\left(\sum_{j=1}^{r} z_j \,\mathrm{Re}\left(P_j(\sigma + it, X)\right) \right) dt\\&= \prod_{p \le X} I_0\left(\sqrt{K_{\boldsymbol{a}}(p, \boldsymbol{z})/p^{2\sigma}} \right) + O\left(\exp\left(-b_2 \frac{\log T}{\log\log T} \right) \right),\end{aligned}$$

where $\mathcal{A} \subset [T, 2T]$ is a set satisfying $\mathrm{meas}(A) \le T\exp(-b_3 \log T/\log\log T)$.

To prove this proposition, we prepare some lemmas.

Lemma 7.1. *Let $\sigma \geq 1/2$ be fixed. Let $X \geq 3$, and T be large. Let $z_1, z_2, \dots, z_r$ be complex numbers. Then, there exists a positive constant $C_1 = C_1(\boldsymbol{a})$ such that, for all $\sigma \geq 1/2$, $k \in \mathbb{Z}_{\geq 1}$, $R > 0$, we have*

$$\frac{1}{T}\int_T^{2T}\left(\sum_{j=1}^{r} z_j \operatorname{Re}\left(P_j(\sigma+it, X)\right)\right)^k dt$$
$$= \frac{k!}{2\pi i}\oint_{|w|=R}\frac{1}{w^{k+1}}\prod_{p\leq X} I_0\left(w\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right)dw + O\left(\frac{(C_1\|\boldsymbol{z}\|X^3)^k}{T}\right).$$

Proof. This lemma can be easily proved by using Lemma 6.8. □

Lemma 7.2. *There exists a positive constant $C_1 = C_1(\boldsymbol{a}, \sigma)$ such that for $X \geq 3$ $w \in \mathbb{C}$, $\boldsymbol{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$ with $\|\boldsymbol{z}\| \leq X^{\sigma}$, we have*

$$\left|\prod_{p\leq X} I_0\left(\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right)\right| \leq \exp\left(C_1\frac{\|\boldsymbol{z}\|^{\frac{1}{\sigma}}}{\log\left(\|\boldsymbol{z}\|+3\right)}\right).$$

Proof. By the definition of $K_{\boldsymbol{a}}(p,\boldsymbol{z})$ and the boundedness of $a_j(p)$, there exists a constant $C = C(\boldsymbol{a}) > 0$ such that

$$\left|I_0\left(\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right)\right| \leq \sum_{\alpha=0}^{\infty}\frac{(C_{\boldsymbol{a}}\|\boldsymbol{z}\|/p^{\sigma})^{2\alpha}}{(2\alpha)!} \leq \exp\left(\frac{C_{\boldsymbol{a}}\|\boldsymbol{z}\|}{p^{\sigma}}\right).$$

By this inequality and using the prime number theorem, we have

$$\left|\prod_{p\leq\|\boldsymbol{z}\|_1^{\frac{1}{\sigma}}} I_0\left(\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right)\right| \leq \exp\left(C\frac{\|\boldsymbol{z}\|_1^{\frac{1}{\sigma}}}{\log\left(\|\boldsymbol{z}\|_1+3\right)}\right)$$

for some $C = C(\boldsymbol{a}, \sigma) > 0$. On the other hand, for $p > \|\boldsymbol{z}\|^{\frac{1}{\sigma}}$, we see that

$$I_0\left(\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right) = 1 + O_{\boldsymbol{a}}\left(\frac{\|\boldsymbol{z}\|_1^2}{p^{2\sigma}}\right).$$

Using this equation and the prime number theorem, we find that

$$\prod_{\|\boldsymbol{z}\|^{\frac{1}{\sigma}}<p\leq X} I_0\left(\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right)$$
$$= \exp\left(\sum_{\|\boldsymbol{z}\|^{\frac{1}{\sigma}}<p\leq X}\log\left(1+O_{\boldsymbol{a}}\left(\frac{\|\boldsymbol{z}\|^2}{p^{2\sigma}}\right)\right)\right)$$
$$\leq \exp\left(C'\sum_{\|\boldsymbol{z}\|^{\frac{1}{\sigma}}<p\leq X}\frac{\|\boldsymbol{z}\|^2}{p^{2\sigma}}\right) \leq \exp\left(C''\frac{\|\boldsymbol{z}\|^{\frac{1}{\sigma}}}{\log\left(\|\boldsymbol{z}\|+3\right)}\right).$$

Thus, by taking $C_1 = \max\{C, C''\}$, we obtain this lemma. □

Lemma 7.3. *Let $L \geq 1$, and let T be large. Put $X = (\log T)^L$. Let $a(p)$ be a bounded complex sequence with $|a(p)| \leq M$. Then there exists a positive constant $C_2 = C_2(\sigma, M)$ such that, for all integer k with $1 \leq k \leq \frac{\log T}{10L \log\log T}$,*

$$\int_T^{2T} \left| \sum_{p \leq X} \frac{a(p)}{p^{\sigma+it}} \right|^{2k} dt \leq T \left(C_2 \frac{k^{1-\sigma}}{(\log (k+3))^{\sigma}} \right)^{2k}.$$

Proof. By applying Lemmas 7.1, 7.2 as $r = 1$, $a_1(p) = a(p)$, $z_1 = 1$, and $\theta_1 = 0$, we can obtain

$$\begin{aligned}
&\frac{1}{T} \int_T^{2T} \left| \sum_{p \leq X} \frac{a(p)}{p^{\sigma+it}} \right|^{2k} dt \\
&= \frac{k!}{2\pi i} \oint_{|w|=R} \frac{1}{w^{k+1}} \prod_{p \leq X} I_0 \left(w \sqrt{a(p)/p^{2\sigma}} \right) dw + O\left(\frac{(CX^3)^k}{T} \right) \\
&\ll \frac{k!}{R^k} \exp\left(C_1 \frac{R^{\frac{1}{\sigma}}}{\log (R+3)} \right) + T^{-1/2}
\end{aligned}$$

for any $0 < R \leq X^{\sigma} = (\log T)^{\sigma L}$, $1 \leq k \leq \frac{\log T}{10L \log\log T}$. Choosing $R = k^{2\sigma-1}(\log k)^{2\sigma}$, this is

$$\ll \left(\frac{k^{1-\sigma}}{(\log k)^{\sigma}} \right)^{2k} \exp\left(C_1' k \left(k^{1-\sigma} \log (k+3) \right) \right) + T^{-1/2} \ll \left(C_2 \frac{k^{1-\sigma}}{(\log (k+3))^{\sigma}} \right)^{2k}$$

for some $C_2 = C_2(\sigma, M)$. □

Lemma 7.4. *Let $L \geq 1$, and let T be large. Put $X = (\log T)^L$. Define the set $\mathcal{A}$ by*

$$\mathcal{A} = \bigcap_{j=1}^{r} \left\{ t \in [T, 2T] \ : \ \left| \mathrm{Re}\left(P_j(\sigma + it, X) \right) \right| \leq \frac{(\log T)^{1-\sigma}}{\log\log T} \right\}. \tag{7.1}$$

Then, there exists a positive number $c_1 = c_1(\boldsymbol{a}, \sigma, L)$ such that

$$\frac{1}{T} \mathrm{meas}([T, 2T] \setminus \mathcal{A}) \leq \exp\left(-c_1 \frac{\log T}{\log\log T} \right).$$

Proof. By Lemma 7.3, there exist positive constants $C_j = C_j(\boldsymbol{a}, \sigma)$, for which

$$\begin{aligned}
&\frac{1}{T} \mathrm{meas} \left\{ t \in [T, 2T] \ : \ \left| \mathrm{Re}\left(P_j(\sigma + it, X) \right) \right| > \frac{(\log T)^{1-\sigma}}{\log\log T} \right\} \\
&\leq \left(C_j \frac{k^{1-\sigma} \log\log T}{(\log k)^{\sigma} (\log T)^{1-\sigma}} \right)^{2k}.
\end{aligned}$$

holds for $2 \le k \le \frac{\log T}{2L \log\log T}$. Hence, we have

$$\begin{aligned}
&\frac{1}{T}\,\mathrm{meas}([T,2T] \setminus \mathcal{A}) \\
&\le \frac{1}{T}\sum_{j=1}^{r} \mathrm{meas}\left\{ t \in [T,2T] \ :\ \left|\mathrm{Re}\left(P_j(\sigma+it,X)\right)\right| > \frac{(\log T)^{1-\sigma}}{\log\log T} \right\} \\
&\le \left(C \frac{k^{1-\sigma}\log\log T}{(\log k)^{\sigma}(\log T)^{1-\sigma}} \right)^{2k},
\end{aligned}$$

where $C = r \cdot \max_{1\le j\le r} C_j$. Thus, we obtain this lemma by choosing $k = [c\log T/\log\log T]$ with $c = c(\boldsymbol{a},\sigma,L)$ a suitably small constant. □

Proof of Proposition 7.2. Let $\mathcal{A}$ be the set defined by (7.1). Let $\delta = \delta(\boldsymbol{a},\sigma,L)$ be a suitably small positive constant to be chosen later. Then we find that

$$\begin{aligned}
&\frac{1}{T}\int_{\mathcal{A}} \exp\left(\sum_{j=1}^{r} z_j \,\mathrm{Re}\left(P_j(\sigma+it,X)\right) \right) dt = \\
&\frac{1}{T}\sum_{0\le k\le Y}\frac{1}{k!}\int_{\mathcal{A}}\left(\sum_{j=1}^{r} z_j \,\mathrm{Re}\left(P_j(\sigma+it,X)\right) \right)^{k} dt + O\left(\sum_{k>Y}\frac{1}{k!}\left(\frac{\|\boldsymbol{z}\|(\log T)^{1-\sigma}}{\log\log T} \right)^{k} \right),
\end{aligned}$$

where $Y = \frac{\log T}{10L\log\log T}$. Here, by using the Stirling formula, this O-term is $\ll \exp\left(-\frac{1}{10L}\frac{\log T}{\log\log T}\right)$ for $\|\boldsymbol{z}\| \le \delta(\log T)^{\sigma}$ if $\delta \le \frac{1}{10e^2 L}$. By using the Cauchy-Schwarz inequality, for $0 \le k \le Y$, we find that

$$\begin{aligned}
&\frac{1}{T}\int_{\mathcal{A}}\left(\sum_{j=1}^{r} z_j \,\mathrm{Re}\left(P_j(\sigma+it,X)\right) \right)^{k} dt \\
&= \frac{1}{T}\int_{T}^{2T}\left(\sum_{j=1}^{r} z_j \,\mathrm{Re}\left(P_j(\sigma+it,X)\right) \right)^{k} dt + \\
&\quad + O\left(\frac{1}{T}(\mathrm{meas}([T,2T]\setminus\mathcal{A}))^{1/2}\left(\int_{T}^{2T}\left| \sum_{j=1}^{r} z_j \,\mathrm{Re}\left(P_j(\sigma+it,X)\right) \right|^{2k} dt \right)^{1/2} \right).
\end{aligned}$$

Using Lemma 7.3 and Lemma 7.4, this O-term is

$$\ll \exp\left(-\frac{c_1}{2}\frac{\log T}{\log\log T}\right)\left(C_2'\|\boldsymbol{z}\|\frac{(k+1)^{1-\sigma}}{(\log(k+3))^{\sigma}} \right)^{k} \le \exp\left((C_2'\delta_1 - \frac{c_1}{2})\frac{\log T}{\log\log T} \right),$$

where c_1 is the same constant as in Lemma 7.4, and C_2' is a positive constant depending on $\boldsymbol{a},\sigma$. Moreover, when $\delta \le \frac{c_1}{4C_2'}$, this right hand side is $\le$

$\exp\left(-\frac{c_1}{4}\frac{\log T}{\log\log T}\right)$. From the above calculations, when $\delta \leq \min\{\frac{1}{10e^2L}, \frac{c_1}{4C_2'}\}$, we have

$$\begin{aligned} &\frac{1}{T}\int_A \exp\left(\sum_{j=1}^r z_j \operatorname{Re}\left(P_j(\sigma+it, X)\right)\right) dt \qquad (7.2)\\ &= \frac{1}{T}\sum_{0\leq k\leq Y}\frac{1}{k!}\int_T^{2T}\left(\sum_{j=1}^r z_j \operatorname{Re}\left(P_j(\sigma+it, X)\right)\right)^k dt + O\left(\exp\left(-c_2\frac{\log T}{\log\log T}\right)\right), \end{aligned}$$

where $c_2 = \min\{\frac{1}{10L}, \frac{c_1}{4}\}$.

Now, by Lemma 7.1, we obtain

$$\begin{aligned} &\frac{1}{T}\sum_{0\leq k\leq Y}\frac{1}{k!}\int_T^{2T}\left(\sum_{j=1}^r z_j \operatorname{Re}\left(P_j(\sigma+it, X)\right)\right)^k dt\\ &= \frac{1}{2\pi i}\oint_{|w|=e}\sum_{0\leq k\leq Y}\frac{1}{w^{k+1}}\prod_{p\leq X} I_0\left(w\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right) dw + O\left(T^{-1/2}\right). \end{aligned}$$

By Lemma 7.2, there exists a positive constant $C_3 = C_3(\boldsymbol{a}, \sigma)$ such that

$$\left|\prod_{p\leq X} I_0\left(w\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right)\right| \leq \exp\left(C_3\delta^{\frac{1}{\sigma}}\frac{\log T}{\log\log T}\right)$$

for $|w| = e$, $\|\boldsymbol{z}\| \leq \delta(\log T)^{\sigma}$. In addition, for $|w| = e$, we see that

$$\left|\sum_{k>Y}\frac{1}{w^{k+1}}\right| \ll \exp\left(-\frac{\log T}{10L\log\log T}\right).$$

Therefore, if $\delta \leq \frac{1}{(20C_3L)^{\sigma}}$, $|w| = e$, it holds that

$$\left|\sum_{k>Y}\frac{1}{w^{k+1}}\prod_{p\leq X} I_0\left(w\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right)\right| \ll \exp\left(-\frac{\log T}{20L\log\log T}\right).$$

Hence, by choosing $\delta = \min\{\frac{1}{10e^2L}, \frac{c_1}{4C_2'}\frac{1}{(20C_3L^{\sigma})}\}$ and by this estimate and (7.2), we have

$$\begin{aligned} &\frac{1}{T}\int_A \exp\left(\sum_{j=1}^r z_j \operatorname{Re}\left(P_j(\sigma+it, X)\right)\right) dt\\ &= \frac{1}{2\pi i}\oint_{|w|=e}\prod_{p\leq X} I_0\left(w\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right)\frac{dw}{w-1} + O\left(\exp\left(-c_3\frac{\log T}{\log\log T}\right)\right), \end{aligned}$$

where $c_3 = \min\{\frac{1}{20L}, \frac{c_1}{4}\}$. This right hand side is equal to

$$\prod_{p\leq X} I_0\left(\sqrt{K_{\boldsymbol{a}}(p,\boldsymbol{z})/p^{2\sigma}}\right) + O\left(\exp\left(-c_3\frac{\log T}{\log\log T}\right)\right),$$

which completes the proof of Proposition 7.2. □

7.3 Distribution functions of Dirichlet polynomials in the strip $\frac{1}{2} < \sigma < 1$

For $z_1, z_2 \in \mathbb{C}$, and a Dirichlet character χ, define

$$K_{\chi,\theta}(p, z_1, z_2) = \left(z_1 e^{-i\theta} + z_2 e^{-i\theta}\chi(p)\right) \times \left(z_1 \overline{e^{-i\theta}} + z_2 \overline{e^{-i\theta}\chi(p)}\right).$$

Then, by Proposition 7.2, there exists a positive constant $b_1 = b_1(\chi, \sigma, L)$ such that, for $\max\{|z_1|, |z_2|\} \le b_1(\log T)^{\sigma}$, we have

$$\begin{aligned} &\frac{1}{T}\int_{\mathcal{A}} \exp\left(z_1 \operatorname{Re}\sum_{p\le X}\frac{e^{-i\theta}}{p^{\sigma+it}} + z_2 \operatorname{Re}\sum_{p\le X}\frac{\chi(p)e^{-i\theta}}{p^{\sigma+it}}\right) dt \\ &= \prod_{p\le X} I_0\left(\sqrt{K_{\chi,\theta}(p, z_1, z_2)/p^{2\sigma}}\right) + O\left(-b_2 \frac{\log T}{\log\log T}\right), \end{aligned} \tag{7.3}$$

where $\operatorname{meas}(\mathcal{A}) \le T\exp(-b_3 \log T/\log\log T)$. Then, we can obtain the following proposition, which plays an important role in the proof of Proposition 7.1.

Proposition 7.3. *Let χ be a quadratic character. Let f_0 be a function with $0 < f_0(x) \le \frac{1}{3}$ and $\lim_{x\to+\infty} f_0(x) = 0$. For any $X \ge 9$, $3 \le x_1, x_2 \le X^{\frac{2\sigma}{3}}$ with $|x_1 - x_2| \le (x_1 + x_2) f_0(x_1 + x_2)$, we have*

$$\begin{aligned} &\prod_{p\le X} I_0\left(\sqrt{K_{\chi,\theta}(p, x_1, x_2)/p^{2\sigma}}\right) \\ &= \exp\left(\frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{2\log(x_1+x_2)}\left(1 + O\left(\frac{1}{\log(x_1+x_2)} + f_0(x_1+x_2)^{\frac{1}{\sigma}}\right)\right)\right). \end{aligned}$$

Here, the implicit constant depends on χ and σ.

To prove this proposition, we prepare two lemmas. Remark that we can prove assertions, similar to these two lemmas, for all primitive not necessarily quadratic characters.

Lemma 7.5. *Let χ be a quadratic character. Put*

$$A_+(y) = \sum_{\substack{p\le y\\ \chi(p)=1}} 1, \quad A_-(y) = \sum_{\substack{p\le y\\ \chi(p)=-1}} 1.$$

There exists a constant $c > 0$ such that, for $y \ge 3$,

$$A_+(y), A_-(y) = \frac{\operatorname{li}(y)}{2} + O_\chi\left(y\exp\left(-c\sqrt{\log y}\right)\right),$$

where $\operatorname{li}(y) := \int_2^y \frac{du}{\log u}$.

Proof. It is well known (see [87, Section 11]) that, for $y \geq 3$,

$$\sum_{p \leq y} \chi(p) = \frac{y^\beta}{\beta} + y \exp\left(-c\sqrt{\log y}\right) \ll_\chi y \exp\left(-c\sqrt{\log y}\right),$$

and

$$\sum_{p \leq y} |\chi(p)| = \pi(y) + O_\chi(1) = \mathrm{li}(y) + O_\chi\left(y \exp\left(-c\sqrt{\log y}\right)\right),$$

where β is an exceptional zero. Thus, by these estimates and

$$A_\pm(y) = \frac{1}{2}\left(\sum_{p \leq y} |\chi(p)| \pm \sum_{p \leq y} \chi(p)\right) + O_\chi(1),$$

which completes the proof of this lemma. □

Lemma 7.6. *Let* $\frac{1}{2} < \sigma < 1$ *be fixed. Let* χ *be a quadratic character. Put*

$$B_+(x, X) := \sum_{\substack{p \leq X \\ \chi(p)=1}} \log I_0\left(\frac{x}{p^\sigma}\right), \quad B_-(x, X) := \sum_{\substack{p \leq X \\ \chi(p)=-1}} \log I_0\left(\frac{x}{p^\sigma}\right),$$

$$c_\pm(\sigma, \chi) = \sum_{\substack{p \\ \chi(p)=\pm 1}} \frac{1}{p^{2\sigma}}$$

For $X \geq 3$ *and* $0 \leq x \leq 2$, *we have*

$$B_\pm(x, X) = \frac{c_\pm(\sigma, \chi)}{4} x^2 + O\left(x^4\right).$$

For $X \geq 3$ *and* $2 \leq x \leq X^{\frac{2\sigma}{3}}$, *we have*

$$B_+(x, X), B_-(x, X) = \frac{G(\sigma) x^{\frac{1}{\sigma}}}{2\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Here, the implicit constants depend on χ *and* σ.

Proof. By the Taylor expansion of I_0, for $0 \leq x \leq 2$, we find that

$$B_\pm(x, X) = \sum_{\substack{p \leq X \\ \chi(p)=\pm 1}} \left(\frac{x^2}{4p^{2\sigma}} + O\left(\frac{x^4}{p^{4\sigma}}\right)\right) = \frac{c_\pm(\sigma, \chi)}{4} x^2 + O\left(x^4\right).$$

In the following, we assume that $x \geq 2$. We write

$$\begin{aligned} B_\pm(x, X) &= \left(\sum_{\substack{p \leq y_0 \\ \chi(p)=\pm 1}} + \sum_{\substack{y_1 < p \leq X \\ \chi(p)=\pm 1}} + \sum_{\substack{y_0 < p \leq y_1 \\ \chi(p)=\pm 1}}\right) \log I_0\left(\frac{x}{p^\sigma}\right) \\ &=: S_1^\pm + S_2^\pm + S_3^\pm, \end{aligned}$$

say. By using partial summation, we find that

$$S_3^{\pm} = -\int_{y_0}^{y_1} A_{\pm}(\xi)\left(\frac{d}{d\xi}\log I_0\left(\frac{x}{\xi^{\sigma}}\right)\right)d\xi + A_{\pm}(y_1)\log I_0\left(\frac{x}{y_1^{\sigma}}\right) \tag{7.4}$$
$$- A_{\pm}(y_0)\log I_0\left(\frac{x}{y_0^{\sigma}}\right).$$

By Lemma 7.5, the integral on the right hand side is equal to

$$-\frac{1}{2}\int_{y_0}^{y_1}\mathrm{li}(\xi)\left(\frac{d}{d\xi}\log I_0\left(\frac{x}{\xi^{\sigma}}\right)\right)d\xi + O\left(\int_{y_0}^{y_1}\xi e^{-c\sqrt{\log\xi}}\left(\frac{d}{d\xi}\log I_0\left(\frac{x}{\xi^{\sigma}}\right)\right)d\xi\right).$$

Note that we used the monotonicity of I_0 in the above deformation. We find that

$$-\int_{y_0}^{y_1}\mathrm{li}(\xi)\left(\frac{d}{d\xi}\log I_0\left(\frac{x}{\xi^{\sigma}}\right)\right)d\xi$$
$$= -\,\mathrm{li}(y_1)\log I_0\left(\frac{x}{y_1^{\sigma}}\right) + \mathrm{li}(y_0)\log I_0\left(\frac{x}{y_0^{\sigma}}\right) + \int_{y_0}^{y_1}\frac{\log I_0\left(\frac{x}{\xi^{\sigma}}\right)}{\log\xi}d\xi,$$

and that

$$\int_{y_0}^{y_1}\xi e^{-c\sqrt{\log\xi}}\left(\frac{d}{d\xi}\log I_0\left(\frac{x}{\xi^{\sigma}}\right)\right)d\xi$$
$$\ll y_1 e^{-c\sqrt{\log y_1}}\log I_0\left(\frac{x}{y_1^{\sigma}}\right) + y_0 e^{-c\sqrt{\log y_0}}\log I_0\left(\frac{x}{y_0^{\sigma}}\right)$$
$$+ \int_{y_0}^{y_1} e^{-c\sqrt{\log\xi}}\log I_0\left(\frac{x}{\xi^{\sigma}}\right)d\xi$$
$$\ll x^2 y_1^{1-2\sigma} e^{-c\sqrt{\log y_1}} + x y_0^{1-\sigma} e^{-c\sqrt{\log y_0}}.$$

Substituting the above estimates to (7.4) and using Lemma 7.5, we obtain

$$S_3^{\pm} = \frac{1}{2}\int_{y_0}^{y_1}\frac{\log I_0\left(\frac{x}{\xi^{\sigma}}\right)}{\log\xi}d\xi + O\left(x^2 y_1^{1-2\sigma} e^{-c\sqrt{\log y_1}} + x y_0^{1-\sigma} e^{-c\sqrt{\log y_0}}\right).$$

By making change of variables $u = \frac{x}{\xi^{\sigma}}$, we have

$$\int_{y_0}^{y_1}\frac{\log I_0\left(\frac{x}{\xi^{\sigma}}\right)}{\log\xi}d\xi = x^{\frac{1}{\sigma}}\int_{x/y_1^{\sigma}}^{x/y_0^{\sigma}}\frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}\log(x/u)}du.$$

For $x^{-1/2} \le u \le x^{1/2}$, it holds that $\frac{1}{\log(x/u)} = \frac{1}{\log x} + O\left(\frac{|\log u|}{(\log x)^2}\right)$. Therefore, the above right hand side is equal to

$$\frac{x^{\frac{1}{\sigma}}}{\log x}\int_{x/y_1^{\sigma}}^{x/y_0^{\sigma}}\frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}}du + O\left(\frac{x^{\frac{1}{\sigma}}}{(\log x)^2}\int_{x/y_1^{\sigma}}^{x/y_0^{\sigma}}\frac{\log I_0(u)|\log u|}{u^{1+\frac{1}{\sigma}}}du\right). \tag{7.5}$$

Moreover, we find that the main term of (7.5) is equal to

$$\begin{aligned}&\frac{x^{\frac{1}{\sigma}}}{\log x}\int_0^\infty \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}}du + O\left(\frac{x^{1/\sigma}}{\log x}\left((x/y_1)^{\frac{2\sigma-1}{\sigma^2}} + (x/y_0)^{\frac{\sigma-1}{\sigma^2}}\right)\right)\\ &= \frac{G(\sigma)x^{\frac{1}{\sigma}}}{\log x} + O\left(\frac{x^{\frac{1}{\sigma}}}{(\log x)^2}\right),\end{aligned}$$

and that the O-term of (7.5) is

$$\ll \frac{x^{\frac{1}{\sigma}}}{(\log x)^2}\int_0^\infty \frac{\log I_0(u)|\log u|}{u^{1+\frac{1}{\sigma}}}du \ll \frac{x^{\frac{1}{\sigma}}}{(\log x)^2}.$$

Hence, choosing $y_0 = x^{\frac{1}{2\sigma}}$, $y_1 = x^{\frac{3}{2\sigma}}$, we have

$$S_3^{\pm} = \frac{G(\sigma)x^{\frac{1}{\sigma}}}{2\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

For $S_1^{\pm}$, by using the inequality $I_0(x/p^\sigma) \le \exp(x/p^\sigma)$, we find that

$$S_1^{\pm} \le \sum_{p\le y_0}\frac{x}{p^\sigma} \ll x \ll \frac{x^{\frac{1}{\sigma}}}{(\log x)^2}.$$

For $S_2^{\pm}$, by using the Taylor expansion of I_0, we find that

$$S_2^{\pm} \ll \sum_{p>y_1}\frac{x^2}{p^{2\sigma}} \ll \frac{x^2}{y_1^{2\sigma-1}} \ll \frac{x^{\frac{1}{\sigma}}}{(\log x)^2}.$$

Thus, we obtain Lemma 7.6. □

Proof of Proposition 7.3. Let f_0 be a function satisfying $0 < f_0(x) \le \frac{1}{3}$ and $\lim_{x\to+\infty} f_0(x) = 0$. Let $3 \le x_1, x_2 \le X^{\frac{2\sigma}{3}}$ with $|x_1 - x_2| \le (x_1 + x_2)f_0(x_1 + x_2)$. Then we can write

$$I_0\left(\sqrt{K_{\chi,\theta}(p, x_1, x_2)/p^{2\sigma}}\right) = I_0\left(\frac{1}{p^\sigma}|x_1 + x_2\chi(p)|\right).$$

We write

$$\begin{aligned}&\sum_{p\le X}\log I_0\left(\sqrt{K_{\chi,\theta}(p, x_1, x_2)/p^{2\sigma}}\right)\\ &= \left(\sum_{\substack{p\le X\\ \chi(p)=1}} + \sum_{\substack{p\le X\\ \chi(p)=-1}} + \sum_{\substack{p\le X\\ \chi(p)=0}}\right)\log I_0\left(\frac{1}{p^\sigma}|x_1 + x_2\chi(p)|\right) =: S_+ + S_- + S_0,\end{aligned}$$

say. By Lemma 7.6, we find that

$$S_+ = \frac{G(\sigma)(x_1 + x_2)^{\frac{1}{\sigma}}}{2\log(x_1 + x_2)}\left(1 + O\left(\frac{1}{\log(x_1 + x_2)}\right)\right).$$

Also, we find that if $|x_1 - x_2| \le 2$,

$$S_- = \frac{c_-(\sigma,\chi)}{4}(x_1 - x_2)^2 + O\left((x_1 - x_2)^4\right) \ll_{\sigma,\chi} \frac{(x_1 + x_2)^{\frac{1}{\sigma}}}{(\log(x_1 + x_2))^2},$$

and that if $2 < |x_1 - x_2| \le \sqrt{x_1 + x_2}$,

$$\begin{aligned} S_- &\le \frac{G(\sigma)|x_1 - x_2|^{\frac{1}{\sigma}}}{2\log(|x_1 - x_2|)}\left(1 + O\left(\frac{1}{\log|x_1 - x_2|}\right)\right) \ll_{\sigma,\chi} \frac{\sqrt{x_1 + x_2}}{\log(x_1 + x_2)} \\ &\ll \frac{(x_1 + x_2)^{\frac{1}{\sigma}}}{(\log(x_1 + x_2))^2}. \end{aligned}$$

Moreover, if $\sqrt{x_1 + x_2} < |x_1 - x_2| \le (x_1 + x_2) f_0(x_1 + x_2)$, we have

$$S_- \le \frac{G(\sigma)|x_1 - x_2|^{\frac{1}{\sigma}}}{2\log(|x_1 - x_2|)}\left(1 + O_{\chi,\sigma}\left(\frac{1}{\log|x_1 - x_2|}\right)\right) \ll_{\sigma,\chi} \frac{(x_1 + x_2)^{\frac{1}{\sigma}} f_0(x_1 + x_2)^{\frac{1}{\sigma}}}{(\log(x_1 + x_2))^2}.$$

Furthermore, we find that

$$S_0 = \sum_{p|q} \log I_0\left(\frac{|x_1|}{p^\sigma}\right) \ll_\chi x_1 \ll \frac{(x_1 + x_2)^{\frac{1}{\sigma}}}{(\log(x_1 + x_2))^2},$$

where q is the conductor of χ. Thus, we obtain

$$\begin{aligned} &\sum_{p \le X} \log I_0\left(\sqrt{K_{\chi,\theta}(p, x_1, x_2)/p^{2\sigma}}\right) \\ &= \frac{G(\sigma)(x_1 + x_2)^{\frac{1}{\sigma}}}{2\log(x_1 + x_2)}\left(1 + O\left(\frac{1}{\log(x_1 + x_2)} + f_0(x_1 + x_2)^{\frac{1}{\sigma}}\right)\right), \end{aligned}$$

which completes the proof of Proposition 7.3. □

Now, we finish the preparation of the proof of Proposition 7.1, and start the proof of the proposition.

Proof of Proposition 7.1. Let f be a positive valued function with $\lim_{x \to +\infty} f(x) = 0$. We may assume that $f(x) \ge \frac{1}{(\log x)^\sigma}$. Let T be large, and $X = (\log T)^L$. Let V_1 be large with $V_1 \le \frac{a_2(\log T)^{1-\sigma}}{\log\log T}$, where $a_2 = a_2(\chi, \sigma, L)$ is a suitably positive constant to be chosen later, and let V_2 be a positive number with $|V_1 - V_2| \le V_1 f(V_1)$. Put

$$\begin{aligned} &\mathscr{T}(T, V_1, V_2; \chi) \\ &:= \left\{t \in \mathcal{A} \,:\, \mathrm{Re}\sum_{p \le X} \frac{e^{-i\theta}}{p^{\sigma+it}} > V_1\right\} \bigcap \left\{t \in \mathcal{A} \,:\, \mathrm{Re}\sum_{p \le X} \frac{e^{-i\theta}\chi(p)}{p^{\sigma+it}} > V_2\right\}. \end{aligned}$$

Then we find that

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x_1v_1+x_2v_2}\operatorname{meas}(\mathscr{T}(T,v_1,v_2,\chi))dv_1dv_2\\&=\frac{1}{x_1x_2}\int_{\mathcal{A}}\exp\left(x_1\operatorname{Re}\sum_{p\le X}\frac{e^{-i\theta}}{p^{\sigma+it}}+x_2\operatorname{Re}\sum_{p\le X}\frac{e^{-i\theta}\chi(p)}{p^{\sigma+it}}\right)dt.\end{aligned}$$

Therefore, by equation (7.3) and Propositions 7.2, 7.3, we have

$$\begin{aligned}&\frac{1}{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x_1v_1+x_2v_2}\operatorname{meas}(\mathscr{T}(T,v_1,v_2,\chi))dv_1dv_2\\&=\exp\left(\frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{2\log(x_1+x_2)}\left(1+O_{\sigma,\chi}\left(f_0(x_1+x_2)^{\frac{1}{\sigma}}\right)\right)\right)\end{aligned}\tag{7.6}$$

for $3\le x_1,x_2\le b_1(\log T)^{\sigma}$, $|x_1-x_2|\le(x_1+x_2)f_0(x_1+x_2)$, where $b_1=b_1(\chi,\sigma,L_2)$ is the same constant as in Proposition 7.2 in the case $\boldsymbol{a}=(\mathbf{1},\chi)$ with $\mathbf{1}$ the identically one function. Here, we decide the parameters x_1 and x_2 as the solutions of the equations

$$V_1=\frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{4\sigma x_1\log(x_1+x_2)},\quad V_2=\frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{4\sigma x_2\log(x_1+x_2)}.\tag{7.7}$$

Then we can find that these x_1, x_2 satisfy the equations

$$\begin{aligned}x_1&=\frac{2^{\frac{2\sigma-1}{1-\sigma}}A(\sigma)}{1-\sigma}(V_1\log V_1)^{\frac{\sigma}{1-\sigma}}\left(1+O_{\sigma}\left(f(V_1)\right)\right),\\x_2&=\frac{2^{\frac{2\sigma-1}{1-\sigma}}A(\sigma)}{1-\sigma}(V_1\log V_1)^{\frac{\sigma}{1-\sigma}}\left(1+O_{\sigma}\left(f(V_1)\right)\right).\end{aligned}$$

We choose $f_0=Bf$ with $B=B(\sigma)$ a sufficiently large constant, and $a_2=a_2(\sigma,\chi,L)$ sufficiently small. Then we find that these x_1, x_2 satisfy $3\le x_1,x_2\le\frac{b_1}{2}(\log T)^{\sigma}$, $|x_1-x_2|\le\frac{1}{2}(x_1+x_2)f_0(x_1+x_2)$.

Now, we divide the range of the integral of (7.6) as follows:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}=&\int_{V_2(1-\delta)}^{V_2(1+\delta)}\int_{V_1(1-\delta)}^{V_1(1+\delta)}+\int_{-\infty}^{V_2(1-\delta)}\int_{V_1(1-\delta)}^{V_1(1+\delta)}+\int_{V_2(1+\delta)}^{\infty}\int_{V_1(1-\delta)}^{V_1(1+\delta)}\\&+\int_{-\infty}^{\infty}\int_{-\infty}^{V_1(1-\delta)}+\int_{-\infty}^{\infty}\int_{V_1(1+\delta)}^{\infty},\end{aligned}\tag{7.8}$$

where $\delta=K_1f_0(x_1+x_2)^{\frac{1}{2\sigma}}$ with $K_1=K_1(\sigma,\chi)$ a suitably large constant to be chosen later. By equation (7.6), we find that

$$\begin{aligned}&\frac{1}{T}\int_{-\infty}^{V_2(1-\delta)}\int_{V_1(1-\delta)}^{V_1(1+\delta)} e^{x_1v_1+x_2v_2}\operatorname{meas}(\mathscr{T}(T,v_1,v_2,\chi))dv_1dv_2\\&\le\frac{1}{T}e^{\delta x_2V_2(1-\delta)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x_1v_1+x_2(1-\delta)v_2}\operatorname{meas}(\mathscr{T}(T,v_1,v_2;\chi))dv_1dv_2\\&=e^{\delta x_2V_2(1-\delta)}\exp\left(\frac{G(\sigma)(x_1+(1-\delta)x_2)^{\frac{1}{\sigma}}}{2\log(x_1+x_2)}\left(1+O_{\sigma,\chi}\left(f_0(x_1+x_2)^{\frac{1}{\sigma}}\right)\right)\right).\end{aligned}\tag{7.9}$$

Remark that we must confirm that the numbers x_1, $x_2(1-\delta)$ satisfy $3 \le x_1, x_2(1-\delta) \le b_1(\log T)^{\sigma}$ and $|x_1 - x_2(1-\delta)| \le (x_1+x_2) f_0(x_1+x_2)$, but these hold for any sufficiently large T, V_1 depending on σ and χ. Using the formulas $x_2 = x_1 + O(x_1 f_0(x_1+x_2))$ and $(1+r)^{\frac{1}{\sigma}} = 1 + \frac{r}{\sigma} + O(r^2)$ with $|r| \le 1$, we see that (7.9) is equal to

$$\begin{aligned}
&\exp\left(\delta x_2 V_2(1-\delta) + \frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{2\log(x_1+x_2)}\left(1 - \frac{\delta}{2\sigma} + O_{\sigma,\chi}\left(f_0(x_1+x_2)\right)\right)\right)\\
&= \exp\left(\frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{2\log(x_1+x_2)}\left(1 - \frac{\delta^2}{2\sigma} + O_{\sigma,\chi}\left(f_0(x_1+x_2)\right)\right)\right).
\end{aligned}$$

Hence, choosing K_1 as a suitably large constant and using equation (7.6), we obtain

$$\begin{aligned}
&\int_{-\infty}^{V_2(1-\delta)} \int_{V_1(1-\delta)}^{V_1(1+\delta)} e^{x_1 v_1 + x_2 v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2\\
&\le \frac{1}{5} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1 v_1 + x_2 v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2.
\end{aligned}$$

Similarly to the above calculations, we can obtain

$$\begin{aligned}
&\frac{1}{T}\int_{V_2(1+\delta)}^{\infty} \int_{V_1(1-\delta)}^{V_1(1+\delta)} e^{x_1 v_1 + x_2 v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2\\
&\le \frac{1}{T} e^{-\delta x_2 V_2(1+\delta)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1 v_1 + x_2(1+\delta) v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2\\
&= \exp\left(\frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{2\log(x_1+x_2)}\left(1 - \frac{\delta^2}{2\sigma} + O_{\sigma,\chi}\left(f_0(x_1+x_2)\right)\right)\right)\\
&\le \frac{1}{5} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1 v_1 + x_2 v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{T}\int_{-\infty}^{\infty} \int_{-\infty}^{V_1(1-\delta)} e^{x_1 v_1 + x_2 v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2\\
&\le \frac{1}{T} e^{\delta x_1 V_1(1+\delta)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1(1+\delta) v_1 + x_2 v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2\\
&= \exp\left(\delta x_1 V_1(1-\delta) + \frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{2\log(x_1+x_2)}\left(1 - \frac{\delta}{2\sigma} + O_{\sigma,\chi}\left(f_0(x_1+x_2)\right)\right)\right)\\
&\le \frac{1}{5} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1 v_1 + x_2 v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{T}\int_{-\infty}^{\infty} \int_{V_1(1+\delta)}^{\infty} e^{x_1 v_1 + x_2 v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2\\
&\le \frac{1}{5} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_1 v_1 + x_2 v_2} \operatorname{meas}(\mathscr{T}(T, v_1, v_2; \chi)) dv_1 dv_2.
\end{aligned}$$

Hence, by these inequalities and equations (7.6), (7.8), we have

$$\frac{1}{T}\int_{V_2(1-\delta)}^{V_2(1+\delta)}\int_{V_1(1-\delta)}^{V_1(1+\delta)} e^{x_1v_1+x_2v_2}\,\mathrm{meas}(\mathscr{T}(T,v_1,v_2;\chi))dv_1dv_2$$
$$= \exp\left(\frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{2\log(x_1+x_2)}\left(1+O_{\sigma,\chi}\left(f_0(x_1+x_2)^{\frac{1}{\sigma}}\right)\right)\right).$$

By this equation and $\int_{V_2(1-\delta)}^{V_2(1+\delta)}\int_{V_1(1-\delta)}^{V_1(1+\delta)} e^{x_1v_1+x_2v_2} = \exp((x_1V_1+x_2V_2)(1+O(\delta)))$, we find that

$$\frac{1}{T}\mathrm{meas}(\mathscr{T}(T,V_1(1+\delta),V_2(1+\delta);\chi))$$
$$\le \exp\left(-\frac{1-\sigma}{\sigma}\frac{G(\sigma)(x_1+x_2)^{\frac{1}{\sigma}}}{2\log(x_1+x_2)}(1+O(\delta))\right)$$
$$\le \frac{1}{T}\mathrm{meas}(\mathscr{T}(T,V_1(1-\delta),V_2(1-\delta);\chi)).$$

In particular, by equations (7.7), the second term is equal to

$$\exp\left(-(1-\sigma)(x_1V_1+x_2V_2)\left(1+O(\delta)\right)\right),$$

and so we have

$$\exp\left(-(1-\sigma)(x_1V_1+x_2V_2)(1+\delta)\left(1+O(\delta)\right)\right)$$
$$\le \frac{1}{T}\mathrm{meas}(\mathscr{T}(T,V_1,V_2;\chi)) \le \exp\left(-(1-\sigma)(x_1V_1+x_2V_2)(1-\delta)\left(1+O(\delta)\right)\right).$$

Therefore, we obtain

$$\frac{1}{T}\mathrm{meas}(\mathscr{T}(T,V_1,V_2;\chi)) = \exp\left(-2^{\frac{\sigma}{1-\sigma}}A(\sigma)V_1^{\frac{1}{1-\sigma}}(\log V_1)^{\frac{\sigma}{1-\sigma}}(1+O(\delta))\right),$$

where $\delta = o(1)$ as $V_1 \to +\infty$. By this equation and the estimate $\mathrm{meas}(A) \ll T\exp(-b_3\log T/\log\log T)$, when a_2 is suitably small, we obtain

$$\frac{1}{T}\mathrm{meas}(\mathscr{S}(T,V_1,V_2;\chi)) = \frac{1}{T}\left(\mathrm{meas}(\mathscr{T}(T,V_1,V_2;\chi)) + O\left(\frac{1}{T}\mathrm{meas}(A)\right)\right)$$
$$= \exp\left(-2^{\frac{\sigma}{1-\sigma}}A(\sigma)V_1^{\frac{1}{1-\sigma}}(\log V_1)^{\frac{\sigma}{1-\sigma}}\left(1+o(1)\right)\right)$$

as $V_1 \to +\infty$. This completes the proof of Proposition 7.1. □

7.4 Proof of dependence of $\zeta(s)$ and $L(s,\chi)$

Proof of Theorem 7.1. Let T be large, and $X = (\log T)^L$ with $L = \frac{10}{2\sigma-1}$. We can use Proposition 6.6 for the Riemann zeta-function and Dirichlet L-functions. Therefore, using the proposition, we obtain

$$\frac{1}{T}\int_T^{2T}\left|\log F(\sigma+it) - \sum_{2\le n\le X}\frac{\Lambda_F(n)}{n^{\sigma+it}\log n}\right|^{2k}dt \le A_2^k k^{4k}T^{(1-2\sigma)\delta_F} + A_2^k k! X^{k(1-2\sigma)}$$

for $1 \leq k \leq \frac{\delta_F}{L} \frac{\log T}{\log\log T}$, where $F(s)$ is $\zeta(s)$ or $L(s,\chi)$. By this inequality, we can easily find that there exists a set $C \subset [T, 2T]$ such that $\mathrm{meas}([T,2T] \setminus C) \leq T \exp\left(-c \log T/\log\log T\right)$, and for all $t \in C$,

$$\left| \log \zeta(\sigma + it) - \sum_{p \leq X} \frac{1}{p^{\sigma+it}} \right| \leq 1 + c,$$

$$\left| \log L(\sigma + it, \chi) - \sum_{p \leq X} \frac{\chi(p)}{p^{\sigma+it}} \right| \leq 1 + c(\chi).$$

Here, $c = \sum_{p^k, k\geq 2} \frac{\Lambda(p^k)}{p^{k\sigma}(\log p^k)}$, and $c(\chi) = \left|\sum_{p^k, k\geq 2} \frac{\Lambda(p^k)\chi(p^k)}{p^{k\sigma}(\log p^k)}\right|$. In particular, when a_1 is sufficiently small, it follows that

$$\mathrm{meas}([T,2T] \setminus C) \leq T \exp\left(-2 \cdot 2^{\frac{\sigma}{1-\sigma}} A(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\sigma}{1-\sigma}}\right). \tag{7.10}$$

Therefore, the right hand side is $\leq K$ with $K = K(\sigma)$ a positive constant. Then, it holds that

$$\begin{aligned} &\mathrm{meas}\left(C \cap \mathcal{S}(T, (V_1 + K, V_2 + K); \chi, \sigma, \theta)\right) \\ &\leq \mathrm{meas}\left\{t \in C \ : \ \mathrm{Re}\, e^{-i\theta} \log \zeta(\sigma + it) > V_1 \text{ and } \mathrm{Re}\, e^{-i\theta} \log L(\sigma + it, \chi) > V_2\right\} \\ s &\leq \mathrm{meas}\left(C \cap \mathcal{S}(T, (V_1 + K, V_2 + K); \chi, \sigma, \theta)\right). \end{aligned}$$

Hence, by these inequalities and Proposition 7.1, we have

$$\begin{aligned} &\frac{1}{T} \mathrm{meas}\left\{t \in C \ : \ \mathrm{Re}\, e^{-i\theta} \log \zeta(\sigma + it) > V_1 \text{ and } \mathrm{Re}\, e^{-i\theta} \log L(\sigma + it, \chi) > V_2\right\} \\ &= \exp\left(-2^{\frac{\sigma}{1-\sigma}} A(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\sigma}{1-\sigma}} (1 + o(1))\right) \end{aligned}$$

as $V_1 \to +\infty$. Thus, by this equation and inequality (7.10), we obtain Theorem 7.1. □

Bibliography

[1] M. Avdispahić and L. Smajlović, On the Selberg orthogonality for automorphic L-functions, *Arch. Math.* **94** (2010), 147–154.

[2] R. J. Backlund, Über die Beziehung zwischen Anwachsen und Nullstellen der Zetafunktion, *Öfversigt Finska Vetensk. Soc.* **61** (1918/1919) no. 9.

[3] R. C. Baker, G. Harman, and J. Pintz, The difference between consecutive primes, II, *Proc. Lond. Math. Soc.* (3) **83** (2001), 532–562.

[4] R. Balasubramanian and R. Raghunathan, Beyond the extended Selberg class: $1 < d_F < 2$, preprint, `arXiv:2011.07525`.

[5] S. Bettin, V. Chandee, and M. Radziwiłł, The mean square of the product of the Riemann zeta-function with Dirichlet polynomials *J. Reine Angew. Math.* **729** (2017), 51–79.

[6] H. Bohr, Zur Theorie der Riemann'schen Zeta-funktion im kritischen Streifen, *Acta Math.* **40** (1916), 67–100.

[7] H. Bohr and R. Courant, Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion, *J. Reine Angew. Math.* **144** (1914), 249–274.

[8] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion, Erste Mitteilung, *Acta Math.* **54** (1930), 1–35; Zweite Mitteilung, ibid. **58** (1932), 1–55.

[9] E. Bombieri and D. A. Hejhal, On the distribution of zeros of linear combinations of Euler products, *Duke Math. J.* **80** no.3 (1995), 821–862.

[10] A. Bondarenko and K. Seip, Large greatest common divisor sums and extreme values of the Riemann zeta-function, *Duke Math. J.* **166** no.9 (2017), 1685–1701.

[11] A. Bondarenko and K. Seip, Extreme values of the Riemann zeta function and its argument, *Math. Ann.* **372** (2018), 999–1015.

[12] E. Carneiro, V. Chandee, and M. B. Milinovich, Bounding $S(t)$ and $S_1(t)$ on the Riemann Hypothesis, *Math. Ann.* **356** (2013), 939–968.

[13] T. Chatterjee and S. Gun, On the zeros of generalized Hurwitz zeta functions, *J. Number Theory* **145** (2014), 352–361.

[14] A. Chirre, Extreme values for $S_n(\sigma, t)$ near the critical line, *J. Number Theory* **200** (2019), 329–352.

[15] A. Chirre and K. Mahatab, Large values of the argument of the Riemann zeta-function and its iterates, preprint, `arXiv:2006.04288`.

[16] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein and N.C. Snaith, Integral moments of L-functions, *Proc. Lond. Math. Soc.* (3) **91** (2005), 33–104.

[17] J. B. Conrey and A. Ghosh, On mean values of the zeta-function, *Mathematika* **31** (1984), 159–161.

[18] J. B. Conrey and A. Ghosh, Mean values of the Riemann zeta-function III, *Proceedings of the Amalfi Conference on Analytic Number Theory* pp. 35–59, 1992.

[19] J. B. Conrey and A. Ghosh, On the Selberg class of Dirichlet series: small degrees, *Duke Math. J.* **72** no.3 (1993), 673–693.

[20] H. Cramér, Über die Nulistellen der Zetafunktion, *Math. Z.* **2** (1918), no. 3-4, 237–241.

[21] H. Davenport, *Multiplicative number theory.* Third edition. Revised and with a preface by Hugh L. Montgomery, Graduate Texts in Mathematics, 74. Springer-Verlag, New York, 2000.

[22] K. Endo and S. Inoue, On the value distribution of iterated integrals of the logarithm of the Riemann zeta-function I: denseness, to appear in *Forum Math*.

[23] K. Endo, S. Inoue, and M. Mine, On the value distribution of iterated integrals of the logarithm of the Riemann zeta-function II: probabilistic aspects, preparation.

[24] D. W. Farmer, S. M. Gonek, and C. P. Hughes, The maximum size of L-functions, *J. Reine Angew. Math.* **609** (2007), 215–236.

[25] K. Ford, B. Green, S. Konyagin, and T. Tao, *Ann. of Math.* (2) **183** (2016), 935–974.

[26] K. Ford, B. Green, S. Konyagin, J. Maynard, and T. Tao, Long gaps between primes, *J. Amer. Math. Soc.* **31** no.1 (2018), 65–105.

[27] K. Ford and A. Zaharescu, Unnormalized differences between zeros of L-functions, *Compos. Math.* (2) **151** (2015), 230–252.

[28] A. Fujii, On the zeros of Dirichlet L-functions. I, *Trans. Amer. Math. Soc.* **196** (1974), 225–235.

[29] A. Fujii, On the zeros of the Riemann zeta-function, *Comment. Math. Univ. St. Paul.* **51** (2002), no.1, 1–17.

[30] R. Garunkštis and J. Steuding, On the roots of the equation $\zeta(s) = a$, *Abh. Math. Semin. Univ. Hambg.* **84** (2014), 1–15.

[31] D. A. Goldston and S. M. Gonek, A note on $S(t)$ and the zeros of the Riemann zeta-function, *Bull. Lond. Math. Soc.* **39** (2007), 482–486.

[32] S. M. Gonek, On negative moments of the Riemann zeta-function, *Mathematika* **36** (1989), 71–88.

[33] S. M. Gonek, C. P. Hughes, and J. P. Keating, A hybrid Euler-Hadamard product for the Riemann zeta-function, *Duke Math. J.* no.3 **136** (2007), 507–549.

[34] A. Good, The square mean of Dirichlet series associated with cusp forms, *Mathematika* **29** no. 2 (1982), 278–295.

[35] A. Granville and K. Soundararajan, The distribution of values of $L(1, \chi_d)$, *Geom. Funct. Anal.* **13**, no. 5 (2003) 992–1028.

[36] A. Granville and K. Soundararajan, Extreme values of $|\zeta(1 + it)|$, *The Riemann Zeta Function and Related Themes: Papers in Honor of Professor K. Ramachandra*, Ramanujan Math. Soc. Lecture Notes Series 2, Mysore, 2006, pp.65–80.

[37] G. H. Hardy and J. E. Littlewood, Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes, *Acta Math.* **41** (1916), 119–196.

[38] G. Harman and K. Matsumoto, Discrepancy estimates for the value-distribution of the Riemann zeta-function, *J. Lond. Math. Soc.* (2) **50** (1994), 17–24.

[39] A. J. Harper, Sharp conditional bounds for moments of the Riemann zeta function, preprint, `arXiv:1305.4618`.

[40] T. Hattori and K. Matsumoto, A limit theorem for Bohr-Jessen's probability measures of the Riemann zeta-function, *J. Reine Angew. Math.* **507** (1999), 219–232.

[41] W. Heap, Moments of the Dedekind zeta function and other non-primitive L-functions, *Math. Proc. Cambridge Philos. Soc.* 2013, 1–29.

[42] W. Heap, M. Radziwiłł, and K. Soundararajan, Sharp upper bounds for fractional moments of the Riemann zeta-function, *Quart. J. Math.* **70** (2019), 1387–1396.

[43] W. Heap and K. Soundararajan, Lower bounds for moments of zeta and L-functions revisited, preprint, `arXiv:2007.13154`.

[44] D. R. Heath-Brown, Fractional moments of the Riemann zeta-function, *J. Lond. Math. Soc.* (2) **24** (1981), 65–78.

[45] C. Heuberger and S. Kropf, Higher dimensional quasi-power theorem and Berry-Esseen inequality, *Monatsh. Math.* **187** (2018), 293–314.

[46] P.-H. Hsu and P.-J. Wong, On Selberg's central limit theorem for Dirichlet L-functions, to appear in *J. Théor. Nombres Bordeaux*.

[47] H. H. Hwang, Large deviations for combinatorial distributions. I: Central limit theorems, *Ann. Appl. Probab.*, **6** (1996), no. 1, 297–319.

[48] A. E. Ingham, Mean-value theorems in the theory of the Riemann zeta-function, *Proc. Lond. Math. Soc.* (2) **27** (1927), no.4, 273–300.

[49] A. E. Ingham, On the difference between consecutive primes, *Quart. J. Math.* **8** (1937), 255–266.

[50] S. Inoue, On the logarithm of the Riemann zeta-function and its iterated integrals, preprint, `arXiv:1909.03643`.

[51] S. Inoue, Extreme values for iterated integrals of the logarithm of the Riemann zeta-function, preprint, `arXiv:2009.04099`.

[52] S. Inoue and J. Li, Joint value distribution of L-functions on the critical line, preparation.

[53] S. Inoue and J. Li, Joint extreme values of $\zeta(s)$ and $L(s, \chi)$ in the strip $\frac{1}{2} < \sigma < 1$, preparation.

[54] B. Jessen and A. Wintner, Distribution functions and The Riemann zeta function, *Trans. Amer. Math. Soc.* **38**, no. 1, 48–88.

[55] D. Joyner, *Distribution Theorems of L-functions*, Longman Scientific & Technical, 1986.

[56] M. Jutila, On the value distribution of the zeta-function on the critical line, *Bull. Lond. Math. Soc.* **15** (1983), 513–518.

[57] J. Kaczorowski and A. Perelli, On the structure of the Selberg class, I: $0 \le d \le 1$, *Acta Math.* **182** (1999), 207–241.

[58] J. Kaczorowski and A. Perelli, On the structure of the Selberg class, IV: basic invariants, *Acta Arith.* **104** (2002), no.2, 97–116.

[59] J. Kaczorowski and A. Perelli, On the structure of the Selberg class, V: $1 < d < 5/3$, *Invent. Math.* **150** (2002), 485–516.

[60] J. Kaczorowski and A. Perelli, On the structure of the Selberg class, VII: $1 < d < 2$, *Ann. of Math.* **173** (2011), 1397–1441.

[61] J. Kaczorowski and A. Perelli, Classification of L-functions of degree 2 and conductor 1, preprint, `arXiv:2009.12329`.

[62] A. A. Karatsuba and S. M. Voronin, The Riemann zeta-function, Translated from the Russian by Neal Koblitz, De Gruyter Expositions in Mathematics **5**, Walter de Gruyter & Co., Berlin, 1992.

[63] J. P. Keating and N. C. Snaith, Random matrix theory and L-functions at $s = 1/2$, *Comm. Math. Phys.* **214** (2000), 91–110.

[64] H. Kim, Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. With appendix 1 by D. Ramakrishnan and appendix 2 by Kim and P. Sarnak, *J. Amer. Math. Soc.* **16** (2003), no.1, 139–183.

[65] E. Kowalski and A. Nikeghbali, Mod-Gaussian convergence and the value distribution of $\zeta(\frac{1}{2}+it)$ and related quantities, *J. Lond. Math. Soc.* (2) **86** (2012), 291–319.

[66] Y. Lamzouri, On the distribution of extreme values of zeta and L-functions in the strip $\frac{1}{2} < \sigma < 1$, *Int. Math. Res. Not. IMRN* **2011**, no.23, 5449–5503.

[67] Y. Lamzouri, S. Lester, and M. Radziwiłł, Discrepancy bounds for the distribution of the Riemann zeta-function and applications, *J. Anal. Math.* **139** no.2 (2019), 453–494.

[68] A. Laurinčikas, A limit theorem for the Riemann zeta-function on the critical line II, *Litovsk. Mat. Sb.* **27** (3) (1987), 91–110.

[69] Y. Lee, T. Nakamura, and Ł. Pańkowski, Selberg's orthonormality conjecture and joint universality of L-functions, *Math. Z.* **286** (2017), 1–18.

[70] E. Lindelöf, Quelques remarques sur la croissance de la function $\zeta(s)$, *Bull. Sci. Math.* **32** (1908) 341–356.

[71] J. E. Littlewood, On the zeros of the Riemann zeta-function, *Proc. Camb. Phil. Soc.* **22** (1924), 295–318.

[72] J. Liu, Y. Wang and Y. Ye, A proof of Selberg's orthogonality for automorphic L-functions, *Manuscripta Math.* **118** (2005), 135–149.

[73] J. Liu and Y. Ye, Selberg's orthogonality conjecture for automorphic L-functions, *Amer. J. Math.* **127** no.4 (2005), 837–849.

[74] J. Liu and Y. Ye, Zeros of automorphic L-functions and noncyclic base change Number Theory, Dev. Math., vol. 15, Springer, New York (2006), 119–152.

[75] M. Loève, *Probability Theory I*, Graduate Texts in Mathematics, Springer-Verlag, 1977.

[76] W. Luo, Zeros of Hecke L-functions associated with cusp forms, *Acta Arith.* **71** (1995), 139–158.

[77] K. Matsumoto, Value-distribution of zeta-functions, *Analytic Number Theory*, Proc. Japanese-French Sympos. held in Tokyo, K. Nagasaka and E. Fouvry (eds.), Lect. Notes in Math. **1434**, Springer-Verlag, 1990, pp. 178–187.

[78] K. Matsumoto, On the theory of M-function, RIMS Kôkyûroku, *Profinite monodromy, Galois representations, and complex functions* **2120**, M. Kaneko (ed.), 2019, pp.153–165, also available at `arXiv:1810.01552`.

[79] K. Matsumoto and Y. Umegaki, On the density function for the value-distribution of automorphic L-functions, *J. Number Theory* **198** (2019), 176–199.

[80] K. Matsumoto and Y. Umegaki, On the value-distribution of symmetric power L-functions, *Proceedings of Ropar Conference*, RMS-Lecture Notes Series no.26, 2020, pp.147–167.

[81] J. Maynard, Small gaps between primes, *Ann. of Math.* (2) **181** (2015), 383–413.

[82] J. Maynard, Large gaps between primes, *Ann. of Math.* (2) **183** (2016), 915–933.

[83] M. B. Milinovich and C. L. Turnage-Butterbaugh. Moments of products of automorphic L-functions, *J. Number Theory* **139** (2014), 175–204.

[84] M. Mine, On certain mean values of logarithmic derivatives of L-functions and the related density functions, *Funct. Approx. Comment. Math.* **61** (2019), 179–199.

[85] M. Mine, personal communication, January 2020.

[86] H. L. Montgomery, Extreme values of the Riemann zeta function, *Comment. Math. Helvetici* **52** (1977), 511–518.

[87] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University press, 2007.

[88] Y. Motohashi, A note on the mean value of the Dedekind zeta function of the quadratic field, *Math. Ann.* **188** (1970), 123–127.

[89] M. R. Murty, *Problems and Analytic Number Theory*, Second Edition, Graduate Texts in Mathematics, Vol. 206, Springer, 2008.

[90] H. Nagoshi and J. Steuding, Universality for L-functions in the Selberg class, *Lith. Math. J.* **50** no.3 (2010), 293–311.

[91] J. Najnudel, Exponential moments of the argument of the Riemann zeta function on the critical line, *Mathematika* **66** (2020), 612–621.

[92] A. Perelli, A survey of the Selberg class of L-functions. II, *Riv. Mat. Univ. Parma* (7) **3*** (2004), 83–118.

[93] A. Perelli, A survey of the Selberg class of L-functions. I, *Milan J. Math.* **73** (2005), 19–52.

[94] H. S. A. Potter, The mean values of certain Dirichlet series I, *Proc. Lond. Math. Soc.* (2) **46** (1940), 467–478.

[95] M. Radziwiłł, Large deviations in Selberg's central limit theorem, `arXiv:1108.5092`.

[96] M. Radziwiłł, The 4.36th Moment of the Riemann zeta-function, *Int. Math. Res. Not. IMRN* **2012** (2011), 4245–4259.

[97] M. Radziwiłł and K. Soundararajan, Continuous lower bounds for moments of zeta and L-functions, *Mathematika* **59** (2013), 119–128.

[98] M. Radziwiłł and K. Soundararajan, Selberg's central limit theorem for $\log|\zeta(\frac{1}{2}+it)|$, *Enseign. Math.* (2) **63** (2017), 1–19.

[99] K. Ramachandra, Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series. II, *Hardy-Ramanujan J.* **3** (1980), 1–24.

[100] K. Ramachandra, Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series. III, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **5** (1980), 145–158.

[101] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsber. Berlin. Akad.* (1859), 671–680.

[102] Z. Rudnick and P. Sarnak, Zeros of principal L-functions and random matrix theory, *Duke Math. J.* no.2 **81** (1996), 269–322.

[103] S. M. Sadikova, On two-dimensional analogues of a inequality of K. G. Esseen and their application to the Central limit theorem, *Teor. Veroyatnost.*, 369–380, 1966.

[104] A. Sankaranarayanan and J. Sengupta. Zero-density estimate of L-functions attached to Maaß forms, *Acta Arith.* (3) **127** (2007), 273–284.

[105] A. Selberg, On the zeros of Riemann's zeta-function, the Number of Zeros of $\zeta(s)$ in the Strip $0 < t < T$, Avhandl. Norske Vid.-Akad. Olso I. Mat.-Naturv. Kl., no.1; Collected Papers, Vol. 1, New York: Springer Verlag. 1989, 85–141.

[106] A. Selberg, On the remainder formula for $N(T)$, the Number of Zeros of $\zeta(s)$ in the Strip $0 < t < T$, Avhandl. Norske Vid.-Akad. Olso I. Mat.-Naturv. Kl., no.1; Collected Papers, Vol. 1, New York: Springer Verlag. 1989, 179–203.

[107] A. Selberg, Contributions to the theory of the Riemann zeta-function, Avhandl. Norske Vid.-Akad. Olso I. Mat.-Naturv. Kl., no.1; Collected Papers, Vol. 1, New York: Springer Verlag. 1989, 214–280.

[108] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, *Proceedings of the Amalfi Conference on Analytic Number Theory*, pp.367–385, 1992.

[109] K. Soundararajan, Mean-values of the Riemann zeta-function, *Mathematika* **42** (1995), 158–174.

[110] K. Soundararajan, Extreme values of zeta and L-functions, *Math. Ann.* **342** (2008), 467–486.

[111] K. Soundararajan, Moments of the Riemann zeta-function, *Ann. of Math.* (2) **170** (2009), 981–993.

[112] J. Steuding, *Value-distribution of L-functions*, Lecture Notes in Mathematics, **1877**, Springer, 2007.

[113] G. Tenenbaum, Sur la distribution conjointe des deux fonctions "nombre de facteurs premiers," *Aeq. Math.* **35** (1988), 55–68.

[114] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Second Edition, Edited and with a preface by D. R. Heath-Brown, The Clarendon Press, Oxford University Press, New York, 1986.

[115] B. Topacogullari, The fourth moment of individual Dirichlet L-functions on the critical line, `arXiv.1909.11517`, to appear in *Math. Z.*

[116] K. M. Tsang, *The distribution of the values of the Riemann zeta function*, PhD thesis, Princeton University, Princeton, NJ, 1984.

[117] K. M. Tsang, Some Ω-theorems for the Riemann zeta-function, *Acta Arith.* **46** (1986), no.4, 369–395.

[118] K. M. Tsang, The large values of the Riemann zeta-function, *Mathematika* **40** (1993), 203–214.

[119] S. M. Voronin, Theorem on the 'universality' of the Riemann zeta-function, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **39** (1975), 475–486 (in Russian). English transl. : *Math. USSR, Izv.* **9** (1975), 443–445.

[120] S. M. Voronin, Ω-theorems in the theory of the Riemann zeta-function, (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **52** (1988), no. 2, 424–436 ; translation in Math. USSR–Izv. **32** (1989), no. 2, 429–442.

[121] G. N. Watson, *A treatise on the theory of Bessel functions*, second ed., Cambridge University Press, London, 1980.

[122] Q. Zhang, Integral mean values of modular L-functions, *J. Number Theory* **115** (2005), no. 1, 100—122.

[123] Q. Zhang, Integral mean values of Maass L-functions, *Int. Math. Res. Not.* **2006**, Art. ID 41417, 19 pp.

[124] Y. Zhang, Bounded gaps between primes, *Ann. of Math.* (2) **179** (2014), 1121–1174.